# A STUDY OF QUASI-LINEAR NONCAPILLARY TWO-PHASE FLOW IN POROUS MEDIA

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## ABSTRACT

This paper presents a general approach to the study of quasi-linear two-phase flow in porous media. It is especially applicable to reservoirs which can be approximated by one-dimensional models.

The technique presented here uses the familiar "Buckley-Leverett calculation procedures". It provides a systematic means of making one-dimensional calculations for situations beyond the scope of the Buckley-Leverett method and which are pertinent to practical reservoir engineering, e.g., performance of heterogeneous reservoirs with many production and injection wells, recovery of attic or cellar oil, etc...

## ACKNOWLEDGMENT

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# INTRODUCTION

The Buckley-Leverett[1] method and Welge's[2] graphical technique for calculating the evolution and advance of saturation profiles in a linear two-phase flow system is well known and widely used. Although improved and expanded techniques have been developed and reported by a number of authors, the simple "Buckley-Leverett, Welge approach" remains the most widely used. Unfortunately, the applicability of the Buckley-Leverett, Welge solution is limited.

For example, the Buckley-Leverett method is limited to a strictly linear system. The displacement process is assumed to take place in a linear core whose physical characteristics are uniform along the direction of flow. In particular, absolute permeability, porosity, cross-sectional area and flow rate are assumed uniform throughout the entire core.

A very satisfactory and general theory was developed by Martin[3] for the case of quasi-linear flow. A quasi-linear system is a system in which flow is, on the whole, one-dimensional but in which absolute permeability, porosity, cross-sectional area, dip angle and flow rate are not uniform along the axis of flow. Martin derived an equation free from the assumptions that (1) the properties of the formation are uniform and that (2) injection and production occur only at an inlet and outlet face. Hence, the equation for saturation derived by Martin is much more general but not as simple as Buckley and Leverett's. However, when capillary effects are neglected, the partial differential equation derived by Martin is also a first order one. Unfortunately, the mathematical apparatus necessary to solve Martin's equation is more elaborate than that required for an understanding of the Buckley-Leverett approach and, consequently, it is not as widely used.

The approach summarized in this paper intends to breach the gap between the two procedures without requiring a mathematical knowledge beyond that necessary for understanding the Buckley-Leverett method, and at the same time, retaining some of the generality achieved in Martin's solution. Specifically, the effects of changing formation properties and flow rates in a quasi-linear system are approximated by a series of blocks of uniform but different properties. Because formation properties and flow characteristics are uniform in each block, the normal Buckley-Leverett approach can be used if proper care is exercised in handling the transition from one block to the next.

## SATURATION EQUATION IN A QUASI-LINEAR SYSTEM

The equations of two-phase fluid flow in a porous media are obtained by applying the law of conservation of mass and Darcy's law for each phase. By combining the mathematical representation of these laws, it is possible to eliminate pressure as a variable and obtain an equation for water saturation:

$$A\phi \frac{\partial S_w}{\partial t} + QF'_w \frac{\partial S_w}{\partial x} = - q_w - f_w \frac{\partial Q}{\partial x} + \lambda_{ro} f_w \Delta\rho g \frac{d(kA\sin\theta)}{dx} \qquad (1)$$

Throughout this paper, the capillary pressure difference between the two phases is ignored so that the same pressure prevails in both phases.

Equation 1 is a first order partial differential equation which can be solved to give $S_w$ and hence $S_o$ at any $t$ and $x$. The equation reduces to the Buckley-Leverett equation with appropriate simplifications.

## GENERALIZED METHOD

The original Buckley-Leverett method of solution is not applicable to the general differential equation that describes saturation behavior in a quasi-linear system (Equation 1). For simplicity this equation is rewritten as:

$$P(x)\,\frac{\partial S_w}{\partial t} + R(x, S_w, t)\,\frac{\partial S_w}{\partial x} = T(x, S_w, t) \tag{2}$$

The exact expressions for $P(x)$, $R(x, S_w, t)$ and $T(x, S_w, t)$ can be obtained by comparison with Equation 1. The solution of Equation 2 can be obtained by reducing it to two ordinary differential equations (the characteristics):

$$\frac{dt}{P(x)} = \frac{dx}{R(x, S_w, t)} = \frac{dS_w}{T(x, S_w, t)} \tag{3}$$

The method of characteristics involves the solution of two ordinary differential equations. The Buckley-Leverett method is consistent with this method even though it appears to involve only one ordinary differential equation. In fact, the second equation is trivial, namely $dS_w = 0$. Therefore, the Buckley-Leverett type of simplification follows immediately whenever Equation 2 is a homogeneous equation, i.e., $T(x, s, t) = 0$, or

$$\frac{kA\sin\theta}{Q} = C(t) \tag{4}$$

Equation 4 implies that the partial differential equation (Equation 2) for saturation will be homogeneous over any region in which the quantity $kA\sin\theta/Q$ is uniform. Therefore, if $kA\sin\theta/Q$ is constant, the Buckley-Leverett type of calculation using fractional flow curves can be carried out. Usually, this is not the case in reservoir problems of physical significance and the fractional flow concept loses most of its usefulness because $F_w$ becomes a function not only of saturation but also of $x$ and $t$.

Thus, we see that quasi-linear flow cannot be treated by the normal Buckley-Leverett method if the properties of the medium are represented as continouous functions of position. However if the formation can be reasonably well represented by a series of blocks of uniform characteristics a generalized Buckley-Leverett method using fractional flow curves can be used. The principal problem is to investigate what happens at boundaries between blocks.

