DISSERTATION

# PERIODIC EXISTENCE THEOREMS IN OPTIMAL CONTROL

submitted by
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for the Degree of Doctor of Philosophy
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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY COLLEEN LIVINGSTON TITLED PERIODIC EXISTENCE THEOREMS IN OPTIMAL CONTROL BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

PERIODIC EXISTENCE THEOREMS IN OPTIMAL CONTROL

This paper considers existence theorems for the optimal control problem

$$\min_{(x,u)} \quad l_o(x(0), x(T)) + \int_0^T f_o(t, x(t), \dot{x}(t), u(t))$$

$$\text{subject to:} \quad \dot{x}(t) = f(t, x(t), u(t))$$

$$(x(0), x(T)) \in \mathcal{B}$$

$$(x(t), u(t)) \in \mathcal{F}$$

$$x \in A_n[0, T]$$

$$u \in \mathcal{L}_m[0, T]$$

Using the results of R.T. Rockafellar this problem is reformulated into a problem in which the control and the constraints are absorbed into the objective function. Rockafellar has established a general existence theorem for such control problems. This existence theorem requires finding bounds on a Hamiltonian function. In his work Rockafellar specialized his results to certain initial value problems.

The central new theorems in this paper specialize Rockafellar's general theorem to periodic problems in optimal control. These theorems are then coupled with results of R.E. Gaines and J. Peterson which give necessary conditions for a finite minimum.

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# Contents

# Chapter 1

# Introduction

This paper considers existence theorems for the optimal control problem

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$$\text{subject to:} \quad \dot{x}(t) = f(t, x(t), u(t))$$

$$(x(0), x(T)) \in \mathcal{B}$$

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$$x \in A_n[0, T]$$

$$u \in \mathcal{L}_m[0, T]$$

The function $x$ is called the state variable, while the function $u$ is the control variable.

Chapter 2 reviews the concepts of a subdifferential, the Fenchel Transform, and a Hamiltonian differential inclusion. In addition, it reviews some foundational existence theorems from analysis. Chapter 3 reformulates the control problem. This reformulation, due to R.T. Rockafellar, results in a problem in which the control and the constraints are absorbed into the objective function. This chapter also briefly establishes the equivalence of the reformulation to the original control problem. Next, a general existence theorem for control problems is stated. Here the minimum may be infinite. This existence theorem requires finding bounds on a Hamiltonian function. The chapter concludes with a few examples which are intended to motivate our thinking about existence theorems.

In Chapter 4 we state two existence theorems due to Rockafellar. Both of these theorems are applicable to certain initial value problems. Next, similar theorems are proved for periodic control problems. These existence theorems are then applied to the set of examples previously considered.

In Chapter 5 necessary conditions for a finite minimum due to R.E. Gaines and J. Peterson are given. These necessary conditions are extended to a more general quadratic Hamiltonian than that considered by Gaines and Peterson. Chapter 6 relates our existence theorems to conditions for finite minimums. Finally, Chapter 7 concludes with an application to a problem from water management.

## Notation

Unless otherwise stated, all integrations are with respect to $t$. The subscript indicating the size of the space often will be omitted.

$C_n[0,T] = \{x : [0,T] \rightarrow \Re^n | x$ is continuous on $[0,T]$ $\}$ with $\|x\|_C = sup_{t\in[0,T]}\|x(t)\|$

$C_n^1[0,T] = \{x : [0,T] \rightarrow \Re^n | x$ and $\dot{x}$ are continuous on $[0,T]$ $\}$

$A_n[0,T] = \{x : [0,T] \rightarrow \Re^n | x$ is absolutely continuous on $[0,T]$ $\}$

$\mathcal{L}_n[0,T] = \{x : [0,T] \rightarrow \Re^n |$ x is Lebesgue measureable on $[0,T]\}$

$\mathcal{L}_n^1[0,T] = \{x : [0,T] \rightarrow \Re^n | x \in \mathcal{L}_n[0,T]$ and $\int_0^T \|x\| < \infty$ $\}$ with $\|x\|_1 = \int_0^T \|x\|$

$\mathcal{L}_n^\infty[0,T] = \{x : [0,T] \rightarrow \Re^n | x \in \mathcal{L}_n[0,T]$ and $\|x\|_\infty < \infty\}$, where $\|x\|_\infty = \text{ess sup}_{t\in[0,T]}\|x(t)\|$

$< x, y > = \sum_{i=1}^n x_i y_i$, that is an inner product in $\Re^n$.

# Chapter 2

# Mathematical Background

This chapter reviews terminology from the study of control problems. In particular, it examines the subdifferential, a tool in the analysis of nondifferentiable real-valued functions; the nature of Hamiltonian systems of differential inclusion; and the Fenchel transform, an extension of the Legendre transform of mechanics. The final section reviews the pertinent analysis. The primary resources for this material are [1], [3], [7], [15], and [16].

## 2.1 Subdifferentials

For the convex function $F(x) = |x|$ the derivative $F'(0)$ is undefined in the traditional sense. Consider the slopes of the those lines which lie on or below the graph of $F$ and which intersect the graph of $F$ at $(0,0)$. These slopes are the elements of $\partial F(0)$, the subdifferential set. Each such slope is called a subgradient. In particular, $\partial F(0) = [-1,1]$.

Let $V$ be a locally convex topological vector space and $V^*$ its topological dual space.

**Definition 1** *Let $F : V \longrightarrow \overline{\Re}$ be a convex function. The subdifferential of $F$ at $\bar{x}$ is denoted by $\partial F(\bar{x})$. We have*

$$x^* \in \partial F(\bar{x}) \text{ if and only if } F(\bar{x}) \text{ is finite and}$$

$$\langle x - \bar{x}, x^* \rangle + F(\bar{x}) \leq F(x) \quad \text{for all } x \in V.$$

*The element $x^*$ is called a subgradient. The function $F$ is said to be subdifferentiable at $\bar{x}$ provided $\partial F(\bar{x}) \neq \emptyset$.*

- Note that $\partial F : V \longrightarrow V^*$, and $\partial F$ is a set-valued function.
- If $F$ has no subgradient at $\bar{x}$, then $\partial F(\bar{x}) = \emptyset$.

The subdifferential of a function $F : \Re \to \overline{\Re}$ at a point $\bar{x}$ can be visualized as the set of slopes of the continuous affine minorants, that is, those lines which lie on or below the graph of $F$ and which intersect that graph at $(\bar{x}, F(\bar{x}))$. In the case of a convex function, the subdifferential consists of values between the left- and right- handed derivatives at a point.

Let $F : V \to \overline{\Re}$. If it exists, the directional derivative of $F$ at $\bar{x}$ in the direction $u$ will be denoted by $DF_u(\bar{x})$ provided the directional derivative exists.

**Proposition 1** *The set $\partial F(\bar{x})$ is closed and convex.*

**Definition 2** *The function $F$ is said to be Gateaux differentiable at $\bar{x}$ provided there exists an $x^* \in V^*$ such that for all $u \in V$, we have $DF_u(\bar{x}) = \langle x^*, u\rangle$. In this case the Gateaux derivative is $x^*$ which is denoted by $F'(\bar{x})$.*

**Proposition 2** *If $F$ is a convex function and Gateaux differentiable at $\bar{x} \in V$ then it is subdifferentiable at $\bar{x}$ and $\partial F(\bar{x}) = \{\nabla F(\bar{x})\}$. Conversely, if $F$ is continuous and finite at $\bar{x} \in V$ and has a unique subgradient then $F$ is Gateaux differentiable at $\bar{x}$ and $\partial F(\bar{x}) = \{\nabla F(\bar{x})\}$.*

**Proposition 3** *Let $F_1 : V \to \Re$ and $F_2 : V \to \Re$. Let $\lambda > 0$. At every $\bar{x} \in V$*

*(a) $\partial(\lambda F)(\bar{x}) = \lambda \partial F(\bar{x})$, and*

*(b) $\partial F_1(\bar{x}) \cup \partial F_2(\bar{x}) \subset \partial(F_1 + F_2)(\bar{x})$.*

**Proposition 4** *Assume $F$ is convex. $F(\bar{x}) = \min_{x \in V} F(x)$ if and only if $0 \in \partial F(\bar{x})$.*

## 2.2 Hamiltonian Systems

Let $H(t, x, p)$ be $C^1$ on $\Re \times \Re^n \times \Re^n$. A Hamiltonian system of ordinary differential equations has the form

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}(t, x, p) \\ \dot{p} &= -\frac{\partial H}{\partial x}(t, x, p). \end{aligned}$$

Hamiltonian systems are particularly tractable in the autonomous case. Clearly the critical points of the function $H(x, p)$ occur where $\dot{x} = \dot{p} = 0$, that is at the fixed points of the dynamical system.

**Theorem 1** *If $H(t, x, p) = H(x, p)$ is autonomous, then the orbits of the Hamiltonian system lie along the level curves of $H$.*

Assume that $H(t,x,p)$ is concave as a function of $x$ and convex as a function of $p$. This will be the case for many of the the Hamiltonian functions which will be considered. We let $\partial_p H(t,x,\bar{p})$ denote the subdifferential of $H$ at $\bar{p}$ for each $(t,x)$. Likewise, we let $-\partial_x H(t,\bar{x},p)$ denote the subdifferential of $-H$ at $\bar{x}$ for each $(t,p)$. More precisely,

$$\partial_p H(t,x,\bar{p}) = \{p^* \in \Re^n | H(t,x,\bar{p}) + \langle p-\bar{p}, p^* \rangle \leq H(t,x,p) \quad \forall p \in \Re^n\}$$

$$-\partial_x H(t,\bar{x},p) = \{x^* \in \Re^n | H(t,\bar{x},p) + \langle \bar{x}-x, x^* \rangle \geq H(t,x,p) \quad \forall x \in \Re^n\}$$

A Hamiltonian system of differential inclusions has the form

$$\dot{x} \in \partial_p H(t,x,p)$$

$$\dot{p} \in -\partial_x H(t,x,p).$$

**Example 1** Consider the Hamiltonian function

$$H(x,p) = \begin{cases} px - p, & \text{for } p < 0 \\ px + p, & \text{for } p \geq 0 \end{cases} = px + |p|$$

The differential inclusion is

$$\dot{x} \in \partial H_p(x,p) = \begin{cases} x - 1, & \text{for } p < 0 \\ [x-1, x+1], & \text{for } p = 0 \\ x + 1, & \text{for } p > 0 \end{cases}$$

$$\dot{p} \in -\partial H_x(x,p) = -p$$

## 2.3 The Fenchel Transform

Consider a strongly convex $C^1$ function $F : \Re \to \Re$, e.g. $F(u) = u^2$. The Legendre Transform of $F$ at $v$ is given by $L(v) = uv - F(u)$, where $v = \nabla F(u)$. At the point $(u, F(u))$, $v$ is the slope of the tangent line to the graph of $F$ and $-L(v)$ is the vertical intercept of this tangent line. The Fenchel Transform extends this definition to general functions. Properties of the Fenchel transform are described extensively in [1], [7] and [15]. We include a few properties for completeness.

**Definition 3** *Let $F : V \to \Re$. The Fenchel transform of $F$ is a function $F^* : \Re^n \to (-\infty, +\infty]$ defined by*

$$F^*(v) = \sup_{u \in dom F} \{\langle v, u \rangle - F(u)\}$$

For $F : \Re \to \bar{\Re}$ recall that $\partial F(\bar{x})$ is the set of the slopes of all the continuous affine minorants at $\bar{x}$. The Fenchel Transform $F^*$ can be thought of as the vertical intercept of one of these minorants as a function of its slope.

**Definition 4** *Let $\Gamma_o(\Re^n)$ be the set of functions $F : \Re^n \to (-\infty, +\infty]$ which are convex and lower semicontinuous and for which*

$$\mathcal{D}(F) = \{x \in \Re^n : F(x) < +\infty\} \neq \emptyset.$$

**Proposition 5** *If $F \in \Gamma_o(\Re^n)$ then $(F^*)^* = F$. This expresses a duality between $F$ and $F^*$.*

**Proposition 6** *If $F \in \Gamma_o(\Re^n)$, then $F^*$ is convex and lower semicontinuous.*

**Proposition 7** *If $F_1 \in \Gamma_o(\Re^n)$ and $F_2 \in \Gamma_o(\Re^n)$ and $F_1 \leq F_2$ then $F_1^* \geq F_2^*$.*

**Proposition 8** *If $F \in \Gamma_o(\Re^n)$, then the following are equivalent:*

*(a) $v \in \partial F(u)$*

*(b) $F(u) + F^*(v) = \langle v, u \rangle$*

*(c) $u \in \partial F^*(v)$.*

The equivalence between (a) and (b) indicates that

$$F^*(v) = \sup_{u \in dom F} \{\langle v, u \rangle - F(u)\}$$

is attained when $v \in \partial F(u)$.

## 2.4 Analysis Background

This section provides a review of the fundamental analytical techniques for demonstrating the existence of a minimizer. This material is described in depth in [15].

**Theorem 2** *Let $f : S \to \Re$. If $f$ is lower semicontinuous and $S$ is a sequentially compact normed space then $f$ attains its minimum on $S$.*

**Definition 5** *Let $f : A[0, T] \to \Re$. A sequence $\{x_n\} \in A[0, T]$ is said to be a minimizing sequence for $f$ provided $f(x_n) \to \inf_x f(x)$ as $n \to \infty$.*

**Definition 6** *The function $f$ is weakly lower semicontinuous at $\bar{x}$ provided*

$$\underline{\lim}_{n \to \infty} f(x_n) \geq f(x_o)$$

*whenever $x_n$ converges weakly to $\bar{x}$.*

**Theorem 3** *If $\{x_k\}$ is a weakly convergent minimizing sequence for $f$ and $f$ is weakly lower semicontinuous, then the weak limit $\bar{x}$ of $\{x_k\}$ minimizes $f$.*

Theorems of Berkowitz and Macki and Strauss ([2] [3] [14]) demonstrate the recurring themes of convexity and a priori bounds on the successful responses. In addition, we point out that in these theorems, both the objective function and the feasible set must satisfy specified requirements. Although the approaches differ in the various theorems, the proofs typically utilize the foundational theorems of existence theory.

# Chapter 3

# Optimization Background

This chapter begins with a statement of the optimal control problem which we will consider. A reformulation via infinite penalties absorbs the control and the constraints. We then demonstrate the equivalence of this reformulated problem and begin to consider related existence theorems. This work is due to R.T. Rockafellar and is processed in detail in [17] and [18].

## 3.1 The Optimal Control Problem

We state a general optimal control problem as

$$\min_{(x,u)} \quad l_o(x(0), x(T)) + \int_0^T f_o(t, x(t), \dot{x}(t), u(t)) \tag{3.1}$$

$$\begin{aligned} \text{subject to:} \quad & \dot{x}(t) = f(t, x(t), u(t)) \\ & (x(0), x(T)) \in \mathcal{B} \\ & (x(t), u(t)) \in \mathcal{F} \\ & x \in A_n[0, T] \\ & u \in \mathcal{L}_m[0, T] \end{aligned}$$

The function $x$ is called the state variable, while the function $u$ is the control variable.

## 3.2 The Infinite Penalty Reformulation

In this section, we follow the reformulation of the control problem by Rockafellar. The use of infinite penalties absorbs the constraints and the boundary condition.

