DISSERTATION

COUNTING ISOGENY CLASSES OF DRINFELD MODULES OVER FINITE FIELDS VIA FROBENIUS DISTRIBUTIONS

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ABSTRACT

COUNTING ISOGENY CLASSES OF DRINFELD MODULES OVER FINITE FIELDS VIA FROBENIUS DISTRIBUTIONS

Classically, the size of an isogeny class of an elliptic curve – or more generally, a principally polarized abelian variety – over a finite field is given by a suitable class number. Gekeler expressed the size of an isogeny class of an elliptic curve over a prime field in terms of a product over all primes of local density functions. These local density functions are what one might expect given a random matrix heuristic. In his proof, Gekeler shows that the product of these factors gives the size of an isogeny class by appealing to class numbers of imaginary quadratic orders. Achter, Altug, Garcia, and Gordon generalized Gekeler's product formula to higher dimensional abelian varieties over prime power fields without the calculation of class numbers. Their proof uses the formula of Langlands and Kottwitz that expresses the size of an isogeny class in terms of adèlic orbital integrals. This dissertation focuses on the function field analog of the same problem. Due to Laumon, one can express the size of an isogeny class of Drinfeld modules over finite fields via adèlic orbital integrals. Meanwhile, Gekeler proved a product formula for rank two Drinfeld modules using a similar argument to that for elliptic curves. We generalize Gekeler's formula to higher rank Drinfeld modules by the direct comparison of Gekeler-style density functions with orbital integrals.

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"If I discern of the saints something that is worthy of praise and admiration, and proceed to examine it in the clear light of truth, I become aware that what makes them appear praiseworthy and admirable really belongs to another." –Bernard of Clairvaux

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DEDICATION

I would like to dedicate this thesis to Jesus through the Immaculate Heart of Mary. May this thesis be my song of ascent, my prayer, "a surge of [my] heart; a simple look turned toward heaven, a cry of recognition and of love, embracing both trial and joy." – Therese of Lisieux

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Chapter 1

Introduction

In the early 20th century, number theorists developed the theory of algebraic curves over finite fields. Artin began attaching zeta functions to specific curves over finite fields, leading to Hasse and Weil proving the analog of the Riemann Hypothesis for general curves. This was an early milestone in arithmetic geometry [15]. Similarly, Carlitz began studying curves over function fields. Instead of attaching zeta functions to these curves, he attached to them an exponential function, and as a result, defined what is now called the *Carlitz module*. By the 1970s, Hayes and Drinfeld began independently working with the Carlitz module to develop explicit class field theory for function fields through a generalization of the Carlitz module, now called *Drinfeld modules*, which, in the function field case, play the role of elliptic curves in more classical number theory [28, Ch. 12, 13]. The Carlitz module is a Drinfeld module of rank one and is analogous to cyclotomic number fields in classical number theory. Similarly, Drinfeld modules are analogous to elliptic curves. On the other hand, in Drinfeld's 1974 paper, *Elliptic modules*, he described a modular variety for Drinfeld modules, which became a key player in the geometric Langlands program. In particular, the cohomology of this Drinfeld modular variety is related to automorphic forms [7, 15].

The analogy between elliptic curves and Drinfeld modules is deep and far-reaching – from construction to classification by isogeny, the story of Drinfeld modules runs parallel to that of elliptic curves. In the case of elliptic curves over finite fields, the trace of Frobenius is a complete invariant under isogeny. Specifically, the Frobenius endomorphism of an elliptic curve E over $\mathbb{F}_q$ has a characteristic polynomial of the form $f(T) = T^2 - aT + q$ where $|a| \leq 2\sqrt{q}$. The coefficient a is called the *trace of Frobenius*, and it counts the number of $\mathbb{F}_q$ rational points of E via the equation $a = q + 1 - \#E(\mathbb{F}_q)$ [31, p. 143-144]. Any two elliptic curves having the same characteristic polynomial of Frobenius are isogenous. Therefore, counting the number of elliptic curves in a given isogeny class over $\mathbb{F}_q$ is equivalent to counting elliptic curves over $\mathbb{F}_q$ having coefficient a in the linear term of the characteristic polynomial of Frobenius. In 2003, Gekeler proved a product formula giving the size of an isogeny class for an elliptic curve over $\mathbb{F}_p$ for some prime p using a random matrix model. The formula is a product over primes ℓ of $\mathbb{Z}$, where for each prime we ask how often does a two-by-two matrix over $\mathbb{Z}_\ell$ have as its characteristic polynomial the characteristic polynomial of Frobenius [11]? Gekeler's formula is the result of relating the weighted cardinality of an isogeny class with class numbers of imaginary quadratic orders over $\mathbb{Z}$ and expressing them through the analytic class number formula. This gives the size of an isogeny class in terms of an L-function, which through Euler expansion gives a product of local terms. Gekeler then goes on to interpret each factor as a local density function to arrive at his final product formula.

Later in [2], it was shown by Achter and Gordon that we can arrive at the same product formula using orbital integrals. This proof is more conceptual in nature. It comes from Langlands' method for relating the cohomology of a Shimura variety to automorphic forms. One tool in the Langlands program is a formula for the size of an isogeny class as a product of adèlic orbital integrals. Achter and Gordon use this result to arrive at their formula by relating Gekeler's numbers to geometric orbital integrals. They then give the relationship between geometric and canonical measures to obtain a formula in terms of canonical orbital integrals. Special consideration is given to the primes at infinity and p, where p is the characteristic of the base field. The product formula can be generalized to higher dimensional abelian varieties using Kottwitz' method for higher dimensional Shimura varieties [1].

Since Drinfeld modules are the function field analog of elliptic curves, we aim to retell this story, with the goal of counting the size of an isogeny class for a Drinfeld module over a finite field. Again, two Drinfeld modules are isogenous if they have the same characteristic polynomial of Frobenius. In 2008, Gekeler proved a new product formula, this time giving the size of an isogeny class of a rank two Drinfeld module [12] over a finite field $\mathbb{F}_q[T]/\mathfrak{p}$. As before, this formula makes use of the probability a random matrix has a given characteristic polynomial. Alternatively, Drinfeld (for rank two) and Laumon (in general) prove that the size of an isogeny class of Drinfeld modules over finite extensions of $\mathbb{F}_q[T]/\mathfrak{p}$ is given via adèlic orbital integrals. In Chapter

2, we review the definitions of Drinfeld modules and discuss isogeny invariants. In Chapter 3, we describe Gekeler's product formula in more detail. In Chapter 4, we review measures on orbits in GL_r necessary for understanding Laumon's formula for the size of an isogeny class, which is provided in Chapter 5. In Chapter 6, we generalize Gekeler's product formula to ordinary rank r Drinfeld modules by direct comparison to adèlic orbital integrals. We define the local *Gekeler ratios*

$$v_{\mathfrak{l}}(\gamma_0) := \lim_{d\to\infty} \lim_{n\to\infty} \frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n)/|\mathfrak{l}|^{n(r-1)}}$$

at each finite place, and

$$v_\infty(\gamma_0) = |D(\gamma_0)|_\infty^{1/2} \frac{\rho_{\mathbf{T}} \cdot |\operatorname{coker}(\alpha)| \cdot f(E_{\tilde{\infty}}/K_\infty)}{q^r \cdot L(1, \sigma_{\mathbf{T}}) \cdot \mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)}$$

at the infinite place. We then prove the following theorem.

Theorem A. *Let ϕ be an ordinary rank r Drinfeld module over the finite field L of degree m over $A/\mathfrak{p}$. Let $\gamma_0 \in GL_r(\mathbb{F}_q[T])$ be a matrix with the same characteristic polynomial as ϕ. The weighted size of the isogeny class of ϕ is*

$$h_\phi^*(L) = |\mathfrak{p}|^{\frac{m(r-1)}{2}} v_\infty(\gamma_0) \prod_{\mathfrak{l}} v_{\mathfrak{l}}(\gamma_0).$$

After this, we demonstrate that when restricted back to the rank two setting, we exactly recover Gekeler's product formula. One major benefit of our proof is that it generalizes Gekeler's result even in the rank two setting to fields larger than $\mathbb{F}_q[T]/\mathfrak{p}$.

It is also worth noting that Gekeler's product formula for rank two Drinfeld modules is stated for arbitrary Drinfeld modules rather than just ordinary ones. Although we deal only with ordinary isogeny classes for the purposes of this paper, Laumon's adèlic orbital integral formula for the size of an isogeny class holds for any Drinfeld module. It is therefore reasonable to expect that small adjustments to our argument will generalize our formula to arbitrary rank r Drinfeld modules.

1.1 Preliminaries

The notation in this paper follows that of Gekeler in [12].

A	$=$	$\mathbb{F}_q[T]$, the polynomial ring over a finite field of characteristic p;
K	$=$	$\mathrm{Frac}\, A = \mathbb{F}_q(T)$, the quotient field of A;
K_∞	$=$	$\mathbb{F}_q((T^{-1}))$, the completion of K at infinity;
$\mathfrak{p}$	$=$	A fixed prime ideal of A.
L	$=$	A field provided with structure of an A-algebra by $\gamma : A \to L$. This means L is either an extension of K or of $A/\mathfrak{p}$, in which case, γ is the $\mod \mathfrak{p}$-map.

In this paper, we will typically take L to be a degree m extension of $A/\mathfrak{p}$, a degree d extension of $\mathbb{F}_q$. The completion above is taken with respect to the corresponding absolute value at infinity. This is denoted by $|\cdot|_\infty$ or $|\cdot|$, and is defined as $|a|_\infty = \#(A/a) = q^{\deg(a)}$ for $a \in A$. By an abuse of notation, $|\cdot|$ may also be the extension of the absolute value of infinity in K, K_∞ or some other extension of K. We will also use the $\mathfrak{l}$-adic absolute value for a prime ideal of $\mathfrak{l}$ of A. These absolute values are defined by $|a|_\mathfrak{l} := q^{-\operatorname{ord}_\mathfrak{l}(a)}$ for prime ideals $\mathfrak{l}$ of A. Here, the ∞-adic valuation of an element $a \in A$ is given by $\nu_\infty(a) := -\deg_T(a)$ while the $\mathfrak{l}$-adic valuation of an element a is $\nu_\mathfrak{l}(a) = \operatorname{ord}_\mathfrak{l}(a)$.

Remark 1.1.1. We use the notation $\mathfrak{v}$ for an arbitrary place of K and the notation $\mathfrak{l}$ for an arbitrary place of K away from ∞.

A *Drinfeld A-module* over L gives an A-structure to the endomorphism ring $\mathrm{End}_L(\mathbb{G}_a)$ of the additive group scheme $\mathbb{G}_a$ over L.

As a scheme, $\mathbb{G}_a$ is the affine line over L, or $\mathrm{Spec}(L[X])$. Since specifying a map of schemes is equivalent to specifying a map on their underlying rings in the opposite direction, the group structure on $\mathbb{G}_a$ is given by the maps

$$\begin{aligned}\tilde{m} : L[X] &\to L[X] \otimes_L L[X],\\ X &\mapsto X \otimes 1 + 1 \otimes X;\\ \tilde{i} : L[X] &\to L[X],\\ X &\mapsto -X;\\ \tilde{e} : L[X] &\to L,\\ X &\mapsto 0.\end{aligned}$$

An *endomorphism* of $\mathbb{G}_a$ is a map from $\mathbb{G}_a$ to itself respecting this group structure. In particular, we can identify an endomorphism f with an additive polynomial by considering the underlying map on $L[X]$. In this way, we identify $\mathrm{End}_L(\mathbb{G}_a)$, with $L_{add}[X]$, where $L_{add}[X]$ denotes the ring of additive polynomials in X.

Since L has positive characteristic, the set of additive polynomials not only includes polynomials of the form $f(X) = aX$ for $a \in L$, but also the Frobenius map given by $\tau : X \to X^q$. By addition and composition, each element of $L_{add}[X]$ is of the form $f(X) = a_0X + a_1X^q + \cdots + a_nX^{q^n}$ for $a \in L$ [28, Proposition 12.2]. After a change of variables from X^q to τ, $L_{\text{add}}[X]$ is isomorphic to the twisted polynomial ring $L\{\tau\}$ with addition and multiplication subject to the commutation rule $\tau a = a^q\tau$. Therefore, a Drinfeld A-module over L gives an A-structure to $\mathrm{End}_L(\mathbb{G}_a) \cong L\{\tau\}$.

Definition 1.1.2. Each non-zero element f in $L\{\tau\}$ can be written uniquely as $f = f_s \circ \tau^{ht(f)}$ where f_s is separable, i.e., has a non-zero constant coefficient. We call the number $ht(f)$ the *height* of f.

Remark 1.1.3. It is important to note that the theory of Drinfeld modules was developed in a more general setting. Specifically, K could be a more complicated function field with A the ring of elements regular away from a place [10, Section 1]. For our purposes, we continue with the assumption that $K = \mathbb{F}_q(T)$ is the function field of the projective line.

Chapter 2

Drinfeld Modules over Finite Fields

2.1 Definition

Recall that L is an A-algebra via the map $\gamma : A \to L$. Following [12, (1.2)], we define a Drinfeld A-module.

Definition 2.1.1. A *Drinfeld A-module*[1] ϕ over L is a map $\phi : A \to L\{\tau\}$ defined by

$$\phi : A \to L\{\tau\},$$
$$a \mapsto \phi_a,$$

such that $\phi_a = \gamma(a) + \sum_{i \geq 1} l_i(a)\tau^i$.

The image

$$\phi_T = \gamma(T) + \sum_{1 \leq i \leq r} l_i(T)\tau^i$$

completely determines ϕ. The *rank* of ϕ is the degree in τ of $\phi(T)$, so $l_r(T)$ is assumed to be non-zero.

Definition 2.1.2. The *characteristic* of a Drinfeld A-module ϕ, char(ϕ), is defined to be the prime ideal $\mathfrak{p} = \ker(\gamma)$ of A. If $\mathfrak{p} = (0)$ then ϕ is said to have *generic characteristic*, otherwise ϕ has *positive characteristic* [24, Definition 2.2].

Remark 2.1.3. Because L has positive characteristic p, ϕ is injective. This is true regardless of whether ϕ has positive or generic characteristic [24, Section 2.3]. Since we are interested in

[1] In this text, we often use the phrase, "Drinfeld module," in place of the more precise language, "Drinfeld A-module." All Drinfeld modules discussed are A-modules.

Drinfeld modules over finite fields, i.e., when L is a finite extension of $A/\mathfrak{p}$, all Drinfeld modules in this paper have characteristic $\mathfrak{p}$.

2.1.1 Morphisms

Let ϕ and ψ be Drinfeld modules over L. Because a Drinfeld module is an A-module structure on the additive group scheme $\mathbb{G}_a$ over L, a morphism of Drinfeld modules must be a morphism of $\mathbb{G}_a$ commuting with the A-module structures associated with ϕ and ψ [10, (1.1)].

Definition 2.1.4. A *morphism* between Drinfeld modules ϕ and ψ is an element u of $\mathrm{End}(\mathbb{G}_a)$ such that $u \circ \phi_a = \psi_a \circ u$ for all a in A; i.e., the following diagram commutes.

$$\begin{array}{ccc} \mathbb{G}_a & \xrightarrow{\phi_a} & \mathbb{G}_a \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ \mathbb{G}_a & \xrightarrow{\psi_a} & \mathbb{G}_a \end{array}$$

An *isomorphism* of Drinfeld modules is an invertible morphism.

Recall that $L\{\tau\}$ is a subset of the set of endomorphisms of the additive group scheme over L subject to $\tau c = c^q \tau$. Since ϕ_a and ψ_a are elements of $L\{\tau\}$ and u commutes with the A-structures on ϕ and ψ, we have that u must be in $L\{\tau\} \subset \mathrm{End}_L(\mathbb{G}_a)$.

Definition 2.1.5. The *kernel* $\ker(u)$ of a morphism $u : \phi \to \psi$ is the kernel of the underlying map from $\mathbb{G}_a$ to itself. As such, the $\ker(u)$ is a group scheme.

Definition 2.1.6. An *endomorphism* of a Drinfeld module is a morphism from ϕ to itself, i.e., an element of $L\{\tau\}$ so that $u \circ \phi_a = \phi_a \circ u$ for all a in A. The set of endomorphisms

$$\mathrm{End}_L(\phi) := \{u \in L\{\tau\} : u\phi_a = \phi_a u \text{ for all } a \text{ in } A\}$$

is a ring under addition and composition.

We will study the endomorphism ring of Drinfeld modules in greater detail in Sections 2.1.2 and 2.3.

Definition 2.1.7. An invertible endomorphism is called an *automorphism.* We denote the group of automorphisms of a Drinfeld module over L by $\mathrm{Aut}_L(\phi)$.

The group of automorphisms for a rank two Drinfeld module ϕ over a finite field L defined by $\phi_T = \gamma(T)g\tau + \Delta\tau^2$ is given by

$$\mathrm{Aut}_L(\phi) = \begin{cases} \mathbb{F}_q^* & \text{if } g \neq 0 \text{ or } L \text{ doesn't contain } \mathbb{F}_{q^2}, \\ \mathbb{F}_{q^2}^* & \text{otherwise.} \end{cases} \tag{2.1}$$

Definition 2.1.8. An *isogeny* is a non-zero morphism $u : \phi \to \psi$. If such a morphism exists, ϕ and ψ necessarily have the same rank. Over the algebraic closure, an isogeny $u : \phi(\overline{L}) \to \psi(\overline{L})$ is surjective such that $\ker(u)(\overline{L})$ is finite. The notation $\phi(\overline{L})$ means the A-module structure on $\mathrm{End}_{\overline{L}}(\mathbb{G}_a)$ induced by $\phi : A \to L\{\tau\}$.

Isogeny is an equivalence relation on the set of all Drinfeld modules over a given A-field L, as seen in the following proposition from [13, Proposition 2.5].

Proposition 2.1.9. *Let ϕ and ψ be Drinfeld A-modules over any field L, and suppose there exists an isogeny $u : \phi \to \psi$. Then there exist an $a \in A$ and an isogeny $v : \psi \to \phi$ with $vu = \phi_a$.*

2.1.2 The Endomorphism Ring

Fix an arbitrary Drinfeld module ϕ over the finite field L. Let

$$\mathrm{End}_L(\phi) := \{u \in L\{\tau\} : u\phi_a = \phi_a u \text{ for all } a \text{ in } A\}$$

be the ring of endomorphisms of ϕ over L. We make the following observations. First, since ϕ is injective, the image $\phi(A)$ can be identified with A in $L\{\tau\}$. According to the ring homomorphism properties of ϕ, we see that since A is commutative with respect to multiplication, the image $\phi(A)$ is also. Equivalently, the image $\phi(A)$ is a commutative subring of the endomorphism ring $\mathrm{End}_L(\phi)$. Second, by definition, $\mathrm{End}_L(\phi)$ is the centralizer of $\phi(A)$ in $L\{\tau\}$. Third, $\mathrm{End}(\phi)$

absorbs the action of A via $\phi(A)$, so it is an A-algebra. Fourth and finally, since $\mathrm{End}_L(\phi) \subseteq L\{\tau\}$, it is torsion free over L [15, Ch. 4.7].

For computational practice, we now consider a rank one Drinfeld module (also called the *Carlitz module*) and examine its endomorphism ring.

Example 2.1.10. Let A be $\mathbb{F}_4[T]$ and fix the prime $\mathfrak{p} = \langle T^2 + \alpha T + 1 \rangle$. Let $L = A/\mathfrak{p} \cong \mathbb{F}_{16}$ and let ϕ be the Drinfeld module

$$\phi : A \to L\{\tau\}, \tag{2.2}$$

$$a \mapsto \phi_a, \tag{2.3}$$

defined by

$$\phi_T = \gamma(T) + \tau = T + \tau \mod \mathfrak{p}.$$

Since an arbitrary element in A is of the form $g(T) = a_0 + a_1 T + \cdots a_k T^k$, with $a_i \in \mathbb{F}_4$, an arbitrary element in the image of ϕ is of the form $\phi_{g(T)} = a_0 + a_1(T + \tau) + \cdots a_k(T + \tau)^k$. The key here is that the coefficients are in $\mathbb{F}_4$, not $\mathbb{F}_{16}$.

Now consider an arbitrary endomorphism of ϕ, given by $u = \sum_{i=0}^{j} l_i \tau^i$ with $l_i \in \mathbb{F}_{16}$. By definition, $u\phi_a = \phi_a u$ for all $a \in \mathbb{F}_4[T]$. Expanding both sides according to the definition of multiplication in $L\{\tau\}$ gives the equality

$$\begin{aligned} l_0 T + (l_0 + l_1 T^4)\tau + \cdots + (l_i + l_{i+1} T^{4^{i+1}})\tau^{i+1} + \cdots + l_m \tau^{m+1} \\ = T l_0 + (l_0^4 + T l_1)\tau + \cdots + (l_{m-1}^4 + T l_m)\tau^m + l_m^4 \tau^{m+1}. \end{aligned}$$

This implies that $l_i^4 \equiv l_i$, for all i. Thus, each l_i is in $\mathbb{F}_4$, not $\mathbb{F}_{16}$. From here we conclude that $u \in \phi(A)$, and thus $\phi(A) = \mathrm{End}(\phi)$. Since ϕ is injective, we really have $\mathrm{End}(\phi) \cong A$.

