DISSERTATION

VOLTERRA SERIES FRACTIONAL MECHANICS

Submitted by

David W. Dreisigmeyer

Electrical and Computer Engineering Department

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Spring 2004

COLORADO STATE UNIVERSITY

March 24, 2004

WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY DAVID W. DREISIGMEYER ENTITLED 'VOLTERRA SERIES FRACTIONAL MECHANICS' BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

Committee on Graduate Work

Richard E. Eykholt Richard E. Eykholt

Louis Scharf

Gerhard Dangelmayr

Peter Young

Advisor

Department Head

ABSTRACT OF DISSERTATION

VOLTERRA SERIES FRACTIONAL MECHANICS

We reexamine the problem of having nonconservative equations of motion arise from the use of a variational principle. That is, we start with an action for our system. The equations of motion are derived by requiring the perturbation of the action to vanish. We seek to be able to include, e.g., dissipation terms in our equations through this method.

Of particular interest to us is including fractional derivative operators in our system's equation. These operators lie 'between' the ordinary integer derivatives. One of their main attributes is that they are non-local operators. By successfully including fractional derivatives in our formalism, we are able to model many non-conservative systems. Previous work in this area has been largely unsatisfactory.

One of our main ideas is to treat actions as Volterra series, which are a generalization of power series to functionals. The kernels in this series are what give rise to the fractional derivatives in our equations of motion. The Volterra series concept is a convenient setting for modelling actions. In particular, it explicitly separates the derivative operators from the system's position function.

Fractional derivatives commonly raise many difficulties when they are used. Our work is no exception to this. The fundamental difficulty for us is that fractional derivatives come in both advanced and retarded forms. This results in a pair of equations when we use the standard variation technique: an advanced and a retarded equation of motion. This causes us to reexamine many of our assumptions when using a variational principle to model systems.

David W. Dreisigmeyer
Electrical and Computer Engineering Department
Colorado State University
Fort Collins, CO 80523
Spring 2004

Preface

One of the fundamental difficulties in analytic mechanics is including friction in the equations of motion. So far, no entirely satisfactory method has been found to handle this situation. Here we take a new look at this problem. Before giving an overview of our work, let us briefly examine some previous efforts.

The problem of having a dissipation term $\dot{q}$ arise in the equation of motion for a system has a long history. Bauer [2] showed that "the equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle". There are loopholes in Bauer's proof, however. One of these is to allow for additional equations of motion to arise. This method was employed by Bateman [1]. He used the Lagrangian

$$L = m\dot{x}\dot{y} + \frac{C}{2}\left(x\dot{y} - \dot{x}y\right) \tag{1}$$

which gives the equations of motion

$$m\ddot{x} + C\dot{x} = 0 \qquad m\ddot{y} - C\dot{y} = 0. \tag{2}$$

Bateman's method is not very general, so we look for other methods to model nonconservative systems.

Caldeira and Leggett [4] suggest recognizing that a dissipative system is coupled to an environment. The environment is modelled as a collection of harmonic oscillators which results in the Lagrangian

$$L = \frac{m}{2}\,\dot{q}^2 - V(q) + \sum_{n=1}^{\infty}\left\{\frac{m_n}{2}\,\dot{q}_n^2 - \frac{m_n\omega_n^2}{2}\,(q_n - q)^2\right\} \qquad (3)$$

where q is the system's coordinate and the q_n's are the environment's coordinates. While the system by itself is nonconservative, the system plus environment is conservative. This procedure does allow the introduction of very general dissipation terms into the system's equation of motion. However, the microscopic modelling of the environment makes (3) much more complex than, say, (1).

In order to overcome the difficulties of the above two procedures, Riewe examined using fractional derivatives in the Lagrangians [16, 17]. This method takes advantage of another loophole in Bauer's proof. Namely, Bauer assumed that all derivatives were integer ordered. Riewe's method has the advantage of not introducing extra coordinates as in (1) and (3). However, it ultimately results in noncausal equations of motion. A rather ad hoc procedure of replacing anticausal with causal operators needs to be used at the end in order to arrive at causal equations of motion.

Our goal in this work is to overcome the failings of the above procedures. Like

Riewe, we will employ fractional derivatives to model nonconservative systems. Our method is designed to avoid the operator replacement that Riewe employed. In the process of doing this, we develop a very convenient method for modelling systems involving operators much more general than fractional derivatives. This comes about by treating the actions as Volterra series. The operators appear as kernels in this series expansion. These kernels can be chosen to model quite general situations. Also, within the Volterra series framework, it is fairly easy to see how classical mechanics can be generalized.

In deriving our equations of motion, we will first employ the standard variational principle of mechanics. Unfortunately, this method does not work. It leads to both an advanced and a retarded equation of motion that the system must satisfy. This leads us to consider an alternate method for deriving the equations of motion. This second procedure is closely related to Caldeira and Leggett's modelling of nonconservative systems. However, instead of the microscopic modelling of the environment via harmonic oscillators, we model the macroscopic effects of the environment on the system with fractional derivatives. So our procedure is simpler and more transparent than Caldeira and Leggett's.

Perhaps a more fundamental question is why the standard variational principle fails. Here the usefulness of the Volterra series framework becomes apparent.

It allows us to examine in detail what properties the kernels must have in order to derive a single, retarded equation of motion. Thus, we can easily consider such novel situations as complex kernels, or kernels and systems defined over $\mathbb{C}$ instead of $\mathbb{R}$. All of this would stem from a desire to make the standard variational principle work due to its elegance. We would then need to give a physical interpretation to any method that succeeds. This is a common situation when fractional derivatives are employed to model a system. They often require us to reexamine our underlying assumptions.

Our work is organized as follows. Chapter **1** is devoted to the background material we need in our latter developments. Fractional derivatives are introduced within the generalized function framework. Treating fractional derivatives as generalized functions is important for two points. First, the distributional treatment of fractional derivatives is the easiest to work with. Also, we can easily treat the fractional derivatives as kernels in our Volterra series. Other definitions of fractional derivatives lack these desirable qualities.

After looking at fractional derivatives, we move on to examining Volterra series. As stated previously, this is a very convenient setting in which to model actions. While Volterra series intuitively make sense, there are some mathematical technicalities that need to be overcome to give them rigorous sense. So we will spend some time looking at Volterra series from a more mathematically precise viewpoint.

Finally, in Chapter **1**, we examine Riewe's fractional mechanics formalism. We will do this by looking at the harmonic oscillator. This will be sufficient to point out the shortcomings of his method as transparently as possible. It is precisely these failings that we wish to overcome.

In order to consolidate that concepts of Chapter **1**, we provide a heuristic overview of all the concepts in Chapter **2**. This should make the central ideas accessible by ignoring any mathematical details that obscure things. In this way we should be well prepared to proceed to our new formalism.

Chapter **3** covers our new fractional mechanics formalism. First we proceed using the standard variational principle to derive our equations of motion. Ultimately this method is unsatisfactory. The reason is that two equations of motion are derived. One of these is the retarded equation that we desire. The other is an advanced equation. When the derivatives in an equation of motion are only second order, both an advanced and a retarded solution are permitted. This is because the advanced and retarded equations are identical. Typically the advanced solutions are suppressed. However, as soon as fractional derivatives are allowed to occur, the advanced and retarded equations of motion are no longer identical. Now there is a definite sense of a time direction in the equations. So while the problem of advanced and retarded equations is not new, it only becomes a difficulty when fractional derivatives occur.

In order to overcome the difficulty of the standard procedure, we introduce an alternate method of deriving the equations of motion. While this procedure initially appears rather ad hoc, it can be given justification via the Caldeira-Leggett model. So we examine the Caldeira-Leggett model in some detail to provide this justification. Our method is more transparent than the Caldeira-Leggett model.

Finally, we return to the standard method for deriving equations of motion. Due to its elegance, we would like this procedure to work. First we look at why this method fails. Then we examine some variations of our actions to try to resolve this dilemma. Currently there is no known way to make the standard variational principle work. However, we are able to eliminate and/or restrict possible candidate solutions to the problem. This is done by putting requirements on the Volterra series' kernels that must be met.

Contents

Chapter 1

Background Material

In Chapter **3** we will present a new formalism for using fractional derivatives in equations of motion. In this chapter we gather together all the mathematical concepts we will need. First we introduce the concept of fractional differentiation in Section **1.1**. Our approach will be to use the distributional form of fractional derivatives, which is probably the most convenient to work with. Next we present Volterra series – a generalization to functionals of the power series concept for functions. We examine different forms of the series representation in Section **1.2.1**. Section **1.2.2** is devoted to a rigorous examination of Volterra series. We also review some related topics that will be important in later developments. Our first taste of fractional mechanics is in Section **1.3** when we review Riewe's formalism. Riewe's fractional mechanics is built around the Feller fractional derivatives, which are developed in Section **1.1.1**. The Feller fractional derivatives are rather strange since they contain the even integer ordered derivatives as special cases but not the odd ones. So, for example, $(d/dt)^2$ would be a Feller derivative while (d/dt) would not be. This creates difficulties for Riewe's fractional mechanics. Correcting these defects is our goal in Chapter **3**.

1.1 Fractional Derivatives

Integer ordered integration and differentiation are well developed fields in mathematics. This is likely due to the fact that they have shown themselves to be somewhat useful in modelling the physical world. Of course mathematicians are wont to not leave well enough alone. Almost as soon as the calculus was developed, Leibniz began to wonder about fractional ordered integration and differentiation. For about three-hundred years the subject was developed by the likes of Riemann and Liouville, among others. No one really thought that fractional operations would be anything more than an area of pure mathematics. (As an aside, the operations were extended to any real order, though the term 'fractional' was retained.)

As often turns out to be the case in mathematics, a seemingly theoretical mathematical subject is found to have important physical applications. Fractional derivatives have shown themselves to be quite useful in applied problems. Indeed, a near renaissance in the use of fractional derivatives has occurred recently. Their applications have been in such diverse fields as nonconservative mechanics [6, 16, 17], control and systems theory [15], quantum mechanics [12, 13] and, hydrology [19], among other areas. At the time of this writing, fractional differentiation can likely be considered an emerging 'hot' topic in applied mathematics.

Our understanding of many problems was significantly impacted by the development of functional analysis. In particular, the theory of generalized functions, or distributions, has had a profound effect on how we address many problems. This also holds in the theory of fractional differentiation [9]. While there are many ways to approach the idea of fractional operations, the distributional theory is probably the most convenient to use [15]. So we will largely concern ourselves with the generalized function definition of fractional operations here.

Let us start to acquaint ourselves with the fractional operations. To begin developing the theory, let us first write down Cauchy's integral formula

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t f(\tau)(t-\tau)^{n-1} d\tau \tag{1.1}$$

where $n > 0$ is an integer, $\Gamma(n)$ is the gamma function, and $a < t$. Equation **(1.1)** is a convolution of $f(t)$ and the function

$$\Phi_n^+(t) := \begin{cases} \frac{1}{\Gamma(n)} t^{n-1} & t > 0 \\ 0 & t \le 0 \end{cases} \tag{1.2}$$

if we set $f(t) \equiv 0$ for $t < a$. So we can rewrite **(1.1)** as

$${}_a\mathbf{I}_t^n[f] = f(t) * \Phi_n^+(t) \tag{1.3}$$

where $*$ is the convolution operation defined by

$$g(t) * h(t) := \int_{-\infty}^{\infty} g(\tau) h(t-\tau) d\tau. \tag{1.4}$$

Equation **(1.3)** will be our stepping stone to generalizing the integer ordered operations to fractional order.

The above procedure works so well for the integers $n > 0$, we want to consider extending it to any real $\alpha > 0$. This is obviously possible, so we let

$${}_a\mathbf{I}_t^\alpha[f] = f(t) * \Phi_\alpha^+(t) \tag{1.5}$$

be the left fractional integral (LFI) of $f(t)$ of order $\alpha > 0$. Everything works fine until we consider the case $\alpha = 0$. We reasonably expect that

$$ {}_a\mathbf{I}_t^0[f] = f(t) \tag{1.6} $$

but, it is not immediately obvious that the integral in (**1.3**) is not divergent. Also, for $-1 < \alpha < 0$, the integral is obviously divergent. It is apparent that treating $f(t)$ and $\Phi_\alpha^+(t)$ as regular functions will not be sufficiently general for our purposes. Instead we will consider them to be distributions, or generalized functions.

The first order of business is to define the convolution operation for distributions. Let $k(t) = g(t) * h(t)$ and $\varphi(t)$ be a test function. Then [9]

$$ \begin{aligned} \langle k, \varphi \rangle &:= \int k(t)\varphi(t)dt \\ &= \int \left\{ \int g(\xi)h(t-\xi)d\xi \right\} \varphi(t)dt \\ &= \int\int g(\xi)h(\eta)\varphi(\xi+\eta)d\xi d\eta. \end{aligned} \tag{1.7} $$

Equation (**1.7**) is meaningful as long as either $g(t)$ or $h(t)$ has bounded support or, $g(t)$ and $h(t)$ are bounded on the same side (e.g., $g(t) \equiv 0$ for $t < t_1$ and $h(t) \equiv 0$ for $t < t_2$). We will always assume that one of these situations is the case. From (**1.7**), it can be seen that the generalization of (**1.4**) is

$$ \langle g * h, \varphi \rangle = \langle g(t), \langle h(\tau), \varphi(t+\tau) \rangle \rangle. \tag{1.8} $$

The convolution operation has the properties

$$g * h = h * g \tag{1.9}$$

$$f * (g * h) = (f * g) * h \tag{1.10}$$

$$D(g * h) = (Dg) * h = g * (Dh) \tag{1.11}$$

where $D(\cdot)$ is the generalized derivative. The generalized derivative of a distribution f is defined by

$$\langle Df, \varphi \rangle := -\langle f, d\varphi/dt \rangle \tag{1.12}$$

where φ is a test function. Remember that the relationship between the generalized and classical derivatives, beginning at $t = a$, is given by [15]

$$D^n f = f^{(n)} + \sum_{k=0}^{n-1} \left[D^{n-k-1} \delta(t-a) \right] f^{(k)}(a) \tag{1.13}$$

where $f^{(n)}$ is the classical derivative.