We define two functions $l(a, b)$ and $K(t, x, v, u)$ via infinite penalties.

$$l(a,b) = \begin{cases} l_o(a,b), & \text{for } (a,b) \in \mathcal{B} \\ +\infty, & \text{otherwise} \end{cases}$$

$$K(t,x,v,u) = \begin{cases} f_o, & \text{for } v = f(t,x,u) \text{ and } (x,u) \in \mathcal{F} \\ +\infty, & \text{otherwise} \end{cases}$$

Define $\Phi(x,u) = l(x(0), x(T)) + \int_0^T K(t,x,\dot{x},u)$ This changes the original optimal control problem to

$$\min_{(x,u)} \Phi(x,u) \tag{3.2}$$

$$\text{subject to: } x \in A_n[0,T]$$

$$u \in \mathcal{L}_m[0,T].$$

The next step of the reformulation involves suppressing the dependence on u. We define the function $L$ as

$$L(t,x,v) = \inf_u K(t,x,v,u). \tag{3.3}$$

Setting $\Psi(x) = l(x(0), x(T)) + \int_0^T L(t,x,\dot{x})$ we consider the reformulated problem

$$\min \psi(x) \tag{3.4}$$

$$\text{subject to: } x \in A_n[0,T]$$

The reformulated problem (3.4) is said to have a minimizer provided there exists $\bar{x} \in A_n[0,T]$ such that $\psi(\bar{x}) \leq \psi(x)$ for all $x \in A_n[0,T]$. In Chapter 5 we will discuss the possibility that $\psi(x) \equiv +\infty$.

The Fenchel transform $H$ of $L$ with respect to $v$ is defined as

$$H(t,x,p) = \sup_{v \in R^n} \{\langle p, v\rangle - L(t,x,v)\}.$$

The function $H$ is called a Hamiltonian function. In the next example we will see that $H$ incorporates both the objective function and the constraints. The study of this Hamiltonian function will lend insight into the existence of a minimizer for $\psi$. Under our later assumptions we will see that, provided $l_o$ is convex, ( 3.4) is akin to a convex problem of Bolza with extended real valued functions permitted.

Next, we introduce an important necessary condition for an optimal control.

**Theorem 4** *If there exists a minimizer for the reformulated problem 3.4, then the following differential inclusion has a solution*

$$\dot{x} \in \partial_p H(t, x, p) \quad \text{a.e.}$$

$$-\dot{p} \in \partial_x H(t, x, p) \quad \text{a.e.}$$

$$(p(0), -p(T)) \in \partial l(x(0), x(T)).$$

This is an extension of the classical Euler-Lagrange equation from the calculus of variations. Proofs can be found in [2] and [18]. We will return to this system of differential inclusions in the fifth chapter where we consider conditions which guarantee finite minimums.

Assuming $l_o = 0$, for an initial value problem with $x(0) = x_o$ we have

$$l(a, b) = \begin{cases} 0, & \text{for } a = x_o \\ +\infty, & \text{otherwise.} \end{cases}$$

and so

$$l(x(0), x(T)) = \begin{cases} 0 & \text{for } x(0) = x_o \\ +\infty, & \text{otherwise.} \end{cases}$$

For a periodic problem with $x(0) = x(T)$ we attain

$$l(a, b) = \begin{cases} 0, & \text{for } a = b \\ +\infty, & \text{otherwise} \end{cases}$$

and so

$$l(x(0), x(T)) = \begin{cases} 0 & \text{for } x(0) = x(T) \\ +\infty, & \text{otherwise.} \end{cases}$$

**Example 2** Consider

$$\min_{(x,u)} \int_0^T f_o(x)$$

$$\text{subject to: } \dot{x}(t) = f(x) + u$$

$$x(0) = x_o$$

$$|u(t)| \leq 1$$

*Reformulating, we obtain*

$$l(a, b) = \begin{cases} 0 & \text{for } a = x_o \\ +\infty, & \text{otherwise} \end{cases}$$

$$K(x, v, u) = \begin{cases} f_o(x), & \text{for } v = f(x) + u \text{ and } |u| \leq 1 \\ +\infty, & \text{otherwise} \end{cases}$$

$$L(x,v) = \inf_u K(t,x,v,u) = \begin{cases} f_o(x), & \text{for } f(x)-1 \le v \le f(x)+1 \\ +\infty, & \text{otherwise} \end{cases}$$

*The Hamiltonian, then, is*

$$H(x,p) = \sup_v \{pv - f_o(x)\} = \begin{cases} p[f(x)-1] - f_o(x), & \text{for } p < 0 \\ p[f(x)+1] - f_o(x), & \text{for } p \ge 0 \end{cases} = pf(x)+|p|-f_o(x)$$

*Theorem 4 states that if there is an optimal control we will have a solution to the following differential inclusion.*

$$\dot{x} \in \partial_p H(t,x,p)(x,p) = \begin{cases} f(x)-1, & \text{for } p < 0 \\ [f(x)-1, f(x)+1], & \text{for } p = 0 \\ f(x)+1, & \text{for } p > 0 \end{cases}$$

$$\dot{p} \in -\partial_x H(t,x,p)(x,p) = -pf'(x) + f_o'(x)$$

$$(p(0), -p(T)) \in \partial l(x(0), x(T)) = \begin{cases} \{(\xi, 0) : \xi \in \Re\}, & \text{for } x(0) = x_o \\ \emptyset, & \text{for } x(0) \ne x_o \end{cases}$$

*We look for orbits in the $(x,-p)$ phase plane which satisfy*

$$(x(0), p(T)) = (x_o, 0).$$

## 3.3 Questions

Certain questions arise when working with the infinite reformulation. In particular we ask the following:

1. Does the reformulation preserve the minimum of the original control problem?
2. When do the functions $K$ and $L$ have finite integrals?
3. Does $\psi$ have a minimum and is it finite?

## 3.4 Assumptions

This section presents assumptions which permit answers to these questions. The information in the remainder of this chapter is covered in detail in [17].

We begin with several definitions and then place five assumptions on our original problem and the reformulation process. Throughout this material assume that

$$F : [0,T] \times \Re^k \to \Re \cup \{+\infty\}.$$

**Definition 7** *A subset of* $[0,T] \times \Re^k$ *is* $\mathcal{L} \times B$ *measurable provided it is in the* $\sigma$*-algebra generated by products of Lebesgue measurable subsets of* $[0,T]$ *and Borel subsets of* $\Re^k$.

**Definition 8** *The function* $F$ *is* $\mathcal{L} \times B$ *measurable provided* $F$ *is measurable in* $(t,z)$ *with respect to the* $\sigma$*-algebra generated by products of Lebesgue measurable subsets of* $[0,T]$ *and Borel subsets of* $\Re^k$.

**Definition 9** *The function* $F$ *is a normal integrand provided that* $F$ *is* $\mathcal{L} \times B$ *measurable and* $F$ *is lower semicontinuous in* $z$ *for a fixed* $t$.

**Definition 10** *The function* $F$ *is proper provided* $F \not\equiv +\infty$.

Assume the control problem (3.1) and the reformulation satisfy the following five assumptions.

**Assumption 1** *i.* $\mathcal{F}$ *is a nonempty, closed set.*

*ii.* $f_o$ *is continuous in* $(x,u)$ *and measurable in* $t$. *This is known as the Caratheodory condition.*

*iii.* $f$ *is continuous in* $(x,u)$ *and measurable in* $t$.

**Assumption 2** *i.* $\mathcal{B}$ *is a nonempty closed set.*

*ii.* $l_o$ *is lower semicontinuous.*

**Assumption 3 (inf-boundedness condition)** *For all bounded sets* $D \subset \Re^n \times \Re^n$ *and any* $\alpha \in \Re$, *for all* $t \in [0,T]$ *the set*

$$C(D,\alpha,t) = \{u \in \Re^m | \exists (x,v) \in D \text{ with } K(t,x,u,v) \leq \alpha\}$$

*is bounded.*

For the control problem (3.1) this translates to

$$C(D,\alpha,t) = \{u \in \Re^m | \exists (x,v) \in D \text{ with } v = f(t,x,u), (x,u) \in \mathcal{F} \text{ and } f_o(t,x,u) \leq \alpha.\}$$

In the proof the Equivalence Theorem below this condition permits a compactness which, when combined with the lower semicontinuity of $K(t,x,v,u)$ in $(x,v,u)$, yields the existence of minimum for the integral of $L(t,x,v,u)$.

**Assumption 4 (Convexity Condition)** *The function* $L$ *is convex in* $v$ *for each* $(t,x)$.

If $K(t,x,v,u)$ is convex in $(v,u)$ then Assumption 4 will be satisfied.

**Assumption 5 (Basic Growth Condition)** *For all* $p \in \Re^n$ *and any bounded subset* $S \subset \Re^n$ *there exists a Lebesgue integrable function* $\phi : [0,T] \to \Re$ *such that*

$$H(t,x,p) \leq \phi(t) \text{ for } t \in [0,T] \text{ a.e. and } x \in S.$$

## 3.5 An Equivalence Theorem

**Lemma 1** *If Assumption 1 holds then $K$ is a proper normal integrand.*

**Lemma 2** *If $F$ is a normal integrand and $z(t)$ is Lebesgue measurable, then $F(t, z(t))$ is Lebesgue measurable.*

This lemma will give the measurability of the functions $K$ and $L$ since the mapping $t \rightarrow (x(t), \dot{x}(t), u(t))$ is measurable in our case.

**Lemma 3** *If Assumption 2 holds then the function $l$ is a proper lower semi-continuous function.*

**Definition 11** *Let $\Gamma : [0, T] \rightarrow \mathcal{P}(\Re^k) = 2^{\Re^k}$. The function $\Gamma$ is a measurable multifunction if $\Gamma(t)$ is closed for each $t$, and the graph of $\Gamma = \{(t, z) | z \in \Gamma(t)\}$ is $\mathcal{L} \times B$ measurable. Here $Dom(\Gamma) = \{t | \Gamma(t) \neq \emptyset\}$.*

**Definition 12** *Let $f : \Re^k \rightarrow \Re \cup \{+\infty\}$. The epigraph of $f$, denoted epi(f) is the set*

$$\{(x, \alpha) \in \Re \times \Re^k | f(x) \leq \alpha\}.$$

For $f : \Re \rightarrow \Re \cup \{+\infty\}$, epi($f$) contains the points lying above the graph of $f$.

**Lemma 4** *The function $f$ described above is lower semicontinuous if and only if epi(f) is closed.*

**Lemma 5** *The function $F$ is a normal integrand if and only if*

$$\Gamma(t) = \{(x, \alpha) \in \Re \times \Re^k | F(t, x) \leq \alpha\} = epiF(t, \cdot)$$

*is a measurable multifunction.*

**Theorem 5** *(Castaing) Let $\Gamma : [0, T] \rightarrow \mathcal{P}(\Re^k)$ be a closed-valued multifunction. Then $\Gamma$ is measurable if and only if there exists a countable sequence of measurable functions $\{z_i\}$ such that $\Gamma(t) = \overline{\{z_i(t)\}}$ for all $t \in dom(\Gamma)$.*

In the proof of the Equivalence Theorem, given any $x \in A[0, T]$ with $\psi(x) \leq +\infty$ a corollary to this theorem allows the selection of a measurable function $u$ so that $\Phi(x, u) = \psi(x)$.

**Lemma 6** *Under assumptions 1 and 3, $L$ is a proper normal integrand, lower semicontinuous in $(x, v)$.*

We now state an Equivalence Theorem due to Rockafellar.

**Theorem 6** *[Equivalence Theorem] Suppose that Assumption 3 (inf-boundedness condition) is satisfied. By the previous lemma $L$ is $\mathcal{L} \times B$ measurable and lower semicontinuous. The infimum in the definition of $L$ is always attained (hence never $-\infty$). Thus, the functional $\psi$ is well-defined.*

*Furthermore for all $x \in A[0,T]$ we have*

$$\psi(x) = \min_{u \in \mathcal{L}[0,T]} \Phi(x,u)$$

*where the minimum is attained by at least one measurable function $u$.*

In this sense the reformulated problem (3.4) is equivalent to (3.2) The function $K$ in Rockafellar's paper is designed to permit the absorption of the constraints via infinite penalties so that (3.1) and (3.2) are equivalent.

## 3.6 An Existence Theorem

Having discussed the equivalence of the reformulation to the original optimal control problem, we now turn our attention to the existence of a minimizer. We begin with a few lemmas.

**Lemma 7** *Assumptions 1 and 3 imply that $H$ is convex and lower semicontinuous in $p$.*

**Lemma 8** *Assumptions 1, 3, and 4 imply that*

$$L(t,x,v) = \sup_{p \in \Re^n} \{\langle p, v\rangle - H(t,x,p)\}.$$

That is, $L$ is the Fenchel transform of $H$ giving a duality relationship between these two functions.

**Lemma 9** *Assumptions 1, 3, 4, and 5 imply that $H$ is upper semicontinuous in $(x,p)$ and $\mathcal{L} \times B$ measurable.*

The following theorem from [17] will play a central role in the existence theorems.

**Theorem 7** *Let $f : [0,T] \times \Re^k \to \Re \cup \{+\infty\}$ be a proper normal integrand and let*

$$g(t, w(t)) = \sup_{z \in \Re^k} \{w \cdot z - f(t,z)\}. \tag{3.5}$$

*(a) If $\int_0^T f(t,z(t))dt < +\infty$ for at least one $z \in \mathcal{L}^\infty$. then for any $w \in \mathcal{L}^1[0,T]$ we have*

$$\int_0^T g(t,w(t))dt = \sup_{z \in \mathcal{L}^\infty} \{\int_0^T [w(t) \cdot z(t) - f(t,z(t))]dt\}. \tag{3.6}$$

*(b) If $\int_0^T f(t, z(t))dt < +\infty$ for every $z \in \Re^k$, then for every $z \in \mathcal{L}^\infty$ and $\beta \in \Re$ the set*

$$S = \{w \in \mathcal{L}^1 : \int_0^T g(t, w(t))dt \leq \beta + \int_0^T \{w(t) \cdot z(t)\}dt \qquad (3.7)$$

*is compact in the weak topology on $\mathcal{L}^1$, and hence bounded in the norm topology on $\mathcal{L}^1$.*

**Lemma 10** *Assumptions 1, 2, 3, 4, and 5 imply that $\psi$ is weakly lower semicontinuous on $A[0,T]$.*

Proof. Let $x_n \rightharpoonup \bar{x}$ weakly. From the basic growth condition

$$H(t, x_n(t), p(t)) \leq \phi(t) \ \forall n \text{ where } \phi \in \mathcal{L}^1[0,T].$$

Hence, $H(t, x_n(t), p(t)) \in \mathcal{L}^1[0,T]$ for all $n$. Since $-H(t, x(t), p(t))$ is lower semicontinuous in $(x, p)$, we have

$$-H(t, \bar{x}, p(t)) \leq \underline{\lim}_{n\to\infty} - H(t, x_n(t), p(t)).$$

By Fatou's lemma,

$$\int_0^T -H(t, \bar{x}, p(t))) \leq \underline{\lim}_{n\to\infty} \int_0^T -H(t, x_n(t), p(t)) \text{ whenever } x_n \rightharpoonup \bar{x} \text{ weakly}.$$

Hence, $\int_0^T -H(t, x(t), p(t))$ is weakly lower semicontinous.

We note that since $\psi$ is weakly lower semicontinuous, proving that $\psi$ has a bounded minimizing sequence will demonstrate the existence of a minimizer for $\psi$. This is the goal of the following two results.