When ϕ is rank one, we easily see that $\mathrm{End}(\phi)$ is a rank one (projective) A-module (see also Example 2.3.4). More generally, if ϕ is a rank r Drinfeld A-module, its endomorphism ring is a projective a A-module of rank $\leq r^2$ (see Theorem 2.3.3). To prove this, we need representations of the endomorphism ring on the $\mathfrak{l}$*-adic Tate module of* ϕ. These representations also allow us to define the characteristic and minimal polynomials of a Drinfeld module.

2.2 The Tate and Dieudonné Modules

2.2.1 The Tate Module

Fix a rank r Drinfeld module ϕ over L. The $\mathfrak{l}$-adic Tate module of ϕ is defined in terms of $\mathfrak{l}$-power *torsion submodules* of ϕ.

Definition 2.2.1. [10, Page 189] Let $\mathfrak{a}$ be an arbitrary ideal of A. The $\mathfrak{a}$-*torsion submodule* of ϕ is

$$_{\mathfrak{a}}\phi := \bigcap_{a\in\mathfrak{a}} \ker(\phi_a).$$

By the definition of ϕ_a as an endomorphism of $\mathbb{G}_a$, the $\mathfrak{a}$-torsion submodule is a group scheme. We consider the $\overline{L}$-points

$$_{\mathfrak{a}}\phi(\overline{L}) = \bigcap_{a\in\mathfrak{a}} \ker(\phi_a(\overline{L})).$$

Via the image $\phi(A)$, $_{\mathfrak{a}}\phi(\overline{L})$ is a finite A-module.

Theorem 2.2.3 (of Drinfeld) as stated in [13, Proposition 2.1] (in the case of L a finite field) and [24, Theorem. 2.5] (in general) gives the structure of the torsion submodules of ϕ. The $\mathfrak{p}$-torsion submodule of ϕ is explained in terms of the *height* of ϕ.

For each $a \in A$, let $ht(\phi_a)$ be the height of $\phi_a \in L\{\tau\}$ as defined in Definition 1.1.2. Remarkably, the function $a \mapsto ht(\phi_a)$ defines a valuation on A equivalent to the $\mathfrak{p}$-adic valuation. For the purposes of this paper, we forego this explanation.

Definition 2.2.2. Let ϕ be a Drinfeld module over a finite field L with characteristic $\mathfrak{p} \neq (0)$ where $\mathfrak{p}$ has degree d. Then the *height* $ht(\phi)$ of ϕ is the number

$$ht(\phi) := \frac{1}{d} \min\{ht(\phi_a) : 0 \neq a \in \mathfrak{p}\}.$$

This is an integer such that $1 \leq ht(\phi) \leq r$.

Theorem 2.2.3. *Let L be the degree m extension of $A/\mathfrak{p}$. Fix a rank r Drinfeld module $\phi : A \to L\{\tau\}$ over L with char$(\phi) = \mathfrak{p} \neq (0)$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be coprime ideals of A. Then:*

(a) The torsion submodule of the ideal $\mathfrak{ab}$ is such that

$${}_{\mathfrak{ab}}\phi = {}_{\mathfrak{a}}\phi \oplus {}_{\mathfrak{b}}\phi.$$

(b) If $\mathfrak{a}$ and $\mathfrak{p}$ are coprime, then the $\mathfrak{a}$-torsion submodule ${}_{\mathfrak{a}}\phi(\overline{L})$ is a free $A/\mathfrak{a}$-module of rank r.

(c) If $\mathfrak{a} = \mathfrak{p}^k$, then the $\mathfrak{a}$-torsion submodule ${}_{\mathfrak{a}}\phi(\overline{L})$ is a free $A/\mathfrak{a}$-module of rank $r - ht(\phi)$.

For the remainder of this section, we fix a prime $\mathfrak{l}$ away from $\mathfrak{p}$ and consider the $\mathfrak{l}$-power torsion submodules of ϕ. Let ${}_{\mathfrak{l}^i}\phi$ and ${}_{\mathfrak{l}^j}\phi$ be two $\mathfrak{l}$-power torsion submodules with $i < j$. Then there is a surjection

$$f_{ij} : {}_{\mathfrak{l}^j}\phi \to {}_{\mathfrak{l}^i}\phi$$

defined by the multiplication by $\mathfrak{l}^{j-i}$ map. These maps form an inverse system.

Definition 2.2.4. Fix a rank r Drinfeld module ϕ over a finite extension of $A/\mathfrak{p}$ and a prime $\mathfrak{l} \neq \mathfrak{p}$. Then the $\mathfrak{l}$-*adic Tate module* attached to a Drinfeld module ϕ over L is

$$T_{\mathfrak{l}}(\phi) := \varprojlim {}_{\mathfrak{l}^n}\phi(\overline{L}).$$

This inverse limit naturally has the structure of an $A_{\mathfrak{l}}$-module of rank r (see Theorem 2.2.3).

For any ideal $\mathfrak{a} \subset A$, the Galois group $\text{Gal}(\overline{L} : L)$ acts on ${}_{\mathfrak{a}}\phi(\overline{L})$. However, since $\mathfrak{l}$ and $\mathfrak{p}$ are coprime, the image $\phi_{\mathfrak{l}^n}$ is separable[2] for all $n \geq 1$. As an A-module, we have

$$ {}_{\mathfrak{l}^n}\phi(\overline{L}) \cong (A/\mathfrak{l}^n)^r. $$

Therefore, the group $\text{Aut}_A({}_{\mathfrak{l}^n}\phi(\overline{L}))$ of automorphisms of ${}_{\mathfrak{l}^n}\phi$ over A is

$$ \text{Aut}_A({}_{\mathfrak{l}^n}\phi(\overline{L})) \cong \text{GL}_r(A/\mathfrak{l}^n). $$

Even more, the separability of $\phi_{\mathfrak{l}^n}$ implies that ${}_{\mathfrak{l}^n}\phi(\overline{L}) = {}_{\mathfrak{l}^n}\phi(L^s)$, where L^s is the separable closure of L. The Galois action commutes with the structure of ${}_{\mathfrak{l}^n}\phi(\overline{L})$ as an $A/\mathfrak{l}^n$-module giving the continuous representation

$$ \varpi : \text{Gal}(L^s : L) \to \text{Aut}_A({}_{\mathfrak{l}^n}\phi(\overline{L})) \cong \text{GL}_r(A/\mathfrak{l}^n). \tag{2.4} $$

Since the inverse limit respects the above structures, there is an induced action of the Galois group on $T_{\mathfrak{l}}(\phi)$ given by

$$ \varpi_{\mathfrak{l}} : \text{Gal}(\overline{L} : L) \to \text{Aut}_A(T_{\mathfrak{l}}(\phi)) \cong \text{GL}_r(A_{\mathfrak{l}}). $$

Similarly, there is a natural ring representation of the endomorphism ring on the Tate module as described in Section 2.3 [15, Ch. 4.10].

The following proposition is an important result about the Tate module we will use in Section 5.1; it states that isogenies between Drinfeld modules ϕ and ψ induce injective maps between their $\mathfrak{l}$-adic Tate modules.

Proposition 2.2.5. *([24, Proposition 3.4]) Let ϕ and ψ be Drinfeld A-modules over L. Then for all prime ideals $\mathfrak{l}$ of A different from ∞ and $\mathfrak{p}$, the natural homomorphism*

[2] The separability of $\phi_{\mathfrak{l}}$ also implies that as a scheme, ${}_{\mathfrak{l}}\phi$ is étale.

$$\mathrm{Hom}_A(\phi,\psi)\otimes_A A_{\mathfrak{l}} \to \mathrm{Hom}_{A_{\mathfrak{l}}}(T_{\mathfrak{l}}(\phi),T_{\mathfrak{l}}(\psi)) \tag{2.5}$$

is injective.

Remark 2.2.6.

Define ${}_{\mathfrak{l}^\infty}\phi$ to be the direct limit

$${}_{\mathfrak{l}^\infty}\phi := \varinjlim {}_{\mathfrak{l}^n}\phi.$$

Then there is an equality

$$T_{\mathfrak{l}}(\phi) = \mathrm{Hom}_{A_{\mathfrak{l}}}(K_{\mathfrak{l}}/A_{\mathfrak{l}}, {}_{\mathfrak{l}^\infty}\phi(\overline{L})). \tag{2.6}$$

This formulation of the Tate module is consistent with the definition of the Dieudonné module, the $\mathfrak{p}$-adic analog of the Tate module.

2.2.2 The Dieudonné Module

We continue to work with a rank r Drinfeld module ϕ over L, the degree m extension of $A/\mathfrak{p}$. In defining a module that keeps track of $\mathfrak{p}$-power torsion, one challenge comes from the fact that the polynomial $\phi_{\mathfrak{p}}$ is not separable. As a consequence, as the $\mathfrak{p}$-power torsion submodules ${}_{\mathfrak{p}^n}\phi$ are not as nice as schemes as the $\mathfrak{l}$-power torsion submodules of the previous section[3]. Dieudonné's theory accommodates for this obstruction. Following [23, Sections 2.4 & 2.5] and [3], we describe the construction of the *Dieudonné* module of ϕ.

Let $K_{\mathfrak{p}}^{\mathrm{un}}$ be the maximal unramified extension of $K_{\mathfrak{p}}$ and $A_{\overline{A/\mathfrak{p}}}$ be the completion of its ring of integers. The ring of integers has the property that $A_{\overline{A/\mathfrak{p}}}/\mathfrak{p}A_{\overline{A/\mathfrak{p}}} = \overline{A/\mathfrak{p}}$. Similarly, let A_L be (the completion of) the ring of integers of the unique unramified degree m extension of $K_{\mathfrak{p}}$. The residue field of A_L is L. Let K_L be the field of fractions of A_L.

[3]In particular, they are not étale.

The arithmetic Frobenius element F of $\mathrm{Gal}(L : A/\mathfrak{p})$ is the m-th iterate of the $q^{\deg_T(\mathfrak{p})}$-power map (this is the same as the map used later in Section 2.3.1). This Frobenius map lifts to $F_\mathfrak{p}$ in the Galois group $\mathrm{Gal}(K_L : K_\mathfrak{p})$.

Definition 2.2.7. ([23, Definition 2.4.1]) A *Dieudonné $A_\mathfrak{p}$-module over L* is a free A_L-module M of finite rank endowed with an injective $F_\mathfrak{p}$-linear map $f : M \to M$ such that the cokernel of f is of finite length over A_L. We denote a Dieudonné $A_\mathfrak{p}$-module by the pair (M, f). The rank of (M, f) is the rank of M as an A_L-module. Similarly, a *Dieudonné $K_\mathfrak{p}$-module over L* is a finite dimensional K_L-vector space N endowed with a bijective $F_\mathfrak{p}$-linear map $f : N \to N$. The rank of (N, f) is the dimension of N as a K_L-vector space. If (M, f) is a Dieudonné $A_\mathfrak{p}$-module over L, then $(K_\mathfrak{p} \otimes_{A_\mathfrak{p}} M, K_\mathfrak{p} \otimes_{A_\mathfrak{p}} f)$ is a Dieudonné $K_\mathfrak{p}$-module over L of the same rank.

Definition 2.2.8. ([23, Definition 2.4.8]) For every $n \geq 1$, let P_n be a finite group scheme over L. Suppose also that P_n has the structure of an $A_\mathfrak{p}$-module. Via the inclusion $A/\mathfrak{p} \subset A_\mathfrak{p}$, we can view the L points of P_n as an $A/\mathfrak{p}$-vector space. A *$\mathfrak{p}$-divisible L-scheme in $A_\mathfrak{p}$-modules* is a direct system of the form

$$P = (P_1 \xrightarrow{i_1} P_2 \xrightarrow{i_2} P_3 \xrightarrow{i_3} \cdots) = \varinjlim P_n,$$

such that for each integer $n \geq 1$, the following properties hold.

(a) As an L-scheme with the structure of an $A/\mathfrak{p}$-vector space, P_n embeds in to finitely many copies of $\mathbb{G}_a$ over $A/\mathfrak{p}$.

(b) The L-scheme P_n is finite of dimension $n \cdot c$ over $A/\mathfrak{p}$ for some integer $c \geq 0$ independent of n (and the order of P_n is $q^{\deg_T(\mathfrak{p}) n \cdot c}$).

(c) The sequence of $A_\mathfrak{p}$-modules

$$0 \to P_n \xrightarrow{i_n} P_{n+1} \xrightarrow{[\mathfrak{p}^n]} P_{n+1},$$

is exact.

The integer c in (b) is called the *rank* of P.

There is an equivalence of categories between Dieudonné $A_{\mathfrak{p}}$-modules and $\mathfrak{p}$-divisible L-schemes in $A_{\mathfrak{p}}$-modules [23, Corollary 2.4.14], which means for every $\mathfrak{p}$-divisible L subscheme, there is a corresponding Dieudonné $A_{\mathfrak{p}}$-module. Given a $\mathfrak{p}$-divisible L scheme, let $H_{\mathfrak{p}}$ be its corresponding Dieudonné $A_{\mathfrak{p}}$-module.

The direct limit

$$ {}_{\mathfrak{p}^{\infty}}\phi = \varinjlim_{n} {}_{\mathfrak{p}^{n}}\phi \tag{2.7} $$

is a $\mathfrak{p}$-divisible L-scheme in $A_{\mathfrak{p}}$-modules of rank r [23, Lemma 2.5.1]. That is, each ${}_{\mathfrak{p}^{n}}\phi$ has dimension $n \cdot r$ over $A/\mathfrak{p}$. *The Dieudonné module of* ϕ is the Dieudonné module attached to ${}_{\mathfrak{p}^{\infty}}\phi$ as a $\mathfrak{p}$-divisible L-scheme in $A_{\mathfrak{p}}$-modules.

Definition 2.2.9. For a Drinfeld A-module ϕ over L of rank r and characteristic $\mathfrak{p}$, *the Dieudonné $A_{\mathfrak{p}}$-module over L* is the pair $(B_{\mathfrak{p}}(\phi), f_{\phi,\mathfrak{p}})$ corresponding to the direct system (2.7), and $f_{\phi,\mathfrak{p}}$ is the map

$$ f_{\phi,\mathfrak{p}} : B_{\mathfrak{p}}(\phi) \to B_{\mathfrak{p}}(\phi), $$

induced by $F_{\mathfrak{p}}$ on ${}_{\mathfrak{p}^{n}}\phi$. Over A_L, $B_{\mathfrak{p}}(\phi)$ has rank r.

Remark 2.2.10. The standard notation for the Dieudonné module of ϕ is $H_{\mathfrak{p}}(\phi)$. Later, we use $H_L(\phi)$ to refer to the L-isogeny class of ϕ, so we instead denote the Dieudonné module by $B_{\mathfrak{p}}(\phi)$ to avoid a confusing clash of notation.

The next lemma gives an important property of the map $f_{\phi,\mathfrak{p}}$.

Lemma 2.2.11. *([23, Lemma 2.5.4]) The Dieudonné $A_{\mathfrak{p}}$-module $(B_{\mathfrak{p}}(\phi), f_{\phi,\mathfrak{p}})$ over L of a Drinfeld A-module ϕ over L has the inclusions*

$$ \mathfrak{p}B_{\mathfrak{p}}(\phi) \subseteq f_{\phi,\mathfrak{p}}(B_{\mathfrak{p}}(\phi)) \subseteq B_{\mathfrak{p}}(\phi), $$

and the equality

$$\dim_L(B_{\mathfrak{p}}(\phi)/f_{\phi,\mathfrak{p}}(B_{\mathfrak{p}}(\phi))) = 1.$$

2.3 The Endomorphism Algebra

The primary object of study in this section is the *endomorphism algebra* $E := \mathrm{End}(\phi) \otimes K$ of a Drinfeld module ϕ. First, we use the $\mathfrak{l}$-adic representation of the endomorphism ring on the Tate module to show that the endomorphism ring of a Drinfeld module is a projective A-module of rank less than or equal to r^2, and that $\mathrm{End}(\phi) \otimes_A K_\infty$ is a division ring. Second, we demonstrate that E is a central division algebra over $K(F)$, where F is the Frobenius endomorphism of ϕ.

To begin, the next lemma gives criteria for when an A-module is projective [13, Lemma 2.6].

Lemma 2.3.1. *Let V be a K_∞-vector space of dimension n.*

(a) There exists a unique topology on V such that each one-dimensional subspace is topologically isomorphic to K_∞.

(b) Let $H \subset V$ be an A-submodule which is discrete with respect to the above topology. Then H is projective over A of rank at most n.

The first point is true because $V \cong K_\infty^n$. Since A is discrete in K_∞, induction on n gives the second point [13, Lemma 2.6].

The next lemma is used in proof of Theorem 2.3.3, and induces a canonical $\mathfrak{l}$-adic representation of the endomorphism ring on the Tate module for a given Drinfeld module ϕ.

Lemma 2.3.2. *Let $\mathfrak{a} \in A \setminus \mathfrak{p}$. Then the canonical representation*

$$i_{\mathfrak{a}} : \mathrm{End}(\phi) \otimes A/\mathfrak{a} \to \mathrm{End}_A({}_{\mathfrak{a}}\phi(\overline{L}))$$

is injective.

For each prime $\mathfrak{l}$, there is a canonical representation $i_{\mathfrak{l}}$ induced on the $\mathfrak{l}$-adic Tate module, given by

$$i_{\mathfrak{l}} : \mathrm{End}(\phi) \otimes A_{\mathfrak{l}} \to \mathrm{End}_{A_{\mathfrak{l}}}(T_{\mathfrak{l}}(\phi)) \cong \mathbf{Mat}_r(A_{\mathfrak{l}}). \tag{2.8}$$

The proof in [13, Lemma 2.7] shows that if an element u in $\mathrm{End}(\phi)$ vanishes on ${}_{\mathfrak{a}}\phi$ (i.e., comes from the kernel of $i_{\mathfrak{a}}$), then it is trivial in $\mathrm{End}(\phi) \otimes A/\mathfrak{a}$ because u must be an element in $\mathfrak{a} \cdot \mathrm{End}(\phi)$.

We now characterize the rank of the endomorphism ring as a projective A-module in the next theorem [13, Theorem. 2.8].

Theorem 2.3.3. *The endomorphism ring* $\mathrm{End}(\phi)$ *is a projective* A*-module of rank less than or equal to* r^2 *(*r *= rank of* ϕ*), and* $\mathrm{End}(\phi) \otimes_A K_\infty$ *is a division ring.*

Here we start by noting that $E := \mathrm{End}(\phi) \otimes_A K$ is the ring of quotients of $\mathrm{End}(\phi)$. The degree map on $\mathrm{End}(\phi) \setminus \{0\}$ taking an element u to $-\deg_\tau(u)$ extends uniquely to a discrete valuation w_∞ on $\mathrm{End}(\phi) \otimes_A K$. When restricted to K, the valuation w_∞ agrees with the valuation ν_∞ on K. The topology on $\mathrm{End}(\phi)$ induced by w_∞ is discrete and agrees with the ∞-adic topology on finite dimensional K-subspaces of $E = \mathrm{End}(\phi) \otimes_A K$. Therefore, the valuation w_∞ extends to $\mathrm{End}(\phi) \otimes K_\infty$. This together with the fact that $E = \mathrm{End}(\phi) \otimes K$ is a division ring implies that $\mathrm{End}(\phi) \otimes K_\infty$ is a division ring.

By Lemma 2.3.1, $\mathrm{End}(\phi)$ is projective over A. For each K_∞-subspace V of dimension n in $\mathrm{End}(\phi) \otimes K_\infty$, we have that $\mathrm{End}(\phi) \cap V$ is projective of rank $\leq n$. There are no K_∞-subspaces of $\mathrm{End}(\phi) \otimes K_\infty$ having dimension larger than r^2, otherwise we would contradict Lemma 2.3.2. See [13, Theorem. 2.8] for more details.

Example 2.3.4. Let ϕ be a rank one Drinfeld module. Then $\mathrm{End}(\phi)$ can either be a rank one or rank zero projective A-module. However, it is not the zero-module since $\phi(A) \subseteq \mathrm{End}(\phi)$. Therefore $\mathrm{End}(\phi) \cong A$.

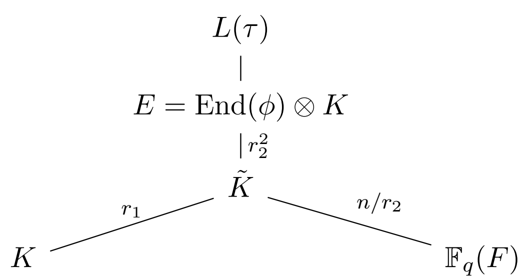
A few facts are central to the upcoming discussion on minimal and characteristic polynomial rings, and the proof of Theorem 2.6.1.

