ON THE ADJUSTMENT OF SIMPLE ATMOSPHERIC CURRENTS

by

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Translated from
Berichte des Deutschen
Wetterdienstes, Nr. 87,
Band 12, 87/3—87/20.

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Translation of

Über die Adaptation einfacher Stromfelder

in der Atmosphäre

von Günter Fischer

(Berichte des Deutschen Wetterdienstes,

Nr. 87, Band 12, pp. 87/3 - 87/20)

Translators' Preface

Dr. Fischer's paper on geostrophic adjustment in a stratified fluid was published in Berichte des Deutschen Wetterdienstes, 12 (87), 1963. It is one of the few papers in the literature on this topic and, except through Blumen's (1972)[1] review, is largely unknown to American meteorologists. It is hoped that this translation will make the paper more widely known. We would like to thank Scott Fulton and Odie Panella for their help in preparing this translation.

Thomas Nehrkorn

Wayne Schubert

March 1980

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[1] Blumen, W., 1972: Geostrophic adjustment. Rev. Geophys. Space Phys., 10, 485-528.

ABSTRACT

A disturbance of geostrophic equilibrium in the form of an unbalanced vortex of finite lateral extension or an unbalanced zonal current of finite width is suddenly injected into upper atmospheric layers between 8 and 16 km at time $t = 0$. We consider the changes of motion, temperature and pressure caused by the initially unbalanced velocity field which seeks to gain a stationary geostrophic equilibrium, and compare the final to the initial state. To this end the linearized hydro-thermodynamical equations are solved on the assumption that the basic state of the horizontally unlimited atmosphere is in rest and the lapse rate of temperature vanishes or, in a second example, is adiabatic. We show that the solutions for these baroclinic disturbances are obtained by a superposition of solutions for barotropic disturbances; these latter are therefore discussed in detail. The following questions are dealt with: which general initial conditions lead to adjustment, which energy transformations take place, how does the lateral and vertical extension of the initial disturbance influence the results, which conclusions can be drawn from the results with respect to the dynamics of the true atmosphere.

CONTENTS

1. Introduction

From the beginning of meteorological research the question of the deviations from the geostrophic equilibrium has attracted great interest. Although the equilibrium between Coriolis force and pressure gradient force is present to a large extent in the free atmosphere of higher latitudes, it cannot exist exactly because no energy transformations would take place. Weather maps show also that velocity and pressure fields are non-stationary in the atmosphere and that air masses are subject to individual accelerations.

Experience shows, however, that the atmosphere of higher latitudes is approximately in geostrophic equilibrium, which is a result of the fact that masses are always trying to restore their equilibrium between Coriolis force and pressure gradient force. The process is called adjustment, after Shaw.

How the adjustment process takes place in the atmosphere, how the different variables change, and finally, how the stationary geostrophic final state is reached will be examined in this work with the aid of simple models. It will be assumed as is generally done that deviations from the geostrophic equilibrium other than those in the friction layer near the surface occur mainly in the upper troposphere. For example, one could consider a cyclonic vortex (a mass of cold air) orginally in geostrophic equilibrium which becomes supergeostrophic by traveling southward in the upper layers of the troposphere and then tries to regain its equilibrium (latitudinal adjustment). Although the Coriolis parameter is assumed to be constant in this work one can suppose a small meridional displacement of the vortex and investigate how the then-ageostrophic vortex assumes its equilibrium at the fixed location. A deviation from the geostrophic equilibrium also has to occur in the inflow and outflow regions of an upper tropospheric frontal zone since the air masses passing through these regions are exposed to a pressure gradient which varies with time. This leads to subgeostrophic winds in the inflow region and supergeostrophic winds in the outflow region. In this case also the air masses will seek to gain their geostrophic equilibrium (gradient adjustment).

Two simple models were selected and treated mathematically to investigate in more detail the two situations described above: 1) a supergeostrophic vortex of finite extent and 2) a supergeostrophic current of finite width. Both perturbations shall occur suddenly and be limited to a relatively thin layer at the tropopause level. Our problem then is to examine how the adjustment of this vortex or current, respectiviely, takes place, how the pressure, temperature and velocity fields change in all layers, what the final geostrophic state looks like and what influence the horizontal and vertical extent of the perturbation has on the solution.

These two models assume a geostrophic imbalance in the initial state which is caused by a perturbation of the velocity field. At first sight it might seem more appropriate to assume, following Stümke (21), an initial perturbation in the temperature and thus in the pressure field as is caused by the differential heating of continent and ocean and of different latitude zones. A thermal model of this kind could perhaps describe the generation of the monsoon circulation. This was not done, though, since the disturbances apparent in the weather maps which occur over short periods of time (on the order of days) appear as perturbations of the velocity fields, where thermal effects evidently play a second order role.

The translation of the preceding models into a halfway comprehensive mathematical system is unfortunately not possible without certain simplifications. Thus, only small deviations from a basic state of rest are assumed. Our theoretical results can be generalized only to a limited extent to a basic state in geostrophic equilibrium, as will be shown later. Another assumption is that the atmosphere is isothermal in the basic state. The influence of the stability on the adjustment processes can be estimated, though, through a comparison with an atmosphere with neutral stratification.

A basically similar model has been treated by Raethjen (14). The differences lie mainly in the mathematical execution; nevertheless, the results are similar. Rossby (17) gave the incentive to theoretical investigations of the adjustment of simple non-geostrophic velocity fields in the ocean, with the wind-produced ocean currents in mind. Continuing the work of Rossby, Cahn (2) computed the adjustment of a current in a barotropic ocean with the aid of the linearized hydrodynamic equations; Bolin (1) followed with a similar investigation in the baroclinic ocean. Fjeldstad (7) described the mathematical basis for the computation of the adjustment with respect to oceanic current.

The latter three works have in common that they solve an initial value problem. This work follows the same approach. At the time $t = 0$ a horizontally and vertically limited perturbation is prescribed in an otherwise horizontally unlimited region, and, since no external forces shall occur, the final state is dependent only on the initial conditions.

2. Equations

In the "basic state" the atmosphere shall be at rest and be stratified isothermally; the deviations from the basic state shall be so small that quadratic terms can be neglected. With

$$\begin{aligned} P(x,y,z,t) &= \overline{P}(z) + \Delta P \equiv \overline{P}(z)(1+\varepsilon) \\ \varrho(x,y,z,t) &= \overline{\varrho}(z) + \Delta \varrho \equiv \overline{\varrho}(z)(1+\sigma) \\ T(x,y,z,t) &= \overline{T} + \Delta T \equiv \overline{T}(1+\tau) \end{aligned} \quad , \qquad [1]$$

where the barred quantities correspond to the basic state and ε, σ, τ as functions of the three space coordinates x, y, z and the time t represent the relative pressure, density and temperature perturbations, the linearized equations can be written as

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -R\overline{T}\frac{\partial \varepsilon}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -R\overline{T}\frac{\partial \varepsilon}{\partial y} \\ -g(\varepsilon-\sigma) &= -R\overline{T}\frac{\partial \varepsilon}{\partial z} \\ \frac{\partial \sigma}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \frac{g}{R\overline{T}}w &= 0 \\ \frac{\partial \varepsilon}{\partial t} - \varkappa\frac{\partial \sigma}{\partial t} + \frac{g}{R\overline{T}}(\varkappa-1)w &= 0 \\ \varepsilon - \sigma &= \tau \end{aligned} \qquad [2]$$

u, v, w are the velocity components in the x, y, z - directions, f is the Coriolis parameter, $\kappa = c_p/c_v$ the ratio of the specific heats, R the gas constant of atmospheric air and g the gravitational acceleration. The first two equations are the dynamic equations of motion, followed by the hydrostatic relation, continuity equation, the adiabatic equation and the gas equation. It is thus assumed that vertical acceleration can be neglected and that hydrostatic equilibrium always prevails; furthermore, it is required that no heat is added to or taken out of the system.

The above equations correspond to a basic state at rest in a coordinate system which is fixed with respect to the earth. With less accuracy, they also describe the perturbation movements relative to a geostrophically adjusted vortex or (zonal) current extending over all layers as a basic state, where the coordinate system is fixed with respect to the rotating basic current; then f represents the rotation of the absolute vortex. In this case additional terms related to the pressure gradient of the basic field would appear in the linearization of the continuity equation as well as the adiabatic equation. To a certain approximation the equations should also describe this case satisfactorily, though, as will be shown later.

In order to arrange the system of equations [2] into a more tractable form, the density perturbation and vertical velocity in the continuity equation are eliminated with the aid of the hydrostatic relation and the adiabatic equation. With

$$w = \frac{R\overline{T}}{g}\frac{\partial \varepsilon}{\partial t} - \frac{\varkappa}{\varkappa - 1}\left(\frac{R\overline{T}}{g}\right)^2 \frac{\partial^2 \varepsilon}{\partial t\, \partial z} \qquad [3a]$$

as well as

$$\sigma = \varepsilon - \frac{R\overline{T}}{g}\frac{\partial \varepsilon}{\partial z} \qquad [3b]$$

and

$$\tau = \frac{R\overline{T}}{g}\frac{\partial \varepsilon}{\partial z} \qquad [3c]$$

[2] yields the differential equations

$$\begin{aligned} &\frac{\partial u}{\partial t} - fv = -R\overline{T}\frac{\partial \varepsilon}{\partial x} \\ &\frac{\partial v}{\partial t} + fu = -R\overline{T}\frac{\partial \varepsilon}{\partial y} \\ &\frac{\partial^2}{\partial t\, \partial z}\left(\frac{\partial \varepsilon}{\partial z} - \frac{g}{R\overline{T}}\varepsilon\right) - \frac{\varkappa - 1}{\varkappa}\left(\frac{g}{R\overline{T}}\right)^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \end{aligned} \qquad [4]$$

3. Boundary and Initial Conditions

The following boundary conditions in z are prescribed: the vertical velocity w vanishes at the ground ($z = 0$) as does the pressure perturbation ε at a yet to be specified height $z = h$. Thus

$$w = 0 \text{ for } z = 0 \quad \text{and } \varepsilon = 0 \text{ for } z = h. \qquad [5]$$

While the vertical velocity necessarily has to vanish at the ground (with a horizontal surface), requiring the pressure perturbation to vanish at some high level is just a "reasonable" assumption.

The latter condition is satisfied by the following expression for ε as well as for u and v which also conforms with the system of equations [4]:

$$(\varepsilon, u, v) = \sum_{k=1}^{\infty} (\varepsilon_k, u_k, v_k)\, e^{\frac{-gh}{2RT}\zeta} \frac{\sin \lambda_k \zeta}{\lambda_k} \qquad [6]$$

where

$$\zeta = 1 - \frac{z}{h}.$$

The coefficients ε_k, u_k, v_k are functions of x, y and t. The eigenvalues λ_k are determined for the condition that the vertical velocity has to vanish at the ground. From [3a] and [6] one obtains the transcendental relationship:

$$\operatorname{tg} \lambda_k = \frac{2\varkappa}{2-\varkappa} \frac{RT}{gh} \lambda_k \qquad k = 1, 2, 3 \ldots \qquad [7][1]$$

For later computations we set $\frac{2\varkappa}{2-\varkappa} \frac{RT}{gh} = 1$; then it follows that $\lambda_1 = 0$ with $\lambda_{k+1} > \lambda_k$.

The above series [6] together with the relation [7] constitutes an "anharmonic Fourier series"; the $\sin \lambda_k \zeta$ form an orthogonal system,

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[1] The abbreviations tg stands for tangent.

through which any arbitrary function can be approximated, especially the initial conditions which will be described in more detail in the following.