**Theorem 8** *For any $r \geq 0$ and $\alpha \in \Re$,*

$$D = \{x \in A[0,T] | \psi(x) \leq \alpha \textit{ and } \|x\|_\infty \leq r\}$$

*is compact in the weak topology on $A[0,T]$.*

Proof. See [17]. Theorem 7 is used in this proof.

**Lemma 11** *If $\{x_n\}$ is a minimizing sequence for $\psi$ and there exists $\beta$ such that $\|x_n\|_\infty \leq \beta$ for all $n$, then there exists a minimizer for $\psi$.*

Proof. Let $\alpha_o = \inf \psi(x)$. If $\alpha_o = +\infty$ then any $\bar{x}$ is optimal. Suppose $\alpha_o < \infty$. By the previous theorem

$$D = \{x \in A[0,T] | \psi(x) \leq \alpha_o + 1 \text{ and } \|x\|_\infty \leq \beta\}$$

is compact in the weak topology on $A[0,T]$. Let $\{x_n\}$ denote a subsequence which lies in $D$. Then that subsequence $\{x_n\}$ forms a bounded minimizing sequence. Since $\psi$ is weakly lower semicontinuous, $\psi$ has a minimizer.

We summarize the above results by stating verbatim the corresponding theorems from Rockafellar. The proofs are given in detail in [17].

**Theorem 9** *[Semicontinuity Theorem] Suppose L and H satisfy the convexity and basic growth conditions, respectively. Then, for all real numbers $\alpha$ and $r$ the set*

$$\{x \in A[0,T] | \psi(x) \leq \alpha, \|x\|_\infty \leq r\}$$

*is compact in the weak topology of $A[0,T]$ and hence also compact as a subset of $C[0,T]$ in the norm topology of $C[0,T]$.*

*In particular, $\psi$ is lower semicontinuous relative to the norms of $A[0,T]$ and $C[0,T]$ and weakly sequentially lower semicontinuous relative to $A[0,T]$.*

**Theorem 10** *[Existence Theorem 1.] Suppose L and H satisfy the convexity and basic growth conditions, respectively. If there exists a minimizing sequence $\{x_n\}$ for $\psi$ and a real number $\beta$ such that $\|x_n\|_\infty \leq \beta$, then there is a subsequence converging in both the norm topology of $C[0,T]$ and the weak topology of $A[0,T]$ to an $x \in A[0,T]$ minimizing $\psi$. That is, there is an $\hat{x}$ which is optimal for $\psi$.*

*In particular, $\psi$ attains its minimum over $A[0,T]$ if there is an $r > 0$ such that*

$$L(t,x,v) < +\infty \text{ implies } |x| < r.$$

The following corollary is central to the proof of our second existence theorem for periodic problems, proved in Chapter 4.

**Corollary 1** *If for all $\alpha \in \Re$ there exists a $\beta : \Re \to \Re$ such that $\psi(x) \leq \alpha$ implies $\|x\|_\infty \leq \beta(\alpha)$ then there exists an $\hat{x}$ which is optimal for $\psi$.*

## 3.7 Periodic Examples

We conclude this chapter with several motivating examples. Each of these one-dimensional periodic examples is free from a control. For each example we discuss the convexity of the objective function, the compactness of the feasible set and the existence of a solution.

**Example 3**

$$\min_x \int_0^T x(t)dt$$

$$\text{subject to: } \dot{x}(t) = 0$$

$$x(0) = x(T)$$

*Here, the objective function is not strictly convex. The feasible set, which consists of the constant functions, is not compact. Clearly there is no minimizer.*

**Example 4**

$$\min_x \int_0^T |x(t)|dt$$

$$\textit{subject to: } \dot{x}(t) = 0$$

$$x(0) = x(T)$$

*Although the feasible set is not compact, the convexity of the objective function gives rise to a minimizer for this problem, namely* $x(t) \equiv 0$.

**Example 5**

$$\min_x \int_0^T x(t)dt$$

$$\textit{subject to: } \dot{x} = \begin{cases} 0, & \text{for } |x| \leq M \\ x - M, & \text{for } x > M \\ x + M, & \text{for } x < -M \end{cases}$$

$$x(0) = x(T)$$

*where* $M > 0$. *Even though the objective function is nonconvex, the compactness of the feasible set* $\{x : |x| \leq M\}$ *permits a minimum, namely* $x(t) \equiv -M$.

**Example 6**

$$\min_x \int_0^T -x(t)dt$$

$$\textit{subject to: } \dot{x} = \begin{cases} 0, & \text{for } x \leq M \\ x - M, & \text{for } x > M \end{cases}$$

$$x(0) = x(T)$$

*where* $M > 0$. *Here the objective function is nonconvex and the feasible set* $\{x : x \leq M\}$ *is not compact, however, a minimizer still exists, namely* $x(t) \equiv M$.

The final example above is particularly interesting, since we have neither a convex objective function nor a compact feasible set, and yet a minimizer still exists. This indicates an interplay between the objective function and the solutions to the differential equation. In the next chapter this interplay will be accessed via a Hamiltonian function which involves both the objective function and the constraints.

# Chapter 4

# Existence Theorems

This chapter begins with two theorems of Rockafellar which apply to certain initial value problems. We then consider general control problems with periodic constraints. Two existence theorems are proved. Examples are given and compared with the usual compactness/convexity conditions.

## 4.1 Initial Value Problems in Optimal Control

The initial value problem in optimal control is given as

$$\min_{(x,u)} \left\{ l_o(x(0), x(T)) + \int_0^T f_o(t, x(t), \dot{x}(t), u(t)) \right\} \tag{4.1}$$

$$\begin{aligned} \text{subject to: } \dot{x}(t) &= f(t, x(t), u(t)) \\ x(0) &= x_o \\ x &\in A_n[0, T] \\ u &\in \mathcal{L}_m[0, T] \end{aligned}$$

Reformulating we obtain

$$\min_{x} \left\{ l(x(0), x(T)) + \int_0^T L(t, x, \dot{x}) \right\} \tag{4.2}$$

$$\text{subject to: } x \in A_n[O, T]$$

$$l(x(0), x(T)) = \begin{cases} l_o & \text{for } x(0) = x_o \\ +\infty, & \text{otherwise} \end{cases}$$

We now state two existence theorems. The theorems give bounds on $H$ which admit the existence of an optimal control. The proofs rely on Corollary 1 of Theorem 10.

**Theorem 11** *Suppose there exist functions*

$$\theta : [0,T] \times [0,+\infty) \to \Re \quad \text{and}$$
$$j : [0,+\infty) \to \Re$$

*and a nonnegative constant* $\eta$ *such that*

$$H(t,x,p) \le \theta(t,\|p\|) \quad \textit{and}$$
$$l(\lambda,\rho) \ge j(\|\lambda\|) - \eta\|\rho\|$$

*where* $\theta$ *is*

*i)* $\mathcal{L} \times \mathcal{B}$ *measurable*

*ii) integrable in* $t$ *for fixed* $s$

*iii) convex, lower semicontinuous and nondecreasing in* $s$.

*and* $j$ *satisfies*

*i)* $j$ *is continuous and nondecreasing in* $s$

*ii)* $\lim_{s\to\infty} \frac{j(s)}{s} = +\infty$

*Then* $\psi$ *attains a minimum.*

Proof: The proof is relatively straighforward, seeking to access Corollary 1 of Theorem 10. (See [17]).

Assume $\alpha \ge \psi(x) = l(x(0),x(T)) + \int_0^T L(t,x,\dot{x})$. Accessing the duality of $L$ and $H$ we have $\langle p,v\rangle - H(t,x,p) \le L(t,x,v)$ for all $p \in \mathcal{L}^\infty$. Therefore

$$\begin{aligned}\alpha &\ge l(x(0),x(T)) + \int_0^T [\langle p(t),\dot{x}(t)\rangle - H(t,x(t),p(t))]dt \text{ for all } p \in \mathcal{L}^\infty \\ &\ge j(\|x(0)\|) - \eta\|x(T)\| + \int_0^T [\langle p(t),\dot{x}(t)\rangle - \theta(t,\|p\|)]dt \text{ for all } p \in \mathcal{L}^\infty.\end{aligned}$$

Let $s > 0$. Setting $p(t) = \frac{s\dot{x}(t)}{\|\dot{x}(t)\|}$ gives $\|p\| = s$ and $\langle p,\dot{x}\rangle = s\|\dot{x}\|$ and so

$$\begin{aligned}\alpha &\ge j(\|x(0)\|) - \eta\|x(T)\| + \int_0^T [s\|\dot{x}(t)\| - \theta(t,s)]dt \\ &\ge j(\|x(0)\|) - \eta\|x(T)\| + \int_0^T g(t,w(t)),\end{aligned}$$

where

$$g : \Re \times \Re \to \Re \text{ is defined by}$$

$$g(t,\xi) = \sup_{s>0}\{s|\xi| - \theta(t,s)\}$$

and $w(t) = \|\dot{x}(t)\|$.

Since

$$\|x\|_\infty \leq \|x(0)\| + \int_0^T \|\dot{x}(t)\| \qquad (*)$$

we have

$$\begin{aligned}
\int_0^T g(t,w(t)) &\leq \alpha - j(\|x(0)\|) + \eta\|x(T)\| \\
&\leq \alpha - j(\|x(0)\|) + \eta\|x\|_\infty \\
&\leq \alpha - j(\|x(0)\|) + \eta\left[\|x(0)\| + \int_0^T \|\dot{x}(t)\|\right] \\
&\leq \alpha + \left[\eta\|x(0)\| - j(\|x(0)\|)\right] + \int_0^T \eta w(t).
\end{aligned}$$

By the properties of $j$ there exists $\gamma \in \Re$ such that $\eta\|x(0)\| - j(\|x(0)\|) \leq \gamma$ and so

$$\int_0^T g(t,w(t))dt \leq \alpha + \gamma + \int_0^T \eta w(t).$$

By Theorem 7 the set of $w(t)$ satisfying this statement have an $\mathcal{L}^1$ bound, that is, $\int_0^T \|\dot{x}(t)\|$ is bounded.

From the above we also have

$$\begin{aligned}
j(\|x(0)\|) - \eta\|x(0)\| &\leq \alpha + \int_0^T \left[\eta w(t) - g(t,w(t))\right] \\
&\leq \alpha + \int_0^T \theta(t,\eta) < +\infty.
\end{aligned}$$

Using the properties of $j$ it is easy to show that $\|x(0)\|$ is bounded. By (*) then $\|x\|_\infty$ is bounded and so, applying Corollary 1, $\psi$ attains its minimum.

Q.E.D.

**Theorem 12** *Assume*

$$H(t,x,p) \leq \theta(t,\|p\|) + \sigma(t)\|x\| + \tau(t)\|p\|\|x\|$$

*where $\sigma$ and $\tau$ are nonnegative integrable real-valued functions and*

$$l(\lambda,\rho) \geq j(\|\lambda\|) - \eta\|\rho\|$$

*where $\theta$, $j$, and $\eta$ are as in the previous theorem. Then $\psi$ attains a minimum.*

Proof: The proof is an extension of the previous proof (See [17]). As before, the goal is to find a bound on $\|x\|_\infty$. We begin with an initial estimate. Define the functions:

$$\begin{aligned} g(t,\xi) &= \sup_{s>0}\{s|\xi| - \theta(t,s)\} \\ w(t) &= max\{0, \|\dot{x}(t)\| - \tau(t)\|x(t)\|\} \\ r(t) &= e^{-\int_0^t \tau(\xi)d\xi}. \end{aligned}$$

Almost everywhere we have

$$\frac{d}{dt}\|x(t)\| = \begin{cases} \dfrac{\langle \dot{x}(t), x(t)\rangle}{\|x(t)\|}, & \text{for } \|x(t)\| > 0 \\ 0, & \text{for } \|x(t)\| = 0 \end{cases}$$

and so

$$\frac{d}{dt}\|x(t)\| - \tau(t)\|x(t)\| \leq w(t).$$

Then,

$$\frac{d}{dt}\left[r(t)\|x(t)\|\right] = r(t)\left[\frac{d}{dt}\|x(t)\| - \tau(t)\|x(t)\|\right] \leq w(t)$$

since $r$ is decreasing. Integrating from 0 to $t$

$$r(t)\|x(t)\| - r(0)\|x(0)\| \leq \int_0^t w(\xi)d\xi$$

and so, taking the supremum over $t \in [0, T]$

$$\|x\|_\infty \leq \frac{1}{r(T)}\left[\|x(0)\| + \int_0^T w(\xi)d\xi\right]. \tag{A}$$

Deducing bounds on $\|x(0)\|$ and $\int_0^T w(\xi)d\xi$ will complete the proof.

As before we assume $\alpha \geq \psi(x) = l(x(0), x(T)) + \int_0^T L(t, x, \dot{x})$. Accessing the duality of $L$ and $H$ we have $\langle p, v\rangle - H(t, x, p) \leq L(t, x, v)$ for all $p \in \mathcal{L}^\infty$. Therefore, for all $p \in \mathcal{L}^\infty$ we have

$$\begin{aligned} \alpha &\geq l(x(0), x(T)) + \int_0^T \left[\langle p(t), \dot{x}(t)\rangle - H(t, x(t), p(t))\right] \\ &\geq j(\|x(0)\|) - \eta\|x(T)\| + \int_0^T \left[\langle p(t), \dot{x}(t)\rangle - \theta(t, \|p\|) - \sigma(t)\|x\| - \tau(t)\|x\|\|p\|\right]. \end{aligned}$$

Let $s > 0$. Setting $p(t) = \dfrac{s\dot{x}(t)}{\|\dot{x}(t)\|}$ gives $\|p\| = s$ and $\langle p, \dot{x}\rangle = s\|\dot{x}\|$ and so

$$\alpha \geq j(\|x(0)\|) - \|x\|_\infty[\eta + \int_0^T \sigma(t)] + \int_0^T \{s[\|\dot{x}(t)\| - r(t)\|x(t)\|] - \theta(t,s)\}.$$

From (A) and the definitions of $w(t)$ and $g(t, w(t))$

$$\begin{aligned}\alpha &\geq j(\|x(0)\|) - \left[\|x(0)\| + \int_0^T w(t)\right]\frac{[\eta + \int_0^T \sigma(t)dt]}{r(T)} + \int_0^T g(t, w(t)) \\ &\geq j(\|x(0)\|) - \bar{s}\|x(0)\| + \int_0^T [g(t, w(t)) - \bar{s}w(t)] \qquad \text{(B)}\end{aligned}$$

where $\bar{s} = \dfrac{\eta + \int_0^T \sigma(t)}{r(T)}$.

By the properties of $j$ there exists $\gamma \in \Re$ such that $\bar{s}\|x(0)\| - j(\|x(0)\|) \leq \gamma$ and so

$$\int_0^T g(t, w(t)) \leq \alpha + \gamma + \bar{s}\int_0^T w(t).$$

By Theorem 7 the set of $w(t)$ satisfying this statement have an $\mathcal{L}^1$ bound, say

$$\int_0^T w(t) \leq D_1. \qquad (C)$$

From (B) and the definition of $g(t, w(t))$

$$j(\|x(0)\|) - \bar{s}\|x(0)\| \leq \alpha + \int_0^T \theta(t, \bar{s}) < +\infty.$$

Via the properties of $j$ we can show this implies that there exists $D_2$ such that

$$\|x(0)\| \leq D_2. \qquad (D)$$

Equations (A), (C) and (D) imply

$$\|x\|_\infty \leq \frac{1}{r(T)}[D_2 + D_1].$$

By Corollary 1 of Theorem 10, $\psi$ attains its minimum.

