The Moist Available Energy of a Conditionally Unstable Atmosphere

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ABSTRACT

The concept of "moist available energy," defined by Lorenz is applied to study the potential energy available for cumulus convection in a conditionally unstable atmosphere. Lorenz's parcel-moving algorithm to determine the moist available energy is shown to be impractical for this problem, and a new algorithm based on mass exchanges is proposed. This new algorithm is applied to the GATE data, to determine the time variations of the moist available energy of the observed tropical atmosphere. Implications for cumulus parameterization are discussed.

1. Introduction

Lorenz (1955) defined the available potential energy (APE) of the atmosphere as the difference between the actual total enthalpy and the minimum total enthalpy that could be achieved by rearranging the mass under reversible adiabatic processes.

This definition can be understood by considering the conservation equation for the total energy of the atmosphere (including the internal, potential, and kinetic energies). According to this equation, the sum of the kinetic energy per unit mass and the enthalpy per unit mass changes in time due to redistribution of mass within the atmosphere, and also due to energy sources and sinks such as radiation, latent heating, and surface exchanges. Here the enthalpy per unit mass is defined as the product of the temperature and the specific heat at constant pressure. Of course, when the total energy equation is integrated over the entire atmosphere, the redistribution term drops out. In the absence of energy sources or sinks, therefore, we find that

$$\frac{\partial}{\partial t}(K+H)=0, \tag{1.1}$$

where K is the total kinetic energy, and H is the total enthalpy.

The total enthalpy can be varied by adiabatically redistributing mass over the globe, and Lorenz pointed out that there exists a particular mass distribution for which H is minimized. According to (1.1), K is maximized for this same state, which Lorenz called the *reference state*. The APE is then defined as the difference between the total enthalpy of the given state and that of the reference state. It thus represents the portion of the nonkinetic energy that is available for conversion into kinetic energy under reversible adiabatic processes.

Lorenz (1978, 1979) extended the concept of APE to the moist atmosphere, by recognizing that moist adiabatic processes are, in fact, adiabatic rather than diabatic. From this point of view, the latent heat of water vapor is a portion of the enthalpy. He presented both graphical and digital algorithms for determining the MAE. The latter was based on rearranging discrete parcels from their configuration in the given state to that in the reference state. He showed that the moist available energy (MAE) is never less than the dry available energy (DAE, synonymous with the dry APE), although the DAE represents the bulk of the total available energy in the global atmosphere. He demonstrated that the MAE increases rapidly as the temperature increases, for fixed relative humidity. He was also able to define a specific MAE, i.e. the contribution of a particular parcel to the global MAE.

The fact that the MAE is an upper bound on the amount of kinetic energy that can be generated by any circulation whatsoever is both a strength and a weakness for the concept of available energy. It is a strength because the concept is completely general. It is a weakness because it is possible that no dynamically realizable circulation can extract all of the available energy.

The concept of available energy is usually applied to statically stable atmospheric states, but it is equally applicable to statically unstable systems. When the atmosphere is everywhere statically stable in the dry sense, the DAE is entirely due to the existence of temperature gradients along isobaric surfaces, i.e., the DAE resides in the horizontal rather than the vertical structure of the atmosphere. The reference state can be reached by rearranging the mass of the system so that the pressure is uniform along isentropic surfaces. The vertical ordering of the isentropic surfaces does not change during this process. For a dry statically unstable system, on the other hand, the reference state can only be reached by vertically reordering the isentropes; the potential temperature decreases upward in the given state, but increases upward in the reference state. As an example, consider a simple system containing two parcels of equal mass. In the given state, parcels with potential temperatures θ_1 and θ_2 reside at pressures p_1 and p_2, respectively. We assume that $\theta_1 < \theta_2$ and $p_1 < p_2$, so that the given state is statically unstable. The enthalpy per unit mass of parcel i is $c_p \theta_i (p_i/p_0)^\kappa$, where p_0 is the reference pressure used in the definition of the potential temperature, and κ is Poisson's constant. If the parcels are interchanged ("swapped"), so that parcel number two goes to pressure p_1 and vice versa, the change in the total enthalpy per unit mass is $c_p (\theta_1 - \theta_2) [(p_2/p_0)^\kappa - (p_1/p_0)^\kappa]$, which is negative. This implies that the total enthalpy is minimized by the swap; the final state is the reference state, and the change in enthalpy given above is the available potential energy of the system.