Considering $\Phi_\alpha^+(t)$ as a generalized function allows us to extend **(1.5)** to any α, where the convolution operation is defined as in **(1.8)**. For $\alpha < 0$, this will define the left fractional derivative (LFD) as

$$\begin{aligned} {}_a\mathbf{D}_t^{-\alpha}[f] &:= {}_a\mathbf{I}_t^{\alpha}[f] \\ &= f(t) * \Phi_\alpha^+(t). \end{aligned} \tag{1.14}$$

In the sequel, we will find it easier to assume $\alpha > 0$ and use the notation

$$ {}_a\mathbf{D}_t^{\alpha}[f] = f(t) * \Phi_{-\alpha}^{+}(t) \qquad {}_a\mathbf{D}_t^{-\alpha}[f] = {}_a\mathbf{I}_t^{\alpha}[f]. \tag{1.15} $$

Also, for reasons that will become apparent shortly, we will often set $f(t) \equiv 0$ for $t < a$ and $t > b$, where $a < b$. We do not want any resulting discontinuities in $f(t)$ at $t = b$ to affect the LFDs. So t must be restricted to the interval $a \leq t < b$ in the LFDs. It would perhaps be better to write **(1.15)** as

$$ {}_{a^-}\mathbf{D}_{t^-}^{\alpha}[f] = \frac{1}{\Gamma(-\alpha)} \int_{a^-}^{t^-} f(\tau)(t-\tau)^{-(\alpha+1)} d\tau. \tag{1.16} $$

To avoid cluttering our notation, we will continue to use the notation in (**1.15**) with the understanding that it formally means (**1.16**).

The distributions $\Phi_{\alpha}^{+}(t)$ have been well studied [9, 15]. Their two most important properties are

$$ \Phi_n^{+}(t) = D^{-n}\delta(t^+) \tag{1.17} $$

for any integer n, and, for any β and γ,

$$ \Phi_{\beta}^{+}(t-c) * \Phi_{\gamma}^{+}(t) = \Phi_{\beta+\gamma}^{+}(t-c) \tag{1.18} $$

where c is a constant. These properties can be heuristically 'proved' by noting that the Fourier transform of $\Phi_{\alpha}^{+}(t)$ is given by [9]

$$ \Phi_{\alpha}^{+}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} ie^{i\pi(\alpha-1)/2}(\omega + i0)^{-\alpha}. \tag{1.19} $$

For α an integer, **(1.17)** is easily seen from **(1.19)**. (We use a t^+ in **(1.17)** to signify the fact that $\Phi_\alpha^+(t)$ is nonzero only for $t > 0$. This observation will prove significant soon.) Equation **(1.18)** follows from the properties of the Fourier transform

$$\Phi_\alpha^+(t-c) * \Phi_\beta^+(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad ie^{i\omega c} e^{i\pi((\alpha+\beta)-1)/2}(\omega + i0)^{-(\alpha+\beta)}. \tag{1.20}$$

Note that **(1.18)** implies

$$ {}_a\mathbf{D}_t^\beta \left[{}_a\mathbf{D}_t^\gamma [f] \right] = {}_a\mathbf{D}_t^{\beta+\gamma} [f] \tag{1.21}$$

$$ {}_a\mathbf{D}_t^\beta \left[{}_a\mathbf{D}_t^{-\beta} [f] \right] = f. \tag{1.22}$$

Now let $0 \leq n-1 \leq \alpha < n$. Then, using **(1.9)** – **(1.11)** and **(1.17)** and **(1.18)**, we have

$$\begin{aligned} {}_a\mathbf{D}_t^\alpha [f] &= f(t) * \Phi_{-\alpha}^+(t) \\ &= f(t) * \left(D^n \Phi_{n-\alpha}^+(t)\right) \\ &= \left(D^n f(t)\right) * \Phi_{n-\alpha}^+(t) \qquad (1.23) \\ &= D^n \left(f(t) * \Phi_{n-\alpha}^+(t)\right). \qquad (1.24) \end{aligned}$$

Equations **(1.23)** and **(1.24)** are the distributional forms of the Caputo and Riemann-Liouville fractional derivative, respectively [15]. These are two alternate ways of defining fractional operators. In the standard definitions of these derivatives, D^n

is replaced with $(d/dt)^n$. Thus, they are not equivalent definitions of the fractional operators in their standard forms. As shown in [15], we have that

$$\left(\frac{d}{dt}\right)^n [f(t) * \Phi^+_{n-\alpha}(t)] = \left[\left(\frac{d}{dt}\right)^n f(t)\right] * \Phi^+_{n-\alpha}(t) + \sum_{k=0}^{n-1} \Phi^+_{k-\alpha+1}(t-a) f^{(k)}(a). \tag{1.25}$$

Letting $\alpha \to n$ gives us

$$_a\mathbf{R}^n_t [f] = {_a\mathbf{C}^n_t} [f] + \sum_{k=0}^{n-1} \delta^{(n-k-1)}(t-a) f^{(k)}(a). \tag{1.26}$$

Comparing **(1.26)** with **(1.13)** shows that the Riemann-Liouville fractional derivative $_a\mathbf{R}^\alpha_t [f]$ is a generalization of the generalized derivative. The Caputo fractional derivative $_a\mathbf{C}^\alpha_t [f]$ is a generalization of the ordinary derivative. The Riemann-Liouville fractional derivative is equivalent to the distributional definition when the latter is interpreted as a finite-part integral [15]. We will largely be unconcerned with these alternate definitions in what follows.

In addition to the left fractional operations, we can also define right fractional operations. If we set $f(t) \equiv 0$ for $t > b$ and define

$$\Phi^-_\alpha(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} (-t)^{\alpha-1} & t < 0 \\ 0 & t \geq 0 \end{cases} \tag{1.27}$$

the right fractional operations are defined by

$$_t\mathbf{D}^\alpha_b [f] := f(t) * \Phi^-_{-\alpha}(t). \tag{1.28}$$

Most of the above observations for the left fractional operations also hold for the right ones. However, (**1.17**) needs to be replaced with

$$\Phi_n^-(t) = (-1)^n D^{-n}\delta(t^-) \tag{1.29}$$

for any integer n. (We use a t^- in (**1.29**) to signify the fact that $\Phi_\alpha^-(t)$ is nonzero only for $t < 0$.) When $f(t) \equiv 0$ for $t < a$ and $t > b$, we do not allow any resulting discontinuities in $f(t)$ at $t = a$ to affect the RFDs. Similar to the case for the LFDs, we will take (**1.28**) as meaning

$$_{t^+}\mathbf{D}_{b^+}^{\alpha}[f] = \frac{1}{\Gamma(-\alpha)} \int_{t^+}^{b^+} f(\tau)(\tau - t)^{-(\alpha+1)} d\tau \tag{1.30}$$

though we will continue to use the notation in (**1.28**). Finally, the Fourier transform of $\Phi_\alpha^-(t)$ is given by [9]

$$\Phi_\alpha^-(t) \stackrel{\mathcal{F}}{\longleftrightarrow} -ie^{-i\pi(\alpha-1)/2}(\omega - i0)^{-\alpha}. \tag{1.31}$$

Note that for the left operations, the 'left' integration limit a determines the allowable functions in the operation ${}_a\mathbf{D}_t^\alpha[f]$. Namely, $f(t)$ must vanish for $t < a$. Also, ${}_a\mathbf{D}_t^\alpha[f]$ is a function of α and t and, a functional of $f(t)$. Similar comments hold for the right operations. Here, the 'right' integration limit b means $f(t) \equiv 0$ for $t > b$.

Now let $f(t)$ be compactly supported on the interval $[a, b]$. Then ${}_a\mathbf{D}_t^\alpha[f] = 0$

whenever $t < a$. However, ${}_a\mathbf{D}_t^\alpha[f]$ does not generally vanish for $t > a$. Thus, the left operations are causal or retarded. That is, they do not 'pre-act' to the input function $f(t)$. They do retain a memory of their reaction with $f(t)$ for all time $t > b$. Conversely, ${}_t\mathbf{D}_b^\alpha[f] = 0$ whenever $t > b$ but, generally, ${}_t\mathbf{D}_b^\alpha[f] \neq 0$ for $t < b$. Hence, the right operations are anti-causal or advanced. So the right operations do pre-act to the input function. The fact that fractional derivatives map functions with compact support to functions that are only bounded on one side is one of the difficulties encountered when using fractional derivatives.

1.1.1 Feller Fractional Derivatives

Love and Young [14] showed that, for fractional integrals, the following integration by parts formula holds

$$\int_a^b {}_a\mathbf{D}_t^{-\alpha}[f]\, g dt = \int_a^b f\, {}_t\mathbf{D}_b^{-\alpha}[g]\, dt \tag{1.32}$$

where $\alpha > 0$ and, $f(t)$ and $g(t)$ are sufficiently nice functions. Riewe suggested that **(1.32)** also holds for fractional derivatives under certain conditions [16, 17]. This is then applied to find quantities of the form

$$\int_a^b {}_a\mathbf{D}_t^\alpha[f]\, {}_a\mathbf{D}_t^\alpha[g]\, dt = \int_a^b f\, {}_t\mathbf{D}_b^\alpha\left[{}_a\mathbf{D}_t^\alpha[g]\right] dt \tag{1.33}$$

in deriving the equations of motion. (We examine Riewe's formalism in more detail in Section **1.3**. For now we are only concerned with **(1.33)** and its derivation/interpretation.) Let us consider **(1.33)** in more detail.

What we are mainly interested in is the generalized function that corresponds to ${}_t\mathbf{D}_b^\alpha\left[{}_a\mathbf{D}_t^\alpha[g]\right]$ when $f(t)$ and $g(t)$ are test functions in **(1.33)**. To begin, let $f(t)$ and $g(t)$ be compactly supported test functions. Then ${}_a\mathbf{D}_t^\alpha[f]$ and ${}_a\mathbf{D}_t^\alpha[g]$ are infinitely differentiable, ordinary functions. So the terms on the left-hand side of **(1.33)** can be understood in the sense of ordinary functions. Now let $n-1 \leq \alpha < n$. Then we have [15]

$$\begin{aligned} {}_a\mathbf{D}_t^\alpha[f] &= \left(\frac{d}{dt}\right)^n {}_a\mathbf{D}_t^{\alpha-n}[f] \\ &= {}_a\mathbf{D}_t^{\alpha-n}\left[f^{(n)}\right] \end{aligned} \tag{1.34}$$

and

$$\begin{aligned} {}_t\mathbf{D}_b^\alpha[g] &= \left(-\frac{d}{dt}\right)^n {}_t\mathbf{D}_b^{\alpha-n}[g] \\ &= {}_t\mathbf{D}_b^{\alpha-n}\left[(-1)^n g^{(n)}\right] \end{aligned} \tag{1.35}$$

when $f(t)$ and $g(t)$ are test functions compactly supported in $a \leq t \leq b$. So **(1.33)** can be written as

$$\int_a^b {}_a\mathbf{D}_t^\alpha [f]\, {}_a\mathbf{D}_t^\alpha [g]\, dt = \int_a^b {}_a\mathbf{D}_t^{\alpha-n} \left[f^{(n)}\right] {}_a\mathbf{D}_t^\alpha [g]\, dt$$
$$= \int_a^b f^{(n)} {}_t\mathbf{D}_b^{\alpha-n} \left[{}_a\mathbf{D}_t^\alpha [g]\right] dt \qquad (1.36)$$

where we used **(1.32)**. Remembering that $f^{(m)}(a) = 0 = f^{(m)}(b)$, $0 \le m \le n$, we use the ordinary integration by parts formula in **(1.36)**, along with **(1.35)**, to arrive at

$$\int_a^b {}_a\mathbf{D}_t^\alpha [f]\, {}_a\mathbf{D}_t^\alpha [g]\, dt = \int_a^b f\, {}_t\mathbf{D}_b^\alpha \left[{}_a\mathbf{D}_t^\alpha [g]\right] dt. \qquad (1.37)$$

So we have verified **(1.33)** with $f(t)$ and $g(t)$ test functions. Our assumptions on $f(t)$ and $g(t)$ can be relaxed but, the above argument will be sufficient for our purposes.