Q.E.D.

Let us briefly examine the role of the requirement on the endpoint element of the objective function:

$$l(\lambda, \rho) \geq j(\|\lambda\|) - \eta\|\rho\|.$$

In both examples, we consider an initial value problem $x(0) = x_o$. Then

$$l(\lambda, \rho) = \begin{cases} l_o(\lambda, \rho) & \text{for } \lambda = x_o \\ +\infty, & \text{otherwise} \end{cases}.$$

It is easy to satisfy the requirement at $\lambda \neq x_o$.

**Example 7** *Assume $l_o(\lambda, \rho) \equiv 0$. Then, letting $j(s) = (s - x_o)^2$ and $\eta = 0$ satisfies the requirement.*

**Example 8** *On the other hand, if $l_o(\lambda, \rho) = -\rho^2$ we would need*

$$l(x(0), \rho) = -\rho^2 \geq j(\|x(0)\|) - \eta\|\rho\|,$$

*which cannot occur on an infinite $\rho$-interval. Notice, however, that if $x(T)$ were constrained to a finite interval, we could select an appropriate $\eta$ to satisfy the requirement.*

## 4.2 Initial Value Problem Examples

We now apply Theorems 11 and 12 to four one- dimensional linear examples. In each example locating the optimal control is straightforward. These four examples demonstrate the success of the preceding existence theorems for initial value problems in optimal control.

**Example 9**

$$\min_x \int_0^T |x(t)|dt$$

$$\textit{subject to:} \quad \dot{x}(t) = u(t)$$

$$x(0) = 0$$

$$|u(t)| \leq 1$$

*Here $H(x,p) = |p| - |x| \leq |p| = \theta(|p|)$ so that Theorem 11 predicts the existence of a minimizer. Clearly the optimal control in this case is $u \equiv 0$.*

**Example 10**

$$\min_x \int_0^T x(t)dt$$

$$\textit{subject to:} \quad \dot{x}(t) = u(t)$$

$$x(0) = 0$$

$$|u(t)| \leq 1$$

*Here $H(x,p) = |p| - x \leq |p| + |x| = \theta(|p|) + |x|$ so that Theorem 12 predicts the existence of a minimizer. The optimal control in this case is $u \equiv -1$.*

**Example 11**

$$\min_x \int_0^T x(t)dt$$

$$\textit{subject to: } \dot{x}(t) = x(t) + u(t)$$

$$x(0) = 0$$

$$|u(t)| \leq 1$$

*Here* $H(x,p) = |p| - x + xp \leq \theta(|p|) + |x| + |x||p|$ *so that Theorem 12 predicts the existence of a minimizer. The optimal control in this case is* $u \equiv -1$.

**Example 12**

$$\min_x \int_0^1 -x^2(t)dt$$

$$\textit{subject to: } \dot{x}(t) = u(t)$$

$$x(0) = 0$$

$$|u(t)| \leq 1$$

*Here* $H(x,p) = |p| + x^2$ *which cannot be bounded by a function of the form* $\theta(|p|) + |x| + |x||p|$. *Hence Theorem 12 does not predict the existence of a minimizer.*

*In this example the Hamiltonian system of differential inclusions is*

$$\dot{x} = \begin{cases} -1, & \text{for } p < 0 \\ [-1,1], & \text{for } p = 0 \\ 1, & \text{for } p > 0 \end{cases}$$

$$\dot{p} = -2x$$

$$(x(0), p(T)) = (0, 0).$$

*The orbit diagram indicates a fixed point at* $(x,p) = (0,0)$ *at which point the objective function has a value of* 0. *If, however,* $x(t) = t$, *the objective function has value* $-1/3$. *This occurs when* $u(t) = 1$ *and is the minimizer.*

## 4.3 Periodic Problems in Optimal Control

In this section we state and prove two existence theorems for periodic problems in optimal control. These theorems are similar to those in the preceding section. The proof of the second periodic theorem, however, differs in the approach to its proof.

We state the periodic optimal control problem as follows:

$$\min_{(x,u)} \quad \left\{l_o(x(0), x(T)) + \int_0^T f_o(t, x(t), \dot{x}(t), u(t))\right\} \tag{4.3}$$

$$\begin{aligned} \text{subject to: } \dot{x}(t) &= f(t, x(t), u(t)) \\ x(0) &= x(T) \\ x &\in A_n[0, T] \\ u &\in \mathcal{L}_m[0, T] \end{aligned}$$

The reformulated problem is

$$\min_x \left\{l(x(0), x(T)) + \int_0^T L(t, x, \dot{x})\right\} \tag{4.4}$$

$$\text{subject to: } x \in A_n[O, T]$$

where

$$l(a, b) = \begin{cases} l_o(a, b), & \text{for } a = b \\ +\infty, & \text{otherwise} \end{cases}$$

**Theorem 13** *Suppose*

*(1) There exists a $C^1$ function $V : \Re^n \rightarrow \Re$, a positive real number $\gamma$ and a real valued function $\delta \in \mathcal{L}^1[0, T]$ such that*

$$H(t, x, -\nabla V(x)) \leq -\gamma\|x\| + \delta(t),$$

*(2) $H(t, x, p) \leq \theta(t, \|p\|) + \sigma\|x\| + \tau\|x\|\|p\|$ where $\sigma$ and $\tau$ are nonnegative real numbers and $\theta$ is*

*i) $\mathcal{L} \times \mathcal{B}$ measurable*

*ii) integrable in t for fixed s*

*iii) convex, lower semicontinuous and nondecreasing in s.*

*(3) $l(\lambda, \rho) \geq -\beta$, where $\beta$ is constant, and*

*(4) $l(\lambda, \rho) < +\infty$ implies $\lambda = \rho$.*

*Then $\psi$ attains a minimum.*

Proof: The proof seeks to apply Corollary 1 by finding bounds on $\|x\|_\infty$ via $\mathcal{L}^1$ bounds on $x$ and on a carefully selected function $w$ which fits into the context of Theorem 7.

Assume $\alpha \geq \psi(x) = \int_0^T L(t, x(t), \dot{x}(t))dt + l(x(0), x(T))$. By (4), then $x(0) = x(T)$ and accessing the duality of $L$ and $H$ we have

$$\langle p, v\rangle - H(t, x, p) \leq L(t, x, v) \text{ for all } p \in \mathcal{L}^{\infty}.$$

Therefore

$$\alpha \geq \int_0^T \langle p(t), \dot{x}(t)\rangle dt - \int_0^T H(t, x(t), p(t))dt - \beta \text{ for all } p \in \mathcal{L}^{\infty}. \qquad (A)$$

Setting $p(t) = -\nabla V(x(t))$ gives:

$$\begin{aligned}
\alpha &\geq \int_0^T \langle -\nabla V(x(t)), \dot{x}(t)\rangle dt - \int_0^T H(t, x(t), -\nabla V(x(t))dt - \beta \\
&\geq -[V(x(T)) - V(x(0))] + \int_0^T \gamma \|x\| dt - \int_0^T \delta(t)dt - \beta \\
&= \gamma \int_0^T \|x\| dt - \int_0^T \delta(t)dt - \beta.
\end{aligned}$$

Thus, we have an $\mathcal{L}^1$ bound for $x(t)$:

$$\int_0^T \|x\| dt \leq \frac{1}{\gamma}\left[\alpha + \int_0^T \delta(t)dt + \beta\right] \qquad (B)$$

We now seek an $\mathcal{L}^1$ bound on $\dot{x}(t)$. We claim that

$$\alpha \geq \int_0^T g(t, w(t))dt - \sigma \int_0^T \|x\| dt - \beta. \qquad (C)$$

where

$$g(t, \xi) : \Re \times \Re \rightarrow \Re \text{ is defined by}$$

$$\begin{aligned}
g(t, \xi) &= \sup_{s>0} [s \cdot \xi - \theta(t, s)] \\
w(t) &= \max[0, \|\dot{x}\| - \tau\|x\|].
\end{aligned}$$

Then, from (A) and (2), for all $p \in \mathcal{L}^{\infty}$ we have

$$\begin{aligned}
\alpha &\geq \int_0^T \left[\langle p(t), \dot{x}(t)\rangle - H(t, x(t), p(t)\right] dt - \beta \\
&\geq \int_0^T \left[\langle p(t), \dot{x}(t)\rangle - \theta(t, \|p\|) - \sigma\|x\| - \tau\|x\|\|p\|\right] dt - \beta..
\end{aligned}$$

or

$$\int_0^T \left[\langle p(t), \dot{x}(t)\rangle - \tau\|x\|\|p\| - \theta(t, \|p\|)\right] dt \leq \alpha + \sigma \int_0^T \|x\| dt + \beta. \qquad (D)$$

There are two cases to consider.

Case 1. If $w(t) = \|\dot{x}\| - \tau\|x\|$ then we let $s > 0$ and set $p(t) = \dfrac{s\dot{x}(t)}{\|\dot{x}(t)\|}$ so that $\|p\| = s$ and $\langle p(t), \dot{x}(t)\rangle = s\|\dot{x}\|$. In (D) then

$$\int_0^T \{s[\|\dot{x}\| - \tau\|x\|] - \theta(t,s)\}dt \leq \alpha + \sigma\int_0^T \|x\|dt + \beta$$

so, taking the supremum over $s > 0$

$$\int_0^T g(t, w(t))dt \leq \alpha + \sigma\int_0^T \|x\|dt + \beta$$

Hence (C) holds and

$$\|\dot{x}\| = w(t) + \tau\|x\|. \qquad (E)$$

Case 2. If $w(t) = 0$ then $g(t, w(t)) = \sup_s\{-\theta(t,s)\} = -\theta(t,0)$. Setting $p = 0$ in (D) we have

$$\int_0^T -\theta(t,0) = \int_0^T g(t, w(t))dt \leq \alpha + \sigma\int_0^T \|x\|dt + \beta$$

implying that (C) holds. In this case $w(t) = 0$ implies

$$\|\dot{x}\| \leq \tau\|x\|. \qquad (F)$$

In both cases (C) holds, and so

$$\int_0^T g(t, w(t))dt \leq \alpha + \sigma\int_0^T \|x\|dt + \beta.$$

Theorem 7 implies that there is an a priori bound on the $\mathcal{L}^1$ norm of $w(t)$ and in (B) we found an $\mathcal{L}^1$ norm for $x(t)$. Hence, (E) and (F) imply that in either case we have a bound on the $\mathcal{L}^1$ norm of $\dot{x}(t)$.

Now $x(t) = x(0) + \int_0^t \dot{x}(s)ds$ so we have

$$\|x(0)\| = \|x(t) - \int_0^t \dot{x}(s)ds\| \leq \|x(t)\| + \int_0^T \|\dot{x}(s)\|ds.$$

Integrating from 0 to $T$

$$\int_0^T \|x(0)\| \leq \int_0^T \|x(t)\| + \int_0^T\int_0^T \|\dot{x}(s)\|dsdw \leq N$$

for some $N > 0$ since both $x$ and $\dot{x}$ have $\mathcal{L}^1$ bounds. Hence $\|x(0)\| \leq \frac{N}{T}$ so we have a bound for $\|x(0)\|$.

Finally, since $x(t) = x(0) + \int_0^t \dot{x}(s)ds$ we have a bound on $\|x\|_\infty$.

Applying Corollary 1 gives the existence of an $\hat{x}$ which is optimal for $\psi$.

Q.E.D.

**Theorem 14** *Suppose there exist functions*

*(a)* $W : \Re^n \to \Re$, *nonnegative and* $C^1$ *with* $W(x) \to \infty$ *as* $\|x\| \to \infty$,

*(b)* $V : \Re^n \to \Re$, *nonnegative and* $C^1$ *with* $V(x) \to \infty$ *as* $\|x\| \to \infty$,

*(c)* $\sigma \in \mathcal{L}^1[0,T]$ *with* $\sigma(t) \geq 0$ *and* $\int_0^T \sigma(t)dt \leq 1$,

*(d)* $Q : \Re^n \to \Re$, $C^1$ *with* $Q(x) \geq -\nu V(x) + \mu$ *where* $\mu \in \Re$ *and* $\nu < 1 - \int_0^T \sigma(t)dt$,

*(e)* $\delta \in \mathcal{L}^1[0,T]$ *with* $\delta(t) \geq 0$,

*(f)* $\gamma \in \mathcal{L}^1[0,T]$ *with* $\gamma(t) \geq 0$,

*(g)* $\theta \in \mathcal{L}^1[0,T]$ *with* $\theta(t) \geq 0$,

*such that*

*(1)* $H(t,x,-\nabla V(x)) \leq -\gamma(t)W(x) + \delta(t)$,

*(2)* $H(t,x,\nabla Q(x))) \leq \sigma(t)V(x) + \theta(t)$,

*and there exists a nonnegative constant* $\beta$ *satisfying*

*(3)* $l(\lambda,\rho) \geq -\beta$, *and*

*(4)* $l(\lambda,\rho) < +\infty$ *implies* $\lambda = \rho$.

*Then* $\psi$ *attains a minimum.*

Proof: Assume that $\psi$ does not attain a minimum. Then, there exists an $\alpha$ and a minimizing sequence $\{x_n\}$ for $\psi$ so that $\|\psi(x_n)\| \leq \alpha$ for all $n$ and $\|x_n\|_\infty \to +\infty$ as $n \to \infty$. Extend all functions of $t$ periodically with period $T$.

Then $x_n(0) = x_n(T)$ and for all $n$ and all $p \in \mathcal{L}^\infty$ we have

$$(*) \qquad \alpha + \beta \geq \int_0^T \langle p, \dot{x}_n \rangle dt - \int_0^T H(t,x_n,p)dt.$$

Fix n and define $x_n(t_{n_1})$ and $x_n(t_{n_2})$ by

$$\min_{t\in[0,T]}(V+Q)(x_n(t)) = (V+Q)(x_n(t_{n_2}))$$

$$\max_{t\in[0,T]} V(x_n(t)) = V(x_n(t_{n_1})).$$

Without loss of generality, assume $t_{n_1} \leq t_{n_2}$ and let $I_n = [t_{n_1}, t_{n_2}]$. Let $I_n^C$ denote the complement of $I_n$ relative to $[0,T]$, that is $I_n^C = [0,t_{n_1}] \cup [t_{n_2},T]$.