## VARIATION IN RESERVOIR PROPERTIES OR/AND FLOW CONDITIONS

To handle this problem, we assume the reservoir is either naturally divided into zones of uniform properties with sharp transition at boundaries or artificially segmented, for calculation purposes, into blocks of uniform characteristics (Figure 1). Then, within each block the velocity at which a saturation travels is obtained from the slope of the tangent to the fractional flow curve and is uniquely and well defined. However, at the boundary the situation is unclear. Due to the discontinuity in reservoir properties, a resulting discontinuity in saturation can be intuitively expected to occur at the boundary. Once created, the discontinuity could either be stationary or propagated into the adjacent blocks. Whatever occurs at the boundary must satisfy the law of conservation of mass and be compatible with the information deduced from the fractional flow curves on either side of the boundary.

Two typical cases are presented to show what may happen at a boundary.

## Encroaching Updip Water Drive

Consider the case of an updip advance of a water-oil contact. This is both a simple and common incidence and we will use this particular case as an introduction to the general approach.

At some distance updip from the original water-oil contact, the properties of the reservoir are assumed to change abruptly. The block in contact with the aquifer is denoted block 1. The next block updip is block 2 and, therefore, water moves through block 1 toward block 2. At the boundary the properties of the reservoir change so that the quantity $kA\sin\theta$ has a different value on each side of the boundary.

We consider the case of an increase in $kA\sin\theta$ at the crossing of the boundary in the direction of flow. Because the density difference

SCHEMATIC STEP APPROXIMATION

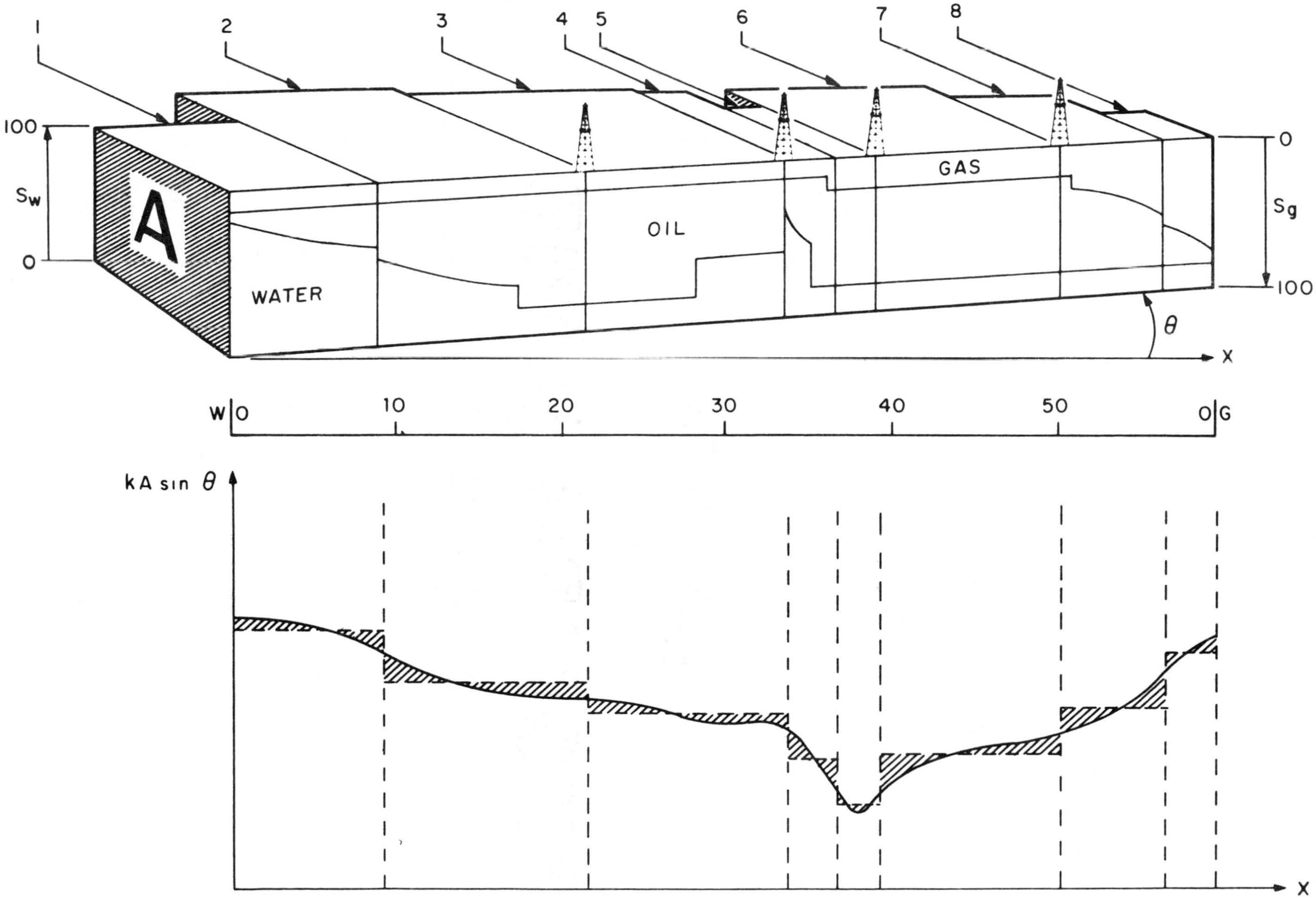



FIGURE 1

in a water-oil system is positive it follows that for any given saturation S:

$$f_w(S) \geq F_w^1(S) \geq F_w^2(S) \tag{5}$$

as shown in Figure 2. Subscripts or superscripts 1 and 2 always refer to blocks 1 and 2. Subscript w referring to water will often be omitted.