At time $t = 0$ a perturbation of the geostrophic equilibrium shall exist in the form of an additional wind field which is superimposed on the basic field. If one denotes the initial state by superscript 0, then the vertical distribution of the vorticity perturbation is described by:

$$\sigma^0 = \tau^0 = w^0 = 0;\quad \varepsilon^0 = D^0 = 0$$

$$C^0(x, y, \zeta) = \begin{cases} \widetilde{C}^0(x, y)\, e^{-\frac{g\,h}{2\,RT}\zeta}(1 - \cos 8\pi\zeta) & \text{for} \quad \frac{1}{2} < \zeta < \frac{3}{4} \\ 0 \quad \text{for } 0 < \zeta < \frac{1}{2} & \text{and} \quad {}^{3}/_{4} < \zeta < 1 \end{cases} \tag{8}$$

where

$$C = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \;.$$

The above initial distribution $C^0(x,y,\zeta)$ can be described by the anharmonic Fourier series according to [6] such that

$$C^0(x, y, \zeta) = \sum_{k=1}^{\infty} C_k^0(x, y)\, e^{-\frac{g\,h}{2\,RT}\zeta}\, \frac{\sin \lambda_k \zeta}{\lambda_k} \,.$$

Because of the orthogonality the coefficients $C_k^0(x,y)$ are determined from

$$C_k^0(x, y) = 2\int_0^1 \frac{C^0(x, y, \zeta)\, e^{\frac{g\,h}{2\,RT}\zeta}\, \lambda_k \sin \lambda_k \zeta}{1 - \frac{\sin 2\lambda_k}{2\lambda_k}}\, d\zeta.$$

The horizontal distribution of the initial vorticity is given by:

$$\widetilde{C}^0(x, y) = \begin{cases} \text{const} \neq 0 & \text{for } x^2 + y^2 = r^2 < a^2 \\ 0 & \text{for } x^2 + y^2 = r^2 > a^2. \end{cases} \tag{9}$$

Thus, the initial conditions show a supergeostrophic perturbation vorticity with rotational symmetry and limited radial extend which extends over just a partial layer in the "middle" of the atmosphere. Since the region in which the vorticity is embedded is assumed to be infinite in the horizontal direction, no boundary conditions are imposed on x and y, or r.

The limitation of the initial vorticity to the region $r < a$ does not mean, however, that the associated velocity is limited to this region, too. The constant vorticity defined here corresponds to an angular velocity in the region $r < a$ which is the same everywhere and equal to exactly half the vorticity. Although the vorticity is zero for $r > a$, the angular velocity doesn't vanish there but approaches zero (beginning at $r = a$) with $1/r^2$. Thus, the initial velocity field extends over a larger area than the initial vorticity field.

The same initial vertical distribution as given in [8] shall be valid for an initially supergeostrophic current extending in the zonal direction (x-direction) with a limited width (the derivatives ∂/∂_x vanish in this case). This results in:

$$\sigma^0 = \tau^0 = w^0 = 0, \ \varepsilon^0 = v^0 = 0$$

$$u^0(y,\zeta) = \begin{cases} \widetilde{u^0}(y)\, e^{-\frac{gh}{2RT}\zeta}(1 - \cos 8\pi\zeta) & \text{for } \frac{1}{2} < \zeta < \frac{3}{4} \\ 0 \text{ for } 0 < \zeta < \frac{1}{2} & \text{and } \frac{3}{4} < \zeta < 1 \end{cases}$$

$$\widetilde{u^0}(y) \begin{cases} \neq 0 & \text{for } |y| < b \\ = 0 & \text{for } |y| > b \end{cases} \qquad [10]$$

Substitution of [6] into the equations [4] results in the following system of equations for the coefficients:

$$\frac{\partial u_k}{\partial t} - f u_k = -R\overline{T}\frac{\partial \varepsilon_k}{\partial x}$$

$$\frac{\partial v_k}{\partial t} + f u_k = -R\overline{T}\frac{\partial \varepsilon_k}{\partial y} \qquad [11]$$

$$\frac{\partial \varepsilon_k}{\partial t} + \frac{\gamma_k^2}{R\overline{T}}\left(\frac{\partial u_k}{\partial x} + \frac{\partial v_k}{\partial y}\right) = 0.$$

Where

$$\gamma_k^2 = 4\,\frac{\varkappa - 1}{\varkappa}\,\frac{R\overline{T}}{1 + \left(\frac{2R\overline{T}}{gh}\lambda_k\right)^2},$$

which corresponds to each eigenvalue λ_k, is the squared phase velocity of long gravity waves which would occur in a non-rotating coordinate system (in the rotating coordinate system dispersion occurs). (See also Hallmann (11).)

The system of equations [11] is identical with the system which describes the motions in an autobarotropic medium, e.g. in a homogeneous incompressible fluid of depth H, if one interprets $R\overline{T}\ \varepsilon_k/g$ as the deviation of the free surface from the basic state and $\gamma_k = \sqrt{gH}$ as the velocity of the gravity waves, or in an atmosphere with indifferent stratification in which all displacements take place adiabatically. The latter case of an "adiabatic atmosphere" will be treated later.

The solution of the system of equations [4] is thus composed of the solutions of the system of equations [11]. For that reason one can limit oneself in the following discussion to barotropic perturbations.

4. Solution for the vortex

In the following it is more convenient to use the relative vorticity C and the horizontal divergence D as dependent variables instead of the wind components. One then obtains the following equations:

$$\frac{\partial C_k}{\partial t} + f D_k = 0$$
$$\frac{\partial D_k}{\partial t} - f C_k = - R\overline{T}\, \nabla^2 \varepsilon_k \qquad [12]$$
$$\frac{\partial \varepsilon_k}{\partial t} + \frac{\gamma_k^2}{R\overline{T}} D_k = 0 \quad , \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The first equation is the vorticity equation; for it to be valid in this form, C_k has to be small compared with f. The first and last equations combine together to form the (linearized) potential vorticity equation, which relates the pressure field and the vorticity field:

$$\frac{1}{f}\frac{\partial C_k}{\partial t} = \frac{R\overline{T}}{\gamma_k^2}\frac{\partial \varepsilon_k}{\partial t} \qquad [13]$$

The system [12] reduces to the following (hyperbolic) differential equation:

$$\left\{\frac{\partial^2}{\partial t^2} + f^2 - \gamma_k^2 \ \nabla^2\right\}\left\{\frac{\partial \varepsilon_k}{\partial t}, \frac{\partial C_k}{\partial t}, D_k\right\} = 0. \qquad [14]$$

The solution of this differential equation, under the initial conditions

$$\varepsilon_k^0 = D_k^0 = 0 \quad , \quad C_k^0 = \frac{1}{f}\left(\frac{\partial D_k}{\partial t}\right)^0 \neq 0$$

is well known. It was shown by Hadamard (8) that the solution of this Cauchy initial value problem (see Fjeldstad (7)) is

$$D_k(x, y, t) = \frac{f}{2\pi\, \gamma_k^2} \iint\limits_S C_k^0(\alpha, \beta)\, \frac{\cos\sqrt{f^2 t^2 - \dfrac{f^2 \varrho^2}{\gamma_k^2}}}{\sqrt{f^2 t^2 - \dfrac{f^2 \varrho^2}{\gamma_k^2}}}\, d\alpha\, d\beta$$

$$[15]$$

$$\frac{1}{f}\left(C_k(x, y, t) - C_k^0(x, y)\right) = \frac{R\overline{T}}{\gamma_k^2}\, \varepsilon_k(x, y, t)$$
$$= - \int_0^t D_k(x, y, \eta)\, d\eta$$

where $\tilde{\rho}^2 = (x-\alpha)^2 + (y-\beta)^2$, and where the integration is performed over the area S for which $\tilde{\rho} \leq \gamma_k t$. If the initial disturbance

$$C_k^0 = \frac{1}{f}\left(\frac{\partial D_k}{\partial t}\right)^0$$

is non-zero only within a region $x^2 + y^2 = r^2 < a^2$, then the solution for a point outside this region will vanish as long as the integration region, i.e. the area of the circle around the point in question with radius $\tilde{\rho}$, does not overlap the disturbance region. This means physically that nothing happens at the point in the undisturbed region until the edge of the initial disturbance which is traveling with the velocity γ_k reaches the point. The edge thus forms a wave front, the velocity of which is independent of the rotation of the coordinate system. The conclusions from this can be found in a work by Rossby (18).

Now let the initial conditions be defined such that

$$C_k^0(r) = \begin{cases} \text{const.} \equiv C_k^* \neq 0 & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}, \quad \varepsilon_k^0 = D_k^0 = 0 \qquad [16]$$

This distribution corresponds completely to the one given in the previous paragraph under [9], except that it is defined for the coefficients here. Based on this assumption the integration for the central point $r = 0$ can be performed easily. One obtains for the horizontal divergence:

$$D_k(r = 0, t) = \begin{cases} C_k^* \sin ft & \text{for } \gamma_k t < a \\ C_k^* \left(\sin ft - \sin \sqrt{f^2 t^2 - \frac{f^2 a^2}{\gamma_k^2}}\right) & \text{for } \gamma_k t > a. \end{cases} \qquad [17]$$

In the center of the initial disturbance the masses perform pure inertial oscillations with the period $2\pi/f$ until the edge of the disturbance which is traveling with the velocity γ_k reaches the central point at time $ft = fa/\gamma_k$. After that oscillations due to inertial and gravity waves occur.

For very large time, strictly speaking for $ft \gg fa/\gamma_k$, the above solution can be written in the following form:

$$D_k \left(r = o, t \gg \frac{a}{\gamma_k}\right) = \frac{f^2 a^2}{2 \gamma_k^2} C_k^* \frac{\cos ft}{ft} \qquad [18]$$

From the point in time $ft = fa/\gamma_k$ on, the radial oscillations of the vortex are thus strongly damped and for large time approach zero with $1/ft$; the same holds also for the pressure and vorticity variations. This damping occurs even without friction by loss of a part of the initial energy to infinity (meteorological noise), while the other part finds its equilibrium. More details will be discussed in the next section, where solutions are obtained for the zonal current.

To compute the time variation of the relative pressure and vortex perturbations the above expressions [17] have to be integrated over time. The final stationary state (superscript ∞) for the central point is then given by:

$$\frac{1}{f}\left(C_k^\infty - C_k^*\right) = \frac{R\bar{T}}{\gamma_k^2} \varepsilon_k^\infty = \frac{C_k^*}{f} \left\{ \left(\cos \frac{fa}{\gamma_k} - 1\right) - f \int_{a/\gamma_k}^{\infty} \left(\sin ft - \sin \sqrt{f^2 t^2 - \frac{f^2 a^2}{\gamma_k^2}} \right) dt \right\} \text{ for } r = o.$$

The integral on the right hand side is easily evaluated by differentiating with respect to the lower limit and then integrating twice. One obtains:

$$\frac{f R\bar{T}}{\gamma_k^2} \varepsilon_k^\infty (r = o) = C_k^* \left\{ \frac{fa}{\gamma_k} K_1\left(\frac{fa}{\gamma_k}\right) - 1 \right\} \qquad [19]$$

$$C_k^\infty (r = o) = C_k^* \frac{fa}{\gamma_k} K_1\left(\frac{fa}{\gamma_k}\right).$$

Here K_n is the order n modified Bessel function of the second kind.