Let

$$p_n(t) = \begin{cases} -\nabla V(x_n(t)), & \text{for } t \in I_n \\ \nabla Q(x_n(t)), & \text{for } t \in I_n^C. \end{cases}$$

Then, in (*), using (1) and (2), we have

$$\begin{aligned}
\alpha+\beta &\geq -\int_{I_n}\langle\nabla V(x_n),\dot{x}_n\rangle+\int_{I_n^C}\langle\nabla Q(x_n),\dot{x}_n\rangle \\
&\quad -\int_{I_n} H(t,x_n,-\nabla V(x_n))dt-\int_{I_n^c} H(t,x_n,\nabla Q(x_n))dt \\
&\geq V(x_n(t_{n_1}))-V(x_n(t_{n_2}))+Q(x_n(T))-Q(x_n(t_{n_2}))+Q(x_n(t_{n_1}))-Q(x_n(0)) \\
&\quad +\int_{I_n}[\gamma(t)W(x_n)-\delta(t)]dt-\int_{I_n^C}[\sigma(t)V(x_n)+\theta(t)]dt
\end{aligned}$$

Rearranging and applying (d) and the definition of $t_{n_1}$ we have
$\alpha+\beta+\int_0^T\delta(t)dt+\int_0^T\theta(t)dt+V(x_n(t_{n_2}))+Q(x_n(t_{n_2}))$

$$\begin{aligned}
&\geq V(x_n(t_{n_1}))+Q(x_n(t_{n_1}))+\int_{I_n}\gamma(t)W(x_n)dt-\int_0^T\sigma(t)V(x_n)dt \\
&\geq V(x_n(t_{n_1}))\left[1-\nu-\int_0^T\sigma(t)dt\right]+\mu.
\end{aligned}$$

Let $\max\limits_{t\in[0,T]} x_n(t)=x_n(\tau_n)$. As $n\to\infty$ we have the following string of implications:

$$\begin{aligned}
\|x_n\|_\infty\to\infty &\Rightarrow \|x_n(\tau_n)\|\to\infty, \\
&\Rightarrow V(x_n(\tau_n))\to\infty, \quad \text{by hypothesis (b)}, \\
&\Rightarrow V(x_n(t_{n_1}))\to\infty, \\
&\Rightarrow (V+Q)(x_n(t_{n_2}))\to\infty \quad \text{by the last inequality}, \\
&\Rightarrow (V+Q)(x_n(t))\to\infty \quad \text{uniformly in } t, \\
&\Rightarrow \|x_n(t)\|\to\infty \quad \text{uniformly in } t, \text{ by the continuity of } V+Q, \\
&\Rightarrow W(x_n(t))\to\infty \quad \text{uniformly in } t, \text{ by hypothesis (a)}.
\end{aligned}$$

Finally, letting $p_n(t)=-\nabla V(x_n(t))$ in (*) gives

$$\alpha+\beta+\int_0^T\delta(t)dt\geq\int_0^T\gamma(t)W(x_n(t))dt$$

The right-hand side of this equation approaches $\infty$ as $n\to\infty$, which contradicts the boundedness of the left-hand side.

Hence, there is a minimizer for $\psi$.

Q.E.D.

**Theorem 15** *Suppose there exist $W, V, \delta, \gamma, \theta, \sigma$and $\beta$ as described above, and a function $Q(x) \leq \nu V(x) + \mu$ where $\mu \in \Re$ and $\nu < 1 - \int_0^T \sigma(t)dt$, satisfying $H(t, x, \nabla V(x)) \leq -\gamma(t)W(x) + \delta(t)$ along with (2), (3), and (4) above. Then $\psi$ attains a minimum.*

Proof: The proof is similar to that for the above theorem, defining $x_n(t_{n_2})$ by

$$\min_{t \in [0,T]} (V - Q)(x_n(t)) = (V - Q)(x_n(t_{n_2})).$$

Q.E.D.

## 4.4 Periodic Problem Examples

In this section we consider numerous examples of periodic problems in optimal control. The first four examples return to the motivating examples discussed at the end of Chapter 3. For each example we discuss the convexity of the objective function, the compactness of the feasible set and how the two periodic theorems apply.

**Example 13**

$$\min_x \int_0^T x(t)dt$$

$$\begin{aligned} \textit{subject to:} \quad \dot{x}(t) &= 0 \\ x(0) &= x(T) \end{aligned}$$

*Also, the function $H(x, p) = -x$, cannot be made less than $-\gamma W(x) + \delta$ for suitable $W$. Consequently we cannot satisfy the conditions of Theorem 14. The Hamiltonian system associated with this problem is*

$$\begin{aligned} \dot{x} &= 0 \\ -\dot{p} &= 1 \end{aligned}$$

$$p(0) = p(T),\ x(0) = x(T)$$

*which also has no solution. Here, the objective functions is non-convex, and the feasible set, $\{x : x \equiv constant\}$ is not compact. There is no solution to this optimization problem.*

**Example 14**

$$\min_x \int_0^T |x(t)|dt$$

$$\text{subject to: } \begin{aligned} \dot{x}(t) &= 0 \\ x(0) &= x(T) \end{aligned}$$

*The function $H(x,p) = -|x(t)|$ easily satisfies the conditions of Theorem 13 by setting $V(x) = x^2$. The Hamiltonian system associated with this problem is*

$$\dot{x} = 0$$

$$\dot{p} = \begin{cases} 1, & \text{for } x < 0 \\ [-1, 1], & \text{for } x = 0 \\ -1, & \text{for } x > 0 \end{cases}$$

$$p(0) = p(T),\ x(0) = x(T)$$

*for which $x = 0$ is a solution. Although the feasible set, $\{x : x \equiv constant\}$ is not compact, the convexity of the objective function gives rise to a minimizer for this problem, namely $x(t) \equiv 0$.*

**Example 15**

$$\min_x \int_0^T x(t)dt$$

$$\text{subject to: } \dot{x} = \begin{cases} 0, & \text{for } |x| \leq M \\ x - M, & \text{for } x > M \\ x + M, & \text{for } x < -M \end{cases}$$

$$x(0) = x(T)$$

*where $M > 0$. The Hamiltonian is*

$$H(x,p) = \begin{cases} -x, & \text{for } |x| \leq M \\ p(x - M) - x, & \text{for } x > M \\ p(x + M) - x, & \text{for } x < -M \end{cases}$$

*Setting $V(x) = x^2$, $Q(x) = 0$ and $W(x) = |x|$ satisfies the conditions of Theorem 14. Even though the objective function is nonconvex, the compactness of the feasible set $\{x : |x| \leq M\}$ permits a minimum, namely $x(t) \equiv -M$.*

**Example 16**

$$\min_x \int_0^T -x(t)dt$$

$$\text{subject to: } \dot{x} = \begin{cases} 0, & \text{for } x \leq M \\ x - M, & \text{for } x > M \end{cases}$$

$$x(0) = x(T)$$

*where $M > 0$. The Hamiltonian is*

$$H(x,p) = \begin{cases} x, & \text{for } x \leq M \\ p(x-M) + x, & \text{for } x > M \end{cases}$$

*Setting $V(x) = x^2$ and $Q(x) = 0$ and $W(x) = |x|$ satisfies the conditions of Theorem 14. Here the objective function is nonconvex and the feasible set $\{x : x \leq M\}$ is not compact, a minimizer still exists, namely $x(t) \equiv M$.*

Recall that in the final example we had neither a convex objective function nor a compact feasible set, and yet, a minimizer still exists. The theorems seem to successfully decipher the interplay between the objective function and the solutions to the differential equation.

We now apply Theorems 13 and 14 to three additional one- dimensional examples each involving a control.

**Example 17**

$$\min_x \int_0^1 |x(t)| dt$$

$$\begin{aligned} \textit{subject to:}\quad \dot{x}(t) &= u \\ x(0) &= x(T) \\ |u(t)| &\leq 1. \end{aligned}$$

*Here $H(x,p) = |p| - |x|$. Setting $V(x) = x^2/4$, we have $H(x, -\nabla V(x)) = -\frac{1}{2}|x|$ and so Theorem 13 predicts the existence of an optimal control, which in this case is $u \equiv 0$.*

**Example 18**

$$\min_x \int_0^1 x(t) dt$$

$$\begin{aligned} \textit{subject to:}\quad \dot{x}(t) &= u \\ x(0) &= x(T) \\ |u(t)| &\leq 1. \end{aligned}$$

*Here, if $u = 0$ then $x$ would be a constant, say $k$ and $\int_0^1 x(t)dt = \int_0^1 k dt = k$ which cannot be minimized. For this problem $H(x,p) = |p| - x$. If Theorem 14 held there would exist appropriate $V(x)$, $W(x)$, $\gamma$ and $\delta$ so that*

$$H(x, -\nabla V(x)) = |-\nabla V(x)| - x \leq -\gamma W(x) + \delta \qquad (*)$$

*which would imply $|-\nabla V(x)| \leq x - \gamma W(x) + \delta$. Since $W(x) \to \infty$ as $|x| \to \infty$, we have $x - \gamma W(x) + \delta \to -\infty$ as $x \to \infty$ which contradicts the positivity of the left hand side of (*). A similar argument shows that Theorem 13 does not apply.*

**Example 19**

$$\min_x \int_0^1 x(t)dt$$

$$\begin{aligned} \textit{subject to:}\quad \dot{x}(t) &= x + u \\ x(0) &= x(T) \\ |u(t)| &\leq 1. \end{aligned}$$

*The set* $u \equiv 1$ *and* $x \equiv -1$ *provides a minimizer. We have*

$$H(x,p) = |p| - x + xp.$$

*Setting* $V(x) = x^2$ *gives* $H(x, -\nabla V(x)) = H(x, 2x) = 2|x| - x - 2x^2$ *which is easily bounded by a function of the form* $-\gamma|x| + \delta$. *Hence Theorem 13 predicts the existence of an optimal control.*

# Chapter 5

# Associated Hamiltonian System

We include this discussion of the Hamiltonian system associated with the periodic problem because when the Hamiltonian system has a solution then the solution to the optimal control problem is finite. The Hamiltonian system also gives us insight into the nature of the solution.

This chapter opens with an example for which Theorem 14 predicts the existence of a minimizer but for which the minimum is $+\infty$. It then considers theorems of Gaines, Gupta and Peterson ([9] [10]) which guarantee finite minimums and extends these theorems to a general quadratic Hamiltonian.

## 5.1 The Periodic Problem

**Example 20** *Consider the one-dimensional example*

$$\min_{(x,u)} \int_0^T x^2$$

$$\begin{aligned} \textit{subject to:}\ \dot{x} &= 1 \\ x(0) &= x(T) \end{aligned}$$

*Here* $H(x,p) = p - x^2$. *Setting* $V = x^2$, $Q = 0$, *and* $W = x^2$ *satisfies the conditions of 14 , and so a minimum for* $\psi$ *is attained. Since the periodic boundary values cannot be satisfied* $\psi = +\infty$ *and hence the minimum is not finite.*

We will consider a general Hamiltonian system which can be associated with the periodic problem in optimal control.

$$\dot{x} \in \partial_p H(t,x,p) = f(t,x,p) \quad \text{a.e. on } [0,T]$$
$$-\dot{p} \in \partial_x H(t,x,p) = g(t,x,p) \quad \text{a.e. on } [0,T] \tag{5.1}$$
$$x(0) = x(T),\ p(0) = p(T)$$

where $[0,T]$ is fixed with $0 < T \leq \infty$.

Assume $f, g : [0,T] \times \Re^n \times \Re^n \to 2^{\Re^n}$ and satisfy the following assumptions:

**Assumption 6** *For all $(t,x,p) \in [0,T] \times \Re^n \times \Re^n$, $f(t,x,p)$ and $g(t,x,p)$ are nonempty, compact and convex subsets of $\Re^n$.*

**Assumption 7** *For all $t \in [0,T]$, $f(t,.,.)$ and $g(t,.,.)$ are upper semicontinuous on $\Re^n \times \Re^n$.*

**Assumption 8** *For all $(x,p) \in \Re^n \times \Re^n$, $f(t,x,p)$ and $g(t,x,p)$ are measurable multifunctions on $[0,T]$.*

**Assumption 9** *If $K \subseteq \Re^n \times \Re^n$ is compact, then there is an $\alpha \in \mathcal{L}_1^1[0,T]$ such that if $(x,p) \in K$, $u \in f(t,x,p)$, $v \in g(t,x,p)$, and $t \in [0,T]$, then*

$$(\|u\|^2 + \|v\|^2)^{1/2} \leq \alpha(t).$$

**Assumption 10** *There are positive constants $a, b, c, d$ and $e$ such that for all $(x,p) \in \Re^n \times \Re^n$, if $u \in f(t,x,p)$, and $v \in g(t,x,p)$ then*

*(i)* $\langle -x, v\rangle + \langle p, u\rangle \geq a(\|u\|^2 + \|v\|^2)^{1/2} - b(\|x\|^2 + \|p\|^2)^{1/2} - c$,

*(ii)* $\langle -x, v\rangle + \langle p, u\rangle \geq d(\|x\|^2 + \|p\|^2)^{1/2} - e$.

**Theorem 16 (Finiteness)** *Under the above assumptions the periodic Hamiltonian system 5.1 possesses a solution. See [10].*

**Theorem 17** *Suppose that Assumptions 1, 3, and 4 of Chapter 3 are satisfied. Also suppose that for any bounded sets $S_1$ and $S_2$ in $\Re^n$ there exists an $\alpha \in \mathcal{L}_1^1[0,T]$ such that*

$$|H(t,x,p)| \leq \alpha(t) \text{ for all } x \in S_1, p \in S_2.$$

*If $(x,p)$ satisfies a periodic Hamiltonian system of differential inclusions, then $\psi(x) < +\infty$.*

Proof. Since $\dot{x} \in \partial_p H(t,x,p)$ we have

$$H(t,x,p) + \langle q - p, \dot{x}\rangle \leq H(t,x,q) \text{ for all } q \in \Re^n.$$

and so

$$\langle p, \dot{x}\rangle - H(t,x,p) \geq \langle q, \dot{x}\rangle - H(t,x,q) \text{ for all } q \in \Re^n. \tag{a}$$

By Lemma 8 of Chapter 3

$$L(t,x,v) = \sup_p\{\langle p,v\rangle - H(t,x,p)\}.$$

Taking the supremum over $q$ on both sides of (a) gives

$$\langle p,\dot{x}\rangle - H(t,x,p) \geq L(t,x,\dot{x}). \qquad (b)$$

By hypothesis

$$-\alpha(t) \leq H(t,x,p).$$

Multiplying by $-1$, adding $\langle p,\dot{x}\rangle$ and taking the supremum over $p$ gives

$$\langle p,\dot{x}\rangle + \alpha(t) \geq L(t,x,\dot{x}). \qquad (c)$$

Integrating (c), since $\alpha \in \mathcal{L}_1^1[0,T]$ we obtain

$$\int_0^T L(t,x,\dot{x}) < +\infty.$$

Since we are assuming the periodic conditions are satisfied we also have $l(x(0),x(T)) < +\infty$. Hence $\psi(x) = l(x(0),x(T)) + \int_0^T L(t,x,\dot{x})$ is finite.

Q.E.D.

The canonical example in the Gaines/Peterson paper [10] is the Hamiltonian $H(x,p) = p^2 - x^2$. We seek to extend this to a class of general quadratic Hamiltonians.