Because block 1 is in complete contact with the aquifer, the saturation profile in block 1 behaves as a normal Buckley-Leverett profile with a velocity that is obtained from the proper values of $F_w^1$. When the Buckley-Leverett front for block 1 reaches the boundary between blocks 1 and 2, water starts moving into block 2. However, the saturation profile in block 2 will not be a normal Buckley-Leverett profile because only water saturations between $S_{wc}$ and $S_a$, shown in Figure 2, are available to move into block 2. As the water drive displaces more and more oil from block 1 to block 2, the saturation of water on the left side of the boundary will gradually increase from $S_a$ to $S_b$, to $S_c$ and so forth. At the instant that any value of saturation, $S_a$ or $S_b$, reaches the boundary on the left side, we assume that the saturation in block 2 is not $S_a$ or $S_b$ but $S_a'$ and $S_b'$. Therefore, the saturation profile undergoes a discontinuity at the boundary. The size of the discontinuity, including zero, will be obtained by considering a material balance at the boundary.

If there is neither injection nor production at the boundary, the total flow rates must be the same on both sides:

$$Q_1 = Q_2 = Q \tag{6}$$

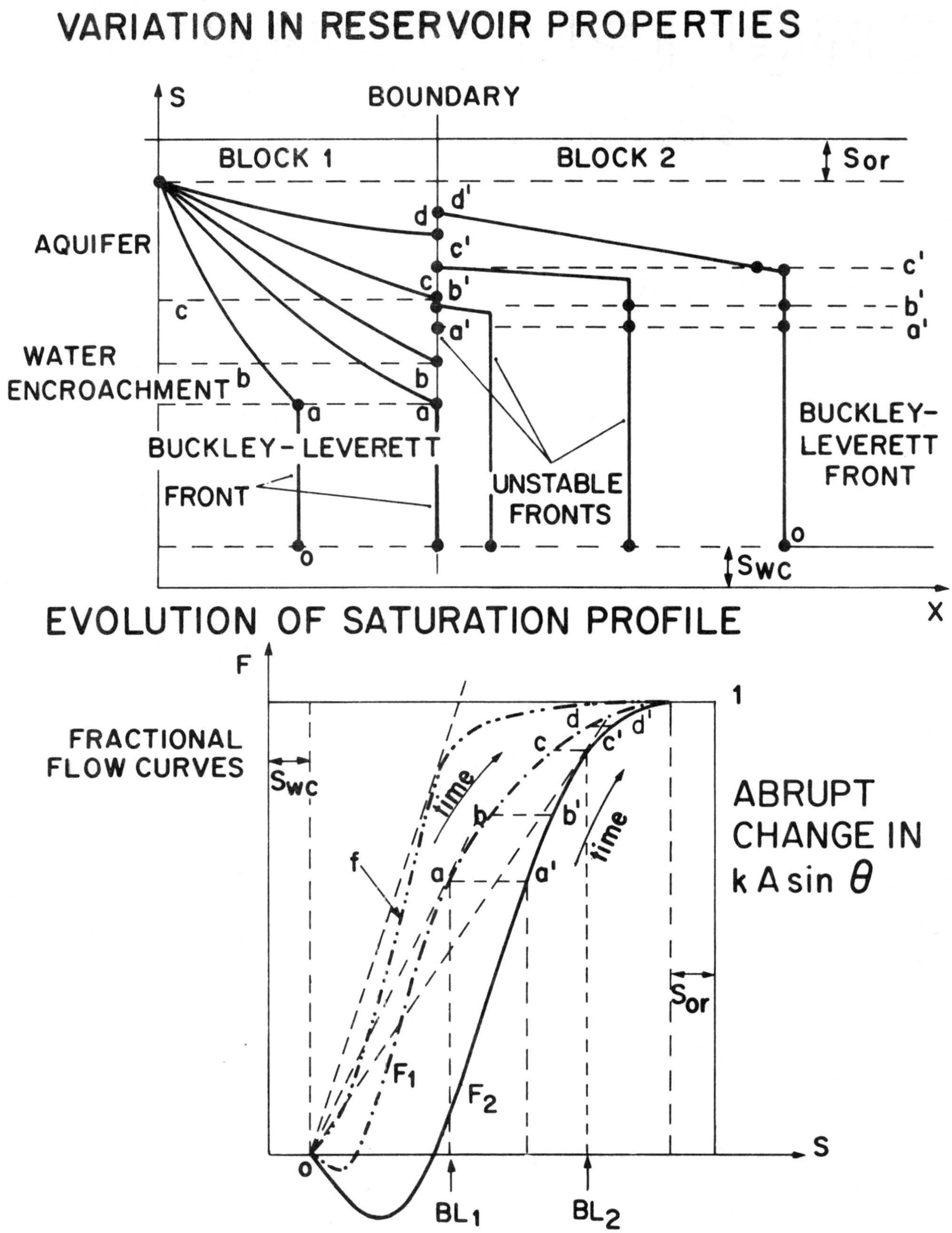

VARIATION IN RESERVOIR PROPERTIES
S
BOUNDARY
BLOCK 1
BLOCK 2
Sor
AQUIFER
WATER
ENCROACHMENT
d
d'
c'
c
b'
a'
b
a
c'
b'
a'
BUCKLEY-LEVERETT
FRONT
UNSTABLE
FRONTS
BUCKLEY-
LEVERETT
FRONT
o
o
Swc
X
EVOLUTION OF SATURATION PROFILE
F
1
FRACTIONAL
FLOW CURVES
Swc
time
time
f
d
d'
c
c'
b
b'
a
a'
ABRUPT
CHANGE IN
k A sin θ
Sor
F1
F2
o
S
BL1
BL2

FIGURE 2

This equation does not mean that the flow rates of oil and water must be conserved on both sides. It only says that the sum of the flow rates of water and oil is conserved at the boundary:

$$F_w^1(S) + F_o^1(S) = F_w^2(S') + F_o^2(S') \tag{7}$$

Because $F_o = 1 - F_w$, Equation 7 is an identity no matter what S and S' may be; Equation 7 alone cannot give the value of the saturation discontinuity.