The moist atmospheres used as examples by Lorenz (1978, 1979) were statically stable everywhere; for such atmospheres, the MAE resides in the horizontal rather than the vertical structure. Lorenz did point out, however, that the existence of conditional instability represents a supply of MAE, and complicates the design of algorithms to determine the MAE.

Consider an idealized atmosphere which is horizontally uniform but conditionally unstable. Since the dry static stability is positive, the DAE is zero, but the MAE is positive. A portion of the air in the reference state must be saturated when the given state is conditionally unstable. The purpose of this paper is to investigate how the MAE for a conditionally unstable sounding compares with the "positive area on the tephigram" and other conventional measures of the potential energy available for cumulus convection.

2. A preliminary look at GATE soundings

We have computed the MAE of GATE soundings, using the moist thermodynamic formulae given by Lorenz (1979). We divided each sounding in to N parcels; initially we used various values of N in the range 9 to 37. We tested several algorithms for finding the reference state for which H is minimized. One of the algorithms is a "swapper", which checks the changes in H that are produced by swapping pairs of parcels, and makes swaps that reduce H until it cannot be reduced further by additional swaps. The second is a "lift-shift" algorithm, which mimics cumulus convection by lifting parcels from low levels to high levels, while shifting all the intervening parcels down by one level. If the resulting sounding has a lower H, it is adopted as a step towards the reference sounding, and further lift-shift possibilities are checked. When no possible lift-shift reduces H, the algorithm terminates. In some cases, both swapper and lift-shift can fail to find the true reference state. For this reason, we also tested a "brute force" method, which checks all possible permutations of the parcels and selects the permutation for which H is minimized. This approach is feasible only when the number of parcels under consideration is less than about 10.

To our initial surprise, the various GATE soundings tested were found to have zero MAE, i.e. the given states were the same as the reference states. An example sounding, for August 30 at 1800 GMT, is given in Fig. 1. It is clear that conditional instability exists, so that MAE *must* be present. At first, a programming error was suspected.

By artificially increasing the specific humidity of the sounding in Fig. 1 to 101% of saturation at every level, and thus increasing the MAE drastically beyond that present in the unmodified sounding, we succeeded in detecting MAE with the lift-shift algorithm.

After further investigation, we determined that the reason that the unmodified GATE soundings were found to have no MAE is straightforward: the parcels that were being moved around, which ranged in size from about 30 mb to about 100 mb, were too massive. The problem with such massive parcels can be understood through the following argument. In real convective situations, the cumulus mass flux is typically on the order of 100 mb per day (e.g. Yanai *et al.*, 1973). This means that 100 mb of boundary-layer mass is carried upward in cumulus towers in a day, while the free-atmospheric environment sinks by a corresponding amount. Experiments with numerical cloud models show that the conditional instability present in real soundings can be released by convection in an hour or so, however (e. g., Soong and Tao, 1980; Dudhia and Moncrieff, 1987; Krueger, 1988). This means that, in the absence of a forcing mechanism to maintain the instability, only about 4 mb (1/24 of 100 mb) of mass can rise to the tropopause before the MAE is exhausted.

By trial and error we have determined that, for the sounding shown in Fig. 1, the minimum number of parcels needed to detect MAE with the lift-shift algorithm is 96, corresponding to a parcel mass of about 10 mb. Several hundred parcels would be needed to obtain an accurate result.

