

Civil Engineering Department
Colorado A. and M. College
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A COMPARATIVE STUDY

OF

MOMENTUM TRANSFER, HEAT TRANSFER, AND VAPOR TRANSFER

PART II

FORCED CONVECTION, TURBULENT CASE

by

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FOREWORD

This report is Part II of a preliminary study in connection with the Wind-tunnel Project under contract with the ONR to be carried out by the Hydraulics Laboratory of the Colorado Agricultural and Mechanical College, Fort Collins, Colorado. Aside from Section F of Chapter IV, which is the work of the present writer, it is entirely a review of existing literature.

Review of much of the existing literature for Chapters II and IV was done by Dr. K. C. Kuo at Fort Collins in the summer of 1949. This work has greatly facilitated the preparation of these two chapters.

Due to the immensely diverse subjects treated in this report, it is impossible to maintain consistency of notations without either violating well-established conventions or making the resulting notations undesirably cumbersome. Efforts have been made to achieve the maximum amount of consistency such that what inconsistencies still remain are unlikely to cause confusion. It may be mentioned here that bars in general denote mean values, single primes denote fluctuations and double primes the standard deviation of these fluctuations. Although primes are also used sometimes to denote differentiations, it is believed that this inconsistency will not introduce confusion. Often, for simplicity, subscripts x and y are used to denote partial differentiation with respect to the particular variable or variables indicated by the subscripts, especially in differential equations. That the subscripts x and y in R_x and R_y occurring in Karman's theory of isotopic turbulence do not denote differentiations is self-evident.

The writer has found it possible to devise a general symbol to denote the various correlation coefficients in the statistical theory of turbulence. The symbol

$$R(\xi, \eta, \rho; \tau)[u', v']$$

denotes the correlation between u' and v' taken at two points whose cartesian coordinates differ by ξ , η , and ζ respectively, and at a time difference τ . That this symbol can be extended to include triple correlations or correlations of even higher orders is obvious. The correlations generally encountered in existing theories, however, do not usually call for such generality of representation. In the first place, the quantities whose correlation is being considered are often understood, and so the brackets with their contents can be omitted. Then as a rule only one of the quantities ξ , η , ζ is different from zero, and from the context it is often obvious which one does not vanish. Thus, if the correlation is between simultaneous quantities, the symbol $R_2(s)$ is often sufficient, where s may denote either ξ , η , or ζ , and the subscript 2 implies simultaneity. Similarly, $R_1(\tau)$ denotes the correlation between two understood quantities taken at the same point at the time interval τ apart. Thus, elements of Kármán's correlation tensor as well as his R_x and R_y belong to the R_2 category. To guard against any possible confusion, let it be demonstrated here that Kármán's R_x is actually $R(\xi, 0, 0; 0) [u', u']$ and his R_y is actually $R(\xi, 0, 0; 0) [v', v']$ or $R(\xi, 0, 0; 0) [w', w']$. In the text, the simplest symbols and the conventional symbols are used as much as possible, the general representation being used only where it cannot be avoided without the risk of incurring confusion.

Since complete consistency of notations cannot be properly achieved anyhow, and since all notations are sufficiently defined in the text, it seems that a table of notations can be advantageously omitted without causing confusion. This has been done in the present report.

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PART II. FORCED CONVECTION, TURBULENT CASE

CHAPTER I. THEORIES OF TURBULENCE

The theories of turbulence can be roughly divided into two categories, namely the transfer theories and the statistical theories, both of which will be presented in detail in the present Chapter. It will be seen that while the transfer theories are now generally considered as unsatisfactory, the statistical theories have not been sufficiently developed for application to practical problems.

A. Transfer Theories

In turbulent flow, the velocity fluctuations introduce apparent turbulent stresses in addition to the stresses due to molecular viscosity. These stresses, however, cannot be related to the distribution of the mean velocity without some assumption the validity of which cannot be ascertained a priori. Boussinesq (75,1887) introduced the eddy viscosity ϵ in analogy to the kinematic viscosity ν . The apparent stresses are then

$$\sigma_x = 2\rho\epsilon\frac{\partial\bar{u}}{\partial x} \quad \tau_{xy} = \rho\epsilon\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right)$$

and seven other elements of the stress tensor, where ρ is the density, \bar{u} and \bar{v} are the mean velocity-components in the x- and y-directions respectively. The Navier-Stokes equation can then be used with ν changed to $\nu + \epsilon$.

The idea of eddy viscosity is oftentimes helpful. It suffers, however, from the deficiency that the spatial distribution of ϵ is not known, except that near the solid boundaries ϵ must vanish and in certain special cases it should be constant. The following transfer theories represent efforts to correlate ϵ with properties of the mean flow.

1. Prandtl's momentum-transfer theory

In analogy to the mean free path of molecules in the kinetic theory of gases, Prandtl (148, 1925) proposed a mixing length for fluid particles

and developed the momentum-transfer theory. Essentially the mixing length signifies the length through which a fluid particle must travel with its original mean velocity before it assumes the mean velocity of its new environment. Denoting the mixing length by λ , one has, for parallel plane motion,

$$-u_1' = \bar{u}(y_1) - \bar{u}(y_1 - \lambda) \quad (v' > 0)$$

where the prime denotes a fluctuational quantity, and the bar denotes a mean quantity,

$$u_2' = \bar{u}(y_1 - \lambda) - \bar{u}(y_1) \quad (v' < 0)$$

and

$$|\overline{u'^2}| = \frac{1}{2}(|u_1'|^2 + |u_2'|^2) = \lambda^2 \left| \left(\frac{\partial \bar{u}}{\partial y} \right) \right|^2 \quad (1)$$

where $\left(\frac{\partial \bar{u}}{\partial y} \right)$, denotes the velocity gradient at the elevation y_1

Since u' and v' are of the order of magnitude,

$$|\overline{v'^2}| = k \cdot \lambda \frac{\partial \bar{u}}{\partial y} \quad (2)$$

The quantity $\overline{u'v'}$, being negative for positive mean velocity gradient, can be put equal to $-c |\overline{u'^2}| |\overline{v'^2}|$. Thus $\overline{u'v'} = -ck \lambda^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2$

The constant ck can be absorbed in the mixing length, and one can write

$$\overline{u'v'} = -\lambda^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2$$

and the shear stress as

$$\tau = -\rho \overline{u'v'} = \rho \lambda^2 \left(\frac{\partial \bar{u}}{\partial y} \right)^2$$

or, more generally,

$$\tau = \rho \lambda^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \frac{\partial \bar{u}}{\partial y} \quad (3)$$

which gives the proper sign to τ for an existing velocity gradient.

Boussinesq's eddy viscosity can be written

$$\epsilon = \lambda^2 \left| \frac{\partial \bar{u}}{\partial y} \right| \quad (4)$$

This affords a connection between ϵ and the mean velocity gradient, but the quantity λ still has to be determined.

It may be noted that since

$$\tau = \rho \overline{u'v'} = \rho \lambda \overline{v'^2} \frac{\partial \bar{u}}{\partial y}$$

by virtue of Eq 1, the relation $\frac{\partial \bar{p}}{\partial y} = \frac{\partial \bar{p}}{\partial x}$ yields

$$\frac{1}{\rho} \frac{d\bar{p}}{dx} = \frac{d}{dy} (\overline{v' \zeta} \frac{d\bar{u}}{dy}) \quad (5)$$

where \bar{p} denotes the mean pressure,

The momentum-transfer theory outlined in the foregoing can be easily extended to cover the axillary-symmetric case.

2. Taylor's vorticity-transfer theory

Consider again the parallel plane flow with a steady mean motion. The equation of motion

$$\frac{\partial u'}{\partial t} + (\bar{u} + u') \frac{\partial u'}{\partial x} + v' \frac{\partial (u + u')}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$

where $p = \bar{p} + p'$ can be written as

$$-\frac{\partial}{\partial x} \left(\frac{\bar{p}}{\rho} + \frac{1}{2} u'^2 + \frac{1}{2} v'^2 + \bar{u} u' \right) = \frac{\partial u'}{\partial t} + v' \frac{\partial \bar{u}}{\partial y} - 2 \overline{v' \zeta'}$$

where

$$\zeta' = \frac{1}{2} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right)$$

Assuming $u'^2 + v'^2$ to be constant in the x-direction and taking mean values with respect to time, one has

$$\frac{1}{\rho} \frac{d\bar{p}}{dx} = 2 \overline{v' \zeta'}$$

If one considers the vorticity to be transported in the same way as the momentum is transported in Pradtl's momentum-transfer theory, then

$$\zeta' = \zeta \frac{d}{dy} \left(\frac{1}{2} \frac{d\bar{u}}{dy} \right)$$

where $\zeta = \frac{1}{2} \frac{d\bar{u}}{dy}$ is the vorticity for the mean motion. Thus one has

$$\frac{1}{\rho} \frac{d\bar{p}}{dx} = \overline{v' \zeta} \frac{d^2 \bar{u}}{dy^2} \quad (6)$$

as compared with Eq 5. Here

$$\zeta = \overline{v' \zeta'} \quad (7)$$

The vorticity-transfer theory is due to G. I. Taylor (191,1915; 194,1932).

3. Taylor's generalized vorticity-transfer theory

Taylor's generalized vorticity-transfer theory (194,1932) was later improved in representation by Goldstein (100,1935). In the following, a combination of the two versions will be presented. Denoting the mean vorticity components as

$$\bar{\xi} = \frac{1}{2} \left(\frac{\partial \bar{w}}{\partial y} - \frac{\partial \bar{v}}{\partial z} \right)$$

$$\bar{\eta} = \frac{1}{2} \left(\frac{\partial \bar{u}}{\partial z} - \frac{\partial \bar{w}}{\partial x} \right)$$

$$\bar{\zeta} = \frac{1}{2} \left(\frac{\partial \bar{v}}{\partial x} - \frac{\partial \bar{u}}{\partial y} \right)$$

and the turbulent vorticity components as

$$\xi' = \frac{1}{2} \left(\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} \right)$$

etc., the equations of motion can be written as (37,145, p.578, eqs 6)

$$x - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} = \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \bar{w} \frac{\partial \bar{u}}{\partial z} + \frac{1}{2} \frac{\partial}{\partial x} (\bar{q}')^2 + 2 (\overline{w' \eta' - v' \zeta'}) \quad (8)$$

where $\bar{q}'^2 = \overline{u'^2 + v'^2 + w'^2}$, and two similar equations. It may be mentioned here that since

$$\bar{u}' = \bar{v}' = \bar{w}' = 0$$

one has

$$\bar{\xi}' = \bar{\eta}' = \bar{\zeta}' = 0$$

Suppose now that a particle which occupied the position (a,b,c) and had the mean vorticity components $\bar{\xi}_0, \bar{\eta}_0, \bar{\zeta}_0$ at time t_0 occupies the position (x, y, z) and has the mean vorticity components $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ at time t. The first vorticity component at (x, y, z), according to the history of the particle, would be equal to (37, 1945, Eqs 3, p. 205)

$$\bar{\xi}_0 \frac{\partial x}{\partial a} + \bar{\eta}_0 \frac{\partial x}{\partial b} + \bar{\zeta}_0 \frac{\partial x}{\partial c}$$

The difference of this quantity from $\bar{\xi}$ is the turbulent part of ξ , hence one has

$$\bar{\xi} + \xi' = \bar{\xi}_0 \frac{\partial x}{\partial a} + \bar{\eta}_0 \frac{\partial x}{\partial b} + \bar{\zeta}_0 \frac{\partial x}{\partial c} \quad (9)$$

and two similar equations for the other two vorticity components.

On the other hand, if the quantities

$$L_1 = x - a, \quad L_2 = y - b, \quad L_3 = z - c \quad (10)$$

are small, the Taylor's expansion of $\bar{\xi}$ gives, approximately,

$$\bar{\xi} = \bar{\xi}_0 + L_1 \frac{\partial \bar{\xi}}{\partial x} + L_2 \frac{\partial \bar{\xi}}{\partial y} + L_3 \frac{\partial \bar{\xi}}{\partial z} \quad (11)$$

From Eqs 9 and 11, one obtains

$$\bar{\xi}' = \bar{\xi}_0 \left(\frac{\partial x}{\partial a} - 1 \right) + \bar{\eta}_0 \frac{\partial x}{\partial b} + \bar{\zeta}_0 \frac{\partial x}{\partial c} - L_1 \frac{\partial \bar{\xi}}{\partial x} - L_2 \frac{\partial \bar{\xi}}{\partial y} - L_3 \frac{\partial \bar{\xi}}{\partial z} \quad (12)$$

The Lagrangian form of the equation of continuity being

$$\frac{\partial (x, y, z)}{\partial (a, b, c)} = 1 \quad (13)$$

or

$$\frac{\partial (a, b, c)}{\partial (x, y, z)} = 1 \quad (14)$$

for incompressible fluids, one has

$$\frac{\partial x}{\partial a} = \frac{\partial (b, c)}{\partial (y, z)} = \frac{\partial b}{\partial y} \frac{\partial c}{\partial z} - \frac{\partial c}{\partial y} \frac{\partial b}{\partial z} = \left(1 - \frac{\partial L_2}{\partial y} \right) \left(1 - \frac{\partial L_3}{\partial z} \right) - \frac{\partial L_3}{\partial y} \frac{\partial L_2}{\partial z} \quad (15)$$

If the second-order terms in the L's and their derivatives are omitted,

Eq 15 becomes

$$\frac{\partial L_1}{\partial x} + \frac{\partial L_2}{\partial y} + \frac{\partial L_3}{\partial z} = 0 \quad (16)$$

and Eq 15 can be written, in view of Eq 16 as

$$\frac{\partial x}{\partial a} = 1 + \frac{\partial L_1}{\partial x} \quad (17)$$

Similarly,

$$\frac{\partial x}{\partial b} = \frac{\partial L_1}{\partial y}, \quad \frac{\partial x}{\partial c} = \frac{\partial L_1}{\partial z} \quad (18)$$

and, in view of Eqs 11, 17 and 18, Eq 12 can be written as

$$\bar{\xi}' = \bar{\xi}_0 \frac{\partial L_1}{\partial x} + \bar{\eta}_0 \frac{\partial L_1}{\partial y} + \bar{\zeta}_0 \frac{\partial L_1}{\partial z} - L_1 \frac{\partial \bar{\xi}}{\partial x} - L_2 \frac{\partial \bar{\xi}}{\partial y} - L_3 \frac{\partial \bar{\xi}}{\partial z} \quad (19)$$

if second order terms in the L's are omitted.

Since the dilatation of the vorticity must vanish by continuity,

$$\frac{\partial \bar{\xi}}{\partial x} + \frac{\partial \bar{\eta}}{\partial y} + \frac{\partial \bar{\zeta}}{\partial z} = 0 \quad (20)$$

and Eq 19 can be written as

$$\bar{\xi} = \frac{\partial}{\partial y} (L_1 \bar{\eta} - L_2 \bar{\zeta}) - \frac{\partial}{\partial z} (L_3 \bar{\xi} - L_1 \bar{\zeta}) \quad (21)$$

Two similar equations can be written for η' and ζ' , so that in vector forms, one has

$$\vec{\omega}' = \text{CURL} (\vec{L} \times \vec{\omega}) \quad (22)$$

where

$$\begin{aligned} \vec{\omega}' &= \zeta' i + \eta' j + \xi' k \\ \vec{L} &= L_1 i + L_2 j + L_3 k \\ \vec{\omega} &= \bar{\xi} i + \bar{\eta} j + \bar{\zeta} k \end{aligned}$$

Eq. 22 embodies the generalized vorticity-transfer theory.

It can be shown that this generalized theory contains both Taylor's vorticity-transfer theory and Prandtl's momentum-transfer theory. When the mean motion is confined to the x-direction, and \bar{u} is a function of y only, one has

$$\xi = -\frac{1}{2} \frac{d\bar{u}}{dy}, \quad \bar{\eta} = \bar{\zeta} = 0$$

Then from Eq 19 and similar equations, one has

$$2 \overline{v' \xi - \omega' \eta'} = \overline{L_2 v'} \frac{d^2 \bar{u}}{dy^2} - \overline{(v' \frac{\partial L_3}{\partial z} - \omega' \frac{\partial L_2}{\partial z})} \frac{d\bar{u}}{dy} \quad (23)$$

or, from Eq 21, on the assumption that $\overline{L_1 v'}$ does not vary with x and that $\overline{L_2 \omega'}$ does not vary with z,

$$2 \overline{v' \xi - \omega' \eta'} = \frac{d}{dy} (L_2 \overline{v'} \frac{d\bar{u}}{dy} - (L_1 \frac{\partial v'}{\partial x} - L_2 \frac{\partial \omega'}{\partial x}) \frac{d\bar{u}}{dy}) \quad (24)$$

where the partial differentiations have been changed to ordinary ones since in this particular instance the corresponding quantities differentiated are functions of y only.

If now the turbulent motion is two-dimensional in the x-y plane, the last term on the right of Eq 23 goes out, and Eq 8 becomes, assuming there is no extraneous force and that $(q')^2$ does not vary with x,

$$\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} = \overline{L_2 v'} \frac{d^2 \bar{u}}{dy^2}$$

which is identical to Eq 6, so that Taylor's vorticity-transfer theory is included as a special case.

On the other hand, if

$$\frac{\partial u'}{\partial x} = \frac{\partial v'}{\partial x} = \frac{\partial w'}{\partial x} = 0 \quad (25)$$

so that lines of particles parallel to the x-axis move as a whole, the last term on the right of Eq 24 goes out, and Eq 8 becomes, under the same assumptions as before

$$\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} = \frac{d}{dy} \left(\overline{L_2 v'} \frac{d\bar{u}}{dy} \right)$$

which is identical to Eq 5 of Prandtl's momentum-transfer theory.

4. Taylor's modified vorticity-transfer theory

If

$$\frac{\partial x}{\partial a} = \frac{\partial y}{\partial b} = \frac{\partial z}{\partial c} = 1$$

$$\frac{\partial x}{\partial b} = \frac{\partial x}{\partial c} = \frac{\partial y}{\partial a} = \dots = \frac{\partial z}{\partial b} = 0$$

then Eq 9 and similar equations yield

$$\bar{\xi} + \xi' = \bar{\xi}_0, \quad \bar{\eta} + \eta' = \bar{\eta}_0, \quad \bar{s} + s' = \bar{s}_0 \quad (26)$$

which express the fact that the vorticity-components are transferable in the sense that heat is transferable.

Equations similar to Eq 21 give

$$\overline{v' s' - w' \eta'} = \overline{L_1 w'} \frac{\partial \bar{\eta}}{\partial x} + \overline{L_2 w'} \frac{\partial \bar{\eta}}{\partial y} + \overline{L_3 w'} \frac{\partial \bar{\eta}}{\partial z} - \overline{L_1 v'} \frac{\partial \bar{s}}{\partial x}$$

$$- \overline{L_2 v'} \frac{\partial \bar{s}}{\partial y} - \overline{L_3 v'} \frac{\partial \bar{s}}{\partial z} \quad (27)$$

This and two other equations obtained by permutation, in conjunction with Eq 18, constitute the modified vorticity-transfer theory (198, 1935 and 1937).

When the turbulence is isotropic, one has

$$\overline{L_1 w'} = \overline{L_2 w'} = \overline{L_1 v'} = \overline{L_3 v'} = \overline{L_2 u'} = \overline{L_3 u'} = 0$$

and

$$\overline{L_1 u'} = \overline{L_2 v'} = \overline{L_3 w'} = K \quad (\text{Say})$$

so that

$$\overline{v' s' - w' \eta'} = K \left(\frac{\partial \bar{\eta}}{\partial z} - \frac{\partial \bar{s}}{\partial y} \right)$$

which, in the case of one-directional flow, is simply

$$\overline{v' s' - w' \eta'} = K \left(\frac{\partial \bar{\eta}}{\partial z} - \frac{\partial \bar{s}}{\partial y} \right) = K \nabla^2 U \quad (28)$$

where ∇^2 is the Laplacian $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.

B. Evaluation of the Transfer Theories

The boundary layer equations of Prandtl are still valid if the kinematic viscosity ν is replaced by $\nu + \epsilon$. In general, the ϵ 's determined by the different transfer theories (with the help of one experimental datum) to fit the experimental data are different. The test of the virtue of these theories lies then in their application to heat transfer or vapor transfer with the ϵ 's thus determined. It will be shown that although certain theories withstand this test in specific cases, none retain their validity in general.

1. Wakes

As an immediate example of the above statement, the temperature distribution in the plane wake of a heated body will be considered. Let the free stream velocity be U , the mean longitudinal velocity in the wake be $U - u$ where u is the deficiency of velocity in the wake, and the mean transverse velocity be v . Assuming

$$\frac{u}{U} = x^{-\frac{1}{2}} f(y) \quad (29)$$

where x is measured along the axis of symmetry from some point in the wake, and $\eta = y x^{-\frac{1}{2}}$, y being measured from the symmetry axis in a perpendicular direction. The equation of continuity

$$\frac{\partial}{\partial x} (U - u) + \frac{\partial v}{\partial y} = 0 \quad (30)$$

can be integrated with the boundary condition $v = 0$ at $y = 0$ to yield

$$\frac{v}{U} = -\frac{1}{2} x^{-1} \eta f(\eta) \quad (31)$$

The equation of motion according to the momentum-transfer theory is

$$-U \frac{\partial u}{\partial x} = -\frac{\partial}{\partial y} \left(\epsilon \frac{\partial u}{\partial y} \right) \quad (32)$$

where the pressure gradient as well as the other quantities of smaller

magnitude are neglected. In terms of η , Eq 32 becomes

$$\frac{1}{2} \frac{d}{dy} (\eta f) = -\frac{d}{dy} (\epsilon f)$$

which when integrated with the boundary condition $f'(0) = 0$, gives

$$\frac{1}{2} \eta f = -\epsilon f'(\eta)$$

or

$$\epsilon = -\frac{\eta f}{2 f'} \quad (33)$$

Now, taking $\eta = ax^{\frac{1}{2}}$, by means of Eq 4

$$\epsilon = -a^2 f' \quad (34)$$

From Eqs 33 and 34

$$\eta f = 2 a^2 (f')^2$$

which can be integrated to

$$(f)^{\frac{1}{2}} = (18 a^2)^{-\frac{1}{2}} \eta^{3/2} + \text{constant}$$

Let u vanish at $\eta = \eta_0$. The above equation can then be written as

$$f' = (18 a^2)^{-\frac{1}{2}} \eta_0^{3/2} (1 - \xi^{3/2})^2 \quad (35)$$

where

$$\xi = \eta / \eta_0$$

If the maximum value of u is denoted by u_{\max} ,

$$u/u_{\max} = (1 - \xi^{3/2})^2 \quad (36)$$

This result is in good agreement with the experimental results of Schlichting (1930) and of Fage and Falkner (1932).

Now let T denote the temperature at any point in the wake, T_0 denote the temperature of the ambient flow, and $\Theta = (T - T_0)/T_0$.

The energy equation is

$$u \frac{\partial \Theta}{\partial x} = \frac{\partial}{\partial y} \left(\epsilon \frac{\partial \Theta}{\partial y} \right) \quad (37)$$

Letting $\Theta = x^{-\frac{1}{2}} \phi(\eta)$, integration of the above equation is

$$\epsilon = -\frac{\eta \phi}{2 \phi'} \quad (38)$$

Comparing Eqs 33 and 38, one has

$$\frac{\phi'}{\phi} = \frac{f'}{F}$$

or

$$\phi(\eta) = \text{const. } f(\eta)$$

which means

$$\Theta/\Theta_{\text{MAX}} = (1 - \xi^{3/2})^2 \quad (39)$$

This, however, is not in good agreement with the experimental results of Fage and Falkner (95,1932).

Now, according to the vorticity-transfer theory, the equation of motion

is

$$-U \frac{\partial u}{\partial x} = -\epsilon \frac{\partial^2 u}{\partial y^2}$$

which in terms of η becomes

$$\frac{1}{2} \frac{d}{d\eta} (\eta f) = -\epsilon f''$$

So that

$$-\epsilon = \frac{f - \eta f'}{2f''} \quad (40)$$

If the same assumption for the mixing length l is made as before Eq 40 can be written as

$$a^2 f'' = \frac{f - \eta f'}{2 f''}$$

which can be integrated into

$$\eta f = Aa^2 (f')^2$$

and

$$f = (9a^2)^{-1} \eta_0^3 (1 - \xi^{3/2})^2 \quad (41)$$

where η_0 and ξ have the same meaning as before. Eqs 35 and 41 are identical.

Also, one has $u/u_{\text{MAX}} = (1 - \xi^{3/2})^2$

which is identical to Eq 36.

This may at first sight create the impression that the vorticity-transfer and momentum-transfer theories always yield the same result. That

this is not true is evident when one calculates the temperature distribution.

With the substitution $\Theta = x^{-\frac{1}{2}} \phi(\eta)$, Eq 37 can again be integrated to yield Eq 38. Eqs 38 and 40 then give

$$\frac{\phi'}{\phi} = \frac{\eta f''}{f + \eta f'}$$

which can be integrated into

$$\log \phi = \int_0^\eta \frac{\eta f'' d\eta}{f + \eta f'} + \text{constant} \quad (42)$$

When Eq 41 is substituted into Eq 42, one obtains

$$\log \phi = \log (1 - \xi^{3/2}) + \text{constant}$$

or

$$\frac{\Theta}{\Theta_{MAX}} = 1 - \xi^{3/2} \quad (43)$$

The experimental results of Fage and Falkner (95, 1932) are in good agreement with Eq 43. The reason for the difference between Eq 39 and 43 lies in the difference of the mixing lengths in the two theories. Since Eqs. 35 and 41 must necessarily be identical, the quantity a in Eq 41 is $\sqrt{2}$ times as large as that in Eq 35. The larger a corresponds to a larger mixing length, and consequently to the curve expressed by Eq 43, which is less concentrated than that expressed by Eq 39.

Calculations for the velocity and temperature distributions in the wake behind a row of heated parallel bars can be found in (19, 1938). Experiments have been carried out by Gran Olsson (140, 1936). The momentum-transfer theory gives the velocity distribution

$$\frac{u}{U} = \frac{\lambda^3}{184.7 \lambda^2} F(y) x^{-1} \quad (44)$$

where λ is the spacing of the bars, λ is the mixing length, and F is determined by the equation

$$y = \frac{\lambda}{(138.5)^{1/3}} \int_F^1 \frac{dF}{(1-F^2)^{1/3}} \quad (45)$$

other quantities having the same meanings as in the case of a single wake.

The vorticity-transfer theory gives

$$\frac{u}{U} = \frac{\lambda^3}{92.4 \lambda^2} F(y) x^{-1} \quad (46)$$

with an λ' being $\sqrt{2}$ times as large as the λ for the momentum-transfer theory. The distributions given by Eqs 44 and 46 are exactly the same, and check with Olsson's experiments very well.

The temperature distribution can be found directly from the velocity distribution according to each theory. The theoretical distribution according to the vorticity-transfer theory is too high as compared with Olsson's experimental results, while that according to the momentum transfer theory is too low. Attempts were made to compute the temperature distribution independently by assuming constant ϵ . The result, being the same according to both theories, is too low. Thus, it appears that neither the momentum-transfer theory nor the vorticity-transfer theory is satisfactory.

The three dimensional wake behind a symmetric heated body has been treated by Goldstein (101,1938) and Tomotika (203,1938) who used the modified vorticity-transfer theory. Experimental results of Hall and Hislop (105,1938) agree well with the modified vorticity-transfer theory except near the edge of the wake.

The discussions on heat wakes apply also to evaporation wakes.

2. Jets

As a second example of the invalidity of the transfer theories, the problem of turbulent jets will be considered.

Let u and v be the longitudinal (in the x -direction) and transverse (in the y -direction) velocity components, respectively, at any point in a plane jet. Under the assumptions that

1. There exists similarity between sections perpendicular to the X -axis.
2. The mixing length and the width of the jet at any section are

proportional to x ,

and with the substitutions

$$\lambda = cx, \quad \eta = y/x \quad \psi = Ax^{\frac{1}{2}}F(\eta)$$

where ψ is the stream-function, the equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial}{\partial y} \left[\lambda^2 \left(\frac{\partial u}{\partial y} \right)^2 \right]$$

can be transformed, by virtue of the relations

$$u = \frac{\partial \psi}{\partial y} = A x^{-\frac{1}{2}} F'$$

$$v = \frac{\partial \psi}{\partial x} = \frac{A}{2} x^{-\frac{1}{2}} (2\eta F' - F)$$

into the equation

$$F'^2 + FF'' = 2c^2 \frac{d}{d\eta} (F'')^2 \quad (47)$$

Eq. 47 can be integrated to give

$$FF' = 2c^2 (F'')^2 \quad (48)$$

the constant of integration being zero since $F(0)$ and $F''(1)$ are both zero.

With the substitution

$$\xi = \eta / (2c^2)^{1/3}$$

Eq 48 can be transformed into

$$FF' = (F'')^2 \quad (49)$$

where the primes now denote differentiation with respect to the new variable ξ

Eq 49 with the boundary conditions $F'(0) = 1$, $F(0) = 0$, and $F' = 0$ at the edge of the jet has been solved by Tollmien (201, 1926) whose result shows that at the edge $\xi = 2.412$. Experiments have been conducted by Förthmann (96, 1934). Comparison of the theory with the experiments show that $c = 0.0165$, and $\lambda/Y = 0.17$ where Y corresponds to $u = \frac{1}{2} u_{\max}$. The agreement is good.

It must be noted here that the vorticity-transfer theory will yield exactly the same velocity distribution. The only difference will lie in the value of c and hence the value of λ . This difference will be reflected in the calculation for the temperature distribution.

The temperature distribution has been carried out by Howarth (109, 1938) both according to the momentum-transfer theory and the vorticity-transfer theory. According to the momentum-transfer theory, the temperature and

velocity distributions are identical, whereas according to the vorticity-transfer theory the temperature-distribution function is the square root of the velocity-distribution function. This situation has been illustrated in the case of wakes treated in the last section. No experimental results for the plane heated jet have been reported.

The velocity distribution for the axially symmetrical jet has been calculated by Tollmien (201,1926) on the basis of the momentum-transfer theory. The result is in good agreement with the experimental data of Ruden (169,1933) and Kuethe (127,1935). On the basis of the vorticity-transfer theory, Howarth (109,1938) has computed the velocity distribution under the assumptions that $\omega \sim r^{-1}$ and that $\omega \sim r^{-3/2}$, there being no real integral corresponding to a constant ω . He also computed the velocity distribution on the basis of the modified vorticity-transfer theory under the assumption of constant ω . All of his calculated results are not in good agreement with the experimental results of Ruden and Kuethe.

For the temperature distribution for the axially symmetrical jet all calculations have been carried out which are analogous to those for axially symmetric wakes (see 19,1938). Small-scale graphs of the velocity- and temperature distributions have been published by Ruden (169,1933). The observed $\frac{\Delta T}{\Delta T_{max}}$ ($\Delta T = T - T_0$, T_0 being the temperature of the environment) is greater than the observed $\frac{u}{u_{max}}$, so that the momentum-transfer theory, according to which the velocity and temperature distributions should be identical, is not valid, although it leads to a satisfactory velocity distribution. The vorticity-transfer theory leads to results completely at variance with experiments. The modified vorticity-transfer theory under the assumptions of isotropic turbulence and constant ω gives a satisfactory temperature distribution, but as has been remarked in the last paragraph, does not give a velocity distribution in agreement with experiments.

The velocity distribution in the mixing region of a parallel stream flowing over a still fluid has been calculated by Tollmien (201, 1926) on the basis of the momentum-transfer theory. The result is in good agreement with experiments. Calculations for the temperature distribution can be found in (19, 1938). The vorticity-transfer theory gives a temperature distribution in better agreement with the experimental data of Ruden (169, 1933) than the momentum-transfer theory.

Remarks on preheated jets apply also to pre-moistened jets.

From the foregoing, it seems that attempts to calculate the temperature distribution from the velocity distribution according to the transfer theories without deeper considerations of the characteristics of turbulent motion have not consistently led to satisfactory results. Consequently the virtue of these transfer theories is doubtful.