Because water flowing in block 2 comes from block 1, the flow rate of water in block 2 cannot exceed that in block 1. Symbolically:

$$Q_w^2 \leq Q_w^1 \quad \text{or} \quad F_w^2(S') \leq F_w^1(S) \tag{8}$$

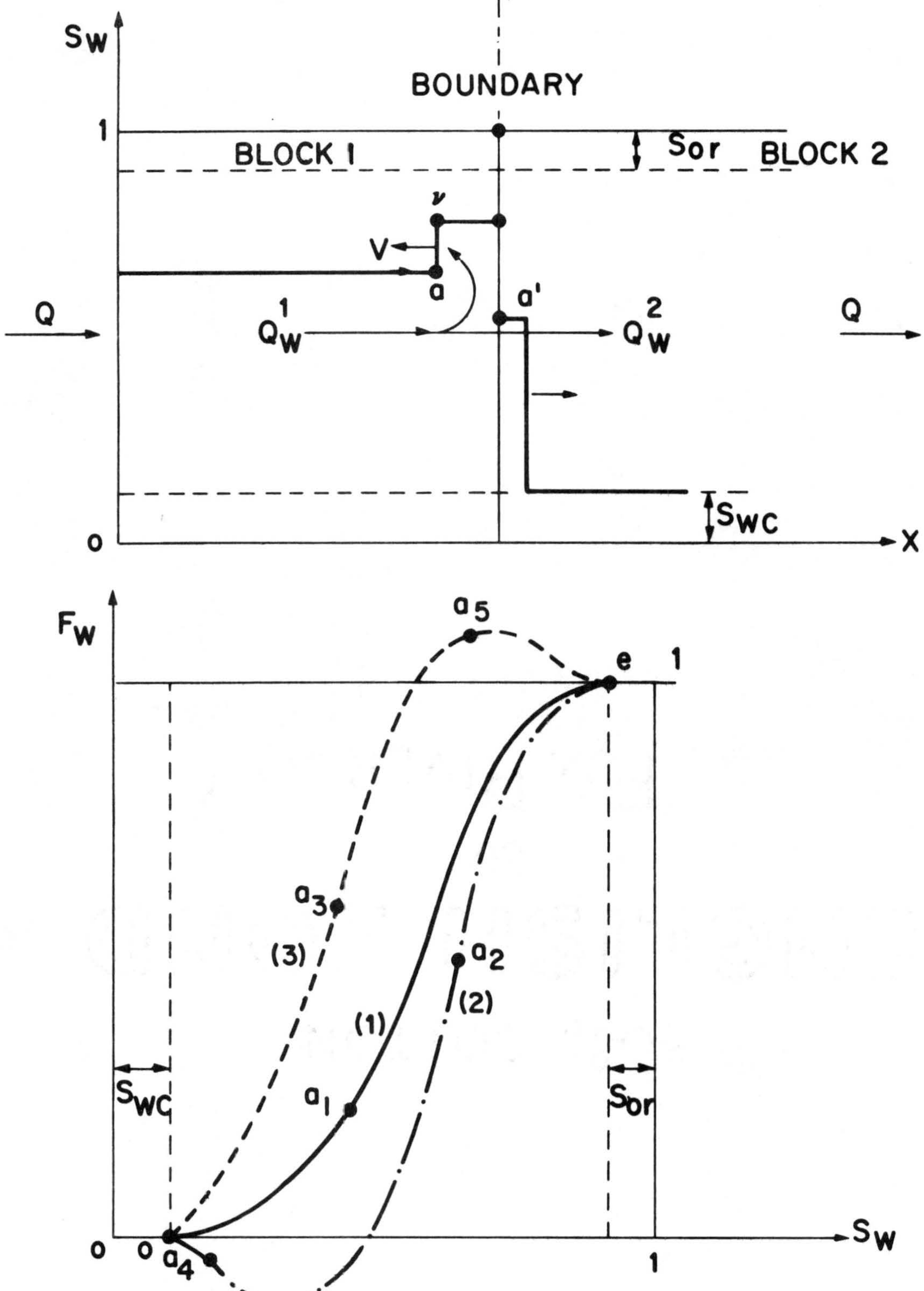
First assuming the case where $Q_w^1 = Q_w^2$, then it follows that $F_w^1(S) = F_w^2(S')$. Consequently the jump from S to S' is given by the equation

$$F_w^1(S) = F_w^2(S') \tag{9}$$

This solution can be obtained graphically (Figure 2), using the fractional flow diagrams, by drawing horizontal lines such as aa', bb' and cc' to obtain values of $S'_a$, $S'_b$, $S'_c$.

Given a saturation profile at an arbitrary initial time $t_o$ (profile oabc, Figure 2), the saturation profile at a later time, t, is easily constructed.