3. A practical algorithm

It is obviously impractical to divide each sounding into several hundred parcels; an alternative approach is as follows. We divide the given sounding into N layers, where N is a manageable number of order 10. Let the mass of layer i be denoted by m_i. Imagine that a system of "pipes" is set up, connecting each layer of the sounding with every other layer. Each pipe allows mass to be transferred adiabatically and reversibly in a single direction. Let the amount of mass transferred *from* layer i *to* layer j be M_{ij}. We will use a prime to denote a variable in the reference state. For an intensive variable A that is conserved under adiabatic reversible processes, we can write

$$m'_i A'_i = m_i A_i + \sum_{j=1}^{N} M_{ji} \hat{A}_j - \sum_{j=1}^{N} M_{ij} \hat{A}_i , \tag{3.1}$$

where $\hat{A}$ denotes a "source" value of A that must be specified. The source value of A represents a typical value of A in the layer from which mass is removed. When mass flows from layer j to layer i, the source value should be characteristic of layer j, and vice versa. This is the reason for stipulating that

Three possible choices of $\hat{A}$ are considered in this paper:

$$\hat{A}_j = A_j , \tag{3.2a}$$

$$\hat{A}_j = A'_j , \tag{3.2b}$$

$$\hat{A}_j = (A_j + A'_j) / 2 \tag{3.2c}$$

we shall refer to these as the "forward," "backward," and "trapezoidal" schemes, respectively. The trapezoidal scheme has the advantage that it is reversible, which is in accord with our wish to consider reversible adiabatic processes.

Putting A ≡1 in (3.1) gives a mass conservation equation:

$$m'_i = m_i + \sum_{j=1}^{N} (M_{ji} - M_{ij}). \tag{3.3}$$

We allow only "eddy" mass exchange, so that

$$m_i' = m_i , \tag{3.4}$$

$$\sum_{j=1}^{N} (M_{ji} - M_{ij}) = 0. \tag{3.5}$$

Of course, the diagonal elements M_{ii} can be set to zero. Using (3.3-3.5), we can simplify (3.1) to

$$\left(m_i + \sum_{j=1}^{N} \frac{1}{2} M_{j,i} \right) (A'_i - A_i) - \sum_{j=1}^{N} \frac{1}{2} M_{j,i} (A'_j - A_j) = \sum_{j=1}^{N} M_{j,i} (A_j - A_i). \tag{3.6}$$

These equations can be used to evaluate the MAE of conditionally unstable soundings. If a set of M_{ij}'s is specified, we can apply (3.6) to determine the changes in the entropy and total mixing ratio of the air. The total enthalpy of the new state can then be evaluated and compared with that of the given state. *We seek the matrix M_{ij} such that the total enthalpy of the final state is minimized, subject to the constraint (3.5).* Of course, we must also restrict ourselves to non-negative *M*'s.

With this method, as with Lorenz's parcel-moving method, a conditionally unstable given state corresponds to a reference state in which some portion of the air is saturated. As a result, the amount of mass lifted from lower levels to upper levels may have to attain a finite minimum value before any decrease in the total enthalpy occurs. It should also be noted that (3.6) can only be used to determine the *average* entropy and total mixing ratio of the adjusted state; the adjusted enthalpy has to be based on these average values.

As an example, consider the simple case N=2. Then (3.5) implies that

$$M_{1,2} = M_{2,1} = M. \tag{3.7}$$

After some manipulation, we find from (3.6) that

$$A'_1 - A_1 = \frac{m_2 M(A_2 - A_1)}{m_1 m_2 + \frac{1}{2} M(m_1 + m_2)}, \tag{3.8}$$

$$A'_2 - A_2 = \frac{- m_1 M(A_2 - A_1)}{m_1 m_2 + \frac{1}{2} M(m_1 + m_2)}. \tag{3.9}$$

For $M \rightarrow \infty$, and if $m_1 = m_2$, (3.8-9) imply that the parcels exchange places: parcel number 1 takes property A_2, and vice versa. The trapezoidal scheme thus gives us a parcel swapper in the limit of large M. This is an attractive property of the scheme.