We still need to interpret ${}_t\mathbf{D}_b^\alpha \left[{}_a\mathbf{D}_t^\alpha [g]\right]$ in **(1.33)**. What we are looking for is some distribution Ψ_γ that corresponds to the $\Phi_\alpha^\pm$. To that end, write **(1.33)** as

$$\int_a^b {}_a\mathbf{D}_t^\alpha [f]\, {}_a\mathbf{D}_t^\alpha [g]\, dt = \int_a^b {}_a\mathbf{D}_t^{\alpha-n} \left[f^{(n)}\right] {}_a\mathbf{D}_t^{\alpha-n} \left[g^{(n)}\right] dt$$
$$= \int_a^b \varphi\, {}_t\mathbf{D}_b^{\alpha-n} \left[{}_a\mathbf{D}_t^{\alpha-n} [\eta]\right] dt \qquad (1.38)$$

where $n-1 \le \alpha < n$, $\varphi(t) := f^{(n)}(t)$ and $\eta(t) := g^{(n)}(t)$. Now we can interpret ${}_t\mathbf{D}_b^{-\beta} \left[{}_a\mathbf{D}_t^{-\beta} [\eta]\right]$, $\beta = n - \alpha$, using ordinary functions. The most convenient way to do this is by using the Fourier transformation. The Fourier transform of $\Phi_\beta^\pm$ is given by

$$\Phi_\beta^\pm(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \pm i e^{\pm i\pi(\beta-1)/2}(\omega \pm i0)^{-\beta}. \tag{1.39}$$

Then,

$$\begin{aligned} {}_t\mathbf{D}_b^{-\beta}\left[{}_a\mathbf{D}_t^{-\beta}[\eta]\right] &= \Phi_\beta^- * \Phi_\beta^+ * \eta \\ &\overset{\mathcal{F}}{\longleftrightarrow} |\omega|^{-2\beta}\eta(\omega). \end{aligned} \tag{1.40}$$

Now define

$$\Psi_{2\beta}(t) := \frac{|t|^{2\beta-1}}{2\cos(\beta\pi)\Gamma(2\beta)} \tag{1.41}$$

$$= \frac{1}{2\cos(\beta\pi)}\left[\Phi_{2\beta}^+(t) + \Phi_{2\beta}^-(t)\right] \tag{1.42}$$

where

$$\Psi_{2\beta}(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad |\omega|^{-2\beta}. \tag{1.43}$$

So **(1.38)** becomes

$$\begin{aligned} \int_a^b {}_a\mathbf{D}_t^\alpha[f]\, {}_a\mathbf{D}_t^\alpha[g]\, dt &= \int_a^b f^{(n)}\left[\Psi_{2(n-\alpha)} * g^{(n)}\right] dt \\ &= \int_a^b f\left\{(-D)^n\left[\Psi_{2(n-\alpha)} * D^n g\right]\right\} dt \\ &= \int_a^b f\left\{\left[(-1)^n D^{2n}\Psi_{2(n-\alpha)}\right] * g\right\} dt. \end{aligned} \tag{1.44}$$

Now we need to show a few properties of the $\Psi_\beta(t)$. From **(1.18)** and **(1.42)** we have

$$\Psi_\delta * \Psi_\gamma \quad = \quad \Psi_{\delta+\gamma}. \tag{1.45}$$

Also, the quantity $\Psi_\beta * g$ will be called a Feller fractional derivative (FFD). We will write this as

$${}_a^t\mathbf{F}_b^\beta[g] \quad := \quad \Psi_\beta * g. \tag{1.46}$$

With n an integer we have

$${}_a^t\mathbf{F}_b^{2n}[g] \quad = \quad (-1)^n g^{(2n)}(t) \tag{1.47}$$

for $a < t < b$, but

$${}_a^t\mathbf{F}_b^{2n+1}[g] \quad \neq \quad \pm g^{(2n+1)}(t). \tag{1.48}$$

Equations **(1.47)** and **(1.48)** show that the even integer ordered derivatives are included in the FFDs while the odd integer ordered derivatives are not.

Returning to **(1.44)**, it follows from **(1.45)** and **(1.47)** that

$$\begin{aligned} \int_a^b {}_a\mathbf{D}_t^\alpha[f]\, {}_a\mathbf{D}_t^\alpha[g]\, dt &= \int_a^b f\left\{\left[(-1)^n D^{2n}\Psi_{2(n-\alpha)}\right] * g\right\} dt \\ &= \int_a^b f\left[\Psi_{-2n} * \Psi_{2n-2\alpha} * g\right] dt \\ &= \int_a^b f\left[\Psi_{-2\alpha} * g\right] dt. \end{aligned} \tag{1.49}$$

This is the form we wanted. Now we can treat the $\Psi_{-2\alpha}$ as distributions that are sampled by the test functions f and g.

The Ψ_β are the central distributions in Riewe's fractional mechanics formalism. From (**1.47**) we see that as long as we restrict our attention to even ordered derivatives, the Ψ_β act satisfactorily. However, (**1.48**) shows that the odd ordered derivatives are not included in the Ψ_β. As we will see, this creates difficulties for Riewe's formalism. These difficulties can be precisely located in the $1/\cos(\beta\pi)$ factor in (**1.42**). By removing this factor, we can remove the main complications in Riewe's fractional mechanics.

1.2 Volterra Series

In Section **1.1** we examined fractional derivatives in the generalized function formalism. We saw that the distributions $\Phi_\alpha^\pm$ and Ψ_β played central roles in the theory of fractional derivatives. We will now examine how we will use these distributions. In short, we are going to treat the $\Phi_\alpha^\pm$ and the Ψ_β as kernels in a Volterra series [18, 20].

The Volterra series concept is a generalization to functionals of the power series of a function. For a multi-dimensional function $g(\mathbf{q})$ we have the power series

$$g(\mathbf{q}) = g(\mathbf{0}) + \sum_i \frac{\partial g}{\partial q_i} q_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 g}{\partial q_i q_j} q_i q_j + \cdots \tag{1.50}$$

where $\mathbf{q} = [q_1, \ldots, q_n]$. Now, as we let $\mathbf{q} \to q(t)$, the function $g(\mathbf{q})$ will become the functional $G[q(t)]$. We would reasonably expect **(1.50)** to generalize to

$$\begin{aligned} G[q] &= K_0^{(s)} + \int_{\tau_1} K_1^{(s)}(\tau_1) q(\tau_1) d\tau_1 + \\ &\frac{1}{2} \int_{\tau_1} \int_{\tau_2} K_2^{(s)}(\tau_1, \tau_2) q(\tau_1) q(\tau_2) d\tau_1 d\tau_2 + \cdots . \end{aligned} \tag{1.51}$$

Equation **(1.51)** is the Volterra series of $G[q(t)]$. The kernels in **(1.51)** are given by, e.g.,

$$K_0^{(s)} = G[0] \tag{1.52}$$

$$K_2^{(s)}(\tau_1, \tau_2) = \frac{\delta^2 G[q]}{\delta q(\tau_1) \delta q(\tau_2)}. \tag{1.53}$$

For our purposes we can let $K_0^{(s)} \equiv 0$ since a non-zero $K_0^{(s)}$ only adds an irrelevant constant to **(1.51)**. Our notation can be tidied up a bit by defining

$$K_n^{(s)} \star q^n := \int_{\tau_1} \cdots \int_{\tau_n} K_n^{(s)}(\tau_1, \ldots, \tau_n) q(\tau_n) \cdots q(\tau_1) d\tau_n \cdots d\tau_1. \tag{1.54}$$

Then **(1.51)**, with $K_0^{(s)} = 0$, can be compactly written as

$$\mathcal{V}[q] = \sum_{n=1}^{\infty} \frac{1}{n!} K_n^{(s)} \star q^n. \tag{1.55}$$

When we use **(1.55)** as the action in our fractional mechanics, we will typically assume that $K_n^{(s)} \equiv 0$ for $n > N$. This is a common situation in mechanics where the potential is a finite polynomial in q. So our actions will usually be of the form

$$\mathcal{V}[q] \;=\; \sum_{n=1}^{N} \frac{1}{n!} K_n^{(s)} \star q^n. \tag{1.56}$$

$\mathcal{V}[q]$ in **(1.56)** is called, in the engineering literature, a polynomial system of degree N [18]. We will continue to call **(1.56)** a Volterra aeries, however. The finite number of terms in **(1.56)** allows us to avoid the convergence issues that **(1.55)** raises. Showing convergence of **(1.55)** is often difficult, so our ability to use **(1.56)** in our subsequent developments is rather convenient [3, 18].

1.2.1 Types of Kernels

From **(1.53)** we can see that the kernels in **(1.55)** are symmetric. That means that interchanging any of the τ_i in $K_n^{(s)}(\tau_1, \ldots, \tau_n)$ does not change the kernel. From our derivation above, the symmetric kernels seem to be the natural choice in the Volterra series. However, we may be given asymmetric kernels and would like to symmetrize them or vice versa. Let us see how this is done.

As motivation, consider the function

$$v(\mathbf{q}) = \frac{1}{2}\sum_{i,j} K_{ij} q_i q_j \tag{1.57}$$

where $\mathbf{q} = [q_1, \ldots, q_n]$. We assume the matrix K_{ij} in **(1.57)** is symmetric, i.e., $K_{ij} = K_{ji}$. Now, it is obvious that **(1.57)** can also be written as

$$\begin{aligned} v(\mathbf{q}) &= \frac{1}{2}\sum_{i} K_{ii} q_i^2 + \sum_{i<j} K_{ij} q_i q_j \\ &= \sum_{i\le j} \widetilde{K}_{ij} q_i q_j \end{aligned} \tag{1.58}$$

where $\widetilde{K}_{ij}$ is a lower triangular matrix, i.e., $\widetilde{K}_{ij} = 0$ when $i > j$. So it seems that we might be able to switch between symmetric kernels $K_n^{(s)}$ and triangular kernels $K_n^{(t)}$ in **(1.56)**. This is in fact possible.

The triangular kernels are given by, e.g.,

$$K_n^{(t)}(\tau_1, \ldots, \tau_n) = 0 \qquad \text{unless } \tau_1 \ge \tau_2 \ge \cdots \ge \tau_n. \tag{1.59}$$

We can symmetrize the triangular kernels. Let σ be a permutation of $1, \ldots, n$. The symmetrization of **(1.59)** is defined as [3]

$$\begin{aligned} \mathrm{sym}K_n^{(t)}(\tau_1, \ldots, \tau_n) &:= \frac{1}{n!}\sum_{\sigma} K_n^{(t)}(\tau_{\sigma 1}, \ldots, \tau_{\sigma n}) \\ &= \frac{1}{n!} K_n^{(s)}(\tau_1, \ldots, \tau_n). \end{aligned} \tag{1.60}$$

One can freely change between the triangular and symmetric kernels in a Volterra series. The $\Phi_\alpha^\pm$ are triangular kernels in the $t\tau$-plane while the Ψ_β are symmetric kernels.

1.2.2 Function Spaces and Other Stuff

In (**1.56**) the kernels will generally be distributions. So the $q(\tau)$ used should be some test function in order to make rigorous sense of (**1.56**). Here we consider a rigorous interpretation of Volterra series. Before we start, let us briefly review some spaces of test functions and their associated distributions [8, 21]. The most commonly encountered space of test functions is $\mathcal{D}_t$, the space of compactly supported, infinitely differentiable functions. The dual space $\mathcal{D}_t^{'}$ is the standard distributions. A larger space of test functions are those of rapid descent, $\mathcal{S}_t$. A function is in $\mathcal{S}_t$ if it is infinitely differentiable and, it and all of its derivatives vanish faster than any power of $|t|^{-1}$ as $|t| \to \infty$. The dual space is the tempered distributions $\mathcal{S}_t^{'}$. A distribution $f(t)$ is in $\mathcal{S}_t^{'}$ if there exists a finite integer N such that

$$\lim_{|t|\to\infty} |t|^{-N} f(t) \quad = \quad 0. \tag{1.61}$$

Our final test function space is $\mathcal{E}_t$, the space of infinitely differentiable functions without any restrictions on their supports. The dual space $\mathcal{E}_t^{'}$ is all those distributions with compact support. We have that

$$\begin{aligned} \mathcal{D}_t \;\subset\; \mathcal{S}_t \;\subset\; \mathcal{E}_t \\ \mathcal{E}_t' \;\subset\; \mathcal{S}_t' \;\subset\; \mathcal{D}_t' \end{aligned} \tag{1.62}$$

and

$$\mathcal{D}_t \;\subset\; \mathcal{S}_t \;\subset\; \mathcal{S}_t' \;\subset\; \mathcal{D}_t' \tag{1.63}$$

where, in **(1.63)**, any space is dense in any space to the right.

One cannot generally define the product of two distributions. That is, if $f(t)$ and $g(t)$ are distributions, $f(t)g(t)$ is not necessarily a distribution. This is one of the main disadvantages of the theory of distributions. It also creates problems for our mechanics formalism. As we explore our fractional mechanics, we will want to define quantities of the form

$$\Phi_\alpha^+(t-\tau)q(t)q(\tau) \tag{1.64}$$

where $q(t)$ is generally a distribution. We need to make sense of **(1.64)** in the Volterra series framework.

First, let us define the infinitely smooth function

$$\widetilde{q}(t) \;:=\; \langle \widehat{q}(\xi), \phi(t-\xi) \rangle \tag{1.65}$$

where $\widehat{q}(\xi)$ is a distribution with support on $a \le \xi \le b$ and $\phi(t)$ is a test function in $\mathcal{D}_t$. Equation **(1.65)** is the regularization of the distribution $\widehat{q}(t)$. By regularization we mean the convolution of a distribution in $\mathcal{D}_t'$ with a test function in $\mathcal{D}_t$. The following theorem makes this precise.

Theorem 1 (Zemanian (1965), Theorem 5.5-1) *Let* $f(t) \in \mathcal{D}_t'$ *and* $\phi(t) \in \mathcal{D}_t$. *Define*

$$\begin{aligned} h(t) &:= f(t) * \phi(t) \\ &= \langle f(\xi), \phi(t-\xi) \rangle \end{aligned} \tag{1.66}$$

where $h(t)$ *is a regular function. Then*

$$\frac{d^n}{dt^n} h(t) = \langle f(\xi), \frac{d^n}{dt^n} \phi(t-\xi) \rangle. \tag{1.67}$$

That is, $h(t)$ *is infinitely differentiable. Hence,* $h(t) \in \mathcal{E}_t$.

Since $\mathcal{D}_t$ is dense in $\mathcal{D}_t'$ (see **(1.63)**), and the convolution operation is continuous [21], we can make $h(t)$ as 'close' to $f(t)$ as we like. That is, we can let $\phi(t) \to \delta(t)$ in **(1.66)**.

