DIMENSIONAL ANALYSIS AS A TOOL
IN
HYDRAULIC DESIGN AND RESEARCH

By

Maurice L. Albertson
Colorado A & M College
Fort Collins, Colorado

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Dimensional analysis has been given treatment in numerous technical papers and books. In spite of this fact, however, the use of dimensional analysis as an aid in the solution of problems in hydraulics has been rather generally misunderstood and therefore the profession as a whole has been slow to adopt it. Contrary to some beliefs, it is not a magic cure-all for problems but rather a convenient and powerful tool to aid in the solution of certain ones. It is neither fool proof nor does it automatically supply a significant function for design purposes. Rather, it depends largely upon the ability of the operator to forecast accurately the variables involved and after making the dimensional analysis to arrange the parameters in the most useful manner.

Perhaps the simplest and most desirable method of finding an answer to hydraulics problems is that of direct mathematical solution. Common examples of the successful use of this method of analysis are the derivation of the Stokes' equation for drag on a submerged object, Poiseuille's equation for laminar flow in pipes, the equation for the hydraulic jump, and the equation for flow through porous media. Unfortuately, however, most problems in hydraulics involve so many variables that direct mathematical solution is out of the question. It is under these circumstances that dimensional analysis can be of the greatest assistance.

There are 6 important advantages to the use of dimensional analysis:

1. It reduces the number of variables involved (generally by three for hydraulics problems) when the Pi-theorem is applied.

2. It expresses in dimensionless terms the functional relationship between the variables involved which apply regardless of the system of units used.

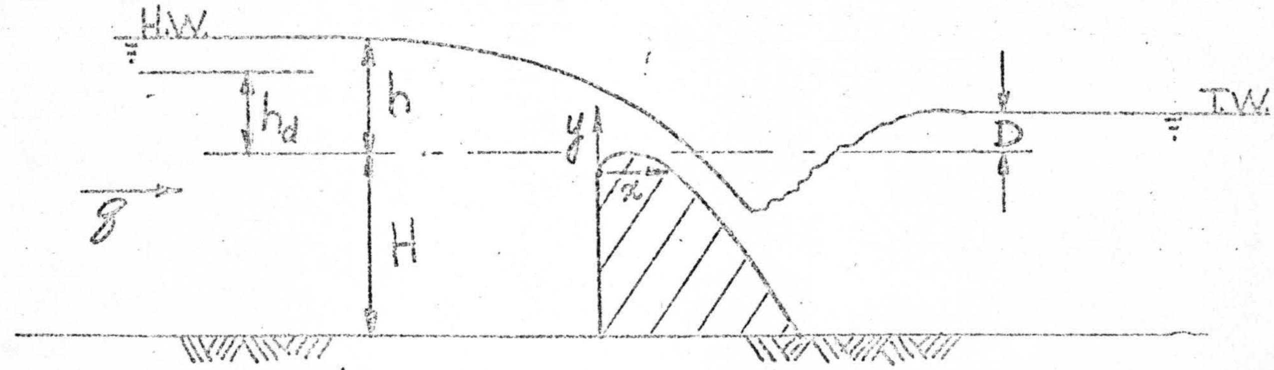
3. If the variables are wisely chosen, the dimensionless parameters can be used to make certain logical deductions about the problem.

4. It sets up the problem so that the research program may be carried out in a systematic and simplified manner.

5. It enables design curves to be developed from either experimental data or direct computation which permit direct (rather than trial-and-error) solution of the problem.

6. It aids in setting up a theoretical equation in a simplified and dimensionless form.

To illustrate the use and significance of each of these advantages of dimensional analysis, two examples will be chosen (a) the design of spillway crests and (b) the determination of apron elevations. From the sketch it is evident that the important variables describing geometry are



the height of the dam H , the head on the dam h , the design head hd , the surface roughness k , and the tailwater elevation D relative to the crest of the dam. The variables describing the flow are the discharge per unit width q and the pressure Δp at a point x, y on the crest. The fluid characteristics are the density ρ , the viscosity μ , and the difference in specific gravity $\Delta\gamma$ across the fluid interface. It the discharge is chosen as the dependent variable then

$$q = \phi_1 (H, h, h_d, D, K, \rho, \mu, \Delta\gamma) \tag{1}$$

If q , ρ, and H are chosen as repeating variables for dimensional analysis, Eq. 1 becomes

$$\frac{q}{\sqrt{g H^3}} = \phi_2 \left(\frac{h}{H}, \frac{h_d}{H}, \frac{D}{H}, \frac{K}{H}, R_e\right) \tag{2}$$

In which $R_e = \frac{q\rho}{\mu}$ is the Reynolds number.

This step illustrates the first advantage of dimensional analysis since the number of parameters to be considered has been reduced from 8 to 5. Likewise, it may be seen that the parameters are all dimensionless so that any system of units may be used provided the same system is used throughout.