We have applied these equations to the idealized two-level "sounding"given in Table 1. In the table, p is pressure, T is temperature, q is mixing ratio, θ is the potential temperature, θ_e is equivalent potential temperature, and θ_{es} is saturation equivalent potential temperature. The two layers are assumed to be of equal thickness, with surface pressure 1000 mb and top pressure 100 mb. Both levels are nearly saturated. The relative humidity at 775 mb is so high that even slight lifting is sufficient to produce condensation. Since the θ_e of the lower layer exceeds the θ_{es} of the upper layer, the sounding is conditionally unstable. Following Lorenz (1979), we assume that entropy and total mixing ratio are conserved. We tried various values of M/m_2., increasing from zero by steps of 0.01. The results are shown in Fig. 2. The upper level becomes saturated with even a one percent injection of air from the lower level. A minimum of the total enthalpy occurs for $M/m_2 = 0.11$; this corresponds to the reference state. The MAE per unit mass is 13.1 J kg^{-1}. In the limit $M/m_2 \rightarrow \infty$, the change in H approaches 1397 J kg^{-1}. Clearly, for this particular case the behavior of the algorithm as $M \rightarrow \infty$ is irrelevant.

4. Penetrators

For N > 2, we need a way to ensure that (3.5) is automatically satisfied. This we can do by introducing "penetrators." A penetrator $P_{i,j}$ consists of a mass flux that penetrates from layer i to layer j, with a compensating, *nonpenetrative, level-by-level return flow* . Each penetrator satisfies (3.5), so any superposition of penetrators also satisfies (3.5). By analogy with (3.1), the change in A_i due to an ensemble of penetrators is given by

$$\begin{aligned} m'_i A'_i - m_i A_i = & \sum_{j=1}^{N} P_{j,i} (\hat{A}_j - \hat{A}_i) \\ & - \sum_{j=1}^{N} P_{i,j} \hat{A}_i + \sum_{j<i} P_{i,j} \hat{A}_{i-1} + \sum_{j>i} P_{i,j} \hat{A}_{i+1} \\ & + \sum_{j<i} \sum_{l>i} P_{j,l} (\hat{A}_{i-1} - \hat{A}_i) + \sum_{j>i} \sum_{l<i} P_{j,l} (\hat{A}_{i+1} - \hat{A}_i). \end{aligned} \tag{4.1}$$

On the right-hand-side of (4.1), the terms on the first line represent the effects of "incoming" penetrators that terminate at level i, those on the second line represent the effects of "outgoing" penetrators that originate at level i, and the terms on the third line represent the effects of penetrators that are "just passing through" level i. Putting $A \equiv 1$, and using

$$P_{i,i} = 0 \tag{4.2}$$

for all i, we find that

$$m'_i - m_i = 0 \tag{4.3}$$

is automatically satisfied, as intended. After some algebraic manipulation, (4.1) can be rewritten as

$$m_i(A'_i - A_i) = \sum_{|j-i|>1} P_{j,i}(\hat{A}_j - \hat{A}_i)$$
$$+ (\hat{A}_{i-1} - \hat{A}_i)\left(Q_{i,i-1} + \sum_{j<i-1} P_{i,j} + \sum_{j<i} \sum_{l>i} P_{j,l}\right)$$
$$+ (\hat{A}_{i+1} - \hat{A}_i)\left(Q_{i,i+1} + \sum_{j>i+1} P_{i,j} + \sum_{j>i} \sum_{l<i} P_{j,l}\right),$$

(4.4)

where

$$Q_{i,i-1} \equiv P_{i,i-1} + P_{i-1,i}.$$

(4.5)

Notice that $P_{i,i-1}$ and $P_{i,i+1}$ do not appear explicitly in (4.4), although they do appear implicitly through $Q_{i,i-1}$ and $Q_{i,i+1}$. The interpretation is straightforward. A penetrator that joins two neighboring layers doesn't really penetrate at all. As a result, the injection of air from i to i+1 with the accompanying return flow from i+1 to i has exactly the same effect as injection from i+1 to i with a return flow from i to i+1. This means that $P_{i+1,i}$ and $P_{i,i+1}$ are redundant; they do the same thing. That is why only their sum, $Q_{i,i+1}$, appears in (4.4).