One obtains the same solution from equation [13] if one requires geostrophic equilibrium for $t \to \infty$. It follows in general from equation [13]:

$$\frac{1}{f} C_k^\infty (r) - \frac{R\bar{T}}{\gamma_k^2} \varepsilon_k^\infty (r) = \frac{1}{f} C_k^o (r) - \frac{R\bar{T}}{\gamma_k^2} \varepsilon_k^o (r) \qquad [20]$$

The geostrophic relation yields:

$$-f\,C_k^{\infty}(r) = -R\bar{T}\,\nabla^2\,\varepsilon_k^{\infty}(r) \qquad [21]$$

This leads to the equation:

$$\nabla^2\,\varepsilon_k^{\infty} - \frac{f^2}{\gamma_k^2}\,\varepsilon_k^{\infty} = \frac{f^2}{R\bar{T}}\left(\frac{1}{f}\,C_k^0 - \frac{R\bar{T}}{\gamma_k^2}\,\varepsilon_k^0\right) \qquad [22]$$

The same equation for the final stationary state is obtained, of course, from the time-independent differential equation [14] after integrating over time from 0 to ∞ with the given initial conditions.

The solution of the foregoing differential equations for the final state of the entire field with $\varepsilon_k^0 = 0$ is

$$-\frac{R\bar{T}}{f}\,\varepsilon_k^{\infty}(r) = K_0\left(\frac{fr}{\gamma_k}\right)\int_0^r C_k^0(\eta)\,I_0\left(\frac{f\eta}{\gamma_k}\right)\eta\,d\eta + I_0\left(\frac{fr}{\gamma_k}\right)\int_r^{\infty} C_k^0(\eta)\,K_0\left(\frac{f\eta}{\gamma_k}\right)\eta\,d\eta \qquad [23]$$

where I_n is the order n modified Bessel function of the first kind.

Considering the special initial conditions [16] it follows from [23] that:

$$-f\,\frac{R\bar{T}}{\gamma_k^2}\,\varepsilon_k^{\infty}(r) = \begin{cases} C_k^{*}\left\{1 - \frac{fa}{\gamma_k}\,I_0\left(\frac{fr}{\gamma_k}\right)K_1\left(\frac{fa}{\gamma_k}\right)\right\} & \text{for } r < a \\ C_k^{*}\,K_0\left(\frac{fr}{\gamma_k}\right)I_1\left(\frac{fa}{\gamma_k}\right)\cdot\frac{fa}{\gamma_k} & \text{for } r > a \end{cases}$$

$$C_k^{\infty}(r) = C_k^0(r) + \frac{fR\bar{T}}{\gamma_k^2}\,\varepsilon_k^{\infty}(r) \qquad [24]$$

$$D_k^{\infty} = 0.$$

For r = 0 the solution is identical to the one previously obtained for the central point.

Properties of the Bessel functions used here:

$$I_1(0) = 0;\quad r\,K_1(r) = 0 \quad \text{for } r \to 0;\quad K_0(r) = 0 \quad \text{for } r \to \infty$$

$$I_0(0) = 1;\quad r\,K_1(r) = 1 \quad \text{for } r \to \infty;\quad K_0(r) \approx \ln\frac{2}{r\cdot 1{,}781\ldots} \quad \text{for } r \ll 1$$

5. Solution for the zonal current

From the system of equations [11], with $\frac{\partial}{\partial x} = 0$, one obtains

$$\left\{ \frac{\partial^2}{\partial t^2} + f^2 - \gamma_k^2 \frac{\partial^2}{\partial y^2} \right\} \left\{ \frac{\partial \varepsilon_k}{\partial t}, \frac{\partial u_k}{\partial t}, v_k \right\} = 0 \qquad [25]$$

With the initial condition

$$\varepsilon_k^0 = v_k^0 = 0 \quad , \; u_k^0(y) = -\frac{1}{f}\left(\frac{\partial v_k}{\partial t}\right)^0 \neq 0$$

there follows the solution

$$v_k(y,t) = -\frac{f}{2\gamma_k} \int_{-\gamma_k t}^{\gamma_k t} J_0\left(\sqrt{f^2 t^2 - \frac{f^2 \eta^2}{\gamma_k^2}}\right) u_k^0(y-\eta)\, d\eta$$

$$u_k(y,t) = f \int_0^t v_k(y,\eta)\, d\eta \qquad [26]$$

$$\varepsilon_k(y,t) = -\frac{\gamma_k^2}{RT} \int_0^t \frac{\partial v_k}{\partial y}(y,\eta)\, d\eta$$

where J_0 is the order zero Bessel function of the first kind.

In spite of the similarities between the differential equation [14] and [25] the solutions are quite different, a situation generally found in solutions of the wave equations for one and two dimensions. However, the solution of [25] is included in the more general solution of [14].

The adjustment of a barotropic zonal current perturbation of constant zonal velocity and width 2b was studied by Rossby (17) and Cahn (2). The initial conditions used in their work were:

$$u_k^0(y) = \begin{cases} \text{const} = u_k^* \neq 0 & \text{for } |y| < b \\ 0 & \text{for } |y| > b \end{cases}, \quad \varepsilon_k^0 = v_k^0 = 0. \qquad [27]$$

Some aspects not explicitly explained in these papers will be discussed briefly in the following.

For the middle line of the zonal current ($y = 0$) the integration to time $ft \ll fb/\gamma_k$ is easily performed again. It turns out that the masses perform pure inertial oscillations with a period of $2\pi/f$ up to that time, just as in the case of the vortex. However, for larger times ($ft >> fb/\gamma_k$) the solutions differ; the oscillations of a particle in the center of the zonal current approach zero as $1/\sqrt{ft}$, wheras the oscillations of a particle in the center of a vortex are damped as 1/ft as was shown before. Thus, the final stationary state of a vortex perturbation will be reached earlier than that of a zonal current perturbation.

The equation for the final geostrophic state is:

$$f^2 \varepsilon_k^{\infty} - \gamma_k^2 \frac{\partial^2 \varepsilon_k^{\infty}}{\partial y^2} = \left(\frac{\partial^2 \varepsilon_k}{\partial t^2}\right)^0 = \frac{f \gamma_k^2}{R\overline{T}} \frac{\partial u_k^0}{\partial y}. \qquad [28]$$

The solution of this differential equation for an arbitrary initial state is:

$$\frac{R\overline{T}}{f} \varepsilon_k^{\infty}(y) = \frac{1}{2} \left\{ e^{\frac{fy}{\gamma_k}} \int_y^{\infty} u_k^0(\eta)\, e^{-\frac{f\eta}{\gamma_k}} d\eta - e^{-\frac{fy}{\gamma_k}} \int_{-\infty}^{y} u_k^0(\eta)\, e^{\frac{f\eta}{\gamma_k}} d\eta \right\}$$

$$-\frac{R\overline{T}}{f} \frac{\partial \varepsilon_k^{\infty}}{\partial y} = u_k^{\infty} \qquad [29]$$

$$\text{or } \frac{1}{f}\left(\frac{\partial u_k^{\infty}}{\partial y} - \frac{\partial u_k^0}{\partial y}\right) = -\frac{R\overline{T}}{\gamma_k^2} \varepsilon_k^{\infty}$$

$$v_k^{\infty} = 0$$

For the special case of the Rossby-Cahn model

$$\left.\begin{aligned} \varepsilon_k^{\infty} &= -\frac{\gamma_k u_k^*}{R\overline{T}} e^{-\frac{fb}{\gamma_k}} \sinh \frac{fy}{\gamma_k} \\ u_k^{\infty} &= u_k^* e^{-\frac{fb}{\gamma_k}} \cosh \frac{fy}{\gamma_k} \end{aligned}\right\} \text{ for } |y| < b \qquad [30]$$

$$\pm \frac{R\overline{T}}{\gamma_k} \varepsilon_k^{\infty} = \pm u_k^{\infty} = -u_k^* e^{-\frac{f|y|}{\gamma_k}} \sinh \frac{fb}{\gamma_k} \quad \text{for } \begin{matrix} y > +b \\ y < -b \end{matrix}$$

It can be seen from these equations that a velocity and pressure field have formed outside the initial current, and that a countercurrent has formed at the edge of the original current.

The energy transformations which took place during the adjustment process are easily computed. From the differential equations one obtains the following expression for the time variation of the kinetic and potential energy per unit mass of the entire field:

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{1}{2}\left[u_k^2+v_k^2+\left(\frac{R\overline{T}}{\gamma_k}\epsilon_k\right)^2\right]dy$$

$$=-\int_{-\infty}^{\infty} R\overline{T}\frac{\partial}{\partial y}(v_k\,\epsilon_k)\,dy. \qquad [31]$$

From there it follows with the initial conditions [27]:

$$\int_{-\infty}^{\infty}\frac{u_k^{0\,2}}{2}\,dy \equiv b\cdot u_k^{*2}=\frac{1}{2}\int_{-\infty}^{\infty}\left\{u_k^{\infty\,2}+\left(\frac{R\overline{T}}{\gamma_k}\epsilon_k^{\infty}\right)^2\right\}dy+$$

$$+R\overline{T}\int_0^{\infty}dt\int_{-\infty}^{\infty}\frac{\partial}{\partial y}(v_k\,\epsilon_k)\,dy. \qquad [32]$$

Here $b\cdot u_k^{*2}$ is the initial energy of the entire field which consists just of the kinetic energy of the zonal current. The final energy of the field consists of kinetic and potential energy; the last term on the right hand side of [31] or [32] does not vanish as it might appear at first sight, but it represents the energy loss to infinity, as will be shown in the next paragraph. If, for instance, only the non-geostrophic component $v_k^o(y)$ were prescribed as the initial disturbance in a limited region, no adjustment would take place, i.e. the final state would be at rest. While the initial energy $\int_{-\infty}^{+\infty}\frac{1}{2}\,v_k^{o\,2}\,dy$ is not equal to zero, the kinetic and potential energy of the final state do vanish. Thus, the last term on the right hand side of [32] has to be identical to the energy lost to infinity; in this special example it is the entire initial energy.

If one computes the energy partitioning for the case of the zonal current perturbation for the different regions for both initial and final state, the following result is obtained:

Initial energy in the entire field

$$E_k^o = u_k^{*2}\cdot b$$

Final kinetic energy in the current

$$= E_k^0 \left[\frac{1}{2} e^{-\frac{2fb}{\gamma_k}} \left(\frac{\gamma_k}{2fb} \sinh \frac{2fb}{\gamma_k} + 1 \right) \right]$$

Final potential energy in the current

$$= E_k^0 \left[\frac{1}{2} e^{-\frac{2fb}{\gamma_k}} \left(\frac{\gamma_k}{2fb} \sinh \frac{2fb}{\gamma_k} - 1 \right) \right]$$

Final kinetic energy outside the current

$$= E_k^o \left[\tfrac{1}{2} e^{-\frac{2fb}{\gamma_k}} \left(\frac{\gamma_k}{fb} \sinh^2 \frac{fb}{\gamma_k} \right) \right] \qquad [33]$$

Final potential energy outside the current

$$= E_k^o \left[\tfrac{1}{2} e^{-\frac{2fb}{\gamma_k}} \left(\frac{\gamma_k}{fb} \sinh^2 \frac{fb}{\gamma_k} \right) \right]$$

Final energy of the entire field

$$E_k^{\infty} = E_k^0 \frac{\gamma_k}{2fb} \left(1 - e^{-\frac{2fb}{\gamma_k}} \right)$$

Energy loss to infinity

$$E_k^o - E_k^{\infty} = E_k^o \left[1 - \frac{\gamma_k}{2fb} + \frac{\gamma_k}{2fb} e^{-\frac{2fb}{\gamma_k}} \right]$$

The already mentioned work of Rossby (17) also gives a discussion of the energy transformations which is limited to the total energy of the initial and final state. Rossby obtains:

$$E_k^{\infty} = E_k^0 \frac{\gamma_k / fb}{1 + \gamma_k / fb + \frac{1}{3} fb / \gamma_k}$$

For small values of the parameter $fb/\gamma_k \ll 1$ the result is practically the same as the one given above. The difference $E_k^o - E_k^{\infty}$ between initial and final energy which is interpreted as an energy loss to infinity here was attributed by Rossby to inertial oscillations which he thought are not damped. Although it did not discuss the energetics, the later work of Cahn then showed that, even without friction, energy dispersion takes place and thus a final state free of oscillations can be reached.