2.3.1 Central Division Algebra

Recall that L is the finite field of degree m over $A/\mathfrak{p}$. Since $A/\mathfrak{p}$ is the degree d extension of $\mathbb{F}_q$, L has q^n elements where $n = d \cdot m$. Let $F = \tau^n : x \mapsto x^{q^n}$ be the Frobenius morphism of L. According to [10, Section 2], the Frobenius morphism F is also an endomorphism of the fixed Drinfeld module ϕ over L (because F commutes with $\phi(A)$ in $L\{\tau\}$). Define $L(\tau)$ to be the division ring of fractions of $L\{\tau\}$. This means that $L(\tau)$ need not be commutative but it is closed under left and right multiplicative inverses. It is central of dimension n^2 over $\mathbb{F}_q(F)$, the field of rational functions in Frobenius F [13, (3.2)]. Between $\mathbb{F}_q(F)$ and $L(\tau)$ is $E := \mathrm{End}(\phi) \otimes K$ and its commutative subfield $\tilde{K} = K(F)$ generated by F. We have already seen in Theorem 2.3.3 that E is a division ring of $\mathrm{End}(\phi)$.

We have the following theorem as stated in [13, Theorem. 3.8] and illustrated in the subsequent diagram using facts about the invariants of central division algebras and completions.

Theorem 2.3.5. *Let $\tilde{K} = K(F)$ be the subfield of $E := \mathrm{End}(\phi) \otimes K$ over K generated by F and $r_1 = [\tilde{K} : K]$ its degree. Then r/r_1 is an integer r_2 and E is a central division algebra over $\tilde{K}$ of degree r_2^2. There exists a unique place $\tilde{\mathfrak{p}}$ of $\tilde{K}$ that divides F and lies above the place $\mathfrak{p}$ of K. The place ∞ of K has a unique extension $\tilde{\infty}$ to $\tilde{K}$. Then E splits at places $\tilde{\mathfrak{l}}$ of $\tilde{K}$ different from $\tilde{\mathfrak{p}}$ and $\tilde{\infty}$, with invariants $1/r_2$, $-1/r_2$ at $\tilde{\mathfrak{p}}$ and $\tilde{\infty}$ respectively.*



2.4 Minimal and Characteristic Polynomials

Keeping the same notation from the previous section, we now define the minimal and characteristic polynomials of ϕ, a rank r Drinfeld module over L, the degree m extension of $A/\mathfrak{p}$.

Definition 2.4.1. As an element of E, F must satisfy a unique monic polynomial $M_\phi(X)$ of minimal degree over $A \subset K$. We call $M_\phi(X)$ the *minimal polynomial of* ϕ.

Definition 2.4.2. Let $\mathfrak{l} \neq \mathfrak{p}$ be a prime of A. Let $i_{\mathfrak{l}}(F) \in \mathrm{End}(T_{\mathfrak{l}}(\phi))$ be the image of F under the representation $i_{\mathfrak{l}}$, as in (2.8). Let $P_\phi(X) \in A[X]$ be the characteristic polynomial of $i_{\mathfrak{l}}(F)$. Provided $\mathfrak{l} \neq \mathfrak{p}$, $P_\phi(X)$ is independent of the choice of $\mathfrak{l}$ and is known as the *characteristic polynomial of* ϕ.

The following theorem describes the constant term of the characteristic polynomial.

Theorem 2.4.3. *([10, Theorem 5.1])*

Let ϕ be a Drinfeld module over the finite field L, and let $P_\phi(X)$ be its characteristic polynomial. Then, there is an equality of ideals

$$(P_\phi(0)) = (\mathfrak{p}^{[L:A/\mathfrak{p}]}).$$

In practice, we use the field norm maps attached to the extensions in Theorem 2.3.5 to compute the minimal and characteristic polynomials of a Drinfeld module, as we see for rank two in Appendix A. We always refer to the norm maps by the extension from which they arise. For example, N_K^L is the norm map of the extension L over K. Presently, we need the composition of the reduced norm $Nrd_{\tilde{K}}^E$ and the field norm $N_K^{\tilde{K}}$. The next lemma relates the minimal and characteristic polynomials more closely (as stated in [13, Lemma 3.13]). Notably, the characteristic polynomial is the r_2 power of the minimal polynomial, where r_2 is the quotient of the rank of ϕ and the degree of the extension of $\tilde{K}$ over K.

Lemma 2.4.4. *Let u be an endomorphism of ϕ, and $\mathfrak{l} \neq \mathfrak{p}$ a prime of A. Then*

(a) $(N_K^{\tilde{K}} \circ Nrd_{\tilde{K}}^E)(u) = \det(i_{\mathfrak{l}}(u))$,

(b) $-v_\infty((N_K^{\tilde{K}} \circ Nrd_{\tilde{K}}^E)(u)) = (-r)^{-1} \deg_\tau((N_K^{\tilde{K}} \circ Nrd_{\tilde{K}}^E)(u)) = -\deg_\tau u$,

(c) $P_\phi(X) = M_\phi(X)^{r_2}$.

Remark 2.4.5. Here, (b) is stated assuming we have fixed $\infty = 1/T$. In general, if ∞ is a prime of degree d_∞, we have $-v_\infty((N_K^{\tilde{K}} \circ Nrd_{\tilde{K}}^E)(u)) = (rd_\infty)^{-1} \deg_\tau((N_K^{\tilde{K}} \circ Nrd_{\tilde{K}}^E)(u)) = (d_\infty)^{-1} \deg_\tau u$.

2.5 Newton Polygons

In [27], Poonen defines *supersingular* and *ordinary* Drinfeld modules in terms of the Newton polygon of the characteristic polynomial at $\mathfrak{p}$. Recall that for a polynomial $P(X) = a_r x^r + a_{r-1}X^{r-1} + \cdots a_1 X + a_0$, to construct the Newton polygon, we plot the points $(i, v_{\mathfrak{p}}(a_i))$. The *Newton polygon* is the convex hull of these points. The slopes of the convex hull characterize the $\mathfrak{p}$-adic valuations of the roots of $P(X)$. If the Newton polygon of $P(X)$ has a segment with slope of $-s$, $P(X)$ has a root with $\mathfrak{p}$-adic valuation s. If $P(X)$ has a root of multiplicity e and valuation s, there are e segments of slope $-s$ in the Newton polygon.

Fix a rank r Drinfeld module ϕ over the finite field L (a degree m extension of $A/\mathfrak{p}$) with characteristic polynomial $P_\phi(X) = X^r - a_{r-1}X^{r-1} + \cdots + (-1)^k a_{r-(1+k)} X^{r-(1+k)} + \cdots (-1)^r a_0$ (viewed as a polynomial in $A[X]$). By Theorem 2.4.3, we know $a_0 = \pm\mathfrak{p}^m$ is a generator of the ideal $(\mathfrak{p}^m)$.

Definition 2.5.1. We say ϕ is *ordinary* if a_1 is non-zero modulo $\mathfrak{p}$. Since the leading coefficient is one, the Newton polygon of an ordinary Drinfeld module takes the shape in Figure 2.1.

Definition 2.5.2. We say that ϕ is *supersingular* if $P_\phi(X) \equiv X^r \mod \mathfrak{p}$.

Remark 2.5.3. Notice that if $r = 1$, the only characteristic polynomial is $P_\phi(X) = X$, which implies ϕ is *both* ordinary and supersingular. If $r = 2$, the characteristic polynomial of ϕ is of the form $P_\phi(X) = X^2 - aX + b$, so may ϕ may be either ordinary or supersingular, but not both. If $r \geq 3$, ϕ could be ordinary, supersingular, or neither. See [10, 15] for equivalent descriptions of supersingular Drinfeld modules.

Remark 2.5.4. Observe that if ϕ is ordinary, the Newton polygon of $P_\phi(X)$ has exactly one segment of slope $-m$, which means that $P_\phi(X)$ has exactly one root of $\mathfrak{p}$-adic valuation m. However, $P_\phi(X) = M_\phi(X)^{r_2}$ as in Lemma 2.4.4. Since this root cannot be a repeated root, $r_2 = 1$. By Theorem 2.3.5, this means that the endomorphism algebra $E = \mathrm{End}(\phi) \otimes K$ of ϕ *is* the commutative subfield $\tilde{K}$.

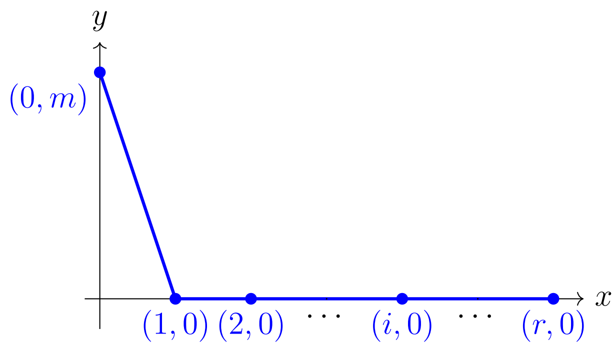


Figure 2.1: The Newton polygon of P_ϕ if ϕ is ordinary

Warning! 2.5.5. In Remark 2.5.4, we see that if ϕ is ordinary, it has a commutative endomorphism algebra. The converse is not true. There are examples of Drinfeld modules with commutative endomorphism algebras which are not ordinary. Goss shows this in Example 4.12.18 of [15]. Let $L = A/\mathfrak{p} = \mathbb{F}_q[T]/T$ be isomorphic to $\mathbb{F}_q$. Suppose ϕ is a rank two Drinfeld over L determined by $\phi_T = \tau^2$. Observe that the Frobenius endomorphism is $F = \tau$, and $[\tilde{K} : K] = 2$. By Theorem 2.3.5, the degree of the endomorphism algebra over $\tilde{K}$ is one, so $\operatorname{End}\phi \otimes K$ is commutative. In this case the characteristic and minimal polynomials are equal $M_\phi(X) = P_\phi(X) = X^2$, and imply that ϕ is supersingular. Here, ϕ is also not ordinary because $a_1 = 0$.

In Appendix A, we include more examples of rank two Drinfeld modules and their characteristic polynomials, which we compute using the method outlined in [12].

2.6 Isogenies

The next theorem allows us to classify Drinfeld modules up to isogeny completely using their endomorphism algebras, characteristic polynomials, and minimal polynomials.

2.6.1 Classification

Theorem 2.6.1. *Let ϕ and ψ be Drinfeld modules over L. The following are equivalent:*

(a) ϕ *and* ψ *are isogenous,*

(b) $\operatorname{End}_L(\phi) \otimes_A K$ *and* $\operatorname{End}_L(\psi) \otimes_A K$ *are isomorphic* K*-algebras,*

(c) $M_{\phi,L} = M_{\psi,L}$,

(d) $P_{\phi,L} = P_{\psi,L}$.

The proof of this theorem can be found in [13, Theorem. 3.13] and [10, Theorem. 3.5]. It follows the steps outlined below.

In summary, we know that (c) and (d) are equivalent because $P_\phi(X) = M_\phi(X)^{r_2}$. Moreover, Theorem 2.3.5 implies that r_2 arising from ϕ is the same as r_2 arising from ψ. To see that (a) implies (c), let $u : \phi \to \psi$ be an isogeny and $u^{-1} : \psi \to \phi$ its dual. By definition, for each $a \in A$,

$$u\phi_a = \psi_a u. \tag{2.9}$$

The Frobenius endomorphism of ϕ, denoted F, must satisfy the minimal polynomial $M_\phi(X) = \sum a_i X^i$, which means $M_\phi(F) = \sum a_i F^i = 0 \in A$. We can view $M_\phi(F)$ as an element of $L\{\tau\}$. Note $F = \tau^n$, and a_i is identified with ϕ_{a_i} via the isomorphism between A and $\phi(A)$. We have

$$M_\phi(F) = \sum \phi_{a_i} F^i = 0.$$

But F is an endomorphism, so it commutes with ϕ_{a_i} for each a_i, yielding

$$M_\phi(F) = \sum F^i \phi_{a_i} = 0.$$

Now, apply the isogeny to $M_\phi(F)$, giving $u(M_\phi(F)) = 0$. On the right, we have $u(M_\phi(F)) = \sum uF^i\phi_{a_i}$. We also know that $uF = Fu$ since F is an endomorphism of both ϕ and ψ, yielding $\sum uF^i\phi_{a_i} = 0$. Using Equation (2.9), we obtain

$$\sum uF^i\phi_{a_i} = \sum F^i u\phi_{a_i} = \sum F^i \psi_{a_i} u = 0.$$

Ultimately, this implies that $M_\psi(F) = \sum F^i \psi_{a_i} = 0$. By definition of the minimal polynomial, this means $M_\phi(X) \mid M_\psi(X)$. Meanwhile, the same argument on the dual isogeny $u^{-1} : \psi \to \phi$ implies $M_\psi \mid M_\phi$. Since both polynomials are monic, they must be equal, i.e., $M_\phi(X) = M_\psi(X)$. To see (c) implies (b), let ϕ and ψ be Drinfeld modules with equal minimal polynomials. Since the minimal polynomials are equal, the commutative subfields of $\mathrm{End}(\phi) \otimes K$ and $\mathrm{End}(\psi) \otimes K$ generated by Frobenius (denoted $\tilde{K}$) are isomorphic. This isomorphism extends to an isomorphism of $\mathrm{End}(\phi) \otimes K$ and $\mathrm{End}(\psi) \otimes K$ by Theorem 2.3.5. Finally, to see that (b) implies (a), let α be an isomorphism of the endomorphism algebras $\mathrm{End}(\phi) \otimes K$ and $\mathrm{End}(\psi) \otimes K$. Each $\mathrm{End}(\phi) \otimes K$ and $\mathrm{End}(\psi) \otimes K$ is a simple subalgebra of $L(\tau)$. The Skolem-Noether Theorem thus implies that there is an invertible u in $L(\tau)$ such that $\alpha(f) = u \circ f \circ u^{-1}$ for each f in $\mathrm{End}(\phi) \otimes K$ [13, Theorem 1.4, 3.13]. Up to a central element, we may assume $u \in L\{\tau\}$, so u defines an L-isogeny $u : \phi \to \psi$, [10, Theorem. 3.5].

Corollary 2.6.2. *All rank* 1 *Drinfeld modules over a fixed* A*-field* L *are isogenous.*

Proof. We know by Theorem 2.3.3 that $\mathrm{End}(\phi)$ is a rank ≤ 1 projective A module, but $\phi(A) \cong A$ and $\phi(A) \subseteq \mathrm{End}(\phi)$. Therefore, $\mathrm{End}(\phi)$ is a rank 1 projective A module – there is only one – and condition (b) holds. □

2.6.2 Isogeny Classes over Finite Fields

As we have already seen, we can specify an isogeny class by choosing a characteristic polynomial. Following Gekeler's notation, we denote an isogeny class for rank two Drinfeld modules over $A/\mathfrak{p}$ by $H(a, b, \mathfrak{p})$ having cardinality $h(a, b, \mathfrak{p})$. The weighted cardinality of an isogeny class is given by

$$h^*(a, b, \mathfrak{p}) = \sum_{\phi \in H(a,b,\mathfrak{p})} \frac{1}{\#\mathrm{Aut}_{A/\mathfrak{p}}(\phi)}, \tag{2.10}$$

where $\mathrm{Aut}_{A/\mathfrak{p}}(\phi)$ is the automorphism group of ϕ over $A/\mathfrak{p}$ [12].

Chapter 3

Gekeler's Product Formula

3.1 Gekeler's Results

In [12], Gekeler proved that the size of an isogeny class of a rank two Drinfeld module over the "prime field" $A/\mathfrak{p}$ is the product over primes of local density functions, which we call *Gekeler ratios*. They are defined as follows.

Definition 3.1.1. For a fixed rank two Drinfeld module ϕ with characteristic polynomial of Frobenius $P_\phi(X) = X^2 - aX + b$ and endomorphism algebra $E = \mathrm{End}(\phi) \otimes K$. By Theorem 2.3.5, the place ∞ of K does not split in E, which means E is a quadratic imaginary extension of K. Define the local *Gekeler ratio* at $\mathfrak{l}$ for each finite place $\mathfrak{l}$ of K to be

$$v_{\mathfrak{l}}(a,b) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{Mat}_2(A/\mathfrak{l}^n) : \mathrm{tr}(\gamma) \equiv a \mod \mathfrak{l}^n, \det(\gamma) \equiv b \mod \mathfrak{l}^n\}}{\#\mathrm{SL}_2(A_{\mathfrak{l}}/\mathfrak{l}^n)/|\mathfrak{l}|^{n(2-1)}}. \tag{3.1}$$

Gekeler defines an additional local factor at the infinite place in terms of the polynomial discriminant $\Delta := a^2 - 4b$ of $P_{\gamma_0}(X) = X^2 - aX + b$ by

$$v_\infty(a,b) = \left|\frac{\Delta}{b}\right|^{1/2} \begin{Bmatrix} \dfrac{2}{(q+1)(q-1)} \\ \dfrac{q^{-1/2}}{q-1} \end{Bmatrix} \text{ if } \infty \begin{Bmatrix} \text{is unramified in } E \\ \text{ramifies in } E \end{Bmatrix}. \tag{3.2}$$

Theorem 3.1.2. *[12, Theorem 8.17] Let $A/\mathfrak{p}$ be a finite field with characteristic other than two. Let ϕ be a rank two Drinfeld module over $A/\mathfrak{p}$ with characteristic polynomial $P_{\gamma_0}(X) = X^2 - aX + b$. Let $H(a,b,\mathfrak{p})$ be the isogeny class of ϕ. Then the weighted size of $H(a,b,\mathfrak{p})$ is*

$$h^*(a,b,\mathfrak{p}) = |\mathfrak{p}|^{1/2} v(a,b) v_\infty(a,b), \tag{3.3}$$

where

$$v(a,b) = \prod_{\mathfrak{l}} v_{\mathfrak{l}}(a,b)$$

is the product over primes of the local Gekeler ratios.

Remark 3.1.3. This is the statement of Gekeler's theorem assuming that $A/\mathfrak{p}$ has characteristic other than two. Gekeler addresses the case when $A/\mathfrak{p}$ has characteristic two separately.

The Frobenius endomorphism generates an *order* C in E. Let B be the integral closure of A in E. Then C is of the form

$$C = B_f = A + fB, \tag{3.4}$$

for some monic $f \in A$, called the *index* of C in B. Another order $B_{f'}$ contains B_f if and only if $f' \mid f$. Recall that a *fractional ideal* $\mathfrak{c}$ of C is a non-zero, finitely generated C-submodule of E. A fractional ideal $\mathfrak{c}$ of C is a *proper fractional ideal* if the set

$$\mathcal{M}(\mathfrak{c}) := \{x \in B : x\mathfrak{c} \subset \mathfrak{c}\}$$

coincides with C. We say two fractional ideals $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are *equivalent* if and only if there is some element $e \in E^\times$ such that $\mathfrak{c}_2 = e \cdot \mathfrak{c}_1$. Let $H(C)$ be the set of equivalence classes of fractional ideals of C with respect to this relation. Then $H(C)$ has finite cardinality $h(C)$, which is called the *class number* of C. Similarly let $H_{prop}(C)$ be the set of equivalence classes of proper fractional ideals of C. This set also has finite cardinality, denoted $h_{prop}(C)$. Note that the unit group $C^\times$ is finite of order is $(q^2 - 1)$ when $A/\mathfrak{p}$ contains $\mathbb{F}_{q^2}$ and $(q-1)$ otherwise.

Define the weighted number $h^*_{prop}(C)$ by

$$h^*_{prop}(C) = \frac{h_{prop}(C)}{\#C^\times}.$$

There is a relationship between $h^*_{prop}(C)$ and $h^*_{prop}(B)$ by

$$h^*_{prop}(C) = |f| \prod_{\mathfrak{l}|f} \left(1 - \chi(\mathfrak{l})|\mathfrak{l}|^{-1}\right) h^*_{prop}(B), \tag{3.5}$$

where χ is the *Dirichlet character* for E/K defined by

$$\chi(\mathfrak{v}) = \begin{cases} 1, \text{ if } \mathfrak{v} \text{ is split,} \\ -1 \text{ if } \mathfrak{v} \text{ is inert,} \\ 0 \text{ if } \mathfrak{v} \text{ is ramified.} \end{cases} \tag{3.6}$$

We also define the weighted class number $h^*(C)$ by

$$h^*(C) = \sum_{\mathfrak{c} \in H(C)} \frac{1}{\#\mathcal{M}(\mathfrak{c})}.$$

One can show

$$h^*(C) = \sum_{f'|f} h^*_{prop}(B_{f'}). \tag{3.7}$$

In combining Equations (3.5) and (3.7), we see that the weighted class numbers $h^*(C)$ can be written as a product over primes of local functions. The core of Gekeler's proof is to show that the weighted number

$$h^*(a, b, \mathfrak{p}) = \sum_{\phi \in H(a,b,\mathfrak{p})} \frac{1}{\#\mathrm{Aut}_{A/\mathfrak{p}}(\phi)} \tag{3.8}$$

is equal to $h^*(C)$ [12, Proposition 6.8]. Combining this fact with (3.5) and (3.7) implies

$$h^*(a, b, \mathfrak{p}) = S(f, B)h^*(B), \tag{3.9}$$

where

$$S(f,B) = \sum_{f'|f} |f'| \prod_{\mathfrak{l} \text{ prime, and } \mathfrak{l}|f'} \left(1 - \chi(\mathfrak{l})|\mathfrak{l}|^{-1}\right). \tag{3.10}$$

Gekeler then uses the analytic class number formula to give the size of an isogeny class in terms of a *Dirichlet L-Function.*

Definition 3.1.4. The *Dirichlet L-function* attached to χ is the product

$$L(s,\chi) = \prod_{\mathfrak{v}} L_{\mathfrak{v}}(s,\chi),$$

of local Dirichlet L-factors

$$L_{\mathfrak{v}}(s,\chi) := (1 - \chi(\mathfrak{v})|\mathfrak{v}|^{-s})^{-1}. \tag{3.11}$$

At $s = 1$, the product converges only conditionally.