## 5.2 The Quadratic Hamiltonian

Consider the general quadratic Hamiltonian

$$H : [0,T] \times \Re^n \times \Re^n \to \Re$$

defined by

$$H(t,x,p) = \langle Ax,x\rangle + \langle Bx,p\rangle + \langle Cp,p\rangle + \langle q,x\rangle + \langle r,p\rangle + s$$

where $A$ and $C$ are symmetric $n \times n$ matrices and $A$ and $C$ are negative and positive definite respectively, $q$ and $r$ are $n \times 1$ matrices and $s \in \Re$. The values of all the variables may depend on $t$. This yields the linear system of $2n$ differential equations

$$\begin{bmatrix} \dot{x} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} B & 2C \\ 2A & B^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} r \\ q \end{bmatrix}.$$

**Theorem 18** *The problem stated above has a periodic solution.*

Proof. Assumptions 1-4 of Theorem 16 are easily satisfied. We define the function $f(x,p)$ by

$$\begin{aligned} f(x,p) &= \langle -x, v\rangle + \langle p, u\rangle \\ &= \langle -2Ax, x\rangle + \langle 2Cp, p\rangle - \langle q, x\rangle + \langle r, p\rangle \end{aligned}$$

The critical point $(\bar{x}, \bar{p})$ of $f(x,p)$ is found by solving

$$\begin{aligned} f_x &= -4Ax - q = 0 \\ f_p &= 4Cp + r = 0. \end{aligned}$$

yielding $\bar{x} = \frac{-1}{4}A^{-1}q$, $\bar{p} = \frac{-1}{4}C^{-1}r$, and $f(\bar{x}, \bar{p}) = \frac{1}{8}\langle A^{-1}q, q\rangle - \frac{1}{8}\langle C^{-1}r, r\rangle$. Since

$$\begin{aligned} f(x,p) - f(\bar{x},\bar{p}) &= \langle -2Ax, x\rangle - \langle q, x\rangle - \frac{1}{8}\langle A^{-1}q, q\rangle + \langle 2Cp, p\rangle + \langle r, p\rangle - \frac{1}{8}\langle C^{-1}r, r\rangle \\ &= \langle -2A(x + \frac{1}{4}A^{-1}q), x + \frac{1}{4}A^{-1}q\rangle + \langle 2C(p + \frac{1}{4}C^{-1}r), p + \frac{1}{4}C^{-1}r\rangle \\ &> 0 \end{aligned}$$

for all $(x,p) \neq (\bar{x}, \bar{p})$ we have that $(\bar{x}, \bar{p})$ is a minimizer for the quadratic function $f(x,p)$. This is sufficient to show that there exist $a, b > 0$ such that

$$\langle -x, v\rangle + \langle p, u\rangle > a(\|x\|^2 + \|p\|^2) - b$$

which satisfies Assumption 5 of Theorem 16. If $a$ and $b$ are functions of $t$ we can select their minimum and maximum values on $[0,T]$.

Q.E.D.

**Theorem 19** *Let $R : \Re^n \to \Re^n$. Let $P : \Re^n \to \Re^n$ be invertible and set $Q = PRP^{-1}$. Consider the linear systems*

*(i) $\dot{x} = Rx + r(t)$, $x(0) = x(T)$ and*

*(ii) $\dot{x} = Qx + Pr(t)$, $x(0) = x(T)$.*

*The function $u$ is a solution to (i) iff $Pu$ is a solution to (ii).*

Proof. Since $u$ is a solution to (i),

$$\begin{aligned} \dot{u} &= Ru + r(t) \\ P\dot{u} &= PR(P^{-1}P)u + Pr(t) \\ (Pu)' &= Q(Pu) + Pr(t) \end{aligned}$$

Hence $Pu$ is a solution to (ii). Note that in this theorem the $n \times n$ matrices are independent of $t$.

Q.E.D.

**Theorem 20** *(a) Iff $D : \Re^n \rightarrow \Re^n$ and $E : \Re^n \rightarrow \Re^n$ are both symmetric. If both $E$ is positive definite and $D + D^T$ is sign definite, then there exists a nonsingular matrix $P : \Re^{2n} \rightarrow \Re^{2n}$ such that*

$$P \begin{bmatrix} D & E \\ 0 & -D^T \end{bmatrix} P^{-1} = \begin{bmatrix} B & C \\ -A & -B^T \end{bmatrix}.$$

*where $A$ is negative definite and symmetric and $C$ is positive definite and symmetric.*

*(b) Similarly, assume $D : \Re^n \rightarrow \Re^n$ and $E : \Re^n \rightarrow \Re^n$ are both symmetric. If both $E$ is negative definite and $D + D^T$ is sign definite, then there exists a nonsingular matrix $P : \Re^{2n} \rightarrow \Re^{2n}$ such that*

$$P \begin{bmatrix} D & 0 \\ -E & -D^T \end{bmatrix} P^{-1} = \begin{bmatrix} B & C \\ -A & -B^T \end{bmatrix}.$$

*where $A$ is negative definite and symmetric and $C$ is positive definite and symmetric.*

Proof: (a) Let $P = \begin{bmatrix} I & 0 \\ aI & bI \end{bmatrix}$. Then $P^{-1} = \frac{1}{b} \begin{bmatrix} bI & 0 \\ -aI & I \end{bmatrix}$. and

$$P \begin{bmatrix} D & E \\ 0 & -D^T \end{bmatrix} P^{-1} = \frac{1}{b} \begin{bmatrix} bD - aE & E \\ ab(D + D^T) - a^2 E & aE - bD^T \end{bmatrix} = \begin{bmatrix} B & C \\ -A & -B^T \end{bmatrix}.$$

Let $b > 0$. Clearly $B$ and $C$ satisfy the requirements. We have

$$A = \frac{1}{b}[a^2 E - ab(D + D^T)].$$

We wish to show that $a$ and $b$ can be selected so that $A$ is negative definite. Assume that $D + D^T$ is negative definite. We must have $a < 0$.

Let $u = min_{\|x\|=1}\{a\langle Ex, x\rangle - b\langle (D + D^T)x, x\rangle\}$. We can select $a$ and $b$ so that $u > 0$. Let $y$ be any vector and let $d = \|y\|$. Then

$$a\langle Ey, y\rangle - b\langle (D + D^T)y, y\rangle = d^2\left[a\left\langle E\frac{y}{d}, \frac{y}{d}\right\rangle - b\left\langle (D + D^T)\frac{y}{d}, \frac{y}{d}\right\rangle\right] \geq d^2 u > 0$$

and so $A$ is negative definite.

In the case where $D + D^T$ is positive definite the proof is similar.

For part (b), use $P = \begin{bmatrix} aI & bI \\ 0 & I \end{bmatrix}$. and proceed similarly.

Q.E.D.

**Theorem 21** *(a) Assume that $D$ and $E$ are symmetric. If $E$ is positive definite and $D + D^T$ is sign definite, then the system*

$$\begin{bmatrix} \dot{x} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} D & E \\ 0 & D^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}$$

*has a periodic solution.*

*(b) Assume that $D$ and $E$ are symmetric. If both $E$ and $D+D^T$ are negative definite, then the system*

$$\begin{bmatrix} \dot{x} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} D & 0 \\ E & D^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix}$$

*has a periodic solution.*

Proof. Rewrite the system as

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} D & E \\ 0 & -D^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} u \\ -v \end{bmatrix}$$

Apply Theorem 20 to obtain a matrix $Q$ satisfying

$$Q = P \begin{bmatrix} D & E \\ 0 & -D^T \end{bmatrix} P^{-1} = \begin{bmatrix} B & C \\ -A & -B^T \end{bmatrix}$$

where $A$ and $C$ are symmetric and negative and positive definite respectively. Let

$$\begin{bmatrix} r \\ q \end{bmatrix} = P \begin{bmatrix} u \\ -v \end{bmatrix}$$

Then the system

$$\begin{bmatrix} \dot{x} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} B & C \\ A & B^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} r \\ -q \end{bmatrix}$$

has a periodic solution by Theorem 16. Apply Theorem 4 to complete the proof. The proof to part (b) is similar.

Q.E.D.

To summarize, if

$$H(t,x,p) = \langle Ax, x\rangle + \langle Bx, p\rangle + \langle Cp, p\rangle + \langle q, x\rangle + \langle r, p\rangle + s$$

the corresponding Hamiltonian system

$$\begin{bmatrix} \dot{x} \\ -\dot{p} \end{bmatrix} = \begin{bmatrix} B & 2C \\ 2A & B^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} r \\ q \end{bmatrix}$$

has a periodic solution in each of the following three cases:

1. $A$ and $C$ are symmetric, $A$ is negative definite, and $C$ is positive definite,
2. $A$ is symmetric and negative definite, $B + B^T$ is sign definite, and $C = 0$,
3. $C$ is symmetric and positive definite, $B + B^T$ is sign definite, and $A = 0$.

# Chapter 6

# Compatibility of the Existence and Finiteness Theorems

This chapter investigates the compatibility of the Periodic Existence Theorems 13 and 14 of Chapter 4 and 16 of Chapter 5.

The relationships between the various elements of our problem are reviewed below. We assume that the hypotheses of Theorem 17 hold.

(1) If $\psi$ has a minimizing pair $(x, p)$ then the associated $(x, u)$ pair minimizes the original optimal control problem. (This minimum need not be finite.) (Theorem 6)

(2) If $\psi$ has a minimizing pair $(x, p)$ and the minimizer is finite, then $(x, p)$ will solve the associated Hamiltonian differential inclusion also. (Theorem 4)

(3) Under certain inequality assumptions the Hamiltonian system of differential inclusions has a solution. (Theorem 16)

(4) If $(x, p)$ solves a periodic system of differential inclusions and $H$ is appropriately bounded then $\psi(x, p)$ must be finite. (Theorem 17)

(5) If $\psi$ attains its minimum and if the associated Hamiltonian system of differential inclusions has a solution, then $\psi$ has a finite minimum. (This results directly from (4).)

We first consider the canonical quadratic example considered by Gaines and Peterson ([10]).

**Example 21**

$$\min_x \int_0^1 (\frac{1}{4}u^2 + x^2)$$

$$\begin{aligned} \textit{subject to: } \dot{x} &= u \\ x(0) &= x(T) \end{aligned}$$

*Here* $H(x,p) = p^2 - x^2 \le p^2 + |x|$. *Setting* $V(x) = \frac{1}{4}x^2$ *gives*

$$H(x, -\nabla V(x)) = H(x, -\frac{1}{2}x) = -\frac{3}{4}x^2 \le -\gamma|x| + \delta$$

*for some* $\gamma$, $\delta > 0$. *and so, by Theorem 13* $\psi$ *attains its minimum. The associated Hamiltonian system is*

$$\begin{aligned} \dot{x} &= 2p \\ -\dot{p} &= -2x \\ x(0) &= x(T) \\ p(0) &= p(T) \end{aligned}$$

*Assumption 4 of the Finiteness Theorem 16 requires the existence of an* $\alpha$ *such that*

$$\sqrt{4p^2 + 4x^2} \le \alpha \textit{ whenever } (x,p) \in K \textit{ where } K \textit{ is compact.}$$

*Assumption 5 requires the existence of positive constants* $a, b, c, d, e$ *such that* $(x, p)$

*(i)* $-x(-2x) + p(2p) = 2x^2 + 2p^2 \ge a\sqrt{4p^2 + 4x^2} - b\sqrt{x^2 + p^2} - c$,

*(ii)* $2x^2 + 2p^2 \ge d\sqrt{x^2 + p^2} - e$.

*These conditions, along with Assumptions 1, 2, 3 of the Finiteness Theorem can be satisfied easily. Here, the Existence and Finiteness Theorems together predict the existence of a finite minimum for* $\psi$.

The next example was discussed previously at the end of Chapter 4.

**Example 22**

$$\min_x \int_0^1 x(t)dt$$

$$\begin{aligned} \textit{subject to: } \dot{x}(t) &= x + u \\ x(0) &= x(T) \\ |u(t)| &\le 1. \end{aligned}$$

*We showed previously that $H(x,p) = |p| - x + xp$, and that $\psi$ attains its minimum. The associated Hamiltonian system is*

$$\dot{x} = \begin{cases} x+1, & \text{for } p > 0 \\ [x-1, x+1], & \text{for } p = 0 \\ x-1, & \text{for } p < 0 \end{cases}$$

$$-\dot{p} = p - 1$$

$$p(0) = p(T),\ x(0) = x(T).$$

*Assumption 5 requires the existence of positive constants $d$ and $e$ such that for all $(x,p)$*

*(ii)* $x \pm p \geq d\sqrt{x^2 + p^2} - e$.

*Setting $p = x$ in (ii) gives*

$$d\sqrt{2}|x| - e \leq \begin{cases} 2x, & \text{for } p \geq 0 \\ 0, & \text{for } p < 0 \end{cases}$$

*which cannot be satisfied for any positive $d$ and $e$. Consequently Theorem 16 does not guarantee that a minimum value of $\psi$ is finite. Nevertheless, $x = -1$, $p = 1$ solves the differential inclusions and hence $\psi$ has a finite minimum. We can see that $(x,u) = (-1,1)$ is the minimizer for the optimal control problem.*

We now consider a more general periodic problem in optimal control.

$$\min_{x,u} \int_0^T [f_o(u(t)) + f_1(x(t))]dt$$

$$\text{subject to:} \quad \begin{aligned} \dot{x}(t) &= f(x(t)) + u \\ x(0) &= x(T) \end{aligned}$$

Reformulating, we obtain

$$l(a,b) = \begin{cases} 0 & \text{for } a = b \\ +\infty, & \text{otherwise} \end{cases}$$

$$K(x,v,u) = \begin{cases} f_o(u) + f_1(x), & \text{for } v = f(x) + u \\ +\infty, & \text{otherwise} \end{cases}$$

$$L(x,v) = \inf_u K(x,v,u) = f_o(v - f(x)) + f_1(x).$$

The Hamiltonian, then, is

$$H(x,p) = \sup_v \{pv - f_o(v - f(x)) - f(x)\}.$$

Differentiating with respect to $v$ shows that this supremum occurs when $v = g(p) + f(x)$ where $g = (f_o')^{-1}$ assuming this inverse exists. Hence,

$$H(x,p) = p[g(p) + f(x)] - f_o(g(p)) - f_1(x).$$

Theorem 4 states that if there is an optimal control and the minimizer is finite, then we will have a solution to the following differential inclusion.

$$\begin{aligned} \dot{x} &\in \partial_p H(t,x,p)(x,p) = g(p) + f(x) \\ -\dot{p} &\in \partial_x H(t,x,p)(x,p) = pf'(x) - f_1'(x) \end{aligned}$$

$$x(0) = x(T) \text{ and } p(0) = p(T).$$

To show that this system has a solution it will suffice to show that there exists a constant solution, that is there exists an $(\bar{x}, \bar{p})$ pair satisfying

(a) $g(p) + f(x) = 0$ and,

(b) $pf'(x) - f_1'(x) = 0$.

From (a) we have

(c) $p = g^{-1}(-f(x)) = f_o'(-f(x))$.

Substituting into (b) gives

(d) $f_1'(x) = f_o'(-f(x))f'(x)$.

If we assume that (d) has a solution, $\bar{x}$, then (c) gives a $\bar{p}$ so that $(\bar{x}, \bar{p})$ satisfy (a) and (b).

We now apply this idea to a specific problem.

**Example 23** *In the preceding set $f_o(u) = u^2$, $f_1(x) = x^2$, and $f(x) = x^3$ to obtain*

$$\min_{x,u} \int_0^T [u^2 + x^2]$$

$$\textit{subject to: } \begin{aligned} \dot{x}(t) &= x^3 + u \\ x(0) &= x(T) \end{aligned}$$

*Then*

$$H(x,p) = \frac{1}{4}p^2 - x^2 + px^3.$$

*Setting $Q(x) = 0$, $V(x) = \frac{1}{2}x^2$, $W(x) = |x|$, $\delta = 2$, $\gamma = 1$, $\theta = 0$, $\sigma = 1/2$, and $\nu = T/4$ will satisfy the conditions of Theorem 14 as follows:*

*(c)* $\nu < 1 - \int_0^T \sigma(t)dt$, *and*

$$Q(x) = 0 \geq \frac{-Tx^2}{8} = -\nu V(x) + \mu.$$

*(1)* $H(x, -\nabla V(x)) = H(x, -x) = -.75x^2 - x^4 \leq -|x| + 2 = -\gamma W(x) + \delta$ *and*

*(2)* $H(x, \nabla Q(x))) = H(x, 0) = -x^2 \leq 0.25x^2 = \sigma V(x) + \theta$.