Under the condition that $fb/\gamma_k << 1$ the following statements can be made from the above expressions: The initial energy inside the zonal current decreases by about $100 \cdot 2fb/\gamma_k\%$ through the adjustment process. The gain in potential energy within the zonal current due to the formation of a pressure field is negligibly small in comparison with the energy loss there, so that it hardly contributes to the energy budget. Approximately half the energy loss of the initial current can be found outside the current, where potential and kinetic energy increased equally; the other half escaped to infinity and is thus lost to the field. Numerical examples will be given in Section 7.

6. Discussion of the solutions

The solution for the center of a vortex as well as a zonal current until time $ft = fa/\gamma_k$ or $ft = fb/\gamma_k$, respectively, yielded, with the special initial conditions for (15) and (26) of the last two sections, pure inertial oscillations with a period of $2\pi/f$, motions which would occur with a time-independent pressure field; the particles begin to describe an inertial circle. Strictly speaking: every particle inside the disturbance region moves on an inertial circle until the edge of the disturbance which is traveling with velocity γ_k reaches the particle. Normally, this period of time is so short that the particles pass over only a small portion of the inertial circle. The pure inertial oscillations are followed by damped oscillations due to gravity-inerta waves. The fact that pure inertial oscillations occur at all is a result of the prescribed form of the initial disturbance. As can be seen, the time evolution of the solutions at a fixed point is a modified image of the initial conditions, which also follows from the theory of the characteristics of hyperbolic differential equations; in the case of a discontinuity at distance a, the behavior of the solution will be discontinuous at time $t = a/\gamma_k$. Since, with the given initial distribution, gravity waves can only form at the edge of the disturbance region, undisturbed inertial oscillation can occur in the inside until the gravity waves have penetrated into the region. Although the initial distributions might appear unnatural, they do not restrict the generality to any great extent; the solutions obtained demonstrate quite clearly the processes taking place during the adjustment.

The initial conditions will be discussed in more detail: One can limit oneself to the more general case of the vortex disturbance. To obtain a final stationary geostrophic state the initial conditions have to be a perturbation of the geostrophic equilibrium, either in the form of an unbalanced wind field or an unbalanced pressure field; in both cases only the perturbation of the equilibrium $\left(\frac{\partial D}{\partial t}\right)^o$ is important. A perturbation horizontal divergence as initial condition alone does not constitute a perturbation of the geostrophic equilibrium, however, and therefore does not lead to adjustment. A perturbation divergence could thus be superimposed on a perturbation vortex without influencing the final stationary state. The initial energy associated with the perturbation divergence is thus completely lost to infinity. For that reason the formation of a cyclonic vortex associated with a high pressure region in the center of the perturbation divergence is avoided, a situation apparently not impossible from [20] with $\varepsilon_k^o = C_k^o = 0$, $(D_k^o \neq 0)$. Only the differential equation [22] which follows from the addition of the geostrophic relation [21] shows that the final state has to be at rest, since there are no edges. What is valid for the perturbation divergence is also valid for the vertical component of the velocity; it is proportional to the variation of the pressure field according to [3a], which in turn is proportional to the horizontal divergence. Thus, prescribing a horizontal divergence or a pressure tendency alone as an initial condition does not lead to adjustment, neither does prescribing a vorticity tendency or a vertical velocity.

If, on the other hand, a supergeostrophic cyclonic vortex is prescribed, the vorticity will decrease with time and, at the same time, the pressure will drop; the vortex then remains cyclonic and the equilibrium with the low pressure area will be reached through damped oscillations. If an unadjusted low pressure area is prescribed initially, The pressure will rise with the simultaneous formation of a cyclonic vortex during the adjustment; the initial low pressure area remains with decreased intensity and is in geostrophic equilibrium with the newly formed cyclonic vortex in the final state. The pressure and vorticity changes are determined in both cases by the potential vorticity equation [13]; the differences between initial and final state are then equal in both cases, if the initial perturbations of the equilibrium $\left(\frac{\partial D}{\partial t}\right)^o$ were equal.

The statement that an initial perturbation divergence cannot produce a geostrophically adjusted field can also be applied to the case where a forced divergence is acting for a finite period of time, and the fields are then left undisturbed. Starting form a situation without pressure or velocity perturbations the (positive) horizontal divergence will from [12] force a pressure drop as well as an anti-cyclonic vortex. Even in the moment when the field is freed from the forced divergence--one can transform this moment into time $t = 0$ for reasons of convenience--the relation $\frac{1}{f} C_k^o = \frac{R\overline{T}}{\gamma_k^2} \varepsilon_k^o$ holds. From equation [22] it can be seen, however, for the same reason given before for the initial divergence, that the final state has to be at rest. To what extent these results can be applied to the dynamic processes in the "non-linear atmosphere" is left for discussion.

7. Examples for barotropic disturbances

The statements of the previous section shall be explained further with a few examples. Fig. 1 shows the time evolution of the horizontal divergence, vorticity and the pressure in the center of a barotropic perturbation vortex with initial conditions [16]; numerical values are $f = 10^{-4}\sec^{-1}$ and $fa/\gamma_k = 0.5$, which would correspond to an initial perturbation radius of 500 km in an ocean 1 km deep or a perturbation radius of 1000 km in an ocean 4 km deep. The equilibrium is then reached when

$$-\frac{1}{fC_k}\frac{\partial D_k}{\partial t} = \frac{R\overline{T}\,\nabla^2 \varepsilon_k}{fC_k} - 1 \equiv \frac{C_{kg}}{C_k} - 1 = 0$$

which, in this model, is practically after 5 hours, since at that time the adjustment is completed to 90%, as can be seen from the graph. After that damped oscillations about the geostrophic equilibrium state start in with relative deviations up to 10% to both sides. The vorticity decreased by 18% during the adjustment. The pure inertial oscillations which occur until time $ft = fa/\gamma_k = 0.5$ do not really materialize, since this time is just 1/13 of the inertial period; all that can be seen is an almost linear increase up to that time, and after that a very rapid decrease under damped oscillations with a period slightly shorter than the inertial period.

To obtain a solution with a given perturbation divergence one just has to substitute D_k by C_k and C_k^o by D_k^o in the first equation of [15]. With accordingly permuted initial conditions [16] the time evolution of vorticity and thus also pressure perturbation in this case would be the same as the time evolution of divergence in Fig. 1 (solid line); the final state is thus at rest, as was discussed earlier.

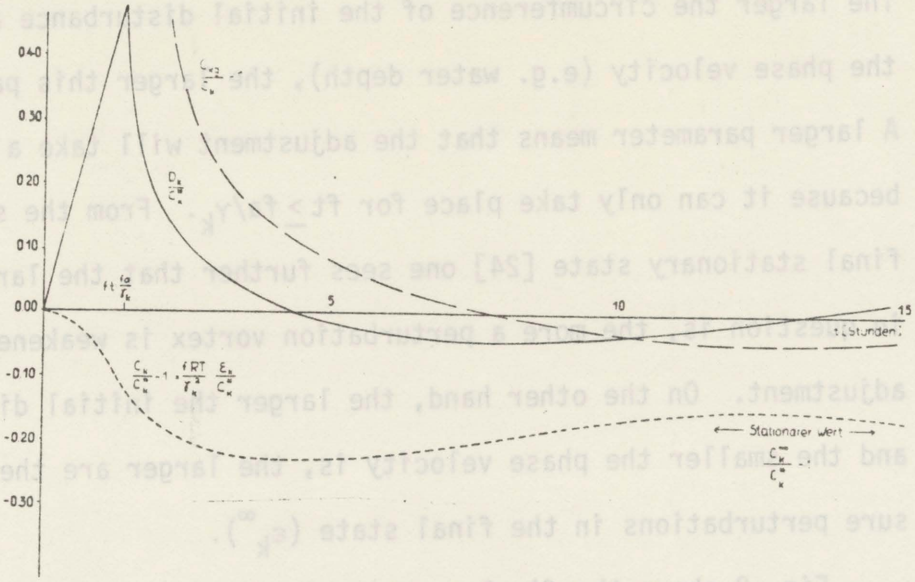

Fig. 1.

Time evolution of some field variables in the center ($r = o$) of an initially supergeostrophic barotropic vortex with constant vorticity $C_k^* \neq 0$ in the region $fr/\gamma_k < fa/\gamma_k = 0.5$ for $t = 0$. Here C_k and D_k are vorticity and horizontal divergence, $C_{kg} = RT/f \; \nabla^2 \varepsilon_k$ is the geostrophic vorticity. Time t is given in hours, and "Stationärer Wert" denotes the stationary value.

The parameter fa/γ_k, which basically corresponds to the square of Raethjen's adjustment parameter, has a decisive influence on the solutions. The larger the circumference of the initial disturbance and the smaller the phase velocity (e.g. water depth), the larger this parameter becomes. A larger parameter means that the adjustment will take a longer time, because it can only take place for $ft \geq fa/\gamma_k$. From the solutions for the final stationary state [24] one sees further that the larger the parameter in question is, the more a perturbation vortex is weakened during the adjustment. On the other hand, the larger the initial disturbance radius and the smaller the phase velocity is, the larger are the relative pressure perturbations in the final state (ε_k^{∞}).

Fig. 2 shows the final geostrophic state of an initially unadjusted vortex. The initial conditions and numerical values are the same as for Fig. 1. One can see that the vortex has weakened only a little and essentially maintained its shape. A region with weak anticyclonic vorticity has formed at the edge of the initial cyclonic perturbation vorticity; this anticyclonic vorticity just outside the edge is equal to the decrease of cyclonic vorticity just inside the edge. The newly formed low pressure area has a larger circumference than the initial perturbation vortex; this is plausible since the initial velocity field, which has a larger circumference than the original perturbation vortex, is determining the formation of the pressure field. To illustrate this fact, initial as well as stationary angular velocity ϕ_k^o and ϕ_k^{∞} have been entered into Fig. 2. It is also possible, of course, to prescribe a vortex with an associated wind field limited to $r < a$, which, for instance, has a constant angular velocity $\phi_k^o = C_k^*/2$ there, but has an angular velocity equal to zero for $r > a$. Such a vortex, again with $fa/\gamma_k = 0.5$,

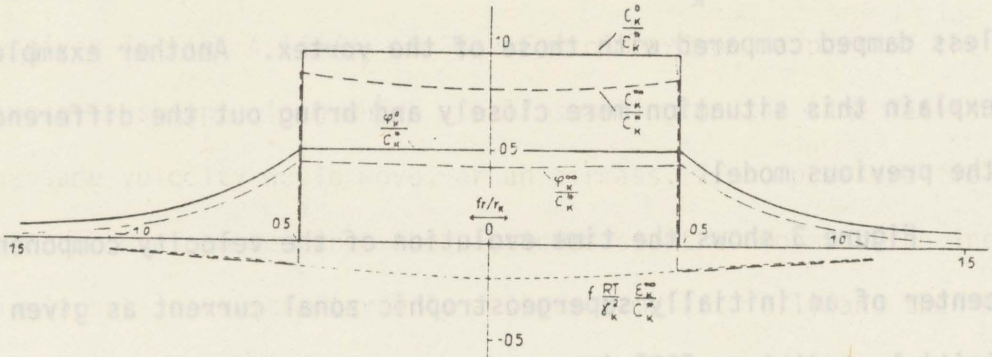

Fig. 2.