Our goal is to find the values of the P's and Q's such that the total enthlapy of the adjusted state is minimized. We must require that the P's and Q's are non-negative, since their source regions have to be specified. For an N-layer sounding, we can define N^2 different values of P. Not all of these are meaningful, however. As indicated in (4.2), the diagonal elements of the P matrix can be set to zero, since they have no effect on the sounding. In addition, the redundancy of the neighboring-layer P's, discussed above, allows us to replace 2(N-1) of the P's by (N - 1) Q's, effectively reducing the number of unknowns by N - 1. The actual number of unknowns is then $N^2 - N - (N - 1) = (N - 1)^2$. In general, there are (N - 1) Q's and (N - 2)(N - 1) P's. Table 2 shows how the numbers of the various unknowns change as N changes. We can anticipate that, in most cases of practical interest, many of these unknowns will turn out to be zero. Unfortunately, there is know obvious way be sure in advance which ones these will be.

5. Discussion

The next question is, how can we determine the P's and Q's for the case of arbitrary N? For the case N = 2, discussed in Section 3 above, we searched a one-dimensional space by brute force to find the value of the single unknown, which we now recognize as a Q. A glance at Table 2 shows that this approach quickly becomes impractical for arbitrary N.

As an alternative, we could use a numerical method to determine the partial derivative H with respect to each of the P's and Q's. The results could be arranged as a matrix. We could then try a linearization approximation, in which this matrix is used to find the values of the P's and Q's that minimize H.

Unfortunately, this approach fails, for two reasons. First, it cannot guarantee that the P's and Q's are non-negative. Second, as is clear from Fig. 2, H varies nonlinearly with the P's and Q's. This nonlinearity arises from the nonlinear dependence of the enthalpy of saturated air on the entropy and total mixing ratio. It is critical for the existence of a *minimum* value of H for finite positive values of the P's and Q's. To see this, consider the *dry* statically unstable case with N = 2, as discussed in the Introduction. If we use the trapezoidal algorithm, we find that there is no minimum of H for finite Q. Instead, H decreases monotonically as Q increases, and is minimized when $Q \to \infty$, i.e. when the parcels are swapped.

The nonlinearity also rules out the use of linear programming methods, which, if they were applicable, could ensure non-negativity of the P's and Q's. It appears that we must employ nonlinear programming, which is relatively unknown territory with few strong theorems.

6. Plans for future work

We plan to explore these matters further by developing an algorithm for the case of arbitrary N, and applying it to real data, beginning with the GATE data. The following questions will be addressed:

1. How do the reference state and the MAE vary with time?

2. How do time changes in the reference state compare to time changes in the given soundings themselves?

3. What time-changes of the MAE and the reference state would occur if the observed large-scale circulations acted alone, without compensating cumulus effects? How do these hypothetical changes in the MAE compare to those that actually occur?

4. For each observation time, what mass exchanges would be required to reach the reference state?

5. Is there a way to allow the reference state to be horizontally inhomogeneous?

6. How do the results change if the "pipes" that carry the mass between layers are endowed with various properties? For example, we might choose to give up the assumption of adiabatic reversible mass transfer, and allow water condensed inside pipes to precipitate out.

7. Besides the absolute minimum of H that denotes the reference state, do local minima ever occur in the mass-exchange space?