The analytic class number formula says

$$h^*(B) = f(E_\infty/K_\infty) q^g L(1,\chi), \tag{3.12}$$

where $f(E_\infty/K_\infty)$ is the inertia degree of ∞ and g is the genus of (the associated algebraic curve of) E. Combining (3.12) with (3.9) implies that the weighted size of the isogeny class $H(a,b,\mathfrak{p})$ is

$$h^*(a,b,\mathfrak{p}) = f(E_\infty/K_\infty) q^g L(1,\chi) S(f,B). \tag{3.13}$$

We can break up the Dirichlet L-function and the factor $S(f,B)$ prime by prime. Let $f = \prod_{\mathfrak{l}} \mathfrak{l}^{m_{\mathfrak{l}}(f)}$ be the prime factorization of f. Then

$$S(f,B) = \prod_{\mathfrak{l}} S(\mathfrak{l}^{m_{\mathfrak{l}}(f)}, B).$$

For all but finitely many $\mathfrak{l}$, the contribution of $S(\mathfrak{l}^{m_{\mathfrak{l}}(f)}, B)$ is trivial. Then we have

$$h^*(a,b,\mathfrak{p}) = f(E_\infty/K_\infty)q^g L_\infty(1,\chi) \prod_{\mathfrak{l} \text{ prime of } A} S(\mathfrak{l}^{m_\mathfrak{l}(f)}, B) L_\mathfrak{l}(1,\chi). \tag{3.14}$$

Finally, Gekeler computes these local L-factors prime by prime and interprets them as the local density functions (3.1) and (3.2).

Remark 3.1.5. Gekeler's formula actually relies on the weighted number

$$h^*(a,b,\mathfrak{p}) := \sum_{\phi \in H(a,b,\mathfrak{p})} \frac{(q-1)}{\#\mathrm{Aut}_{A/\mathfrak{p}}(\phi)}.$$

As a result, his formula differs from the one stated here by a factor of $(q-1)$.

3.2 Generalizing to Higher Rank

Recently, Karemaker, Katen, and Papikian showed in [19, Theorem 5.3] that the number of isomorphism classes in an isogeny class of a Drinfeld module of any rank is bounded below by a sum of class numbers, lending credence to the idea that Gekeler's product formula should generalize to higher rank Drinfeld modules. However, we recover Gekeler's product formula and generalize it to higher rank Drinfeld modules without appealing to class numbers. Instead, our method compares Gekeler ratios[4] directly to orbital integrals. Another upshot of our method is that our result holds not only $A/\mathfrak{p}$, but also for any finite extension L over $A/\mathfrak{p}$. To accommodate this generalization, we will use the notation $H_\phi(L)$ to represent the isogeny class of the Drinfeld module ϕ over L and

$$h^*_\phi(L) = \sum_{\psi \in H_\phi(L)} \frac{1}{\#\mathrm{Aut}_L(\psi)} \tag{3.15}$$

for its weighted cardinality.

Finally, just as Gekeler's result requires separate treatment for fields with even characteristic, the computation of our global factor does also. We omit this case in the comparison between our formula and Gekeler's for the purposes of this text.

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[4] Or rather the generalization of Gekeler ratios to higher rank.

Chapter 4

Measures on Reductive Groups and Their Orbits

In this chapter, we describe the relationship between different choices of measures on the orbit of a regular, semisimple element γ_0 in $\mathrm{GL}_r(K_\mathfrak{v})$ where $\mathfrak{v}$ is an arbitrary place of K. This allows us to explain the adèlic orbital integrals of Laumon's formula in Chapter 5. More importantly, converting between different measures allows us to relate Laumon's adèlic orbital integrals to Gekeler ratios for higher rank Drinfeld modules in Chapter 6.

For locally compact topological groups, Haar measure assigns a translation invariant volume to subsets, unique up to scaling by a positive constant. Unfortunately, finding the correct conversion factors between two Haar measures on groups over non-Archimedean local fields is not always straightforward. One difficulty – addressed by Julia Gordon in [14] – arises from the two different ways to normalize a measure on such a group. One may either work with a measure arising from a specified differential form, or normalize a measure by specifying the volume of a specific compact subgroup. The crux of the argument of the main theorem in Chapter 6 relies on correctly converting between different normalizations of Haar measures on the orbit of a particular element in GL_r. One choice of measure on the orbit of γ_0 – the geometric measure – comes from the description of the orbit of γ_0 via its characteristic polynomial. Alternatively, we can think of the orbit of γ_0 as the quotient of GL_r by the centralizer of γ_0. Because γ_0 is regular and semisimple, its centralizer is an algebraic torus. In both cases, we must understand $\mathfrak{v}$-adic integration on the algebraic group GL_r. In the second case, we must also understand measures on algebraic tori.

4.1 Volumes as Point Counts

As a first example of a volume form on a variety over a local field, consider the affine line $\mathbb{A}^1$ over $K_\mathfrak{v}$. The choice of coordinate x gives the differential form of top degree $\mathrm{d}x$ and associated measure $|\mathrm{d}x|_\mathfrak{v}$ such that for any subset $S \subset \mathbb{A}^1$, we have

$$\mu_{|dx|_{\mathfrak{v}}}(S) = \int_S |dx|_{\mathfrak{v}}.$$

We declare that this measure gives volume one to the compact set $A_{\mathfrak{v}}$. This is analogous to the standard normalization of the measure dx on the affine line over $\mathbb{R}$ such that $\int_0^1 dx = 1$ [14].

Weil introduced p-adic integration on a d-dimensional variety $\mathbf{V}$. As in the above example, he defined a measure $|d\omega_{\mathbf{V}}|$ on $\mathbf{V}$ over a local field in terms of a non-vanishing top degree differential form. The volume of the set of integral points with respect to the measure $|d\omega_{\mathbf{V}}|$ is related to a point count. More generally, let $\mathcal{X}$ be a *smooth* scheme over $A_{\mathfrak{v}}$ and denote the residue field of $A_{\mathfrak{v}}$ by $A/\mathfrak{v}$. In the introduction, we declared $|\mathfrak{v}| := \#A/\mathfrak{v}$. Let ω be and a top degree non-vanishing differential form over $A_{\mathfrak{v}}$. Then the volume of $\mathcal{X}(A)$ with respect to the associated measure $|d\omega|_{\mathfrak{v}}$ is

$$\mathrm{vol}_{|d\omega|_{\mathfrak{v}}}(\mathcal{X}(A)) = \frac{\#\mathcal{X}(\kappa_{K_{\mathfrak{v}}})}{|\mathfrak{v}|^{\dim(\mathcal{X})}}. \tag{4.1}$$

Thanks to Serre, Oesterlé, and Veys a similar relationship between volumes and point counts applies to not necessarily smooth sets [32]. In particular, let Y be a closed analytic subset of $\mathbb{Z}_p^r$ of dimension d and Y_d be the open set of smooth points of Y of dimension d. Serre and Oesterlé showed that there is a unique measure μ^{SO} (we call this the Serre-Oesterlé measure) on Y which is concentrated on Y_d. Let $U \subset Y_d$ be an open ball. The measure μ^{SO} is such that for any bianalytic isometric bijection φ from U to a ball B of $\mathbb{Z}_p^r$, the restricted measure $\mu^{SO}|_U$ is the image of $\mu|_B$ by φ^{-1} where μ is the normalized Haar measure on $\mathbb{Z}_p^d$. It turns out that for closed, but not necessarily smooth subsets, the following result of Oesterlé holds.

Theorem 4.1.1 (Oesterlé, as stated in [32])**.** *Let Y be a closed, analytic subset of $\mathbb{Z}_p^r$ of dimension d. Take $N_n(Y)$ to be the cardinality of the image of Y under the natural map $\mathbb{Z}_p^r \to (\mathbb{Z}/p^n\mathbb{Z})^r$. Then*

$$\lim_{n\to\infty} \frac{N_n(Y)}{p^{nd}} = \mu^{SO}(Y).$$

Veys generalized (4.1.1) to arbitrary subanalytic sets in $\mathbb{Z}_p^r$ [32, Theorem 2.2]. We may replace $\mathbb{Z}_p$ in Theorem 4.1.1 with any complete local field.

Because volumes and point counts are intimately related, we recall Hensel's Lemma for complete rings, which allows us to lift roots of a polynomial modulo a prime to roots modulo higher powers of the same prime. The proof, along with the generalization to non-smooth systems of equations, is included in Appendix B.

Lemma 4.1.2 (Hensel's Lemma)**.** *Let $\mathcal{X}$ be a smooth scheme of dimension d, over a complete local ring $A_\mathfrak{v}$, with maximal ideal $\mathfrak{v}$ and residue field $A/\mathfrak{v}$. If $|\mathfrak{v}| = \#A/\mathfrak{v}$, then each point modulo $\mathfrak{v}^n$ lifts to exactly $|\mathfrak{v}|^d$ solutions modulo $\mathfrak{v}^{n+1}$. That is*

$$\#\mathcal{X}(A/\mathfrak{v}^{n+1}A) = |\mathfrak{v}|^{nd}\#\mathcal{X}(A/\mathfrak{v}A). \tag{4.2}$$

4.1.1 Reductive Groups

As earlier stated, for locally compact topological groups, the choice of measure (or rather normalization of the Haar measure) depends on the choice of differential form or a compact subgroup. Locally compact groups have a *canonical* choice – for example, $A_\mathfrak{v}$ is the canonical compact subgroup of the affine line over $K_\mathfrak{v}$. However, in a general reductive group, there may be more than one conjugacy class of maximal compact subgroup, each giving rise to different normalizations [14, 2.3].

Even still, Gross defines a "canonical" top-degree differential form ω_{GL_r} on a reductive group G over $K_\mathfrak{v}$ analogous to the differential form dx on the affine line above [16]. When G is *split* over the local field $K_\mathfrak{v}$, integrating the corresponding measure $|d\omega_{\mathrm{GL}_r}|_\mathfrak{v}$ gives

$$\int_{\mathrm{GL}_r(A_\mathfrak{v})} |d\omega_{\mathrm{GL}_r}|_\mathfrak{v} = \frac{\#\mathrm{GL}_r(\kappa_{K_\mathfrak{v}})}{|\mathfrak{v}|^{\dim(G)}}, \tag{4.3}$$

which coincides with the Serre-Oesterlé measure for smooth schemes. Observe that $|d\omega_{\mathrm{GL}_r}|_\mathfrak{v}$ has not been normalized to give volume one to the integral points, an important point in Proposition 6.3.1.

4.1.2 Tori

An algebraic torus is a reductive group with a canonical compact subgroup that can be used to normalize the Haar measure.

Definition 4.1.3. Let K be any field with algebraic closure $\overline{K}$. An *algebraic torus* over K is an algebraic group $\mathbf{T}$ defined over K, which over $\overline{K}$ is isomorphic to a finite number of copies of the multiplicative group. That is, there exists a number $s \geq 1$ such that as algebraic groups,

$$\mathbf{T}_{\overline{K}} \cong \mathbb{G}_{m,\overline{K}}^s.$$

We call s the *the rank of* $\mathbf{T}$. For an extension $K \subset E$ of K, we say that $\mathbf{T}$ is E-split if

$$\mathbf{T}_E = \mathbb{G}_{m,E}^s.$$

Let $\mathbf{T}$ be an arbitrary torus with rank s over $K_\mathfrak{v}$. To define a differential form on $\mathbf{T}$, we start with a choice of coordinates on $\mathbf{T}$. Equivalently, we choose a basis of the character group $X^*(\mathbf{T})$. Recall that the character group for an arbitrary reductive group is defined in the following way.

Definition 4.1.4. Let G be a reductive group over an arbitrary field K. Then the *character group* $X^*(G)$ of G is the group of representations

$$X^*(G) = \{\chi : G_{\overline{K}} \to \mathbb{G}_{m,\overline{K}}\}.$$

For the rank s torus $\mathbf{T}$, the character group $X^*(\mathbf{T})$ is generated by s characters $\chi_1, \cdots, \chi_s$. If $\mathbf{T}$ is $K_\mathfrak{v}$-split, this set of coordinates is defined over $K_\mathfrak{v}$, otherwise it is defined over the splitting field of $\mathbf{T}$. The top degree non-vanishing differential form is

$$\omega_\mathbf{T} = \frac{d\chi_1}{\chi_1} \wedge \cdots \wedge \frac{d\chi_s}{\chi_s}. \tag{4.4}$$

The unique maximal compact subgroup of $\mathbf{T}(K_\mathfrak{v})$ is

$$\mathbf{T}^c = \{t \in \mathbf{T}(K_\mathfrak{v}) : |\chi(t)|_\mathfrak{v} = 1 \text{ for } \chi \in X^*(\mathbf{T})_{K_\mathfrak{v}}\}.$$

The $\mathfrak{v}$-adic topology implies that $\mathbf{T}^c = \mathbf{T}(A_\mathfrak{v})$. In [14], Gordon explains the volume of $\mathbf{T}^c$ with respect to the measure $|d\omega_\mathbf{T}|_\mathfrak{v}$ coming from this differential form.

When $\mathbf{T}$ is the *restriction of scalars* of $\mathbb{G}_m$, the volume depends on the ramification of the extension. We recall the definition of restriction of scalars here.

Definition 4.1.5. Let E/K be a finite extension of fields and V a scheme over E. Then $\mathrm{Res}_{E/K}(V)$ is a functor from the category of K-schemes to the category of sets. Explicitly, let S be a K-scheme. Then

$$\mathrm{Res}_{E/K}(V)(S) = V(S \times_K E).$$

Under certain assumptions, $\mathrm{Res}_{E/K}(V)$ is representable. The variety that represents this functor is called the *restriction of scalars* of V. The restriction of scalars of a commutative group variety $\mathbb{G}$ is a commutative group variety over K of dimension $[E : K] \dim_E(\mathbb{G})$. If $\mathbb{G}$ is a torus, $\mathrm{Res}_{E/K}(\mathbb{G})$ is again a torus. Explicitly, the restriction of scalars $\mathbf{T} = \mathrm{Res}_{E/K}(\mathbb{G}_m)$ is a torus over K of dimension $[E : K]$, and $\mathrm{Res}_{E/K}(\mathbb{G}_m)(K) \cong E^\times$.

Let $E/K_\mathfrak{v}$ be a degree two extension and $\mathbf{T} = \mathrm{Res}_{E/K}(\mathbb{G}_m)$. The following result allows us to compute $\mathrm{vol}_{\omega_\mathbf{T}}(\mathbf{T}^c)$ explicitly, provided that the characteristic of K is not two. We use this result in Section 6.7 to compare our formula for the size of an ordinary isogeny class with Gekeler's.

Lemma 4.1.6. *[14, Example 2.3] Let $E/K_{\mathfrak{v}}$ be a degree two extension of fields having characteristic other than two. Let $|\mathfrak{v}|$ be the size of the residue field $\kappa_{K_{\mathfrak{v}}}$. Finally, let $\mathbf{T} := Res_{E_{\mathfrak{v}}/K_{\mathfrak{v}}}(\mathbb{G}_m)$. Then*

$$\mathrm{vol}_{|\omega_{\mathbf{T}}|_{\mathfrak{v}}}(\mathbf{T}^c) = \begin{cases} \left(\frac{|\mathfrak{v}|-1}{|\mathfrak{v}|}\right)^2 & \mathbf{T} \textit{ split}, \\ \frac{(|\mathfrak{v}|-1)(|\mathfrak{v}|+1)}{|\mathfrak{v}|^2} & \mathbf{T} \textit{ non-split unramified}, \\ \frac{(|\mathfrak{v}|-1)}{|\mathfrak{v}|^{3/2}} & \mathbf{T} \textit{ non-split ramified}. \end{cases} \tag{4.5}$$

More generally, the volume of $\mathbf{T}$ is related to the *Artin L-function*. We include the definition here because it also appears in the *Tamagawa measure* in Chapter 6. Suppose for a moment that $\mathbf{T}$ is an arbitrary torus over $K_{\mathfrak{v}}$ with splitting field E. The Galois group $\mathrm{Gal}(E/K_{\mathfrak{v}})$ naturally acts on the group of characters $X^*(\mathbf{T})$. The Galois group also acts transitively on the set of equivalence classes of places in E over $\mathfrak{v}$. For a fixed extension $\mathfrak{w}$ of $\mathfrak{v}$ to E, the *decomposition group* is the subgroup $\mathrm{Gal}_{\mathfrak{w}} \subseteq \mathrm{Gal}(E/K_{\mathfrak{v}})$ that stabilizes the equivalence class $[\mathfrak{w}]$. The *inertia* group is the subgroup $I_{\mathfrak{w}} \subseteq \mathrm{Gal}_{\mathfrak{w}}$ of elements that act trivially on the residue field $\kappa_{E_{\mathfrak{w}}}$. Then there is an exact sequence

$$1 \to I_{\mathfrak{w}} \to \mathrm{Gal}(E/K_{\mathfrak{v}}) \to \mathrm{Gal}(\kappa_{E_{\mathfrak{w}}}/\kappa_{K_{\mathfrak{v}}}) \to 1,$$

which gives a natural action of $\mathrm{Gal}(\kappa_{E_{\mathfrak{w}}}/\kappa_{K_{\mathfrak{v}}})$ on the set $X^*(\mathbf{T})^{I_{\mathfrak{w}}}$ of inertia invariants of the character lattice. Since the residue fields $\kappa_{E_{\mathfrak{w}}}$ and $\kappa_{K_{\mathfrak{v}}}$ are finite, the Galois group $\mathrm{Gal}(\kappa_{E_{\mathfrak{w}}}/\kappa_{K_{\mathfrak{v}}})$ is cyclic and generated by the Frobenius automorphism $\mathrm{Fr}_{\mathfrak{w}}$ of $\kappa_{E_{\mathfrak{w}}}$. Let $d_I := \mathrm{rank}(X^*(\mathbf{T})^{I_{\mathfrak{w}}})$. The action of interest is given by the representation

$$\sigma_{\mathbf{T}} : \mathrm{Gal}(\kappa_{E_{\mathfrak{w}}}/\kappa_{K_{\mathfrak{v}}}) \to \mathrm{Aut}_{\mathbb{Z}}(X^*(\mathbf{T})^{I_{\mathfrak{w}}}) \cong \mathrm{GL}_{d_I}(\mathbb{Z}). \tag{4.6}$$

Definition 4.1.7. The *Artin L-factor* at $\mathfrak{v}$ associated with the representation $\sigma_{\mathbf{T}}$ is

$$L_{\mathfrak{v}}(s,\sigma_{\mathbf{T}}) = \det\left(I_{d_I} - \frac{\sigma_{\mathbf{T}}(\mathrm{Fr}_{\mathfrak{w}})}{|\mathfrak{v}|^s}\right)^{-1}, \tag{4.7}$$

where I_{d_I} is the identity matrix of size d_I. The *Artin L-function* is the product

$$L(s,\sigma_{\mathbf{T}}) = \prod_{\mathfrak{v}} L_{\mathfrak{v}}(s,\sigma_{\mathbf{T}}).$$

In a general setting such as this, there is another canonical compact subgroup T^0 of $\mathbf{T}$. The subgroup T^0 is defined by the *(weak) Néron model* of $\mathbf{T}$, a scheme $\mathcal{T}$ over $A_{\mathfrak{v}}$ with generic fiber $\mathbf{T}$. Then the $A_{\mathfrak{v}}$-points of the identity component provide a canonical compact subgroup T^0 of $\mathbf{T}(K_{\mathfrak{v}})$. Equipped with this model, Gross defines a canonical form $\omega_{\mathbf{T}}^{can}$ on $\mathbf{T}$, which in general differs from $\omega_{\mathbf{T}}$, as described by Gross. This canonical measure is such that

$$\mathrm{vol}_{|d\omega_{\mathbf{T}}^{can}|}(T^0) = \frac{\#\mathcal{T}^0_{\kappa_{K_{\mathfrak{v}}}}(\mathbb{F}_q)}{|\mathfrak{v}|^{\dim(\mathbf{T})}} = L_{\mathfrak{v}}(1,\sigma_T)^{-1}. \tag{4.8}$$

We use this fact to obtain the volume of $\mathbf{T}^c$ with respect to the measure $|\omega_{\mathbf{T}}|_{\mathfrak{v}}$. There are two cases: either $\mathbf{T}$ splits over an *unramified* extension of $K_{\mathfrak{v}}$, or it splits over a ramified extension. In the former case, the subgroups T^0 and $\mathbf{T}^c$ coincide, so

$$\mathrm{vol}_{|d\omega_{\mathbf{T}}|}(\mathbf{T}^c) = \mathrm{vol}_{|d\omega_{\mathbf{T}}^{can}|}(T^0) = \frac{\#\mathcal{T}^0_{\kappa_{K_{\mathfrak{v}}}}(\mathbb{F}_q)}{|\mathfrak{v}|^{\dim(\mathbf{T})}} = L_{\mathfrak{v}}(1,\sigma_T)^{-1}. \tag{4.9}$$

In the latter case, when $\mathbf{T}$ splits over a ramified extension of $K_{\mathfrak{v}}$, T^0 is a finite index subgroup in $\mathbf{T}^c$. Bitan computes this index in [4]. In this setting, we also need to multiply the canonical measure by a factor of $\sqrt{\Delta_{E/K}}$ to obtain the measure $d\omega_{\mathbf{T}}$. Together, this implies

$$\mathrm{vol}_{|d\omega_{\mathbf{T}}|}(\mathbf{T}^c) = [\mathbf{T}^c : T^0]\sqrt{|\Delta_{E/K}|}\,\mathrm{vol}_{|d\omega_{\mathbf{T}}^{can}|}(T^0). \tag{4.10}$$

4.2 Measures on Orbits of GL_r

In this section, we describe two different methods for obtaining a measure on the orbit of an element γ in GL_r. As earlier mentioned, the first method is to identify the orbit of γ with the quotient of GL_r by the centralizer of γ. Explicitly, GL_r acts on itself by conjugation. By definition, this action is transitive on the conjugacy class, i.e., orbit, of γ. Thus, the conjugacy class is identified with the quotient of GL_r by the stabilizer of γ; however, this stabilizer is the centralizer of γ. The second method works well for regular semisimple elements; the orbits of such elements are determined by characteristic polynomial. We can construct a measure on such an orbit using this identification. The resulting measure, called the geometric measure, is a convenient choice for the purposes of this project because the local density functions in the style of Gekeler are written in terms of characteristic polynomial.