C. Statistical Theories

1. Taylor's theory

The statistical theory had its beginning as early as 1915, when Taylor (191) published his "Eddy Motion in the Atmosphere" which showed that turbulent motion is capable of diffusing heat and other diffusible properties throughout the fluid in much the same way that molecular agitation gives rise to molecular diffusion. This paper is followed by another by the same author in 1921 (193) in which diffusion by continuous movements was treated mathematically in the Lagrangian manner. Fourteen years later in 1935 and 1936, the same author published his "Statistical Theory of Turbulence" (197), in which the scale and the degree of isotropic turbulence as well as the mean square of the turbulent pressure gradient in isotropically turbulent flow were correlated with the size of the mesh of the grid in the wind tunnel. These correlations have been excellently verified by experiments in the U.S., Germany, and England.

(a) Fundamental considerations. Suppose that one observes the value $p_1, p_2 \dots p_n$ of a quantity p at a large number of successive times $t_1, t_2 \dots t_n$, and suppose that the mean of the squares of $p_1, p_2, \dots p_n$, denoted by

$$\overline{p^2} = \frac{p_1^2 + p_2^2 + \dots + p_n^2}{n}$$

as well as $\left(\frac{d^n p}{dt^n}\right)^2$ for any n is constant. If further one observes the values $p_1 + \delta p_1, p_2 + \delta p_2, \dots p_n + \delta p_n$, at times $t_1 + \delta t, t_2 + \delta t, \dots t_n + \delta t$, where δt is a small interval of time, then since $\overline{p^2}$ is constant, it must be equal to (to the first order)

$$\begin{aligned} & \frac{1}{n} \left\{ \left(p_1 + \frac{dp_1}{dt} \delta t \right)^2 + \left(p_2 + \frac{dp_2}{dt} \delta t \right)^2 + \dots + \left(p_n + \frac{dp_n}{dt} \delta t \right)^2 \right\} \\ &= \overline{p^2} + 2 \overline{p \frac{dp}{dt}} \delta t \end{aligned} \quad (50)$$

It appears therefore that the quantity in the square bracket can be differentiated. The constancy of $\overline{p^2}$ then requires that

$$\overline{p \frac{dp}{dt}} = 0 \quad (51)$$

Differentiating Eq 51, one obtains

$$\overline{p \frac{d^2 p}{dt^2}} + \overline{\left(\frac{dp}{dt} \right)^2} = 0 \quad (52)$$

Hence the correlation of p and $\frac{d^2 p}{dt^2}$ is, by definition and by virtue of Eq 52,

$$R_1 \left[p, \frac{d^2 p}{dt^2} \right] = \frac{\overline{p \frac{d^2 p}{dt^2}}}{\overline{p^2} \left(\overline{\frac{d^2 p}{dt^2}} \right)^2} = - \frac{\overline{\left(\frac{dp}{dt} \right)^2}}{\overline{p^2} \left(\overline{\frac{d^2 p}{dt^2}} \right)^2} \quad (53)$$

which shows that the correlation between p and its second derivative with respect to time must be negative.

Similarly, since $\overline{\left(\frac{dp}{dt} \right)^2}$ is constant,

$$\overline{\frac{dp}{dt} \frac{d^2 p}{dt^2}} = 0 \quad (54)$$

Differentiating,

$$\overline{\frac{dp}{dt} \frac{d^3p}{dt^3}} + \overline{\frac{d^2p}{dt^2}} = 0 \quad (55)$$

And differentiating Eq 52,

$$p \frac{d^3p}{dt^3} + 2 \frac{dp}{dt} \frac{d^2p}{dt^2} = 0$$

So

$$p \frac{d^3p}{dt^3} = 0 \quad (56)$$

Differentiating Eq 56, one has, by virtue of Eq 55,

$$p \frac{d^4p}{dt^4} - \left(\frac{d^2p}{dt^2} \right)^2 = 0 \quad (57)$$

Proceeding in this way, it can be shown that

$$p \frac{d^{2n}p}{dt^{2n}} = (-1)^n \left(\frac{d^n p}{dt^n} \right)^2 \quad (58)$$

$$p \frac{d^{2n+1}p}{dt^{2n+1}} = 0 \quad (59)$$

In analysing an actual curve, it may be tedious to obtain the standard deviations of p and its derivatives. Taylor offers, however, another method of defining the statistical properties of the curve which is equivalent to that given above, but which is likely to be more manageable. In presenting this method, Eqs 58 and 59 will be of use.

Suppose that one takes, as before, the values $p_1, p_2 \dots p_n$ at a large number of times t_1, t_2, \dots, t_n . Let these values of p be correlated with the values p_1', p_2', \dots, p_n' at times $t_1 + \xi, t_2 + \xi, \dots, t_n + \xi$, where ξ is a finite interval of time which may be positive or negative. Let the coefficient of correlation so found be $R_1(\xi)$. Then $R_1(\xi)$ must evidently be a function of ξ .

If p_t is the value of p at time t , and $p_{t+\xi}$ that at time $t + \xi$, then by definition

$$\overline{p_t p_{t+\xi}} = R_1(\xi) \sqrt{\overline{p_t^2}} \sqrt{\overline{p_{t+\xi}^2}}$$

and since by hypothesis $\overline{p^2}$ is constant,

$$R_1(\xi) = \overline{p_t p_{t+\xi}} / \overline{p^2} \quad (60)$$

Now expand $p_{t+\xi}$ in powers of ξ ,

$$p_{t+\xi} = p_t + \xi \frac{dp}{dt} + \frac{\xi^2}{2!} \frac{d^2 p}{dt^2} + \dots \quad (61)$$

Hence

$$\overline{p_t p_{t+\xi}} = \overline{p_t^2} + \xi \overline{p \frac{dp}{dt}} + \frac{\xi^2}{2!} \overline{p \frac{d^2 p}{dt^2}} + \dots$$

and, by virtue of Eqs 58 and 59, and the definition of $R_1(\xi)$,

$$R_1(\xi) = 1 - \frac{\xi^2}{2!} \frac{\overline{\left(\frac{dp}{dt}\right)^2}}{\overline{p^2}} + \dots + (-1)^n \frac{\xi^{2n}}{2n!} \frac{\overline{\left(\frac{d^n p}{dt^n}\right)^2}}{\overline{p^2}} + \dots \quad (62)$$

It will be seen that $R_1(\xi)$ is an even function of ξ , as might have been expected.

Now let the quantity p under consideration be the fluctuational velocity component v^1 in the y -direction (which, in case $v = 0$, is just the instantaneous velocity v). Then

$$\overline{v_t^1 v_{\xi}^1} = R_1(\xi - t) \overline{v_t^1{}^2}$$

If $\overline{v^1{}^2}$ is constant, one has, since $R_1(\xi - t)$ is an even function $\xi - t$,

$$\begin{aligned} \int_0^t \overline{v_t^1 v_{\xi}^1} d\xi &= \overline{v^1{}^2} \int_0^t R_1(\xi - t) d\xi = \overline{v^1{}^2} \int_0^t R_1(t - \xi) d\xi \\ &= \overline{v^1{}^2} \int_t^0 -R_1(t - \xi) d(t - \xi) = \overline{v^1{}^2} \int_0^t R_1(\xi) d\xi \end{aligned} \quad (63)$$

Also,

$$\int_0^t \overline{v_t^1 v_{\xi}^1} d\xi = \overline{v_t^1} \int_0^t \overline{v_{\xi}^1} d\xi = \overline{v_t^1} Y = \frac{1}{2} \frac{d}{dt} Y^2 \quad (64)$$

So

$$\overline{Y^2} = 2 \overline{v^1{}^2} \int_0^T \int_0^t R_1(\xi) d\xi dt \quad (65)$$

where Y is the distance traversed by a particle in time T in the y -direction, \overline{v} being assumed to be zero.

When T is so small that $R_1(\xi)$ does not differ appreciably from 1 during the interval T , Eq 65 becomes

$$\begin{aligned} \overline{Y^2} &= v'^2 \cdot T^2 \\ \text{or } \sqrt{\overline{Y^2}} &= v'' \cdot T \end{aligned} \quad (66)$$

where $v'' = \sqrt{v'^2}$. Eq 66 states that the standard deviation of a particle from its initial position is proportional to T when T is small.

The correlation coefficient $R_1(\xi)$ can be expected to fall to zero for large values of ξ . On the assumption that

$$\lim_{t \rightarrow \infty} \int_0^t R_1(\xi) d\xi = I \quad (\text{finite})$$

a time T_1 can be defined such that for $T > T_1$,

$$\int_0^T R_1(\xi) d\xi = I$$

Then, for $T > T_1$ after the beginning of the motion,

$$\frac{d}{dt} \overline{Y^2} = 2 \overline{v'^2} I \quad (67)$$

so that $\overline{Y^2}$ increases at a uniform rate. In the limit when $\overline{Y^2}$ is large

$$\sqrt{\overline{Y^2}} = v'' \sqrt{2IT} \quad (68)$$

so that the standard deviation of Y is proportional to the square root of T .

The trends of Eqs 66 and 68 have been verified by Richardson (160,1920) who performed some experiments on the diffusion of smoke emitted from a fixed point in a wind.

From Eq 67

$$\overline{Y v'} = \overline{v'^2} T$$

Hence, utilizing Eq 68, the correlation coefficient of Y and v' is, for very large T ,

$$R_1 [Y, v'] = \frac{\overline{Y v'}}{(\overline{Y^2})^{\frac{1}{2}} v''} = \sqrt{\frac{I}{2T}} \quad (69)$$

If $\overline{Y^2}$ is measured, the equation

$$\frac{d^2}{dt^2} \overline{Y^2} = 2 \overline{v'^2} R_1(\xi) \quad (70)$$

which is a consequence of Eqs 63 and 64, permits $R_1(\xi)$ to be computed.

(b) Theory of isotropic turbulence. From Eqs 63 and 64,

$$\frac{1}{2} \frac{d}{dt} \overline{Y^2} = \overline{v^2} \int_0^T R_1(\xi) d\xi \quad (70)$$

where bars again indicate mean values, one can define a length λ_1 , such that

$$\lambda_1 \sqrt{\overline{v'^2}} = \overline{v'^2} \int_0^T R_1(\xi) d\xi = \frac{1}{2} \frac{d}{dt} (\overline{Y^2}) \quad (71)$$

It will be seen from Eq 71 that the length λ_1 defined as

$$\lambda_1 = \sqrt{\overline{v'^2}} \int_0^T R_1(\xi) d\xi \quad (72)$$

bears the same relationship to diffusion by turbulent motion that the mean free path does to molecular diffusion. In this sense it is very similar to the mixing-length λ of Prandtl, but with the important difference that the hypothetical process of mixing involved in Prandtl's theory does not occur in its definition.

The length λ_1 can be considered as the mean free path of particles in turbulent motion, in the Lagrangian system. It is also possible to define a length λ_2 which will indicate the scale of turbulence in the Eulerian system. If one imagines that the correlation R_y between the values of u' (fluctuational velocity component perpendicular to v') at two points distant y apart in the direction of y has been determined for various values of y , and that R_y falls to zero when $y \geq Y$, then λ_2 can be defined as

$$\lambda_2 = \int_0^\infty R_2(y) dy = \int_0^Y R_2(y) dy \quad (73)$$

The length λ_2 may be taken as a possible definition of the average size of the eddies.

The lengths λ_1 and λ_2 can be computed if $\int_0^t R_1(\xi) d\xi$ and $R_2(y)$ are measured, the former of which can be obtained by measuring $\frac{d}{dt} \overline{Y^2}$ and the latter of which can be obtained by measuring $\overline{u^2}$ and $\overline{u_0' u'(y)}$ by hot wire anemometers, and also by an electric dynamometer in the case of $\overline{u_0' u'(y)}$.

Yet another method due to Prandtl is to pass the currents from the two hot wires (at distant y apart) through coils which cause deflections of a spot of light in two directions at right angles to one another. If the two hot wires are identical and so close that the correlation is nearly 1.0, the spot of light moves over a very elongated elliptic area, the long axis of which is at 45° to the deflections caused by either of the wires in the absence of disturbances from the other. By measuring the ratio of the principal axes of the elliptical blackened areas produced on a photographic plate by the moving spot of light during a prolonged exposure, it is possible to calculate R_y . This method is specially suitable for measurements when the correlation is very high, i.e., $1 - R_2(y)$ is small. For small correlations the electric dynamometer method gives better results.

It may be mentioned that since at small t

$$\sqrt{Y^2} = v'' t$$

and since $t = \frac{x}{U}$ where U is the mean velocity of flow,

$$\sqrt{\frac{Y^2}{x}} = \frac{v''}{U} \quad (74)$$

at small x , which permits the measurement of the degree of turbulence by measuring the standard deviation of Y at distance x .

Now the diffusion phenomenon will be studied when the turbulence is decaying. Since $\sqrt{v'^2}$ is now not constant, the diffusion equation should be Eq 64 instead of Eq 70, or, by an easy transformation,

$$\frac{1}{2} \frac{d}{dt} \overline{Y^2} = \overline{v_t' \int_0^t v_{t-\xi}' d\xi} = \overline{v_t' \int_0^t v_{t-\xi}' d\xi} \quad (75)$$

where $Y = \int_0^t v_{t-\xi}' d\xi = \int_0^t v_{t-\xi}' d\xi$

Writing v_t'' for $\sqrt{v_t'^2}$, $v_{t-\xi}''$ for $\sqrt{v_{t-\xi}'^2}$, and $R_1[v_t', v_{t-\xi}']$ for the coefficient of correlation between v_t' and $v_{t-\xi}'$, one has

$$\frac{1}{2} \frac{d}{dt} \overline{Y^2} = v_t'' \int_0^t v_{t-\xi}'' R_1[v_t', v_{t-\xi}'] d\xi \quad (76)$$

When v'' is not constant, it is not possible to proceed beyond Eq 76, but the existing experimental evidence seems to indicate that turbulent diffusion is proportional to the mean speed, and that if matter from a concentrated source is diffused over an area down-stream from the source, an increase in the speed of the whole system (i.e. proportional increases in turbulent and mean speed) leaves the distribution of matter in space unchanged, though the absolute concentration is reduced. The condition that this may be so is that $R_1[v'_t, v'_{t-\zeta}]$ is a function of ζ only where

$$d\zeta = v'' d\xi = (v''/U) dx \quad (77)$$

and $x = Ut$ is the distance down-stream from the source.

The equation which represents the lateral spread of matter or heat from a concentrated source is therefore

$$\frac{1}{2} \frac{U}{v''} \frac{d}{dx} (\overline{Y^2}) = \int_0^\zeta R_3(\zeta) d\zeta \quad (78)$$

where

$$\zeta = \int_0^x \frac{v''}{U} dx$$

and $R_3(\zeta)$ is the correlation between the velocities of a particle at times t_1 and t_2 when $\zeta = \int_{t_1}^{t_2} v'' dt$. If $R_3(\zeta)$ falls to zero at a finite value of ζ , say $\zeta = \zeta_1$, and remains zero for all greater values of ζ ,

$\int_0^{\zeta_1} R_3(\zeta) d\zeta$ is finite. If l_ζ be written for $\int_0^{\zeta_1} R_3(\zeta) d\zeta$ then

Eq 78 becomes, for sufficiently large x ,

$$\frac{1}{2} \frac{U}{v''} \frac{d}{dx} (\overline{Y^2}) = l_\zeta \quad (79)$$

This is the same expression as Eq 71 found for turbulence which is not decaying.

Eq 78 may be expressed in the form

$$\frac{1}{2} \frac{d}{d\zeta} (\overline{Y^2}) = \int_0^\zeta R_3(\zeta) d\zeta \quad (80)$$

When ζ is small so that essentially $R_3(\zeta) \doteq 1$ over the range from 0 to

x , Eq 80 gives $\frac{1}{2} \frac{d}{d\zeta} (\overline{Y^2}) = \zeta$ (81)

from which, by integration,

$$Y^2 = \zeta^2 \text{ or } \sqrt{Y^2} = \zeta \quad (82)$$

When the turbulence is constant $\zeta = x v''/U$ so that Eq 82 reduces to Eq 74. If the turbulence is not constant and if Y^2 and v''/U are measured at a number of values of x , then both ζ and $\frac{1}{2} \frac{U}{v''} \frac{d}{dx} (\overline{Y^2})$ can be found.

Thus $\int_0^{\zeta} R_3(\zeta) d\zeta$ can be plotted against ζ and $R_3(\zeta)$ can be found graphically from this experimental curve.

One now proceeds to study the dissipation of energy, the principal agents in which are the eddies of very small scale. The rate of dissipation of energy at any instant depends only on the viscosity and the instantaneous velocity in the following way (37,1945, p.580) if the mean velocity U is constant and is in the x -direction: ($u'_x = \frac{\partial u'}{\partial x}$ etc.)

$$\overline{W} = \mu \left\{ 2 \overline{u'_x{}^2} + 2 \overline{v'_y{}^2} + 2 \overline{w'_z{}^2} + \overline{(v'_x + u'_y)^2} \right. \\ \left. + \overline{(w'_y + v'_z)^2} + \overline{(u'_z + w'_x)^2} \right\} \quad (83)$$

If, therefore, the representation of the essential statistical properties of the velocity field can be expressed by the R_y curve and similar correlation curves it must be possible to deduce from them the rate of dissipation of energy. This would in general involve a complicated analysis, but the problem can be much simplified if the field of turbulent flow is assumed to be isotropic, in which case

$$\overline{u'_x{}^2} = \overline{v'_y{}^2} = \overline{w'_z{}^2}$$

$$\overline{u'_y{}^2} = \overline{u'_z{}^2} = \overline{v'_x{}^2} = \overline{v'_z{}^2} = \overline{w'_x{}^2} = \overline{w'_y{}^2}$$

and

$$\overline{v'_x u'_y} = \overline{w'_y v'_z} = \overline{u'_z w'_x}$$

so that

$$\frac{\overline{W}}{M} = 6(\overline{u_x'^2} + \overline{u_y'^2} + \overline{v_x' u_y'}) \tag{84}$$

Consider the most general possible expression for the mean value of any quadratic function of the nine first order partial derivatives of the velocity components. The following table will show the number of terms in each of the ten different groups, the total number of terms being $\frac{9(8)}{2} + 9 = 45$.

Type	Symbol	No. of Terms in Group	Total
$\overline{u_x'^2}$	a1	3	45
$\overline{u_x' u_y'}$	a2	6	
$\overline{u_y'^2}$	a3	6	
$\overline{u_y' u_z'}$	a4	3	
$\overline{u_x' v_x'}$	a5	6	
$\overline{u_x' v_y'}$	a6	3	
$\overline{u_x' v_z'}$	a7	6	
$\overline{u_y' v_x'}$	a8	3	
$\overline{u_y' v_z'}$	a9	6	
$\overline{u_z' v_z'}$	a10	3	

From the equation of continuity and the condition of isotropy it can be obtained that

$$a_1 = -2a_6 \quad (85)$$

On rotating the reference axes by 45° in different ways, the condition of isotropy of turbulence furnishes

$$a_2 = a_4 = a_5 = a_7 = a_9 = a_{10} = 0 \quad (86)$$

$$a_1 - a_3 - a_6 - a_8 = 0 \quad (87)$$

One more equation is needed in order that all the non-vanishing a 's can be expressed in terms of one of them, a_3 , say. This can be obtained by noting that (37, 1945, p. 581)

$$\begin{aligned} \iiint \frac{\overline{W}}{\mu} dx dy dz &= \iiint (\xi^2 + \eta^2 + \zeta^2) dx dy dz - \iint \frac{\partial}{\partial n} (q'^2) dS \\ &+ 2 \iint \begin{vmatrix} l' & m' & n' \\ u' & v' & w' \\ \xi & \eta & \zeta \end{vmatrix} dS \end{aligned} \quad (88)$$

where

$$\begin{aligned} \xi &= w'_y - v'_z && \text{etc.} \\ q'^2 &= u'^2 + v'^2 + w'^2 \end{aligned}$$

n is measured in the direction of the normal to the surface S enclosing the volume, and the integrals are taken over S and the enclosed volume. If S is large compared with the scale of the turbulence, the surface integrals are small compared with the volume integrals and can be neglected. Hence, taking the mean values of all quantities in Eq 88,

$$\frac{\overline{W}}{\mu} = \overline{\xi^2} + \overline{\eta^2} + \overline{\zeta^2} = 6a_3 - 6a_8 \quad (89)$$

Eqs 84 and 89 then give

$$6a_1 + 6a_3 + 6a_8 = 6a_3 - 6a_8$$

or

$$a_1 - 2a_8 = 0 \quad (90)$$

Using Eqs 85, 87, and 90, one has

$$a_1 = \frac{1}{2}a_3 = -2a_6 = -2a_8 \quad (91)$$

and

$$\frac{\overline{W}}{\mu} = 7.5 a_3 = 7.5 \overline{\left(\frac{\partial u'}{\partial y}\right)^2} \quad (92)$$

From Eq 62, one has in this case

$$R_y = 1 - \frac{1}{2!} \frac{y^2}{u^2} \overline{\left(\frac{\partial u'}{\partial y}\right)^2} + \frac{1}{4!} \frac{y^4}{u^2} \overline{\left(\frac{\partial^2 u'}{\partial y^2}\right)^2} + \dots \quad (93)$$

from which

$$\overline{\left(\frac{\partial u'}{\partial y}\right)^2} = 2 \overline{u^2} \lim_{y \rightarrow 0} \left(\frac{1 - R_2(y)}{y^2} \right) \quad (94)$$

Defining λ_1^2 as the radius of curvature of the R_y curve at $y = 0$, one has

$$\frac{1}{\lambda_1^2} = 2 \lim_{y \rightarrow 0} \left(\frac{1 - R_2(y)}{y^2} \right)$$

or, on putting $\lambda_1 = \lambda / \sqrt{2}$

$$\frac{1}{\lambda^2} = \lim_{y \rightarrow 0} \left(\frac{1 - R_2(y)}{y^2} \right) \quad (95)$$

A physical interpretation of λ may be found by describing the parabola which touches the $R_2(y)$ curve of the origin. This parabola will cut the axis $R_2(y) = 0$ at the point $y = \lambda$. λ may be regarded roughly as a measure of the diameters of the smallest eddies which are responsible for the dissipation of energy.

Combining Eqs 92, 94, and 95:

$$\overline{W} = 15 \mu \overline{u'^2} / \lambda^2 \quad (96)$$

If \overline{W} , $\overline{u'^2}$, and λ^2 can be measured independently, the last equation offers a check on the theory. It may be noticed also that if the Reynold's stresses in geometrically similar fields of flow are proportional to $\overline{u'^2} = u''^2$, \overline{W} is proportional to u''^3 , and λ to $(u'')^{-\frac{1}{2}}$. Since λ^{-2} is proportional to the curvature of the R_y curve, the latter must be proportional to u'' . In the limit of very high values of u'' , the $R_2(y)$ curve may be expected to have a pointed top.

The question may be asked about the relation between λ and λ , where λ is some linear dimension defining the scale of the turbulence system. So far as changes in linear dimensions, velocity, and density are concerned:

$$W = \text{constant} \cdot \left| \frac{\rho u''^3}{\lambda} \right| \quad (97)$$

Combining Eqs 96 and 97

$$\frac{\lambda^2}{\lambda^2} = C \frac{\rho}{\lambda u''}$$

Taking λ as the mesh length M ,

$$\frac{\lambda}{M} = A \sqrt{\frac{\rho}{\lambda u''}} \quad (98)$$

where A is assumed by Taylor to be an absolute constant for all grids of a definite type, e.g., for all square-mesh grids or honeycombs.

One is now in a position to predict the way in which turbulence may be expected to decay when a definite scale has been given to it as the air stream passes through a regular grid or honeycomb.

The rate of loss of kinetic energy of the turbulence per unit volume is, in isotropic turbulence,

$$-\frac{1}{2} \rho U \frac{d}{dx} (\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) = -\frac{3}{2} \rho U \frac{d}{dx} (\overline{u'^2})$$

This must be equal to the rate of dissipation \overline{W} , so that

$$-\frac{3}{2} \rho U \frac{d}{dx} (\overline{u'^2}) = 15 \mu \overline{u'^2} / \lambda^2$$

which becomes, by virtue of Eq 98,

$$-\frac{U}{u''^2} \frac{d}{dx} (u''^2) = \frac{10}{A^2 M}$$

Integrating,

$$\frac{U}{u''} = \frac{5x}{A^2 M} + \text{constant} \quad (99)$$

That is, U/u'' increase linearly with x

The space gradients of pressure fluctuations in isotropically turbulent flow will now be investigated. Remembering the variable ξ defined by Eq 77, Eq 62 gives, on substituting v' for the general quantity p and ξ

for t :

$$\overline{\left(\frac{Dv'}{D\xi}\right)^2} = 2 v''^2 \lim_{\xi \rightarrow 0} \left(\frac{1 - R_3(\xi)}{\xi^2} \right) \quad (100)$$

where $Dv'/D\xi$ is the rate of change of v' with respect to ξ as the particle moves downstream with mean velocity U , and v''^2 is written for $\overline{v'^2}$. If v'' is constant (non-decaying),

$$\frac{Dv'}{D\xi} = \frac{Dv'}{D\left(\frac{v''}{U} x\right)} = \frac{1}{v''} \frac{Dv'}{Dt}$$

and

$$\overline{\left(\frac{Dv'}{Dt}\right)^2} = 2v''^4 \lim_{\xi \rightarrow 0} \left(\frac{1 - R_3(\xi)}{\xi^2} \right) = 2 \frac{v''^4}{\lambda_\xi^2} \quad (101)$$

Now Dv'/Dt is simply the acceleration of a particle in the direction y expressed in the Lagrangian manner. The Lagrangian equation of motion in the y -direction is in this case:

$$-\frac{1}{\rho} \frac{\partial p'}{\partial y} = \frac{Dv'}{Dt} \quad (102)$$

Thus the last two equations give

$$\overline{\left(\frac{\partial p'}{\partial y}\right)^2} = 2\rho^2 \frac{v''^4}{\lambda_\xi^2} \quad (103)$$

This is an important result because the disturbing effect of turbulence on the laminar boundary layer at the surface of a solid moving in a stream of fluid might be ascribed to the pressure gradients which accompany turbulent motion.

It is natural to inquire about the relation of λ_ξ and λ . In the Eulerian system,

$$-\frac{1}{\rho} \frac{\partial p'}{\partial y} = \frac{\partial v'}{\partial t} + \frac{1}{2} \frac{\partial}{\partial y} (u'^2 + v'^2 + w'^2) - w' \left(\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} \right) + u' \left(\frac{\partial w'}{\partial x} - \frac{\partial u'}{\partial y} \right)$$

Taylor made the assumption that $\frac{1}{\rho} \sqrt{\overline{\left(\frac{\partial p'}{\partial y}\right)^2}}$ is of the same order of magnitude

as

$$\frac{1}{2} \sqrt{\overline{\left[\frac{\partial}{\partial y} (q')^2 \right]^2}}$$

which in isotropic turbulence is equal to

$$\frac{3}{2} \sqrt{\overline{\left(\frac{\partial v'}{\partial y} \right)^2}}$$

which in turn must be of the same order of magnitude as

which in turn must be of the same order of magnitude as

$$\sqrt{v'^2 \left(\frac{\partial v'}{\partial y}\right)^2}$$

Taylor therefore assumed that in isotropic turbulence

$$\sqrt{\left(\frac{\partial p'}{\partial y}\right)^2} = 3 B \rho \sqrt{v'^2 \left(\frac{\partial v'}{\partial y}\right)^2} \quad (104)$$

where B is a constant which is expected to be of order of magnitude unity.

From Eqs 103 and 104,

$$B^2 = \frac{2 v''^4}{9 \lambda_s^2 v'^2 \left(\frac{\partial v'}{\partial y}\right)^2} \quad (105)$$

where

$$\left(\frac{\partial v'}{\partial y}\right)^2 = a_1 = \frac{1}{2} a_3 = \frac{1}{2} \left(\frac{\partial u'}{\partial y}\right)^2 = \frac{v''^2}{\lambda^2} \quad (106)$$

by Eqs 91, 94 and 95. Thus

$$B^2 = \frac{2}{9} \left(\frac{\lambda^2}{\lambda_s^2}\right) \quad (107)$$

The assumption represented by Eq 104 is therefore equivalent to the assumption that λ_s is constant multiple of λ .

Experiments quoted in Taylor's paper (197,1935) showed that

$$\lambda_1 = 0.1 M$$

$$\lambda_2 = 0.2 M$$

and verified Eqs 96, 98 and 99, the value of A being found to be approximately

2. Few experiments have been done on the pressure fluctuations. The set of

experiments (175,1935) gives $B = 0.94$, so that Eq 104 is equivalent to

$\lambda = 2\lambda_s$ approximately. All these experimental results are subject to the restriction that $\lambda u''/v$, the Reynolds number of turbulence, is greater than some number which must be determined by experiment.

The essential features of diffusion in isotropic (non-decaying) turbulence are, according to Taylor's theory:

1. For time intervals which are small in comparison with the ratio of λ_1 to $u'' = \sqrt{u'^2}$, the diffusing quantity N spreads at a uniform rate proportional to u'' , and the rate does not depend on λ_1 .

2. For time intervals which are large in comparison with the ratio of λ_1 to u'' , the diffusing quantity N spreads in accordance with the usual diffusion equation

$$\frac{DN}{Dt} = \frac{\partial}{\partial x} \left(K \frac{\partial N}{\partial x} \right) + \frac{\partial}{\partial y} \left(K \frac{\partial N}{\partial y} \right) + \frac{\partial}{\partial z} \left(K \frac{\partial N}{\partial z} \right)$$

with a constant coefficient of diffusion K equal to $\lambda_1 u''$, λ_1 being defined as $\int_0^{\infty} R_1(\xi) d\xi$

3. For intermediate intervals, the diffusion depends on $R_1(\xi)$.

If the turbulence is decaying, similar conclusions may be drawn, according to Taylor's theory, by replacing λ_1 with λ_λ and by redefining K as $\lambda_\lambda u''$, where u'' is now varying and λ_λ is defined as $\int_0^{\infty} R_3(\zeta) d\zeta$, ζ being defined as

$$\zeta = \int_0^x \frac{v''}{U} dx$$

and U being the mean velocity which is in the x -direction.

For later developments of Taylor's theory, see (200, 1938), and (104, 1938).

2. von Kármán's theory

In isotropic turbulence the correlation tensor has spherical symmetry and the several components are functions only of the distance r between the two points, and of the time t . Denote by u'_1, v'_1, w'_1 and u'_2, v'_2, w'_2 the components of the velocity fluctuations at the points P_1 and P_2 having coordinates $(x'_1, 0, 0)$ and $(x'_2, 0, 0)$ respectively. Suppose that $\overline{u'^2_1}, \overline{v'^2_1}, \overline{w'^2_1}$, which by isotropy are equal, are independent of position and equal to $\overline{u'^2_2}$. Then

$$\overline{u'^2_1} = \overline{v'^2_1} = \overline{w'^2_1} = \overline{u'^2_2} = \overline{v'^2_2} = \overline{w'^2_2} = \overline{u'^2}$$

The correlation coefficients $R_{22} = \overline{v'_1 v'_2} / \overline{u'^2}$ and $R_{33} = \overline{w'_1 w'_2} / \overline{u'^2}$ will be identical because of isotropy and will be a function $g(r,t)$ of r and t . The correlation coefficient $R_{11} = \overline{u'_1 u'_2} / \overline{u'^2}$ will be a function $f(r,t)$. All the other correlation coefficients can be shown to be zero by rotations and reflections, remembering the isotropy of turbulence. Thus the correlation tensor is, for the particular points chosen:

$$\begin{pmatrix} f(r,t) & 0 & 0 \\ 0 & g(r,t) & 0 \\ 0 & 0 & g(r,t) \end{pmatrix}$$

which can be decomposed to

$$[f(r,t) - g(r,t)] \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + g(r,t) I$$

where I is the unit tensor. If a rotation is given to the coordinate axes, such that the points $(x'_1, 0, 0)$ and $(x'_2, 0, 0)$ assume their new coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) , it can be readily seen from the methods of tensor calculus that the correlation tensor will assume the form in the new coordinate system:

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \frac{f(r,t) - g(r,t)}{r^2} \begin{pmatrix} X^2 & XY & XZ \\ YX & Y^2 & YZ \\ ZX & ZY & Z^2 \end{pmatrix} + g(r,t) I \quad (108)$$

where $X = x_2 - x_1$, $Y = y_2 - y_1$, $Z = z_2 - z_1$, $r^2 = X^2 + Y^2 + Z^2 = (x'_2 - x'_1)^2$, and where the first subscript of R corresponds to the velocity component at (x_1, y_1, z_1) , the second to those at (x_2, y_2, z_2) , the subscripts 1, 2, 3 of the R 's correspond to the velocity components u' , v' , w' .