*Thus, we expect a minimizer for this problem.*

*The associated Hamiltonian system is*

$$\begin{aligned} \dot{x} &= \frac{1}{2}p + x^3 \\ -\dot{p} &= -2x + 3px^2 \\ x(0) &= x(T) \\ p(0) &= p(T) \end{aligned}$$

*Assumption 5 of Theorem 16 requires the existence of positive constants d and e such that for all* $(x, p)$

$$2x^2 + 0.5p^2 - 2px^3 \geq d\sqrt{x^2 + p^2} - e.$$

*Setting* $p = x$, *we can see that (ii) cannot satisfied for very large values of* $x$. *since*

$$\lim_{x \to -\infty} 2.5x^2 - 2x^4 = -\infty \text{ but}$$

$$\lim_{x \to +\infty} d\sqrt{2}|x| - 3 = +\infty.$$

*Hence the finiteness of the minimizer is not guaranteed by Theorem 16.*

*Nevertheless,* $(x, p) = (0, 0)$ *solves the periodic Hamiltonian system and so* $\psi(0)$ *is finite. Therefore, since there is a minimum, the minimum for* $\psi$ *must indeed be finite. This solution, placed into the constraints generates a control* $u \equiv 0$.

To summarize, in the previous two examples, an Existence Theorem predicts a minimizer, but the Finiteness Theorem does not guarantee a finite minimum, despite the fact that the minimum is indeed finite! Consequently, the search for compatibility is narrowed to even more specific examples.

Consider the problem for which $f_o(u) = 0$. Note that $f_0'$ is not invertible so the formulation preceding example 23 does not apply. We have, however, considered this example previously in Chapter 3.

$$\min_{x,u} \int_0^T f_1(x)dt$$

$$
\begin{aligned}
\text{subject to: } \dot{x} &= f(x) + u \\
x(0) &= x(T) \\
|u| &\leq 1
\end{aligned}
$$

**Theorem 22** *Assume $f(x)$ and $f_1(x)$ are both invertible with $f$ and $f_1$ strictly increasing and decreasing, respectively. If $1 \in R(f)$ and $f'(f^{-1}(1)) \neq 0$ and $f_1'(f^{-1}(1) \neq 0$ then the Hamiltonian system of differential inclusions associated with the above problem has a constant solution.*

Proof. From Example 1 in Chapter 3 the Hamiltonian is

$$H(x, p) = pf(x) + |p| - f_1(x)$$

and the associated differential inclusion is

$$\dot{x} = \begin{cases} f(x) - 1, & \text{for } p < 0 \\ [f(x) - 1, f(x) + 1], & \text{for } p = 0 \\ f(x) + 1, & \text{for } p > 0 \end{cases}$$

$$-\dot{p} = pf'(x) - f_1'(x)$$

$$x(0) = x(T) \text{ and } p(0) = p(T).$$

A constant solution $(\bar{x}, \bar{p})$ must satisfy

$$
\begin{aligned}
\dot{x} &= 0 \\
-\dot{p} &= 0.
\end{aligned}
$$

If $-\dot{p} = 0$ then $\bar{p} = \frac{f_1'(\bar{x})}{f'(\bar{x})} < 0$ from the monotonicity. Therefore $\dot{x} = f(\bar{x}) - 1 = 0$ and so $\bar{x} = f^{-1}(1)$. Similarly, if $f$ and $f_1$ are both decreasing or both increasing and $-1 \in R(f)$ and $f'(f^1(1)) \neq 0$ and $f_1'(f^1(1) \neq 0$, then the Hamiltonian system has a constant solution.

Q.E.D.

**Theorem 23** *The above optimal control problem has a solution provided there exist $\gamma$, $\delta > 0$, $\sigma$ and $\theta > 0$ with $0 \leq \sigma T < 1$ and $\gamma T \geq 0$ such that*

*(1)* $|x| - xf(x) - f_1(x) \leq -\gamma|x| + \delta$

*(2)* $-f_1(x) \leq \frac{1}{2}\sigma x^2 + \theta$.

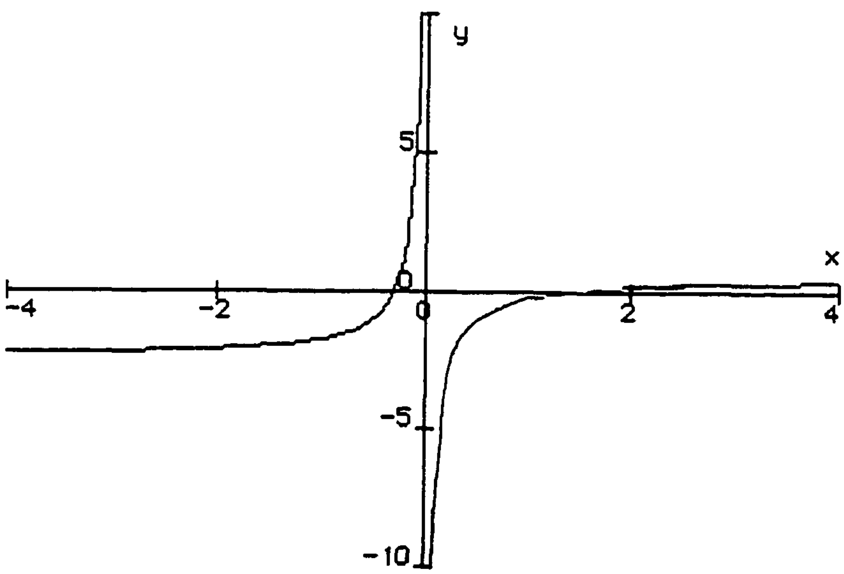
Proof. Apply Theorem 14 using $V(x) = \frac{1}{2}x^2$, $W(x) = |x|$, and $Q(x) = 0$.

Q.E.D.

**Example 24** *If $f_1(x) = x$ then (1) in Theorem 23 requires $f(x)$ to satisfy*

$$\begin{aligned} f(x) &\leq -(\gamma+2) - \frac{\delta}{x}. x < 0 \\ f(x) &\geq \gamma - \frac{\delta}{x}. x > 0 \end{aligned}$$

*where $\gamma$, $\delta > 0$ and $\gamma T \geq 0$. Using $T = 1$, $\gamma = \frac{1}{2}$ and $\delta = \frac{3}{4}$, the graph of $f(x)$ must lie in the region shown below. Clearly there are increasing functions $f(x)$ whose graphs lie in this region and which satisfy the hypotheses of Theorem 22.*




Consider the case where $f_o(u) = u^2$.

$$\min_{x,u} \int_0^T [u^2 + f_1(x)]$$

$$\begin{aligned} \text{subject to:} \quad \dot{x} &= f(x) + u \\ x(0) &= x(T) \end{aligned}$$

**Theorem 24** *The above optimal control problem has a solution provided there exist* $\gamma$, $\delta > 0$, $\sigma$ *and* $\theta > 0$ *with* $0 \le \sigma T \le 1$ *and* $\gamma T \ge 0$ *such that*

*(1)* $\frac{1}{4}x^2 - xf(x) - f_1(x) \le -\gamma|x| + \delta$

*(2)* $-f_1(x) \le \frac{1}{2}\sigma x^2 + \theta$.

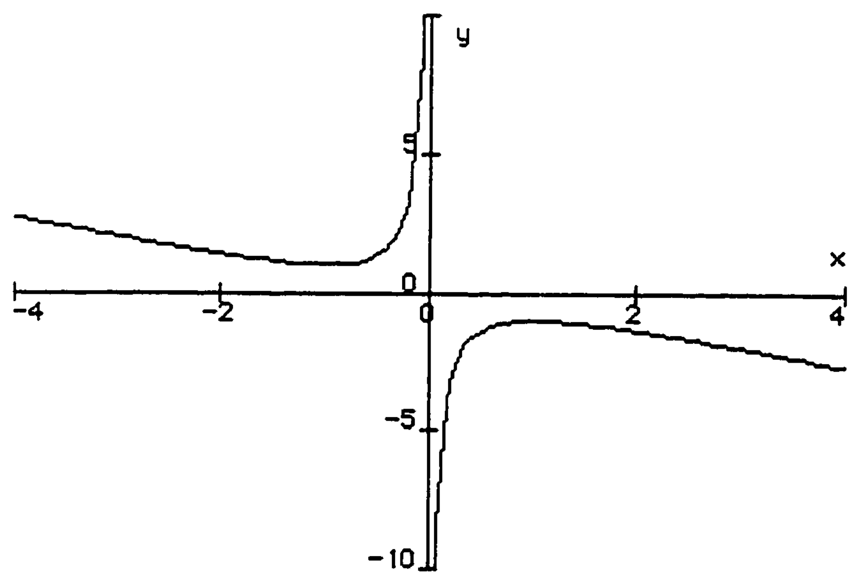
Proof. $H(x,p) = \frac{1}{4}p^2 + pf(x) - f_1(x)$. Apply Theorem 14 using $V(x) = \frac{1}{2}x^2$, $W(x) = |x|$, and $Q(x) = 0$.

Q.E.D.

**Example 25** *From (2)* $f_1(x)$ *can be any linear or nonnegative function. Computations of (1) indicate that choosing* $f_1(x) = kx$, $k > 0$ *or choosing an* $f_1(x) = 0$ *for* $x < 0$ *prevent the use of functions* $f(x)$ *for which* $lim_{x \to -\infty} f(x) = +\infty$. *Using* $f_1 = x^2$ *for all* $x$ *requires* $f(x)$ *to satisfy*

$$\begin{aligned} f(x) &\le \frac{-3}{4}x - \gamma - \frac{\delta}{x}, x < 0 \\ f(x) &\ge \frac{-3}{4}x + \gamma - \frac{\delta}{x}, x \ge 0 \end{aligned}$$

*which permits greater flexibility in a selection of* $f(x)$. *For example, using* $T = 1$, $\gamma = \frac{1}{2}$ *and* $\delta = \frac{3}{4}$, *the graph of* $f(x)$ *must lie in the shaded region below.*



**Theorem 25** *For the problem*

$$\min_{x,u} \int_0^T (u^2 + x^2)$$

$$\begin{aligned} \textit{subject to:} \quad \dot{x} &= f(x) + u \\ x(0) &= x(T) \end{aligned}$$

*If $f(x)$ is a polynomial for which $\psi$ has a minimizer, then the optimal control problem has a finite solution.*

Proof. In this problem

$$H(x,p) = \frac{1}{4}p^2 - x^2 + pf(x)$$

which yields the system

$$\begin{aligned} \dot{x}(t) &= \frac{1}{2}p + f(x) \\ -\dot{p}(t) &= pf'(x) - 2x \\ x(0) &= x(T) \\ p(0) &= p(T) \end{aligned}$$

This system has a constant solution $\bar{p} = -2f(\bar{x})$ where $\bar{x}$ solves $f(x)f'(x) = -x$. The function $f(x)f'(x)$ is either an odd polynomial with a positive lead coefficient or it is the zero polynomial. In either case the equation $f(x)f'(x) = -x$ has a solution. Since $(\bar{x}, \bar{p})$ solves the system, by Theorem 17 $\psi(\bar{x}, \bar{p})$ is finite and so the minimum for $\psi$ and hence for the optimal control problem is finite.

Q.E.D.

To summarize, in the problem

$$\min_{x,u} \int_0^T [f_o(u(t)) + f_1(x(t))]dt$$

$$\begin{aligned} \text{subject to:} \quad \dot{x}(t) &= f(x(t)) + u \\ x(0) &= x(T) \end{aligned}$$

we saw two examples for which the Existence Theorems predict a minimizer, but for which the Finiteness Theorem fails to predict a finite minimizer. In the final two theorems, we focused on the cases $f_o(u) = 0$ and $f_o(u) = u^2$ and demonstrated restrictions on $f(x)$ and $f_1(x)$ which suffice to allow the Finiteness Theorem to predict a finite minimum.

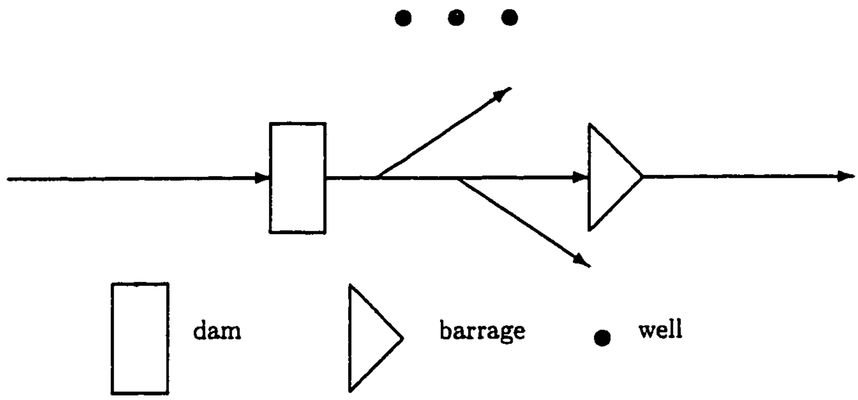
# Chapter 7

# Water Management Example

## 7.1 The Water Management Problem

We conclude with a periodic example involving an irrigation system. After explaining the problem we will apply the second periodic theorem, Theorem 14 and examine the problem in light of the finiteness theorems developed in Chapter 5.

Consider the following diagram of a simple water management problem. The arrows represent the direction of the flow of the water. The diagram indicates a reservoir before the dam, a reservoir between a dam and a barrage, two canals draining from the second reservoir, and three wells.




We let the control and state be given by

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \text{Volume over dam} \\ \text{Volume through canals and over barrage} \\ \text{Volume out wells} \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \text{Volume before dam} \\ \text{Volume between dam and barrage} \end{bmatrix}$$

We will let $T$ represent one time period. We seek a model satisfying the following:

(a) The states and the controls have volume units.

(b) The controls should be as close as possible to some target controls. The target will be given by

$$\Lambda = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

(c) The state system should satisfy a mass-balance equation.

(d) The states at the beginning of each time period should be equivalent. This imposes a periodicity on the states.

(e) Both the states and the controls are bounded. (Our model will not incorporate these bounds.)

## 7.2 Linear Case

In addition to the above, we make the following assumptions:

(1.) The inflow from the river is a constant $F$.

(2.) Seepage, evaporation, plant use, etc. affect water levels. Let $f$ represent the effect on the water volume before the dam, and $g$ represent the effect on the water volume between the dam and the barrage. We assume these are linear functions given by

$$f(x_1) = Mx_1$$

$$g(x_1, x_2) = -Nx_1 + Px_2$$

where $M, N, P \geq 0$, so $f$ is increasing in $x_1$ while $g$ is increasing in $x_2$ but decreasing in $x_1$. Net loss out of the reservoirs will mean $f, g > 0$ while net gain into the the reservoirs will be indicated by $f, g < 0$.