We discuss only the setting for particular assumptions about γ needed in this work; it is a regular semisimple element of GL_r, its centralizer is the restriction of scalars torus $\mathbf{T} = \mathrm{Res}_{E/K}(\mathbb{G}_m)$ for a degree r extension of K, and its characteristic polynomial has the desired Newton polygon structure. We continue to work over the completion $K_{\mathfrak{v}}$ for an arbitrary place $\mathfrak{v}$ of K. As a set, the orbit of $\gamma \in \mathrm{GL}_r(K_{\mathfrak{v}})$ is $\mathbf{T}(K_{\mathfrak{v}})\backslash\mathrm{GL}_r(K_{\mathfrak{v}})$.

Definition 4.2.1. Let $|d\omega_{\mathrm{GL}_r}|$ and $|d\omega_{\mathbf{T}}|$ be the measures from the respective top-degree non-vanishing differential forms on GL_r and $\mathbf{T}$ normalized as above. Then the quotient measure

$$|d\omega_{\mathbf{T}\backslash\mathrm{GL}_r}| := \frac{|d\omega_{\mathrm{GL}_r}|}{|d\omega_{\mathbf{T}}|} \tag{4.11}$$

is a measure on the orbit of γ identified with the quotient space $\mathbf{T}(K_{\mathfrak{v}})\backslash\mathrm{GL}_r(K_{\mathfrak{v}})$.

4.2.1 The Geometric Measure

The geometric measure is constructed by decomposing the space $\mathrm{GL}_r(K_{\mathfrak{v}})$ according to characteristic polynomials. Define $c : \mathrm{GL}_r(K_{\mathfrak{v}}) \to C(K_{\mathfrak{v}})$ to be the characteristic polynomial map, where $C \cong \mathbb{A}^{r-1} \times \mathbb{G}_m$ is the space of characteristic polynomials. The canonical differential form ω_C on C corresponds to the product measure

$$|d\omega_C| = |dx_1|_{\mathfrak{v}} \wedge \cdots \wedge |dx_{r-1}|_{\mathfrak{v}} \wedge \frac{|dt|_{\mathfrak{v}}}{|t|_{\mathfrak{v}}}.$$

For γ_0 regular and semisimple, the inverse image $c^{-1}(c(\gamma_0))$ is the orbit of γ_0 under conjugation because, in GL_r, two regular elements are conjugate if and only if they have the same characteristic polynomial. We define the differential form $\omega^{geom}_{c^{-1}(c(\gamma_0))}$ on the orbit so that the differential form ω_{GL_r} on GL_r factors as the wedge product

$$\omega_{\mathrm{GL}_r} = \omega^{geom}_{c^{-1}(c(\gamma_0))} \wedge \omega_C. \tag{4.12}$$

Integrating $|\omega^{geom}_{c^{-1}(c(\gamma_0))}|$ defines a measure μ^{geom}_γ on the orbit of γ_0 in GL_r. Let

$$P_{\gamma_0}(X) = X^r - a_{r-1}X^{r-1} + \cdots + (-1)^k a_{r-(1+k)} X^{r-(1+k)} + \cdots (-1)^r a_0$$

be the characteristic polynomial of γ_0. For each $n \geq 1$, define a subset $\widetilde{U}_n(\gamma_0) \subset C$ by

$$\widetilde{U}_n(\gamma_0) := \{(b_{r-1}, \cdots, b_0) \in C : b_i \equiv a_i \mod \mathfrak{v}^n\}.$$

With respect to the $\mathfrak{v}$-adic topology, $\widetilde{U}_n(\gamma_0)$ is a neighborhood of $c(\gamma_0)$. Then, $c^{-1}(\widetilde{U}_n(\gamma_0))$ is a neighborhood of the orbit of γ_0. Let $B \subseteq \mathrm{Mat}_r(A_{\mathfrak{v}}) \cap \mathrm{GL}_r(K_{\mathfrak{v}})$ be an open subset. Theorem 4.1.1 together with (4.12) gives that the volume of $B \cap c^{-1}(c(\gamma_0))$ is

$$\mathrm{vol}_{|\mu^{geom}_\gamma|}(B \cap c^{-1}(c(\gamma_0)) = \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(B \cap c^{-1}(\widetilde{U}_n))}{\mathrm{vol}_{|d\omega_C|}(\widetilde{U}_n)}. \tag{4.13}$$

4.2.2 Orbital Integrals and Moving Between Measures

The present task is to explain how orbital integrals for these two measures differ.

Definition 4.2.2. Let $\star \in \{\mu^{geom}_\gamma, |d\omega_{\mathbf{T}\backslash\mathrm{GL}_r}|\}$ be a measure on the orbit of a regular semisimple element $\gamma \in \mathrm{GL}_r(K_{\mathfrak{v}})$ and f a function on $\mathrm{GL}_r(K_{\mathfrak{v}})$ with compact support. Then the *orbital integral of f with respect to this measure is*

$$O_\gamma^\star(f) = \int_{\mathbf{T}(K_\mathfrak{v})\backslash \mathrm{GL}_r(K_\mathfrak{v})} f(h^{-1}\gamma h)\star(h).$$

The relationship between measures depends on the specific orbit of interest. In general, the measures differ by a factor involving the *Weyl discriminant*. The Weyl discriminant $D(\gamma_0)$ of a regular semisimple element γ_0 of a reductive group over an arbitrary field K is defined in terms of its eigenvalues $\{\lambda_i \in \overline{K}\}$ in the standard representation. For GL_r, we have

$$D(\gamma_0) := \prod_{1\leq i,j\leq r, i\neq j} \left(1 - \frac{\lambda_i}{\lambda_j}\right). \tag{4.14}$$

With this definition, we can rewrite the Weyl discriminant in terms of the *polynomial discriminant* Δ_{γ_0} of the characteristic polynomial $P_{\gamma_0}(X)$ of γ_0. Since P_{γ_0} is monic with roots $\{\lambda_i \in \overline{K}\}$, by definition, $\Delta_{\gamma_0} = (-1)^{\frac{r(r-1)}{2}} \prod_{i\neq j}(\lambda_j - \lambda_i)$. Therefore,

$$\begin{aligned} D(\gamma_0) &= \prod_{1\leq i,j\leq r, i\neq j} \left(1 - \frac{\lambda_i}{\lambda_j}\right) = \prod_{i\neq j} \frac{1}{\lambda_j} (\lambda_j - \lambda_i) \\ &= \prod_{i\neq j} \frac{1}{\lambda_j} (\lambda_j - \lambda_i) = \prod_{i\neq j} \frac{1}{\lambda_j} \prod_{i\neq j} (\lambda_j - \lambda_i) \\ &= \prod_j \left(\frac{1}{\lambda_j}\right)^{(r-1)} \prod_{i\neq j} (\lambda_j - \lambda_i). \end{aligned} \tag{4.15}$$

Since $\prod_j (\lambda_j) = \det(\gamma_0)$, we have

$$D(\gamma_0) = \det(\gamma_0)^{-(r-1)} (-1)^{-\frac{r(r-1)}{2}} \Delta_{\gamma_0}. \tag{4.16}$$

To convert between the two orbital integrals, we use the relationship

$$O_{\gamma_0}^{geom}(f) = |D(\gamma_0)|_\mathfrak{v}^{1/2} |\det(\gamma_0)|_\mathfrak{v}^{-\frac{r-1}{2}} \cdot O_{\gamma_0}^{d\omega_{\mathbf{T}\backslash G}}(f), \tag{4.17}$$

as explained in [14, Example 3.17] for the group GSp_{2r}.

Remark 4.2.3. We use (4.17) in the proof of Proposition 6.3.1. In that setting, there is a factor of $\mathrm{vol}_{\mu^{SO}_{\mathrm{GL}_r}}(\mathrm{GL}_r(A_{\mathfrak{v}}))$ we omit here because we are working with an unnormalized measure on GL_r.

Chapter 5

Laumon's Orbital Integral Formula

The broad motivating force behind Langlands' program is the relationship between classical algebraic number theory, such as Artin's reciprocity and Galois groups of number fields, with the theory of automorphic forms. One known result in Langlands' program is that the cohomology of Shimura varieties is related to automorphic forms over number fields. Many Shimura varieties parameterize isomorphism classes of abelian varieties with additional data, and all Shimura varieties are determined by a reductive group. In the case of elliptic curves, Shimura varieties are classical modular curves. Similarly, there is a Drinfeld modular variety, controlled by the reductive group $\mathrm{GL}_r(K)$, where K is a function field over a finite field. The proof that the cohomology of Shimura varieties or Drinfeld modular varieties can be given in terms automorphic forms (developed by Ihara, Langlands, Kottwitz, and others for abelian varieties, and Drinfeld and Laumon for Drinfeld modules) is outlined by the following steps. First, use the Grothendieck-Lefschetz trace formula to express the trace of the virtual modules coming from the cohomology group of the modular variety in terms of a number of fixed points. Second, compare the fixed points with the geometric side of the Arthur-Selberg trace formula. Third, compute the spectral side of the Arthur-Selberg trace formula explicitly. Fourth, the identification between the two sides gives the desired description in terms of automorphic forms. The result we are interested – that the size of an isogeny class can be expressed in terms of orbital integrals – is a key tool used along the way. Specifically, in Chapter Three of [23], Laumon uses Drinfeld's description of the set of rank r Drinfeld modules of a given positive characteristic to obtain a formula for the number of fixed points of the action of (a power of) Frobenius and a Hecke correspondence on this set in terms of an adèlic orbital integral and a twisted orbital integral. The "Fundamental Lemma" allows Laumon to replace the twisted orbital integral with and ordinary one, although we state Laumon's formula with twisted orbital integrals. Drinfeld and Laumon gave their descriptions of an isogeny class with *level structure*, but following

Langlands and Kottwitz for abelian varieties, we explain the adèlic orbital integral formula with the trivial level structure.

5.0.1 The Adèles

Let $\mathbb{A}$ denote the adèles of K. The adèle ring of the function field K is the (restricted) product over places $\mathfrak{v}$ of completions $K_\mathfrak{v}$ with respect to $A_\mathfrak{v}$. That is, $\mathbb{A}$ is the set of tuples $(a_\mathfrak{v})$ where $a_\mathfrak{v}$ is in $K_\mathfrak{v}$ for all places of K and in $A_\mathfrak{v}$ for almost all places. With the usual adèlic topology, generated by restricted open neighborhoods $\prod_{\mathfrak{v}\in V} U_\mathfrak{v} \times \prod_{\mathfrak{v}\notin V} A_\mathfrak{v}$ as V ranges over finite sets of places of K, $\mathbb{A}$ is a compact ring. The topological group $\mathrm{GL}_r(\mathbb{A})$ is the set of (tuples of) $r \times r$ matrices $(g_\mathfrak{v})$ where $g_\mathfrak{v}$ is a matrix in $\mathrm{GL}_r(K_\mathfrak{v})$ for all $\mathfrak{v}$ and in $\mathrm{GL}_r(A_\mathfrak{v})$ for all but finitely many places. The topology on $\mathrm{GL}_r(\mathbb{A})$ is induced by the topology on $\mathbb{A}$. The ring of finite adèles $\mathbb{A}_f$ is defined in the same way, excluding the place at infinity. Define an open compact subring of $\mathbb{A}_f$ by $\hat{\mathbf{A}} := \prod_\mathfrak{l} A_\mathfrak{l}$. Similarly, let $\mathbb{A}_f^\mathfrak{p}$ be the finite prime to $\mathfrak{p}$ adèles, and $\hat{\mathbf{A}}^\mathfrak{p}$ its ring of integers.

Recall the notation from Section 2.2.2: $K_\mathfrak{p}^{\mathrm{un}}$ is the maximal unramified extension of $K_\mathfrak{p}$ and $A_{\overline{A/\mathfrak{p}}}$ is the completion of its ring of integers with $A_{\overline{A/\mathfrak{p}}}/\mathfrak{p}A_{\overline{A/\mathfrak{p}}} = \overline{A/\mathfrak{p}}$. Recall also that A_L is the ring of integers of the unique unramified degree m extension of $K_\mathfrak{p}$ with residue field L and field of fractions K_L. Finally, recall that F is the arithmetic Frobenius automorphism of L with lift $F_\mathfrak{p}$ in $\mathrm{Gal}(K_L : K_\mathfrak{p})$.

5.1 Laumon's Formula

In Theorems 2.6.1 and 2.3.5, we saw that the isogeny class of a Drinfeld module ϕ over a finite field extension L of $A/\mathfrak{p}$ is exactly determined by its endomorphism algebra $E = \mathrm{End}(\phi) \otimes K$, and that the center $\tilde{K} = K(F)$ of E is a field generated by the Frobenius endomorphism of ϕ. We also saw that there are unique places $\tilde{\mathfrak{p}}$ and $\tilde{\infty}$ lying above $\mathfrak{p}$ and ∞ of K in $\tilde{K}$. Finally, recall that the field L has $|\mathfrak{p}|^m = q^{m\deg(\mathfrak{p})}$ elements. The next proposition and corollary provide a convenient algebraic method for indexing isogeny classes of Drinfeld modules. First, we need the following definition.

Definition 5.1.1. We say a pair $(\tilde{K}, \Pi)$ where $\tilde{K}$ is a finite field extension of K and $\Pi \in \tilde{K}$ is a *Weil* $(K, \infty, \mathfrak{p})$*-pair of rank* r *over the finite field* L if

(a) $[\tilde{K} : K]$ divides r,

(b) $\tilde{K} \otimes K_\infty$ is a field,

(c) if $|\cdot|_\infty$ is the natural extension of $|\cdot|_\infty$ on K_∞ to $\tilde{K} \otimes K_\infty$, we have $|\Pi|_\infty = (q^{m \deg(\mathfrak{p})})^{1/r}$,

(d) there exists one and only one place $\tilde{\mathfrak{p}} \neq \tilde{\infty}$ of $\tilde{K}$ such that $\mathrm{ord}_{\tilde{\mathfrak{p}}}(\Pi) \neq 0$ and moreover $\tilde{\mathfrak{p}}$ divides $\mathfrak{p}$ (and $\tilde{\infty}$ is the unique place of $\tilde{K}$ over ∞),

(e) $\tilde{K}$ is generated by Π over K.

Proposition 5.1.2. *([23, Proposition 2.2.2.iv]) The map from the set of isogeny classes of Drinfeld* A*-modules of rank* r *over* L *to the set of isomorphism classes of Weil* $(K, \infty, \mathfrak{p})$*-pairs of rank* r *over* L *which maps the isogeny class of* ϕ *to the isomorphism class of* $(K(F), F)$ *for the Frobenius endomorphism* F *of* ϕ *is bijective.*

With this proposition, we can name an isogeny class by naming an appropriate Weil $(K, \infty, \mathfrak{p})$-pair. However, this identification is not sufficient for obtaining the double coset description of an isogeny class we need. To each Weil $(K, \infty, \mathfrak{p})$-pair of rank r over a finite extension of $A/\mathfrak{p}$, we can attach a $(K, \infty, \mathfrak{p})$*-type,* which is defined as follows.

Definition 5.1.3. Let $(\tilde{K}, \tilde{\mathfrak{p}})$ be a pair where $\tilde{K}$ is any finite field extension of K and $\tilde{\mathfrak{p}}$ is a place of $\tilde{K}$ dividing $\mathfrak{p}$. We say that $(\tilde{K}, \tilde{\mathfrak{p}})$ is a $(K, \infty, \mathfrak{p})$*-type of rank* r if

(a) $[\tilde{K} : K]$ divides r;

(b) $\tilde{K} \otimes_K K_\infty$ is a field,

(c) $\tilde{K} = K(\Pi)$ for each integral element $\Pi \in \tilde{K}$ such that Π only has a zero at $\tilde{\mathfrak{p}}$, i.e., $\mathrm{ord}_{\tilde{\infty}}(\Pi) < 0$, $\mathrm{ord}_{\tilde{\mathfrak{p}}}(\Pi) > 0$, $\mathrm{ord}_{\tilde{\mathfrak{o}}}(\Pi) = 0$ for any place $\tilde{\mathfrak{o}} \neq \tilde{\infty}, \tilde{\mathfrak{p}}$. (Note that $\tilde{\infty}$ is the only place of $\tilde{K}$ over ∞.)

Let $(\tilde{K}, F)$ be a Weil-$(K, \infty, \mathfrak{p})$-pair of rank r over a finite extension L of $A/\mathfrak{p}$ (perhaps coming from a Drinfeld module with Frobenius endomorphism F). For each integer $s \geq 1$, consider the field intermediate field $K \subset K(F^s) \subset \tilde{K}$. Define $\tilde{K}'$ by

$$\tilde{K}' := \cap_s K(F^s).$$

Then the $(K, \infty, \mathfrak{p})$-type of rank r attached to $(\tilde{K}', F)$ is the pair $(\tilde{K}', \tilde{\mathfrak{p}}')$, where $\tilde{\mathfrak{p}}'$ is the place of $\tilde{K}'$ induced by $\tilde{\mathfrak{p}}$. The argument that this is indeed a $(K, \infty, \mathfrak{p})$-type is simple but irrelevant to the current exposition. We refer interested readers to [23, Page 27].

Corollary 5.1.4. *([23, Corollary 2.2.3]) There is a bijection between the set of isogeny classes of rank r Drinfeld modules over $\overline{A/\mathfrak{p}}$ and the set of isomorphism classes of $(K, \infty, \mathfrak{p})$-types of rank r. Moreover, if $(\tilde{F}, \tilde{\mathfrak{p}})$ is the $(K, \infty, \mathfrak{p})$ type associated to a Drinfeld module ϕ of rank r over $A/\mathfrak{p}$, then $\mathrm{End}(\phi) \otimes_K K$ is the unique central division algebra over $\tilde{K}$ whose non-zero invariants are $-[\tilde{K} : K]/r$ at $\tilde{\infty}$ and $[\tilde{K} : K]/r$ at $\mathfrak{p}$.*

Let ϕ be a Drinfeld module over any finite field $\tilde{L}$ containing L. We can (bijectively) attach ϕ to the Weil $(K, \infty, \mathfrak{p})$-pair $(K(F), F)$ of rank r over $\tilde{L}$. This pair has the associated $(K, \infty, \mathfrak{p})$-type $(\tilde{K}, \tilde{\mathfrak{p}})$ as above. The isomorphism class of $(\tilde{F}, \tilde{\mathfrak{p}})$ is determined by the isogeny class of ϕ. Indexing by $(K, \infty, \mathfrak{p})$-type not only gives a convenient naming scheme for isogeny classes but also allows us to describe them via fixed points of the correct action on a double coset space.