Initial state (solid line) and final state (dashed line) of a barotropic perturbation vortex with rotational symmetry and constant vorticity $C_k^* \neq 0$ in the region $fr/\gamma_k < fa/\gamma_k = 0.5$ for $t = 0$. The graphs show the following variables normalized by C_k^*: initial vorticity $C_k^0(r)$ and angular velocity $\phi_k^0(r)$, vorticity and angular velocity after completed adjustment $C_k^\infty(r)$ and $\phi_k^\infty(r)$, as well as the corresponding pressure field $(fR\bar{T}/\gamma_k^2)\varepsilon_k^\infty$.

would be weakened by just 6%, and the adjustment would proceed considerably faster, it would be practically finished at time $ft = fa/\gamma_k$.

The case of an ageostrophic zonal current is similar, except that the parameter fb/γ_k governs the solutions and that the oscillations are less damped compared with those of the vortex. Another example will explain this situation more closely and bring out the differences from the previous model.

Figure 3 shows the time evolution of the velocity components in the center of an initially supergeostrophic zonal current as given by the initial conditions [27] (the pressure perturbation always remains zero in the center of the zonal current). The parameter fb/γ_k was chosen to be 0.1 here, which would correspond to a width of about 500 km in an ocean 6 km deep, or a width of 1000 km in an ocean 25 km deep. As can be seen, the oscillations go on for a longer time in spite of the samller parameter, and thus they are less strongly damped than in the case of the vortex. As the curves for

$$\frac{-R\bar{T}\frac{\partial \varepsilon_k}{\partial y}}{f\,u_k} - 1 \equiv \frac{u_{kg}}{u_k} - 1$$

(relative deviation from the geostrophic equilibrium) show, the masses perform oscillations around the equilibrium state for $ft > 0.1$, but with such a small amplitude that one can say the adjustment is practically completed at time $ft = fb/\ {}_k = 0.1$. An analogous result was also found for a vortex with a wind field limited to the region $r < a$. The curve for $\frac{u_k}{u_k^*} - 1$ indicates the relative weakening of the (zonal) velocity during the adjustment as well as the meridional displacement (southward displacement for supergeostrophic west wind on the Northern hemisphere) of the center of the zonal current $(y = 0)$ in units of fy/u_k^*. This

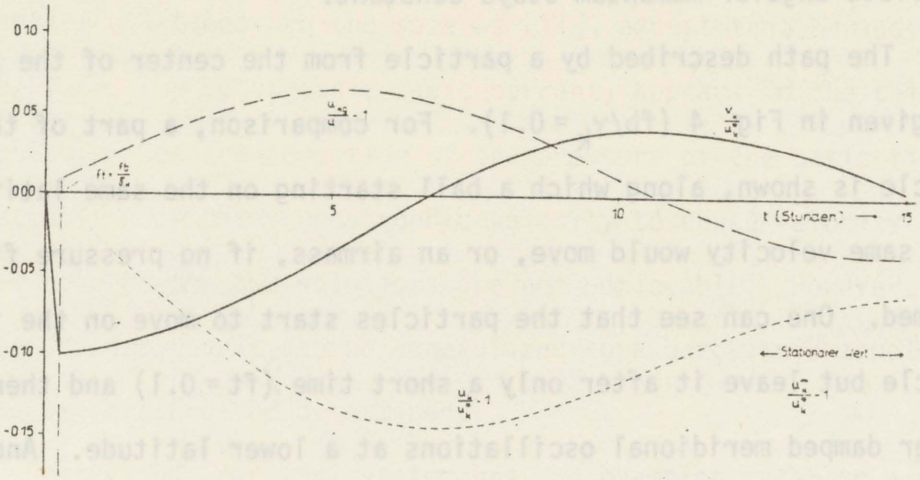

Fig. 3.

Time evolution of some field variables in the center ($y = 0$) of an initially supergeostrophic zonal current with constant zonal velocity $u_k^* \neq 0$ in the region $f|y|/\gamma_k < fb/\gamma_k = 0.1$. Here u_k, v_k are the horizontal velocity components, $u_{kg} = -(R\bar{T}/f)(\partial\varepsilon_k/\partial y)$ the geostrophic velocity. Time is given in hours, and "Stationärer Wert" denotes the stationary value.

follows form the first equation of system [11] (with $\frac{\partial}{\partial x} = 0$) after integration over time. This equation expresses in simplified form that the absolute angular momentum stays constant.

The path described by a particle from the center of the zonal current is given in Fig. 4 ($fb/\gamma_k = 0.1$). For comparison, a part of the inertial circle is shown, along which a ball starting on the same latitude with the same velocity would move, or an airmass, if no pressure field were formed. One can see that the particles start to move on the inertial circle but leave it after only a short time ($ft = 0.1$) and then move zonally under damped meridional oscillations at a lower latitude. Another example of the particle movement ($ft = 1.0$) can be found in the already mentioned work of Cahn.

The energy transformations that have taken place during the adjustment will be briefly discussed with the aid of this model. The different contributions to the energy budget will be normalized by the initial energy of the entire field. With $fb/\gamma_k = 0.1$ [33] yields the following situation: in the final state an energy of 82.42% remains in the zonal current; 82.15% of that is kinetic energy, the rest potential energy. The increase of potential energy in the zonal current is thus very small compared to the total loss. This total loss of the zonal current of 17.58% has to be compared with an energy gain outside the zonal current of 8.22%, which is distributed evenly between potential and kinetic energy. The percentage of energy which has been actually lost to infinity is 9.36%. At the beginning it was stated that the equations [2] also describe in an approximate fashion motions relative to a barotropic geostrophic basic state. This statement is valid if this (zonal) basic velocity is small compared to the phase velocity γ_k. In the case of barotropic disturbances

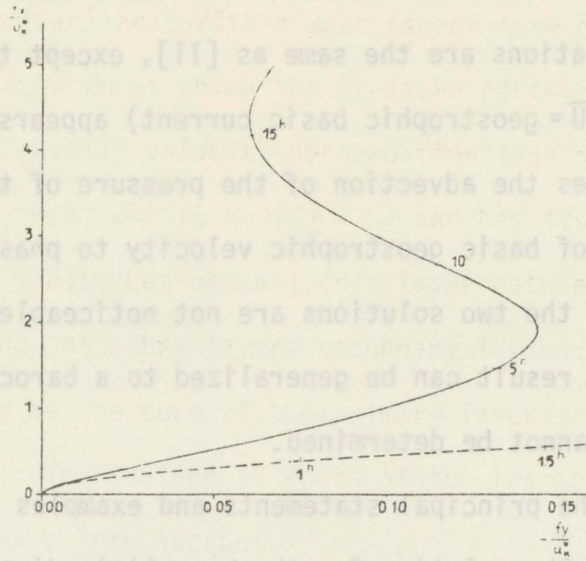

Fig. 4.

Trajectory of a particle initially at the center (y = 0) of the zonal current (solid line) and inertial circle (dashed line). Conditions are the same as for Fig. 3.

it is possible to solve the differential equations, if an infinitely wide, zonal and constant velocity field is given as the basic current, on which an ageostrophic zonal current with limited width is superimposed. The differential equations are the same as [11], except that an additional term $-(f\bar{U}/RT)\cdot v_k$ ($\bar{U}$ = geostrophic basic current) appears in the third equation, which describes the advection of the pressure of the basic field. Up to a ratio 1/10 of basic geostrophic velocity to phase velocity the differences between the two solutions are not noticeable. However, the extent to which this result can be generalized to a baroclinic zonal current disturbance cannot be determined.

So far the principal statements and examples refer to a barotropic disturbance. The solution for the baroclinic disturbances in a compressible atmosphere are composed of the solutions for barotropic disturbances, as was shown. The solution with the smallest eigenvalue $\lambda_1 = 0$, or equivalently with the largest phase velocity $\gamma_1 = 2\sqrt{(1-1/\kappa)R\bar{T}}$ corresponds to an initial vertical distribution proportional to $\zeta e^{-\frac{gh}{R\bar{T}}\zeta}$, a distribution which leaves the sign of all perturbation quantities in all layers $(0 \leq z < h)$ unchanged. Such a vortex extending over all layers with this initial distribution would have the same vertical structure in the final geostrophic state. Its solution with respect to the horizontal would be totally analogous to that of an homogeneous ocean of depth $4(1-1/\kappa)(R\bar{T}/g)$. The examples given in the previous chapter which referred to a barotropic disturbance are thus also applicable to, among others, this special non-barotropic all-layer-disturbance which occurs in the atmosphere.

If this disturbance is only prescribed in a partial layer of the atmosphere, further terms of the Fourier series [6] have to be added to represent the initial vertical distribution. The smaller the vertical extent of the initial disturbance is, the larger is the contribution of the higher order terms and the later the series converges. Since the higher order terms are associated with larger eigenvalues and thus with smaller phase velocities the following conclusion, which is important for the adjustment in the atmosphere, can be drawn: the smaller the vertical extent of the initial disturbance, the longer it takes for the adjustment, the more the initial disturbance is weakened, and the smaller are the relative pressure changes. Thus, the vertical extent of a baroclinic disturbance has an analogous influence on the adjustment as the depth (phase velocity) has in the case of a barotropic disturbance which was discussed in the previous chapter.

8. Description of the results

For the numerical computation of the adjustment we set $\frac{2\kappa}{2-\kappa}\frac{R\overline{T}}{gh} = 1$, as was mentioned earlier. With a mean temperature of $\overline{T} = 250°K$ this results in a height h of 32 km at which the pressure perturbations are required to vanish. The initial velocity perturbation thus occupies the layer between 8 km ($\zeta = 6/8$) and 16 km ($\zeta = 4/8$) and has its maximum a little above 12 km ($\zeta = 5/8$). Let us call this layer between 8 km and 16 km the primary layer and the other layers secondary layers; the height 12 km shall be defined as the core of the primary layer. To represent the initial vertical distribution 17 roots of the transcendental equation [7] had to be taken into account.

For the case of the vortex model the maximum vorticity at the 12 km level was assumed to be $C° = 10^{-6} \sec^{-1}$ and the radial extent was set at 500 km. The total width of the zonal current shall be $2b = 500$ km, where a linear decrease of the current velocity from the center $y = 0$ to both sides has been assumed for reasons of better graphical representation; the initial velocity for $y = 0$ in the core of the primary layer shall be $u° = 1$ m/sec. Other details can easily be seen in the figures.