Using $\widetilde{q}(t)$ instead of $q(t)$ in **(1.64)**, we have that

$$\Phi_\alpha^+(t-\tau)\widetilde{q}(t)\widetilde{q}(\tau) \tag{1.68}$$

is a distribution [21]. We can sample **(1.68)** by

$$\langle \Phi_\alpha^+(t-\tau)\widetilde{q}(t)\widetilde{q}(\tau), \eta(t)\eta(\tau) \rangle \tag{1.69}$$

where $\eta(t)$ is in $\mathcal{D}_t$ with support on, e.g., $a \leq t \leq b$. Looking ahead a little, let us consider **(1.69)** as an action. Then we would want $\eta(t)$ to be the indicator function for $a \leq t \leq b$:

$$\eta(t) = \begin{cases} 1 & \text{, if } a \leq t \leq b \\ 0 & \text{, else.} \end{cases} \tag{1.70}$$

But then $\eta(t)$ would not be in $\mathcal{D}_t$. We can rearrange **(1.69)** slightly to find

$$\begin{aligned} \langle \Phi_\alpha^+(t-\tau)\widetilde{q}(t)\widetilde{q}(\tau), \eta(t)\eta(\tau)\rangle &= \langle \Phi_\alpha^+(t-\tau)\eta(t)\eta(\tau), \widetilde{q}(t)\widetilde{q}(\tau)\rangle \\ &= \langle \widehat{\Phi}_\alpha^+(t-\tau), \widetilde{q}(t)\widetilde{q}(\tau)\rangle \\ &:= \mathcal{V}[\widetilde{q}(t)] \end{aligned} \tag{1.71}$$

where

$$\widehat{\Phi}_\alpha^+(t-\tau) := \Phi_\alpha^+(t-\tau)\eta(t)\eta(\tau). \tag{1.72}$$

Notice that $\widehat{\Phi}_\alpha^+(t-\tau)$ is compactly supported on the region $a \leq t, \tau \leq b$. So $\widehat{\Phi}_\alpha^+(t-\tau) \in \mathcal{E}'_{t,\tau}$. Also, $\widetilde{q}(t)\widetilde{q}(\tau) \in \mathcal{E}_{t,\tau}$. Now we can let $\eta(t)$ be the indicator function for $a \leq t \leq b$ and give meaning to **(1.69)** via **(1.71)**.

To take the functional derivative of **(1.71)**, we will need an integration by parts formula. Observe that

$$\langle \Phi_\alpha^+(t-\tau), \phi(t,\tau)\rangle = \langle \Phi_\alpha^-(\tau-t), \phi(t,\tau)\rangle \tag{1.73}$$

for any $\phi(t,\tau) \in \mathcal{D}_{t,\tau}$. It follows that

$$\langle \widehat{\Phi}_\alpha^+(t-\tau), \widehat{\phi}(t,\tau)\rangle = \langle \widehat{\Phi}_\alpha^-(\tau-t), \widehat{\phi}(t,\tau)\rangle \tag{1.74}$$

where $\widehat{\phi}(t,\tau)$ is any function in $\mathcal{E}_{t,\tau}$. Equation **(1.74)** is our integration by parts formula (compare with **(1.32)**).

Returning to **(1.71)**, let us now perturb $\widetilde{q}(t)$ by a test function $hx_\rho(t-\rho)$, where h is infinitesimal, $a < \rho < b$ and $x_\rho(t-\rho)$ is compactly supported on the interval $a \leq t \leq b$. (This will generally require choosing a different $x_\rho(t-\rho)$ for every ρ. One should imagine $x_\rho(t-\rho)$ as a test function concentrated around ρ. For simplicity, we will take $\int x_\rho(t-\rho)\, dt = 1$.) Then the functional derivative of **(1.71)**, after letting $h \to 0$, is given by

$$\begin{aligned} \frac{\delta \mathcal{V}[\widetilde{q}(t)]}{\delta \widetilde{q}(\rho)} &= \langle \widehat{\Phi}_\alpha^+(t-\tau), \widetilde{q}(t) x_\rho(\tau-\rho)\rangle + \\ &\quad \langle \widehat{\Phi}_\alpha^+(t-\tau), x_\rho(t-\rho)\widetilde{q}(\tau)\rangle. \end{aligned} \tag{1.75}$$

We will focus our attention on the second term on the right-hand side of **(1.75)**.

Using **(1.65)** and **(1.74)**, we have

$$\begin{aligned}
\langle\widehat{\Phi}_\alpha^+(t-\tau), x_\rho(t-\rho)\widetilde{q}(\tau)\rangle &= \langle\langle\widehat{\Phi}_\alpha^+(t-\tau), x_\rho(t-\rho)\rangle, \langle\widehat{q}(\xi), \phi(\tau-\xi)\rangle\rangle \\
&= \langle\langle\widehat{\Phi}_\alpha^-(\tau-t), x_\rho(t-\rho)\rangle, \langle\widehat{q}(\xi), \phi(\tau-\xi)\rangle\rangle \\
&= \langle\langle\widehat{\Phi}_\alpha^-(t-\rho), x_\rho(\tau-t)\rangle, \langle\widehat{q}(\xi), \phi(\tau-\xi)\rangle\rangle.
\end{aligned} \tag{1.76}$$

Equation **(1.76)** can be understood using only ordinary functions because both of the quantities $\langle\widehat{\Phi}_\alpha^-(t-\rho), x_\rho(\tau-t)\rangle$ and $\langle\widehat{q}(\xi), \phi(\tau-\xi)\rangle$ are infinitely smooth functions of τ, being regularizations of $\widehat{\Phi}_\alpha^-(t-\rho)$ (a shifted version of $\widehat{\Phi}_\alpha^-(t)$) and $\widehat{q}(\xi)$, respectively.

Due to the continuity of the convolution operation and the density of $\mathcal{D}_t$ in $\mathcal{D}_t'$, we can let $x_\rho(t) \longrightarrow \delta(t)$ in **(1.76)** to have

$$\langle\widehat{\Phi}_\alpha^+(t-\tau), x_\rho(t-\rho)\widetilde{q}(\tau)\rangle \longrightarrow \langle\widehat{\Phi}_\alpha^-(\rho-\tau), \widetilde{q}(\tau)\rangle. \tag{1.77}$$

A similar analysis shows that

$$\langle\widehat{\Phi}_\alpha^+(t-\tau), \widetilde{q}(t)x_\rho(\tau-\rho)\rangle \longrightarrow \langle\widehat{\Phi}_\alpha^-(\rho-t), \widetilde{q}(t)\rangle. \tag{1.78}$$

Then **(1.75)** becomes

$$\frac{\delta\mathcal{V}[\widetilde{q}(t)]}{\delta\widetilde{q}(\rho)} = \langle\widehat{\Phi}_\alpha^-(\rho-t), \widetilde{q}(t)\rangle + \langle\widehat{\Phi}_\alpha^+(\rho-\tau), \widetilde{q}(\tau)\rangle. \tag{1.79}$$

The result in **(1.79)** is complicated by the fact that $\widehat{\Phi}_\alpha^+(t-\tau)$ in **(1.71)** is triangular. If we had a symmetric kernel, $\widehat{K}_2^{(s)}(t,\tau)$, in **(1.71)** we would have found that

$$\begin{aligned} \frac{\delta \widehat{K}_2^{(s)}[\widetilde{q}(t)]}{\delta \widetilde{q}(\rho)} &= 2\widehat{K}_2^{(s)} \star \widetilde{q} \\ &= 2\langle \widehat{K}_2^{(s)}(\rho,\tau), \widetilde{q}(\tau)\rangle. \end{aligned} \tag{1.80}$$

Equation **(1.80)** is a special case of the more general formula [20]

$$\begin{aligned} \frac{\delta \widehat{K}_n^{(s)} \star \widetilde{q}^{\,n}}{\delta \widetilde{q}(\rho)} &= n\widehat{K}_n^{(s)} \star \widetilde{q}^{\,n-1} \\ &= n\langle \widehat{K}_n^{(s)}(\rho,\tau_2,\ldots,\tau_n), \widetilde{q}(\tau_2)\cdots\widetilde{q}(\tau_n)\rangle. \end{aligned} \tag{1.81}$$

Equations **(1.79)** and **(1.80)** will be important for our fractional mechanics. As we saw in Section **1.2.1**, we can switch between triangular and symmetric kernels in our actions. Thus, **(1.79)** is the variation of the action with a triangular kernel and **(1.80)** the variation with a symmetric kernel. In **(1.79)** the retarded, $\widehat{\Phi}_\alpha^+(\rho-\tau)$, and advanced, $\widehat{\Phi}_\alpha^-(\rho-t)$, parts of the variation are explicitly shown. In **(1.80)** they are combined in the $\widehat{K}_2^{(s)}(\rho,\tau)$.

Of more importance to us is letting $x_\rho(t) \to \delta(t)$ and $\phi(t) \to \delta(t)$ in **(1.76)**. Again, we can approximate this situation as closely as we like because $\mathcal{D}_t$ is dense in $\mathcal{D}_t'$ and the convolution operation is continuous. So we will take **(1.76)** as meaning

$$\begin{aligned}\langle\widehat{\Phi}_\alpha^+(t-\tau), x_\rho(t-\rho)\widetilde{q}(\tau)\rangle &\longrightarrow \langle\widehat{\Phi}_\alpha^-(\tau-\rho), \widehat{q}(\tau)\rangle \\ &= \widehat{\Phi}_\alpha^+(\tau) * \widehat{q}(\tau)\mid_{\tau=\rho} \qquad (1.82)\end{aligned}$$

in the classical limit where $a < \rho < b$. A similar result would follow from the first term of the right-hand side of **(1.75)** or, if we had used $\widehat{\Phi}_\alpha^-(t-\tau)$ or $\widehat{\Psi}_{2\alpha}(t-\tau)$ in **(1.71)** instead of $\widehat{\Phi}_\alpha^+(t-\tau)$.

For our latter developments, it will be useful to summarize what we have done. We started with some system, $q(t)$, that interacts with an environment, $\widehat{\Phi}_\alpha^+(t)$, over some time period, $a \le t \le b$. Then we force the system and the environment to vanish outside of $a \le t \le b$. This is because whatever the system and environment do outside their time period of interaction should not affect their (classical) interaction. So now we have $\widehat{q}(t)$ and $\widehat{\Phi}_\alpha^+(t)$. Due to some mathematical technicalities, we need to smooth (i.e., regularize) $\widehat{q}(t)$ and $\widehat{\Phi}_\alpha^+(t)$. (This regularization can be imagined as a local averaging.) We eventually end up with $\widetilde{q}(t)$ and $\widetilde{\Phi}_\alpha^-(t-\rho) := \langle\widehat{\Phi}_\alpha^-(\tau-\rho), x_\rho(t-\tau)\rangle$ in **(1.76)**. In order to arrive at **(1.82)**, we need to let this regularization become 'sharp'. That is, for the classical equations of motion, we let $\widetilde{q}(t) \to \widehat{q}(t)$ and $\widetilde{\Phi}_\alpha^-(t-\rho) \to \widehat{\Phi}_\alpha^-(t-\rho)$. Now, integrating $\widehat{q}(t)\widehat{\Phi}_\alpha^-(t-\rho)$ over t is the same as evaluating $\widehat{\Phi}_\alpha^+(t) * \widehat{q}(t)$ at

$t = \rho$ where $a < \rho < b$. Using this procedure, we will end up with our ordinary fractional differential equation for our classical equation of motion, as will be seen shortly.

Having taken care of the mathematical formalities, we will use a sloppy, though intuitive, notation from now on. That is, we will drop the 'hats', 'tildes', etc. So our actions will be written as in (**1.56**)

$$\mathcal{V}[q] = \sum_{n=1}^{N} \frac{1}{n!} K_n^{(s)} \star q^n \tag{1.83}$$

even though $K_n^{(s)}$ in (**1.83**) should have a 'hat' and q should have a 'tilde', see (**1.71**). Also, if we take the functional derivative of (**1.83**), we will use the notation, see (**1.81**),

$$\begin{aligned} \frac{\delta K_n^{(s)} \star q^n}{\delta q(t)} &= n K_n^{(s)} \star q^{n-1} \\ &:= n \langle K_n^{(s)}(t, \tau_2, \ldots, \tau_n), q(\tau_2) \cdots q(\tau_n) \rangle \end{aligned} \tag{1.84}$$

so that (**1.83**) would have the functional derivative

$$\frac{\delta \mathcal{V}[q]}{\delta q(t)} = \sum_{n=1}^{N} \frac{1}{(n-1)!} K_n^{(s)} \star q^{n-1}. \tag{1.85}$$

The above notation should not cause any confusion as long as we remember that it can be justified in the classical case.

1.3 Riewe's Fractional Lagrangian Mechanics

In classical mechanics, the Lagrangian typically contains terms that only involve the position $q(t)$ and its first time derivative $\dot{q}(t)$. Generalized mechanics is the branch of classical mechanics that deals with the inclusion of higher ordered derivatives of $q(t)$ in the Lagrangian. However, all of the derivatives in generalized mechanics are assumed to be integer ordered. Riewe generalized generalized mechanics by allowing the inclusion of fractional derivatives in the Lagrangian [16, 17]. (Instead of calling Riewe's formalism generalized generalized mechanics, we will refer to it as fractional mechanics.) While Riewe's method has some complications, which we attempt to overcome in Chapter 3, the basic idea is worthwhile to pursue. We will restrict our attention to Riewe's fractional Lagrangian mechanics. This is the form that is most relevant to our formalism developed in Chapter 3. (Riewe does develop a Hamiltonian formulation in [16, 17]. We will not examine this here.)

The first thing we need to do is restate Riewe's formalism in the language of generalized functions and Volterra series. All of the work in [16, 17] was done using the notation in **(1.33)**, where the derivatives are understood in the sense of Riemann-Liouville (see **(1.24)** and the following discussion.) We can write **(1.33)** as

$$\int_a^b {}_a\mathbf{D}_t^{\alpha}\left[q\right] {}_a\mathbf{D}_t^{\alpha}\left[q\right]\, dt \;=\; \langle \Psi_{-2\alpha}(t-\tau), q(t)q(\tau)\rangle. \tag{1.86}$$

The right-hand side of (**1.86**) will be our basic building block for Riewe's fractional mechanics.