Let $H_{(\tilde{K},\tilde{\mathfrak{p}})}(L)$ be the set of isomorphism classes of rank r Drinfeld A-modules over L in the isogeny class corresponding to the type $(\tilde{K}, \tilde{\mathfrak{p}})$. Laumon expresses the weighted cardinality

$$h^*_{(\tilde{K},\tilde{\mathfrak{p}})}(L) := \sum_{\psi \in H_{(\tilde{K},\tilde{\mathfrak{p}})}(L)} \frac{1}{\#\mathrm{Aut}_L(\psi)} \tag{5.1}$$

of the isogeny class associated to $(\tilde{K}, \tilde{\mathfrak{p}})$ in terms of adèlic orbital integrals. Chapters 7 and 8 of [9] also provide a detailed outline of how to obtain the adèlic orbital integral formula with level structure. Section 3 of [19] repackages parts of Laumon's description for Drinfeld modules with trivial level structure. In the case of abelian varieties, we refer to [29, Chapter 5], which is based on Kottwitz's formula in [20, 21].

Laumon begins by describing an isogeny class over $\overline{A/\mathfrak{p}}$ as a double coset space in $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})$. Fix a rank r Drinfeld module ϕ_0 with isogeny class corresponding to type $(\tilde{K}, \tilde{\mathfrak{p}})$ by specifying the endomorphism algebra $E = \mathrm{End}(\phi_0) \otimes_A K$. Let E^{opp} be its opposite ring. Let $F_{\mathfrak{p}}$ be the Frobenius map on $\mathrm{Gal}(K_L : K_{\mathfrak{p}})$ as in Section 2.2.2. Let $\mathrm{Frob}_{\mathfrak{p}}$ be the geometric Frobenius automorphism of L. Let

$$\epsilon_{\mathfrak{p}} = \begin{pmatrix} \mathfrak{p} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

and define $\overline{M}$ to be the double coset $\mathrm{GL}_r(A_{\overline{A/\mathfrak{p}}})\epsilon_{\mathfrak{p}}\mathrm{GL}_r(A_{\overline{A/\mathfrak{p}}})$. Let $h_{\mathfrak{p}}$ be an arbitrary element of $\mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}})$. By Section 2.7 of [23], there is a $\mathrm{Frob}_{\mathfrak{p}}$-equivariant bijection between the double coset space

$$(E^{opp})^{\times} \backslash [\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})/\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}}) \times (\{h_{\mathfrak{p}} \in \mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}}) : h_{\mathfrak{p}}^{-1}\epsilon_{\mathfrak{p}}F_{\mathfrak{p}}(h_{\mathfrak{p}}) \in \overline{M}\}/\mathrm{GL}_r(A_{\overline{A/\mathfrak{p}}}))] \quad (5.2)$$

and the isogeny class $H_{(\tilde{K},\tilde{\mathfrak{p}})}(\overline{A/\mathfrak{p}})$. This bijection relies on identifying isogenies with lattices in the product of all the finite, prime to $\mathfrak{p}$, $\mathfrak{l}$-adic Tate modules and the Dieudonné $A_{\mathfrak{p}}$-module (see Sections 2.6 and 2.7 of [23]).

There is an action on $\mathbb{M}_r(\overline{A/\mathfrak{p}})$, the set of *all* isomorphism classes of rank r Drinfeld modules over $\overline{A/\mathfrak{p}}$, by the geometric Frobenius $\mathrm{Frob}_{\mathfrak{p}}$ automorphism of L. The set $\mathbb{M}_r(\overline{A/\mathfrak{p}})^{\mathrm{Frob}_{\mathfrak{p}}^m}$ fixed by the m-th iterate of the geometric Frobenius automorphism is the set $\mathbb{M}_r(L)$ of isomorphism classes of rank r Drinfeld A-modules over L. This Frobenius power also preserves isogeny classes, and the number of fixed points of the action on the isogeny class $H_{(\tilde{K},\tilde{\mathfrak{p}})}(\overline{A/\mathfrak{p}})$ is the size of the isogeny class $H_{(\tilde{K},\tilde{\mathfrak{p}})}(L)$ over L.

The action of $\mathrm{Frob}_\mathfrak{p}$ on the isogeny class $H_{(\tilde{K},\tilde{\mathfrak{p}})}(\overline{A/\mathfrak{p}})$ is induced by the identity on $\mathrm{GL}_r(\mathbb{A}_f^\mathfrak{p})$ and by $h_\mathfrak{p} \mapsto \epsilon_\mathfrak{p} F_\mathfrak{p}(h_\mathfrak{p})$ on $\mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}})$. The given type has endomorphism algebra E. Each fixed point is associated to the conjugacy class of an element $\delta \in (E^{opp})^\times$ which is m-*admissible.*

Definition 5.1.5. We say an element δ is m-*admissible* if there exists an element $h_\mathfrak{p}^\delta \in \mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}})$ such that

$$(h_\mathfrak{p}^\delta)^{-1}\delta^{-1}N_m(\epsilon_\mathfrak{p})F_\mathfrak{p}^m(h_\mathfrak{p}^\delta) = 1,$$

where

$$N_m(h') = h' \cdot F_\mathfrak{p}(h') \cdots F_\mathfrak{p}^{m-1}(h')$$

for all $h' \in \mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}})$.

Set $\gamma_m^\delta = (h_\mathfrak{p}^\delta)^{-1}\epsilon_\mathfrak{p} F_\mathfrak{p}(h_\mathfrak{p}^\delta)$; this is an element of $\mathrm{GL}_r(K_L)$. Let

$$(E^{opp})_\delta^\times = \delta' \in (E^{opp})^\times | \delta'\delta = \delta\delta'\}$$

be the centralizer of δ. The number of fixed points associated to each such conjugacy class is the number of double classes

$$(E^{opp})_\delta^\times [h\mathrm{GL}_r(\hat{\mathbf{A}}^\mathfrak{p}), h_\mathfrak{p}^\delta h_m \mathrm{GL}_r(A_L))], \tag{5.3}$$

with $h \in \mathrm{GL}_r(\mathbb{A}_f^\mathfrak{p})$ and $h_m \in \mathrm{GL}_r(K_L)$ such that

$$\begin{cases} (h)^{-1}\delta h \in \mathrm{GL}_r(\hat{\mathbf{A}}^\mathfrak{p})\delta\mathrm{GL}_r(\hat{\mathbf{A}}^\mathfrak{p}), \\ (h_m)^{-1}\gamma_m^\delta F_\mathfrak{p}(h_m) \in \mathrm{GL}_r(A_L)\epsilon_\mathfrak{p}\mathrm{GL}_r(A_L). \end{cases}$$

Let $(E^{opp})_\natural^\times$ be a set of representatives of the conjugacy classes in $(E^{opp})^\times$. The size of an isogeny class over L is thus the sum over m-admissible δ's of $(E^{opp})_\natural^\times$ of the number of classes ((5.3)) associated to δ [23, Proposition 3.2.7]. The number of such h and h_m can be computed by integrating characteristic functions of the orbit of δ and the twisted orbit of γ_m^δ, which is where the orbital integrals of Laumon's formula come from (see (5.4) and (5.5)). The global volume factor is a consequence of Weyl integration (see [21, Page 432]).

5.1.1 The Orbital Integrals

While we do not fully explain measures on the orbits of GL_r until Chapter 4, we presently introduce the measures used in Laumon's integrals. First, suppose dh is a Haar measure on $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})$ normalized so that

$$\int_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})} dh = \mathrm{vol}_{dh}(\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})) = 1.$$

Similarly, let dh_m be the Haar measure on $\mathrm{GL}_r(K_L)$ normalized so that

$$\int_{\mathrm{GL}_r(A_L)} dh_m = \mathrm{vol}_{dh_m}(\mathrm{GL}_r(A_L)) = 1.$$

Let $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})_\delta$ be the centralizer of δ in $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})$, and $\mathrm{GL}_{r,\gamma_m^\delta}^{F_{\mathfrak{p}}}(K_L)$ be the $F_{\mathfrak{p}}$-centralizer of γ_m^δ in $\mathrm{GL}_r(K_L)$. These centralizers admit arbitrary Haar measures $dh_{f,\delta}^{\mathfrak{p}}$ and $dh_{m,\delta}$ respectively. We identify the orbit of δ in $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})$ with $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})_\delta\backslash\mathrm{GL}_r(A_f^{\mathfrak{p}})$ and the orbit of γ_m^δ in $\mathrm{GL}_r(K_L)$ with $\mathrm{GL}_{r,\gamma_m^\delta}^{F_{\mathfrak{p}}}(K_{\mathfrak{p}})\backslash\mathrm{GL}_r(K_L)$. Via these identifications, the orbit of δ is a measure space with respect to the quotient measure $\frac{dh}{dh_{f,\delta}^{\mathfrak{p}}}$. In the same way, the twisted orbit of γ_m^δ has quotient measure $\frac{dh_m}{dh_{m,\delta}}$.

Let $\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}$ be the characteristic function of $\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})$. Define $S_m := \mathrm{GL}_r(A_L)\epsilon_{\mathfrak{p}}\mathrm{GL}_r(A_L)$, and let $\mathbb{1}_{S_m}$ be its characteristic function. Consider the orbital integral

$$O_\delta(\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}) := \int_{\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})_\delta\backslash\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})} \mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}(h^{-1}\delta h)\frac{dh}{dh_{f,\delta}^{\mathfrak{p}}}, \tag{5.4}$$

and the twisted orbital integral

$$TO_{\gamma_m^\delta}(\mathbb{1}_{S_m}) := \int_{\mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma_m^\delta}(K_\mathfrak{p})\backslash \mathrm{GL}_r(K_L)} \mathbb{1}_{S_m}((h_m)^{-1}\gamma_m^\delta F_\mathfrak{p}(h_m))\frac{dh_m}{dh_{m,\delta}}. \tag{5.5}$$

Remark 5.1.6. The integrals make sense because the functions $\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^\mathfrak{p})}$ and $\mathbb{1}_{S_m}$ are non-negative. By [23, 4.8.9], they are absolutely convergent.

The centralizer $(E^{opp})^\times_\delta$ embeds into both the set of prime-to-$\mathfrak{p}$ finite adèlic points of the centralizer $\mathrm{GL}_r(\mathbb{A}_f^\mathfrak{p})_\delta$ and the set of $K_\mathfrak{p}$ points of the centralizer $\mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma_m^\delta}(K_\mathfrak{p})$ by the inclusions

$$(E^{opp})^\times_\delta \subset ((\mathbb{A}_f^\mathfrak{p} \otimes_K E)^{opp})^\times_\delta \subset \mathrm{GL}_r(\mathbb{A}_f^\mathfrak{p})_\delta, \tag{5.6}$$

$$(E^{opp})^\times_\delta \subset ((K_\mathfrak{p} \otimes_K E)^{opp})^\times_\delta \hookrightarrow \mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma_m^\delta}(K_\mathfrak{p}). \tag{5.7}$$

This induces the embedding $(E^{opp})^\times_\delta \hookrightarrow (\mathrm{GL}_r(\mathbb{A}_f^\mathfrak{p})_\delta \times \mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma_m^\delta}(K_\mathfrak{p}))$. The quotient space

$$(E^{opp})^\times_\delta\backslash(\mathrm{GL}_r(\mathbb{A}_f^\mathfrak{p})_\delta \times \mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma_m^\delta}(K_\mathfrak{p})) \tag{5.8}$$

has the Haar measure

$$\mu_\delta := \frac{dh^\mathfrak{p}_{f,\delta} \times dh_{m,\delta}}{d\mu_K}, \tag{5.9}$$

where $d\mu_K$ is the counting measure on $(E^{opp})^\times_\delta$. The volume of the set (5.8) with respect to the measure μ_δ is finite, as a consequence of (5.11) and [33, 3.1.1]

Remark 5.1.7. The inclusion (5.7) is non-trivial; it is the composition of the inclusion

$$((K_\mathfrak{p} \otimes_K E)^{opp})^\times_\delta \subset \{h_\mathfrak{p} \in \mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}}) : \epsilon_\mathfrak{p} F_\mathfrak{p}(h_\mathfrak{p}) = h_\mathfrak{p}\epsilon_\mathfrak{p}\}_\delta$$

and the isomorphism

$$\{h_{\mathfrak{p}} \in \mathrm{GL}_r(K_{\overline{A/\mathfrak{p}}}) : \epsilon_{\mathfrak{p}} F_{\mathfrak{p}}(h_{\mathfrak{p}}) = h_{\mathfrak{p}} \epsilon_{\mathfrak{p}}\}_\delta \cong \mathrm{GL}^{F_{\mathfrak{p}}}_{r,\gamma_m^\delta}(K_{\mathfrak{p}}).$$

It turns out that the product of the above orbital integrals with the volume of (5.8) with respect to μ_δ exactly counts the number of classes of the form (5.3) for each m-admissible δ.

Proposition 5.1.8. *([23, Proposition 3.3.3]) For a fixed type $(\tilde{K}, \tilde{\mathfrak{p}})$, the size of the isogeny class $H_{(\tilde{K},\tilde{\mathfrak{p}})}(L)$, where L is a degree m extension of $A/\mathfrak{p}$, is the sum over m-admissible δ's of the product*

$$\mathrm{vol}_{\mu_\delta}((E^{opp})^\times_\delta \backslash (GL_r(\mathbb{A}^{\mathfrak{p}}_f)_\delta \times GL_{r,\gamma^\delta_m}(K_{\mathfrak{p}}))) \cdot O_\delta(\mathbb{1}_{GL_r(\mathbb{A}^{\mathfrak{p}}_f)}) \cdot TO_{\gamma^\delta_m}(\mathbb{1}_{S_m}). \tag{5.10}$$

(This product is independent of the choices of the Haar measures $dh^{\mathfrak{p}}_{f,\delta}$ and $dh_{m,\delta}$.)

Unfortunately, this formula is not useful for the purposes of this paper; by a transfer of conjugacy classes, Laumon obtains the size of an isogeny class as a single term involving orbital integrals.

Laumon also simplifies the global term $\mathrm{vol}_{\mu_\delta}((E^{opp})^\times_\delta \backslash (\mathrm{GL}_r(\mathbb{A}^{\mathfrak{p}}_f)_\delta \times \mathrm{GL}_{r,\gamma^\delta_m}(K_{\mathfrak{p}})))$. By [23, Proposition 3.3.4], the element δ is m-admissible, which implies that the embeddings (5.6) and (5.7) are actually isomorphisms. Therefore, we have the equality of volumes

$$\mathrm{vol}_{\mu_\delta}((E^{opp})^\times_\delta \backslash (\mathrm{GL}_r(\mathbb{A}^{\mathfrak{p}}_f)_\delta \times \mathrm{GL}^{F_{\mathfrak{p}}}_{r,\gamma^\delta_m}(K_{\mathfrak{p}}))) = \mathrm{vol}_{\mu'_\delta}((E^{opp})^\times_\delta \backslash ((\mathbb{A}_f \otimes_K E)^{opp})^\times_\delta), \tag{5.11}$$

where $\mu'_\delta := \frac{dh_{f,\delta}}{d\mu_K}$ is the quotient of the measure $dh_{f,\delta}$ – a measure on $((\mathbb{A}_f \otimes_K E)^{opp})^\times_\delta$ induced by $dh^{\mathfrak{p}}_{f,\delta} \times dh_{m,\delta}$ – by the counting measure $d\mu_K$.

5.1.2 Transfer of Conjugacy Classes

By a transfer of conjugacy classes as in Section 3.4 of [23], we can obtain the size of an isogeny class, not as the sum of terms involving orbital integrals, but as a single term involving orbital integrals.

Definition 5.1.9. An element $\gamma \in \mathrm{GL}_r(K)$ is said to be *elliptic* if the K subalgebra $K[\gamma]$ of $\mathrm{Mat}_r(K)$ (the set of $r \times r$ matrices over K) generated by γ is a field, i.e., if the minimal polynomial of γ is irreducible over K. An element $\gamma \in \mathrm{GL}_r(K)$ is said to be *elliptic at infinity* if and only if $K_\infty \otimes_K K[\gamma]$ is a field. This means that there is only one place ∞' of $K[\gamma]$ dividing the place ∞ of K.

Definition 5.1.10. An element $\gamma \in \mathrm{GL}_r(K)$ is said to be *m-admissible at the place* $\mathfrak{p}$ if and only if $\mathrm{ord}_{\mathfrak{p}}(\det(\gamma)) = m$ and there is a place $\mathfrak{p}'$ of $K' = K[\gamma]$ dividing $\mathfrak{p}$ such that $\mathrm{ord}_{\mathfrak{o}'}(\gamma) = 0$ for all other places $\mathfrak{o}' \neq \mathfrak{p}'$ of $K[\gamma]$ which divide $\mathfrak{p}$.

Let γ be an element m-admissible at $\mathfrak{p}$, ∞' the place of K' dividing ∞, and $\mathfrak{p}'$ the place dividing $\mathfrak{p}$. Choose an element Π in K' such that $\mathrm{ord}_{\infty'}(\Pi) \neq 0$, $\mathrm{ord}_{\mathfrak{p}'}(\Pi) \neq 0$, and $\mathrm{ord}_{\mathfrak{o}'}(\Pi) = 0$ for all other places $\mathfrak{o}'$ of K'. Define a field $\tilde{K} = \cap_{s>0} K'[\Pi^s] \subset K'$, and let $\tilde{\infty}$ and $\tilde{\mathfrak{p}}$ be the restrictions of ∞' and $\mathfrak{p}'$ to $\tilde{K}$. By (3.4) of [23], the pair $(\tilde{K}, \tilde{\mathfrak{p}})$ is a $(K, \infty, \mathfrak{p})$-type. We have thus attached to each m-admissible γ an isogeny class. Let E be the unique-up to-isomorphism central division algebra over $\tilde{K}$ with invariants $-[\tilde{K} : K]/r$ at $\tilde{\infty}$, $[\tilde{F} : F]/r$ at $\tilde{\mathfrak{p}}$ and zero at all other places. Because $K' \otimes_{\tilde{K}} \tilde{K}_{\tilde{\infty}}$ and $K' \otimes_{\tilde{K}} \tilde{K}_{\tilde{\mathfrak{p}}}$ are fields, there is at least one embedding $K' \to E^{opp}$ taking γ to an element δ which is m-admissible at $\mathfrak{p}$. Any two such embeddings are conjugate in $(E^{opp})^\times$, so the image δ of γ is well-defined up to conjugacy. Let $\mathrm{GL}_r(K)_{\natural,ell}$ be the set of representatives of the elliptic conjugacy classes.

Proposition 5.1.11. *([23, Proposition 3.4.2]) There is a bijection*

$$\begin{aligned} \{\gamma \in GL_r(K)_{\natural,ell} : \gamma \textit{ is elliptic at } \infty \textit{ and } m\textit{-admissible at } \mathfrak{p}\} \\ \to \{((\tilde{K}, \tilde{\mathfrak{p}}), \delta) : \delta \in (E^{opp})^\times_\natural \textit{ is } m\textit{-admissible at } \mathfrak{p}\} \end{aligned} \tag{5.12}$$

well-defined up to isomorphism of $(\tilde{K}, \tilde{\mathfrak{p}})$, and δ is the conjugacy class associated to $(\tilde{K}, \tilde{\mathfrak{p}})$ as above.

Laumon's proof of Proposition 5.1.11 relies on constructing an inverse to the map (5.12). Given a pair $((\tilde{K}, \tilde{\mathfrak{p}}), \delta)$, we obtain an elliptic element $\gamma \in \mathrm{GL}_r(K)$ which is elliptic at ∞ and

m-admissible at $\mathfrak{p}$ thusly. First choose an embedding of K-algebras $\tilde{K} \hookrightarrow \mathrm{Mat}_r(K)$ by choosing a basis of $\tilde{K}$ over K – this gives an isomorphism between $\mathrm{Mat}_r(K)$ and the set $\mathrm{End}_K(\tilde{K}^{d/[\tilde{K}:K]})$. All choices of embeddings are conjugate. Then the centralizer $\mathrm{Mat}_r(K)_{\tilde{K}}$ can be identified with $M_{r/[\tilde{K}:K]}(\tilde{K})$. Recall that E^{opp} is a central division algebra over $\tilde{K}$ of dimension $(r/[\tilde{K}:K])^2$ as seen in Corollary 5.1.4. There is an injective map from the set of conjugacy classes in E^{opp} to the set of conjugacy classes in $\mathrm{Mat}_r(K)_{\tilde{K}}$ defined by

$$\delta_1 \mapsto \tilde{K}[\delta_1] \hookrightarrow M_{r/[\tilde{K}:K]}(\tilde{K}) \cong \mathrm{Mat}_r(K)_{\tilde{K}}. \tag{5.13}$$

The embedding of $\tilde{K}[\delta_1]$ into $M_{r/[\tilde{K}:K]}$ is well-defined because $\tilde{K}[\delta_1]$ is a field over $\tilde{K}$ of degree dividing $r/[\tilde{K}:K]$ which only depends on the conjugacy class of δ_1. Next, let $\gamma_{\tilde{K}} \in \mathrm{GL}_r(K)_{\tilde{K}}$ be a representative of the image of the conjugacy class of δ by (5.13). As an element of $\mathrm{GL}_r(K)$, $\gamma_{\tilde{K}}$ is elliptic, elliptic at ∞, and m-admissible at $\mathfrak{p}$.