The equation of continuity is

$$\frac{\partial u'_2}{\partial x_2} + \frac{\partial v'_2}{\partial y_2} + \frac{\partial w'_2}{\partial z_2} = 0$$

Multiplication by $u'_1 / \overline{u'^2}$ which is independent of x_2, t_2, z_2 , gives

$$\frac{\partial R_{11}}{\partial X} + \frac{\partial R_{12}}{\partial Y} + \frac{\partial R_{13}}{\partial Z} = 0 \quad (109)$$

Taking the R's from Eq 108, one obtains

$$X \left[2(f-g) + r \left(\frac{\partial f}{\partial r} \right) \right] = 0$$

Since S is arbitrary,

$$2f(r,t) - 2g(r,t) = -r \frac{\partial f(r,t)}{\partial r} \quad (110)$$

Defining

$$L = \int_0^\infty R_y dx = \int_0^\infty g(r,t) dr \quad (111)$$

$$L_x = \int_0^\infty R_x dx = \int_0^\infty f(r,t) dr \quad (112)$$

where $R_y \equiv g(r,t)$, $R_x \equiv f(r,t)$, one has

$$L - L_x = \frac{1}{2} \int_0^\infty r \frac{\partial f}{\partial r} dr = \frac{1}{2} \int_0^\infty x \frac{\partial R_x}{\partial x} dx = -\frac{1}{2} \int_0^\infty R_x dx = -\frac{L_x}{2}$$

or $2L = L_x \quad (112)$

Since f and g are even functions of r,

$$f = 1 + f''_0 r^2/2 + \dots \quad (114)$$

$$g = 1 + g''_0 r^2/2 - \dots \quad (115)$$

where the quantities f''_0 , g''_0 , etc. are functions of time only.

From Eq 110, $2f''_0 = g''_0$, whence for small values of r,

$$\begin{aligned} R &= \left[1 + (g''_0/2) r^2 \right] I + (f''_0 - g''_0/2) \vec{r} \vec{r} \\ &= (1 + f''_0 r^2) I - \frac{f''_0}{2} \vec{r} \vec{r} \end{aligned} \quad (116)$$

The second derivatives are, for very small r,

$$\frac{\partial^2 R_{11}}{\partial X^2} = \frac{\partial^2 R_{22}}{\partial Y^2} = \frac{\partial^2 R_{33}}{\partial Z^2} = f''_0 \quad (117)$$

$$\frac{\partial^2 R_{11}}{\partial Y^2} = \frac{\partial^2 R_{11}}{\partial Z^2} = \text{similar terms by permutation} = 2 f''_0 \quad (118)$$

$$\frac{\partial^2 R_{12}}{\partial X \partial Y} = \text{similar terms by permutation} = -\frac{f''_0}{2} \quad (119)$$

All others, e.g. $\frac{\partial^2 R_{12}}{\partial x \partial z}$ etc. are zero

Von Kármán (121, 1937) points out that the correlation tensor is of the same form as the stress tensor for a continuous medium when there is spherical symmetry. In the analogy $f(r)$ corresponds to the principal radial stress at any point, $g(r)$ to the principal transverse stress, and the several R 's to the stress components over planes normal to the coordinate axes. The relation between f and g given by the continuity equation corresponds to the condition for equilibrium of the stresses.

Eq 110 has been checked experimentally at the National Physical Laboratory (200, 1938).

The correlations of the derivatives of the velocity components are found by noting that the velocity components at point 2 are independent of the coordinates of point 1, and vice versa. Thus, one has

$$\overline{v'_2 \frac{\partial u'_1}{\partial x_1}} = \frac{\partial \overline{(u'_1 v'_2)}}{\partial x_1} = \overline{u'^2} \frac{\partial R_{12}}{\partial x_1} = -\overline{u'^2} \frac{\partial R_{12}}{\partial x}$$

and

$$\frac{\partial u'_1}{\partial x_1} \frac{\partial v'_2}{\partial y_2} = \frac{\partial}{\partial y_2} \left(\overline{v'_2 \frac{\partial u'_1}{\partial x_1}} \right) = -\overline{u'^2} \frac{\partial^2 R_{12}}{\partial x \partial y}$$

So that, letting points 1 and 2 coincide, one has

$$\overline{\frac{\partial u'}{\partial x} \frac{\partial v'}{\partial y}} = \overline{u'_x v'_y} = -\overline{u'^2} \left(\frac{\partial^2 R_{12}}{\partial x \partial y} \right)_{x=y=0} = \frac{f''}{2} \overline{u'^2} = \frac{f''}{2} u''^2 \quad (120)$$

by virtue of Eq 119. By similar reasoning,

(121)

$$\left(\frac{\partial u'}{\partial x} \right)^2 = \left(\frac{\partial v'}{\partial y} \right)^2 = \left(\frac{\partial w'}{\partial z} \right)^2 = -u''^2 f''$$

$$\left(\frac{\partial u'}{\partial y} \right)^2 = \text{similar terms by permutation} = -2 u''^2 f'' \quad (121a)$$

and

$$\overline{\frac{\partial v'}{\partial x} \frac{\partial u'}{\partial y}} = \overline{\frac{\partial w'}{\partial y} \frac{\partial v'}{\partial z}} = \overline{\frac{\partial u'}{\partial z} \frac{\partial w'}{\partial x}} = \frac{u''^2}{2} f'' \quad (122)$$

Thus the relations between the quantities a_1, a_3, a_6 and a_8 in Taylor's theory are found in a simpler and more elegant way.

Since g''_0 is the curvature of the g -curve or the R_y -curve, from Eq 95 one sees that

$$g''_0 = \frac{2}{\lambda^2}$$

and since $2 f''_0 = g''_0$,

$$f''_0 = \frac{1}{\lambda^2} \quad (123)$$

so that all the mean values of the products of the derivatives can be expressed as multiples of u''^2/λ^2

To investigate the propagation of the correlation with time, assume that the velocity fluctuations satisfy the Navier-Stokes, equations of motion, such that

$$\frac{\partial u'_i}{\partial t} + u'_i \frac{\partial u'_i}{\partial x_i} + v'_i \frac{\partial u'_i}{\partial y_i} + w'_i \frac{\partial u'_i}{\partial z_i} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \nabla^2 u'_i \quad (124)$$

Multiplying by u''_2 and taking mean values, one has, on the assumption that the triple correlations as well as the term involving the pressure are zero,

$$\frac{\partial}{\partial t} (R_{11} u''^2) = 2\nu u''^2 \nabla^2 R_{11}$$

Identical equations for the other elements of the correlation tensor can be obtained, and all of these equations including the above one can be replaced by the single equation

$$\frac{\partial}{\partial t} (f u''^2) = 2\nu u''^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) \quad (125)$$

But $f(0,t) = 1$ for all time, and $\left(\frac{\partial f}{\partial r}\right)_{r=0} = 0$ so Eq 125 becomes

$$\frac{\partial u''^2}{\partial t} = -10\nu u''^2 f''_0 = -10\nu u''^2/\lambda^2 \quad (126)$$

where $\frac{d}{dt}$ has replaced $\frac{\partial}{\partial t}$ because u''^2 is a function of time only.

Eliminating u''^2 from Eqs 125 and 126:

$$\frac{\partial f}{\partial t} = 2\nu \left[\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} + \frac{5}{\lambda^2} f \right] \quad (127)$$

This equation determines $f(r,t)$ for all subsequent times, if f is given at $t = 0$ for all values of r .

If the initial shape of the correlation function $f(r,t)$ is arbitrary its shape will in general change with time. However, there are special cases in which the shape of the correlation function remains similar. This will occur if $f(r,t)$ is a function of the dimensionless variable $\xi = r/\sqrt{\nu t}$ only.

Then Eq 127 is reduced to

$$\frac{d^2 f}{d\xi^2} + \left(\frac{4}{\xi} + \frac{\xi}{4} \right) \frac{df}{d\xi} + \frac{5\nu t}{\lambda^2} f = 0 \quad (128)$$

From this equation, on making $\xi \rightarrow 0$, one has, since f is an even function of ξ ,

$$5 \left(\frac{d^2 f}{d\xi^2} \right)_{\xi=0} = -\frac{5\nu t}{\lambda^2}$$

or

$$\frac{\nu t}{\lambda^2} = - \left(\frac{d^2 f}{d\xi^2} \right)_{\xi=0} = \alpha, \quad \text{say.}$$

Thus

$$\lambda^2 = \frac{1}{\alpha} \nu t \quad (129)$$

which means λ^2 increases linearly with time, if the shape of the correlation function remains similar. The numerical factor α , which determines the rate of increase, is given by the initial shape of f .

One is now in a position to discuss the decay of turbulence, under the restriction that the shape of f remains similar. From Eqs 126 and 129

$$\frac{du''^2}{dt} = -10\alpha \frac{u''^2}{t}$$

i.e.

$$u''^2 = \frac{C}{t^{10\alpha}} \quad (C = \text{constant}) \quad (130)$$

Supposing the fluid to be moving with uniform speed U in the direction of the x -axis. If $t = t_0$ when $x = 0$ so that $t = t_0 + \frac{x}{U}$,

Eq 130 gives

$$u''^2 = u_0''^2 / \left(1 + \frac{x}{U t_0} \right)^{10\alpha} \quad (131)$$

With λ_0 corresponding to t_0 , Eq 129 gives

$$\lambda^2 = \alpha \nu \left(t_0 + \frac{x}{U} \right) = \lambda_0^2 + \alpha \nu \frac{x}{U} = \lambda_0^2 \left(1 + \frac{x}{U t_0} \right) \quad (132)$$

G. I. Taylor's result contained in Eq 99 can be written as

$$\frac{1}{u''} = \frac{1}{u''_0} + \text{const.} \frac{x}{U}$$

This is a special case of Eq 131, with $\alpha = \frac{1}{5}$. Also Eqs 131 and 132 give

$$\frac{u''}{u''_0} \frac{\lambda^2}{\lambda_0^2} = \left(1 + \frac{x}{U t_0} \right)^{1-5\alpha}$$

According to Taylor's assumption expressed in Eq 98, $u'' \lambda^2 / \nu$ should be constant. This is true again only when $\alpha = \frac{1}{5}$

Later (1938), von Kármán and Howarth (124 and 125) also considered the triple correlations

$$h(r,t) = \frac{\overline{v'_1{}^2 u'_2}}{u''^3}$$

$$h_1(r,t) = \frac{\overline{u'_1{}^2 u'_2}}{u''^3}$$

$$h_2(r,t) = \frac{\overline{u'_1 v'_1 u'_2}}{u''^3}$$

where the two points 1 and 2 are lined along the direction of u' . He then showed that the general triple correlation tensor T is a function of X, Y, Z , and t , that in isotropic turbulence T is expressible in terms of $h(r,t)$, $h_1(r,t)$, and $h_2(r,t)$, and that the development of these functions in powers of r begins with the r^3 term. The equation of continuity permits the expression of h_1 and h_2 in terms of h by the relations:

$$h_1 = -2h \quad (133)$$

$$h_2 = -h - \frac{r}{2} \frac{dh}{dr} \quad (134)$$

Thus the tensor T can be expressed solely in terms of the scalar function $h(r,t)$.

To investigate the propagation of the correlation with time, the fluctuations are assumed to satisfy the equations of motion, namely,

$$\frac{\partial u_1'}{\partial t} + u_1' \frac{\partial u_1'}{\partial x_1} + v_1' \frac{\partial u_1'}{\partial y_1} + w_1' \frac{\partial u_1'}{\partial z_1} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_1} + \nu \nabla^2 u_1' \quad (135)$$

and two other equations obtained by permutation. Multiplying this equation by u_2' and introducing $X, Y,$ and $Z,$ and taking mean

$$\overline{u_2' \frac{\partial u_1'}{\partial t}} - \frac{\partial \overline{(u_1'^2 u_2')}}{\partial X} - \frac{\partial \overline{(u_1' v_1' u_2')}}{\partial Y} - \frac{\partial \overline{(u_1' w_1' u_2')}}{\partial Z} = -\frac{1}{\rho} \overline{\frac{\partial p'}{\partial x_1} u_2'} + \nu \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \right) \overline{u_1' u_2'} \quad (136)$$

By a similar procedure:

$$\overline{u_1' \frac{\partial u_2'}{\partial t}} + \frac{\partial \overline{(u_2'^2 u_1')}}{\partial X} + \frac{\partial \overline{(u_2' v_2' u_1')}}{\partial Y} + \frac{\partial \overline{(u_2' w_2' u_1')}}{\partial Z} = -\frac{1}{\rho} \overline{\frac{\partial p'}{\partial x_2} u_1'} + \nu \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \right) \overline{u_1' u_2'} \quad (137)$$

Von Kármán showed that the pressure terms vanish. Adding the last two equations and introducing the correlation coefficient $R_{11},$ (remembering that $\overline{u_1'^2 u_2'} = -\overline{u_2'^2 u_1'},$ etc.), one has:

$$\frac{\partial}{\partial t} (\overline{u_1'^2} R_{11}) - 2 \left[\frac{\partial}{\partial X} \overline{(u_1'^2 u_2')} + \frac{\partial}{\partial Y} \overline{(u_1' v_1' u_2')} + \frac{\partial}{\partial Z} \overline{(u_1' w_1' u_2')} \right] = 2\nu \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \right) \overline{u_1' u_2'} \quad (138)$$

This equation may be expressed in terms of the functions $f, g, h_1, h_2,$ and $h.$ Remembering the relation between h_1, h_2 and $h,$ a relation between f and h is obtained.

$$\frac{\partial (f u''^2)}{\partial t} + 2 u''^3 \left(\frac{\partial h}{\partial r} + \frac{4h}{r} \right) = 2\nu u''^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{4}{r} \frac{\partial f}{\partial r} \right) \quad (139)$$

This equation expresses the change of f with t but cannot be solved without some knowledge of the function $h.$

3. Taylor's spectrum theory

"The description of turbulence in terms of intensity and scale resembles the description of the molecular motion of a gas by temperature and mean free path. A more detailed picture can be obtained by considering the dis-

tribution of energy among eddies of different sizes, or more conveniently the distribution of energy with frequency. Just as a beam of white light may be separated into a spectrum by the action of a prism or grating, the electric current produced by a hot wire anemometer subjected to the speed fluctuations may be analyzed by means of electric filters into a spectrum." -- Dryden (92, 1943).

The spectrum analysis offers a good field for the application of Fourier transforms. As far as the present writer is aware, G. I. Taylor (199, 1938) was the first to develop a spectrum theory of turbulence.

With $\overline{u'^2} = u''^2$ expressed in the form

$$u''^2 = u''^2 \int_0^{\infty} F(n) dn \quad (140)$$

where $F(n)$ is the contribution from frequencies between n and $n+dn$ and where

$$\int_0^{\infty} F(n) dn = 1 \quad (141)$$

The function $F(n)$ plotted against n gives the spectrum curve. Expressing u' in the form of a Fourier integral,

$$u' = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} u' \cos \omega (t-t') dt' d\omega \quad (142)$$

where $\omega = 2\pi n$ is the angular velocity associated with n , and writing

$$I_1 = \frac{1}{2\pi} \int_{-T}^T u' \cos \omega t dt = \frac{1}{2\pi} \int_{-T}^T u' \cos 2n\pi t dt \quad (143)$$

$$I_2 = \frac{1}{2\pi} \int_{-T}^T u' \sin \omega t dt = \frac{1}{2\pi} \int_{-T}^T u' \sin 2n\pi t dt \quad (144)$$

one seeks to express $F(n)$ in terms of I_1 and I_2 . This can be done in a simple manner by breaking $u'(t)$ into two parts, one being an even and the other an odd function of t .

$$u'(t) = E(t) + O(t)$$

where

$$E(t) = \frac{u'(t) + u'(-t)}{2}; \quad O(t) = \frac{u'(t) - u'(-t)}{2} \quad (145)$$

The Fourier cosine transform for the even function $E(t)$ is

$$\begin{aligned} g(\omega) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} E(t) \cos \omega t \, dt \\ &= \sqrt{\frac{1}{2\pi}} \int_0^{\infty} [u'(t) + u'(-t)] \cos \omega t \, dt \\ &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} u'(t) \cos \omega t \, dt = \sqrt{2\pi} \lim_{T \rightarrow \infty} I_1 \end{aligned} \quad (146)$$

Similarly, the Fourier sine transform of the odd function $O(t)$ is:

$$h(\omega) = \sqrt{2\pi} \lim_{T \rightarrow \infty} I_2 \quad (147)$$

Now, according to a theorem in Fourier integrals which corresponds to the Parseval's theorem in Fourier series,

$$\int_0^{\infty} E^2 dt = \lim_{T \rightarrow \infty} \int_0^T E^2 dt = \int_0^{\infty} g^2(\omega) d\omega = 2\pi \lim_{T \rightarrow \infty} \int_0^T I_1^2 d\omega \quad (148)$$

Similarly

$$\int_0^{\infty} [O(t)]^2 dt = 2\pi \lim_{T \rightarrow \infty} \int_0^T I_2^2 d\omega \quad (149)$$

But

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (E^2 + O^2) dt &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{[u'(t)]^2 + [u'(-t)]^2}{2} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [u'(t)]^2 dt = u''^2 \end{aligned} \quad (150)$$

So by virtue of the last three equations,

$$\begin{aligned} u''^2 &= 2\pi \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (I_1^2 + I_2^2) d\omega = 2\pi \int_0^{\infty} \lim_{T \rightarrow \infty} \left(\frac{I_1^2 + I_2^2}{T} \right) d\omega \\ &= 4\pi^2 \int_0^{\infty} \lim_{T \rightarrow \infty} \left(\frac{I_1^2 + I_2^2}{T} \right) dn \end{aligned} \quad (151)$$

Comparing Eq 151 and 140

$$F(n) = \frac{4\pi^2}{u''^2} \lim_{T \rightarrow \infty} \frac{I_1^2 + I_2^2}{T} \quad (152)$$

When the fluctuations are superposed on a stream of mean velocity U and are small compared with U , the changes in u' at a fixed point may be regarded as due to the passage of a fixed turbulent pattern over the point, that is, it may be assumed that

$$u' = \phi(t) = \phi(x/U) \quad (153)$$

where x is measured upstream from the fixed point, and $\phi(x/U)$ may be considered to be the instantaneous space distribution of u' at time $t = 0$.

The correlation R_x between the fluctuations at time t and $t + x/U$ is defined by

$$R_x = \frac{\overline{\phi(t) \phi(t + x/U)}}{u'^2} = \lim_{T \rightarrow \infty} \frac{\int_{-T}^T \phi(t) \phi(t + x/U) dt}{2T u'^2} \quad (154)$$

A theorem in the theory of Fourier transform states that if $g(\omega)$ and $h(\omega)$ are the Fourier cosine transforms, respectively, of the even functions $E_1(t)$ and $E_2(t)$, then

$$\int_0^{\infty} g(\omega) h(\omega) d\omega = \int_0^{\infty} E_1(t) E_2(t) dt \quad (155)$$

A similar theorem exists for the odd functions. By expressing $\phi(t)$ and $\phi(t + x/U)$ as sums of an even and an odd function, it can be proved by virtue of these theorems that

$$\int_{-\infty}^{\infty} \phi(t) \phi(t + x/U) dt = 8 \pi^2 \int_0^{\infty} (I_1^2 + I_2^2) \cos(2\pi nx/U) dn \quad (156)$$

from which one obtains, with Eq 152 and 154,

$$R_x = \int_0^{\infty} F(n) \cos(2\pi nx/U) dn \quad (157)$$

and

$$F(n) = (4/U) \int_0^{\infty} R_x \cos(2\pi nx/U) dx \quad (158)$$

In other words, the correlation coefficient R_x and $UF(n)/\sqrt{8\pi}$ are Fourier cosine transforms. If either is measured, the other may be computed. The correlation coefficient R_x is actually the function f in von Kármán's theory, so that the length λ is related to R_x by

$$\frac{1}{\lambda^2} = 2 \lim_{x \rightarrow 0} \frac{1 - R_x}{x^2} \quad (159)$$

When n and x are small

$$\cos \frac{2\pi nx}{U} \doteq 1 - \frac{2\pi^2 n^2 x^2}{U^2}$$

Hence

$$\frac{1}{\lambda^2} = \frac{4\pi^2}{U^2} \int_0^{\infty} n^2 F(n) dn \quad (160)$$

If the turbulence is self-preserving, the shape of the correlation curve is a function of the Reynolds number of the turbulence. Hence the spectrum curve is also a function of the Reynolds number of the turbulence. Introducing the longitudinal scale

$$L_x = \int_0^{\infty} R_x \, dx$$

in Eq 160

$$\frac{L_x^2}{\lambda^2} = 4 \pi^2 \int_0^{\infty} \left(\frac{nL_x}{U} \right)^2 \frac{UF(n)}{L_x} \, d \left(\frac{nL_x}{U} \right) \quad (161)$$

and in Eq 158,

$$\frac{UF(n)}{L_x} = 4 \int_0^{\infty} R_x \cos \left(\frac{2 \pi nL_x}{U} \frac{x}{L_x} \right) d \left(\frac{x}{L_x} \right) \quad (162)$$

both of which are expressed in the dimensionless parameters $\frac{\lambda}{L_x}$,

nL_x/U , $UF(n)/L_x$, x/L_x , and R_x . The mean speed U enters only in fixing the frequency scale. Typical curves of $UF(n)/L_x$ vs. nL_x/U obtained by the National Bureau of Standards and by the National Physical Laboratory show, however, that the relation between the two variables is not independent of U at large values of nL_x/U . Since from Eq 161 the value of L_x/λ is determined largely by the value of $UF(n)/L_x$ at large values of nL_x/U , L_x/λ must not be independent of U , as is known to be true. Indeed it was from this established dependence of L_x/λ on U that the scatter of the $UF(n)/L_x$ vs. nL_x/U curves at large values of nL_x/U was attributed to the influence of U .

When the Reynolds number of turbulence is large, λ/L_x becomes small. Experimental measurements show that both R_x and R_y curves approach exponential curves, and Eq 162 for the corresponding spectrum curve becomes:

$$\frac{UF(n)}{L_x} = \frac{4}{1 + 4 \pi^2 n^2 L_x^2 / U^2} \quad (163)$$

This can be used to give a reference spectrum curve to compare with

experimental data. As U decreases, λ increases, and the departures from Eq 163 at large values of nL_x/U become greater. The changes in the total energy of the fluctuations associated with these departures in the spectrum at high frequencies are extremely small.

Using Eq 163, it is possible to compute the effect of varying the cut-off frequency of the measuring equipment on the measured value of the energy of the fluctuation. If the equipment passes high frequencies but cuts off sharply at a lower frequency n_0 , the measured total energy is

$$\frac{1}{2} \rho u''^2 \int_{n_0 L_x / U}^{\infty} \frac{4(L_x / U) dn}{1 + 4\pi^2 n^2 L_x^2 / U^2} = \left(1 - \frac{4}{2\pi} \tan^{-1} \frac{2\pi n_0 L_x}{U}\right) \frac{1}{2} \rho u''^2 \quad (164)$$

Similarly, if the equipment passes low frequencies but cuts off sharply at a higher frequency n_h , the measured total energy is

$$\frac{4}{2\pi} \tan^{-1} \left(\frac{2\pi n_h L_x}{U} \right) \cdot \frac{1}{2} \rho u''^2 \quad (165)$$

In Taylor's discussion of discontinuous random motion (193, 1921) the exponential function was found to be the limiting form of the correlation coefficient of small paths. Therefore, the fact that the correlation curves are of the exponential type at high Reynolds numbers of turbulence can be interpreted as meaning that, at such high Reynolds number of turbulence, turbulence is a phenomenon of pure chance.

4. Burgers' spectrum theory

Due to the limitations on the extent of the present work, it is impossible to include all of the numerous and beautiful works of Professor Burgers. The following sections pertaining to the spectrum of turbulence are taken from his lecture notes issued while he was visiting at the California Institute of Technology between December 1950 and May 1951. For his other works one is referred to (77, 1929; 77a, 1948; 77b, 1950; 77c, 1950).

(a) The instantaneous spectrum of a turbulent field. Restricting to a single coordinate y , the velocity of fluctuation $v'(y, t)$ in the Eulerian description can be represented as (at a single instant)

$$v' = \int_{-\infty}^{\infty} \varphi(k) e^{iky} dk \quad (166)$$

from which the amplitude function $\varphi(k)$ is obtained:

$$\varphi(k) = \frac{1}{2\pi} \int_{-M}^M v'(y) e^{-iky} dy \quad (167)$$

where only values of v' in the region $-M \leq y \leq M$ are considered. Since v' is real, the amplitude function must satisfy the relation

$$\varphi(-k) = \varphi^*(k) \quad (168)$$

where the asterisk denotes the complex conjugate. The amplitude function is in fact a function of k and of time t . One will, for the time being, restrict one's considerations to the single instant.

To obtain the Eulerian correlation function $R_2(\eta) = R(\eta, 0)$ (where the zero means the time interval is zero and consequently the correlation is between simultaneous values only), one forms

$$v'_1 v'_2 = \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \varphi(k_1) \varphi(k_2) e^{i(k_1+k_2)y + ik_2\eta} \quad (169)$$

Taking mean over $-M_1 \leq y \leq M_1$ and using $M_1 > M$ ($\frac{1}{M}$ is used since $v'=0$ for $|y|>M$)

$$\overline{v'_1 v'_2} = \frac{1}{M} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \varphi(k_1) \varphi(k_2) e^{ik_2\eta} \frac{\sin(k_1+k_2)M_1}{k_1+k_2} \quad (170)$$

Writing k for k_1 and k_3 for $k_1 + k_2$, and using Eqs 167 and 168:

$$\begin{aligned} \overline{v'_1 v'_2} &= \frac{1}{M} \int_{-\infty}^{\infty} dk \varphi(k) e^{-ik\eta} \int_{-\infty}^{\infty} dk_3 \varphi(k_3 - k) e^{ik_3\eta} \frac{\sin k_3 M_1}{k_3} \\ &= \frac{1}{2\pi M} \int_{-\infty}^{\infty} dk \varphi(k) e^{-ik\eta} \int_{-\infty}^{\infty} dk_3 e^{ik_3\eta} \frac{\sin k_3 M_1}{k_3} \int_{-M}^M v'(y) e^{-i(k_3-k)y} dy \\ &\rightarrow \frac{\pi}{M} \int_{-\infty}^{\infty} dk \varphi(k) e^{-ik\eta} \varphi(k) \\ &\rightarrow \frac{2\pi}{M} \int_0^{\infty} \varphi(k) \varphi^*(k) \cos k\eta dk \quad \text{as } M_1 \rightarrow \infty \quad (171) \end{aligned}$$

Since $\lim_{M_1 \rightarrow \infty} \int_{-\infty}^{\infty} F(k_3) (\sin k_3 M_1 / k_3) dk_3 = \pi F(0)$ and $\varphi(-k) = \varphi^*(k)$

Hence

$$R_2(\eta) = \frac{\overline{v_1' v_2'}}{\overline{v'^2}} = \frac{1}{\overline{v'^2}} \int_0^\infty \Gamma(k) \cos k\eta dk \quad (172)$$

where

$$\Gamma(k) = \frac{2\pi}{M} \varphi(k) \varphi^*(k) \quad (173)$$

Since $R_2(\eta)$, and therefore $\Gamma(k)$, should by nature be independent of M , $\varphi(k)$ must be proportional to $M^{\frac{1}{2}}$.

For $\eta = 0$,

$$\overline{v'^2} = \int_0^\infty \Gamma(k) dk \quad (174)$$

so that the kinetic energy of turbulence is

$$E = \frac{1}{2} \int_0^\infty \Gamma(k) dk \quad (175)$$

and the function $\Gamma(k)$ gives the energy spectrum of the turbulence.

If $R_2(\eta)$ is known, $\Gamma(k)$ can be obtained from the inversion of Eq 172:

$$\Gamma(k) = \frac{2\overline{v'^2}}{\pi} \int_0^\infty R_2(\eta) \cos k\eta d\eta \quad (176)$$

It must not be forgotten that Eqs 166 to 175 refer to a single instant and the energy spectrum is an instantaneous spectrum.

(b) Spectrum of a homogeneous and stationary turbulence field. The

expression for $v'(y,t)$ can be generalized to

$$v' = \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega f(k, \omega) e^{i(ky + \omega t)} \quad (177)$$

This representation shall be valid for $-M < y < +M$; $-T < t < T$, outside of this domain v' is assumed to vanish.

The complete Eulerian correlation function can be shown to be (the derivation being the same as that for Eq 171)

$$\begin{aligned} R(\eta, \tau) &= \frac{\pi^2}{MT\overline{v'^2}} \int_{-\infty}^{\infty} d\omega f(k, \omega) f(-k, -\omega) e^{-i(k\eta + \omega\tau)} \\ &= \frac{1}{\overline{v'^2}} \int_0^\infty dk \int_0^\infty d\omega [F(k, \omega) \cos(k\eta + \omega\tau) + G(k, \omega) \cos(k\eta - \omega\tau)] \end{aligned} \quad (178)$$

where $\overline{v'^2} = \overline{v''^2}$, and

$$F(k, \omega) = \frac{2\pi^2}{MT} f(k, \omega) f(-k, -\omega) \quad (179)$$

$$G(k, \omega) = \frac{2\pi^2}{MT} f(k, -\omega) f(-k, \omega) \quad (180)$$

For $\tau = 0$, $R(\eta, t)$ becomes $R(\eta, 0)$ or $R_2(\eta)$ and

$$\Gamma(k) = \int_0^\infty d\omega [F(k, \omega) + G(k, \omega)] \quad (181)$$

For $\eta = 0$

$$R_1(\tau) = R(0, \tau) = \frac{1}{v''^2} \int_0^\infty d\omega \Gamma^*(\omega) \cos \omega \tau \quad (182)$$

where

$$\Gamma^*(\omega) = \int_0^\infty dk [F(k, \omega) + G(k, \omega)] \quad (183)$$

The inverse of Eq 182 is

$$\Gamma^*(\omega) = \frac{2v''^2}{\pi} \int_0^\infty d\tau R_1(\tau) \cos \omega \tau \quad (184)$$

The function $\Gamma^*(\omega)$ gives the energy spectrum with reference to frequencies in time, whereas $\Gamma(k)$ gives the spectrum with reference to frequencies in space.

(c) Physical interpretation of the relation between the spectrum and the correlation function. The electrical signal obtained from the hot-wire anemometer, after having been amplified, is passed through an electric filter. The transmitted signal can be applied to a thermo-cross, by means of which its mean square can be determined. The relation between the incoming electric signal $v'(t)$ and the outgoing signal $y(t)$ is

$$\frac{d^2y}{dt^2} + 2p\omega \frac{dy}{dt} + \omega^2 y = \alpha \frac{dv'}{dt} \quad (185)$$

where w , p , and α are quantities depending on the circuit. It has been supposed that the incoming signal operates through induction, so that only $\frac{dv'}{dt}$ appears in Eq 185.

When, in particular,

$$v'(t) = Ae^{int}$$

one has

$$y(t) = \frac{A \alpha n e^{int}}{(\omega^2 - n^2) + 2ipn\omega}$$

Hence, if $p \ll 1$, the only frequencies which are transmitted through the filter are those which differ only slightly from ω . The band of transmitted frequencies has a width proportional to ωp . If one takes $n = \omega$, the above equation becomes

$$y(t) = \frac{\alpha}{2p\omega} A e^{int}$$

so that in order to have a constant scale factor, one must take α proportional to ω .