Instead of examining the most general situation, we will instead look at a few examples. These will illustrate the advantages of the notation in (**1.86**) as well as the difficulties inherent in Riewe's method. We start with the harmonic oscillator (HO), which we will first examine using the standard techniques. The HO's equation of motion is

$$m\ddot{q} + m\omega^2 q \;=\; 0 \tag{1.87}$$

over some time period $a < t < b$. Typically we would use the Lagrangian

$$L_{HO}(q, \dot{q}) \;=\; \frac{m}{2}\,(\dot{q})^2 - \frac{m\omega^2}{2}\,q^2 \tag{1.88}$$

to derive (**1.87**). How do we go about doing this? First, we form the action

$$\mathcal{V}_{HO}[q] \;:=\; \int_a^b L_{HO}(q, \dot{q})\, dt. \tag{1.89}$$

Next, we perturb $q(t)$ in (**1.89**) by an infinitesimal function $\eta(t)$ that vanishes outside $a < t < b$. This gives us

$$\begin{aligned}\delta \mathcal{V}_{HO}[q] &= \delta \int_a^b L_{HO}(q,\dot{q})\,dt \\ &= \int_a^b [L_{HO}(q+\eta,\dot{q}+\dot{\eta}) - L_{HO}(q,\dot{q})]\,dt. \end{aligned} \tag{1.90}$$

Expanding the perturbed Lagrangian in **(1.90)**

$$L_{HO}(q+\eta,\dot{q}+\dot{\eta}) \approx L_{HO}(q,\dot{q}) + \frac{\partial L_{HO}}{\partial q}\,\eta + \frac{\partial L_{HO}}{\partial \dot{q}}\,\dot{\eta} \tag{1.91}$$

and using this in **(1.90)** results in

$$\begin{aligned}\delta \mathcal{V}_{HO}[q] &\approx \int_a^b \left[\frac{\partial L_{HO}}{\partial q}\,\eta + \frac{\partial L_{HO}}{\partial \dot{q}}\,\dot{\eta}\right] dt \\ &= \int_a^b \eta \left[\frac{\partial L_{HO}}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_{HO}}{\partial \dot{q}}\right)\right] dt. \end{aligned} \tag{1.92}$$

Requiring **(1.92)** to vanish is equivalent to requiring the bracketed term to vanish since $\eta(t)$ is arbitrary. So we have

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_{HO}}{\partial \dot{q}}\right) - \frac{\partial L_{HO}}{\partial q} &= m\ddot{q} + m\omega^2 q \\ &= 0 \end{aligned} \tag{1.93}$$

when $\delta \mathcal{V}_{HO}[q] = 0$. The left-hand side of **(1.93)** is the one-dimensional Euler-Lagrange equation of motion. The right-hand side is the HO's equation of motion.

Now, there is an alternate Lagrangian we could have used to derive **(1.87)**. Let

$$L'_{HO}(q,\ddot{q}) = -\frac{m}{2}\,q\ddot{q} - \frac{m\omega^2}{2}\,q^2. \tag{1.94}$$

The Euler-Lagrange equation associated with (**1.94**) is given by [10]

$$\frac{d^2}{dt^2}\left(\frac{\partial L'_{HO}}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L'_{HO}}{\partial \dot{q}}\right) + \frac{\partial L'_{HO}}{\partial q} = 0. \tag{1.95}$$

Using (**1.94**) in (**1.95**) results in (**1.87**) again. (The equations in (**1.94**) and (**1.95**) fall into the subject of generalized mechanics.) The Lagrangians in (**1.88**) and (**1.94**) are not equivalent, even though they both lead to the same equation of motion. Notice, however, that

$$\begin{aligned} \mathcal{V}_{HO}[q] &= \int_a^b L_{HO}(q,\dot{q})\,dt \\ &= \int_a^b \left[\frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2\right] dt \end{aligned} \tag{1.96}$$

and

$$\begin{aligned} \mathcal{V}'_{HO}[q] &= \int_a^b L'_{HO}(q,\ddot{q})\,dt \\ &= \int_a^b \left[-\frac{m}{2}q\ddot{q} - \frac{m\omega^2}{2}q^2\right] \\ &= C + \int_a^b \left[\frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2\right] dt \end{aligned} \tag{1.97}$$

where $C = -m/2(q\dot{q})|_a^b$, which can be considered a constant since $\eta(a) = \eta(b) = 0 = \dot{\eta}(a) = \dot{\eta}(b)$. So, up to a constant, the actions associated with the Lagrangians in (**1.88**) and (**1.94**) are the same. The constant in (**1.97**) can be discarded without affecting our analysis, being the K_0 term of the Volterra series $\mathcal{V}'_{HO}$.

The above suggests that the action is more fundamental than the Lagrangian. In fact, it seems as though we are dealing with an equivalence class of actions where

$$\mathcal{V}_{HO} \sim \mathcal{V}_{HO}^{'} \quad \Leftrightarrow \quad \mathcal{V}_{HO} = A\left[C + \mathcal{V}_{HO}^{'}\right] \tag{1.98}$$

with $C \in \mathbb{R}$ and $A \in \mathbb{R}\backslash 0$. This is why we let $K_0 = 0$ in our Volterra series. Using the notation in (**1.86**), we can write our representative example of the action associated with the HO as

$$\mathcal{V}_{HO}[q] \;\; = \;\; -\frac{m}{2}\,\langle \Psi_{-2}(t-\tau) - \omega^2\Psi_0(t-\tau), q(t)q(\tau)\rangle. \tag{1.99}$$

By using only an action, we will not be concerned with any particular Lagrangian that leads to that action.

Notice that the $\Psi_{-\beta}(t-\tau)$ in (**1.99**) are all even order, i.e., $\beta = 2n$ with n an integer. It follows from (**1.17**), (**1.29**) and (**1.47**) that we can interchange the $\Psi_{-\beta}$ with $\Phi^{\pm}_{-\beta}$ in (**1.99**), up to a sign on the $\Phi^{\pm}_{-\beta}$. How do we decide which kernel to use in (**1.99**)? Because of the use of (**1.33**) in deriving the equations of motion, Riewe was naturally lead to use the $\Psi_{-\beta}(t-\tau)$. This is not an unreasonable choice. In particular, the $\Psi_{-\beta}(t-\tau)$ are symmetric under an interchange of t and τ, while the $\Phi^{\pm}_{-\beta}(t,\tau)$, being triangular kernels, are not symmetric under this interchange.

Notice, however, that

$$\frac{1}{2}\left[\Phi^{+}_{-\beta}(t-\tau)+\Phi^{-}_{-\beta}(t-\tau)\right] \tag{1.100}$$

is also symmetric under an interchange of t and τ

$$\begin{aligned}\frac{1}{2}\left[\Phi^{+}_{-\beta}(t-\tau)+\Phi^{-}_{-\beta}(t-\tau)\right] &\xleftrightarrow{t\leftrightarrow\tau} \frac{1}{2}\left[\Phi^{+}_{-\beta}(\tau-t)+\Phi^{-}_{-\beta}(\tau-t)\right]\\ &= \frac{1}{2}\left[\Phi^{-}_{-\beta}(t-\tau)+\Phi^{+}_{-\beta}(t-\tau)\right].\end{aligned} \tag{1.101}$$

(Compare **(1.100)** with **(1.42)**. Notice the lack of the $1/\cos(\beta\pi)$ term.) In any case, Riewe used the $\Psi_{-\beta}(t-\tau)$ in the kernels and it is this situation we wish to examine further. Let us move on to our next example.

We will now consider the nonconservative HO. A typical example of this situation is the HO with friction

$$m\ddot{q}+mC\dot{q}+m\omega^2 q \;=\; 0. \tag{1.102}$$

The action associated with the nonconservative HO is

$$\mathcal{V}[q] \;=\; -\frac{m}{2}\left\langle \Psi_{-2}(t-\tau)-C\Psi_{-\gamma}(t-\tau)-\omega^2\Psi_0(t-\tau), q(t)q(\tau)\right\rangle \tag{1.103}$$

where we only used the $\Psi_{-\beta}(t-\tau)$ in our kernel and $0<\gamma<2$. From **(1.48)** we see that it is impossible to have **(1.102)** result from **(1.103)**. Specifically, our equation of motion from **(1.103)** would be

$$
\begin{aligned}
-m{}_a^t\mathbf{F}_b^2[q] + mC{}_a^t\mathbf{F}_b^\gamma[q] + m\omega^2{}_a^t\mathbf{F}_b^0[q] &= m\ddot{q} + mC{}_a^t\mathbf{F}_b^\gamma[q] + m\omega^2 q \\
&= 0 \qquad (1.104)
\end{aligned}
$$

for $a < t < b$.

Notice the appearance of the FFD in **(1.104)**. This creates difficulties because the FFD is an acausal operator. In order to have a strictly causal equation of motion, Riewe suggests considering an infinitesimal time interval, e.g., $[0, 2\varepsilon]$, and then replacing all RFDs with LFDs. This seems unsatisfactory because fractional operators have a memory due to their nonlocal (in time) nature. By restricting the time interval to an infinitesimal duration, Riewe is effectively erasing this memory. Also, it is questionable if this will provide an accurate approximation. For example, with our time interval $[0, 2\varepsilon]$, let

$$
\begin{aligned}
f(t) &= \delta(t-\varepsilon) \\
&= \Phi_0^+(t-\varepsilon) \qquad (1.105) \\
&= \Psi_0(t-\varepsilon).
\end{aligned}
$$

Then,

$$
{}_a\mathbf{D}_t^{2\alpha}[f] = \Phi_{-2\alpha}^+(t-\varepsilon) \qquad (1.106)
$$

but

$$ {}_a^t\mathbf{F}_b^{2\alpha}[f] \;=\; \Psi_{-2\alpha}(t-\varepsilon). \tag{1.107} $$

Now, let $\alpha = 1/2$. Then $\Phi_{-1}^{+}(t-\varepsilon) = \dot{\delta}(t-\varepsilon) \neq \Psi_{-1}(t-\varepsilon)$.

If we blindly follow the above procedure for **(1.104)** with $\alpha = 1/2$, we have

$$ \begin{aligned} -m\,{}_a^t\mathbf{F}_b^2[q] + mC\,{}_a^t\mathbf{F}_b^1[q] + m\omega^2\,{}_a^t\mathbf{F}_b^0[q] \;\longrightarrow\; & -m\,{}_a\mathbf{D}_t^2\,[q] + mC\,{}_a\mathbf{D}_t^1\,[q] \\ & +m\omega^2\,{}_a\mathbf{D}_t^0\,[q]\,. \end{aligned} \tag{1.108} $$

So our equation of motion would be

$$ m\ddot{q} - mC\dot{q} - m\omega^2 q \;=\; 0 \tag{1.109} $$

for $a < t < b$. If we changed the sign on $\Psi_{-2}(t-\tau)$ in **(1.103)**, then **(1.109)** would become **(1.102)** by using Riewe's procedure. However, the arguments above call into question if this is a good idea. Also, if $C = 0$ in **(1.104)**, i.e., the standard HO, then the equation in of motion in **(1.104)** is correct without any need to replace operators. So Riewe's formalism is unsatisfactory, as this example shows.

The above sign problem can be avoided by letting the coefficients in our action to be complex. We also redefine the $\Psi_\alpha(t)$ operators to have α dependent phase factors. These are chosen such that the resulting equation of motion will only have real coefficients with the correct signs. We can the replace the FFDs with LFDs and have the correct signs on our various terms. This procedure was not suggested by Riewe. Also, it does not avoid the final replacement of FFDs with LFDs.

1.4 A Few Notes

We are now done with reviewing the background material. In Chapter **2** we summarize both what we have seen so far and, what we plan to do in Chapter **3**. However, this would be a good place to make a few things explicit. In Section **1.1** we saw four different definitions of fractional derivatives: distributional, Riemann-Liouville, Caputo and Feller. We will only use the distributional definition in what follows. This choice is strictly a matter of convenience since the distributional approach is the easiest to work with and, fits most naturally into the Volterra series framework. Riewe's use of the Feller fractional derivatives ultimately is unsatisfactory. As (**1.48**) shows, we would be unable to include odd integer ordered derivatives in our equations of motion if we used Feller derivatives. As we saw above, this deficiency makes it impossible to model even the simple nonconservative HO in Riewe's formalism. This is the major shortcoming in Riewe's method which we correct in Chapter **3**.

Chapter 2

Interlude

We have seen some rather complicated mathematics up to now. Unfortunately, the technicalities can obscure what are rather simple ideas. Here we give a heuristic overview of where we have been and where we'll be going. This should hopefully make the main ideas of our fractional mechanics concrete.

In Section **1.1** we introduced the fractional derivative operations via $\Phi_\alpha^\pm$ and Ψ_α. We view these as generalized functions defined over some multi-dimensional space. To make matters easy, we restrict our attention to the two-dimensional case here (i.e., the $t\tau$-plane). This would be sufficient to handle the HO, for example. We then have three distinct generalized function: $\Phi_\alpha^+(t-\tau) = \Phi_\alpha^-(\tau-t)$,

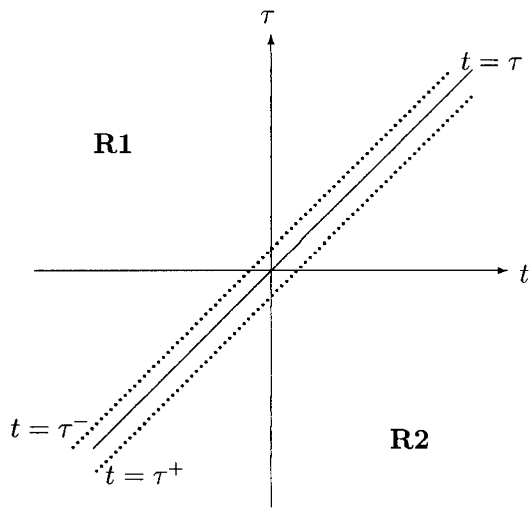

Figure 2.1: The supports of $\Phi^+_\alpha(t-\tau)$, $\Phi^+_\alpha(\tau-t)$ and $\Psi_\alpha(t-\tau)$.