Let us first consider the results for the adjustment of a vortex. Fig. 5 is a diagram of isopleths of horizontal divergence at $r = 0$. The prevailing imbalance initially forces the masses of the primary layer out, whereas no horizontal compensating motions are apparent in the secondary layers at first. This results in a mass deficit averaged over all layers, which will generally lead to a pressure drop. Later on, in this model after about half an hour, a contrary motion (horizontal convergence) starts to develop in the secondary layers which compensates the mass loss of the initial layer. After about 6 hours the divergent

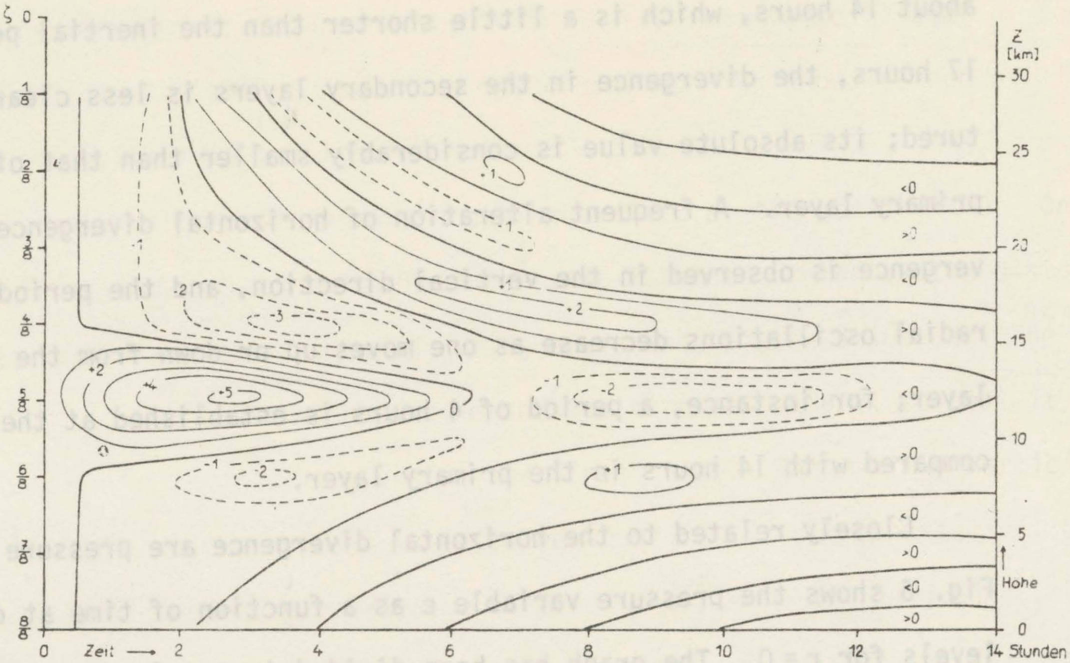

Fig. 5.

Diagram of isopleths of horizontal divergence $10^7 D$ (sec^{-1}) in the center ($r = 0$) of an initially supergeostrophic vortex, the vorticity of which at $t = 0$ is non-zero only in the region $r < a = 500$ km and $4/8 < \zeta < 6/8$. The abscissa shows time in hours, the ordinate height.

component of motion reverses in the primary layer. While the primary layer exhibits clearly structured radial oscillations with a period of about 14 hours, which is a little shorter than the inertial period of 17 hours, the divergence in the secondary layers is less clearly structured; its absolute value is considerably smaller than that of the primary layer. A frequent alteration of horizontal divergence and convergence is observed in the vertical direction, and the periods of the radial oscillations decrease as one moves up or down from the primary layer; for instance, a period of 4 hours is established at the ground compared with 14 hours in the primary layer.

Closely related to the horizontal divergence are pressure changes. Fig. 6 shows the pressure variable ε as a function of time at different levels for $r = 0$. The graph has been divided in two for graphical reasons: "Unten" (below) refers to the pressure variations in and below the primary layer, "Oben" (above) denotes the pressure variations in and above the primary layer. Initially the pressure drops everywhere, which could already be concluded from the vertical distribution of the horizontal divergence. Aside from the fact that the relative pressure changes in the primary layer are by far the greatest, the pressure minimum in the other layers will be reached earlier with increasing distance from the primary layer. At the ground ($\zeta = 8/8$) for instance, the first pressure minimum appears after only one hour, whereas it can be seen after 5 hours in the core of the primary layer ($\zeta = 5/8$). The pressure oscillations, just like the radial oscillations, thus have a much shorter period at the ground than in the primary layer; the pressure oscillations are strongly damped, especially in the secondary layers. The distance between the curves is proportional to the temperature perturbation τ in the

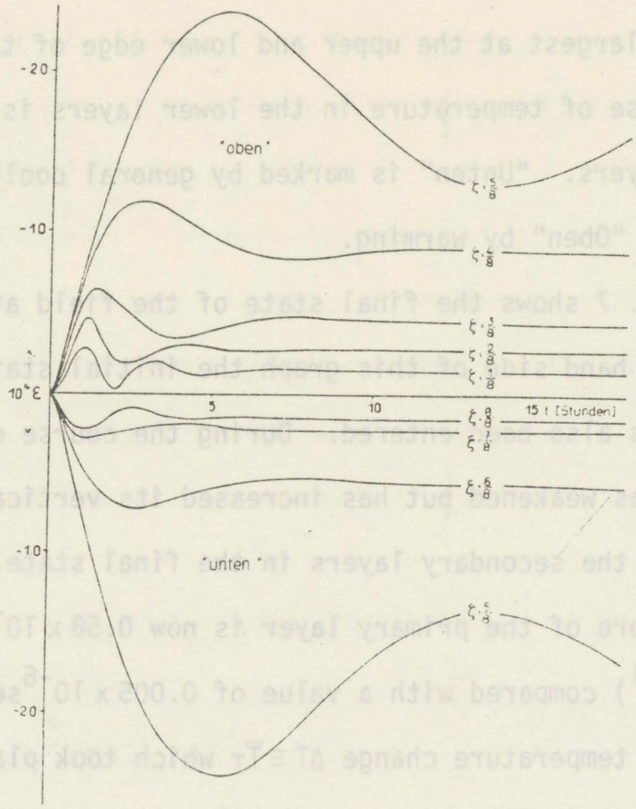

Fig. 6.

Time evolution of the relative pressure perturbation ε at different levels ζ in the center of an initially supergeostrophic vortex (see also Fig. 5).

intermediate layer [3c]. From there one sees the temperature perturbations are the largest at the upper and lower edge of the primary layer, and that the course of temperature in the lower layers is inverse to that of the upper layers. "Unten" is marked by general cooling under damped oscillations, "Oben" by warming.

Fig. 7 shows the final state of the field at the center $r = 0$. On the left hand side of this graph the initial state of the vortex disturbance has also been entered. During the course of the adjustment the vortex has weakened but has increased its vertical extent, so that it also occupies the secondary layers in the final state. The maximum vorticity in the core of the primary layer is now $0.58 \times 10^{-6} sec^{-1}$ (initially $1.00 \times 10^{-6} sec^{-1}$) compared with a value of $0.005 \times 10^{-6} sec^{-1}$ at the ground.

The temperature change $\Delta T \equiv \overline{T}\tau$ which took place during the adjustment can be seen in the center of Fig. 7. One sees that the temperature dropped in the lower layers and increased by about the same amount in the upper layers. These temperature changes can be attributed to the most part to adiabatic lifting (lower layers) or sinking (upper layers), respectively. The effect of the local pressure changes on the temperature is small, however. At the ground, where the vertical velocity is supposed to vanish, the temperature change is caused by the pressure decrease there alone. A maximum temperature decrease of about 0.05°C at the edges of the primary layer has to be compared with a change of about 0.001°C at the ground.

The right half of Fig. 7 shows the pressure fall which took place during the adjustment. The relative pressure changes are by far the greatest in the primary layer, their absolute value is about 12 times larger than at the ground (1.70×10^{-4} versus 0.14×10^{-4}).

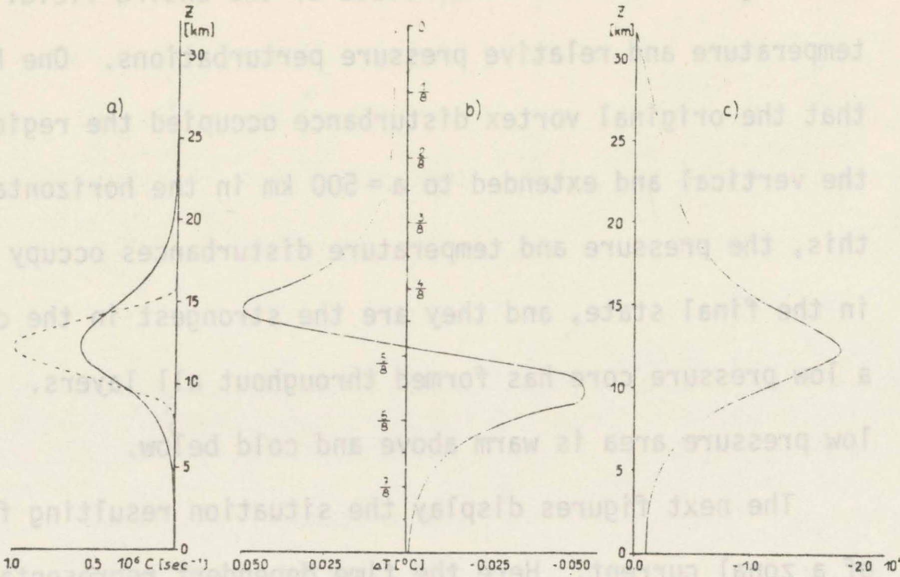

Fig. 7.

Vertical distribution of some field variables in the center of an initially supergeostrophic vortex: a) vorticity, initial state dashed line, final state solid line; b) final state of the temperature perturbation; c) final state of the relative pressure perturbation.

Fig. 8 shows the final state of the entire field. Plotted are the temperature and relative pressure perturbations. One has to remember that the original vortex disturbance occupied the region $1/4 < \zeta < 3/4$ in the vertical and extended to $a = 500$ km in the horizontal. Compared to this, the pressure and temperature disturbances occupy a larger space in the final state, and they are the strongest in the center $r = 0$, where a low pressure core has formed throughout all layers. The newly formed low pressure area is warm above and cold below.

The next figures display the situation resulting from the adjustment of a zonal current. Here the time dependent representation of the different perturbation values was omitted and only the final geostrophic state was computed.

In Fig. 9 the initial state of the velocity field is plotted to the right (the wind is blowing into the plane defined by the paper), as well as the final state to the left, where only half of the field is shown because of the symmetry about $y = 0$. A considerable weakening of the initial current is apparent in the final state, the maximum velocity decreased from 1 m/sec to 0.64 m/sec. On the other hand, an expansion of the current to the secondary layers took place. At both lateral edges of the initial current small countercurrents have formed, which occupy only the middle layers.

Fig. 10 shows the corresponding pressure field. During the adjustment, the pressure did not change at the center of the zonal current. Looking in the direction of the current, a high pressure area has formed to the right and a low pressure area to the left with centers at about $y = 3/4\,b$. The pressure changes are again essentially limited to the primary layer. The relative pressure changes have a maximum of 1.48×10^{-4} and decrease to 0.05×10^{-4} at the ground.

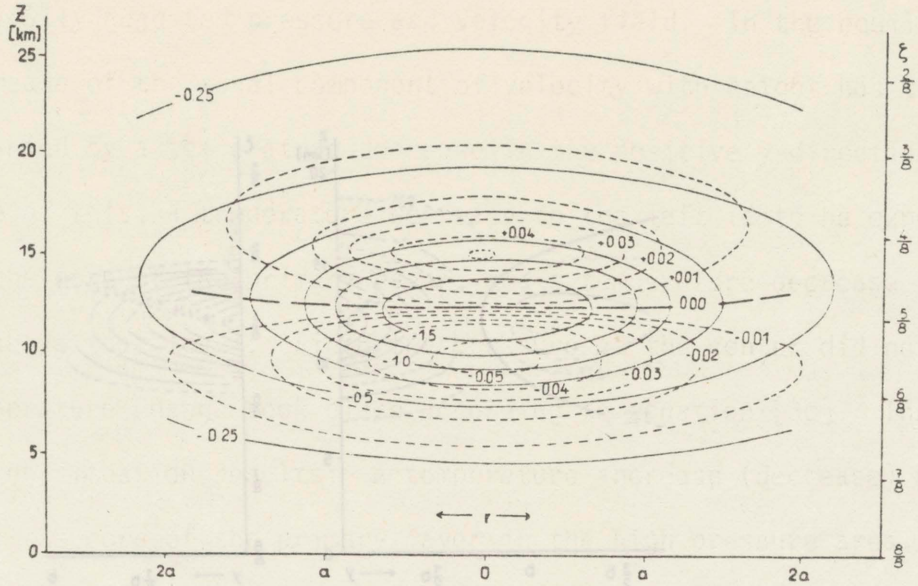

Fig. 8.