REFERENCES

Arakawa, A., 1969: Parameterization of cumulus convection. *Proc. WMO/IUGG Symp. on Numerical Weather Prediction,* Tokyo, 26 November-4 December, 1968, Japan Meteor. Agency, IV, 8, 1-6.

Arakawa, A., and W. H. Schubert, 1974: The interaction of a cumulus cloud ensemble with the large-scale environment, Part I. *J. Atmos Sci.*, **31**, 674-701.

Arakawa, A., and J.-M. Chen, 1987: Closure assumptions in the cumulus parameterization problem. *Short and Medium Range Numerical Weather Prediction.* Collection of Papers Presented at the WMO/IUGG Symposium on Numerical Weather Prediction, Tokyo, 4-8 August 1986, Special Vol., *J. Meteor. Soc. Japan.*

Dudhia, J., and W. M. Moncrieff, 1987: A numerical simulation of quasistationary tropical convective bands. *Quart. J. Roy. Meteor. Soc.*, **113**, 929-967.

Houze, R. A., Jr., and A. K. Betts, 1981: Convection in GATE. *Rev. Geophys. and Space Phys.*, **19**, 541-576.

Krueger, S. K., 1988: Numerical simulation of tropical cumulus clouds and their interaction with the subcloud layer. Submitted to *J. Atmos. Sci.*

Lord, S. J., 1978: Development and observational verification of a cumulus cloud parameterization. Ph.D. Dissertation, University of California, Los Angeles, 339 pp.

Lord, S. J., 1982: Interaction of a cumulus cloud ensemble with the large-scale environment. Part III. Semi-prognostic test of the Arakawa- Schubert cumulus parameterization. *J. Atmos Sci.* , **39**, 88-103.

Lord, S. J., and A. Arakawa, 1980: Interaction of a cumulus cloud ensemble with the large-scale environment. Part II. *J. Atmos Sci.*, **37**, 2677-2692.

Lord, S. J., W. C. Chao, and A. Arakawa, 1982: Interaction of cumulus cloud ensemble with the large-scale environment. Part IV. The discrete model. *J. Atmos Sci.*, **39**, 104-113.

Lorenz, E. N., 1955: Available potential energy and the maintenance of the general circulation. *Tellus,* **7**, 157-167.

Lorenz, E. N., 1978: Available energy and the maintenance of a moist circulation. *Tellus,*, **30**, 15-31.

Lorenz, E. N., 1979: Numerical evaluation of moist available energy. *Tellus*, **31**, 230-235.

Soong, S.-T., and W.-K. Tao, 1980: Response of deep tropical cumulus clouds to mesoscale processes. *J. Atmos. Sci.*, **37**, 2016-2034.

Yanai, M., S. K. Esbensen, and J.-H. Chu, 1973: Determination of bulk properties of tropical cloud clusters from large-scale heat and moisture budgets. *J. Atmos. Sci.*, **30**, 611-627.

Table 1

p, mb	T, K	θ, K	q, g/kg	θ_e, K	θ_{es}, K
325	250	344.78	1.78	350.86	351.04
775	290	311.93	15.70	354.71	354.89

Table 2

Number of layers	Number of P's	Number of Q's	Total Number of Unknowns
2	0	1	1
3	2	2	4
4	6	3	9
5	12	4	16
6	20	5	25
7	30	6	36
8	42	7	49
9	56	8	64
10	72	9	81