For arbitrary $v'(t)$, the particular solution of Eq 184 can be derived in the following way. First, it is easy to show that the complementary solution is, with ϵ denoting an arbitrary angle,

$$y(t) = C e^{-p\omega t} \cos(\omega t \sqrt{1-p^2} + \epsilon) = C e^{-p\omega t} \cos f(t)$$

where

$$f(t) = \omega t \sqrt{1-p^2} + \epsilon$$

Try the particular solution

$$y(t) = A \int_0^\infty dt_1 v'(t-t_1) e^{-p\omega t_1} \cos f(t_1) \quad (186)$$

Then

$$y'(t) = -A \int_0^\infty dt_1 \frac{dv'(t-t_1)}{dt_1} e^{-p\omega t_1} \cos f(t_1)$$

and integrating by parts,

$$y'(t) = A v' \cos \epsilon - A \int_0^\infty dt_1 v'(t-t_1) e^{-p\omega t_1} [p\omega \cos f(t_1) + \omega \sqrt{1-p^2} \sin f(t_1)]$$

Further

$$y''(t) = A \frac{dv'}{dt} \cos \epsilon - A v' (p\omega \cos \epsilon + \omega \sqrt{1-p^2} \sin \epsilon) + A \int_0^\infty dt_1 v'(t-t_1) e^{-p\omega t_1} [(2p^2\omega^2 - \omega^2) \cos f(t_1) + 2\omega^2 p \sqrt{1-p^2} \sin f(t_1)]$$

so

$$\frac{d^2 y}{dt^2} + 2p\omega \frac{dy}{dt} + \omega^2 y = A \frac{dv'}{dt} \cos \epsilon + A \omega v' (p \cos \epsilon - \sqrt{1-p^2} \sin \epsilon)$$

This should be equal to $\alpha \frac{dv'}{dt}$ in order that Eq 185 is satisfied, so that

$$\sin \epsilon = p \quad A = \frac{\alpha}{\cos \epsilon} = \frac{\alpha}{\sqrt{1-p^2}}$$

Thus the particular solution sought is

$$y(t) = \frac{\alpha}{\sqrt{1-p^2}} \int_0^\infty dt_1 v'(t-t_1) e^{-p\omega t_1} \cos(\omega t \sqrt{1-p^2} + \sin^{-1} p) \quad (187)$$

Since when $\frac{dv'}{dt} = 0$ the outgoing signal $y(t)$ is identically zero, the complimentary solution is eliminated from consideration,

The correlation function for the outgoing signal is then

$$\overline{y(t)y(t+\tau)} = \frac{\alpha^2}{1-p^2} \int_0^\infty dt \int_0^\infty dt_2 \overline{v'(t-t_1)v'(t+\tau-t_2)} e^{-p\omega(t_1+t_2)} \cos f(t_1) \cos f(t_2) \quad (188)$$

where

$$f(t) = \omega t \sqrt{1-p^2} + \sin^{-1} p$$

Eq 187 can be transformed by introducing $\beta = t_2 + t_1$, and $\delta = t_2 - t_1$ to the following form

$$\overline{y(t)y(t+\tau)} = \frac{\alpha^2 v'^2}{4p\omega\sqrt{1-p^2}} \int_0^\infty d\delta [R_1(\tau+\delta) + R_2(\tau-\delta)] e^{-p\omega\delta} \cos f(\delta) \quad (189)$$

where

$$R_1(t) = \frac{\overline{v'(t)v'(t-t)}}{v'^2}$$

Putting $\tau = 0$ and letting p be so small that

$$\cos \epsilon = \sqrt{1-p^2} \doteq 1$$

one obtains

$$\overline{y^2} = \frac{\alpha^2 v'^2}{2p\omega} \int_0^\infty d\delta R_1(\delta) e^{-p\omega\delta} \cos(\omega\delta + \epsilon) \quad (190)$$

If now one writes

$$\Gamma_I(\omega) = \int_0^\infty d\tau R_1(\tau) e^{-p\omega\tau} \cos \omega\tau$$

$$\Gamma_{II}(\omega) = \int_0^\infty d\tau R_1(\tau) e^{-p\omega\tau} \sin \omega\tau$$

then from Eq 182, for $p \ll 1$, $\Gamma_I(\omega)$ will differ only slightly from $\Gamma^*(\omega)$,

so long as ω is not so large that $p\omega\tau$ will become comparable to 1 in the

range where $R_1(t)$ has not yet dropped to zero. Usually one may expect that

$\Gamma_{II} < \Gamma_I$. If one now arranges the circuit so that α is proportional to $\omega\sqrt{p}$, one has

$$\overline{y^2} \doteq \text{constant} \cdot [\Gamma_I(\omega) - p\Gamma_{II}(\omega)] \quad (190a)$$

so that if p is sufficiently small (i.e. the filter is sufficiently selective)

$\overline{y^2}$ is nearly proportional to $\Gamma_I(\omega)$ or to $\Gamma^*(\omega)$.

(d) Heat transfer. Vapor transfer in a turbulent medium follows much the same laws as momentum transfer. Certain details of heat transfer, however, require separate consideration.

If the temperature of an element of volume is T , the transport of heat, per unit area and in unit time, is given by $c_v \overline{\rho w T}$ where c_v is the specific heat at constant volume and w is the instantaneous velocity of the particle in the z -direction, or its fluctuations if the mean value of w is zero.

At the same time, work is done by the pressure to the amount

$$\overline{p w} = R \overline{\rho w T}$$

by virtue of the equation of state $p = R \rho T$, R being the gas constant.

Hence the total transport of energy is given by

$$Q = c_p \overline{\rho w T}$$

where $c_p = c_v + R$ is the specific heat at constant pressure.

The temperature of an element of volume, during its random movement, does not only change through conduction but also in consequence of expansion or contraction as it comes into regions of different pressure. There will be a systematic effect connected with the mean pressure gradient in the field, which itself is connected with gravity. For adiabatic processes,

$$T \sim p^{\frac{\gamma-1}{\gamma}}$$

where $\gamma = c_p/c_v$. From the above proportionality one has

$$\begin{aligned} \frac{d\bar{T}}{dt} &= \frac{\gamma-1}{\gamma} \frac{\bar{T}}{\bar{p}} \frac{d\bar{p}}{dt} = \frac{\gamma-1}{\gamma} \frac{\bar{T}}{\bar{p}} w \frac{d\bar{p}}{dz} = \frac{\gamma-1}{\gamma} g \frac{\bar{\rho} \cdot \bar{T}}{\bar{p}} w \\ &= -\frac{\gamma-1}{\gamma R} g w = -\Gamma w \end{aligned}$$

where $\Gamma = -\frac{\gamma-1}{\gamma R} g$ is the adiabatic lapse rate of the temperature, R being the gas constant, and \bar{T} the mean value of T .

Denoting by T the absolute temperature of a particle and \bar{T} the temperature of its surrounding, the rate of change of T is

$$\frac{dT}{dt} = \lambda(\bar{T} - T) - \Gamma w + \psi \quad (191)$$

where λ is a coefficient inversely proportional to the conductivity of the fluid, and ψ takes into account the influences of radiation or of condensation.

Denoting by b the quantity $\frac{dT}{dz}$, the mean temperature at the level where the particle finds itself at the instant $t - t_1$ is

$$\bar{T}(t - t_1) = c + bz_p - b \int_0^{t_1} w(t - t_2) dt_2$$

where z_p denotes the elevation at which the particle finds itself at time t .

Eq 191 can then be integrated to yield

$$T - \bar{T} = -(b + \Gamma) \int_0^\infty dt_1 e^{-\lambda t_1} w(t - t_1) + \int_0^\infty d\lambda e^{-\lambda t_1} \psi(t - t_1)$$

The net transport of heat across the horizontal plane containing z_p is then

$$H = -(b + \Gamma) c_p \bar{\rho} \int_0^\infty dt_1 e^{-\lambda t_1} \overline{w(t-t_1) w(t)} + c_p \bar{\rho} \int_0^\infty dt_1 e^{-\lambda t_1} \overline{\psi(t-t_1) w(t)}$$

in which $\bar{\rho}$ has been taken out of the averaging sign since the error introduced by doing can be neglected. The last equation shows, qualitatively at least, the effect of conductivity on the transport of heat.

5. A remark on Reichardt's inductive method

In 1942, Hans Reichardt at Göttingen published a paper (156) describing an inductive method for studying the characteristics of free turbulence. Essentially, the velocity components are decomposed into two parts; the mean value and the fluctuation. In this way the Navier-Stokes equations of motion and the equation of continuity, coupled with the assumption of similarity, provide certain relationships involving the mean values and the fluctuations. By measuring the primary quantities, all the other quantities of interest can be computed. This method does not seek to predict as the transfer theories, nor does it reveal the microscopic structure of the turbulence as the statistical theories since only mean values are measured, and consequently is in nature largely empirical.

CHAPTER II. REYNOLDS' ANALOGY AND ITS EXTENSIONS

According to Reynolds (1874) there is a complete analogy between momentum transfer and the transfer of heat or of vapor in turbulent flow. In Reynolds' analogy, the effect of the boundary layer (which consists of the laminar sub-layer and the buffer zone) is not considered, thus introducing errors which are not negligible when the Prandtl number is large. Extensions of Reynolds' analogy by various research workers represent different ways of improving the analogy by taking the boundary layer into account. Since among other things pressure gradient has not been considered in these extensions, they in turn have their limitations. This will be pointed out in detail after Reynolds' analogy and its extensions are presented in the following.

A. Reynolds' Analogy

It has already been remarked that if the eddy viscosity is adopted, the quantity $\epsilon + \nu$ should replace ν in the equations of motion and the quantity $\nu/\sigma \epsilon$ should replace $\frac{\nu}{\sigma}$ (σ being the Prandtl number) in the energy equation or the equation of diffusion. Observing that ϵ is very much larger than ν or $\frac{\nu}{\sigma}$, Reynolds (1874) made the statement that the velocity-distribution is the same as the distribution of the quantity under diffusion, i.e. there exists a complete analogy between momentum transfer, heat transfer, and vapor transfer. That this is not always true was remarked in Part I. Reynolds' analogy does not hold when there is a pressure gradient, and when the conditions $\epsilon \gg \nu$ and $\epsilon \gg \frac{\nu}{\sigma}$ are not satisfied throughout the fluid. Thus, Reynolds' analogy is invalid near a solid boundary.

In order to explain the Reynolds analogy in more detail, the following two equations for parallel mean motion will be considered:

$$\tau = \mu \frac{du}{dy} - \rho \overline{u'v'} \quad (192)$$

$$q = -k \frac{dT}{dy} + c \rho \overline{v'T'} \quad (193)$$

where q denotes the rate of heat transfer per unit area per unit time, k is the thermal conductivity, c is the specific heat (at constant pressure for gases), T is the temperature, primes denote fluctuations, and bars have been omitted from the mean quantities. Adopting ϵ , Eqs 192 and 193 can be written as

$$\tau/\rho = (\nu + \epsilon) \frac{du}{dy} \quad (194)$$

$$q/c\rho = \left(\frac{\nu}{\sigma} + \epsilon\right) \frac{dT}{dy} \quad (195)$$

where $\frac{\nu}{\sigma} = \alpha = k/c\rho$ is the thermal diffusivity. It is of course not clear a priori that ϵ should be the same in Eqs 194 and 195, but preliminary experiments with two-dimensional air flow by Corcoran, Roudebush, and Sage (80, 1947) seem to confirm that the same ϵ can be used.

If then $\nu = \alpha$ (or $\sigma = 1$), and the physical situations are completely similar, Eqs 194 and 195 guarantee the analogy between τ and q/c , and between u and T , and give

$$(u_2 - u_1)/\tau = c(T_2 - T_1)/q \quad (196)$$

where u_1 and u_2 are two velocities and T_1 and T_2 the corresponding temperatures. The condition $\nu = \alpha$ can be replaced by the conditions $\epsilon \gg \nu$ and $\epsilon \gg \alpha$.

Let the dimensionless coefficients C_f and C_H be defined by the following:

$$\tau = C_f \left(\frac{\rho}{2} U^2\right) \quad (197)$$

$$q = C_H \rho c U \Delta T \quad (198)$$

where U is a representative velocity and ΔT a representative temperature difference. Eq 196 becomes, since the velocity at the boundary is zero,

which yields

$$C_H = C_f / 2 \quad (199)$$

This important formula has two important consequences

- (i) If $\tau \sim U^n$, then $q \sim U^{n-1}$
- (ii) Roughness increases the friction and heat transfer in the same ratio.

B. Taylor-Prandtl's Extension

In case ν and α are very different, as in the case of liquids, and when their effects cannot be neglected, Reynolds' analogy will be longer be valid without some extension. The first attempt was made by G. I. Taylor (192, 1916). Denoting by T_0 the wall temperature and by u_δ and T_δ the velocity and temperature at the outer edge of the laminar sub-layer, one obtains from Eqs 194 and 195 by omitting ϵ :

$$u_\delta = \frac{\tau}{\rho} \int_0^\delta \frac{dy}{\nu}$$

$$T_\delta - T_0 = \frac{q}{c\rho} \int_0^\delta \frac{\sigma dy}{\nu}$$

and therefore

$$\frac{u_\delta}{\tau} = \frac{c(T_\delta - T_0)}{q\sigma} \quad (200)$$

In the turbulent zone, Eq 196 provides

$$(U - u_\delta) / \tau = c(T_m - T_\delta) / q_f \quad (201)$$

when U and T_m are the mean values of u and T over the pipe section.

Eliminating T_δ between Eqs 200 and 201, one obtains

$$\frac{c(T_m - T_0)}{q_f} = \frac{U}{\tau} + (\sigma - 1) \frac{u_\delta}{\tau}$$

or

$$\frac{1}{C_H} = \frac{2}{C_f} \left[1 + \frac{u_\delta}{U} (\sigma - 1) \right] \quad (202)$$

This result was obtained by Taylor in 1916 (192). Taylor, starting from an equation ^{on} eddy formation by H. A. Lorentz, obtained $u_\delta / U = 0.38$ for flat surfaces, but 0.56 from Stanton and Pannell's experiments for the flow in tubes.

The weak point in the above equation is the ratio u_δ/U , about which there is a whole literature, see (192, 1916), (150, 1928), (112, 1933), (74, 1936), (107, 1937) and (38, 1942). Prandtl (150, 1928) took

$$u_\delta/U = BR^{-1/8} \quad (203)$$

with $B = 1.74$ (but considered $B = 1.1$ to 1.2 as better), and obtained

$$C_H = (C_f/2) \frac{1}{1 + 1.74 R^{-1/8} (\sigma - 1)} \quad (204)$$

Setting $\frac{C_f}{2} = 0.04 R^{-1/4}$, he further obtained

$$Nu = \frac{0.04 R^{3/4} \sigma}{1 + 1.74 R^{-1/8} (\sigma - 1)} \quad (205)$$

where $Nu = \frac{hL}{k}$ is the Nusselt number, L being a length, k being the thermal conductivity, and h , the coefficient of heat transfer, being equal to $q/\Delta T$. It can be easily checked from Eq 198 that $Nu = C_H R \sigma$.

In general, it can be assumed that $u_\delta/(\tau/\rho)^{1/2}$ is a universal constant for smooth surfaces. On the other hand,

$$U/(\tau/\rho)^{1/2} = (2/C_f)^{1/2}$$

So

$$u_\delta/U = \text{constant}$$

and

$$1/C_H = (2/C_f) + \text{constant} (2/C_f)^{1/2} (\sigma - 1) \quad (206)$$

It may be noted that in the Taylor-Prandtl analogy, $\epsilon \neq 0$ at the junction of the laminar and turbulent regions, so that there is a discontinuity of the quantity $\nu + \epsilon$ at that place.

Experiments show that Eqs 204 and 206 are good only for $\sigma \approx 1$. For larger values of σ , the increasing discrepancy is to be avoided by taking account of the transition zone between the laminar and turbulent regions.

C. von Kármán's Extension

Assuming that $u/(\tau/\rho)^{1/2}$ is a universal function of $\eta = (\tau/\rho)^{1/2} y/\nu$:

$$u/(\tau/\rho)^{1/2} = f(\eta)$$

von Kármán (35, 1939) obtained

$$\frac{du}{dy} = (\tau/\rho)^{\frac{1}{2}} f'(\eta) \frac{(\tau/\rho)^{\frac{1}{2}}}{\nu} = \frac{\tau}{\rho\nu} f'(\eta) \quad (207)$$

or

$$\frac{\tau}{\rho \frac{du}{dy}} = \frac{\nu}{f'(\eta)} \quad (208)$$

On the other hand, from Eq 194 follows

$$\tau/\rho \frac{du}{dy} = \nu + \epsilon \quad (209)$$

Thus, from the last two equations, one obtains

$$\epsilon = \nu (1/f'(\eta) - 1) \quad (210)$$

for the transition zone.

Assuming constant τ , Eq 207 leads to

$$u_{\delta} = (\tau/\rho)^{\frac{1}{2}} \int_0^{\eta_{\delta}} f'(\eta) d\eta \quad (211)$$

Similarly

$$T_{\delta} - T_0 = \frac{q_0}{c\rho(\tau/\rho)^{\frac{1}{2}}} \int_0^{\eta_{\delta}} \frac{d\eta}{\frac{1}{\sigma} - 1 + \frac{1}{f'(\eta)}} \quad (212)$$

where η_{δ} is somewhat arbitrary, but is the same for similar flows. Since the integral in Eq 211 is a pure number and that in Eq 212 a function of σ only, one obtains

$$\frac{c\rho(T_{\delta} - T_0)}{q_0} - \frac{\rho u_{\delta}}{\tau} = \left(\frac{\rho}{\tau}\right)^{\frac{1}{2}} [B(\sigma) - A] \quad (213)$$

For the turbulent zone, Reynolds' analogy gives

$$\frac{\rho(U - u_{\delta})}{\tau} = \frac{c\rho(T_m - T_{\delta})}{q_0} \quad (214)$$

From the last two equations it follows that

$$\frac{c\rho(T_m - T_0)}{q_0} - \frac{\rho U}{\tau} = \left(\frac{\rho}{\tau}\right)^{\frac{1}{2}} [B(\sigma) - A] \quad (215)$$

or

$$\frac{1}{c_H} = \frac{2}{c_f} + \left(\frac{2}{c_f}\right)^{\frac{1}{2}} [B(\sigma) - A] \quad (216)$$

In order to determine $B(\sigma) - A$, it must be noted that

(i) in the laminar sub-layer,

$$u/(\tau/\rho)^{\frac{1}{2}} = \eta \quad (217)$$

(ii) in the turbulent zone, the velocity has the usual logarithmic

distribution

$$u/(\tau/\rho)^{\frac{1}{2}} = 5.5 + 2.5 \ln \eta \quad (218)$$

(iii) in the transition zone, the velocity distribution must be continuous with those in the laminar and turbulent zones.

von Kármán used

$$u/(\tau/\rho)^{\frac{1}{2}} = 5 \left[1 + \ln\left(\frac{\eta}{5}\right) \right] \quad (219)$$

in the transition zone to connect Eq 217 at $\eta = 5$ with the same slope and Eq 218 at $\eta = 30$ with a discontinuity in the slope, giving rise to a discontinuity in $\nu + \epsilon$ at the same place, by Eq 209. This discontinuity, however, occurs at a larger η (and larger $\nu + \epsilon$) than the $\eta (= 13)$ at which the jump in $\nu + \epsilon$ occurs in the Taylor-Prandtl extension of Reynolds' analogy. Since only the reciprocals of $\nu + \epsilon$ and $\frac{\nu}{\sigma} + \epsilon$ occur in the integration of u and T from Eqs 194 and 195, the discontinuity in von Kármán's extension is less detrimental.

Substituting Eqs 217 and 219 into 211 and 212, one finds

$$A = 5(1 + \ln 6). \quad B = 5 \left(6 + \ln(1 + 56) \right)$$

Hence

$$\frac{1}{C_H} = \frac{2}{C_f} + 5 \left(\frac{2}{C_f} \right)^{\frac{1}{2}} \left\{ \sigma - 1 + \ln \left[1 + \frac{5}{6}(\sigma - 1) \right] \right\} \quad (220)$$

For $\sigma = 1$

$$\frac{1}{C_H} = \frac{2}{C_f}$$

as is to be expected from Reynolds' analogy. For values of σ very near 1, expansion of the logarithmic function gives, approximately,

$$\frac{1}{C_H} = \frac{2}{C_f} + 5 \left(\frac{2}{C_f} \right)^{\frac{1}{2}} (\sigma - 1)$$

which is Eq 206 of the Taylor-Prandtl extension.

Taking $\frac{2}{C_f} = 0.04 R^{-\frac{1}{4}}$, one obtains from Eq 220:

$$Nu = \frac{0.04 R^{\frac{3}{4}} \sigma}{1 + R^{-1/8} \left\{ \sigma - 1 + \ln \left[1 + \frac{5}{6}(\sigma - 1) \right] \right\}} \quad (221)$$

Experimental results of Dittus and Boeter (87, 193) gave the following

average equation for cooling and heating:

$$N_{u} = 0.0254 R^{0.8} G^{-0.35} \quad (222)$$

In comparison with Eq 222, Eq 205 gives results altogether too low, while Eq 221 shows very good agreement for $G \leq 10$.

For very large G , the asymptotic forms of Eqs 205 and 221 are respectively:

$$N_{u} = 0.023 R^{7/8}$$

$$N_{u} = 0.104 R^{7/8}$$

No experiments for very large G have been reported to check the validity of the last two equations.

For modifications of von Kármán's conversion formula, see the works of Reichardt (155, 1940), who remeasured the universal velocity distribution in the transition zone, of Boelter and co-authors (72, 1941), and of Martinelli (133, 1947).

D. Hofmann's Extension

Based on Nikuradse's data, E. Hofmann (27, 1940) assumed the following velocity profile:

$$\begin{aligned} \varphi &= \eta & \text{for } 0 \leq \eta \leq 2 \\ \varphi &= \eta - 0.00173 (\eta - 2)^3 & \text{for } 2 \leq \eta \leq 14.3 \\ \varphi &= 3 + 7 \log_{10} \eta & \text{for } 14.3 \leq \eta \leq 25 \end{aligned} \quad (223)$$

where $\varphi = u / v_*$, η being the same as in von Kármán's extension, and $v_* = \sqrt{\tau / \rho}$ is the shear velocity. The thickness of the boundary layer corresponds to $\eta = 25$. Comparison of von Kármán's and Hofmann's velocity distributions for turbulent flow with Nikuradse's data is shown in Fig. 1.

Resolving the shear into the turbulent and laminar components:

$$\tau = \tau_t + \tau_l$$

one has

$$\tau_t / \tau = 1 - \tau_2 / \tau = 1 - \frac{\nu}{v_*^2} \frac{\partial u}{\partial y}$$

on the assumption that τ is equal to the shear stress τ_0 on the wall throughout the boundary layer near the wall. But from Eqs 233,

$$\frac{\partial u}{\partial y} = \frac{v_*^2}{\nu} f(\eta)$$

where

$$f(\eta) = 1 \quad 0 \leq \eta \leq 2$$

$$f(\eta) = 0.9792 + 0.208\eta - 0.00519\eta^2 \quad 2 \leq \eta \leq 14.3$$

$$f(\eta) = 3.04/\eta \quad 14.3 \leq \eta \leq 25$$

Therefore

$$\tau_t / \tau = 1 - f(\eta)$$

with the $f(\eta)$ defined as above.

By the mixing-length theory of Prandtl,

$$\rho v_*^2 = \tau = \tau_2 + \tau_t = \rho l^2 \left(\frac{\partial u}{\partial y} \right)^2 + \rho \nu \frac{\partial u}{\partial y} = \rho \left[l^2 \frac{v_*^2}{\nu^2} f^2(\eta) + \nu \frac{v_*^2}{\nu} f(\eta) \right]$$

so that

$$\frac{l^2 v_*^2}{\nu^2} = \frac{1 - f(\eta)}{f^2(\eta)}$$

Similarly, denoting by q the rate of vapor transfer per unit time per unit area, and by Φ the difference $c - c_0$ where c and c_0 are the vapor concentrations at any point and at the surface,

$$\begin{aligned} \frac{q}{\rho} &= \frac{q_t}{\rho} + \frac{q_1}{\rho} = l^2 \frac{\partial u}{\partial y} \frac{\partial \Phi}{\partial y} + \frac{\nu}{\sigma} \frac{\partial \Phi}{\partial y} = \frac{\nu}{\sigma} \left(\sigma \frac{l^2 v_*^2}{\nu^2} f(\eta) + 1 \right) \frac{\partial \Phi}{\partial y} \\ &= \frac{\nu}{\sigma} \left(\sigma \frac{1 - f(\eta)}{f(\eta)} + 1 \right) \frac{\partial \Phi}{\partial y} = \frac{\nu}{\sigma} \frac{\sigma - f(\eta)(\sigma - 1)}{f(\eta)} \frac{\partial \Phi}{\partial y} \end{aligned}$$

(In heat transfer ρ should be replaced by ρc_p and Φ should stand for $T - T_0$). Introducing the dimensionless variable $\psi = \Phi v_* \rho / q$, the last equation becomes (In heat transfer ψ should be $\Phi v_* \rho c_p / q$)

$$\frac{\partial \psi}{\partial \eta} = \frac{\sigma f(\eta)}{\sigma - f(\eta)(\sigma - 1)}$$

Integrating, section by section, one has the following cases:

Case 1. $\sigma = 1$

$$\psi = \varphi = \begin{cases} \eta & 0 \leq \eta \leq 2 \\ \eta - 0.00173 (\eta - 2)^3 & 2 \leq \eta \leq 14.3 \\ 3 + 7 \log_{10} \eta & 14.3 \leq \eta \leq 25 \end{cases}$$

$$0 \leq \eta \leq 2$$

$$2 \leq \eta \leq 14.3$$

$$14.3 \leq \eta \leq 25$$

Case 2. $\sigma > 1$

$$\psi = \begin{cases} \sigma \eta & 0 \leq \eta \leq 2 \\ \sigma \left[2 - \frac{\eta-2}{\sigma-1} + \frac{13.88\sigma}{(\sigma-1)^{3/2}} \tan^{-1} (0.072(\eta-2)\sqrt{\sigma-1}) \right] & 2 \leq \eta \leq 14.3 \\ 7 \log_{10} \frac{\eta+3.04(1/\sigma-1)}{14.3+3.04(1/\sigma-1)} + \frac{\sigma}{\sigma-1} \left[2\sigma - 14.3 + \frac{13.88\sigma}{\sqrt{\sigma-1}} \tan^{-1} (0.886\sqrt{\sigma-1}) \right] & 14.3 \leq \eta \leq 25 \end{cases}$$

Case 3. $\sigma < 1$

$$\psi = \begin{cases} \sigma \eta & 0 \leq \eta \leq 2 \\ \sigma \left[2 + \frac{\eta-2}{1-\sigma} - \frac{15.98\sigma}{(1-\sigma)^{3/2}} \log_{10} \frac{1+0.072(\eta-2)\sqrt{1-\sigma}}{1-0.072(\eta-2)\sqrt{1-\sigma}} \right] & 2 \leq \eta \leq 14.3 \\ 7 \log_{10} \frac{\eta+3.04(1/\sigma-1)}{14.3+3.04(1/\sigma-1)} + \frac{\sigma}{1-\sigma} \left[14.3 - 2\sigma - \frac{15.98\sigma}{\sqrt{1-\sigma}} \log_{10} \frac{1+0.886\sqrt{1-\sigma}}{1-0.886\sqrt{1-\sigma}} \right] & 14.3 \leq \eta \leq 25 \end{cases}$$

When $\eta = 25$,

$$\psi_{25} = \begin{cases} 12.79 (= \psi_{25}) & \sigma = 1 \\ 7 \log_{10} \frac{7.22\sigma + 1}{3.705\sigma + 1} + \frac{\sigma}{\sigma-1} \left[2\sigma - 14.3 + \frac{13.88\sigma}{\sqrt{\sigma-1}} \tan^{-1} (0.886\sqrt{\sigma-1}) \right] & \sigma > 1 \\ 7 \log_{10} \frac{7.22\sigma + 1}{3.705\sigma + 1} + \frac{\sigma}{\sigma-1} \left[14.3 - 2\sigma - \frac{15.98\sigma}{\sqrt{1-\sigma}} \log_{10} \frac{1+0.886\sqrt{1-\sigma}}{1-0.886\sqrt{1-\sigma}} \right] & \sigma < 1 \end{cases} \quad (224)$$

and

$$u_{25} = \psi_{25} v_* = 12.79 v_*$$

In the fully turbulent zone $\eta > 25$, by Reynolds' analogy

$$\frac{\Phi_m - \Phi_{25}}{q} = \frac{U - u_{25}}{\tau} = \frac{U}{\tau} - 12.79 \frac{v_*}{\tau} \quad (225)$$

where c_m and U are the mean concentration and the mean velocity over the pipe section. The above equation may be written:

$$\frac{\Phi_m}{q} = \frac{U}{\tau} + (\psi_{25} - 12.79) \frac{v_*}{\tau}$$

from which follows, after multiplication by ρU

$$\frac{\rho U \Phi_m}{q} = \frac{\rho U^2}{\tau} + \sqrt{\frac{\rho U^2}{\tau}} (\psi_{25} - 12.79)$$

or, remembering the definition of C_f and $C_e = q / (\rho U \Phi_m)$,

$$C_e = \frac{C_f/2}{1 + (C_f/2)^{1/2} (\psi_{25} - 12.79)} \quad (226)$$

where ψ_{25} is given by Eq 224.

E. Mattioli's Extension

Mattioli (134, 1940) started from the stress tensor, used several empirical relations, and finally arrived at the conversion formula

$$C = \frac{C_f/2}{1 + (C_f/2)^{1/2} (9.46 \sigma^{0.77} - 2.46 \frac{\ln \sigma}{\sigma - 1} - 7)} \quad (227)$$

where C may be, of course, C_H or C_e . This formula agrees very well with those of von Kármán and of Hofmann for $\sigma < 1$, as shown in Fig. 2. Eq 220, 226 and 227 are all of the form

$$C = \frac{C_f/2}{1 + \sqrt{C_f/2} f(\sigma)}$$

It is $f(\sigma)$ that is plotted in Fig. 2.

F. Conversion Formulas Based on Experiments

Kraussold' (126, 1933) correlated the data available at the time of his work and recommended for heating of fluid in pipe flow the formula

$$Nu = 0.024 R^{0.8} \sigma^{0.37} \quad (228a)$$

and for cooling

$$Nu = 0.024 R^{0.8} \sigma^{-0.30} \quad (228b)$$

The results of Dittus-Boelter are expressed by Eq 222, which does not differ very much from the above two equations. Since it can be easily verified that

$$C = Nu / (R \sigma)$$

one has, by taking the exponent of σ as 0.35,

$$C = \text{constant } R^{-0.2} \sigma^{-0.65} \approx \text{constant } R^{-0.2} \sigma^{-2/3}$$

from which

$$C_{\sigma=1} = C_{\sigma=1} = \text{same constant } \bar{R}^{0.2}$$

Thus experimental data yield the conversion formula

$$C = \frac{C_f}{2} \sigma^{-2/3} \quad (229)$$

It is interesting to recall at this moment Pohlhausen's solution for

the laminar heat transfer from a heated plate, His result gave

$$Nu = 0.664 R^{1/2} \sigma^{-1/3}$$

so that

$$C = Nu / (R\sigma) = 0.664 R^{-1/2} \sigma^{-2/3}$$

$$C_{f/2} = 0.664 R^{-1/2}$$

and

$$C = (C_{f/2}) \sigma^{-2/3}$$

which is identical with Eq 229. This agreement must be considered as a coincidence, however, since only laminar flow is considered in Pohlhausen's solution.

G. Agreement of the Conversion Formulas

Since the conversion formulas of von Kármán, of Hofmann, and of Mattioli are more recent and refined than those of Reynolds and Taylor-Prandtl, only Eqs 221, 226 and 227 will be compared with Eqs 228a and 228b, which express the experimental data of Kraussold. The result is shown in Fig. 3. For $\sigma < 1$ the conversion formulas do not differ much from each other, as shown in Fig. 2. Furthermore, they are in good agreement with Eqs 222, 228a and 228b.