Distribution of the temperature perturbation and the relative pressure perturbation after completed adjustment of an initially supergeostrophic vortex; dashed lines: temperature perturbation (°C); solid lines: relative pressure perturbation (10^4 ε); (a = 500 km).

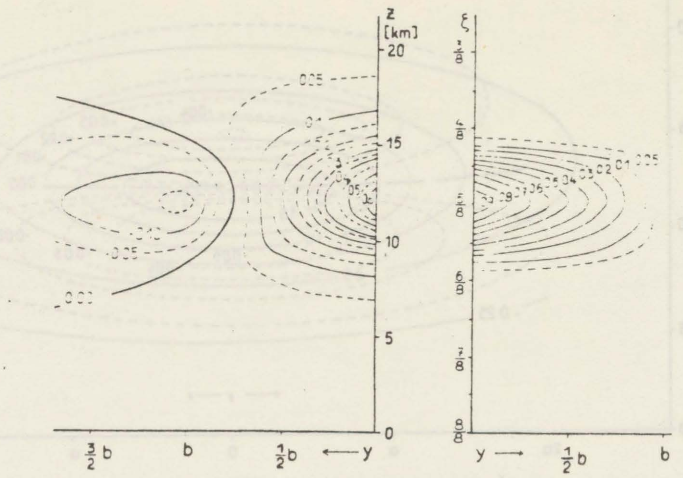

Fig. 9.

Distribution of the zonal velocity u (m/sec) at time t = 0 (right hand side) and after completed adjustment (left hand side) with 2b = 500 km.

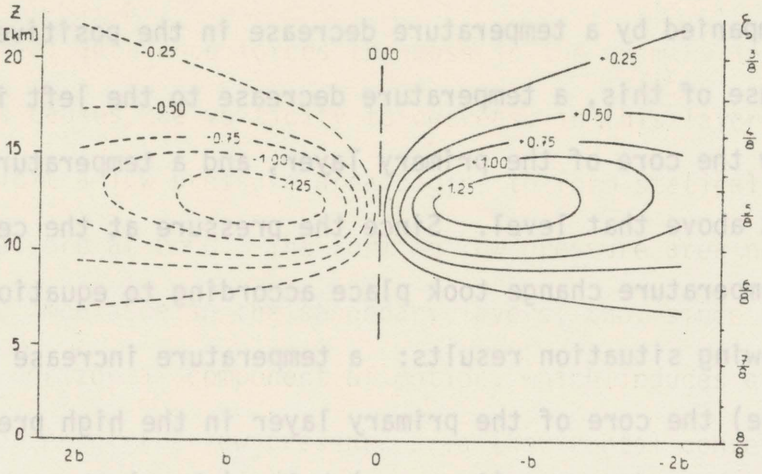

Fig. 10.

Distribution of the relative pressure perturbation (10^4 ε) after completed adjustment of the zonal current of Fig. 9 (2b = 500 km).

Fig. 11 shows the temperature field in equilibrium with the geostrophically adjusted pressure and velocity field. In the equilibrium, an increase of the zonal component of velocity with height has to be accompanied by a temperature decrease in the positive y-direction. Because of this, a temperature decrease to the left is to be expected below the core of the primary layer, and a temperature decrease to the right above that level. Since the pressure at the center did not change, no temperature change took place according to equation [3c]. Thus, the following situation results: a temperature increase (decrease) below (above) the core of the primary layer in the high pressure area, and a temperature decrease (increase) below (above) the core of the primary layer in the low pressure area, with maximum absolute values of 0.05°C.

The changes of pressure and temperature which took place during the adjustment of the zonal current exhibit great similarity with the interdiurnal pressure and temperature changes as they are observed in the atmosphere mainly in the region of influence of a high-tropospheric jet-stream. This fact will be discussed in more detail in the last section.

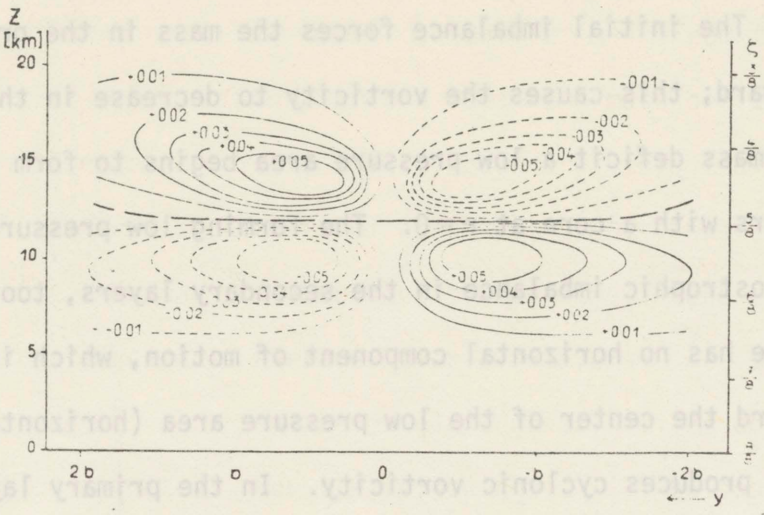

Fig. 11

Distribution of the temperature perturbation (°C) after completed adjustment of the zonal current of Fig. 9 (2b = 500 km).

9. Explanation of the results

The mechanism of adjustment can be deduced form the obtained results without any difficulty. Let us consider the vortex model first.

The initial imbalance forces the mass in the primary layer to move outward; this causes the vorticity to decrease in this layer, and due to the mass deficit a low pressure area begins to form statically in all layers with a core at $r = 0$. The forming low pressure area now produces a geostrophic imbalance in the secondary layers, too, since the mass there has no horizontal component of motion, which induces a movement toward the center of the low pressure area (horizontal convergence) and thus produces cyclonic vorticity. In the primary layer the processes taking place initially are: horizontal divergence → pressure fall and vorticity decrease. On the other hand, for the secondary layer the results indicate: pressure fall → horizontal convergence → formation of a cyclonic vortex. The counteracting divergences in the secondary layers have the result that the mass loss of the primary layer is compensated (dynamic compensation); as a consequence, the surface pressure stops falling. Furthermore, the vertical motions associated with the counteracting divergences induce a temperature decrease in the lower layer--underneath the core of the primary layer--and a temperature increase in the upper layers. This temperature decrease in the lower layers, which also takes place when the dynamic compensation stops the pressure fall at the ground, results in the continued pressure fall outside the surface layer. According to the barometric height formula [3c], the absolute value of the relative pressure change increases with height and reaches a maximum in the core of the primary layer. Above the core of the primary layer, the relative pressure changes decrease with height due to the warming and vanish at height h as required by the boundary condition.

As time goes on the mass inflow in the secondary layers and through the top dominates the weakening mass outflow in the primary layer; the surface pressure rises and the pressure increase slowly expands into higher layers. Under damped oscillations a final state is reached which is qualitatively equivalent to the state of the system after the first time steps: A weakened cyclonic vortex in geostrophic equilibrium with a newly formed low pressure area in the primary layer, and in the secondary layers a newly formed low pressure area in geostrophic equilibrium with a newly formed cyclonic vortex; cooling underneath, warming above the core of the primary layer. These results are qualitatively the same as those obtained in section 3 from the barotropic equations [11]: in the primary layer the adjustment takes place because of an initially supergeostrophic vortex, in the secondary layers an imbalanced pressure field appears as an initial condition, as it were.

The above schematic picture of the processes taking place during adjustment was basically derived by Raethjen (14) under simplified assumptions. However, this pattern appears only during the first time steps (until 2 hours) of the model described here, since later on the counteracting divergences are not limited to the primary layer on the one hand and the secondary layers on the other hand, but horizontal divergence and convergence occur alternately at different levels.

How large is the mass deficit resulting from the adjustment process of a vertical column with crossectional area of 1 m^2 extending from the ground to the height h at $r = 0$; how much mass--this is the essence of the question--has flown through the lateral boundaries on the one hand and the top boundary on the other hand? This question can easily be answered with the aid of the tendency equation:

$$-\frac{\partial}{\partial t} \Delta P_0 = g \int_0^h \bar{\varrho}\, D dz + g (\bar{\varrho} w)_h \qquad [34]$$

where

$$(\bar{\varrho} w)_h = \frac{\varkappa}{\varkappa - 1} \bar{\varrho}_h R \frac{\partial \Delta T_h}{\partial t}$$

(vertical mass flux through the upper boundary $z = h$). From the surface pressure change $\Delta P_o \equiv P_o \varepsilon_o = 0.0145$ mb one obtains a mass deficit of about 150 g/m^2. 5 g/m^2 entered through the top boundary; this leaves a mass flux out of the lateral boundaries of about 155 g/m^2. Thus almost the entire mass deficit is caused by outflow through the lateral boundaries, i.e. a horizontal divergence averaged over all layers.

The adjustment of a current takes place under similar conditions as the adjustment of a vortex. For the following it will be assumed that the current initially posesses a westerly velocity. Due to the Coriolis force all moving particles are displaced to the right at first, since no balanced pressure field is present yet. Hence, the entire current is shifted to the South, where the magnitude of the shift is dependent on the local velocity. The center of the current then experiences the strongest displacement; this requires horizontal convergence ($\partial v/\partial y < 0$) in the southern half of the primary layer and horizontal divergence ($\partial v/\partial y > 0$) in the northern half; in the primary layer the associated mass inflow or outflow, respectively, which is initially not compensated by counteracting motions in the secondary layers produces a pressure increase in the South and a pressure decrease in the North in all layers. The compensating motions which follow lead to warming in the lower part of the forming high pressure area and the upper part of the low pressure area and to cooling in the upper part of the high pressure area and the lower part of the low pressure area. With the shift to the south the air parcels lose westerly momentum or acquire easterly momentum; i.e., the west wind in the center of the current decreases while at the lateral edges an easterly wind is forming (in the primary layer). This meridional displacement does not occur uniformly during the adjustment but in the form of damped oscillations; every point of the current describes a motion similar to that in Fig. 4. The meridional component produced by the initial imbalance leads to the weakening of the zonal component. The meridional component partially builds up the pressure field and the rest is lost to infinity as meteorological noise.

The foregoing explanation was based on the acting forces and the resulting accelerations; the same result is obtained by considering derived quantities, vorticity and horizontal divergence, as was done for the vortex model. The explanations mentioned there can be directly applied to the situation of the current if one keeps in mind that the northern part initially has cyclonic vorticity and the southern part anticyclonic vorticity.

The stability of the basic flow without a doubt plays an important role during adjustment since to a large extent it determines the temperature changes associated with the vertical motions; the disturbances whould expand less from the primary layer into the secondary layers with increasing stability. On the other hand, no temperature changes should be caused by vertical motions in the case of an indifferent stratification, and the disturbances should be distributed evenly from the primary layer during the adjustment. This statement will be examined briefly in the following.