By properties of γ, $(\tilde{K}, \tilde{\mathfrak{p}})$, and δ, we have the following proposition.

Proposition 5.1.12. *([23, Proposition 3.4.3])*

Let $\gamma \in GL_r(K)_{\natural,ell}$ be elliptic at ∞ and m-admissible at $\mathfrak{p}$. Let $((\tilde{K}, \tilde{\mathfrak{p}}), \delta)$ be its image under (5.12). *Then $K[\gamma] \subset Mat_r(K)$ is a field isomorphic over K to the field $K' = K[\delta] \subset E^{opp}$. We have $\tilde{K} \subset K'$ and there are unique places ∞', and $\mathfrak{p}'$ of K' dividing the places $\tilde{\infty}$ and $\tilde{\mathfrak{p}}$ of $\tilde{K}$ respectively. The centralizer $Mat_r(K)_\gamma$ is isomorphic to $M_{r'}(K')$ where $r' = r/[K':K]$ as a K'-algebra and the centralizer $E' = (E^{opp})_\delta$ of δ in E^{opp} is a central division algebra over K' with invariants $1/r'$ at ∞', $-1/r'$ at $\mathfrak{p}'$, and 0 elsewhere.*

We use this proposition later to explicitly describe the fields and centralizers in our particular setting. For now, it implies that γ and δ are conjugate in $\mathrm{Mat}_r(\mathbb{A}_f^{\mathfrak{p}})$ and that $N_m(\gamma_m^\delta)$ is conjugate to γ in $\mathrm{Mat}_r(K_L)$. Remember that for each δ, we have fixed an $h_\mathfrak{p}^\delta$ such that

$$(h_\mathfrak{p}^\delta)^{-1}\delta^{-1}N_m(\epsilon_\mathfrak{p})F_\mathfrak{p}^m(h_\mathfrak{p}^\delta) = 1$$

and

$$\gamma_m^\delta = (h_\mathfrak{p}^\delta)^{-1}\epsilon_\mathfrak{p} F_\mathfrak{p}(h_\mathfrak{p}^\delta).$$

We may now write the size of the isogeny class of the rank r Drinfeld module ϕ over the finite field L associated to the $(K,\infty,\mathfrak{p})$-type $(\tilde{K},\tilde{\mathfrak{p}})$ depending only on γ (and its image δ up to conjugation).

Proposition 5.1.13. *([23, Proposition 3.4.6]) Let $\gamma \in GL_r(K)_{\natural,ell}$ be elliptic, elliptic at ∞, and m-admissible at $\mathfrak{p}$. Let $((\tilde{K},\tilde{\mathfrak{p}}),\delta)$ be its image under* (5.12)*, and E the corresponding division algebra over $\tilde{F}$. Then size of the isogeny class $H_{(\tilde{K},\tilde{\mathfrak{p}})}(L)$ is the product of*

$$\mathrm{vol}_{\mu'_\delta}((E^{opp})^\times_\delta \backslash ((\mathbb{A}_f \otimes_K E)^{opp})^\times_\delta), \tag{5.14}$$

where μ'_δ is the measure in Equation (5.11) *induced by the product of measures in the next two factors, of*

$$\int_{GL_r(\mathbb{A}_f^\mathfrak{p})_\gamma \backslash GL_r(\mathbb{A}_f^\mathfrak{p})} \mathbb{1}_{GL_r(\hat{\mathbf{A}}^\mathfrak{p})}(h^{-1}\gamma h)\frac{dh}{dh_{f,\gamma}^\mathfrak{p}}, \tag{5.15}$$

where $dh_{f,\gamma}^\mathfrak{p}$ is the Haar measure on $GL_r(\mathbb{A}_f^\mathfrak{p})_\gamma$ induced by $dh_{f,\delta}^\mathfrak{p}$ (γ and δ are conjugate in $GL_r(\mathbb{A}_f^\mathfrak{p})$), and of

$$\int_{GL^{F_\mathfrak{p}}_{r,\gamma_m^\delta}(K_\mathfrak{p})\backslash GL_r(K_L)} \mathbb{1}_{S_m}((h_m)^{-1}\gamma_m^\delta F_\mathfrak{p}(h_m))\frac{dh_m}{dh_{m,\delta}}. \tag{5.16}$$

Explicitly, $dh_{f,\gamma}^\mathfrak{p}$ and $dh_{m,\delta}$ are arbitrary Haar measures on their respective centralizers, and μ'_δ is the measure $\frac{dh_{f,\gamma}^\mathfrak{p}\times dh_{m,\delta}}{d\mu_K}$. Further, the number in the set $\mathbb{M}_r(L)$ of all isomorphism classes of rank r Drinfeld A-modules over L is equal to the sum over the $\gamma's$ in $GL_r(K)$ which are elliptic and m-admissible at $\mathfrak{p}$ of this product.

Remarks 5.1.14. (a) The volume term at the beginning is actually finite, regardless of the structure of the endomorphism algebra E due to the compactness of $(E^{opp})^\times_\delta \backslash ((\mathbb{A}_f \otimes_K E)^{opp})^\times_\delta$, proven by Weil in [33, 3.1.1].

(b) In case it was not made abundantly clear in the exposition, γ and δ have the same characteristic polynomial – they are conjugate in $\mathrm{GL}_r(\mathbb{A}_f^{\mathfrak{p}})$. Further, due to the relationship between Weil pairs, types, and isogeny classes, the characteristic polynomial of γ coincides with the characteristic polynomial of the Frobenius endomorphism of the underlying isogeny class of Drinfeld modules.

(c) As we saw in Equation (5.11), the measure μ'_δ is not exactly equal to $\frac{dh^{\mathfrak{p}}_{f,\gamma} \times dh_{m,\gamma}}{d\mu_K}$; however, the measure $dh^{\mathfrak{p}}_{f,\gamma} \times dh_{m,\delta}$ induces a measure on the centralizer $((\mathbb{A}_f \otimes_K E)^{opp})^\times_\delta$.

5.2 Our Setting

Suppose ϕ is an ordinary Drinfeld module over L, the degree m extension of $A/\mathfrak{p}$, with type $(\tilde{K}, \tilde{\mathfrak{p}})$. We have seen that this gives matrices δ and γ having the same characteristic polynomial as ϕ. As we saw in Remark 2.5.4, the characteristic polynomial $P_\phi(X)$ has exactly one root of p-adic valuation m, and the characteristic and minimal polynomials coincide. Therefore, δ is a regular semisimple element. Because δ and γ are conjugate in GL_r, γ is also a regular semisimple element, and the centralizer $(\mathrm{GL}_r)_\gamma$ is the algebraic torus $\mathbf{T} = \mathrm{Res}_{E/k}(\mathbb{G}_m)$ (review Definition 4.1.3). Even further, the corresponding endomorphism algebra $E := \mathrm{End}(\phi) \otimes_K K$ is commutative, the degree of D is $[D : K(F)] = 1$ (where F is the Frobenius endomorphism), and $[D : K] = r$. With all this in mind, we can move away from the language of types and denote the isogeny class of ϕ by $H_\phi(L)$ having weighted cardinality $h^*_\phi(L)$.

Explicitly, observe that for the K-points of the centralizer $\mathrm{GL}_r(K)_\gamma$, i.e., of the torus $\mathbf{T}$, we have

$$\mathrm{GL}_r(K)_\gamma = \mathbf{T}(K) = E^\times = (E^{opp})^\times_\delta. \tag{5.17}$$

By the definition of restriction of scalars, $\mathbf{T} = \mathrm{Res}_{E/K}(\mathbb{G}_m)$, an r-dimensional variety over K such that as sets, $\mathbf{T}(K) = E^\times$. This is also an obvious consequence of Proposition 5.1.12 which implies that $K' = K[\delta] \cong K(F) = E$, i.e., that $r' = r/[K' : K] = 1$, and $\mathrm{Mat}_r(K)_\gamma \cong M_1(E) = E$.

Remarks 5.2.1. (a) In general, the twisted centralizer $\mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma^\delta_m}(K_\mathfrak{p})$ is an *inner twist* of $\mathrm{GL}_r(K_\mathfrak{p})_\gamma = \mathbf{T}(K_\mathfrak{p})$. Since a torus admits no nontrivial inner twists, in our case, this twisted centralizer is isomorphic to $\mathbf{T}(K_\mathfrak{p})$.

(b) Fix arbitrary Haar measures $dh_{f,\gamma}$ and $dh_{m,\delta}$ on $\mathrm{GL}_r(\mathbb{A}^\mathfrak{p}_f)_\gamma$ and $\mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma^\delta_m}(K_\mathfrak{p})$ as above. Due to Equation (5.17), and the first point of this remark, the fixed measures are literally measures on $\mathbf{T}(\mathbb{A}^\mathfrak{p}_f)$ and $\mathbf{T}(K_\mathfrak{p})$.

(c) The product $dh_{f,\gamma} \times dh_{m,\delta}$ is a measure on $\mathbf{T}(\mathbb{A}_f)$. The quotient of $dh_{f,\gamma} \times dh_{m,\delta}$ by the counting measure $d\mu_K$ on $\mathbf{T}(K)$ gives the measure $\mu = \frac{dh_{f,\gamma} \times dh_{m,\delta}}{d\mu_K}$.

Equipped with these simplifications, the formula for the weighted size of the isogeny class of ϕ is thus

$$h^*_\phi(L) = \mathrm{vol}_\mu(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)) \cdot \int_{\mathrm{GL}_r(\mathbb{A}^\mathfrak{p}_f)_\gamma\backslash\mathrm{GL}_r(\mathbb{A}^\mathfrak{p}_f)} \mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^\mathfrak{p})}(h^{-1}\gamma h)\frac{dh}{dh_{f,\gamma}} \cdot \int_{\mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma^\delta_m}(K_\mathfrak{p})\backslash\mathrm{GL}_r(K_L)} \mathbb{1}_{S_m}((h_m)^{-1}\gamma^\delta_m F_\mathfrak{p}(h_m))\frac{dh_m}{dh_{m,\delta}}. \tag{5.18}$$

As a reminder, the measures dh and dh_m on $\mathrm{GL}_r(\mathbb{A}^\mathfrak{p}_f)$ and $\mathrm{GL}_r(K_L)$ are *not* arbitrary; they are normalized to give volume one to the compact subgroups $\mathrm{GL}_r(\hat{\mathbf{A}}_p)$ and $\mathrm{GL}_r(A_L)$.

As a bit of foreshadowing, we will relate these orbital integrals to Gekeler ratios at each prime. This works in part because we can decompose the measure $\frac{dh}{dh_{f,\gamma}}$ as the product of local measures $\frac{dh_\mathfrak{l}}{dh_{\mathfrak{l},\gamma}}$ on the orbit of γ in $\mathrm{GL}_r(K_\mathfrak{l})$ for each place $\mathfrak{l} \neq \mathfrak{p}, \infty$ to obtain orbital integrals at each finite place. In this decomposition, the local measure $dh_\mathfrak{l}$ on $\mathrm{GL}_r(K_\mathfrak{l})$ is normalized to give volume 1 to $\mathrm{GL}_r(A_\mathfrak{l})$. Similarly, the local Haar measure $dh_{\mathfrak{l},\gamma}$ is an arbitrary Haar measure on $\mathrm{GL}_r(K_\mathfrak{l})_\gamma$. To relate the twisted orbital integral to a Gekeler ratio at $\mathfrak{p}$, we require the Fundamental Lemma.

5.3 The Fundamental Lemma

In Chapter 4 of [23], Laumon proves the Fundamental Lemma comparing the twisted orbital integral of (5.16) with an ordinary orbital integral, which is much easier to write as a Gekeler ratio. In Remark 5.2.1, we saw that the twisted centralizer is related to the usual centralizer by *inner twisting,* though we did not define it. Sparing some technical details, an inner twisting between group schemes (with respect to a common field of definition) is an isomorphism over the separable closure which interacts well with the absolute Galois group. The inner twist induces an isomorphism between exterior algebras, and thus gives a transfer of measures coming from differential forms. Naively, it seems reasonable that integration on a twisted orbit should be related to integration the usual orbit of an element.

The integrand of Laumon's twisted orbital integral is a function with input values defined over K_L, while the related ordinary orbital integral involves a function with inputs defined over $K_\mathfrak{p}$. A base change between the *Hecke algebras*

$$\begin{cases} \mathcal{H} = \mathcal{C}_c^\infty(\mathrm{GL}_r(A_\mathfrak{p})\backslash \mathrm{GL}_r(K_\mathfrak{p})/\mathrm{GL}_r(A_\mathfrak{p})) \\ \mathcal{H}_m = \mathcal{C}_c^\infty(\mathrm{GL}_r(A_L)\backslash \mathrm{GL}_r(K_L)/\mathrm{GL}_r(A_L)) \end{cases}$$

allows us to compare functions defined over K_L with functions over $K_\mathfrak{p}$. A concrete tool used to write the base change of specific functions in a Hecke algebra is the *Satake transform*, which expresses functions via polynomials over $\mathbb{Q}$. By definition, $\mathcal{H}$ is the collection of $\mathrm{GL}_r(A_\mathfrak{p})$ invariant functions $f : \mathrm{GL}_r(K_\mathfrak{p}) \to \mathbb{Q}$ with compact support, together with addition and the convolution product

$$(f_1 * f_2)(\gamma) = \int_{\mathrm{GL}_r(K_\mathfrak{p})} f_1(h) f_2(h^{-1}\gamma) dh,$$

where dh is the Haar measure normalized to give volume one to $\mathrm{GL}_r(A_\mathfrak{p})$. We adapt this definition appropriately for $\mathcal{H}_m$ and a Haar measure dh_m on $\mathrm{GL}_r(K_L)$ normalized to give $\mathrm{GL}_r(A_L)$ volume one.

Define $\mathcal{A} := \mathbb{Q}[\sqrt{q}, \frac{1}{\sqrt{q}}]$. Consider the Cartan decomposition of $\mathrm{GL}_r(K_\mathfrak{p})$, which asserts

$$\mathrm{GL}_r(K_\mathfrak{p}) = \bigcup_{e_i \geq e_{i+1}} \mathrm{GL}_r(A_\mathfrak{p}) \begin{pmatrix} \mathfrak{p}^{e_1} & 0 & \cdots & 0 \\ 0 & \mathfrak{p}^{e_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{p}^{e_r} \end{pmatrix} \mathrm{GL}_r(A_\mathfrak{p}). \tag{5.19}$$

In order to define the Satake transform of a function $f \in \mathcal{H}$, we first define a polynomial

$$\varsigma_z : \mathrm{GL}_r(K_\mathfrak{p}) \to \mathcal{A}[z_1, z_1^{-1}, \cdots, z_r, z_r^{-1}]$$

which is right $\mathrm{GL}_r(A_\mathfrak{p})$-invariant as follows. Now consider the Iwasawa decomposition of $\mathrm{GL}_r(K_\mathfrak{p})$

$$\mathrm{GL}_r(K_\mathfrak{p}) = B(K_\mathfrak{p})\mathrm{GL}_r(A_\mathfrak{p}),$$

where $B(K_\mathfrak{p})$ is the subgroup of upper triangular matrices. Let $B(A_\mathfrak{p}) = B(K_\mathfrak{p}) \cap \mathrm{GL}_r(A_\mathfrak{p})$. To define ς_z, it is sufficient to define a polynomial $\varsigma_z|_{B(K_\mathfrak{p})}$ which is right $B(A_\mathfrak{p})$-invariant. Let $g \in B(K_\mathfrak{p})$ be a matrix of the form

$$g = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,r} \\ 0 & b_{2,2} & \cdots & b_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{r,r} \end{pmatrix}.$$

Then define the *modulus character* $\delta_{B(K_\mathfrak{p})}$ by

$$\delta_{B(K_\mathfrak{p})}(g) = \prod_{i<j} \left| \frac{b_{i,i}}{b_{j,j}} \right|_\mathfrak{p}.$$

Similarly, define a quasi-character

$$\chi_z : B(K_\mathfrak{p}) \to \mathbb{Q}[z_1, z_1^{-1}, \cdots, z_r, z_r^{-1}]$$

by

$$\chi_z(g) = z_1^{\nu_{\mathfrak{p}}(b_{1,1})} \cdots z_r^{\nu_{\mathfrak{p}}(b_{r,r})},$$

where $v_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation. Now define $\varsigma_z|_{B(K_{\mathfrak{p}})}$ by

$$\varsigma_z|_{B(K_{\mathfrak{p}})} = \delta_{B(K_{\mathfrak{p}})}^{1/2} \chi_z. \tag{5.20}$$

Definition 5.3.1. The *Satake transform* $(-\vee)$ identifies a function f in $\mathcal{H}$ (resp. f_m in $\mathcal{H}_m$) with a polynomial $f^\vee(z)$ in $\mathcal{A}[z_1, z_1^{-1}, \cdots, z_r, z_r^{-1}]$, where

$$f^\vee(z) = \int_{\mathrm{GL}_r(K_{\mathfrak{p}})} f(h)\alpha_z(h)dh,$$

and dh is the measure normalized to give $\mathrm{GL}_r(A_{\mathfrak{p}})$ volume one. In fact,

$$(-\vee) : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A}[z_1, z_1^{-1}, \cdots, z_r, z_r^{-1}]$$

is an isomorphism.

The *base change homomorphism* is the map $b : \mathcal{A}_m \otimes \mathcal{H}_m \to \mathcal{A} \otimes \mathcal{H}$ such that $b|_{\mathcal{A}_m} = \mathbb{Q}[(\sqrt{q})^m, \frac{1}{(\sqrt{q})^m}] \hookrightarrow \mathcal{A}$. The compatibility between the base change and the Satake transform is given by

$$b(f_m)^\vee(z_1, \cdots, z_r) = f_m^\vee(z_1^m, \cdots, z_r^m).$$

Rather than explicitly stating the base change for a general function $f_m \in \mathcal{A}_m \otimes \mathcal{H}_m$, we now explain the base change $b(\mathbb{1}_{S_m})$ of the integrand $\mathbb{1}_{S_m}$ of Laumon's twisted orbital integral. Recall that $\mathbb{1}_{S_m} : \mathrm{GL}_r(K_L) \to \{0, 1\}$ is the characteristic function of $\mathrm{GL}_r(A_L)\epsilon_{\mathfrak{p}}\mathrm{GL}_r(A_L)$. Equipped with the Satake transform

$$\mathbb{1}_{S_m}^\vee = q^{m(r-1)/2}(z_1 + \cdots + z_r)$$

of $\mathbb{1}_{S_m}$, the next Proposition and subsequent Corollary allow us to explicitly define $b(\mathbb{1}_{S_m})$, and therefore obtain a function over $K_\mathfrak{p}$ for which we can write an ordinary orbital integral.