$\Phi^+_\alpha(\tau-t) = \Phi^-_\alpha(t-\tau)$ and $\Psi_\alpha(t-\tau)$. In Figure **2.1** we show the $t\tau$-plane divided by the $t = \tau$ line. The distribution $\Phi^+_\alpha(t-\tau)$ is non-zero only in the $R2$ region (i.e., $t > \tau$). Alternately, $\Phi^+_\alpha(\tau-t)$ is non-zero only in $R1$ (i.e., $t < \tau$). Consider now the case when $\alpha = -n$, where n is a non-negative integer. Then (see **(1.17)** and **(1.29)**)

$$\Phi^+_{-n}(t-\tau) = \left(\frac{d}{dt}\right)^n \delta(t-\tau^+) \tag{2.1}$$

$$\Phi^-_{-n}(t-\tau) = \left(-\frac{d}{dt}\right)^n \delta(t-\tau^-). \tag{2.2}$$

So, instead of taking the $\Phi^\pm_{-n}(t,\tau)$ as lying on the $t = \tau$ line, we take them as lying on one of the $t = \tau^\pm$ lines. This distinction is not too crucial if n is even. For n

odd, the difference in sign between (**2.1**) and (**2.2**) makes the distinction important. Finally, the $\Psi_\alpha(t-\tau)$ are defined on the whole $t\tau$-plane and are symmetric around the $t = \tau$ line.

So how do we use the distributions $\Phi_\alpha^+(t-\tau)$, $\Phi_\alpha^+(\tau-t)$ and $\Psi_\alpha(t-\tau)$? The action is treated as a Volterra series and the above distributions are used to construct the kernels of the series. The introduction of Volterra series into the fractional mechanics formalism provides a very nice framework. For example, a finite term action may be a truncation of an infinite term Volterra series. This is analogous to approximating a function by the first few terms of its power series. Also, the Volterra series concept provides a way of seeing how to expand classical mechanics. Generalized and fractional mechanics are two generalizations of classical mechanics that fit nicely into the Volterra series framework. There are undoubtedly others.

We have modelled the environment as some combination of the fractional operators. This environment is assumed to be defined over the entire $t\tau$-plane. That is, we take the environment as unchanging and eternal. In the Volterra series we also have a system q that is acted on by the kernels. A typical term in the

Volterra series would be

$$\int \int K_2(t, \tau)q(t)q(\tau)dtd\tau. \tag{2.3}$$

The quantity $q(t)q(\tau)$ is also assumed to be defined over the whole $t\tau$-plane. So the system q is also taken to be eternal. In Figure **2.1**, we imagine 'laying down' two distributions. One of these is the Volterra series kernel (i.e., the environment) $K_2(t, \tau)$ and the other is the system $q(t)q(\tau)$. For the action in **(2.3)**, though, we do not usually take the integrals as extending over all of $\mathbb{R}$. Instead, we restrict the integrals to some finite time period like $a \leq t, \tau \leq b$. This means that the system interacts with the environment only for $a \leq t \leq b$. In Section **1.2.2**, this led to the need to regularize, 'de-regularize', etc. All of this was because we cannot generally define the product of two distributions, as we are implicitly doing in **(2.3)**. Ignoring all of the mathematical technicalities, what we essentially end up doing is setting $K_2(t, \tau)$ and $q(t)q(\tau)$ equal to zero outside of the interaction region I in Figure **2.2**.

Let us motivate this a little. Imagine shooting a pellet through a styrofoam target. We are only concerned with how the pellet's velocity changes as it passes through the target. Of course the pellet (the system) and the target (the environment) existed before their interaction. But we do not need to concern ourselves

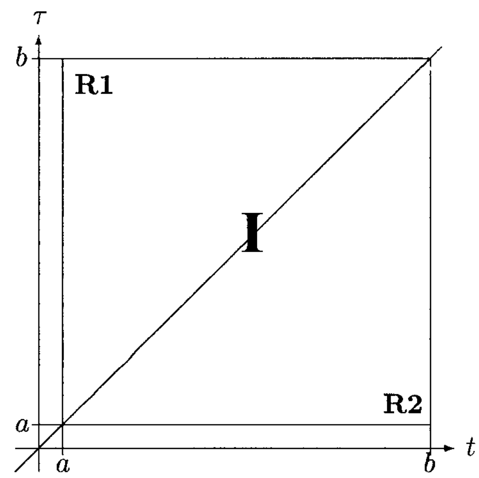

Figure 2.2: Regions of support for the restricted kernels.

with what the pellet or target did prior to their interaction as long as we can specify their states during the interaction period. So we can ignore their evolutions outside of I in Figure **2.2**. This is what we are doing above. While we can imagine the system and environment as existing for all time, what really concerns us is their time period of interaction.

Let $K_2(t, \tau)$ in **(2.3)** be given by $\Phi_\alpha^+(t-\tau)$, which is only defined on $R2$ in I in Figure **2.2**. Looking at the quantity

$$\int_a^b \Phi_\alpha^+(t-\tau) q(\tau) d\tau \tag{2.4}$$

we see that this is, roughly, the LFD of $q(\tau)$ over $a \leq \tau \leq b$. However, the environment only retains a memory of $q(\tau)$ that begins at the start of their interaction at $\tau = a$. Also, the environment's effect on q's evolution ends at $\tau = b$, the end of the interaction period. This is why any equation of motion for $q(\tau)$ is only over some specified time interval.

Since $\Phi_\alpha^+(t-\tau) = \Phi_\alpha^-(\tau - t)$, we could have equally well written

$$\int_a^b \Phi_\alpha^-(\tau - t)q(t)dt \tag{2.5}$$

instead of (**2.4**). Of course (**2.5**) is the RFD of $q(t)$. This is the fundamental difficulty of working with fractional operations. Any equation of motion that involves only LFDs has a dual that involves only RFDs. If α is not an even ordered integer, these two equations of motion are not equal. Classical and generalized mechanics only deal with even ordered derivatives, so this difficulty has been largely overlooked. (Why does α have to be an even ordered integer? First, it needs to be an integer so that it is localized around the $t = \tau$ line. Next, it needs to be even so that no sign difference arises between the LFD and the RFD, see (**1.17**) and (**1.29**).)

Riewe attempted to generalize generalized mechanics in his papers [16, 17]. In doing this he utilized the $\Psi_\alpha(t-\tau)$ as the $K_2(t,\tau)$ kernel in (**2.3**). As we stated previously, generalized mechanics only deals with even ordered derivatives in the equations of motion. In (**1.47**) we saw that

$$ {}_a^t\mathbf{F}_b^{2n}[q] \;=\; (-1)^n q^{(2n)}(t). \tag{2.6} $$

So the $\Psi_\alpha(t-\tau)$ are sufficiently general to handle generalized mechanics. However, (**1.48**) shows that

$$ {}_a^t\mathbf{F}_b^{2n+1}[q] \;\neq\; \pm q^{(2n+1)}(t). \tag{2.7} $$

Hence, we cannot include odd ordered derivatives in our equations of motion by utilizing the $\Psi_\alpha(t-\tau)$. Riewe's formalism is certainly a generalization of generalized mechanics. Since we cannot include a $\dot{q}$ term in our equations of motion, for example, we need to question if using the $\Psi_\alpha(t-\tau)$ as kernels in our Volterra series is wise. We will instead use the $\Phi_\alpha^{\pm}(t,\tau)$ as our kernels. As alluded to above (see also (**1.79**)), we will then have a causal equation of motion arise. We will also have an anti-causal equation of motion. This dual equation arises, as we saw in (**2.5**), because $\Phi_\alpha^{+}(t-\tau) = \Phi_\alpha^{-}(\tau-t)$ in the $t\tau$-plane. This will be made clearer shortly.

Now we will move on to our new fractional mechanics formalism. In Chapter **3** we will see how to overcome the limitations of Riewe's method. In particular, we can have a $\dot{q}$ term arise in our equations of motion. This is not trivial and suggests that our approach is a major improvement over Riewe's. There is a fundamental difficulty with working with fractional derivatives, however. Namely, we need to deal with advanced effects directly. Riewe attempted to sweep this under the carpet by changing RFDs to LFDs. As commented on in Section **1.3**, this is not a wise approach. These advanced effects should not be considered a fatal defect of the theory. Rather, they should be taken as an indication that we do not completely know how to model the physical world. This seems to be a recurring theme wherever fractional derivatives are used. They force us to confront and question our assumptions.

Chapter 3

Volterra Series Fractional Mechanics

Here we present our new fractional mechanics formalism. This will allow us to correct the flaws in Riewe's formalism. For example, we will be able to include odd integer ordered derivatives in our equations of motion. In Section **3.1** we derive our equations of motion using a procedure that is typically used in analytical mechanics. However, a problem immediately presents itself. Namely, we have two equations of motion arise: a retarded equation and an advanced one. So in Section **3.2** we present an alternate procedure that allows us to derive a single, retarded equation of motion. This method seems rather artificial, so we need to examine what is going on in more detail. We do this in Section **3.3**, where we examine the Caldeira-Leggett model. This allows us to give some justification to the procedure developed in Section **3.2**. It also deepens our understanding of how the equations of motion arise. A discussion of our research is given in Section **3.4**, including an extension of Bauer's corollary [2].

3.1 The Standard Development

Now we are prepared to examine our new fractional mechanics formalism. We have previously stated that we will not use the $\Psi_\alpha(t-\tau)$ in our action's kernels. Instead, the $\Phi^{\pm}_{\alpha}(t,\tau)$ will be employed. We will begin by examining the nonconservative HO. We then progressively generalize this action to new situations.

Instead of using the action in **(1.103)** for the nonconservative HO, let us begin with

$$\mathcal{V}_2[q] \;=\; \frac{m}{2}\,\langle \Phi^{+}_{-2}(t-\tau) + C\Phi^{+}_{-\gamma}(t-\tau) + \omega^2\Phi^{+}_{0}(t-\tau), q(t)q(\tau)\rangle. \quad (3.1)$$

We follow the usual procedure of perturbing $q(t)$ by a test function $\eta(t)$ that is compactly supported on $a \leq t \leq b$. From **(3.1)** this results in

$$\begin{aligned}
\delta\mathcal{V}_2[q] &= \underbrace{\frac{m}{2}\left\langle \Phi^+_{-2}(t-\tau) + C\Phi^+_{-\gamma}(t-\tau) + \omega^2\Phi^+_0(t-\tau), \eta(t)q(\tau)\right\rangle}_{\textbf{retarded}} + \\
&\quad \underbrace{\frac{m}{2}\left\langle \Phi^-_{-2}(\tau-t) + C\Phi^-_{-\gamma}(\tau-t) + \omega^2\Phi^-_0(\tau-t), q(t)\eta(\tau)\right\rangle}_{\textbf{advanced}}. \quad (3.2)
\end{aligned}$$

From **(3.2)** we see that $\delta\mathcal{V}_2[q]$ has two separate components: an advanced part and a retarded part. As usual, we will require the perturbed action to vanish. However, in order to accomplish this, we require that the advanced and retarded parts of $\delta\mathcal{V}_2[q]$ vanish separately. This leads to the two equations of motion

$$\left[m\Phi^+_{-2}(t) + mC\Phi^+_{-\gamma}(t) + m\omega^2\Phi^+_0(t)\right] * q(t) = 0 \ \textbf{(retarded)} \qquad (3.3)$$

$$\left[m\Phi^-_{-2}(t) + mC\Phi^-_{-\gamma}(t) + m\omega^2\Phi^-_0(t)\right] * q(t) = 0 \ \textbf{(advanced)}. \qquad (3.4)$$

Equation **(3.3)** is what we want for our nonconservative HO's equation of motion. However, we also 'pick up' the advanced equation in **(3.4)**. For now, we will treat **(3.3)** and **(3.4)** as independent equations that can be solved separately. Since our world appears to be causal, **(3.3)** is the equation we are most interested in.

(The physical significance of **(3.4)** is a rather difficult question. Notice that the difficulty of an advanced equation of motion only arises when we have derivatives in **(3.3)** and **(3.4)** that are not even integer ordered. So, if $C = 0$, **(3.3)** and **(3.4)** are identical. This is a general result that may help explain why the advanced and retarded equation were not a significant difficulty in classical and generalized mechanics – until Riewe's work.)

It is easy to show that **(3.3)** and **(3.4)** would also have resulted if we used $\Phi^+_\alpha(\tau - t)$ in **(3.1)**. It follows that we could also use the symmetric kernels in **(1.100)**

$$\Phi^s_\alpha(t,\tau) \;:=\; \frac{1}{2}\left[\Phi^+_\alpha(t-\tau) + \Phi^+_\alpha(\tau - t)\right] \tag{3.5}$$

in **(3.1)** and still arrived at **(3.3)** and **(3.4)**. From now on, the symmetric kernels in **(3.5)** will be used in our actions. We will then have an advanced and a retarded equation of motion arise, as in **(3.3)** and **(3.4)**. Only the retarded equation of motion will be taken as having any physical meaning in the classical case. Finally, notice that our kernel $\Phi^s_\alpha(t,\tau)$ differs from Riewe's $\Psi_\alpha(t-\tau)$ in **(1.42)** only by the $1/\cos(\alpha\pi)$ factor.