In the case of neutral stratification of the basic field in [1] and the following equations $\overline{T}$ must be replaced by $\overline{T}(z) = \overline{T}_o - \Gamma z$, where $\overline{T}_o$ is the temperature at the ground and Γ = g/cp the adiabatic temperature gradient; in equations [2] the last term drops out of the adiabatic equation, and the continuity equation takes the form:

$$\frac{\partial\sigma}{\partial t} + D + \frac{1}{\bar\varrho}\frac{\partial \bar\varrho w}{\partial z} = 0.$$

From the adiabatic equation and the hydrostatic relation one obtains the following final stationary state of the mass field for an initial velocity disturbance:

$$\frac{\partial}{\partial z}\left(\bar T(z)\cdot \varepsilon^{\infty}\right) = 0$$

From this it follows further that the stationary velocity field is also not dependent on z, since the final state is required to be in geostrophic equilibrium. It should be noted here that the assumption of hydrostatic equilibrium only has to be required for the final state.

The final state of the entire field can also be derived. It is obviously not possible to use the same boundary conditions as in the isothermal model, since the condition ε=o at z=h would be in contradiction with the result [35] that T(z) ε is independent of height. This can be resolved by assuming that the vertical mass flux not only vanishes at the ground but also at the top of the "adiabatic atmosphere" with height h* = To/Γ~26km. After some manipulations one obtains the following final state for the (auto barotropic) "adiabatic atmosphere" for an initial velocity disturbance C_1° (x,y,z):

$$f^2(\bar T\varepsilon^{\infty}) - RT_0\nabla^2(\bar T\varepsilon^{\infty}) = \frac{gf}{R}\int_0^{h^*}\frac{\bar\varrho}{\bar\varrho_0}C^0\,dz \qquad [36]$$

$$C^{\infty} = \frac{R}{f}\nabla^2(\bar T\varepsilon^{\infty})$$

The solution of this differential equation was already discussed in section 4 [23]. The only assumption involved in the derivation of the above equation is that the final state is in hydrostatic equilibrium; this is important since, before reaching the stationary state, vertical accelerations can be expected to be so large (especially during the first time steps of this "adiabatic" model) that hydrostatic equilibrium should not be assumed. The disturbances are thus evenly distributed over all layers during adjustment so that in the final state the velocity is independent of height. The temperature changes during adjustment, but only as a consequence of local pressure changes; the temperature changes thus have the same sign at every height.

Let us now determine the influence of some parameters. The influence of the depth of the layer was already discussed (section 8). The influence of the radial extent of the initial disturbance on the adjustment process basically follows the considerations of the barotropic model: The larger the horizontal extent is, the larger are the changes of the field, in the primary layer as well as in the secondary layers, and the later the final geostrophic state will be reached. With a stable stratification of the basic field a perturbation vortex with small diameter will practically be confined to the primary layer, it will be weakened only slightly during adjustment and the pressure and temperature changes will remain small. The larger the vortex, the larger are the divergences, the pressure and temperature changes, and thus the larger the influence on the other layers. This applies analogously to the lateral extent of a current.

10. Conclusions

Comparing the results obtained in the previous sections with the processes in the atmosphere, certain similarities can be detected. For example, it is observed frequently that a mass of cold air traveling south at high levels increases its intensity: the vorticity increases and the tropospheric air mass continues to cool. The primary effect of the southward shift is the increase in relative vorticity, the tendency of the pressure field to adjust to the changed vorticity field being just a secondary effect; this leads to all the consequences described here, including the cooling of the lower layers.

In the adjustment model of section 8, a supergeostrophic vortex with a maximum vorticity of $C = 10^{-6}$ sec^{-1} was assumed; this vorticity would occur without the associated pressure field if an air mass at middle latitudes at the tropopause level were shifted to the South by approximately 60 km. If the vortex which formed this way corresponded approximately to the model described here, then the tropospheric cooling due to the adjustment would be about 0.02-0.03°C in the case of half-adiabatic stratification, and the warming in the stratosphere with iso thermal stratification would be about 0.05°C. Considerable meridional displacements are thus necessary to obtain an observable effect. The cooling of the troposphere is without a doubt not only due to the adjustment, but is associated with the vertical motions caused by surface friction. As a consequence of the supergeostrophic vortex at the tropopause level a low pressure area also forms at the ground which will have a velocity component directed toward the center of the low pressure system due to friction, which in turn will induce an ascending air motion for reasons of continuity and thus cooling in the low. How strongly the low at the ground will develop depends on the stability of the troposphere; it should appear stronger with decreasing stability of the troposphere.

In this context it might be possible to explain why over warm ocean areas - e.g. the Mediterranean - low pressure areas suddenly form at the ground in the winter when a cold air mass is slowly moving south aloft. As long as the cold air mass was over mainland cyclogenesis at the ground was inhibited by the very stable stratification there. It is not until the relatively warm sea is reached that cyclogenesis occurs at the ground due to the decrease in stability.

Even more interesting are the connections between the adjustment of a zonal current and the processes in the frontal zone. These connections will be briefly described in the following, although much of this might already be known since it can be directly derived from the equations without complicated computations.

Since the parcels move faster or slower, respectively, than the pressure field, supergeostrophic winds, i.e. Coriolis forces larger than the pressure forces, will dominate in the outflow of a frontal zone; conversely, subgeostrophic winds will dominate in the inflow region of a frontal zone. The processes in these situations of course can not be directly compared with the models since in the atmosphere these fields persist for longer periods of time, which means adjustment and perturbation of the equilibrium go on constantly. The reason for these continuing disturbances can not be explained from the models, since all external forces such as friction, advection and dependence of the Coriolis parameter on latitude were neglected, and, what is the most important, a barotropic basic field was assumed which does not correspond to the actual conditions. It is still not useless to develop simple models since they do reproduce much of what can be observed in nature. It should not be too disturbing, then, that in the atmosphere adjustment takes place constantly and thus the compensating motions develop continuously while in the model they only lasted for a short time. Qualitatively one should be able to obtain a correct conceptual idea of the processes in the atmosphere through a comparison with the corresponding results of the model computations (Fig. 9, 10, 11).

If one considers the inflow region and outflow of a high tropospheric frontal zone as one system, then the following processes should take place due to the gradient adjustment. In the inflow region pressure differences are decreased and wind velocities increased in the "primary layer" (upper troposphere): conversely, intensification of pressure differences and a decrease of velocity takes place in the outflow. Associated with this is an ageostrophic motion from high to low pressure in the inflow region, and from low to high pressure in the outflow. This distribution of the ageostrophic components was also documented empirically by Murray and Daniels (12). Due to the vertical motions induced by adjustment the temperature falls in the inflow region in the lower layers of the high pressure area (on the right hand side looking in the direction of the current) and rises in the low pressure region; the temperature changes in the outflow are exactly opposite. The temperature changes in the upper layers (lower stratosphere) are inverse to those in the lower layers. We will further discuss this point later in connection with the divergence theory. Energetically, the pressure does the work necessary to increase the kinetic energy downstream in the inflow region, causing the internal energy to increase in the high pressure region and to decrease in the low pressure region (in the troposphere). The circulation that develops between high and low pressure is thus thermally direct: it works as a heat engine. Conversely, in the outflow the kinetic energy is decreased by the amount that is necessary to build up the pressure field and thus increase the internal energy in the high pressure region and decrease it in the low pressure area. Assuming symmetric conditions would result in the compensation of the increase of kinetic energy and decrease of potential energy in the inflow region by decrease of kinetic energy and increase of potential energy of the same magnitude in the outflow. If no energy is supplied externally the system inflow region-outflow is energetically balanced; what is gained on one side is lost on the other and vice versa, and the whole quasi-steady system could move slowly down stream. Symmetry can not exist, however, because during the process of obtaining geostrophic equilibrium a continuous energy dissipation - not only through friction - takes place in the inflow region as well as in the outflow. To obtain a quasi-steady state as is often observed in nature, the gain of kinetic energy in the inflow region has to be larger than the loss in the outflow, i.e. the pressure field has to do more work in the inflow region than it gains energy in the outflow. This is only possible if in the inflow region the temperature field constantly builds up the pressure field, e.g. by confluence of air masses of different temperatures as was assumed by Namias and Clapp (13) in their confluence theory. The driving mechanism then lies in the thermodynamic energy, which is only reasonable; the thermally direct circulations and thus also subgeostrophic winds then have to dominate in the mean.

In contrast to this are the statistical findings of Faust (3, 4, 5, 6) who found that the system inflow region and outflow of the high tropospheric frontal zone is supergeostrophic in the mean and thus works as a cooling machine. This result followed from considerations of the mean interdiurnal pressure and temperature changes in the frontal zone as well as from the determination of the mean ageostrophic component. Exactly as in the model of the supergeostrophic zonal current, it followed empirically that in the mean the pressure falls on the left hand side of the frontal zone and rises on the right hand side and that an ageostrophic component toward the high pressure dominates. Faust further compared the relative pressure changes in the upper troposphere with those at the ground as they depend on stability; he found further, in analogy to our results, that the relative pressure changes at the ground get larger relative to those aloft with decreasing stability. Faust concluded from this that the origin of the pressure changes is in the upper layers.

It is not easy to explain why the frontal zone should be supergeostrophic in the mean, since this finding is in contradiction with the general idea that the kinetic energy is supplied by the pressure field; the theoretical investigations by Hollman (10) and others which try to explain a supergeostrophic frontal zone are not quite convincing, as a generally satisfactory theory of the formation and maintenance of the frontal zone does not yet exist.

Since we discussed the motions in the inflow region and the outflow of a frontal zone in such detail, we can not fail to mention the divergence theory. Scherhag (20) found that in the outflow cyclogenesis at the ground occurs fairly often. This phenomenon was and still is explained by the fact that the divergence of the isobars in the outflow is associated with actual divergence; a pressure fall at the ground then has to occur when the mass divergence in the outflow level dominates (34). This divergence theory is thus a purely kinematic-static theory which does not refer to dynamic processes. If one wanted to interpret it dynamically, one could not regard the predominant horizontal divergence of the outflow level as the primary cause of the surface pressure changes, because as was shown in section 7 it should not be possible for a primary horizontal divergence to produce a pressure system which is almost in geostrophic equilibrium. From a dynamical point of view, the cause could only lie in the adjustment processes of the mass flowing into the outflow, and horizontal divergences would occur as a secondary effect which influence the pressure fields of all layers as well as change the vorticity or the velocity field toward a geostrophic equilibrium. Let us consider the example of the initially supergeostrophic zonal current again. On its left-hand side horizontal divergence formed due to the adjustment in the primary layer; this caused a decrease of cyclonic shear vorticity in the primary layer as well as a pressure fall extending over all layers, followed by the formation of negative mean horizontal divergences in the secondary layers which led to the formation of cyclonic shear vorticity there, so that in the end a low pressure region was built up which extended over all layers and which was in geostrophic equilibrium. The temperature fell below and rose above the core of the primary layer. Applied to the outflow region, qualitatively similar processes should be expected on its cyclonic side, if one calls the upper troposphere the primary layer.

The question of whether true horizontal divergence actually does occur in the outflow was addressed by Reuter (19). The horizontal divergence was determined there from the vorticity field under the assumption of steady state. The result, which is based on just one case from the weather map, shows that in the outflow the decrease of vorticity in the direction of the current and thus positive horizontal divergence is mainly concentrated in the left half, and that the strongest pressure fall at the ground can be observed there, too. From this result Reuter concludes that in this example actual horizontal divergence does exist in the outflow which, when averaged over all layers, produces a mass deficit and thus leads to a pressure fall.

This work was motivated by a jet-stream theory published by Raethjen (15, 16) which was empirically confirmed by Höflich (16). In this theory a wind disturbance caused by convective overturning in the trough region of the jet stream was also assumed. The air masses which are slowed down by friction and convection are accelerated toward low pressure and describe an anticyclonically curved perturbation path. In this theory the trajectories of the disturbed air masses play a major role. Unfortunately the "linearized" equations used here are simplified to such a degree that the connection with the anti-cyclonic jet-stream-trajectories is not reflected in them.

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