If, on the other hand, only a fraction of the water reaching the boundary from block 1 proceeds into block 2, the remaining fraction must accumulate at the boundary and counterflow (see Figure 3). The




# FLOW BEHAVIOR AT A BOUNDARY

CASE 1

FIGURE 3

velocity at which the counterflow front, $a_\nu$, propagates is negative. The expression for the velocity is given by the usual formula

$$V = \frac{\left[F_w^1(S_\nu) - F_w^1(S_a)\right]}{S_\nu - S_a} \cdot \frac{Q}{A\phi} \tag{10}$$

Due to the fact that water accumulates at the boundary, $S_\nu$ is necessarily greater than $S_a$. For $V$ to be negative, given an incoming saturation $S_a$, there must exist a saturation $S_\nu \geq S_a$ such that $F_w^1(S_\nu) \leq F_w^1(S_a)$.

The typical fractional flow curves of Figure 3 indicate that in most cases such a saturation does not exist, which means that no accumulation induced counterflow will take place. In this instance, the flow rate of both phases is preserved across the boundary

$$Q_w^1 = Q_w^2 \qquad \text{or} \qquad F_w^1(S) = F_w^2(S') \tag{11}$$

Equation 11 is the proper boundary condition for the case of an encroaching water drive, which is therefore resolved.

However, it is possible to find a saturation $S_\nu$ if the two following conditions are fulfilled: (1) the shape of the fractional-flow curve does allow counterflow, i.e., $F$ is either negative or greater than one over a saturation interval and (2) the incoming saturation $S_a$ falls within this interval. With the incoming saturations $a_1$, $a_2$, $a_3$ of Figure 3, no accumulation induced counterflow is possible. With the incoming saturations $a_4$ and $a_5$ a range of possible saturations $S_\nu$ can be found. In either instance because the incoming saturation $S_a$

already represents a counterflowing situation, the accumulation induced front will not be counterflowing. It will propagate in the overall direction of flow (see Figure 4). The conditions to be satisfied by $S_\nu$ are:

$$S_\nu \geq S_a \qquad F_w^1(S_\nu) \geq F_w^1(S_a)$$

which express the fact that accumulation takes place at the boundary, and

$$F_w^1(S_\nu) \leq 0$$

which expresses the fact that the flow rate of water in block 2 comes from block 1. For an incoming saturation a (Figure 5), the point $\nu$ may lie anywhere between m and n .

A further limitation on the range of $S_\nu$ may come from the shape of the fractional flow curve in block 2 (Figure 5). If the fractional flow function $F_w^2$ in block 2 is as represented by curve 1 (Figure 5) no additional restriction on the range of $S_\nu$ is imposed because, given $S_\nu$ in the interval $(S_m, S_n)$, it is always possible to find S' such that Equation 11 is satisfied. In the case of curve 2, satisfaction of Equation 11 is possible only if the point $\nu$ lies between the points p and n . In the cases of curves 3 and 4, the only solution for point $\nu$ is point n . In the latter cases, the "transmission coefficient" at the boundary is zero.

Central Injection of Water or Gas

The case of injection of water (or gas) in the center of the formation is somewhat different from that of an encroaching water