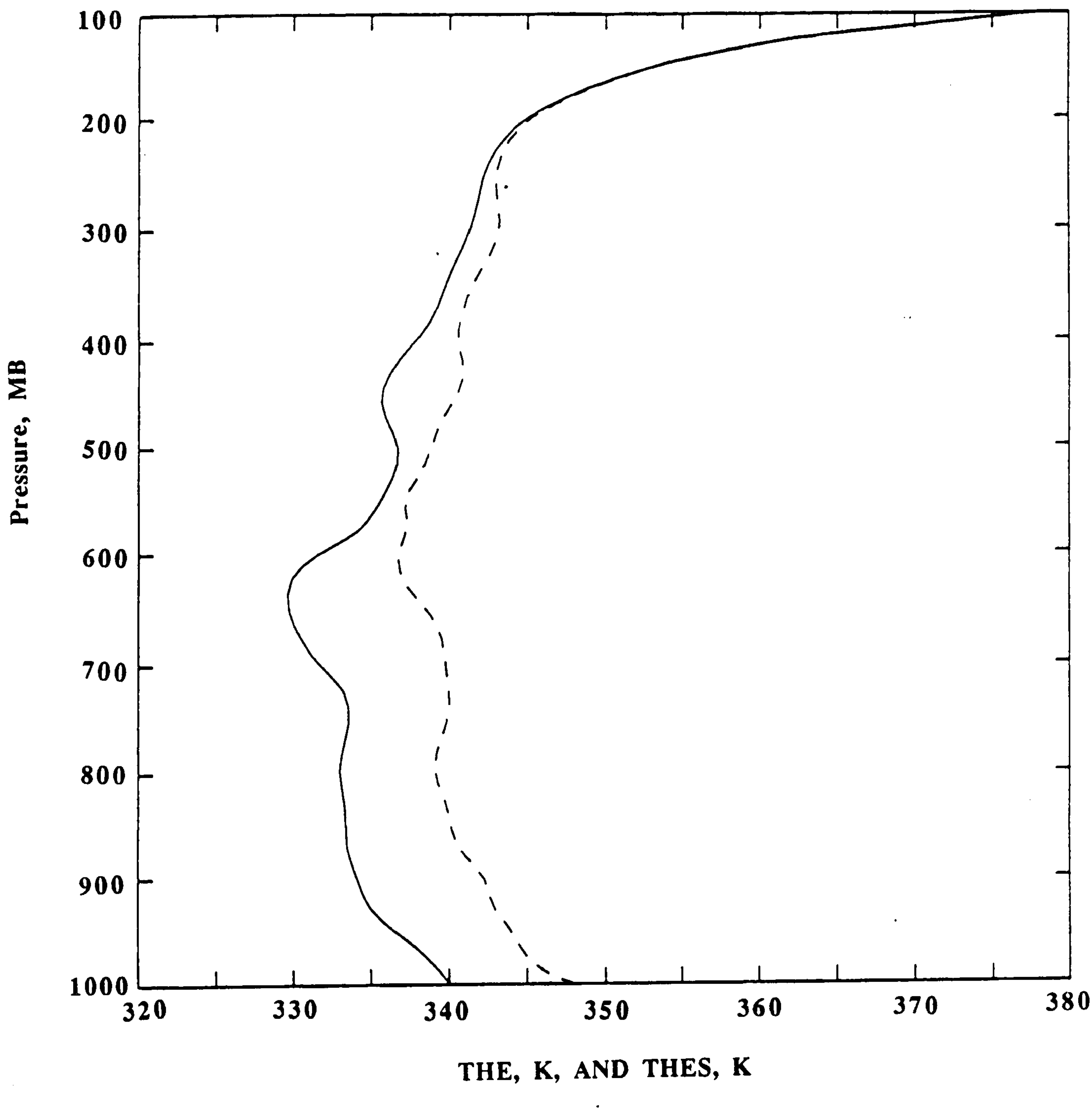

Figure 1: Observed soundings of equivalent potential temperature (solid line) and saturation equivalent potential temperature (dashed line) for GATE Phase III, August 30 at 1800 GMT.