In plotting Eqs 226 and 227, it should be noted that

$$C = Nu / (R\sigma)$$

and that, according to Blasius

$$C_{f/2} = \tau / (\rho U^2) = \left(\frac{V^*}{U} \right)^2 = 0.0395 R^{-1/4}$$

so that Eq 226 can be written

$$Nu = \frac{0.0395 R^{3/4} \sigma}{1 + 0.199 R^{1/8} (\psi_{25} - 12.79)} \quad (226a)$$

Similarly, Eq 227 can be written in a corresponding form for the plotting of

Fig. 3:

$$N_u = \frac{0.0395 R^{3/4} \sigma}{1 + 0.199 R^{-1/8} (9.46 \sigma^{0.77} - 2.46 \frac{\ln \sigma}{\sigma - 1} - 7)} \quad (227a)$$

The Blasius drag formula has been used because Kraussold's experiments were performed in a Reynolds-number range in which the Blasius drag formula is valid. For higher R ($10^6 < R < 10^9$), the following drag formulas should be used:

$$C_f = 0.074 R^{-1/5} \quad (\text{Power Law})$$

$$C_f = \frac{0.472}{(\log_{10} R)^{2.58}} \quad (\text{Schlichting})$$

$$C_f = \frac{0.427}{(-0.407 + \log_{10} R)^{2.64}} \quad (\text{Schultz-Grunow})$$

H. von Kármán's Conversion Formulas
for Evaporation at High Reynolds Numbers

For Reynolds numbers in the range $10^6 < R < 10^9$, substitution of the last three drag formulas and $\sigma = 0.6$ in Eq 220 yields the following three conversion formulas:

$$\frac{1}{C_e} = 27.0 R^{1/5} - 21.0 R^{1/10} \quad (220a)$$

$$\frac{1}{C_e} = 4.24 (\log_{10} R)^{2.58} - 8.30 (\log_{10} R)^{1.29} \quad (220b)$$

$$\frac{1}{C_e} = 4.69 (-0.407 + \log_{10} R)^{2.64} - 8.74 (-0.407 + \log_{10} R)^{1.32} \quad (220c)$$

the last two of which can be used for even higher R and can be replaced by the following two, respectively, with very little deviations:

$$C_e = \frac{0.503}{(\log_{10} R)^{2.88}} \quad (220d)$$

$$C_e = \frac{0.1355}{(-1.275 + \log_{10} R)^{2.495}} \quad (220e)$$

I. Remarks

As has been noted, Eq 196 is valid only when the physical situations for momentum transfer and for heat transfer are strictly similar; in particular there shall not be any pressure gradient if there are no corresponding heat sources in the field of flow. Since in deriving Eqs 201, 214 and 225 a similar equation in Mattioli's extension, Eq 196 has been utilized, the validity of these equations is subject to the same restrictions. Furthermore, strictly speaking, if $\frac{dP}{dx} \neq 0$ these equations cannot really be derived from Eq 196, since the mean values of u , T , or Φ are taken over the entire pipe section which contains a boundary layer where Eq 196 is not valid. However, since the layer is thin for sufficiently large Reynolds numbers, the errors involved are small for ordinary temperature or concentration differences. In the case of flat plates, since the ambient values of u and of T or Φ are used instead of their mean values, this difficulty does not occur. Consequently, the application of Reynolds analogy or of its extensions to heat or vapor transfer in pipe flow lacks a sound theoretical basis, and any verification resulting from such an application should be treated with reserve. Taylor himself is aware of this limitation of Reynolds' analogy, and has criticized (63, 1930) the work of Eagle and Ferguson (93, 1930) on this basis. Jakob states in a footnote on page 506 of his book (32, 1949) that "Deviations from similarity (between the distribution of velocity and that of temperature) occur, but are not very significant in most cases." Therefore, theoretical justification is guaranteed only when one is applying the conversion formulas to cases where the pressure gradient is zero or negligible, as in the case of flat plates, provided the empirical formulas for pipe flow utilized in these conversion formulas apply also to these cases.

It may be mentioned that the assumption of constant shear and τ throughout the boundary layer is exactly satisfied only in the case of flat plates. If the empirical formulas for the velocity distribution in pipe flow can be applied to plates, the conversion formulas can certainly be applied thereto with much more justification. Of course, in the definition of C_f and C_H or C_e , the ambient velocity, temperature and vapor concentration should be used instead of the mean values used in the case of pipe flow.

Chapter III. RATIONAL SOLUTIONS OF SPECIFIC PROBLEMS IN
TURBULENT FLOW

The rational solutions for wakes and jets having been presented in Chapter I, one now turns to consider some simple cases of turbulent flow where solid boundaries are present, among which one has notably the flow in pipes and the flow along flat plates.

A. Smooth Boundary

As a preliminary, one considers first the general case of flow along a smooth flat wall, and endeavors to find the characteristics of turbulent flow near the boundary. Measuring x in the flow direction and along the wall, and y in a direction perpendicular thereto, it can be assumed that near the wall the mixing-length is proportional to y :

$$l = ky \quad (230)$$

where k is to be determined from experiment. Assuming further that the shear stress τ is constant in the entire flow region, the shear velocity defined as

$$v_* = \sqrt{\frac{|\tau|}{\rho}}$$

is also constant. If one neglects the laminar friction, then according to Eq 3:

$$v^2 = k^2 y^2 \left(\frac{du}{dy} \right)^2$$

or

$$\frac{du}{dy} = \frac{v_*}{ky}$$

integration of which yields

$$u = \frac{v_*}{k} \ln y + \text{constant} \quad (231)$$

The constant of integration in the above equation is to be determined from the condition that $u = 0$ for $y = y_0$ where y_0 is different from zero since Eq 231 cannot be expected to apply up to the wall, near which there exists

a laminar sub-layer. Thus

$$u = \frac{v_*}{k} (\ln y - \ln y_0)$$

If the yet unknown distance y_0 is set proportional to ν/v_* , there results

$$u = \frac{v_*}{k} \left(\ln \frac{y v_*}{\nu} - \ln \beta \right)$$

where β is the number of proportionality and $k = 0.4$ from experiments.

The last equation can be put in the dimensionless form

$$\frac{u}{v_*} = A \ln \frac{y v_*}{\nu} + B = A \ln \eta + B \quad (232)$$

where $\eta = y v_* / \nu$. This logarithmic velocity distribution is valid near the wall at large Reynolds numbers. For smaller Reynolds numbers, where the laminar friction also has an influence, tests gave the power law for the velocity distribution:

$$\frac{u}{v_*} = c \eta^n$$

B. Smooth Pipe

Let d be the diameter of the pipe. Consider a length L at the ends of which the pressures are denoted by p_1 and p_2 . Let the mean velocity be U and the dimensionless pipe resistance coefficient λ be defined by the equation:

$$\frac{p_1 - p_2}{L} = \frac{\lambda}{d} \frac{\rho}{2} U^2$$

Blasius (71, 1911) gave from experiments at moderate Reynolds numbers

$$\lambda = 0.3164 \left(\frac{U d}{\nu} \right)^{-1/4} \quad (233)$$

from which the shear stress at the wall is

$$\tau_0 = \frac{p_1 - p_2}{L} \frac{d}{4} = \frac{\lambda}{8} \rho U^2 = 0.03955 \rho U^{7/4} \nu^{1/4} d^{-1/4} = \rho v_*^2 \quad (234)$$

where v_* is now defined to be $\sqrt{\tau_0/\rho}$, since the shear stress τ is not constant over the pipe cross-section. From the above equation one obtains

$$\frac{U}{v_*} = 6.99 \left(\frac{v_* r}{\nu} \right)^{1/7} \quad (235)$$

where r is the radius of the pipe.

Based on the measurements of Nikuradse (136, 1932), the maximum velocity u_{\max} in the pipe is equal to $1.25 U$. Hence

$$\frac{u_{\max}}{v_*} = 8.74 \left(\frac{v_* r}{\nu} \right)^{1/7} \quad (236)$$

If the last equation is assumed to be valid for any distance y from the wall, one obtains

$$\frac{u}{v_*} = 8.74 \left(\frac{v_* y}{\nu} \right)^{1/7} = 8.74 \eta^{1/7} \quad (237)$$

This is the so-called seventh-power law for the velocity distribution; and is in good agreement with the measurements of Nikuradse (136) up to a Reynolds number of 10^5 , within the range $5 \leq \eta \leq 30$.

From Eq 237,

$$v_* = 0.150 u^{7/8} \left(\frac{\nu}{y} \right)^{1/8} \quad (238)$$

$$v_0 = \rho v_*^2 = 0.0225 \rho u^{7/4} \left(\frac{\nu}{y} \right)^{1/4} \quad (239)$$

As the Reynolds number increases, the power decreases from $1/7$.

Nikuradse (136, 1932) has found the law

$$\frac{u}{v_*} = 2.5 \ln (v_* y / \nu) + 5.5 \quad (240)$$

for $4(10)^3 \leq R \leq 3.24(10)^6$. The ranges of validity of the logarithmic and the seventh-power laws overlap somewhat. It should be noted that Eq 240 agrees in form with Eq 232, in spite of the assumption of constant shear stress made in deriving the latter. Comparing them, one has $k = 0.400$ and $\beta = 0.111$.

From Eq 240 it can be shown by integration that

$$U = v_* \left(2.5 \ln \frac{r v_*}{\nu} + 1.75 \right) \quad (241)$$

From Eq 234,

$$\lambda = 8 \left(\frac{v_*}{U} \right)^2 \quad (242)$$

and from the last two equations

$$\frac{1}{\sqrt{\lambda}} = 2.035 \log_{10} \left(\frac{U d}{\nu} \sqrt{\lambda} \right) - 0.91 \quad (243)$$

while Nikuradse's measurements give a slightly different formula,

$$\frac{1}{\sqrt{\lambda}} = 2.0 \log_{10} \left(\frac{Ud}{\nu} \sqrt{\lambda} \right) - 0.80 \quad (244)$$

From Eq 234 one obtains $\lambda = 4 C_f$, and the above equation can be written as

$$\frac{1}{\sqrt{C_f}} = 4.0 \log_{10} (R \sqrt{C_f}) - 0.40 \quad (245)$$

which is the universal law for smooth pipes.

Eq. 239, which is the Blasius law of resistance based on the seventh-power law of velocity distribution, is in good agreement with Eq 245, which is the resistance law based on the logarithmic law of velocity distribution, up to a Reynolds number 10^5 . Beyond this Reynolds number deviation occurs. Since the logarithmic velocity distribution holds for any arbitrarily large values of the Reynolds number, Eq 245 holds for the same values and is therefore applicable to smooth pipes for the entire range of the Reynolds number within which the flow is turbulent. It is called the universal resistance law for smooth pipes.

As has been mentioned in the Remark of Chapter II, the conversion formulas provided by the Reynolds analogy and its various forms to calculate the heat transfer, strictly speaking, do not apply to the usual problems of pipe flow.

There is abundant experimental data in connection with heat transfer in turbulent flow in pipes. Eq 222 is a good representative of the experimental formulas, in which the power of the Reynolds number is always 0.8, but the power of the Prandtl number varies from 0.3 to 0.4, and the coefficient also vary slightly. For highly viscous fluids, McAdams (38, 1942) recommended the following formula based on the data of Sieder and Tate (182, 1936):

$$Nu_{ca} = 0.027 R_a^{0.8} \sigma_a^{\frac{1}{3}} \left(\frac{\mu_a}{\mu_s} \right)^{0.14} \quad (246)$$

where the subscripts s and a refer to the surface temperature and the arithmetic mean of the entrance temperature t_0 and the exit temperature t_L after the exist fluid is thoroughly mixed. When the influence of the starting length cannot be neglected, the last equation should be replaced by the following one given by Nusselt (139, 1931) based on the experiments of Burbach (76, 1930):

$$Nu_a = 0.036 R_a^{0.8} \sigma_a^{\frac{1}{3}} \left(\frac{\mu_a}{\mu_s} \right)^{0.14} \left(\frac{D}{L} \right)^{1/18} \quad (247)$$

The influence is much smaller than in the laminar range, where

$$Nu_a = 1.86 \left(Pr_a \right)^{\frac{1}{3}} \left(\frac{\mu_a}{\mu_s} \right)^{0.14} \left(\frac{D}{L} \right)^{1/3} \quad (248)$$

According to Cholette (79, 1948), the coefficient 0.1 should be used instead of the 1/18 used in Eq 247.

The local heat-transfer coefficients for different hydrodynamic and thermal states of establishment in pipe flow were given semi-theoretically by Latzko (128, 1921). His formulas have recently been checked by Boeter, Young, and Iverson (73, 1948). The agreement is generally quite good.

The problem of evaporation in pipe flow does not usually occur in practice.

C. Rough Pipe

Since the thickness δ_l of the laminar sublayer is proportional to the effectiveness of roughness of a certain grain size k , which depends on k/δ_l , must depend on kv_*/ν . Extensive work by Nikuradse (137, 1933) who used sand roughness (k_s) in his experiments has shown that

1. When $0 \leq k_s v_*/\nu \leq 5$, where k_s is the grain size of the sand roughness, or, equivalently, for small Reynolds numbers, all roughnesses lie within the laminar sublayer. In this case roughness does not increase the drag which depends on the Reynolds number alone, and the pipe is hydraulically smooth.

2. when $k_s v_* / \nu \geq 70$, or when the Reynolds number is very large, all roughnesses project from the laminar sub-layer. A purely square law of form drag applies. The drag now depends only on the relative roughness k_s/r and not on the Reynolds number, and the roughness is fully developed.

3. when $5 \leq k_s v_* / \nu \leq 70$, or when the Reynolds number is medium, the drag will depend on both k_s/r and the Reynolds number.

For fully developed roughness flow, Nikuradse found

$$u = v_* (2.5 \ln \frac{y}{k_s} + 8.5) \quad (249)$$

integration of which gives

$$u = v_* (2.5 \ln \frac{r}{k_s} + 4.75) \quad (250)$$

Hence

$$\lambda = 8 \left(\frac{v_*}{u} \right)^2 = \frac{8}{(2.5 \ln \frac{r}{k_s} + 4.75)^2} = \frac{1}{(2.0 \log_{10} \frac{r}{k_s} + 1.68)^2} \quad (251)$$

Direct measurements of v_* and u by Nikuradse gave

$$\lambda = \frac{1}{(2.0 \log_{10} \frac{r}{k_s} + 1.74)^2} \quad (252)$$

No experiment on heat transfer in turbulent flow through rough pipes with special emphasis on the influence of roughness has been reported. But if the conversion formulas apply approximately to smooth pipes, they should also apply approximately to rough pipes.

D. Smooth Plate

Assume that the flow is turbulent from the start. The boundary-layer thickness $\delta(x)$ increases with x (measured along the plate) and corresponds to the radius of the pipe in pipe flow. The free stream velocity U corresponds to the maximum velocity u_{\max} in the pipe. Denoting by $W(x)$ the drag per unit width of the plate up to a point x , the momentum equation yields

$$\begin{aligned}
 W(x) &= \int_0^x \tau_0(x) dx = \rho \int_0^{\delta(x)} u(U-u) du \\
 &= \rho U^2 \delta(x) \int_0^1 \frac{u}{U} \left(1 - \frac{u}{U}\right) d\left(\frac{y}{\delta}\right)
 \end{aligned}
 \tag{253}$$

the symbols used in the equation having the same meanings as in Blasius flow treated in Report I. Assuming the seventh-power law

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{1/7} \tag{254}$$

Eq. 253 becomes

$$W(x) = \frac{7}{72} \rho U^2 \delta(x) \tag{255}$$

from which

$$\tau_0 = \frac{dW(x)}{dx} = \frac{7}{72} \rho U^2 \frac{d\delta(x)}{dx} \tag{256}$$

On the other hand, by analogy to pipe flow, the following equation can be assumed:

$$\tau_0 = 0.0225 \rho U^{7/4} \left(\frac{\nu}{\delta}\right)^{1/4} \tag{257}$$

Hence the differential equation for $\delta(x)$ is obtained

$$\frac{7}{72} \rho U^2 \frac{d\delta}{dx} = 0.0225 \rho U^{7/4} \left(\frac{\nu}{\delta}\right)^{1/4} \tag{258}$$

Integration and simplification of the above equation yields

$$\delta(x) = \frac{0.37x}{R_x^{1/5}} \tag{259}$$

where $R_x = Ux/\nu$. Thus $\delta(x) \sim x^{4/5}$ for the turbulent boundary layer,

while for the laminar boundary layer $\delta(x) \sim x^{1/2}$.

Substitution in Eq 255 yields

$$W(x) = C_f \frac{\rho}{2} U^2 x \tag{260}$$

where

$$C_f = 0.072 R_x^{-1/5} \tag{261}$$

Comparison with tests results shows that for $5(10^5) < R_x < 10^7$,

$$C_f = 0.074 R_x^{-1/5} \tag{262}$$

If the initial laminar flow on the front part of the plate is taken into consideration, then according to Prandtl

$$C_f = 0.074 R_x^{-1/5} - 1700/R_x \tag{263}$$

for the same range of R_x .

For larger Reynolds numbers the logarithmic velocity distribution must be assumed. Taking

$$\frac{u}{v_*} = 2.49 \ln(1 + 8.93 \eta) \quad (264)$$

which is a slightly modified form of

$$\frac{u}{v_*} = 2.5 \eta + 5.5 \quad (265)$$

where $\eta = v_* y/\nu$, it may be shown that the momentum equation yields the interpolation formula

$$C_f = \frac{0.472}{(\log_{10} R_x)^{2.58}} \quad (266)$$

Though the method of derivation applies to arbitrarily large values of R_x , the interpolation formula itself is valid only in the range $10^6 < R_x < 10^9$. Comparison with test results shows that the agreement improves if Eq 266 is slightly changed:

$$C_f = \frac{0.455}{(\log_{10} R_x)^{2.58}} \quad (10^6 < R_x < 10^9) \quad (267)$$

which is the Prandtl-Schlichting plate drag law.

The laminar approach length may again be taken into consideration by subtracting the same amount before, thus

$$C_f = \frac{0.455}{(\log_{10} R_x)^{2.58}} - \frac{1700}{R_x} \quad (10^6 < R_x < 10^9) \quad (268)$$

The limit 10^9 is high enough for practical purposes.

Very recently Schultz-Grunow (176, 1940) measured the velocity distribution in the plate boundary layer. Based on the measurements the following interpolation formula was proposed:

$$C_f = \frac{0.427}{(0.407 + \log_{10} R_x)^{2.64}} \quad (10^6 < R_x < 10^9) \quad (269)$$

which, however, does not differ much from the Prandtl-Schlichting law.

The corresponding axially symmetrical problem, that is, the turbulent

boundary-layer problem for a body of revolution at zero incidence, was treated by C. B. Millikan (135, 1932). The seventh-power law of the velocity distribution was taken as basis. Application to the general case has not yet been made.

If the plate is heated so that the difference between the temperature of the plate and that of the free stream is $T_s - T_o = \Delta T$. Then assuming a seventh-power law for the temperature distribution in analogy to the velocity distribution:

$$T - T_o = \Delta T \left(\frac{y}{\delta} \right)^{\frac{1}{7}} \quad (270)$$

Latzko (128, 1921) utilized Eq 259 and the principle of continuity to arrive at the relation

$$Nu_x = 0.036 \sigma R_x^{0.8} \quad (271)$$

Eq 270 is of course valid only when the seventh-power law holds for the velocity distribution and only when the Prandtl number is unity. Therefore it differs from von Kármán's conversion formula (Eq 221) which is based on the logarithmic distribution of velocity, and any value of σ . However, the influence of σ on the validity of Eq 270 is not great if σ does not differ appreciably from unity, as in the case of air. Hence Eq 271 can be expected to hold if σ is near unity and the velocity distribution follows the seventh-power law.

Jacob and Dow (33, 1946) took into consideration the influence of the unheated length of approach x_o . Their experimental results on air can be represented by the formula

$$Nu_x = 0.0280 R_x^{0.8} \left[1 + 0.40 \left(\frac{x_o}{x} \right)^{2.75} \right] \quad (272)$$

where $Nu_x = hx/k_a$, h being the coefficient of heat transfer, k_a being the thermal conductivity of air at the mean temperature $(T_s + T_o)/2$.

The experimental work of Juerges' (p.556 and p. 557, 32) gave a formula similar to the last one but with 0.0280 changed to 0.0322, the difference being possibly due to side effects. The experimental work of Elias (14,1929) gave an exponent 0.9 for the Reynolds number. Jakob (32, 1949) believes this might be due to the fact that the flow was changing from laminar to turbulent over a considerable portion of Elias plate, during which transition the exponent of R is higher than both exponents 0.5 and 0.8 for laminar and turbulent flows respectively.

While the great majority of experimental work in heat convection has been done on pipe flow, what experiments have been done in evaporation have been exclusively performed on plates and cylinders. Aside from those which have been superceded by later and more careful ones, the earliest extensive work on air was done by Carl Rohwer (166, 1931) under controlled and natural conditions. Denoting by E the evaporation in inches per 24 hours, e_s the saturated vapor pressure at the temperature of the water surface, in inches of mercury, e_d the vapor pressure of the ambient air, in the same units, U the mean wind velocity in miles per hour at a specified small height above the ground and B the mean barometer reading, in inches of mercury at $32^\circ F$, he obtained under controlled conditions

$$E = (0.44 + 0.118 U)(e_s - e_d) \quad (273)$$

and under natural conditions,

$$E = (1.465 - 0.0186 B)(0.44 + 0.118 U)(e_s - e_d) \quad (274)$$

for reservoirs of diameters less than 9 ft, and

$$E = 0.77 (1.465 - 0.0186 B)(0.44 + 0.118 U)(e_s - e_d) \quad (275)$$

for reservoirs of larger diameters. The limit of validity which has been taken at 9 ft. is of course somewhat arbitrary. It may be mentioned that Eqs 273 and 274 were obtained from experiments on a 3 ft square evaporation tank and Eq 275 was obtained from experiments on a reservoir of 85 ft

diameter, and that experiments at other locations have been used in obtaining Eqs 274 and 275. Edge effect was not considered. For the reservoir the rim depth is so small compared with the diameter that its effect can certainly be neglected.

The experiments made by Shepherd, Hadlock and Brewer (178, 1938) on evaporation from a free water surface and from saturated sand in pans 1 ft square placed in a wind tunnel gave data that can be represented by the formula

$$N_u = 0.103 R^{0.75} \quad (276)$$

Where N_u is now the Nusselt number of evaporation. Hichox (106, 1939) correlated Rohwer's data with the above formula. On reducing Rohwer's results (Eq 273) to a relation between N_u and R , he could find very little agreement between Eqs 273 and 276, but believed the formula

$$N_u = 0.1 R^{0.75} \quad (277)$$

should not be very far from the actual relationship.

In 1935, Powell and Griffiths (146) published their experiments on evaporation from a plane surface. In a later paper (145, 1940), Powell showed that his 1935 results can be arranged to indicate very convincingly the relationship between the Nusselt number of evaporation and the Reynolds number. In the 1940 paper he also published some other very interesting results. For a wetted cylinder of diameter 4.36 cm placed along the direction of wind, his experimental results can be summarized by the formula

$$\frac{e l}{p_w - p_a} = 3.17 \times 10^{-8} (U l)^{0.8} \quad (278)$$

where e is the rate of evaporation in grams per sec per sq cm, p_w and p_a are vapor pressures on the wetted surface and in the ambient air, respectively, in mm mercury, l is the exposed wetted length in cm, and U is the wind velocity in cm per sec. The starting unwetted length is 9 cm. The above formula describes Powell's 1935 results for plates within a maximum error of

seven percent. Hoffman's work on heat transfer from cylinders (26, 1935) gave a formula similar to the above one.

From experiments on circular discs with a diameter ranging from 5.4 to 22.1 cm, all facing wind, Powell obtained the formula

$$\frac{e d}{p_w - p_a} = 3.3 \times 10^{-7} (Ud)^{0.65} \quad (279)$$

and showed that the difference in the rates of evaporation from discs facing windward, facing leeward, and tangential to wind has a maximum value at $Ud = 500$ sq cm per sec (no smaller Ud was recorded) of 40 percent based on the windward evaporation (largest) but decreases to 17 percent for $Ud = 10,000$ sq cm per sec. Further experiments on rectangular plates showed that the maximum evaporation is obtained when the wetted surface is facing leeward and when the plate makes an angle of from 10° to 40° or more with the cross-wind direction, depending on the dimensions of the plate and possibly on the Reynolds number. Powell also showed that when a baffle is erected at the start of the wetted cylinder placed along the wind, its effect is a maximum when the ratio h/l is an optimum, where h is the height of the baffle measured in the same unit as l .

The most recent work on evaporation from a plane surface was done by Albertson (3, 1948). Denoting by x and x' respectively the distances of a point from the hydrodynamical and evaporation leading edges, his results showed that in the range of validity of the power law for the velocity distribution,

$$C_e = \frac{Q}{\Delta c U_{x'}} = \frac{N_{yx'}}{R_{x'} \sigma} = 0.067 \left(\frac{x'}{x}\right)^{-0.37} (Pe_{x'})^{\frac{4}{5}} \left(\frac{x'}{x}\right)^{\frac{1}{5}} - 1 \quad (280)$$

where Q is the rate of evaporation per unit width from the surface up to the wetted length x' , Δc is the difference between saturated concentration and the actual concentration of vapor in the ambient air, and the subscript x' means the dimensionless parameter concerned is based on x' . If the

logarithmic law of velocity distribution is valid, his results showed

$$C_e^{-\frac{1}{2}} = \left[6.15 \left(\frac{x'}{x} \right)^{-0.144} \log_{10} C_e P e_{x'} \right] - 2 \quad (281)$$

The Péclet number in Eqs 280 and 281 may be converted easily into the Reynolds' number. Finally, denoting by B the Péclet number based on x' and on the shear velocity $\sqrt{\tau_0/\rho}$, he obtained

$$C_e S = 0.5 B^{4/5} \quad (282)$$

for the laminar and the turbulent cases alike. Comparison of Albertson's data with the Eqs 220a, 220b, and 220c based on von Kármán's conversion formula is shown in Fig. 4, where the Reynolds number is based on the length $\sqrt{xx'}$ and the ambient velocity.

Evaporation from a plane surface was also investigated by Huss (110, 1940) and Wade (204, 1942). Evaporation from Lake Hefner near Oklahoma City is now being investigated by the U.S.B.R., the U.S.G.S., the U.S. Weather Bureau, and other governmental agencies. Since it is known that the bottom of Lake Hefner does not leak water, and since the occasional inflow of the streams feeding the lake can be measured accurately, Lake Hefner serves as a prototype-sized evaporation pan, and furnishes an ideal site for evaporation studies.

E. Rough Plate

The conversion from pipe resistance to the plate drag may be carried out in much the same manner as previously described for the smooth plate. Based on Nikuradse's data on rough pipes and starting from the equation for pipe

$$u = v_* \left(2.5 \ln \frac{y}{k_s} + B \right) \quad (283)$$

where B has been determined by Nikuradse (137, 1933) to be a function of $v_* k_s/\nu$. Prandtl and Schlichting (153, 1934) arrived at a relation between the drag coefficient C_f and the Reynolds number UL/ν , with L/k_s as a parameter. When the Reynolds number is large enough, C_f depends only on

L/k_s :

$$C_f = (1.89 + 1.62 \log_{10} \frac{L}{k_s})^{-2.5} \quad (284)$$

No work on heat transfer or evaporation from a rough plate has been reported. However, the conversion formulas of von Kármán, Hofmann, and Mattioli are expected to give fairly accurate results.

F. Free Turbulence

The study of free turbulence is generally divided into three groups, namely, jets, wakes, and the free jet boundary. Although the first two groups have been discussed in some detail in chapter I, in connection with the transfer theories of turbulence, certain characteristics of their spread warrant further discussion. The free jet boundary will also be discussed.

1. Jets

Although the velocity distribution in plane and round jets must be obtained experimentally (No theory can guarantee a correct prediction--see Chapter I), the salient characteristics of spread can be obtained from the equation of motion on the assumptions of constant pressure and of similarity for different sections.

For the two-dimensional jet, the equation of motion is

$$uu_x + vv_y = \frac{1}{\rho} \tau_y \quad (285)$$

where τ denotes the shear stress and the subscripts denote differentiation.

Integrating with respect to y from $-\infty$ to $+\infty$, and remembering that τ

vanishes at both limits and that $u_x + v_y = 0$, one has

$$\begin{aligned} \int_{-\infty}^{\infty} u u_x dy + \int_{-\infty}^{\infty} v u_y dy &= \int_{-\infty}^{\infty} u u_x dy + v u \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} u u_x dy \\ &= \frac{d}{dx} \int_{-\infty}^{\infty} u^2 dy = 0 \end{aligned}$$

or

$$\int_{-\infty}^{\infty} \rho u^2 dy = \text{constant} = M \quad (286)$$

where M is the constant momentum flux. Similarly, by multiplying Eq 285 by ρu and integrating, one has

$$\frac{\rho}{3} \frac{d}{dx} \int_{-\infty}^{\infty} u^3 dy + \rho \int_{-\infty}^{\infty} u v u_y dy = \int_{-\infty}^{\infty} \tau u dy = \tau u \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \tau u_y dy \quad (287)$$

or

$$\frac{\rho}{3} \frac{d}{dx} \int_{-\infty}^{\infty} u^3 dy + \frac{\rho}{2} u^2 v \Big|_{-\infty}^{\infty} + \frac{\rho}{2} \int_{-\infty}^{\infty} u^2 u_x dy = \frac{d}{dx} \int_{-\infty}^{\infty} \frac{1}{2} u^3 dy = - \int_{-\infty}^{\infty} \tau u_y dy$$

which expresses the rate of decay of kinetic energy as a result of the dissipation by shear (first into turbulent kinetic energy and finally through viscosity into heat).

The assumption of dynamic similarity requires that

$$\frac{u}{u_{\max}} = f(\eta) \quad (288)$$

in which

$$\eta = \frac{y}{b}$$

where b is some characteristic lateral jet dimension used as a reference length, and that

$$\tau = \rho u_{\max}^2 g(\eta) \quad (289)$$

Substituting Eqs 288 and 289 into 286 and 287, one obtains the following simultaneous equations for u_{\max} and s :

$$u_{\max}^2 b = M / \rho I_1 \quad (290)$$

$$\frac{d}{dx} (u_{\max}^3 b) = -2 \frac{I_3}{I_2} u_{\max}^3 \quad (291)$$

where

$$I_1 \equiv \int_{-\infty}^{\infty} f^2 d\eta$$

$$I_2 \equiv \int_{-\infty}^{\infty} f^3 d\eta$$

$$I_3 \equiv \int_{-\infty}^{\infty} g f' d\eta$$

Simultaneous solution of Eqs 290 and 291 gives (remembering that

$u_{\max} \rightarrow \infty$ as $x \rightarrow 0$ for infinitesimal slit)

$$U_{max} = \sqrt{\frac{M I_2}{4 \rho I_1 I_2 x}} \quad (292)$$

and

$$b = \frac{4 I_3}{I_2} x \quad (293)$$

so that u_{max} is proportional to $x^{-\frac{1}{2}}$ and b to x .

The foregoing analysis is after Corrsin, who gave a similar analysis for the three-dimensional jet in the discussion (80h, 1948) of a paper by Albertson, Jensen, Dai, and Rouse (68a, 1948), with the results

$$U_{max} \sim \sqrt{\frac{M}{\rho}} \frac{1}{x} \quad (294)$$

and

$$b \sim x \quad (295)$$

where M denotes the total momentum flux of the round jet.

Eqs 292 and 293 and proportionalities 294 and 295 are verified by Albertson and co-authors (68a), who found that for the plane jet the velocity distribution follows the Gaussian law:

$$\log_{10} u \sqrt{\frac{\rho x}{M}} = 0.36 - 1.84 \frac{y^2}{x^2}$$

which gives

$$u_{max} = 2.28 \sqrt{\frac{M}{\rho x}}$$

and

$$b = 0.52 x$$

where b is the standard deviation of the velocity distribution curve (which represents the Gaussian function), and that for the round jet the velocity distribution is again Gaussian:

$$\log_{10} \sqrt{\frac{\pi \rho}{4M}} u x = 0.79 - 33 \frac{r^2}{x^2}$$

which gives

$$U_{max} = 6.2 \sqrt{\frac{4M}{\pi \rho}} \frac{1}{x}$$

and

$$b = 0.12 x$$

where b has the same meaning.