# FLOW BEHAVIOR AT A BOUNDARY (CASE 2)

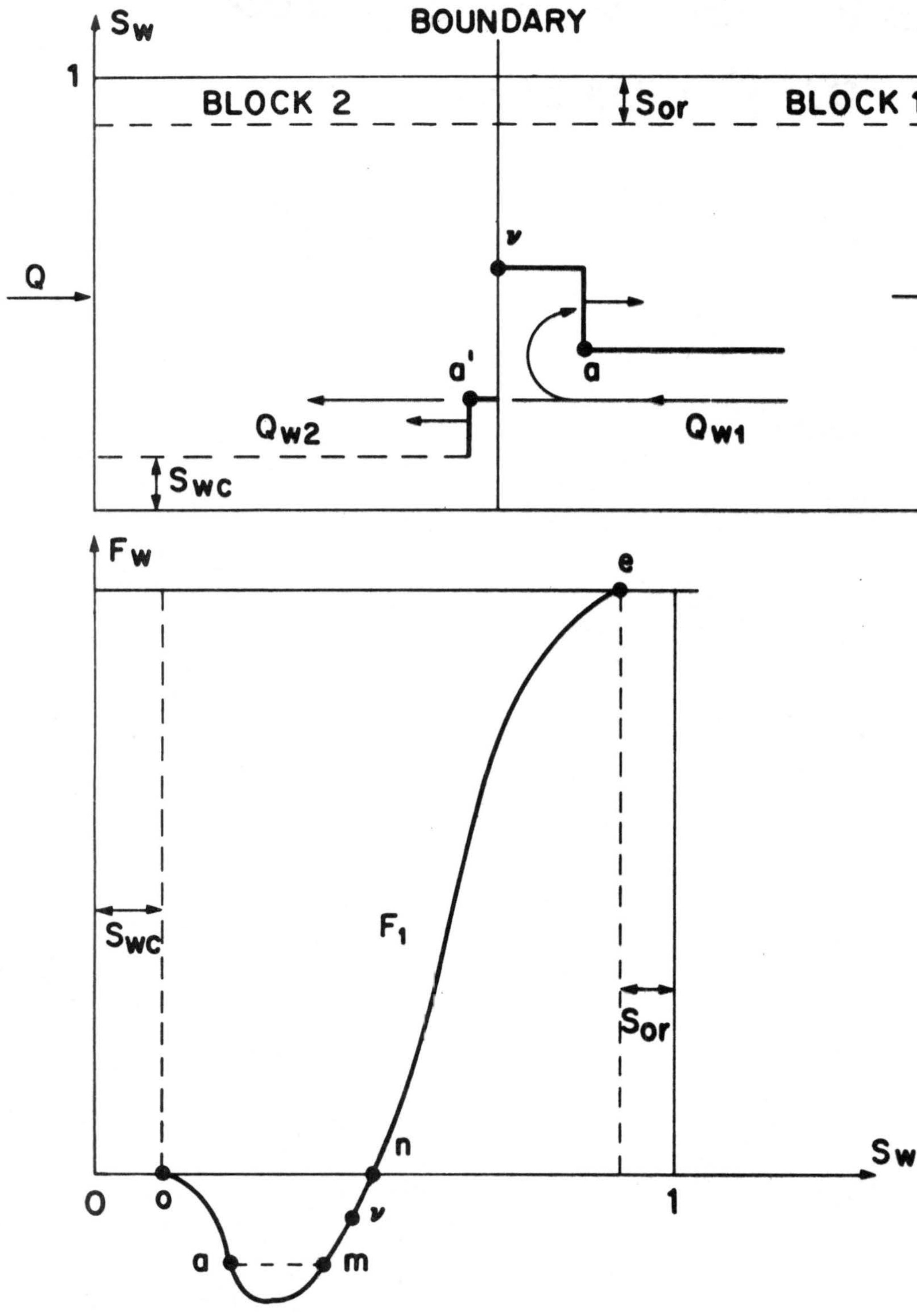
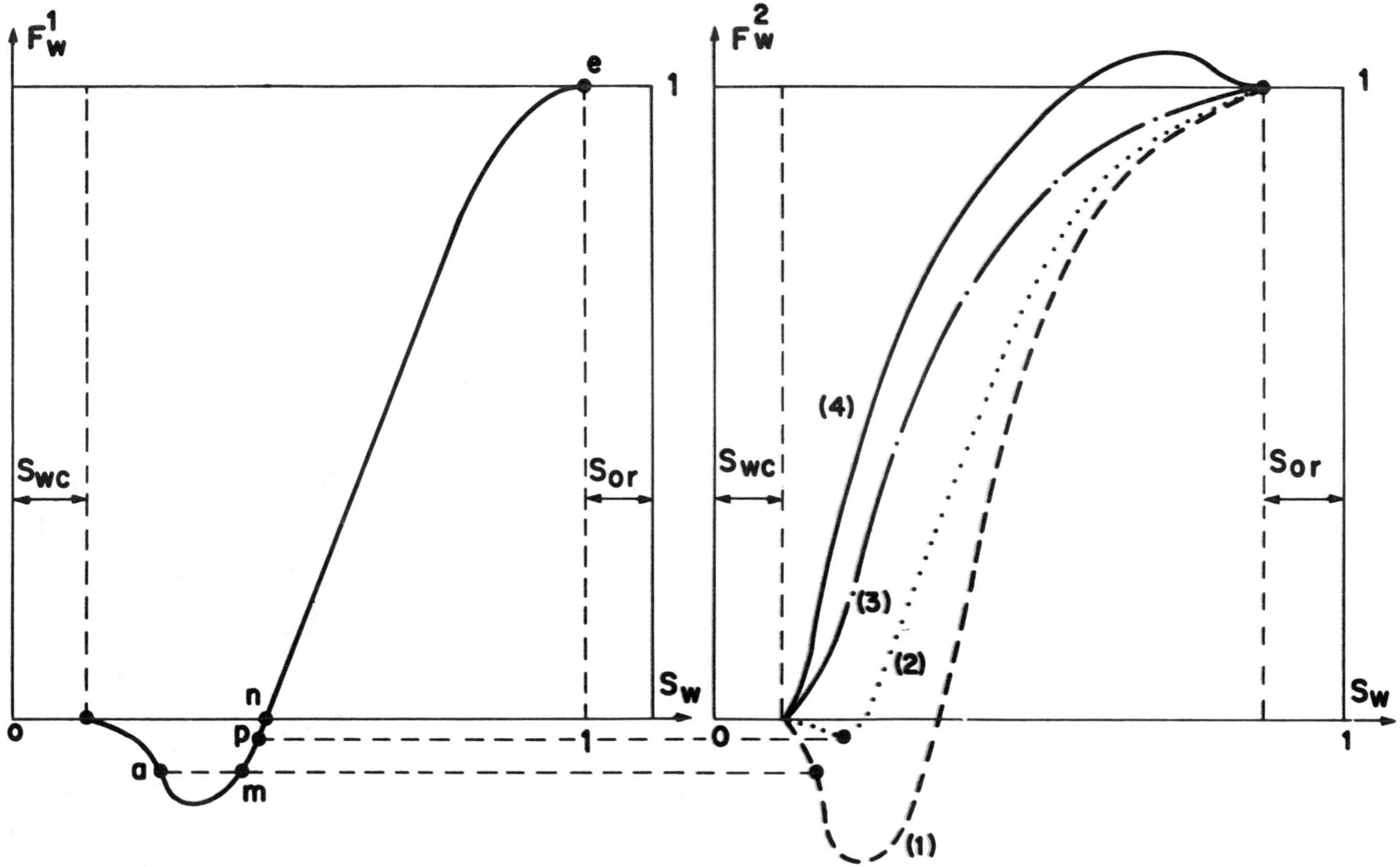