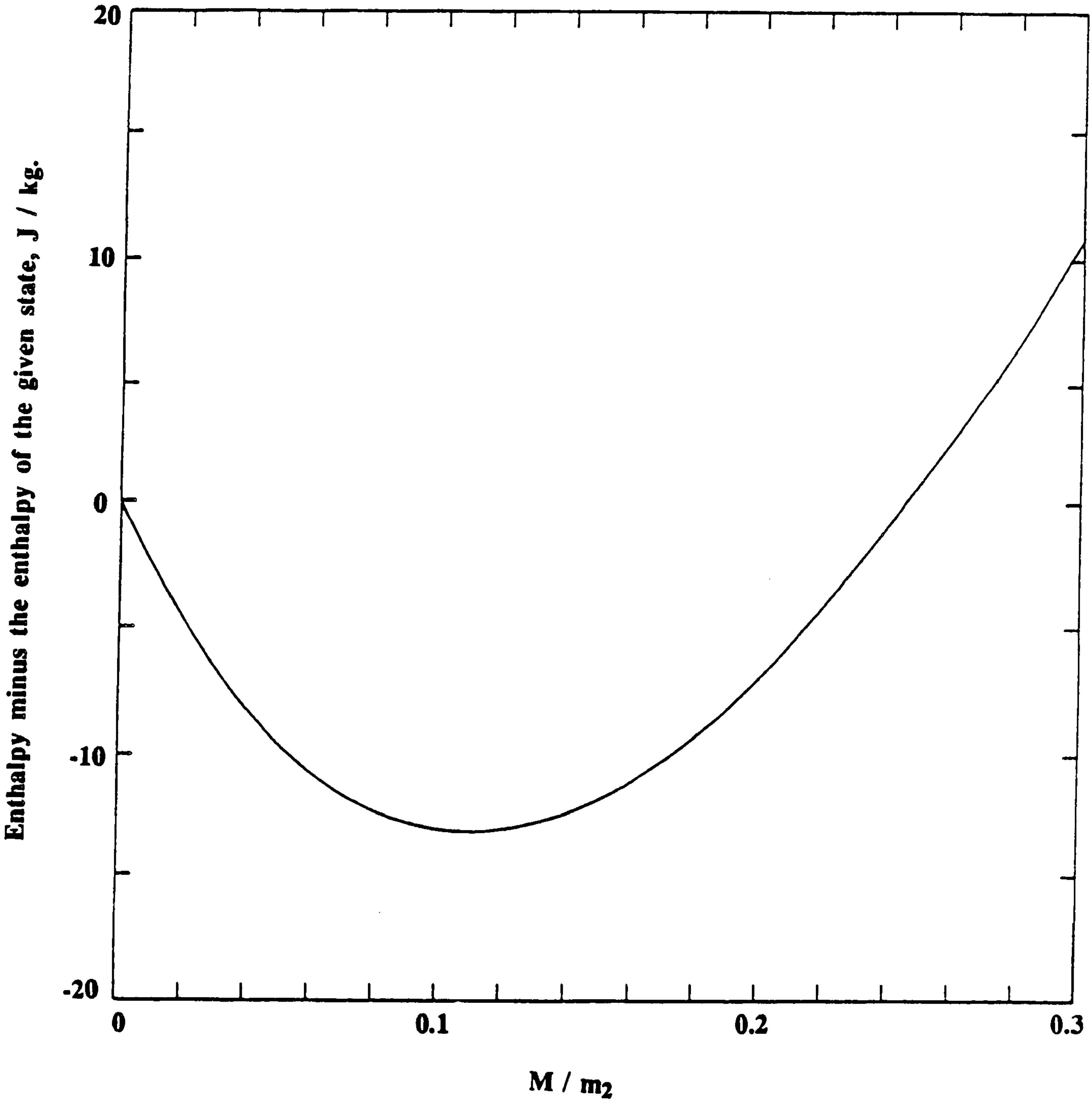

Figure 2: Departure of the total enthalpy from that of the given state, plotted as a function of M / m_2, for the two-level "sounding" given in Table 1. Here M is the amount of mass exchanged between the layers, and m_2 is the mass of the lower layer.