Now we'll look at the driven, nonconservative HO. Using the notation in **(1.54)**, we form the action

$$\mathcal{V}_2'[q] \;=\; f \star q - \frac{m}{2}\left[\Phi^s_{-2} + C\Phi^s_{-\gamma} + \omega^2\Phi^s_0\right] \star q^2. \tag{3.6}$$

The functional derivative of **(3.6)** is given by, see **(1.85)**,

$$\frac{\delta \mathcal{V}_2'[q]}{\delta q} \;=\; f - m\left[\Phi^s_{-2} + C\Phi^s_{-\gamma} + \omega^2\Phi^s_0\right] \star q. \tag{3.7}$$

Using **(3.5)** in **(3.7)** and requiring the advanced and retarded parts to vanish separately gives us

$$\left[m\Phi^{+}_{-2}(t) + mC\Phi^{+}_{-\gamma}(t) + m\omega^2\Phi^{+}_{0}(t)\right] * q(t) \quad = \quad f(t)\ \textbf{(retarded)} \tag{3.8}$$

$$\left[m\Phi^{-}_{-2}(t) + mC\Phi^{-}_{-\gamma}(t) + m\omega^2\Phi^{-}_{0}(t)\right] * q(t) \quad = \quad f(t)\ \textbf{(advanced)}. \tag{3.9}$$

This is a general result. Any forcing function will be given by the $K_1(t)$ term in the Volterra series representation of the action.

Of course the world is not modelled by harmonic oscillators alone. We need to extend the above procedure to higher ordered potentials. The potential in **(3.6)** will be taken as the $m\omega^2\Phi^s_0(t,\tau)$ term. This is essentially a delta function located on the $t = \tau$ line. For a q^3 term in our action, we would want the $K_3(\tau_1, \tau_2, \tau_3)$ kernel to also be a 'delta function' located on the $\tau_1 = \tau_2 = \tau_3$ line. This can be done by letting

$$K_3(\tau_1, \tau_2, \tau_3) \quad = \quad -C_3\Phi^s_0(\tau_1, \tau_2)\Phi^s_0(\tau_2, \tau_3) \tag{3.10}$$

where C_3 is a constant. Then we can form the action

$$\mathcal{V}_3[q] \quad = \quad K_1 \star q^1 + \frac{1}{2}\, K_2 \star q^2 + \frac{1}{6}\, K_3 \star q^3 \tag{3.11}$$

where

$$K_1(\tau_1) = f(\tau_1) \tag{3.12}$$

$$K_2(\tau_1,\tau_2) = -m\left[\Phi^s_{-2}(\tau_1,\tau_2) + C\Phi^s_{-\gamma}(\tau_1,\tau_2) + \omega^2\Phi^s_0(\tau_1,\tau_2)\right]. \tag{3.13}$$

The functional derivative of **(3.11)** is

$$\frac{\delta\mathcal{V}_3[q]}{\delta q} = K_1 + K_2 \star q + \frac{1}{2}\,K_3 \star q^2. \tag{3.14}$$

Using **(3.10)**, **(3.12)** and **(3.13)** in **(3.14)** gives us

$$\begin{aligned}\frac{\delta\mathcal{V}_3[q]}{\delta q} &= \underbrace{\frac{1}{2}\left\{f - m\left[\Phi^+_{-2} + C\Phi^+_{-\gamma}\right] * q - \left[m\omega^2 q + C_3 q^2\right]\right\}}_{\textbf{retarded}} + \\ &\quad \underbrace{\frac{1}{2}\left\{f - m\left[\Phi^-_{-2} + C\Phi^-_{-\gamma}\right] * q - \left[m\omega^2 q + C_3 q^2\right]\right\}}_{\textbf{advanced}}\end{aligned} \tag{3.15}$$

which results in the equations of motion

$$m\left[\Phi^+_{-2} + C\Phi^+_{-\gamma}\right] * q + \left[m\omega^2 q + C_3 q^2\right] = f\ \textbf{(retarded)} \tag{3.16}$$

$$m\left[\Phi^-_{-2} + C\Phi^-_{-\gamma}\right] * q + \left[m\omega^2 q + C_3 q^2\right] = f\ \textbf{(advanced)} \tag{3.17}$$

using our usual procedure. This method can be further extended to higher ordered potentials by using

$$K_n(\tau_1,\ldots,\tau_n) = -C_n\Phi^s_0(\tau_1,\tau_2)\cdots\Phi^s_0(\tau_{n-1},\tau_n). \tag{3.18}$$

In all of the above, we have only used the $K_2(\tau_1, \tau_2)$ kernel as in **(3.13)**. We can further extend our formalism by allowing the inclusion of derivatives of order higher than two. So $K_2(\tau_1, \tau_2)$ will generally be given by

$$K_2(\tau_1, \tau_2) \;=\; \frac{1}{2}\sum_{n=1}^{N} C_n \Phi^s_{-\alpha_n}(\tau_1, \tau_2) \tag{3.19}$$

where the C_n are constants, N is a positive finite integer and,

$$0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_N < \infty. \tag{3.20}$$

We could, of course, use other distributions than the $\Phi^s_{-\alpha_n}(\tau_1, \tau_2)$ in **(3.19)**. Also, it should be possible to allow $N = \infty$. However, this raises convergence issues so we will not consider this situation. Finally, the α_n in **(3.20)** could be allowed to be less than zero. Since $\Phi^s_\alpha(\tau_1, \tau_2)$ leads to a fractional integral when $\alpha > 0$, this situation would give us fractional diffeo-integral equations of motion. We will also not concern ourselves with this.

So now a general action will be given by the Volterra series

$$\mathcal{V}[q] \;=\; \sum_{n=1}^{\infty} \frac{1}{n!} K_n \star q^n \tag{3.21}$$

where $K_1(\tau_1)$ is given by **(3.12)**, $K_2(\tau_1, \tau_2)$ by **(3.19)** and, $K_n(\tau_1, \ldots, \tau_n)$ by **(3.18)** when $n > 2$. (In **(3.21)** we assume that the series

$$\textstyle\sum_{n=3}^{\infty} \frac{1}{n!} K_n \star q^n \tag{3.22}$$

converges.) All of Riewe's actions are contained in (3.21), except for the fact that we used the $\Phi^s_\alpha(\tau_1, \tau_2)$ instead of the $\Psi_\alpha(\tau_1 - \tau_2)$. It follows that generalized and classical Lagrangian mechanics are also contained in (3.21) with an appropriate choice of the K_n.

3.2 An Alternate Development

For generalized and classical Lagrangian mechanics, the derivatives in the equations of motion are all even integer ordered. Hence, there will be no difference between the advanced and retarded equations of motion. The difficulty of nonequivalent advanced and retarded equations only arises when we consider arbitrary fractional derivatives in (3.19), i.e., when we allow $a_n \neq 2m$, where m is an integer. There is an alternate development which can lead to only retarded equations of motion, which we now examine.

Write the action in (3.21) as

$$\mathcal{V}[q] = \langle q(\tau_1), [K_1(\tau_1) + \langle q(\tau_2), [K_2(\tau_1, \tau_2) + \ldots]\rangle \cdots]\rangle \qquad (3.23)$$

where we'll take the action as having a finite number of terms. Now, we can freely change all of the symmetric kernels in (3.23) to triangular kernels (see Section

1.2.1). This doesn't affect the value of $\mathcal{V}[q]$ so we have essentially done nothing so far. However, when we perturb $q(\tau_1)$ by $\eta(\tau_1)$, we will not perturb the $q(\tau_n)$ by $\eta(\tau_n)$ for $n > 1$. Then

$$\delta\mathcal{V}[q] \;=\; \langle\eta(\tau_1),[K_1^{(t)}(\tau_1)+\langle q(\tau_2),[K_2^{(t)}(\tau_1,\tau_2)+\ldots]\rangle\cdots]\rangle. \quad (3.24)$$

Requiring **(3.24)** to vanish for an arbitrary $\eta(\tau_1)$ gives us

$$\langle q(\tau_2),[K_2^{(t)}(\tau_1,\tau_2)+\ldots]\rangle\cdots]\rangle \;=\; -K_1(\tau_1). \quad (3.25)$$

With $K_n^{(t)}(\tau_1,\ldots\tau_n)$ being an appropriate triangularization of $K_n(\tau_1,\ldots\tau_n)$, **(3.25)** will result in a purely causal (i.e., retarded) equation of motion.

Let us examine the above procedure for the driven, nonconservative HO. We'll start with the action in **(3.6)**

$$\mathcal{V}_2'[q] \;=\; f\star q-\frac{m}{2}\left[\Phi_{-2}^s+C\Phi_{-\gamma}^s+\omega^2\Phi_0^s\right]\star q^2. \quad (3.26)$$

Now we appropriately triangularize the kernels in **(3.26)** to get

$$\begin{aligned}\mathcal{V}_r[q] \;=\;& \langle q(\tau_1),[f(\tau_1)+\langle q(\tau_2),-m[\Phi_{-2}^+(\tau_1-\tau_2)+C\Phi_{-\gamma}^+(\tau_1-\tau_2)\\ &+\omega^2\Phi_0^+(\tau_1-\tau_2)]]\rangle]\rangle. \end{aligned}\quad (3.27)$$

Perturbing $q(\tau_1)$ by $\eta(\tau_1)$ and requiring the action's variation to vanish gives us

$$\langle q(\tau_2),m[\Phi_{-2}^+(t-\tau_2)+C\Phi_{-\gamma}^+(t-\tau_2)+\omega^2\Phi_0^+(t-\tau_2)]\rangle \;=\; f(t). \quad (3.28)$$

This is the retarded equation of motion for $q(t)$ that we want. We could also have used the $\Phi^{-}_{-\alpha}(\tau_1, \tau_2)$ or the $\Phi^{s}_{-\alpha}(\tau_1, \tau_2)$ in (**3.26**). Then we would have an advanced or an acausal equation of motion for $q(t)$, respectively. So the procedure developed here is actually more general than the one in Section **3.1**.

The above procedure, as advertised, can lead to a single retarded equation of motion. The triangularization of the kernels in (**3.23**) is not a great cause of concern since it does not affect the value of $\mathcal{V}[q]$. We do need to be careful to choose the correct triangularization so that (**3.25**) is a retarded equation, however. This amounts to *a priori* assuming how the environment will react to $q(t)$. Only perturbing $q(\tau_1)$ in (**3.24**) is more troublesome. We will now examine the Caldeira-Leggett model to provide a justification for our alternate development.

3.3 The Caldeira-Leggett Model

Caldeira and Leggett presented in [4] a model of nonconservative systems. the basic idea is to couple a system to an environment modelled as a collection of HOs. While the system itself may be dissipative, the combined system and environment is conservative. The environmental variables are then removed to arrive at an equation of motion for the system only.

The Lagrangian for the Caldeira-Leggett model is given by

$$L = L_S + L_E + L_C \tag{3.29}$$

where L_S is the system's Lagrangian

$$L_S = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 \tag{3.30}$$

and L_E is the environment's Lagrangian

$$L_E = \sum_{n=1}^{\infty}\left[\frac{m_n}{2}\dot{q}_n^2 - \frac{m_n\omega_n^2}{2}q_n^2\right]. \tag{3.31}$$

For the coupling between the system and environment, we take L_C as

$$L_C = q\sum_{n=1}^{\infty} c_n q_n. \tag{3.32}$$

The c_n are the coupling constants. It proves sufficient to let $c_n = m_n\omega_n^2$ [11]. So L is given by

$$L = \frac{m}{2}\dot{q}^2 - \frac{m\omega^2}{2}q^2 + \sum_{n=1}^{\infty}\left[\frac{m_n}{2}\dot{q}_n^2 - \frac{m_n\omega_n^2}{2}q_n^2 + m_n\omega_n^2 q q_n\right] \tag{3.33}$$

which results in the equations of motion

$$m\ddot{q} - \sum_{n=1}^{\infty} m_n\omega_n^2 q_n + m\omega^2 q = 0 \tag{3.34}$$

$$m_n\ddot{q}_n + m_n\omega_n^2 q_n - m_n\omega_n^2 q = 0 \tag{3.35}$$

for the Caldeira-Leggett model.

Notice that (3.34) and (3.35) are invariant under time reversal. So we have no sense of which direction time is flowing in. If we allow for acausality in the q_n and, assume that their motions are completely determined by q, we can write

$$q_n(t) \quad = \quad \left[\frac{\omega_n}{2}\,\sin(\omega_n|t|)\right] * q(t). \tag{3.36}$$

Using (3.36) in (3.34) results in

$$m\ddot{q} + \frac{1}{2}\left[-\sum_{n=1}^{\infty} m_n\omega_n^3 \sin(\omega_n|t|)\right] * q + m\omega^2 q \quad = \quad 0. \tag{3.37}$$

Now, the q_n are in $\mathcal{S}_t^{'}$. We'll take q as being in $\mathcal{E}_t^{'} \subset \mathcal{S}_t^{'}$ so that (3.37) only holds for $-\infty < a < t < b < \infty$. What we are doing physically is only allowing q to interact with the environment for a finite time period. The q goes on to do whatever it does after this time period. The environment, however, retains a memory of its interaction with q, which is why it is in $\mathcal{S}_t^{'}$ but not $\mathcal{E}_t^{'}$.