FIGURE 4




INFLUENCE OF DOWNSTREAM
CONDITIONS ON UPSTREAM FLOW BEHAVIOR

FIGURE 5

drive. In a water drive by encroachment from the aquifer, the flow of water is in a single direction, the updip direction. When water is injected in the center of the formation into an updip moving stream of oil, some of the water may move downdip whereas the remainder moves updip. For the purpose of illustration, water injection is considered at an injection rate $\Delta Q_w$, which is negative due to our algebraic convention that a negative production rate is an injection rate. With this sign convention, expression of conservation of mass at the injection boundary yields:

$$Q_1 = \Delta Q_w + Q_2 \tag{12}$$

We assume that $Q_1$ is fixed by upstream conditions. For a given injection rate $\Delta Q_w$, Equation 12 determines the downstream flow rate, $Q_2$. Part of the water injection rate flows updip: $\Delta Q_w^2$ (positive), part flows downdip: $\Delta Q_w^1$ (negative). Because no oil is produced at the boundary of blocks 1 and 2 and because oil cannot counterflow in a downdip direction, the flow rate of oil must be the same on either side:

$$Q_1 (1 - F_w^1) = Q_2 (1 - F_w^2) \quad . \tag{13}$$

Satisfaction of Equation 13 requires a saturation discontinuity. Given a value S for the water saturation on the upstream side of the boundary, block 1, the saturation S' on the downstream side is found by obtaining a solution of:

$$F_2(S') = \frac{Q_1}{Q_2} F_1(S) + \left(\frac{-\Delta Q_w}{Q_1}\right) \tag{14}$$

The right-hand side being known, the value of $F_2(S')$ is readily calculated and from the curve of $F_2$, $S'$ is immediately deduced. The solution for a pair of values is indicated: (a, b). If a downdip flow rate of water $\Delta Q_w^1$ is selected a priori the value of S is determined by the relation:

$$\Delta Q_w^1 = F_1(S)Q_1 . \tag{15}$$

Then the value of $S'$ is obtained from Equation 14 or by the equivalent graphical construction. Naturally $\Delta Q_w^1$ cannot be smaller than minimum of $F_1 Q_1$. But it can have any arbitrary value between 0 and (minimum of $F_1$) x $Q_1$. The evolution of the saturation profile is shown on Figure 6 for a priori selected value of $\Delta Q_w^1$, corresponding to point a on the $F_1$ curve.

A limited range of possible solutions has thus been obtained within which the solution must lie; that is, under these circumstances $S_a$ is not a unique solution but must be chosen arbitrarily within limits. However, once $S_a$ is defined, $S_b$ can be solved uniquely.

## CONCLUSIONS

A general procedure to predict the evolution of a saturation profile with time in a reservoir that can be considered grossly linear was described. This approach can be considered as a compromise approach between the "normal" Buckley-Leverett-Welge method and the more general method of Martin. The most important conclusion stemming from this material is that the petroleum engineer can