Under these assumptions, we arrive at the following model:

$$\min_{(x,u)} \int_0^T (u - \Lambda) \cdot (u - \Lambda)$$

$$\text{subject to } \dot{x}(t) = Au + Bx + q$$

where

$$q = \begin{bmatrix} F \\ 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & -d \\ 1 & -1 & -e \end{bmatrix}, B = \begin{bmatrix} -M & 0 \\ N & -P \end{bmatrix}.$$

$$x(0) = x(T)$$

Reformulating

$$K(x, v, u) = \begin{cases} (u - \Lambda) \cdot (u - \Lambda), & \text{for } v = Au + Bx + q \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$L(x, v) = \inf_u K(x, v, u).$$

This is equivalent to the quadratic programming problem

$$\min_u \frac{1}{2} u^T Q u - c^T u + \Lambda^T \Lambda$$

$$\text{subject to } Au = b$$

where

$$Q = 2I, \; b = v - Bx - q = \begin{bmatrix} v_1 - F + Mx_1 \\ v_2 - Nx_1 + Px_2 \end{bmatrix}, c = -2\Lambda.$$

The minimizer for this problem is

$$u = Q^{-1}A^T(AQ^{-1}A^T)^{-1}[AQ^{-1}c + b] - Q^{-1}c.$$

Using Maple V to compute the Hamiltonian we obtain

$$\begin{aligned} H(x,p) &= (F - Mx_1 - t_1 - dt_3)p_1 + (Nx_1 - Px_2 + t_1 - t_2 - et_3)p_2 \\ &\quad + \left(\frac{1}{2}de - \frac{1}{2}\right)p_1p_2 + \left(\frac{1}{4}d^2 + \frac{1}{4}\right)p_1^2 + \left(\frac{1}{4}e^2 + \frac{1}{2}\right)p_2^2. \end{aligned}$$

We first examine this problem in light of the second periodic theorem. Setting $V(x) = \varepsilon x_1^2 + \varepsilon x_2^2$ where $\varepsilon > 0$ and defining

$$\begin{aligned} h(x) &= H(x, \nabla V(x)) \\ &= [-2M\varepsilon + d^2\varepsilon^2 + \varepsilon^2]x_1^2 + [-2P\varepsilon + e^2\varepsilon^2 + 2\varepsilon^2]x_2^2 + [2N\varepsilon + 2de\varepsilon^2 - 2\varepsilon^2]x_1x_2 \\ &\quad + [2F\varepsilon - 2\varepsilon t_1 - 2\varepsilon dt_3]x_1 + [2\varepsilon t_1 - 2\varepsilon t_2 - 2\varepsilon et_3]x_2 \end{aligned}$$

Computing second partial derivatives and assuming that $\varepsilon < min\{\frac{2M}{d^2+1}, \frac{2P}{e^2+2}, \frac{N}{1-de}\}$ gives

$$h_{11} = 2\varepsilon[-2M + d^2\varepsilon + \varepsilon] < 0$$

$$h_{22} = 2\varepsilon[-2P + e^2\varepsilon + 2\varepsilon] < 0$$

$$h_{12} = 2\varepsilon[N + de\varepsilon - \varepsilon] > 0.$$

In order for $h$ to attain a maximum we must also have a positive determinant for the Hessian, that is

$$h_{11}h_{22} - h_{12}^2 > 0.$$

This would be the case provided $-h_{11} > h_{12}$ and $-h_{22} > h_{12}$. That is, we must have

$$2M > N + (d^2 + de)\varepsilon$$

and

$$2P > N + (de + e^2 + 1)\varepsilon.$$

If $2M > N$ and $2P > N$, by choosing $\varepsilon$ small enough we can satisfy these requirements. These restrictions are reasonable in light of the physical meaning of $M$, $N$, and $P$. Hence, this choice of $V$ gives a maximum for $h(x)$ and so we can satisfy the existence requirement

$$H(x, \nabla V(x)) < -\gamma W(x) + \delta.$$

Additionally, if $Q(x) = 0$, we can satisfy

$$H(x, \nabla Q(x)) < \sigma V(x) + \theta$$

and thus fully meet the conditions of the 2nd periodic theorem.

Next, we consider the finiteness of the optimal solution by examining the four dimensional system of differential equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ -\dot{p}_1 \\ -\dot{p}_2 \end{bmatrix} = \begin{bmatrix} -M & 0 & \frac{1}{2}+\frac{1}{2}d^2 & \frac{1}{2}de-\frac{1}{2} \\ N & -P & \frac{1}{2}de-\frac{1}{2} & \frac{1}{2}e^2+1 \\ 0 & 0 & -M & N \\ 0 & 0 & 0 & -P \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} F-t_1-dt_3 \\ t_1-t_2-et_3 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{0} & \mathbf{B^T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix} + \begin{bmatrix} F-t_1-dt_3 \\ t_1-t_2-et_3 \\ 0 \\ 0 \end{bmatrix}$$

Since

$$\begin{aligned} x^T\mathbf{C}x &= x^T \begin{bmatrix} \frac{1}{2}d^2 & \frac{1}{2}de-\frac{1}{2} \\ \frac{1}{2}de-\frac{1}{2} & \frac{1}{2}e^2+1 \end{bmatrix} x \\ &= (\frac{1}{2}+\frac{1}{2}d^2)x_1^2 + (de-1)x_1x_2 + (\frac{1}{2}e^2+1)x_2^2 \\ &= (\frac{d}{\sqrt{2}}x_1 + \frac{e}{\sqrt{2}}x_2)^2 + (\frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2)^2 + \frac{1}{2}x_2^2 > 0, \end{aligned}$$

the matrix $\mathbf{C}$ is positive definite.

Likewise, we must also show that $B + B^T$ is sign definite. If we continue to assume that $2M > N$ and $2P > N$ then

$$\begin{aligned} x^T(\mathbf{B}+\mathbf{B^T})x &= x^T \begin{bmatrix} -2M & N \\ N & -2P \end{bmatrix} x \\ &= -2Mx_1^2 + Nx_1x_2 - 2Px_2^2 \\ &< -N(x_1-x_2)^2 < 0, \end{aligned}$$

Hence,by the results of Chapter 5 show that the Hamiltonian system of differential inclusions has a system has a solution. The hypothesis of Theorem 17 are satisfied by $H$ and so the problem has a finite minimum.

## 7.3 Nonlinear Case

We next consider a situation where $f$ and $g$ are nonlinear functions. We make the following assumptions, akin to the linear case:

(1) The inflow from the river is a constant $F$.

(2) Seepage, evaporation, plant use, etc. affect water levels. Let $f$ represent the effect on the water volume before the dam, and $g$ represent the effect on the water volume between the dam and the barrage. We assume these are nonlinear functions, with the following physically reasonable conditions:

(i.) As in the linear case $f$ is increasing in $x_1$ while $g$ is increasing in $x_2$ but decreasing in $x_1$.

(ii.) Net loss out of the reservoirs will mean $f, g > 0$ while net gain into the reservoirs will be indicated by $f, g < 0$.

(iii.) Since $x_1$ affects the change in volume before the dam more than it does the volume between the dam and the barrage we can reasonably assume that $f'(x_1) > -g_1(x_1, x_2)$. Likewise, since the volume between the dam and the barrage is affected more by $x_2$ than by $x_1$ we can assume $g_2(x_1, x_2) > -g_1(x_1, x_2)$.

(3) $g(x_1, x_2) = \overline{g}(x_1) + \bar{g}(x_2)$

Under these assumptions, we arrive at the following model:

$$\min_{(x,u)} \int_0^T (u - \Lambda) \cdot (u - \Lambda)$$

$$\text{subject to } \dot{x}(t) = Au + B(x) + q$$

where

$$q = \begin{bmatrix} F \\ 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & -d \\ 1 & -1 & -e \end{bmatrix}, B(x) = \begin{bmatrix} -f(x_1) \\ -g(x_1, x_2) \end{bmatrix}.$$

$$x(T) = x(0)$$

Reformulating

$$K(x, v, u) = \begin{cases} (u - \Lambda) \cdot (u - \Lambda), & \text{for } v = Au + B(x) + q \\ +\infty, & \text{otherwise} \end{cases}$$

and

$$L(x, v) = \inf_u K(x, v, u).$$

This is equivalent to the quadratic programming problem

$$\min_u \frac{1}{2} u^T Q u - c^T u + \Lambda^T \Lambda$$

$$\text{subject to } Au = b$$

where

$$Q = 2I, \; b = v - Bx - q = \begin{bmatrix} v_1 - F + f(x_1) \\ v_2 + g(x_1, x_2) \end{bmatrix}, \; c = -2\Lambda.$$

The minimizer for this problem is

$$u = Q^{-1} A^T (AQ^{-1} A^T)^{-1} [AQ^{-1} c + b] - Q^{-1} c.$$

The Hamiltonian, then, is

$$\begin{aligned} H(x,p) &= [F - t_1 - dt_3 - f(x_1)]p_1 + [t_1 - t_2 - et_3 - g(x_1,x_2)]p_2 \\ &\quad + \left(\frac{1}{2}de - \frac{1}{2}\right) p_1 p_2 + \left(\frac{1}{4}d^2 + \frac{1}{4}\right) p_1^2 + \left(\frac{1}{4}e^2 + \frac{1}{2}\right) p_2^2. \end{aligned}$$

We first examine this problem in light of the second periodic theorem. Setting $V(x) = \varepsilon x_1^2 + \varepsilon x_2^2$ where $\varepsilon > 0$ and defining

$$\begin{aligned} h(x) &= H(x, \nabla V(x)) \\ &= 2\varepsilon\Big\{ \left(\frac{1}{2}d^2 + \frac{1}{2}\right) \varepsilon x_1^2 + \left(\frac{1}{2}e^2 + \frac{1}{2}\right) 2\varepsilon x_2^2 + (de-1)2\varepsilon x_1 x_2 \\ &\quad [F - t_1 - dt_3 - f(x_1)]x_1 + [t_1 - t_2 - et_3 - g(x_1,x_2)]x_2 \Big\} \end{aligned}$$

Computing second partial derivatives and assuming that

$$\varepsilon < min\left\{ \frac{2f'(x_1) + x_1 f''(x_1) + x_2 g_{11}(x_1,x_2)}{d^2+1}, \frac{2g_2(x_1,x_2) + x_2 g_{22}(x_1,x_2)}{e^2+2}, \frac{-x_2 g_{12}(x_1,x_2) - g_1(x_1,x_2)}{1-de} \right\}$$

gives

$$\begin{aligned} h_{11} &= -2\varepsilon[2f'(x_1) + x_1 f''(x_1) + x_2 g_{11}(x_1,x_2) - (d^2+1)\varepsilon] < 0 \\ h_{22} &= -2\varepsilon[2g_2(x_1,x_2) + x_2 g_{22}(x_1,x_2) - (e^2+2)\varepsilon] < 0 \\ h_{12} &= 2\varepsilon[-x_2 g_{12}(x_1,x_2) - g_1(x_1,x_2) + (de-1)\varepsilon] > 0. \end{aligned}$$

In order for $h$ to attain a maximum we must also have a positive determinant for the Hessian, that is

$$h_{11}h_{22} - h_{12}^2 > 0.$$

This would be the case provided $-h_{11} > h_{12}$ and $-h_{22} > h_{12}$. That is, we must select $\varepsilon > 0$ so that

$$2f'(x_1) > -g_1(x_1,x_2) - x_1 f''(x_1) - x_2[g_{11}(x_1,x_2) + g_{12}(x_1,x_2)] + (d^2 + de)\varepsilon$$

and

$$2g_2(x_1,x_2) > -g_1(x_1,x_2) - x_2[g_{12}(x_1,x_2) + g_{22}(x_1,x_2)] + (de + e^2 + 1)\varepsilon.$$

From condition (iii.) above, if the second partial derivatives are small enough and appropriately bounded we can select $\varepsilon$ small enough to satisfy these two equations. Hence, this choice of $V$ gives a maximum for $h(x)$ and so we can satisfy the existence requirement

$$H(x, \nabla V(x)) < -\gamma W(x) + \delta.$$

Additionally, if $Q(x) = 0$, we can satisfy

$$H(x, \nabla Q(x)) < \sigma V(x) + \theta$$

and thus fully satisfy the conditions of the 2nd periodic theorem.

Simplifying the five conditions, $\varepsilon > 0$ must satisfy

(a) $\varepsilon < \dfrac{2f'(x_1) + x_1 f''(x_1) + x_2 g_{11}(x_1, x_2)}{d^2 + 1}$

(b) $\varepsilon < \dfrac{2g_2(x_1, x_2) + x_2 g_{22}(x_1, x_2)}{e^2 + 2}$

(c) $\varepsilon < \dfrac{-g_1(x_1, x_2)}{1 - de}$

(d) $\varepsilon < \dfrac{2f'(x_1) + x_1 f''(x_1) + x_2 g_{11}(x_1, x_2) + g_1(x_1, x_2)}{d^2 + de}$

(e) $\varepsilon < \dfrac{2g_2(x_1, x_2) + x_2 g_{22}(x_1, x_2) + g_1(x_1, x_2)}{de + e^2 + 1}$

The denominators in (a)-(e) are always positive. Hence there exists an $\varepsilon > 0$ satisfying (a)-(e) provided the numerator in each inequality is positive. This is always the case in (c) since $g$ is decreasing in $x_1$. If $f(x_1), \overline{g}(x_1)$, and $\tilde{g}(x_2)$ are all concave up then (a) and (b) are easily satisified. In this case, by assumptions (2i) and (2iii) we also can satisfy (d) and (e).

**Example 26** *Consider the above nonlinear problem with*

$$f(x_1) = x_1^2$$

$$g(x_1, x_2) = -\frac{1}{2}x_1 + x_2^2.$$

*These choices of $f$ and $g$ satisfy assumption (2i) but (2iii) is satisfied only for $x_1 > \frac{1}{4}$ and $x_2 > \frac{1}{4}$ and so (d) and (e) are not immediately satisfied. In this example we must be able to select $\varepsilon > 0$ so that*

*(a)* $\varepsilon < \dfrac{6x_1}{d^2 + 1}$.

*(b)* $\varepsilon < \dfrac{6x_2}{e^2 + 2}$.

*(c)* $\varepsilon < \frac{1}{2(1-de)}$.

*(d)* $\varepsilon < \frac{12x_1 - 1}{2(d^2+de)}$.

*(e)* $\varepsilon < \frac{12x_2 - 1}{2(e^2+de+1)}$.

*These equations must hold for all* $(x_1, x_2)$ *and hence, to satisfy (d) and (e) we must have* $x_1 > \frac{1}{12}$ *and* $x_2 > \frac{1}{12}$ *These restrictions prevent the volumes in either reservoir from getting too small. Hence, we woud expect to have a finite minimum in this case.*

Returning to inequalities (a) - (e), we note that (a) and (d) both have positive numerators if

$$x_1 f''(x_1) + x_2 g_{11}(x_1, x_2) > -2f'(x_1) - g_1(x_1, x_2)$$

while the inequalities (b) and (e) both have positive numerators if

$$g_{22}(x_1, x_2) > \frac{-g_1(x_1, x_2) - 2g_2(x_1, x_2)}{x_2}.$$

By our earlier assumption (2iii), the right hand sides of these two inequalities are negative. Hence $f, \bar{g}$, and $\bar{g}$ may all be concave down to some extent.

**Example 27** *Consider the nonlinear problem with*

$$f(x_1) = \sqrt{x_1}$$

$$g(x_1, x_2) = -kx_1 + \sqrt{x_2}$$

*where* $k < 1$ *is small. Here, assumption (2i) is satisfied, and assumption (2iii) is satisfied provided* $x_1 < \frac{1}{4k^2}$ *and* $x_2 < \frac{1}{4k^2}$ *which is reasonable for small enough values of* $k$*. This implies that* $k < \frac{1}{2\sqrt{x_{1max}}}$ *and* $k < 12\sqrt{x_{2max}}$*. This is placing an upper bound on the volume in the reservoirs. Such a choice of* $k$ *guarantees a positive numerator for (a)(b)(d)(e). Hence we expect to have a finite minimum.*

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