If we let the bracketed term in (3.37) be the half-range Fourier sine series for $mC[\Phi_{-\gamma}^{+}(t) + \Phi_{-\gamma}^{-}(t)]$, our equation of motion for q is

$$m\ddot{q} + \frac{mC}{2}\left({}_a\mathbf{D}_t^{\gamma}[q] + {}_t\mathbf{D}_b^{\gamma}[q]\right) + m\omega^2 q \quad = \quad 0. \tag{3.38}$$

So if the environment acts acausally, we have the acausal equation of motion in (3.38). If the environment was purely causal, (3.38) becomes

$$m\ddot{q} + mC\,{}_a\mathbf{D}_t^{\gamma}[q] + m\omega^2 q \quad = \quad 0. \tag{3.39}$$

Similarly, for an anti-causal environment we would have

$$m\ddot{q} + m_t \mathbf{D}_b^\gamma [q] + m\omega^2 q \;=\; 0. \tag{3.40}$$

Comparing **(3.38)** – **(3.40)** with **(3.2)** – **(3.4)**, we see that the action

$$\begin{aligned} \mathcal{V} \;=\; & \int \Bigg(\frac{m}{2}\,\dot{q}^2 - \frac{m\omega^2}{2}\,q^2 + \\ & \sum_{n=1}^{\infty} \left[\frac{m_n}{2}\,\dot{q}_n^2 - \frac{m_n\omega_n^2}{2}\,q_n^2 + m_n\omega_n^2 q q_n \right] \Bigg)\, dt \end{aligned} \tag{3.41}$$

could be used instead of **(3.1)** for a nonconservative HO. Equation **(3.1)** is much more concise but, the above helps to shed light on our fractional mechanics. When we use **(3.41)** for our action, we end up with the time reversal invariant equations in **(3.34)** and **(3.35)**. If we assume these are retarded equations, **(3.39)** results for q (see also **(3.3)**). Conversely, if **(3.34)** and **(3.35)** are taken as advanced equations, q's evolution is given by **(3.40)** (see **(3.4)**). Finally, acausality for the environment gives us the acausal equation in **(3.38)**.

Instead of allowing the q_n to only give rise to the fractional derivatives in **(3.38)**, we will allow them to completely determine q's evolution. Instead of **(3.33)**, our Lagrangian will be

$$L \;=\; \sum_{n=1}^{\infty} \left[\frac{m_n}{2}\,\dot{q}_n^2 - \frac{m_n\omega_n^2}{2}\,q_n^2 + m_n\omega_n^2 q q_n \right]. \tag{3.42}$$

Then **(3.34)** and **(3.35)** become

$$\sum_{n=1}^{\infty} m_n \omega_n^2 q_n = 0 \tag{3.43}$$

$$m_n \ddot{q}_n + m_n \omega_n^2 q_n = m_n \omega_n^2 q \tag{3.44}$$

respectively. To have **(3.38)** result from **(3.43)** we would need

$$\sum_{n=1}^{\infty} m_n \omega_n^2 q_n = \left[m\Phi^s_{-2}(t) + mC\Phi^s_{-\gamma}(t) + m\omega^2\Phi^s_0(t)\right] * q. \tag{3.45}$$

From **(3.45)** we see that

$$\int \sum_{n=1}^{\infty} m_n \omega_n^2 q_n q dt \tag{3.46}$$

is essentially our action in **(3.1)**. This, of course, assumes the environment acts acausally.

So it seems as though we have solved our problems. If we use the Lagrangian in **(3.42)** we can arrive at acausal, causal and anti-causal equations of motion for q. All that we have really done though is to move the problem 'down a step'. Instead of an acausal variation of the action as in Section **3.1**, we have a time reversal invariant equation of motion in **(3.44)**. Instead of separating the two parts of the action's variation, we have to choose which type of solution to **(3.44)** to accept. By only allowing causal solutions to **(3.44)**, we really haven't done anything different than in Section **3.2**. However, the Caldeira-Leggett model does show us a few things. First, we see how the kernels used in the Volterra series arise from the environment. Also, it clarifies the methods in Sections **3.1** and **3.2**. In Section **3.1** we should perhaps require the entire variation of the action to vanish. This would be equivalent to assuming an acausal environment. Also, when deriving **(3.43)** we only perturbed $q(t)$ in **(3.42)**. This is the same thing we did in Section **3.2**, though there we first assumed if the environment was going to be causal, anti-causal or acausal. So the Caldeira-Leggett can be used to provide some justification for the method presented in Section **3.2**.

3.4 Discussion

In Section **1.3** we demonstrated the shortcomings of Riewe's method. In particular, it is impossible to include a $\dot{q}$ term in the equations of motion using the $\Psi_\alpha(t)$ for our kernels. This led us to try using the $\Phi_\alpha^\pm(t)$ in our Volterra series. We were able to demonstrate a method in Section **3.2** that can lead to a single, retarded equation of motion. Some justification for this procedure is provided by the Caldeira-Leggett model. This is not an insignificant development. We can now derive purely causal equations of motion that contain arbitrary fractional derivatives. This is something that cannot be achieved using Riewe's formalism.

Due to its elegance and wide acceptance, we would really want the method in Section **3.1** to work. Bauer showed that

Theorem 2 (Bauer (1931), Corollary 1) *The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle.*

Riewe correctly pointed out that the proof of Bauer's theorem implicitly relies on the assumption of only using integer ordered derivatives [17]. It was hoped that this loophole could be exploited by employing fractional derivatives [6, 16, 17]. However, our introduction of the Volterra series concept allows us to easily extend Bauer's theorem, effectively closing this loophole [7].

Theorem 3 (Dreisigmeyer and Young (2004)) *There does not exist a* $K(t,\tau)$, $t,\tau \in \mathbb{R}$, *such that the variation of the quantity*

$$\mathcal{V}[q] = \int K(t,\tau)q(t)q(\tau)dtd\tau \tag{3.47}$$

will result in ${}_a\mathbf{D}_t^{\alpha}[q]$ *for* $\alpha \neq 2n$, n *an integer.*

PROOF. *The variation of* $\mathcal{V}[q]$ *is given by*

$$\frac{\delta\mathcal{V}[q]}{\delta q(\rho)} = [K(\rho,t) + K(t,\rho)] \star q(t). \tag{3.48}$$

We will assume that

$$[K(\rho,t)+K(t,\rho)]\star q(t) \;=\; \Phi^{+}_{-\alpha}(\rho-t)\star q(t) \tag{3.49}$$

and arrive at a contradiction. We require that (3.49) holds for every $q(t)$. Then we must have

$$[K(\rho,t)+K(t,\rho)] \;=\; \Phi^{+}_{-\alpha}(\rho-t). \tag{3.50}$$

Interchanging ρ and t in (3.50) gives us

$$[K(\rho,t)+K(t,\rho)] \;=\; \Phi^{-}_{-\alpha}(\rho-t). \tag{3.51}$$

Hence, unless $\Phi^{\pm}_{-\alpha}(\rho-t)$ is symmetric in ρ and t, (3.50) and (3.51) cannot both hold. That is, unless $\alpha=2n$, n an integer, there does not exist a $K(t,\tau)$, $t,\tau\in\mathbb{R}$, such that (3.49) holds.■

This is a significant expansion of Bauer's theorem. It shows that Riewe's method in [16, 17] and our method in Section **3.1** are doomed to failure.

In Theorem **3** we allowed $K(t,\tau)$ to be complex but, we assumed that $q(t)=q(\tau)$ when $t=\tau$. Let us see what can happen when we relax this assumption. Define $y(\tau):=q(\tau)$ and consider the quantity

$$\mathcal{V}[q,y] \;=\; \int K_2^{(s)}(t,\tau)q(t)y(\tau)dtd\tau. \tag{3.52}$$

Notice we are using a symmetric kernel in (**3.52**). From (**3.52**) we have

$$\frac{\delta \mathcal{V}[q,y]}{\delta q(\rho)} = K_2^{(s)}(\rho,\tau) \star y(\tau) \tag{3.53}$$

$$= K_2^{(s)}(\tau,\rho) \star y(\tau) \tag{3.54}$$

and

$$\frac{\delta \mathcal{V}[q,y]}{\delta y(\rho)} = K_2^{(s)}(t,\rho) \star q(t) \tag{3.55}$$

$$= K_2^{(s)}(\rho,t) \star q(t). \tag{3.56}$$

From (**3.53**) – (**3.56**), we see that having a symmetric kernel in (**3.52**) forces $q(t) = y(\tau)$ for $t = \tau$. So we need to make the kernel in (**3.52**) triangular. Then we would have

$$\frac{\delta \mathcal{V}[q,y]}{\delta q(\rho)} = K_2^{(t)}(\rho,\tau) \star y(\tau) \tag{3.57}$$

and

$$\frac{\delta \mathcal{V}[q,y]}{\delta y(\rho)} = K_2^{(t)}(t,\rho) \star q(t). \tag{3.58}$$

This is nothing other than an extension of Bateman's method [1], see (**1**) and (**2**). It is also what we implicitly did in Section **3.2**.

A possible physical interpretation of (**3.57**) and (**3.58**) is that $q(t)$ and $y(\tau)$ are two systems travelling oppositely through time. By taking the kernel as triangular, we are forcing the environment to travel the same way through time as the system it is acting on. So, for example, if $K_2^{(t)}(t,\tau) = \Phi_{-\alpha}^{+}(t-\tau)$, we have

$$\frac{\delta \mathcal{V}[q,y]}{\delta q(\rho)} = {}_a\mathbf{D}_\rho^\alpha \, [y] \tag{3.59}$$

and

$$\frac{\delta \mathcal{V}[q,y]}{\delta y(\rho)} = {}_\rho\mathbf{D}_b^\alpha \, [q] \, . \tag{3.60}$$

In **(3.59)** we would take the system and environment (i.e., the 'universe') as travelling forward through time. In **(3.60)** we take the 'universe' as travelling backwards through time. The direction of time itself is determined by some external observer. There are a few objections that can be raised against this method. First, we saw in Section **1.2.1** that we can freely change between a symmetric and a triangular kernel in **(3.52)** without affecting $\mathcal{V}$'s value. It is somewhat unsatisfactory that what is essentially the same action can lead to the situation in **(3.53)** – **(3.56)** or in **(3.57)** – **(3.58)** depending on if we define the action with a symmetric or triangular kernel, respectively. Also, the path integral formulation of quantum mechanics only depends on the action. So, for quantum mechanics, we can always use a symmetric kernel and take $q(t) = y(\tau)$ for $t = \tau$. Next, the Volterra series framework assumes that $q(t) = y(\tau)$ when $t = \tau$. This assumption is violated in the procedure above. Finally, it is not obvious how an observer can travel oppositely through time versus the 'universe'. Overall, it seems better to take $q(t) = y(\tau)$ for $t = \tau$ and, to reject the above method.

Notice the positive contributions of the Volterra series framework in the above. First, it allowed us to easily extend Bauer's result in Theorem **2** to the result in Theorem **3**. It also allows us to reject, with some confidence, the idea of letting $q(t) \neq q(\tau)$ when $t = \tau$ in **(3.47)**. So the positive contributions of using the Volterra series concept have mainly been expressed in the negative results for the possible forms an action can assume. However, our framework can also guide our future research. Assuming we want the method in Section **3.1** to ultimately be successful, how could we generalize **(3.47)** to make this happen? One possibility is presented in [5] which builds on the results presented here.

Bibliography

[1] H. Bateman. On dissipative systems and related variational principles. *Physical Review*, 38:815–819, 1931.

[2] P. S. Bauer. Dissipative dynamical systems I. *Proceedings of the National Academy of Sciences*, 17:311–314, 1931.

[3] S. Boyd, L. O. Chua, and C. A. Decoer. Analytical foundations of Volterra series. *IMA Journal of Mathematical Control and Information*, 1:243–282, 1984.

[4] A. O. Caldeira and A. J. Leggett. Quantum tunnelling in a dissipative system. *Annals of Physics, N.Y.*, 149:374–456, 1983.

[5] D. W. Dreisigmeyer and P. M. Young. Nonconservative Lagrangian mechanics II: purely causal equations of motion. Preprint physics/0402056.

[6] D. W. Dreisigmeyer and P. M. Young. Nonconservative Lagrangian mechanics: a generalized function approach. *Journal of Physics A*, 36:8297–8310, 2003.

[7] D. W. Dreisigmeyer and P. M. Young. Extending Bauer's corollary to fractional derivatives. *Journal of Physics A*, 37:L117–L121, 2004.

[8] R. Estrada and R. P. Kanwal. *Singular Integral Equations*. Birkhäuser, 2000.

[9] I. M. Gelfand and G. E. Shilov. *Generalized Functions I: Properties and Operations*. Academic Press, 1964.

[10] H. Goldstein. *Classical Mechanics*. Addison-Wesley, 1980.

[11] G.-L. Ingold. Path integrals and their applications to dissipative quantum systems. Preprint quant-ph/0208026.

[12] N. Laskin. Fractional quantum mechanics. *Physical Review E*, 62:3135–3145, 2000.

[13] N. Laskin. Fractional Schrödinger equation. *Physical Review E*, 66:056108(7), 2002.

[14] E. R. Love and L. C. Young. On fractional integration by parts. *Proceedings of the London Mathematical Society*, 44:1–35, 1938.

[15] I. Podlubny. *Fractional Differential Equations*. Academic Press, 1999.

[16] F. Riewe. Nonconservative Lagrangian and Hamiltonian mechanics. *Physical Review E*, 53:1890–1898, 1996.

[17] F. Riewe. Mechanics with fractional derivatives. *Physical Review E*, 55:3581–3592, 1997.

[18] W. J. Rugh. *Nonlinear Systems Theory*. John Hopkins, 1981. Web version prepared in 2002 available at www.ece.jhu.edu/ rugh/volterra/book.pdf.

[19] R. Schumer, D. A. Benson, M. M. Meerschaert, and S. W. Wheatcraft. Eulerian derivation of the fractional advection-dispersion equation. *Journal of Contaminant Hydrology*, 48:69–88, 2001.

[20] C. F. Stevens. *The Six Core Theories of Modern Physics*. MIT Press, 1995.

[21] A. H. Zemanian. *Distribution Theory and Transform Analysis*. McGraw-Hill, 1965.