Exactly the same method of analysis can be applied to heated jets, with the results that for the plane jet (H denoting the heat flux per unit length and c_p the specific heat at constant pressure)

$$T_{max} - T_0 \sim \sqrt{\frac{H}{\rho c_p x}} \quad b' \sim x$$

and that for the round jet (H now denoting total heat flux)

$$T_{max} - T_0 \sim \sqrt{\frac{H}{\rho c_p}} \frac{1}{x} \quad b' \sim x$$

where T_0 is the ambient temperature, T_{max} is the temperature on the center-line, and b' is some characteristic lateral dimension of the heat jet used as a reference length. In the analysis of the heat jet, the energy equation instead of the equation of motion should be used.

†. Strictly similar remarks apply to the vapor jets.

2. Wakes

For the plane turbulent wake, the equation of motion is approximately

$$Uu_x = \frac{1}{\rho} \tau_y \quad (296)$$

where U is the ambient velocity (in the x -direction), and u is the difference between U and the actual velocity. Integrating Eq 296, one has

$$\rho U \frac{d}{dx} \int_{-\infty}^{\infty} u dy = 0$$

or

$$\rho U \int_{-\infty}^{\infty} u dy = W \quad (297)$$

where W is the total drag per unit width on the two-dimensional body whose wake is under investigation. Multiplying Eq 296 by u and integrating, one has

$$\frac{1}{2} \rho U \frac{d}{dx} \int_{-\infty}^{\infty} u^2 dy = - \int_{-\infty}^{\infty} \tau u_y dy \quad (298)$$

If now one assumes dynamic similarity, one has

$$\frac{u}{U_{max}} = f(\eta)$$

where

$$\eta = \frac{y}{b}$$

and

$$\tau = \rho u_{max}^2 g(\eta)$$

Eqs 297 and 298 then become

$$u_{max} b = \frac{W}{\rho U I_1} \quad (299)$$

$$\frac{d}{dx} (u_{max}^2 b) = - \frac{2 u_{max}^3}{U} \frac{I_3}{I_2} \quad (300)$$

where

$$I_1 = \int_{-\infty}^{\infty} f(\eta) d\eta$$

$$I_2 = \int_{-\infty}^{\infty} f^2 d\eta$$

$$I_3 = \int_{-\infty}^{\infty} g f' d\eta$$

Solving Eqs 299 and 300 simultaneously, one has

$$u_{max} = \sqrt{\frac{M I_2}{4 \rho I_1 I_3 x}}$$

$$b = \sqrt{\frac{4 M I_3 x}{\rho U^2 I_1 I_2}}$$

For the three-dimensional wake, one works with the equation

$$U u_x = \frac{1}{\rho r} (\tau r)_r$$

and obtains

$$u_{max} \sim \left(\frac{W U}{\rho x^2} \right)^{\frac{1}{3}}$$

and

$$b \sim \left(\frac{W x}{\rho U^2} \right)^{\frac{1}{3}}$$

where \bar{W} is now the total drag on the three-dimensional body.

A similar analysis can be given to heat wakes and vapor wakes, using the energy equation and the diffusion equation respectively. It can be shown

that $T_{\max} - T_0$ varies with x in the same manner as u_{\max} , and S' varies with x in the same manner as S , for both plane and axially symmetric wakes. An analogous remark applies to vapor wakes.

3. Free jet boundary

It is sufficient to discuss the case where a free fluid stream with uniform velocity U mixes with the same fluid at rest. Integrating Eq 285 between the boundaries of the mixing region, one has

$$\int_{-\alpha b}^b \frac{\partial \rho u^2}{\partial x} dy = 0 \quad (301)$$

where b measures the spread of the mixing region into the free stream, and αb measures that into the fluid at rest. Due to similarity for different sections, the number α is constant.

Since b is a function of x , Eq 301 can be written

$$\frac{d}{dx} \int_{-\alpha b}^b \rho u^2 dy - \rho U^2 \frac{db}{dx} = 0 \quad (302)$$

from which it is clear why finite instead of infinite limits are used for the integral, since otherwise the integral will not converge. There is, however, another way of overcoming this difficulty as will be discussed later.

With

$$u = Uf(\eta) \quad (303)$$

and

$$\eta = y/b \quad (304)$$

Eq 302 can be written

$$\left(\int_{-\alpha}^1 f^2 d\eta - 1 \right) \frac{db}{dx} = 0$$

Discarding the trivial solution $\frac{db}{dx} = 0$, one has

$$\int_{-\alpha}^1 f^2 d\eta = 1 \quad (305)$$

which determines α .

Multiplication of Eq 285 by ρu and integration gives

$$\frac{1}{2} \int_{-\alpha b}^b \frac{\partial}{\partial x} (\rho u^3) dy = - \int_{-\alpha b}^b \tau u_y dy \quad (306)$$

With

$$\tau = \rho U^2 g(\eta) \quad (307)$$

this becomes

$$\frac{db}{dx} \left(\int_{-\alpha}^{\infty} f^3 d\eta - 1 \right) = -2 \int_{-\alpha}^{\infty} g f' d\eta$$

which means $\frac{db}{dx}$ is constant and $b \sim x$

The foregoing analysis is due to H. W. Liepmann and J. Lanter.

There is another way of arriving at the results where infinite limits are used.

Since infinite limits have been used for jets and wakes, for the sake of uniformity it is thought desirable to present the alternative analysis.

The equation corresponding to Eq 301 is

$$\int_{-\infty}^{\infty} \frac{\partial \rho u^2}{\partial x} dy = 0$$

which by virtue of Eqs 303 and 304, can be written

$$-2\rho U^2 \frac{db}{dx} \int_{-\infty}^{\infty} f f' \eta d\eta$$

Discarding the trivial case $\frac{db}{dx} = 0$, one has

$$I_1 = \int_{-\infty}^{\infty} f f' \eta d\eta = 0$$

which imposes a condition on f and in particular determines α in Lipemann and Laufer's analysis, and is obviously equivalent to Eq 305.

Eq 306 together with Eqs 303, 304 and 307 yields

$$\frac{db}{dx} = \frac{2 I_3}{3 I_2} = \text{constant}$$

where

$$I_2 = \int_{-\infty}^{\infty} f^2 f' \eta d\eta$$

$$I_3 = \int_{-\infty}^{\infty} g f' d\eta$$

Hence $b \sim x$. The convergence of I_1 , I_2 , and I_3 depends of course on the manner in which $f' \rightarrow 0$ as $\eta \rightarrow \infty$, and since this requires only that for large η

$$f' = o\left(\frac{1}{\eta^2}\right)$$

while ordinarily f' is supposed to vanish exponentially as $\eta \rightarrow \infty$, the integrals can be safely considered as convergent.

A similar analysis can be given for the spread of heat or of vapor when the free stream is heated or has a higher content of vapor.

G. The Turbulent Boundary-Layer in
Accelerated and Retarded Flow

Since the equations governing turbulent flow have not been properly formulated, there exists no method, exact or approximate, for the computation of velocity profiles in the turbulent boundary layer attached to arbitrarily shaped bodies. However, the momentum equation furnishes an integral condition that must be satisfied. This integral condition and some empirical relations obtained from systematic experimentation constitute the equations by means of which the important unknowns can be computed. In the following Grushwitz's (102, 1931) method will be presented.

Grushwitz made the assumption that the velocity profiles of the turbulent boundary-layer for pressure drop and rise can be represented as a one-parameter family, if u/U is plotted against y/δ , where δ is the momentum thickness defined by

$$U^2 \delta = \int_0^{\delta} u(U-u) dy$$

δ being the boundary-layer thickness. For the definition of δ , δ_p , and δ^* (which will be used in the following), reference is made to Report I, p. 34 and p. 35.

As form parameter one selects

$$\eta = 1 - \left(\frac{u(\eta \delta)}{u_1} \right)^2$$

where $u(\eta \delta)$ denotes the velocity u at $y = \eta \delta$, and u_1 as before denotes the potential velocity just outside of the boundary layer, all in a direction along the solid boundary. That η actually is a serviceable form

parameter has been experimentally justified by Gruschwitz. Gruschwitz found from his experiments that the turbulent separation point is given by

$$\eta = 0.8$$

The form parameter η is analogous to the parameter λ of Fohlhausen for the laminar boundary-layer. A considerable difference exists between η and λ , however, since for the laminar boundary-layer the following analytical relation exists

$$\lambda = \frac{\delta_0^2}{\nu} \frac{dU}{dx} \quad (308)$$

while such a relation is thus far lacking for the turbulent boundary-layer, as in general an analytical expression for the turbulent velocity profiles does not exist. One needs therefore an empirical relation equivalent to Eq 308.

Since, as has been remarked at the beginning of this chapter, the equations governing turbulent flow have not been properly formulated, the velocity distribution cannot be determined in an analytic way, and the calculation will be limited to the determination of the four characteristics of the turbulent boundary-layer, namely η , τ_0 , δ^* and ϑ

Precisely as for the laminar boundary layer, the momentum theorem yields the first equation

$$\frac{\tau_0}{\rho U^2} = \frac{d\vartheta}{dx} + \left(1 + \frac{1}{2} \frac{\delta^*}{\vartheta}\right) \frac{\vartheta}{U^2} \frac{dU^2}{dx}$$

The second equation is yielded by the equation

$$\frac{\delta^*}{\vartheta} = H(\eta)$$

obtained by Gruschwitz by evaluation of the measured velocity profiles (102, 1931) and regarded as generally valid.

The empirical relation between $H = \delta^*/\vartheta$ and η found by Gruschwitz may also be represented analytically, according to Pretsch (154, 1938), who set up a power law of the form

$$\frac{u}{u_1} = \left(\frac{y}{\delta}\right)^n = z^n$$

with $n = 1/6, 1/7, 1/8, \dots$, according to the experiments so far. As the

Reynolds number increases, n will decrease and the velocity distribution will finally approach the logarithmic form. But as long as a power representation is adequate, the following will be true no matter what is the actual value of n . From the definition of δ^* (see Report I, p. 34), one has

$$\frac{\delta^*}{\delta} = \int_{y/\delta=0}^1 \left(1 - \frac{u}{u_1}\right) d\frac{y}{\delta} = \int_0^1 (1 - z^n) dz = \frac{n}{n+1}$$

Furthermore

$$\frac{v^2}{\delta} = \int_{y/\delta=0}^1 \frac{u}{u_1} \left(1 - \frac{u}{u_1}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 z^n (1 - z^n) dz = \frac{n}{(n+1)(2n+1)} \tag{309}$$

Hence, from the last two equations, one obtains by division,

$$H = \frac{\delta^*}{v^2} = 2n+1 \tag{310}$$

Substituting the last equation into Eq 309:

$$\frac{v^2}{\delta} = \frac{H-1}{H(H+1)}$$

Then from the definition of η :

$$\eta = 1 - \left(\frac{u(\eta)}{u_1}\right)^2 = 1 - \left(\frac{v^2}{\delta}\right)^{2n} = 1 - \left[\frac{H-1}{H(H+1)}\right]^{H-1} \tag{311}$$

This result is in perfect agreement with Gruschwitz's experimental data, and is tabulated below:

H	1	1.1	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
η	0	0.270	0.404	0.573	0.688	0.772	0.833	0.881	0.916	0.941	0.959	0.972

The third equation was empirically derived by Gruschwitz from his measurements. He considered that the energy variation of a particle moving parallel to the wall at the distance $y = \eta$ is a function of $u(\eta)$, u_1 , v^2 , η . Dimension considerations suggest the following relation:

$$\frac{v^2}{\delta} \frac{d\eta}{dx} = F(\eta, R)$$

where $q = \rho u_1^2 / 2$, $g_1 = \rho + \rho [u(\eta)]^2 / 2$, and R is the

Reynolds number. The test results show no dependence on R , and that

$$\frac{v^2}{g} \frac{dg_1}{dx} = 0.00894 \eta - 0.00461 \quad (312)$$

Furthermore,

$$g_0 - g_1 = p + \frac{\rho}{2} u_1^2 - p - \frac{\rho}{2} [u(v^2)]^2 = \frac{\rho}{2} u_1^2 \left[1 - \left(\frac{u(v^2)}{u_1} \right)^2 \right] = g \eta$$

where $g_0 = p + \rho u_1^2 / 2 = \text{constant}$. On putting $g \eta = \mathcal{S}$, one has therefore

$$\frac{dg_1}{dx} = - \frac{d\mathcal{S}}{dx}$$

and, substituting into Eq 312,

$$v^2 \frac{d\mathcal{S}}{dx} = -0.00894 \mathcal{S} + 0.00461 q \quad (313)$$

The fourth equation is still missing and is replaced by an estimation of

τ_0 . According to the calculations for the plate in longitudinal flow,

$$\frac{\tau_0}{\rho u_1^2} = 0.0225 \left(\frac{u_1 v^2}{\nu} \right)^{-\frac{1}{4}} = 0.0225 (R_S)^{-\frac{1}{4}}$$

If one assumes what is true for the seventh-power velocity distribution,

namely,

$$\delta^* = \frac{1}{8} \delta, \quad v = \frac{7}{72} \delta$$

one can write Eq 311 in the following manner:

$$\frac{\tau_0}{\rho u_1^2} = 0.01338 (R_{S^*})^{-\frac{1}{4}} = 0.01256 (R_v)^{-\frac{1}{4}} \quad (314)$$

Thus, for the calculation of v and η (or of v^2 and \mathcal{S}), one must solve simultaneously

$$\frac{d\mathcal{S}}{dx} + 0.00894 \frac{\mathcal{S}}{v^2} = 0.0461 \frac{g}{v^2}$$

$$\frac{dv^2}{dx} + \left(1 + \frac{H}{2} \right) \frac{v^2}{g} \frac{dg}{dx} = \frac{\tau_0}{\rho u_1^2}$$

where $q = \rho u_1^2 / 2$, H , $\tau_0 / \rho u_1^2$ are functions of x , of $\eta = \mathcal{S} / g$ and of v^2 and x , respectively, q being calculated by any method applicable to potential flow, H and $\tau_0 / \rho u_1^2$ respectively from Eqs 311 and 312. If the dimensionless parameters

$$x' = \frac{x}{t}, \quad v' = \frac{v}{t}, \quad \mathcal{S}' = \frac{\mathcal{S}}{\frac{\rho}{2} U^2} = \left(\frac{u_1}{U} \right)^2 \eta$$

are used, where t is a representative length and U the velocity of approach of the free stream, one has the dimensionless equations:

$$\frac{d\mathcal{S}'}{dx'} + 0.00894 \frac{\mathcal{S}'}{\mathcal{V}'} = 0.00461 \left(\frac{u_1}{U}\right)^2 \frac{1}{\mathcal{V}'}$$

$$\frac{d\mathcal{V}'}{dx'} + 2\left(1 + \frac{H}{2}\right) \mathcal{V}' \frac{U}{u_1} \frac{d\left(\frac{u_1}{U}\right)}{dx'} = \frac{\tau_0}{\rho u_1^2}$$

In solving the above equations the initial values of the unknowns have to be determined. For \mathcal{V}' the initial value is taken to be that for the laminar boundary-layer at the transition point, and \mathcal{S}' is determined from the initial value of η , which is taken by Gruschwitz to be 0.1. According to Gruschwitz, a different initial value of η will make very little difference. With these initial values the equations may be solved by the isocline method. A first approximation for \mathcal{V}' is obtained by first solving the second equation with constant values for $\tau_0/\rho u_1^2$ and H :

$$\frac{\tau_0}{\rho u_1^2} = 0.002 \qquad H = 1.5$$

This first approximation $\mathcal{V}'_1(x')$ is then substituted into the first equation to obtain a solution $\mathcal{S}'_1(x')$, which gives a first approximation for η and hence for $H(\eta)$. Also τ_0 can be improved with \mathcal{V}'_1 according to Eq 314. These newly obtained values of H and τ_0 are then substituted into the second equation to find $\mathcal{V}'_2(x')$; and the process continues until the differences between two successive values of \mathcal{V} and of \mathcal{S} become insignificant. The method converges so well that the answer is essentially attained in the second approximation.

According to Czuber the isocline method for the solution of the differential equations can be applied in the present case in the following manner. Both differential equations having the form

$$\frac{dy}{dx} + f(x)y = g(x)$$

they have the property that all line elements on a straight line

$x = \text{constant}$ radiate from one point, as can be easily shown. The coordinates of this pole are:

$$\xi = x + \frac{1}{f(x)} ; \quad \eta = \frac{g(x)}{f(x)}$$

Thus one has only to calculate a sufficient number of these poles and can then easily draw the integral curve.

The separation point is given by $\eta = 0.8$

It should be mentioned that the calculation for the turbulent boundary-layer must be performed separately for each $R = Ut/\nu$, whereas only one calculation is necessary for the laminar boundary-layer. The reasons are, first, that the transition point travels with R , and second, that the initial value of ν/t further varies with R at the transition point since there

$$\frac{\nu}{t} \sqrt{R} = \text{constant}$$

for the laminar boundary-layer, as can be shown by making the laminar boundary-layer equation not only dimensionless, but also free from R .

It must be noted that the values obtained for τ_0 becomes incorrect in the neighborhood of the separation point. At the separation point τ_0 should be zero, whereas according to Eq 314. τ_0 is never zero.

In the case of flat plates, $q(x) = \text{constant}$, and Eq 313 can be written

$$\nu \frac{d\eta}{dx} = -0.00894\eta + 0.00461 \quad (315)$$

A trivial solution is

$$\eta = \frac{0.00461}{0.00894} = 0.516$$

Since the initial value of η is 0.1, and since according to Eq 315 $d\eta/dx > 0$, η must approach 0.516 asymptotically from below. If the velocity profile follows the seventh-power law, $\eta = 0.487$, as can be computed from Eqs 310 and 311. The profile attained asymptotically for

uniform pressure therefore almost agrees with the seventh-power law that was previously applied to the plate in longitudinal flow.

For examples of calculation, see (172, 1942).

The turbulent heat boundary layer for arbitrarily shaped bodies has been treated by Kalikhman (115, 1946) who considered not only compressibility but also the variations of density and viscosity with temperature. The method is too complicated to be presented here. Suffice it to mention that it involved the process of successive approximation and the use of the concepts of eddy viscosity and mixing length, and of Eq 230. For the turbulent vapor boundary layer the variations of the physical constants can be neglected in ordinary cases, and a method of calculation analogous to that of Gruschwitz for the turbulent flow boundary layer can be developed.

Chapter IV. MASS-TRANSFER IN THE ATMOSPHERE

A. Sutton's Theory for Still Air

When there is no mean motion in the atmosphere, the Lagrangian correlation coefficient can be assumed to be

$$R_{\xi} = \left(\frac{a}{u''_{\xi}} \right)^n \quad (u''_{\xi} > 1)$$

according to O. G. Sutton (186, 1932), a being a constant length and u'' being the root mean square of the velocity fluctuations. As $\xi \rightarrow 0$, Sutton assumed $R_{\xi} \rightarrow 1$, but did not specify the way in which it does so. Substituting the above form of R in Eq 65 for the case $n < 1$, one obtains, on replacing T by t in the final result,

$$\bar{X}^2 = \frac{2a^n}{(1-n)(2-n)} (u''t)^{2-n} = \frac{1}{2} b^2 (u''t)^{2-n}$$

It should be noted that the error committed in the above expression is small only when the contribution of R_{ξ} at small values of ξ is negligible. This is true only for values of n less than 1. For such values it is immediately seen that \bar{X}^2 increases as a power of t higher than the first, thus taking into account the observed increase of the effective eddies with time. However, if $n < 1$, the integral $\int_0^{\infty} R_{\xi} d\xi$ will diverge. This is a defect of Sutton's theory.

The concentration of matter in space resulting from an instantaneous point source is required to satisfy the following conditions

- (i) as $t \rightarrow \infty$, $c \rightarrow 0$
- (ii) as $t \rightarrow 0$, $c \rightarrow 0$ except at the origin
- (iii) $\bar{X}^2 = \frac{b^2}{2} (u''t)^{2-n}$, where $\sqrt{\bar{X}^2}$ is the standard deviation of the concentration-distribution curve.
- (iv) Total amount of matter = constant = Q

These conditions are satisfied by the solution

$$c = \frac{Q}{\pi^{\frac{3}{2}} b^3 (u''t)^{3m/2}} \exp\left(-\frac{r^2}{b^2 (u''t)^m}\right) \quad (316)$$

where $m = 2-n$.

Solutions for continuous sources are obtained from the above equation by integration. Noteworthy is the fact that the concentration has not been required to satisfy the differential equation of diffusion.

B. Sutton's Theory for Uni-Directional Wind and Smooth Surfaces

1. Correlation and interchange coefficients

When there is wind and the wind is in a certain direction, the Lagrangian correlation coefficient can be modified to the following form, according to O. G. Sutton (186, 1934):

$$R_{\xi} = \left(\frac{\nu_1}{\nu_1 + w'^2 \xi} \right)^n \tag{317}$$

where ν has not yet been identified with the kinematic viscosity, and where n is a function of the thermal gradient in the vertical direction and of the surface-roughness, of which no more than an empirical determination is at present possible. Using Eq 317, one now has, with T_1 denoting the time interval beyond which R_{ξ} is negligible,

$$\overline{\nu'^2} = \nu'^2 \int_0^{T_1} \left(\frac{\nu_1}{\nu_1 + w'^2 \xi} \right)^n d\xi = \frac{\nu_1^n}{1-n} \left[(\nu_1 + w'^2 T_0)^{1-n} - \nu_1^{1-n} \right] \approx \frac{\nu_1^n}{1-n} (w'^2 T_0)^{1-n} \tag{318}$$

the terms neglected being of the order of ν_1 at the most. This approximation really amounts to neglecting molecular forces in comparison with eddy forces. Substituting Eq 317 into Eq 65 and changing Y into Z , one has further

$$\overline{Z^2} = \frac{2 \nu_1^n (w'^2 T)^{2-n}}{(1-n)(2-n) w'^2}$$

Since $\overline{Z^2}$ is positive and since it must increase with time, Sutton concludes that n must be between zero and one in magnitude. This will, however, make the integral of R_{ξ} with respect to ξ from zero to infinity divergent, and will again constitute a defect of Sutton's theory.

It does not appear possible to give a rigorous expression for $w''^2 T_1$ in terms of u and z unless one knows how w'' depends upon the boundary conditions and the stability of the motion, and the manner in which u depends upon the height. A fair approximation, however, can be obtained by using the ideas of Prandtl and of Kármán. It has been shown by Hesselberg and Bjordkal (195a, 1929) that the distribution of the velocity fluctuation w' is Gaussian. Writing

$$|\overline{w'}| = \lambda \left| \frac{d\bar{u}}{dz} \right| \tag{319}$$

and using the well-known relationship for a variable with a Gaussian distribution

$$\overline{w''^2} = \overline{w'^2} = \frac{\pi}{2} (\overline{|w'|})^2$$

one has

$$\overline{w''^2} = \frac{\pi}{2} \lambda^2 \left(\frac{d\bar{u}}{dz} \right)^2 \tag{320}$$

where ordinary differentiation is used because \bar{u} is supposed to vary only with z in Sutton's theory. The time T_1 , corresponding to λ , is given by

$$T_1 = \int_z^{z+\lambda} \frac{dz}{w'} \tag{321}$$

Replacing w' by its mean absolute value $|\overline{w'}|$, one has, approximately,

$$T_1 = \frac{\lambda}{|\overline{w'}|} = \left(\left| \frac{d\bar{u}}{dz} \right| \right)^{-1} \tag{322}$$

Hence

$$\overline{w''^2} t_0 = \frac{1}{2} \pi \lambda^2 \left| \frac{d\bar{u}}{dz} \right| \tag{323}$$

It has been shown by Kármán (118, 1930) that the assumption that eddy velocities at different points are dynamically similar leads to a particularly simple expression for λ . Making this assumption, one has Kármán's expression

$$\lambda = k \frac{d\bar{u}}{dz} / \frac{d^2\bar{u}}{dz^2} \tag{324}$$

where k is Kármán's universal constant which is approximately equal to 0.4.

Thus finally

$$\overline{w''^2} T_1 = 0.08 \pi \left| \frac{d\bar{u}}{dz} \right|^3 \left| \frac{d^2\bar{u}}{dz^2} \right|^{-2} \tag{325}$$

approximately. The interchange coefficient is then

$$A(z) = \rho \bar{w} l = \frac{\rho \nu_1^n}{1-n} (w''^2 T_1)^{1-n} = \frac{(0.251)^{1-n}}{1-n} \rho \nu_1^n \left[\left| \frac{d\bar{u}}{dz} \right|^3 \left| \frac{d^2\bar{u}}{dz^2} \right|^{-2} \right]^{1-n} \quad (326)$$

2. Variation of wind with height

Assuming a power function for the velocity distribution

$$\bar{u} = \bar{u}_1 \left(\frac{z}{z_1} \right)^{1/n} \quad (327)$$

where $u_1 = u(z_1)$, one can deduce that since the flow is parallel and the pressure does not vary in the direction of the wind

$$\tau = A(z) \frac{d\bar{u}}{dz} = \text{constant} \quad (328)$$

in the lower layers of the atmosphere. Substituting Eq 326 for A(z) in the above equation, one has

$$\frac{(0.251)^{1-n}}{1-n} \rho \nu_1^n \left(\frac{d\bar{u}}{dz} \right)^{4-3n} \left(\frac{d^2\bar{u}}{dz^2} \right)^{2n-2} = \text{Constant} \quad (329)$$

From Eqs 327 and 329, it can be easily shown that $q = (2-n)/n$ and

$$\bar{u} = \bar{u}_1 \left(\frac{z}{z_1} \right)^{n/2-n} \quad (330)$$

Actual measurements show that $0 < n < 1$.

In problems involving flow in pipes and wind-tunnels, thermal influences are absent, and n is a function of the Reynolds number and the roughness alone. Experiments on smooth pipes up to a Reynolds number of 10^5 show that the velocity distribution follows the seventh-power law:

$$\bar{u} = u_1 \left(\frac{z}{z_1} \right)^{1/7} \quad (331)$$

corresponding to $n = 1/4$. Substituting Eqs 326 and 331 and $n = \frac{1}{4}$ in Eq 328 one has

$$\tau = \frac{0.020}{z_1^{1/4}} \rho \nu_1^{1/4} \bar{u}_1^{7/4}$$

The experimental result is

$$\tau = \frac{0.023}{z_1^{1/4}} \rho \nu^{1/4} \bar{u}_1^{7/4}$$

where ν is the kinematic viscosity. This enables one to identify ν_1 with ν .

3. Evaporation from natural water bodies

After dealing with the wind structure, one is now in a position to study the evaporation from water surfaces occurring in nature. The aim will be the determination of the effect of the size and the shape of the water body and the effect of the mean wind velocity on the rate of evaporation.

To fix ideas, let the evaporation surface be level with the ground, and let it be of such dimensions that it produces no sensible variations in the normal wind structure, and also such that the increase in the vapor content of the air, as it flows over the surface, is never large enough to affect the rate of evaporation from the leeward side of the surface. In practice this latter limitation, owing to the rapid mass exchange taking place normally over the surface, would exclude from consideration only very large water surfaces.

Substituting Eq 330 in Eq 326, the exchange coefficient is

$$A(z) = \rho a \bar{u}_1^{1-n} z^{\frac{2(1-n)}{2-n}} \tag{332}$$

where

$$a = \left| \frac{(0.251)^{1-n} (2-n)^{1-n} n^{1-n}}{(1-n)(2n-2)^{2(1-n)}} \right| \nu^n z^{\frac{n^2-n}{2-n}} \tag{333}$$

depends only on n , z , and ν , and can be treated as a constant. Substituting Eq 332 into the diffusion equation

$$\bar{u} \frac{\partial \Omega}{\partial x} = \frac{1}{\rho} \frac{\partial}{\partial z} \left[A(z) \frac{\partial \Omega}{\partial z} \right] \tag{334}$$

where $\Omega = c - c_0$ is the excess of vapor concentration at any point over that of the ambient air, one obtains

$$\begin{aligned} \frac{(\bar{u}_1)^n}{a} z^{\frac{n}{2-n}} \frac{\partial \Omega}{\partial x} &= \frac{\partial}{\partial z} \left[z^{\frac{2(1-n)}{2-n}} \frac{\partial \Omega}{\partial z} \right] \\ \text{or, writing } m = n/(2-n) &\approx 1/8, \\ \frac{(\bar{u}_1)^n}{a} \frac{\partial \Omega}{\partial x} &= z^{-m} \frac{\partial}{\partial z} \left[z^{1-m} \frac{\partial \Omega}{\partial z} \right] \end{aligned} \tag{335}$$

the boundary conditions being, with x_0 denoting the length of the surface in the direction of the wind,

(i) $\lim_{z \rightarrow 0} \Omega(x, z) = \Omega_0 = c_s - c_0$, c_s being the value of c at $z = 0$.
 (0 ≤ x ≤ x₀)

(ii) $\lim_{z \rightarrow 0} \Omega(x, z) = 0$ (0 ≤ x ≤ x₀)

(iii) $\lim_{x \rightarrow 0} \Omega(x, z) = 0$ (0 < z)

One now makes the transformations

$$\phi = \frac{\Omega}{\Omega_0}, \quad \xi = \frac{x}{x_0}, \quad \zeta = \left(\frac{u_1 n}{a x_0}\right)^{\frac{1}{2}} z^{m + \frac{1}{2}}$$

such that Eq 335 becomes

$$\frac{\partial \phi}{\partial \zeta} = \frac{\partial^2 \phi}{\partial \zeta^2} + \frac{1}{(2m+1)\zeta} \frac{\partial \phi}{\partial \zeta} \tag{336}$$

with the boundary conditions

(i) $\lim_{\zeta \rightarrow 0} \phi(\xi, \zeta) = 1$ (0 < ξ ≤ 1)

(ii) $\lim_{\zeta \rightarrow \infty} \phi(\xi, \zeta) = 0$ (0 ≤ ξ ≤ 1)

(iii) $\lim_{\xi \rightarrow 0} \phi(\xi, \zeta) = 0$ (0 < ζ)

The evaporation surface is now defined by

$$0 \leq \xi \leq 1, \quad \zeta = 0$$

In Eq 336 write

$$\phi(\xi, \zeta) = \zeta^p \psi(\xi, \zeta)$$

where $p = m/(2m+1) \leq n/(2+n)$, so that $0 < p < 1/3$ since $0 < n < 1$. Then the equation for ψ is

$$\frac{\partial \psi}{\partial \zeta} = \frac{\partial^2 \psi}{\partial \zeta^2} + \frac{1}{\zeta} \frac{\partial \psi}{\partial \zeta} - \frac{p^2}{\zeta^2} \psi$$

the solution of which is

$$\psi(\xi, \zeta) = \frac{C}{\xi - h} \text{Exp}\left(-\frac{\zeta^2 + 4\alpha^2}{4(\xi - h)}\right) K_p\left(\frac{\alpha \zeta}{\xi - h}\right)$$

for every h, α , and C , where K_p is the modified Bessel function of the second kind. Taking $h = 0$, and choosing C as a suitable function of α , one is led to the expression

$$\phi(\xi, \zeta) = \zeta^p \psi(\xi, \zeta) = \frac{2 \tan p\pi}{\pi} \zeta^p \int_0^\infty \frac{(2\alpha)^{1-p}}{\xi} \text{Exp}\left(-\frac{\zeta^2 + 4\alpha^2}{4\xi}\right) K_p\left(\frac{\alpha \zeta}{\xi}\right) d\alpha$$

Using this, one can show by means of the expansion of $\omega^p k_p(\omega)$ near $\omega = 0$, and the asymptotic expansion of $K_p(\omega)$ for large ω , that the boundary

conditions for ϕ are satisfied,

The total evaporation per unit width over the length x_0 is then

$$E = \int_0^{\infty} \bar{u} \Omega dz = \frac{\Omega_0 \bar{u}_1}{z_1^m} \int_0^{\infty} z^m \phi(1, \xi) dz = \frac{2 \Omega_0 \bar{u}_1}{(2m+1) z_1^m (ax_0)^{\frac{m+1}{2}}} \int_0^{\infty} \phi(1, \xi) \xi^{\frac{1}{2m+1}} d\xi$$

$$= \frac{\Omega_0}{z_1^m} u_1^{1 - \frac{n(m+1)}{2m+1}} (ax_0)^{\frac{m+1}{2m+1}} K \quad (337)$$

where

$$K = \frac{2}{2m+1} \int_0^{\infty} \phi(1, \xi) \xi^{\frac{1}{2m+1}} d\xi \quad (338)$$

is a constant independent of u_1 , a , and x_0 . The expressions Sutton obtained for E and K are different from the above, and seem to be in error. Fortunately, the discrepancies do not invalidate the experimental verifications which Sutton cited for his theory, as will be presented in the next section.