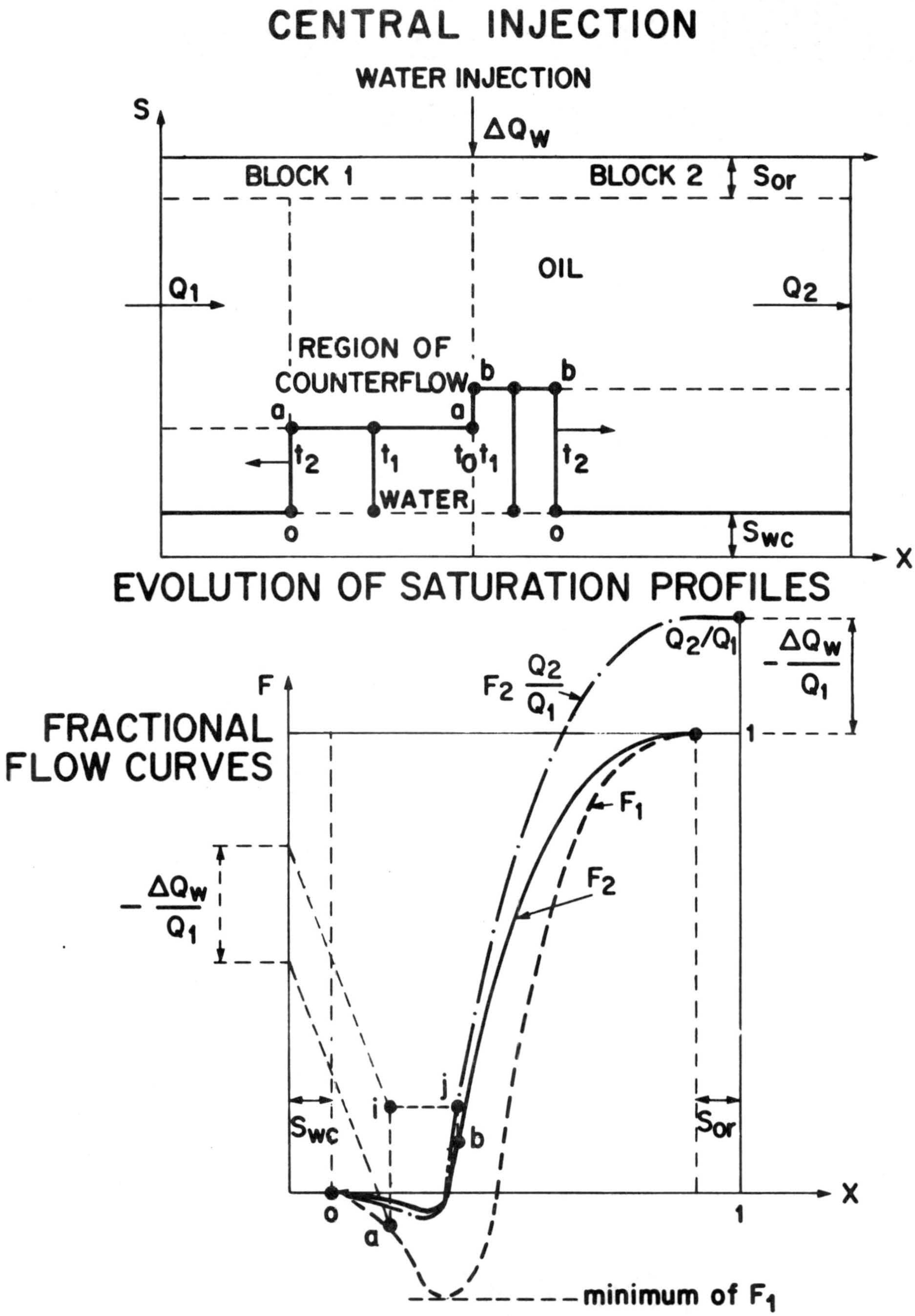

CENTRAL INJECTION
WATER INJECTION
S
ΔQw
BLOCK 1
BLOCK 2
Sor
OIL
Q1
Q2
REGION OF COUNTERFLOW
a
b
t2
t1
t0
WATER
o
Swc
X
EVOLUTION OF SATURATION PROFILES
F
FRACTIONAL FLOW CURVES
Q2/Q1
F2 Q2/Q1
-ΔQw/Q1
1
F1
F2
i
j
Swc
Sor
minimum of F1

FIGURE 6

solve fairly complicated problems in a relatively straightforward way by relying on already acquired knowledge of the normal Buckley-Leverett-Welge methods. The approach is systematic and allows one to break quite complicated reservoir systems into blocks that can be handled by "routine" methods.

Some degree of uncertainty exists in the solution whenever counterflow can occur. From a practical point of view, this may not be significant. From an academic point of view, it is not satisfactory. To eliminate the uncertainty in the solution, other factors (e.g., capillarity) must be considered.

# LIST OF SYMBOLS

| Symbol | Definition |
|---|---|
| a, b, c, d, e, m, n, o, $\nu$ | Various points on the saturation profiles and the fractional flow curves |
| $f$ | Fractional flow function when both capillary and gravity terms are neglected |
| $f_i$, $f_w$ | $f$ for fluid, $i$ for water |
| $g$ | Acceleration of gravity |
| $k$ | Absolute permeability |
| $k_{ri}$, $k_{ro}$, $k_{rw}$ | Relative permeability to fluid $i$, to oil, to water |
| $p$ | Pressure |
| $q$, $q_o$, $q_w$ | Production $(> o)$ or injection $(< o)$ flow rate linear density distribution, for oil, for water |
| $t$ | Time |
| $t_1, t_2 \ldots t_n \ldots$ | Successive time periods in the evolution of a saturation profile |
| $t_o$ | Initial time |
| $x$ | Curvilinear abscissa along the mean streamline |
| $x_f$ | Abscissa of front |
| $x_o$ | Original location of saturation $S$ |
| $x_s$ | Location of saturation $S$ at time $t$ |
| $A$ | Cross-sectional area |
| $F$ | Fractional flow function when only capillary terms are neglected |
| $F_i$, $F_o$, $F_w$ | $F$ for fluid $i$, for oil, for water |

LIST OF SYMBOLS - Continued

| Symbol | Definition |
|---|---|
| $F^1$, $F^2$ | $F$ in block 1, in block 2 |
| $F^+$, $F^-$ | $F$ on the two sides of a front |
| $F_{BL}$ | $F$ at the Buckley-Leverett saturation |
| $F'$ | Derivative of $F$ with respect to $S$ |
| $Q$ | Total flow rate >0 in the positive $x$ direction |
| $Q_o$, $Q_w$ | Flow rate of oil, of water |
| $Q^+$, $Q^-$ | Flow rates on the two sides of a front |
| $Q^1$, $Q^2$ | Flow rate in block 1, in block 2 |
| $S$ | Saturation |
| $S_i$, $S_o$, $S_w$ | Saturation of fluid $i$, of oil, of water |
| $S^+$, $S^-$ | Saturations on the two sides of a front |
| $S_{wc}$ | Connate water saturation |
| $S_{or}$ | Residual oil saturation |
| $S_{BL}$ | Saturation behind the Buckley-Leverett front |
| $S_I$ | Saturation at an inflection point of the fractional flow function |
| $(S_a, S'_a)(S, S')$ | Corresponding saturations on the two sides of a boundary |
| $S_\nu$ | Accumulation induced counterflow saturation |
| $V$ | Velocity of a front |
| $\theta$ | Dip angle |
| $\lambda$ | Mobility |

## LIST OF SYMBOLS - Continued

| Symbol | Definition |
|---|---|
| $\lambda_{ro}$, $\lambda_{rw}$ | Relative mobility of oil, water |
| $\mu_o$, $\mu_w$ | Viscosity, of oil, of water |
| $\rho_o$, $\rho_w$ | Specific mass for oil, water |
| $\Delta\rho$ | $\rho_w - \rho_o$ for a water-oil system |
| $\Delta Q$ | Production (>0) or injection (< 0) flow rate |
| $\Delta Q_w$ | Water injection flow rate |
| $\Delta Q_w^1$, $\Delta Q_w^2$ | Water injection flow rate in block 1, in block 2 |
| $\phi$ | Porosity |

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