To illustrate the third advantage of dimensional analysis, each parameter must be considered individually:

a. Reynolds number R_e indicates to what extent the forces of viscosity enter the problem. Because the flow is rapidly converging as it passes over the crest and it reaches critical velocity before any appreciable amount of boundary layer is developed, the viscous forces are probably of relatively little importance in studying the discharge or pressure distribution over the crest. As the flow passes on down the spillway, however, the boundary layer development is appreciable so that R_e cannot be ignored when studies in this region are being made.

b. The relative roughness $\frac{K}{H}$ of the crest is also of little importance because, like the viscous forces, its influence is small when the flow is rapidly converging. It is a very important part of the boundary layer development and again like R_e is important when evaluating the flow down the spillway.

c. The submergence parameter $\frac{D}{H}$ expresses the height of the tailwater relative to the height of the dam. A more significant parameter would be $\frac{D}{h}$ which then expresses the degree of submergence in terms of the head on the dam. Obviously, as $\frac{D}{h}$ approaches 1.0 the discharge must approach zero.

d. The design parameter $\frac{h_d}{H}$ expresses the geometric characteristics of the crest. If this ratio is known then the general shape of the crest is known.

e. The head ratio $\frac{h}{H}$ might be in more significant form if expressed as $\frac{h}{h_d}$ which states the relationship between the design head and the actual head on the crest. As this ratio approaches 1.0, the discharge coefficient and pressure distribution should approach those values found for the weir.

f. The discharge parameter $\frac{q}{\sqrt{g H^3}}$ is more significant if rearranged as $\frac{q}{\sqrt{g h_d^3}}$ or $\frac{q}{\sqrt{g h^3}}$ because it then becomes a discharge coefficient more nearly like those used for flow over weirs.

The fourth advantage to dimensional analysis is that it sets up the problem so that the research program may be carried out in a systematic and simplified manner. If the problem is considered in terms of the dimensionless parameters then it is necessary only to vary these parameters regardless of the absolute value of the individual variables within the parameters.

If the assumption is made that model crests are to be tested in a systematic manner in a flume which is 3 ft high and is limited by available discharge to approximately $h = 1.5$ ft as a maximum, then the research program could be outlined as follows:

a. Assume that $\frac{K}{H}$ and R_e in Eq. 1 are relatively unimportant and rearrange the other parameters so that

$$\frac{q}{\sqrt{g h_d^3}} = \phi_3\left(\frac{h}{h_d}, \frac{h_d}{H}, \frac{D}{h_d}\right) \tag{3}$$

b. Vary $\frac{h}{h_d}$ from 0.1 to 2.0

$\frac{h_d}{H}$ from 0.1 to 2.0

$\frac{D}{h_d}$ from 0 to 2.0

c. Because of the possible influence of surface tension, it is desirable to have the head over the crest be not less than 0.2 ft and the height of the crest not less than 0.5 ft. (The influence of surface tension could have been considered in the original function which then would have included a Weber number.)

d. From the foregoing limits which have been established, Eq. 3 may be evaluated experimentally by use of from three to six crests as shown in the following table:

H	h_d	h_d/H	h/h_d	D/h_d
2.0	0.2	0.1	1.0 - 7.5	0 - 5.0
2.0	1.0	0.5	0.2 - 1.5	0 - 1.0
1.0	0.5	0.5	0.4 - 3.0	0 - 4.0
1.0	1.0	1.0	0.2 - 1.5	0 - 2.0
0.5	0.5	1.0	0.4 - 3.0	0 - 5.0
0.5	1.0	2.0	0.2 - 1.5	0 - 2.5

Having $\frac{h_d}{H}$ equal 0.5 for crests of two different heights and 1.0 for crests of two different heights, serves to check the principle of similitude and thereby varify the values of $\frac{h_d}{H}$ equal to 0.1 and 2.0. By making an average of approximately 15 carefully selected tests on each crest, in which $\frac{D}{h_d}$ is controlled for various approximate discharges, it is possible to evaluate Eq. 3 over its entire usable range with a total of not more than 100 experiments.

Once the experimental data have been obtained, the function may be plotted with $\frac{q}{\sqrt{g h_d^3}}$ as the ordinate, $\frac{h}{h_d}$ as the abscissa, $\frac{D}{h_d}$ as the third variable, and $\frac{h_d}{H}$ as the fourth variable — the latter being a constant for any given plot. This method of plotting results in four different graphs with a family of curves representing constant values of D/h_d on each graph. Although this method of plotting permits direct solution for discharge Q, the head h, the height of the dam H, or the submergence D, it does not permit direct solution for the design head because h_d appears in more than one parameter. To remedy this situation a less significant but useful system of graphs may be plotted in which h_d is replaced by H in each parameter except h_d/H. Then h_d appears in only one parameter and a direct solution is possible.

The foregoing paragraphs illustrate the method of determining the relationship between the discharge and the various other variables. A similar procedure may be used to determine the pressure distribution over each crest by substituting Δp for any one of the variables in Eq. 3 (say h) and adding the coordinates x and y.

$$\frac{\Delta p \, H^2}{\rho g^2} = \phi_4\left(\frac{D}{H}, \frac{h_d}{H}, \frac{x}{H}, \frac{y}{H}\right) \qquad (4)$$

Eq. 4 permits solving directly for Δp at any point (x, y) or for h_d which may be necessary in order not to exceed the allowable Δp at a given point. This function may be rearranged so that a direct solution may be obtained for any one of the other variables.

The second illustration in which dimensional analysis may be used advantageously is that of solving directly for the apron elevation which is required in order that the jump height will not exceed the tailwater elevation (see the accompanying paper). This illustrates in particular the third, fifth, and sixth advantages.