Since $m = n/(2-n)$ one has

$$\frac{m+1}{2m+1} = \frac{2}{2+n}, \quad 1 - \frac{n(m+1)}{2m+1} = \frac{2-n}{2+n}$$

Hence

$$E = \Omega_0 z_1^{-\frac{n}{2-n}} \bar{u}_1^{\frac{2-n}{2+n}} (ax_0)^{\frac{2}{2+n}} K \quad (339)$$

A few special examples may now be mentioned. The results are obtained by direct integration of Eq 339, remembering that lateral diffusion is neglected.

(i) Rectangular lake

$$E = \Omega_0 z_1^{-\frac{n}{2-n}} \bar{u}_1^{\frac{2-n}{2+n}} (ax_0)^{\frac{2}{2+n}} y_0 K \quad (340)$$

(ii) Elliptic lake, One has,

$$dE = \Omega_0 z_1^{-\frac{n}{2-n}} \bar{u}_1^{\frac{2-n}{2+n}} (ax)^{\frac{2}{2+n}} K dy$$

Then writing (r_1 being the semi-axis down wind, r_2 being that crosswind)

$$x = 2r_1 \cos \theta, \quad y = r_2 \sin \theta$$

one has

$$E = \Omega_0 z_1^{-\frac{n}{2-n}} \bar{u}_1^{\frac{2-n}{2+n}} (2a)^{\frac{2}{2+n}} r_1^{\frac{2}{2+n}} r_2 K K' \quad (341)$$

where

$$K' = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta)^{\frac{2}{2+n}} \cos\theta d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2\theta)^{\frac{1}{2+n}} \cos\theta d\theta = \int_{-1}^1 (1 - \xi^2)^{\frac{1}{2+n}} d\xi$$

(iii) Circular lake with radius r_0 . Putting $r_1 = r_2$ in Eq 341, one has

$$E = \Omega_0 Z^{-\frac{n}{2+n}} \bar{U}_m^{\frac{2-n}{2+n}} (2a)^{\frac{2}{2+n}} r^{\frac{4+n}{2+n}} \quad (343)$$

In general, for similar shapes,

$$E \sim L^{\frac{4+n}{2+n}}$$

where L is a representative length. It must be noted that Sutton's expression for E in Case (ii) above is based on the wrong expression $\chi = r_1(1 + \cos\theta)$, and is in error. The same applies to case (iii). Eqs. 341 to 343 are the corrected forms. Again, Sutton's errors do not invalidate his deduction

$$E \sim L^{\frac{4+n}{2+n}}$$

4. Comparison with experiments and observation.

Concerning the effect of size, experiments were performed by Gellenkamp (99a, 1919) by rotating the evaporation surface on a cross at a low speed so that the relative velocity at any point is between $\frac{1}{2}$ and 1 m per sec. At these low speeds the flow is laminar, and Sutton claims $n = 1$. According to theory $E \sim \chi_0^{2/3} y_0$ and $E \sim r^{1.67}$ for the rectangular and the circular lakes respectively. Experimental results show, respectively $E \sim \chi_0^{0.6} y_0$ and $E \sim r^{1.6}$ indicating good agreement with theory. Sutton also claims that the experiments of Thomas and Ferguson (200a, 1917) support his theory. He has not, however, cleared the difficulty introduced by $n = 1$ in the case of Gellenkamp's experiments. When $n = 1$, Eqs 326 and 332 are invalid, and the theory breaks down.

Concerning the effect of wind velocity, the wind-tunnel experiments of Himus (106a) give $E \sim \bar{U}_m^{0.77}$, and those of Hine (106b, 1924) on nitrobenzene, toluene, m-xylene, and chlorobenzene give $E \sim \bar{U}_m^{0.78}$ rather conclusively. Here $n = \frac{1}{4}$, and the theory gives $E \sim \bar{U}_m^{7/9} \sim \bar{U}_m^{7/9} \doteq \bar{U}_m^{0.78}$, in good agreement with the experiments.

5. Later developments

Lateral diffusion which was neglected by O. G. Sutton has been considered by Davies (82, 1947); 83, 1950). The weak point of Davies' theory lies in the uncertainty of the magnitude of the lateral diffusivity relative to that of the vertical diffusivity. Davies' expression for the lateral diffusivity has not been undisputably verified by experiment.

O. G. Sutton's equation of diffusion, Eq 335, has been thoroughly discussed from a mathematical point of view by W. G. L. Sutton (187, 1943).

C. Pasquill's Modification of Sutton's Theory

Pasquill (143, 1943) changed the ν in Sutton's theory just presented to K , the vapor diffusivity, and obtained $(\sigma = \frac{\nu}{K})$

$$E_p = \sigma^{\frac{2n}{2+n}} E_s$$

the subscripts P and S denote the authors according to whom the E is evaluated.

In comparison with the experiments which Pasquill carried out in the wind-tunnel (for which $n = \frac{1}{4}$), E_s is too large and E_p too small, though the deviations are both small.

Kuo, who reviewed the works of Sutton and Pasquill while working for the present project, suggested using $\sqrt{\nu^2/\sigma} = \sqrt{\nu K}$ instead of ν or K . Kuo's suggestion gives good agreement with Pasquill's experiments, as shown in Table 1.

D. Sutton's Theory for Rough Surfaces

1. Velocity Profile

It has been shown before that the velocity distribution over a smooth surface can be adequately represented by the logarithmic formula

$$\frac{\bar{U}}{V_*} = 2.5 \ln\left(\frac{V_* Z}{\nu}\right) + 5.5 = 2.5 \left(\frac{9V_* Z}{\nu}\right) \tag{344}$$

It seems that for a rough surface the following analogous formula can be used:

$$\frac{\bar{U}}{V_*} = 2.5 \ln\left(\frac{Z}{Z_0}\right) \tag{345}$$

where z_0 is the constant of integration determined by

$$\bar{U} = 0 \quad \text{at} \quad Z = Z_0$$

and is usually called the roughness length. Measurements in sand pipes show that $z_0 = \frac{k_s}{30}$ where k_s is the average diameter of the sand grains.

According to Nikuradse, the surface may be considered as

aerodynamically smooth for $\frac{V_* k_s}{\nu} < 4$ or $\frac{V_* z_0}{\nu} < 0.13$

in the transition zone for $4 < \frac{V_* k_s}{\nu} < 75$ or $0.13 < \frac{V_* z_0}{\nu} < 2.5$

fully rough for $\frac{V_* k_s}{\nu} > 75$ or $\frac{V_* z_0}{\nu} > 2.5$

An alternative form for the velocity distribution due to Rossby and Montgomery

(166a, 1935) is $\frac{\bar{u}}{u_*} = 2.5 \ln \left(\frac{z + z_0}{z_0} \right)$ (346)

Both Eqs 345 and 346 do not reduce to Eq 344 as z_0 approaches zero. O. G.

Sutton (186a, 1949) proposed the following form which will reduce to

Eqs 344 and 345 depending on the roughness of the surface.

$$\frac{\bar{u}}{V_*} = 2.5 \ln \left(\frac{V_* z}{N + \nu/9} \right) \quad (347)$$

and the following one

$$\frac{\bar{u}}{V_*} = 2.5 \ln \left(\frac{V_* z + N}{N + \nu/9} \right) \quad (348)$$

reducible to Eqs 344 and 346.

According to Schlichting, the surface is

smooth for $N < 0.13\nu \doteq 0.02 \text{ cm}^2 \text{ sec}^{-1}$

rough for $N > 2.5\nu \doteq 0.4 \text{ cm}^2 \text{ sec}^{-1}$

When the surface is fully rough, the velocity profile can be approximated

by the power form

$$\bar{u} = \frac{\bar{u}_1}{\left[\left(\frac{z_1}{z_0} \right)^\rho - 1 \right]} \left[\left(\frac{z}{z_0} \right)^\rho - 1 \right] \quad (349)$$

or by the following form corresponding to Eq 346:

$$\bar{u}_1 = \frac{\bar{u}_1}{\left[\left(\frac{z_1 + z_0}{z_0} \right)^\rho - 1 \right]} \left[\left(\frac{z + z_0}{z_0} \right)^\rho - 1 \right] \quad (350)$$

The kinematic viscosity may be introduced to take care of the transition from

smooth to rough flow. When z_0 is small, the above two formulas reduce to the

usual power form

$$\bar{u} = \bar{u}_1 \left(\frac{z}{z_0} \right)^\rho \quad (351)$$

It may be noted that the logarithmic profiles as well as their approximations are not adequate for z_0 greater than 2 or 3 meters, according to observations, and that z_0 will not only increase with roughness but also increase with the rate of inversion in the atmosphere.

For the Lagrangian correlation coefficient, Sutton proposed the form

$$R_{\xi} = \left(\frac{N + \nu}{N + \nu + \frac{w' z \xi}{g}} \right)^n \quad (n > 0) \quad (3E2)$$

where ordinarily $N \gg \nu$. An inconsistency of the theory lies in the fact that in the above equation $N + \nu$ corresponds to ν' or ν in Eq 317, while in Eq 347 $N + \frac{\nu}{g}$ corresponds to $\frac{\nu}{g}$ in Eq 344, and in regard to Eq 348 a correspondence cannot be established.

Provided that N is not excessively large, say not greater than $10^3 \text{ cm}^2 \text{ sec}^{-1}$, the development of the present theory proceeds exactly as in the original smooth surface development presented in B, the condition of constant shear again leading to the relation

$$p = n / (2 - n)$$

2. Vertical diffusivity

Following the same development as in the case of smooth surface, the vertical diffusivity is

$$A(z) = \frac{(\pi/2 k^2)^{1-n} N^n}{1-n} \left[\left(\frac{d\bar{u}}{dz} \right)^3 \left(\frac{d^2\bar{u}}{dz^2} \right)^{-2} \right]^{1-n}$$

where k is again Kármán's constant. Substituting Eq 350 with $p=n/(2-n)$ into the equation above,

$$A(z) = \frac{a(n) N^n \bar{u}_1^{1-n}}{\left[(z_1 + z_0)^{n/2-n} - z_0^{n/2-n} \right]^{1-n}} (z + z_0)^{\frac{2(1-n)}{2-n}} \quad (353)$$

where

$$a(n) = \frac{(\pi/2 k^2)^{1-n} n^{1-n} (2-n)^{1-n}}{(1-n)(2-2n)^{1-n}}$$

from which one has the following table:

n	0.1	0.15	0.20	0.25	0.30
a(n)	0.025	0.050	0.086	0.138	0.212

3. Diffusion in two dimensions.

Substituting Eqs 350 and 353 into Eq 334, one has

$$\frac{\bar{u}_1}{\left[\left(\frac{z+z_0}{z_0}\right)^{n/2-n-1}\right]} \left[\left(\frac{z+z_0}{z_0}\right)^{n/2-n-1} \right] \frac{\partial \Omega}{\partial x} = \frac{a(n) N^n \bar{u}_1^{1-n}}{\left[\left(\frac{z+z_0}{z_0}\right)^{n/2-n} - z_0^{n/2-n} \right]^{1-n}} \frac{\partial}{\partial z} \left\{ \left(\frac{z+z_0}{z_0}\right)^{2(1-n)/2-n} \frac{\partial \Omega}{\partial z} \right\} \quad (354)$$

which for uniform \bar{u} becomes

$$\bar{u}_1 \frac{\partial \Omega}{\partial x} = \frac{a(n) N^n \bar{u}_1^{1-n}}{\left[\left(\frac{z+z_0}{z_0}\right)^{n/2-n} - z_0^{n/2-n} \right]^{1-n}} \frac{\partial}{\partial z} \left(z^{\frac{2(1-n)}{2-n}} \frac{\partial \Omega}{\partial z} \right) \quad (355)$$

and for the power distribution of \bar{u} becomes

$$\bar{u}_1 \left(\frac{z}{z_0}\right)^m \frac{\partial \Omega}{\partial x} = \frac{a(n) N^n \bar{u}_1^{1-n}}{\left[\left(\frac{z+z_0}{z_0}\right)^{n/2-n} - z_0^{n/2-n} \right]^{1-n}} \frac{\partial}{\partial z} \left(z^{\frac{2(1-n)}{2-n}} \frac{\partial \Omega}{\partial z} \right) \quad (356)$$

For a steady constant line source, the boundary conditions are

- (i) $\Omega \rightarrow 0$ as $x \rightarrow \infty$ for $z \geq 0$
- (ii) $A \frac{\partial \Omega}{\partial z} \rightarrow 0$ as $z \rightarrow 0$ for $x > 0$ (impervious ground)
- (iii) $\Omega \rightarrow \infty$ at $x = z = 0$

and the continuity condition is

$$(iv) \int_0^{\infty} \bar{u} \Omega dz = Q = \text{strength of source, for all } x > 0.$$

Writing

$$A(z) = \alpha \bar{u}_1^{1-n} (z+z_0)^{\frac{2(1-n)}{2-n}} = \alpha \bar{u}_1^{1-n} z^{\frac{2(1-n)}{2-n}} \quad (357)$$

the solutions of Eqs 355 and 356 are, to a satisfactory degree of approximation, respectively:

$$\Omega(x, z) = \frac{Q}{\left(\frac{z}{z_0}\right)^{2-n} \Gamma\left(\frac{4-n}{2}\right) \alpha^{\frac{1}{2}(2-n)} \bar{u}_1^{1-n-\frac{1}{2}n^2} z^{\frac{1}{2}(2-n)}} \exp\left\{-\frac{\bar{u}_1^n z^{\frac{2}{2-n}}}{\left(\frac{z}{z_0}\right)^2 \alpha x}\right\} \quad (358)$$

$$\Omega(x, z) = \frac{B(m, n, \alpha, \bar{u}_1)}{x^{(m+1)(2-n)/(2+2m-nn)}} \exp\left\{-\frac{\bar{u}_1^n z^{m+\frac{2}{2-n}}}{\left(m+\frac{2}{2-n}\right)^2 \alpha x z_1^m}\right\} \quad (359)$$

where B denotes a somewhat complicated expression whose value is easily obtained by applying condition (iv).

Measurements at Porton, England, were quoted by Sutton to support Eqs 358 and 359. It should be noted that the value n in Sutton's theory is always obtained from the velocity profile, even though in one of the approximate

solutions the velocity is assumed to be constant. Sutton did not mention how this constant velocity was to be chosen.

4. Approximate formulae for 3-dimensional diffusion

Sutton assumed the lateral and vertical diffusivities to be

$$C_y^2 = \frac{4N^n}{(1-n)(2-n)\bar{u}^n} g_y^{2(1-n)} \quad (360)$$

$$C_z^2 = \frac{4N^n}{(1-n)(2-n)\bar{u}^n} g_z^{2(1-n)} = (g_z/g_y)^2 C_y^2 \quad (361)$$

where

$$g_y = \sqrt{v'^2/\bar{u}^2}, \quad g_z = \sqrt{w'^2/\bar{u}^2}$$

Assuming constant \bar{u} , but again obtaining n from the velocity profile,

the approximate solutions for a point source and a line source are, respectively,

$$\Omega(x, y, z) = \frac{Q}{\pi C_y C_z \bar{u} x^{2-n}} \exp\left\{-x^{n-2} (y^2/C_y^2 + z^2/C_z^2)\right\} \quad (362)$$

$$\Omega(x, z) = \frac{Q}{\sqrt{\pi} C_z \bar{u} x^{1-n/2}} \exp\left\{-x^{n-2} z^2/C_z^2\right\} \quad (363)$$

Measurements made at Porton on diffusion from a point source were quoted by Sutton to support his theory. It should be noticed however that Eqs 360 and 361 are not dimensionally correct.

5. Experimental verification of the form of the Lagrangian correlation coefficient.

Measurements made by P. A. Sheppard at Porton in 1936 showed that the form of Eq 352 was verified with $n = 0.15$, $N = 100 \text{ cm}^2 \text{ sec}^{-1}$, and $\bar{u}'^2 = 6510 \text{ cm}^2 \text{ sec}^{-2}$ at $z_1 = 2\text{m}$.

Integration of Eq 352 between the time limits 0 and 2 gives

$$A(z_1) = \bar{u}'^2 \int_0^2 R_\xi d\xi = \frac{N^n}{1-n} (2\bar{u}'^2)^{1-n} \quad (364)$$

at $z_1 = 2\text{m}$, neglecting a term of higher order in N as compared with the term retained. With the measured n , N , and \bar{u}'^2 , $A(z_1)$ can be computed from Eq 364 to be $8100 \text{ cm}^2 \text{ sec}^{-1}$. On the other hand, with the measured \bar{u}'^2 and R_ξ , $A(z_1)$ can be computed from the same equation to be $8600 \text{ cm}^2 \text{ sec}^{-1}$. This

verification is, however, quite superfluous after the form of the Lagrangian correlation coefficient has been verified. It may be remarked in this connection that any verification of R_{ξ}^0 is more meaningful if n and N are obtained independently from the velocity profile.

E. Frost's Theory

1. Mass-transfer from a plane boundary.

Assuming

$$\lambda = z^{1-m} z_0^m \quad (0 < m < 1) \quad (365)$$

where z is the height. z_0 is a length that bears some relation to the roughness of the surface considered, and λ is Prandtl's mixing length. Frost obtained (97, 1946)

$$E = \lambda^2 \frac{\partial \bar{u}}{\partial z} = z^{2-2m} z_0^{2m} \frac{\partial \bar{u}}{\partial z} \quad (366)$$

$$\tau = \rho z^{2-2m} z_0 \left(\frac{\partial \bar{u}}{\partial z} \right)^2 \quad (367)$$

It has been shown in 1933 by Ertel from observations that the quantity τ/ρ is practically constant for $z < 100$ ft. Hence by integrating the last equation

$$\bar{u} = \frac{1}{m} \sqrt{\frac{\tau}{\rho}} \left(\frac{z}{z_0} \right)^m \quad (368)$$

or

$$\bar{u} = \bar{u}_1 \left(\frac{z}{z_1} \right)^m \quad (369)$$

where u_1 corresponds to a standard height z_1 . This is the usual power law and m is not far from 1/7. Using the above expression for \bar{u} , the expressions for E and τ become

$$E = m z^{1-m} z_0^{2m} \bar{u}_1 z_1^{-m} \quad (370)$$

$$\tau = \rho m^2 z_0^{2m} \bar{u}_1^2 z_1^{-2m} \quad (371)$$

For $z_1 = 15$ meters, $m = 1/7$, the measurements of Rossby-Wust over the rough ocean surfaces gave $\tau = 2.6 \times 10^{-3} \rho \bar{u}^2$

which corresponds to $z_0 = 1$ cm. Over the plain, for $z_1 = 30$ ft. and $m = 1/7$, Taylor's measurements gave $\tau = 2.7 \times 10^{-3} \rho \bar{u}_1^2$ corresponding to $z_0 = 2.6$ cm.

Observations of Sverdrup (1936) of wind velocities, temperatures and vapor pressures at various heights over a snow surface showed that eddy viscosity and the eddy diffusivities, thermal or vapor, obey the same power law and moreover are identical.

With ϵ given by Eq(370) the diffusion equation

$$\bar{u} \frac{\partial (c-c_0)}{\partial x} = \frac{\partial}{\partial z} \left(\epsilon \frac{\partial (c-c_0)}{\partial z} \right) \quad (372)$$

can be solved. By the substitution

$$\xi = \frac{z^{2m+1}}{m z_0^{2m+1} x} \quad (373)$$

Eq 372 with ϵ given by Eq 370 becomes

$$(2m+1)^2 \xi \frac{d^2(c-c_0)}{d\xi^2} + [(2m+1)(m+1) + \xi] \frac{d(c-c_0)}{d\xi} = 0 \quad (374)$$

A first integration gives

$$\frac{d(c-c_0)}{d\xi} = -(c_s - c_0) G \xi^{-\frac{m+1}{2m+1}} e^{-\frac{\xi}{(2m+1)^2}} \quad (375)$$

where G is a constant. A second integration gives

$$c - c_0 = (c_s - c_0) G \int_{\xi}^{\infty} \xi^{-\frac{m+1}{2m+1}} e^{-\frac{\xi}{(2m+1)^2}} d\xi \quad (376)$$

where G is determined by the boundary condition $c = c_s$ when $\xi = 0$

to be

$$G = \left[(2m+1)^{\frac{2m}{2m+1}} \Gamma\left(\frac{m}{2m+1}\right) \right]^{-1} \quad (377)$$

where Γ denotes the gamma-function. The result can be simplified by the transformation

$$\xi = (2m+1)^2 \eta \quad (378)$$

to the form

$$c - c_0 = \frac{(c_s - c_0) \int_{\eta}^{\infty} \eta^{-\frac{m+1}{2m+1}} e^{-\eta} d\eta}{\Gamma\left(\frac{m}{2m+1}\right)} \quad (379)$$

Table 2 gives various values of $(c-c_0)/(c_s-c_0)$ for different values of η .

For $m = 1/7$, $z_0 = 1$ cm.

$$\eta = \frac{4.23 z^{9/7}}{x}$$

Significant is the fact that $(c-c_0)/(c_s-c_0)$ is a function of η alone and does not depend on the magnitude of the ambient velocity. One cannot expect this to be true when the ambient velocity is very small.

From the solution, the local rate of evaporation is

$$e(x) = \lim_{z \rightarrow 0} \left(-\epsilon \frac{\partial c}{\partial z} \right) = \frac{(2m+1) \frac{1}{2^{m+1}} x^{-\frac{m}{2m+1}} (m z_0^{2m})^{\frac{m+1}{2m+1}} \bar{u}_1 z_1^{-m}}{\Gamma\left(\frac{m}{2m+1}\right)} (c_s - c_0) \quad (380)$$

and the rate of evaporation from a strip of length x_0 and unit width is

$$E = \int_0^{x_0} e(x) dx = \frac{(2m+1) \frac{1}{2^{m+1}} (m x_0 z_0^{2m})^{\frac{m+1}{2m+1}} \bar{u}_1 z_1^{-m}}{(m+1) \Gamma\left(\frac{m}{2m+1}\right)} (c_s - c_0) \quad (381)$$

When $m = 1/7$, $z_0 = 1$ cm.

$$E = 2.85 \times 10^{-2} x^{8/9} (c_s - c_0) \bar{u}_1 z_1^{1/7}$$

For small values of η , Eq 379 can be approximated by

$$c - c_0 = (c_s - c_0) \left[1 - \frac{\eta \frac{m}{2^{m+1}}}{\Gamma\left(1 + \frac{m}{2m+1}\right)} \right] \quad (382)$$

or

$$c_s - c = (c_s - c_0) \frac{z^m}{\left[(2m+1)^2 m z_0^{2m} \right]^{\frac{m}{2m+1}} \Gamma\left(1 + \frac{m}{2m+1}\right)} \quad (383)$$

From the above equation and Eq 380,

$$e(x) = \frac{m^2 z_0^{2m} \bar{u}_1 (c_s - c_0)}{z_1^{2m}} \quad (384)$$

2. Mass-transfer from an infinite line source

The diffusion equation is (with $\Omega = c - c_0$)

$$z^m \frac{\partial \Omega}{\partial x} = a_1 \frac{\partial}{\partial z} \left(z^{1-m} \frac{\partial \Omega}{\partial z} \right) \quad (385)$$

where $a_1 = m z_0^{2m}$, z_0 being the characteristic roughness of the surface. With x measured from the line source, the boundary conditions are

(i) $\Omega \rightarrow 0$ as $x \rightarrow 0$ for $z \neq 0$

(ii) $\Omega \rightarrow 0$ as $z \rightarrow \infty$

and the continuity relation for an impermeable ground surface is

(iii) $\int_0^{\infty} \bar{u} \Omega dz = Q = \text{constant}$

With the transformation

$$\Omega = \chi^{-\frac{1+m}{1+2m}} f(\theta) \tag{386}$$

where

$$\theta = \frac{z^{2m+1}}{(2m+1)^2 a_1 \chi}$$

Eq 385 becomes

$$\theta f'' + \left(\theta + \frac{1+m}{1+2m}\right) f' + \frac{1+m}{1+2m} f = 0 \tag{387}$$

or

$$\left(\theta \frac{d}{d\theta} + \frac{1+m}{1+2m}\right) \left(\frac{d}{d\theta} + 1\right) f = 0$$

whence

$$f(\theta) = B e^{-\theta} + C e^{-\theta} \int e^{\theta} \theta^{-\frac{1+m}{1+2m}} d\theta$$

B. C. (i) and (ii) require that $\Omega \rightarrow 0$ when $\theta \rightarrow \infty$, hence $C = 0$, and

$$\Omega = B \chi^{-\frac{m+1}{2m+1}} e^{-\theta} \tag{388}$$

where B must satisfy the equation

$$Q = \int_0^{\infty} B e^{-\theta} \chi^{-\frac{m+1}{2m+1}} \bar{u}_1 z_1^{-m} d\left(\frac{z^{m+1}}{m+1}\right) = B \bar{u}_1 z_1^{-m} (2m+1)^{\frac{1}{2m+1}} a_1^{\frac{m+1}{2m+1}} \Gamma\left(\frac{m+1}{2m+1}\right)$$

Frost claims that observations in England support the validity of Eq 388, with the distribution of Ω independent of the wind velocity.

With the solution for the line source given by Eq 388, the distribution of Ω on land after a sea crossing can be obtained by integration to be

$$\Omega = \frac{\Omega_0}{\Gamma\left(\frac{m}{2m+1}\right) \Gamma\left(\frac{m+1}{2m+1}\right)} \int_0^x \lambda^{-\frac{m}{2m+1}} (x-\lambda) \exp\left(-\frac{z^{2m+1}}{(2m+1)^2 a_1 (x-\lambda)}\right) d\lambda \tag{389}$$

where $\Omega_0 = c_s - c_0$.

For $z = 0$, Eq 389 can be reduced by the transformations

$$x = b\xi, \quad \lambda = b\xi\eta$$

to the following form

$$\Omega = \frac{\Omega_0}{\Gamma\left(\frac{m}{2m+1}\right)\Gamma\left(\frac{m+1}{2m+1}\right)} \int_{\frac{1}{5}}^1 \eta^{-\frac{m}{2m+1}} (1-\eta)^{-\frac{m+1}{2m+1}} d\eta$$

$$= \Omega_0 \left[1 - \frac{1}{\Gamma\left(\frac{m}{2m+1}\right)\Gamma\left(\frac{m+1}{2m+1}\right)} \int_{\frac{1}{5}}^1 \eta^{-\frac{m}{2m+1}} (1-\eta)^{-\frac{m+1}{2m+1}} d\eta \right] \quad (390)$$

For $m = 1/7$,

$$\Omega = \Omega_0 \left[1 - \frac{1}{\Gamma\left(\frac{1}{15}\right)\Gamma\left(\frac{8}{15}\right)} \int_{\frac{1}{5}}^1 \eta^{-\frac{1}{9}} (1-\eta)^{-\frac{8}{9}} d\eta \right]$$

which, on writing $\xi = 1-\eta$, becomes

$$\Omega = \Omega_0 \left[1 - \frac{1}{\frac{1}{9}\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)} \int_0^{1-\frac{1}{5}} \xi^{-\frac{1}{9}} (1-\xi)^{-\frac{8}{9}} d\xi \right]$$

If one writes further $\xi = 1+\epsilon$, one has

$$\Omega \doteq \Omega_0 \left[1 - \frac{\epsilon^{\frac{1}{9}}}{\frac{1}{9}\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{8}{9}\right)} \right] \doteq \Omega_0 (1 - 0.98\epsilon^{\frac{1}{9}}) \quad (391)$$

where $\epsilon = (x-b)/b$. Frost gave an example showing that Eq 391 is 6.3% in error, and attributed this error to the lack of consideration of the land roughness. In the following section land roughness will be considered.

3. Distribution of water vapor over land after a sea-crossing

From Eq 382, after a sea-crossing of length b , Ω is given as a function of z to a high degree of approximation by

$$\Omega = \Omega_0 \left[1 - \frac{z^{2m}}{\Gamma\left(1 + \frac{m}{2m+1}\right) [(2m+1)^2 ab]^{\frac{m}{2m+1}}} \right] \quad (392)$$

where $a = mz_0^{2m}$. Differentiation of the above equation with respect to z gives the following equation for the intersection of land and sea:

$$-K(z) \frac{\partial \Omega}{\partial z} = D\Omega_0 \quad (393)$$

where

$$D = \frac{m a u_0 z_0^{-m}}{\Gamma\left(1 + \frac{m}{2m+1}\right) [(2m+1)^2 ab]^{\frac{m}{2m+1}}} = \text{constant}$$

Over land, the differential equation to be satisfied is

$$z^m \frac{\partial \Omega}{\partial x} = a_1 \frac{\partial}{\partial z} \left(z^{1-m} \frac{\partial \Omega}{\partial z} \right) \quad (394)$$

where $a_1 = m z_0^{2m}$. In this equation x is measured downwind from the leading edge of the land surface,

Writing

$$S = -a_1 \bar{u}_1 z_1^{-m} z^{1-m} \frac{\partial \Omega}{\partial z} \tag{395}$$

the diffusion equation can be written as

$$\frac{\partial S}{\partial x} = a_1 z^{1-m} \frac{\partial}{\partial z} \left(z^{-m} \frac{\partial S}{\partial z} \right) \tag{396}$$

with the boundary conditions

- (i) $S \rightarrow D \Omega_0$ as $x \rightarrow 0$
- (ii) $S \rightarrow 0$ as $z \rightarrow 0$ for $x > 0$

Make the transformation

$$\theta = \frac{z^{2m+1}}{(2m+1)^2 a_1 x} \tag{397}$$

Eq 396 becomes

$$\theta S'' + \left(\theta + \frac{m}{2m+1} \right) S' = 0 \tag{398}$$

the solution of which is

$$S = \frac{D \Omega_0}{\Gamma\left(\frac{m+1}{2m+1}\right)} \int_0^\theta e^{-\theta} \theta^{-\frac{m}{2m+1}} d\theta \tag{399}$$

with all the boundary conditions satisfied. From Eqs 395 and 398 one has

$$\frac{\partial \Omega}{\partial z} = - \frac{D \Omega_0 z^{m-1}}{a_1 \bar{u}_1 z_1^{-m} \Gamma\left(\frac{m+1}{2m+1}\right)} \int_0^\theta e^{-\theta} \theta^{-\frac{m}{2m+1}} d\theta \tag{400}$$

integration (by parts) of which gives

$$\begin{aligned} \Omega &= \Omega_0 \left\{ 1 - \frac{D}{m a_1 \bar{u}_1 z_1^{-m} \Gamma\left(\frac{m+1}{2m+1}\right)} \left[z^m \int_0^\theta e^{-\theta} \theta^{-\frac{m}{2m+1}} d\theta + e^{-\theta} [(2m+1)^2 a_1 x]^{\frac{m}{2m+1}} \right] \right\} \\ &= \Omega_0 \left\{ 1 - \frac{a z^m \int_0^\theta e^{-\theta} \theta^{-\frac{m}{2m+1}} d\theta + e^{-\theta} [(2m+1)^2 a_1 x]^{\frac{m}{2m+1}}}{a_1 \Gamma\left(\frac{m+1}{2m+1}\right) \Gamma\left(1 + \frac{m}{2m+1}\right) [(2m+1)^2 a_1 x]^{\frac{m}{2m+1}}} \right\} \\ &= \Omega_0 \left\{ 1 - \left(\frac{a}{a_1}\right)^{\frac{m+1}{2m+1}} \left(\frac{x}{b}\right)^{\frac{m}{2m+1}} \left[\frac{e^{-\theta} + \theta^{\frac{m}{2m+1}} \int_0^\theta e^{-\theta} \theta^{-\frac{m}{2m+1}} d\theta}{\Gamma\left(1 + \frac{m}{2m+1}\right) \Gamma\left(\frac{m+1}{2m+1}\right)} \right] \right\} \end{aligned} \tag{401}$$

With $m = 1/7$

$$\Omega = \Omega_0 \left\{ 1 - 0.98 \left(\frac{x}{b} \right)^{\frac{1}{9}} \left(\frac{z_0}{z_0'} \right)^{\frac{16}{63}} \left[e^{-\theta} + \theta^{\frac{1}{9}} \int_0^{\theta} e^{-\theta} \theta^{-\frac{1}{9}} d\theta \right] \right\} \quad (402)$$

For small values of θ the last equation becomes

$$\Omega = \Omega_0 \left\{ 1 - 0.98 \left(\frac{x}{b} \right)^{\frac{1}{9}} \left(\frac{z_0}{z_0'} \right)^{\frac{16}{63}} \left(1 + \frac{\theta}{8} \right) \right\}$$

For the same example which Frost used in the previous section, and with $z_0' = 2.6$ cm, the computed Ω at 1.2 m above the ground is 7.4 g/cu m, exactly the same as that measured. This verification, of course, will be meaningful only if z_0' is independently obtained from the velocity profile over the land surface. As the wind sweeps from sea to land the wind profile must show a gradual change in the value of the roughness form z_0 to z_0' . This has not been considered by Frost. The foregoing verification therefore should not be treated as conclusive.

The most serious defect of Frost's theory, however lies in the adequacy of B. C. (i) of Eq 396, in view of the fact that Eqs 392 and 393 are only valid for small values of z . In this connection it is worthwhile to note one peculiarity of Eq 393 which constitutes B. C. (i). The equation implies that the vapor transfer at different elevations is the same, hence that the concentration will be independent of X , contradicting Frost's own theory since Eq 379 clearly shows the dependence of Ω on η and hence on x , and showing the inadequacy of Eq 382 or Eq 392 as an approximation to the distribution of vapor concentration.

F. A Generalization of Sutton's Theory

Sutton's theory can be generalized by considering the m and n occurring in the diffusion equation

$$z^m \frac{\partial c}{\partial x} = D \frac{\partial}{\partial z} \left(z^n \frac{\partial c}{\partial y} \right) \quad (403)$$

where

$$D = \frac{A_1 z_1^{m-n}}{\bar{u}_1}$$

as independent. The validity of the solutions then depends only on the adequacy of the power-function representations of u and $A(z)$, and not on the validity of Sutton's theory.

As a special feature of the present development, dimensional considerations will be utilized in search for a similarity-solution (Ähnlichkeitslösung). These, in conjunction with considerations of the powers of x , A_1 , \bar{u}_1 , and z_1 , will afford in a systematic way the most adequate transformations to be made in the cases treated in the following sections (1) and (2), such that the solutions will be the simplest.

1. Diffusion from a line source embedded in a smooth surface.

Eq 403 is to be solved with the following boundary conditions

- (i) $\frac{\partial c}{\partial z} = 0$ at $z = 0$
- (ii) $c \rightarrow c_0$ as $z \rightarrow \infty$
- (iii) $c \rightarrow c_0$ as $x \rightarrow 0$ for $z > 0$

and with the continuity equation

$$(iv) \int_0^{\infty} \bar{u} (c - c_0) dz = Q = \text{constant}$$

where c_0 is the ambient vapor concentration, and Q is the strength of the line source per unit length. B. C. (i) stipulates that the ground is impervious to vapor. The pertinent variables are, in this case, the following:

$$c, c_0, Q, A_1, \bar{u}_1, z_1, x, z.$$

A dimensional analysis yields the relationship

$$\frac{c - c_0}{c_0} = F\left(\frac{Q}{A_1 c_0}, \frac{\bar{u}_1 x}{A_1}, \frac{z_1}{x}, \frac{z}{x}\right)$$

To obtain a similarity solution, assume

$$\Omega = \frac{c - c_0}{c_0} = \frac{Q}{A_1 c_0} \left(\frac{\bar{u}_1 x}{A_1}\right)^{10} \left(\frac{z_1}{x}\right)^6 \left[\left(\frac{\bar{u}_1 x}{A_1}\right)^r \left(\frac{z_1}{x}\right)^5 \left(\frac{z}{x}\right)\right] \quad (404)$$

where the exponents p , q , r , and s are to be determined.

Substituting Eq 404 in Eq 403 and demanding equal powers in A_1 , z_1 , \bar{u}_1 , and x , one has the values of r and s as follows:

$$r = \frac{1}{m-n+2}, \quad s = \frac{n-m}{m-n+2}$$

which are independent of the values of p and q . The values of p and q are obtained from condition (iv) the satisfaction of which requires that

$$p = \frac{n-1}{m-n+2} \quad q = \frac{n+m}{m-n+2}$$

Thus, a similarity solution is possible with the transformation

$$\Omega = \frac{Q}{A_1 C_0} \left(\frac{A_1^{1-n} z_1^{n+m}}{\bar{u}_1^{1-n} x^{m+1}} \right)^{\frac{1}{m-n+2}} f \left[\left(\frac{\bar{u}_1 z_1^{n-m}}{A_1 x} \right)^{\frac{1}{m-n+2}} \xi \right] \quad (405)$$

Substituting Eq 405 into 403, one obtains, after cancelling terms, the dimensionless equation

$$-\frac{m+1}{m-n+2} \eta^m f - \frac{1}{m-n+2} \eta^{m+1} f' = \frac{d}{d\eta} (\eta^n f') \quad (406)$$

where the primes denote differentiation with respect to the new variable

$$\eta = \left(\frac{u_1 z_1^{n-m}}{A_1 x} \right)^{\frac{1}{m-n+2}} \xi \quad (407)$$

The boundary conditions become

(i) $f(0) = 0$

(ii) and (iii) $f(\infty) = 0$ if $m-n+2 > 0$

and the continuity condition becomes

(iv) $\int_0^\infty \eta^m f d\eta = 1$ (408)

A first integration of Eq 406 yields

$$-\frac{1}{m-n+2} \eta^{m+1} f = \eta^n f'$$

the constant of integration being zero since both f and f' are finite at $\eta = 0$. A second integration gives

$$f = K \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2}\right) \quad (409)$$

where K is determined from Eq 408 and is given by

$$K^{-1} = \int_0^{\infty} \eta^m \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2}\right) d\eta = (m-n+2)^{\frac{m+n}{m-n+2}} \Gamma\left(\frac{m+1}{m-n+2}\right) \quad (410)$$

This integral is convergent since $m > 0$. Eqs 405, 409, and 410 then constitute the solution.

2. Diffusion from a smooth surface.

Denoting by c_s the saturated vapor concentration at the evaporation surface, Eq 403 is in this case to be solved with the following boundary conditions:

- (i) $c = c_s$ at $z = 0$
- (ii) $c = c_0$ at $z = \infty$
- (iii) $c = c_0$ at $x = 0$ for $z > 0$

In this case the parameter containing the unknown c is

$$\Theta = \frac{c_s - c}{c_s - c_0} \quad (411)$$

and the parameter $Q/(A_1 c_0)$ is eliminated from consideration. Since the vapor flux is no longer constant, the integral condition (iv) of the last section no longer exists. Hence, the values of p and q , which are determined by that condition, shall now be chosen to satisfy the boundary conditions listed above. Since B.C. (ii) requires that $\Theta = 1$ at $y = \infty$ irrespective of the values of A_1 , z_1 , \bar{u}_1 , and x , the values of p and q must both be zero.

Substituting therefore

$$\Theta = h(\eta) \quad (412)$$

into Eq 403 where c can be replaced by Θ , one has

$$-\frac{\eta^{m+1}h'}{m-n+2} = \eta^n h'' + n\eta^{n-1}h' \tag{413}$$

where the primes denote differentiation with respect to η which is defined by Eq 407. The boundary conditions are now

(i) $h(0) = 0$

(ii) and (iii) $h(\infty) = 1$ if $m - n + 2 > 0$

Eq 413 may be written

$$-\frac{\eta^{m-n+1}}{m-n+2} - \frac{n}{\eta} = \frac{h''}{h'}$$

a first integration of which gives

$$h' = B\eta^{-n} \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2}\right) \tag{414}$$

A second integration yields the solution

$$h = B \int_0^\eta \eta^{-n} \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2}\right) d\eta = B \Gamma\left(\frac{-n+1}{m-n+2}, \frac{\eta^{m-n+2}}{(m-n+2)^2}\right) (m-n+2)^{\frac{-m-n}{m-n+2}} \tag{415}$$

where B is determined by $h(\infty) = 1$ and is given by

$$B^{-1} = \int_0^\infty \eta^{-n} \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2}\right) d\eta = \Gamma\left(\frac{-n+1}{m-n+2}\right) (m-n+2)^{\frac{-m-n}{m-n+2}} \tag{416}$$

Eqs 411, 412, 415, and 416 then constitute the solution, so long as

$n > 1$ so that the integrals in Eqs 415 and 416 exist. Eq 415 may be written

$$h = \Gamma\left(\frac{-n+1}{m-n+2}, \frac{\eta^{m-n+2}}{(m-n+2)^2}\right) / \Gamma\left(\frac{-n+1}{m-n+2}\right) \tag{417}$$

3. Vapor concentration in the wake of an evaporating surface

The solution for this case can be obtained from the results of the last two sections by realizing that the evaporating surface can be considered as a collection of line sources the strengths of which are determined from the local rates of evaporation and that both the evaporation surface and the dry surface can then be considered as impervious to vapor.

The strength of the elemental line source at $x = \lambda$ is

$$(C_s - C_o) \frac{A_1}{z_1^n} \lim_{z \rightarrow 0} \left(z^n \frac{\partial h}{\partial z} \right) d\lambda = (C_s - C_o) G \lim_{\eta \rightarrow 0} (\eta^n h') = (C_s - C_o) G B d\lambda \quad (418)$$

where

$$G = \left(\frac{A_1^{m+1} \bar{u}_1^{1-n}}{\lambda^{1-n} z_1^{m+n}} \right)^{\frac{1}{m-n+2}}$$

Substituting the expression in Eq 417 for Q in Eq 405 one has

$$d\Omega = d\left(\frac{c - c_o}{c_o}\right) = \frac{C_s - C_o}{C_o} B \left(\frac{1}{\lambda^{1-n} (\chi - \lambda)^{m+1}} \right)^{\frac{1}{m-n+2}} f \left[\left(\frac{\bar{u}_1 z_1^{n-m}}{A_1 (\chi - \lambda)} \right)^{\frac{1}{m-n+2}} \frac{1}{z} \right] d\lambda \quad (419)$$

as the contribution of the elemental line-source to the value of

$\Omega = (c - c_o)/c_o$ at any value of x . If the evaporation surface has a downwind length b , then for $x > b$ one has, on integrating Eq 419

$$\frac{c - c_o}{C_s - C_o} = B \int_0^b \left(\frac{1}{\lambda^{1-n} (\chi - \lambda)^{m+1}} \right)^{\frac{1}{m-n+2}} f(\xi) d\lambda \quad (420)$$

where

$$\xi = \left(\frac{\bar{u}_1 z_1^{n-m}}{A_1 (\chi - \lambda)} \right)^{\frac{1}{m-n+2}} \frac{1}{z} \quad (421)$$

and where f is given by Eq 409.

4. Diffusion in Couette flow

As an example for the theory developed in the previous sections, one considers the diffusion in Couette flow. Here $m = 1$, $n = 0$, and $A = \alpha =$ constant, where α is the vapor diffusivity.

The solution for the case of line-source is, from Eqs 405, 407, 409, and 410,

$$\frac{c - c_o}{C_o} = \frac{Q}{\alpha C_o} \left(\frac{\alpha z_1}{\bar{u}_1 \chi^2} \right)^{\frac{1}{3}} K \exp\left(-\frac{\eta^3}{9}\right) = \frac{Q}{\alpha C_o} \left(\frac{\alpha z_1}{\bar{u}_1 \chi^2} \right)^{\frac{1}{3}} \frac{\exp\left(-\frac{\eta^3}{9}\right)}{3^{\frac{1}{3}} \Gamma\left(\frac{2}{3}\right)} \quad (422)$$

where

$$\eta = \left(\frac{\bar{u}_1}{z_1 \alpha \chi} \right)^{\frac{1}{3}} z \quad (423)$$

The solution for the evaporating surface is, from Eqs 411, 412, 415, and 416:

$$\frac{c_s - c}{c_s - c_0} = B \int_0^\eta e^{-\frac{\eta^3}{9}} d\eta = \frac{\Gamma(\frac{1}{3}, \frac{\eta^3}{9})}{\Gamma(\frac{1}{3})} \quad (424)$$

where η is given by Eq 423.

From Eqs 409, 416, 419, and 420, the vapor concentration in the wake of the evaporating surface of length b is given by

$$\frac{c - c_0}{c_s - c_0} = BK \int_0^b \left(\frac{1}{\lambda(x-\lambda)^2} \right)^{\frac{1}{3}} e^{-\frac{\xi^3}{9}} d\lambda = \frac{1}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \int_0^b \left(\frac{1}{\lambda(x-\lambda)^2} \right)^{\frac{1}{3}} e^{-\frac{\xi^3}{9}} d\lambda \quad (425)$$

where

$$\xi = \left(\frac{\bar{u}_1}{z, \alpha(x-\lambda)} \right)^{\frac{1}{3}} z \quad (426)$$

It should be noted that since in this case $n = 0$, $A(0) = \alpha$ is different from zero and the singularity at $\eta = 0$ for h' in Eq 414 does not exist, so that there is no objections to the theory based on physical considerations.

5. Remarks

It should be noted that along any generalized parabola

$$z = a x^{\frac{1}{m-n+2}}$$

the value for $(c_s - c)/(c_s - c_0)$ is the same for the case of diffusion from a smooth surface, and that on any two generalized parabolas of the above type the values of $(c - c_0)/c_0$ bear the same ratio for any value of x for the case of diffusion from a line-source. These facts provide the reason why the related solutions are called similarity-solutions. The solution represented by Eq 420, however, does not belong to the similarity-solution category.

G. Thorntwaite-Holzman's Theory

Assuming the mixing length λ to be proportional to the height z :

$$\lambda = k z$$

where $k = 0.40$ is Von Kármán's constant, Thorntwaite (200 b, 1942)

obtained, according to Prandtl's theory of momentum transfer:

$$\tau = \rho (k z)^2 \left(\frac{du}{dz} \right)^2 \tag{427}$$

with the eddy viscosity

$$\epsilon = (k z)^2 \left(\frac{du}{dz} \right) \tag{428}$$

Integrating Eq 427 by assuming τ constant, between two elevations z_1 ,

and z_2 :

$$u_2 - u_1 = \frac{1}{k} \sqrt{\frac{\tau}{\rho}} \ln \frac{z_2}{z_1}$$

where u_1 and u_2 are the mean velocities at z_1 and z_2 , respectively.

Thus

$$\sqrt{\frac{\tau}{\rho}} = k \frac{u_2 - u_1}{\ln \frac{z_2}{z_1}} \tag{429}$$

and from Eqs 427 to 429

$$\epsilon = \rho (k z) \sqrt{\frac{\tau}{\rho}} = \frac{\rho k^2 z (u_2 - u_1)}{\ln \frac{z_2}{z_1}}$$

Having obtained the expression for ϵ , the rate of evaporation is

$$E = -\epsilon \frac{dc}{dz}$$

from which one obtains

$$\frac{dc}{dz} = -\frac{E}{\epsilon} = -\frac{E \ln \frac{z_2}{z_1}}{k^2 z (u_2 - u_1)}$$

so that, by integration,

$$c_1 - c_2 = \frac{E \ln \frac{z_2}{z_1}}{k^2 (u_2 - u_1)} \ln \frac{z_2}{z_1}$$

and

$$E = \frac{k^2 (c_1 - c_2) (u_2 - u_1)}{\left(\ln \frac{z_2}{z_1} \right)^2} \tag{430}$$

where c_1 and c_2 correspond to the elevations of z_1 and z_2 and have the same dimension as ρ . If c_1 and c_2 are taken at two different elevations z_1' and z_2' , then the denominator in the above formula should be replaced by $\ln(\frac{z_2}{z_1}) \ln(\frac{z_2'}{z_1'})$.

From the above formula, the rate of evaporation in inches per hour is

$$E = \frac{1.34 k^2 P (e_1 - c_2)(u_2 - u_1)}{(t + 459.4)(\ln z_2 - \ln z_1)^2}$$

where P is the pressure in inches mercury, and t is in degree Fahrenheit.

The vapor pressure e or the absolute humidity ρ_w are connected with c by

$$c = 0.622 e/p$$

$$c = \frac{\rho_w}{\rho_a}$$

Where ρ_a is the density of air. When e and ρ_w are used instead of c , the $1.34 (c_1 - c_2)$ should be respectively replaced by $833 (e_1 - e_2)$ and $0.063 (\rho_{w1} - \rho_{w2})$

If the height of vegetation is considerable, assume a height d such that $u_d = 0$. Then the log-law becomes

$$u_z - u_1 = \frac{1}{k} \sqrt{\frac{\tau}{\rho}} \ln \frac{z-d}{z_1-d}$$

and

$$\frac{u_2 - u_1}{u_3 - u_1} = \frac{\ln(z_2 - d) - \ln(z_1 - d)}{\ln(z_3 - d) - \ln(z_1 - d)}$$

so that d can be determined by measuring u at three elevations. Eqs 430 and 431 can then be applied by considering the ground to be situated at $z = d$, i.e. by reducing all elevations by d .

The foregoing is for stable conditions. For unstable conditions, Pasquill (1944, 1949) obtained

$$E = \frac{(1-\beta)^2 k^2 z_0^{2(1-\beta)} (c_1 - c_2)(u_2 - u_1)}{(z_2^{1-\beta} - z_1^{1-\beta})^2} \tag{431}$$

where $z_0 = 0.25 \text{ cm}$ and β is given by

$$\frac{u_{150 \text{ cm}}}{u_{37.5 \text{ cm}}} = \frac{150^{1-\beta} - 0.25^{1-\beta}}{37.5^{1-\beta} - 0.25^{1-\beta}}$$

Eq 431 is believed to give results within 20% in error.

Chapter V. CONCLUDING REMARKS TO PART II

A. The theory of turbulence is still in the formulation stage. Before the theory is conclusively formulated, deductive mathematical solutions for the three kinds of transfer are impossible. Future research will first be toward, and then depend upon, a conclusive theory of turbulence which naturally includes the theory of anisotropic turbulence.

B. The conversion formulas are based on measurements in pipe flow, where, strictly speaking, the Reynolds analogy does not apply (even to the turbulent core). Since the Reynolds analogy has been used in the derivation of these formulas, their validity for pipe flow is really not theoretically justified. However, if the measured velocity profile in pipe flow applies to flow along a smooth plate, these formulas can be applied to plates with justification.

C. Although the effect of roughness on heat transfer has not been investigated, the conversion formulas may be expected to be applicable, especially for plates.

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Table 1

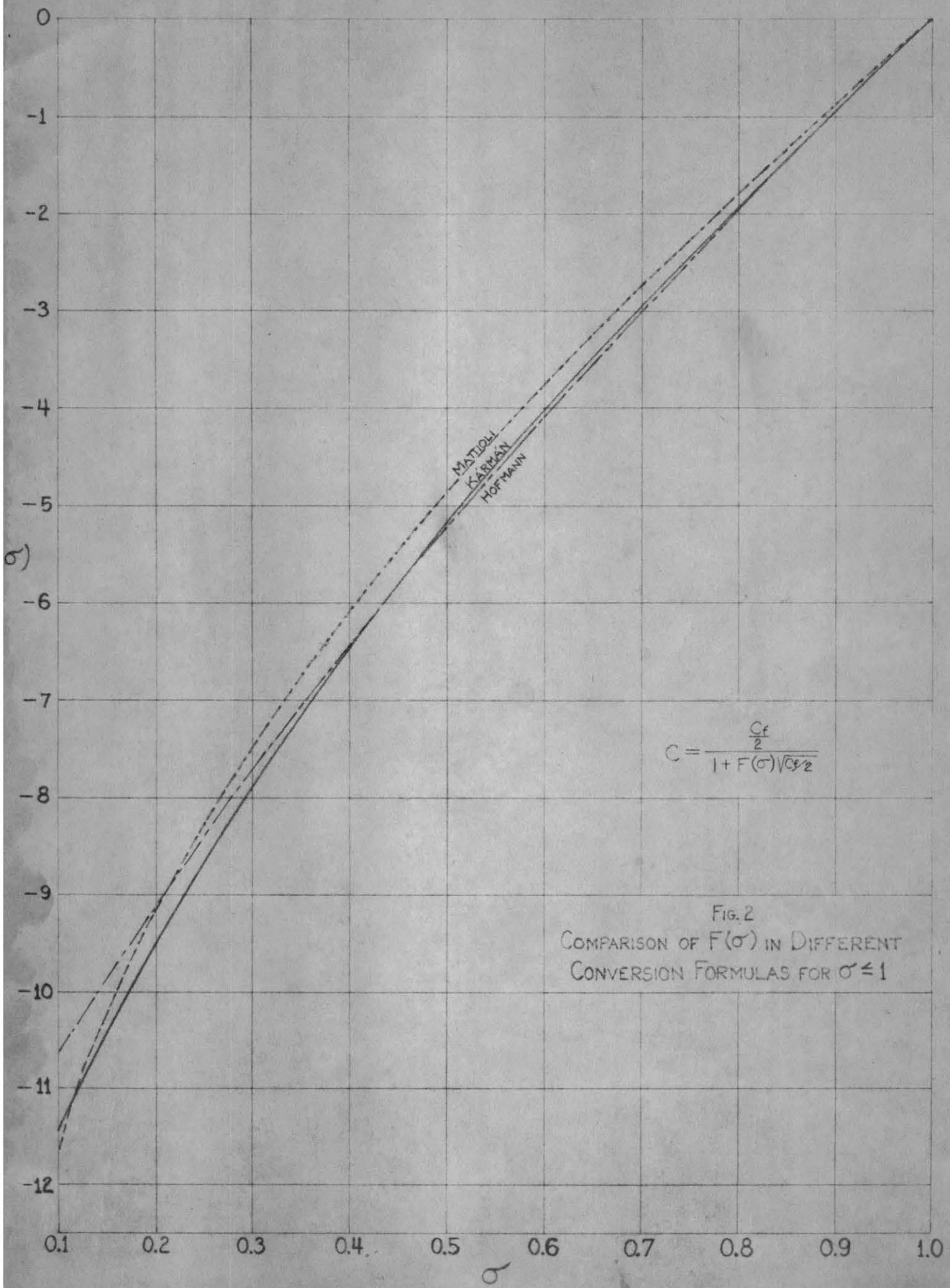
\bar{u}_{1cm} (m/sec)	1	2	3	4	5	6	7	8	9
E_s (Sutton)	0.18	0.18	0.67	0.84	1.00	1.16	1.31	1.45	1.59
E_p (Pasquill)	0.24	0.41	0.58	0.72	0.86	1.00	1.11	1.25	1.37
F_k (Kuo)	0.26	0.44	0.62	0.77	0.92	1.06	1.21	1.33	1.46
Observed (Pasquill)	0.25	0.43	0.60	0.76	0.92	1.06	1.20	1.34	1.48

Table 2

η	$\frac{C-C_0}{C_s-C_0}$	η	$\frac{C-C_0}{C_s-C_0}$	η	$\frac{C-C_0}{C_s-C_0}$
0	1.000	0.030	0.287	1.500	0.013
10 ⁻¹²	0.951	0.040	0.265	1.600	0.011
10 ⁻¹¹	0.937	0.050	0.247	1.700	0.010
10 ⁻¹⁰	0.918	0.060	0.232	1.800	0.008
10 ⁻⁹	0.894	0.070	0.220	1.900	0.007
10 ⁻⁸	0.864	0.080	0.209	2.000	0.006
10 ⁻⁷	0.824	0.090	0.199		
10 ⁻⁶	0.772	0.100	0.190		
10 ⁻⁵	0.706	0.200	0.134		
10 ⁻⁴	0.621	0.300	0.102		
10 ⁻³	0.510	0.400	0.081		
0.002	0.471	0.500	0.065		
0.003	0.446	0.600	0.054		
0.004	0.429	0.700	0.045		
0.005	0.414	0.800	0.038		
0.006	0.402	0.900	0.032		
0.007	0.372	1.000	0.027		
0.008	0.383	1.100	0.023		
0.009	0.375	1.200	0.020		
0.010	0.368	1.300	0.017		
0.020	0.318	1.400	0.015		



FIG. 1 COMPARISON OF KARMAN'S AND HOFMANN'S VELOCITY DISTRIBUTIONS FOR TURBULENT FLOW WITH NIKURADSE'S DATA



$$C = \frac{\frac{C_f}{2}}{1 + F(\sigma)\sqrt{C_f/2}}$$

FIG. 2
COMPARISON OF $F(\sigma)$ IN DIFFERENT
CONVERSION FORMULAS FOR $\sigma \leq 1$

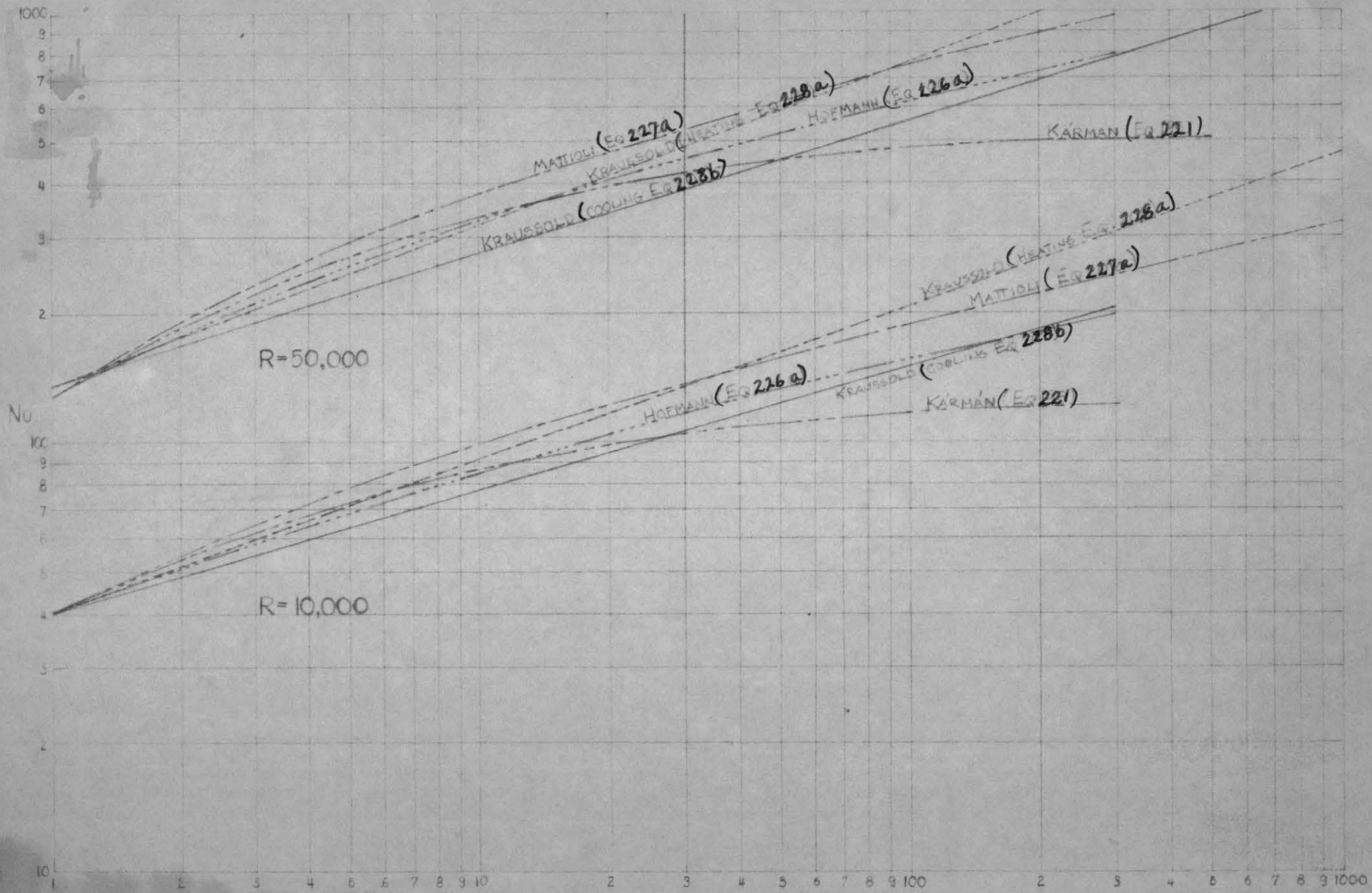


FIG. 3 COMPARISON OF EQS 195, 200a AND 201a WITH KRAUSSOLD'S DATA FOR PIPES

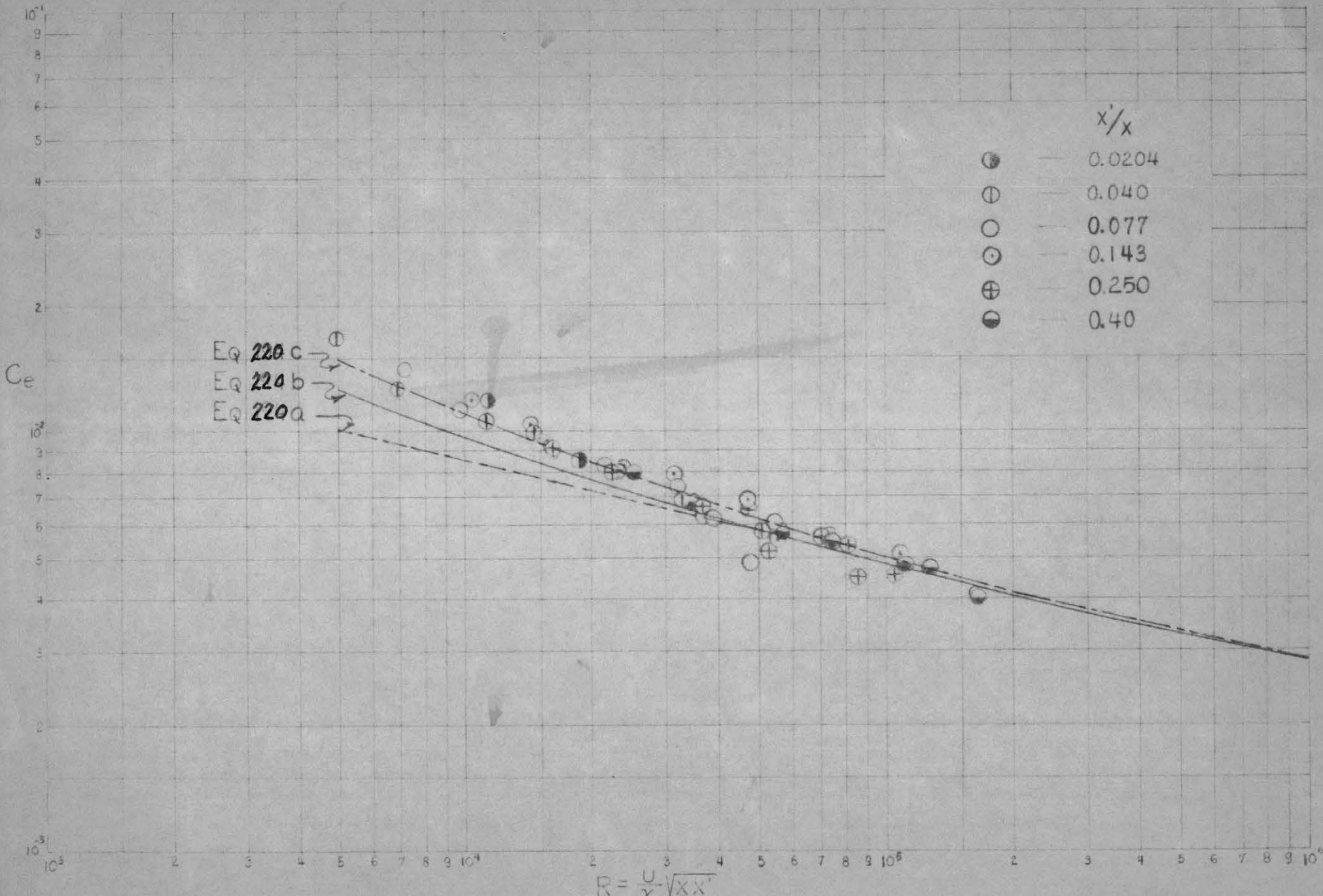


FIG 4 COMPARISON OF ALBERTSON'S DATA WITH EQS 194a, 194b AND 194c