First, suppose P is a standard *parabolic subgroup of GL$_r$*. This means $B \subset P \subset \mathrm{GL}_r$ where B is defined above. Then P has a standard *Levi decomposition* $P = MN$, where M is the Levi subgroup containing the torus of diagonal entries, and N is the unipotent radical of P. Let dm and dn be Haar measures on M and N which have been normalized so that $\mathrm{vol}_{dm}(M(A_\mathfrak{p})) = 1$ and $\mathrm{vol}_{dn}(N(A_\mathfrak{p})) = 1$ respectively. We have the Iwasawa decomposition

$$\mathrm{GL}_r(K_\mathfrak{p}) = M(K_\mathfrak{p})N(K_\mathfrak{p})K_\mathfrak{p}.$$

We can thus integrate any function $f : \mathrm{GL}_r(K_\mathfrak{p}) \to \mathbb{C}$ with compact support over $\mathrm{GL}_r(K_\mathfrak{p})$ by

$$\int_{\mathrm{GL}_r(K_\mathfrak{p})} f(h)dh = \int_{M(K_\mathfrak{p})} \int_{N(K_\mathfrak{p})} \int_{K_\mathfrak{p}} f(mnx)dxdndm,$$

where dx is the Haar measure on $K_\mathfrak{p}$ normalized to give $A_\mathfrak{p}$ volume one. For each such subgroup M, define the Hecke algebra

$$\mathcal{H}_M = C_c^\infty(M(A_\mathfrak{p})\backslash M(K_\mathfrak{p})/M(A_\mathfrak{p})).$$

Then for any function $f \in \mathcal{A} \otimes \mathcal{H}$, we can define a function $f^P \in \mathcal{A} \otimes \mathcal{H}_M$ called the *constant term of f along the parabolic subgroup P* as follows. Let $m \in M(K_\mathfrak{p})$ be any element, and let $\delta_{P(K_\mathfrak{p})}$ be the modulus character of $P(K_\mathfrak{p})$. Then define

$$f^P(m) := \delta^{1/2}_{P(K_\mathfrak{p})}(m) \int_{N(K_\mathfrak{p})} f(mn)dn. \tag{5.21}$$

Proposition 5.3.2. *([23, Proposition 4.2.5]) For each integer $r \geq 1$, define*

$$s_r : GL_r(K_\mathfrak{p}) \to \mathbb{Z}$$

by

$$s_r(\gamma) = (1-q)(1-q^2)\cdots(1-q^{\varrho-1})$$

if $\gamma \in \mathit{Mat}_r(A_\mathfrak{p}) \cap \mathit{GL}_r(K_\mathfrak{p})$ *and* $v_\mathfrak{p}(\det(\gamma)) = m$*, where* ϱ *is the nullity (i.e., the dimension over* $K_\mathfrak{p}/\mathfrak{p}$ *of the kernel) of the matrix* $\overline{\gamma} \in \mathit{Mat}_r(K_\mathfrak{p}/\mathfrak{p})$ *obtained by reducing* γ *modulo* $\mathfrak{p}$*, and by*

$$s_r(\gamma) = 0$$

otherwise. Then for each standard parabolic subgroup $P = MN$ *of* $G = \mathit{GL}_r$ *given by the partition* $r = r_1 + \cdots + r_t$*, and for each*

$$m = (\gamma_1, \cdots, \gamma_t) \in \mathit{GL}_{r_1}(K_\mathfrak{p}) \times \cdots \times \mathit{GL}_{r_t}(K_\mathfrak{p}),$$

the constant term along P *of* (s_r) *is*

$$(s_r)^P(m) = \sum_{j=1}^{t} q^{m(r-r_j)/2} s_{r_j}(\gamma_j) \prod_{k=1,k\neq j}^{t} \mathbb{1}_{GL_{r_k}(A_\mathfrak{p})}(\gamma_k).$$

Here, $\mathbb{1}_{GL_{r_k}(A_\mathfrak{p})}$ *is the characteristic function of* $\mathit{GL}_{r_k}(A_\mathfrak{p}) \subset \mathit{GL}_{r_k}(K_\mathfrak{p})$.

Corollary 5.3.3. *([23, Corollary 4.2.6]) The Satake transform of* s_r *is*

$$(s_r)^\vee(z) = q^{m(r-1)/2}(z_1^m + \cdots + z_r^m)$$

and consequently, s_r *is the base change* $b(\mathbb{1}_{S_m}) \in \mathcal{A} \otimes \mathcal{H}$*.*

Warning! 5.3.4. At the beginning of this section, we allude that the Fundamental Lemma relates twisted orbital integrals for a function $f_m \in \mathcal{H}_m$ to ordinary orbital integrals for $b(f_m) \in \mathcal{H}$. While this is true, and we will soon show that the twisted orbital integral with integrand $\mathbb{1}_{S_m}$ is related to an ordinary orbital integral with integrand $b(\mathbb{1}_{S_m})$, the Fundamental Lemma only relates orbits and twisted orbits of *closed* elements in GL_r.

Definition 5.3.5. An element $\gamma \in \mathrm{GL}_r(K_\mathfrak{p})$ is said to be *closed* if its orbit in $\mathrm{GL}_r(K)$ under conjugation is a closed subset with respect to the $\mathfrak{p}$-adic topology.

Laumon shows that an element is closed if and only if it semisimple [23, Corollary 4.3.3]. By the Newton polygon assumptions in our setting, we know γ is semisimple. Since elliptic elements are semisimple, we could use the Fundamental Lemma to relate orbits and twisted orbits for isogeny classes with any Newton polygon structure.

As before, for an element $\gamma_m \in \mathrm{GL}_r(K_L)$, consider the norm

$$N_m(\gamma_m) = \gamma_m \cdot F_\mathfrak{p}(\gamma_m) \cdots F_\mathfrak{p}^{m-1}(\gamma_m).$$

Two elements $\gamma_m^{'}$ and $\gamma_m^{''}$ are $F_\mathfrak{p}$-conjugate if there is a $g_m \in \mathrm{GL}_r(K_L)$ such that

$$\gamma_m^{''} = g_m^{-1} \gamma_m^{'} F_\mathfrak{p}(g_m).$$

It turns out that for every $\gamma_m \in \mathrm{GL}_r(K_L)$, $N_m(\gamma_m)$ is conjugate to an element up to conjugacy in $\mathrm{GL}_r(K_\mathfrak{p})$, depending only on the $F_\mathfrak{p}$-conjugacy class of $\gamma_m \in \mathrm{GL}_r(K_L)$.

Definition 5.3.6. We say that an element $\gamma_m \in \mathrm{GL}_r(K_L)$ is $F_\mathfrak{p}$*-closed* if the image of the norm $N_m(\gamma_m)$ is conjugate in $\mathrm{GL}_r(K_L)$ to a closed element (see Definition 5.3.5) in $\mathrm{GL}(K_\mathfrak{p})$. This happens if and only if $N_m(\gamma_m)$ is closed in $\mathrm{GL}_r(K_L)$ because K_L is a separable extension of $K_\mathfrak{p}$.

For such $F_\mathfrak{p}$-closed elements, the twisted centralizer $\mathrm{GL}_{r,\gamma_m}^{F_\mathfrak{p}}(K_\mathfrak{p})$ of γ_m is an *inner twist* of the centralizer $\mathrm{GL}_r(K_\mathfrak{p})_\gamma$. As earlier stated, this means there is a transfer of Haar measures between the two centralizers. In particular, if $N_m(\gamma_m) = \gamma \in \mathrm{GL}_r(K_\mathfrak{p})$, and $d\gamma_m^{F_\mathfrak{p}}$ is the transfer of a Haar measure $d\gamma$ on the centralizer $\mathrm{GL}_r(K_\mathfrak{p})_\gamma$, we will use the notation

$$TO_{\gamma_m}(f_m, d\gamma) = TO_{\gamma_m}(f_m, d\gamma_m^{F_\mathfrak{p}}).$$

Finally, there is a notion of m-admissibility as in the previous section. If $\gamma \in \mathrm{GL}_r(K_\mathfrak{p})$ is closed and m-admissible, there is a $\gamma_m \in \mathrm{GL}_r(K_L)$ such that $\gamma = N_m(\gamma_m)$ [23, Lemma 4.5.2]. Then we have the following theorem.

Theorem 5.3.7. *(The Fundamental Lemma, [23, Theorem 4.5.5]) Let $\mathbb{1}_{S_m} \in \mathcal{H}_m$ be the Hecke function with Satake transform*

$$q^{m(r-1)/2}(z_1 + \cdots + z_r)$$

and let

$$f = b(\mathbb{1}_{S_m}) \in \mathcal{H}$$

be its base change, i.e., the Hecke function with Satake transform

$$f^\vee(z) = q^{m(r-1)/2}(z_1^m + \cdots + z_r^m).$$

Let γ be a closed element in GL$_r(K_\mathfrak{p})$ and dh_γ be a Haar measure on the centralizer $G_\gamma(K_\mathfrak{p})$. Let $dh_\mathfrak{p}$ be the Haar measure on GL$_r(K_\mathfrak{p})$ normalized to give the ring of integers volume one. Then the orbital integral

$$O_\gamma(f, dh_\gamma) := \int_{G_\gamma(K_\mathfrak{p})\backslash GL_r(K_\mathfrak{p})} f(h^{-1}\gamma h)\frac{dh_\mathfrak{p}}{dh_\gamma}$$

is zero unless γ is m-admissible. Let γ_m be a closed element of GL$_r(K_L)$ and let $dh_{\gamma_m}^{F_\mathfrak{p}}$ be a Haar measure on the $F_\mathfrak{p}$-centralizer of γ_m in GL$_r(K_L)$. Let h_m be the Haar measure on GL$_r(K_L)$ normalized to give the set of A_L-points volume one. Then the twisted orbital integral

$$TO_{\gamma_m}(\mathbb{1}_{S_m}, dh_{\gamma_m}^{F_\mathfrak{p}}) := \int_{G_{r,\gamma_m}^{F_\mathfrak{p}}(K_\mathfrak{p})\backslash GL_r(K_L)} \mathbb{1}_{S_m}(h_m^{-1}\gamma_m F_\mathfrak{p}(h_m))\frac{dh_m}{dh_{\gamma_m}^{F_\mathfrak{p}}}$$

is zero unless γ_m is m-admissible.

Moreover, if γ is an m-admissible element of GL$_r(K_\mathfrak{p})$ and if γ_m is an m-admissible $F_\mathfrak{p}$-closed element of GL$_r(K_L)$, such that

$$N_m(\gamma_m) = \gamma,$$

then for any Haar measure dh_γ on $G_\gamma(K_\mathfrak{p})$ we have

$$O_\gamma(f, dh_\gamma) = \epsilon(\gamma) TO_{\gamma_m}(\mathbb{1}_{S_m}, dh_\gamma), \tag{5.22}$$

where

$$\epsilon(\gamma) = (-1)^{r'-1} \tag{5.23}$$

for r' defined as in Proposition 5.1.12.

Remarks 5.3.8. (a) The Fundamental Lemma is an essential result to the main purpose of this project. In Lemma 6.4.1, we show that our Gekeler ratio at $\mathfrak{p}$ can be written in terms of the twisted orbital integral of Equation (5.18). As we saw above, in the setting of this dissertation, $r' = 1$, so the sign function satisfies $\epsilon(\gamma) = 1$. With this understanding, we may omit the sign function in Lemma 6.4.1.

(b) The function $f = b(\mathbb{1}_{S_m})$ is exactly the function s_r of Proposition 5.3.2.

Chapter 6

Main Results

In this chapter, we assume the same setting as described in Section 5.2. By choosing measures carefully, defining Gekeler ratios at each place and relating them to both ordinary and twisted orbital integrals, and computing the global volume form, we ultimately show that Laumon's formula for the size of an ordinary rank r Drinfeld module can be expressed as a product of local density functions, extending Gekeler's product formula to higher rank Drinfeld modules.

6.1 Choice of Measure

At every place $\mathfrak{v}$ of K, we define a local measure on $\mathbf{T}(K_{\mathfrak{v}})$ by

$$d\omega_{\mathbf{T},\mathfrak{v}}^{Tama} := L_{\mathfrak{v}}(1, \sigma_{\mathbf{T}}) d\omega_{\mathbf{T},\mathfrak{v}}, \tag{6.1}$$

where $d\omega_{\mathbf{T},\mathfrak{v}}$ is the measure on $\mathbf{T}(K_{\mathfrak{v}})$ coming from the differential form $\omega_{\mathbf{T},\mathfrak{v}}$ defined in terms of characters as in Chapter 4. By taking the product over finite places prime to $\mathfrak{p}$ we obtain the Haar measure

$$d\omega_{\mathbf{T},f}^{Tama,\mathfrak{p}} := \prod_{\mathfrak{l}\neq\infty,\mathfrak{p}} L_{\mathfrak{l}}(1, \sigma_{\mathbf{T}}) d\omega_{\mathbf{T},\mathfrak{l}} \tag{6.2}$$

on $\mathbf{T}(\mathbb{A}_f^{\mathfrak{p}})$. Similarly, the product measure

$$d\omega_{\mathbf{T},f}^{Tama} := d\omega_{\mathbf{T},f}^{Tama,p} \times d\omega_{\mathbf{T},\mathfrak{p}}^{Tama}$$

is a Haar measure on $\mathbf{T}(\mathbb{A}_f)$. By an abuse of notation, we also refer to the quotient measure $\frac{d\omega_{\mathbf{T},f}^{Tama}}{d\mu_K}$ on $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)$ as $d\omega_{\mathbf{T},f}^{Tama}$. With this choice, Laumon's formula for the size of an ordinary isogeny class reads

$$h^*_\phi(L) = \mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},f}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)) \cdot \int_{\mathbf{T}(\mathbb{A}^{\mathfrak{p}}_f)\backslash \mathrm{GL}_r(\mathbb{A}^{\mathfrak{p}}_f)} \mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}(h^{-1}\gamma h)\frac{dh}{d\omega^{Tama,\mathfrak{p}}_{\mathbf{T},f}}$$
$$\cdot \int_{\mathrm{GL}^{F_\mathfrak{p}}_{r,\gamma_m}(K_\mathfrak{p})\backslash \mathrm{GL}_r(K_L)} \mathbb{1}_{S_m}((h_m)^{-1}\gamma_m F_\mathfrak{p}(h_m))\frac{dh_m}{d\omega^{Tama}_{\mathbf{T},\mathfrak{p}}}. \tag{6.3}$$

We use the shorthand notation $TO^{Tama}_{\gamma_0}(\mathbb{1}_{S_m})$ to mean the twisted orbital integral in equation (6.3) where the *measure on $GL_r(K_L)$ has been normalized to give volume on to the ring of integers* Similarly, $O^{Tama,\mathfrak{p}}_{\gamma_0,f}(\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})})$ is the ordinary orbital integral on the finite prime-to-$\mathfrak{p}$ adèlic points.

6.2 Gekeler Ratios

We now define Gekeler ratios which generalize the ratios in Theorem 3.1.2 to arbitrary rank Drinfeld modules, and allow us to relate them to orbital integrals. One difficulty in relating Gekeler's numbers in Theorem 3.1.2 directly to a geometric orbital integral arises in the case that the characteristic polynomial fails to remain square-free modulo $\mathfrak{l}$. In particular, if $\gamma \in \mathrm{GL}_r(A)$ is a regular semisimple element, then for all but finitely many places, the orbit under conjugation of γ in $\mathrm{GL}_r(K_\mathfrak{l})$ is equal to the orbit of γ in $\mathrm{GL}_r(A_\mathfrak{l})$. However, if the projection of γ in $\mathrm{GL}_r(A/\mathfrak{l})$ is *not* regular, then the set of integral points of the orbit of γ in $\mathrm{GL}_r(K_\mathfrak{l})$ consists of several $\mathrm{GL}_r(A_\mathfrak{l})$-orbits. The number of integral orbits is bounded and detected at a finite truncation level (see Lemma 6.2.3). We will use this to define a new kind of conjugacy in Section 6.2.1, which coincides with the formula in Theorem 3.1.2 under the appropriate assumptions.

In this section, we define *Gekeler ratios* for every finite place $\mathfrak{l}$ of K, and for Drinfeld modules of arbitrary rank. Recall we have defined the ∞-adic absolute value by $|\mathfrak{l}| = \#(A/\mathfrak{l}) = q^{\deg_T(\mathfrak{l})}$. To begin, fix a regular semisimple element γ_0 in $\mathrm{GL}_r(K)$ with $P_{\gamma_0}(X) = P_{\phi_0}(X)$ for some ordinary rank r Drinfeld module ϕ_0 over the finite field L, a degree m extension of $A/\mathfrak{p}$. By assumption on the Newton polygon structure of ϕ_0, we know that $r-1$ of the roots of P_{γ_0} are $\mathfrak{p}$-adic units and one root has $\mathfrak{p}$-adic valuation m. We also know that $|\det(\gamma_0)|_\mathfrak{p} = |\mathfrak{p}|^{-m}$. These assumptions are equivalent to those used in Equation (5.18), which we will use as the starting point for the comparison to a Gekeler-like product. At the end of this chapter, we compare Gekeler's formula for rank two Drinfeld modules directly to our formula. In that comparison, we will simplify the global terms using the additional assumption that E/K is a geometric extension (i.e., the field of constants of E is still $\mathbb{F}_q$). We continue with the earlier stated assumption that K has characteristic other than two in order to simplify the computation of $\mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)$.

6.2.1 Quasi-Conjugacy

Let $\pi_n : A_{\mathfrak{l}} \to A_{\mathfrak{l}}/\mathfrak{l}^n$ be the truncation map, and define the set of integral matrices in $\mathrm{GL}_r(K_{\mathfrak{l}})$ by $M(A_{\mathfrak{l}}) := \mathrm{GL}_r(K_{\mathfrak{l}}) \cap \mathrm{Mat}_r(A_{\mathfrak{l}})$. Elements in $M(A_{\mathfrak{l}})$ may have determinants which are not $\mathfrak{l}$-adic units. For each $d \geq 0$, define

$$M(A_{\mathfrak{l}})_d := \{D \in M(A_{\mathfrak{l}}) : \mathrm{ord}_{\mathfrak{l}}(\det(D)) \leq d\},$$

so that while elements in $M(A_{\mathfrak{l}})$ may not be invertible, we can control "how far from invertible" they are. Similarly, let $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d := \{\overline{D} \in M(A_{\mathfrak{l}}/\mathfrak{l}^n) : \mathrm{ord}_{\mathfrak{l}}(\det(\overline{D})) \leq d\}$ be the image of $M(A_{\mathfrak{l}})_d$ under the truncation map.

Definition 6.2.1. (Quasi-conjugacy) We say that an element $\overline{\gamma}$ in $M(A_{\mathfrak{l}}/\mathfrak{l}^n)$ is $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$*-conjugate* to $\pi_n(\gamma_0)$ if there exists $\overline{D}$ in $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$ such that $\overline{D}\overline{\gamma} = \pi_n(\gamma_0)\overline{D}$. We write $\overline{\gamma} \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d} \pi_n(\gamma_0)$.

Observe that if $n < d$ then $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$ contains elements with determinant zero. However, when $n > d$, $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$ does not include elements with determinant zero.

Remark 6.2.2. Fix $d_0 \geq 0$, and suppose $\gamma \in M(A_{\mathfrak{l}})$ is $M(A_{\mathfrak{l}})_{d_0}$-conjugate to γ_0. Then since $M(A_{\mathfrak{l}})_d$ contains $M(A_{\mathfrak{l}})_{d_0}$ for all $d > d_0$, γ is $M(A_{\mathfrak{l}})_d$-conjugate to γ_0. A somewhat less trivial observation is that if $\overline{\gamma}$ is $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_{d_0}$-conjugate to $\pi_n(\gamma_0)$, then for $d > d_0$, $\overline{\gamma}$ is $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$-conjugate to $\pi_n(\gamma_0)$, provided $n \gg d$.

For $\gamma_0 \in M(A_{\mathfrak{l}})$, define

$$Q_{(d,n)}(\gamma_0) = \{\gamma \in M(A_{\mathfrak{l}}/\mathfrak{l}^n) : \gamma \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d} \pi_n(\gamma_0)\}$$

to be the $\sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d}$-conjugacy class of $\pi_n(\gamma_0)$. Similarly, let $\widetilde{Q}_{(d,n)}(\gamma_0) = \pi_n^{-1}(Q_{(d,n)}(\gamma_0))$ be the set of lifts of $Q_{(d,n)}(\gamma_0)$ to $A_{\mathfrak{l}}$. When γ_0 is an element of $\mathrm{GL}_r(A_{\mathfrak{l}}) \subset M(A_{\mathfrak{l}})$, the quasi-conjugacy class is given by $Q_{(d,n)}(\gamma_0) = \{\gamma \in \mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n) : \gamma \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n_d} \pi_n(\gamma_0)\}$. Lemma 6.2.3, proven in [1] for $\mathrm{GSp}_{2g}(\mathbb{Z}_{\mathfrak{l}})$ rather than for $\mathrm{GL}_r(A_{\mathfrak{l}})$, allows us to describe the intersection of $\mathrm{GL}_r(A_{\mathfrak{l}})$ with neighborhoods of the orbit ${}^{\mathrm{GL}_r(K_{\mathfrak{l}})}\gamma_0$ in terms of $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$-conjugacy.

Lemma 6.2.3 ([1, Lemma 3.2]). *Suppose $\gamma_0 \in GL_r(A_{\mathfrak{l}})$ is regular semisimple. There exists an integer $e = e(\gamma_0)$ such that, if $n \gg 0$ and $d > e$, then for $\gamma \in GL_r(A_{\mathfrak{l}}/\ell^n)$, the following are equivalent:*

(a) $\gamma \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d} \gamma_0 \mod \mathfrak{l}^n$,

(b) here exists some $\widetilde{\gamma} \in GL_r(A_{\mathfrak{l}})$ such that $\widetilde{\gamma} \mod \mathfrak{l}^n = \gamma$ and $\widetilde{\gamma} \sim_{GL_r(K_{\mathfrak{l}})} \gamma_0$.

The statement is also true replacing $GL_r(A_{\mathfrak{l}})$ with $M(A_{\mathfrak{l}})$ everywhere

Proof. (As in [1, Lemma 3.2].) The intersection of $\mathrm{GL}_r(A_{\mathfrak{l}})$ with the rational orbit ${}^{\mathrm{GL}_r(K_{\mathfrak{l}})}\gamma_0$ consists of finitely many $\mathrm{GL}_r(A_{\mathfrak{l}})$-orbits ($\gamma_0$ is regular semisimple, so $\mathrm{GL}_r(A_{\mathfrak{l}}) \cap {}^{\mathrm{GL}_r(K_{\mathfrak{l}})}\gamma_0$ is compact and the $\mathrm{GL}_r(A_{\mathfrak{l}})$-orbits are open in the intersection). Let $g_1, \cdots, g_s$ be representatives of these orbits. Each g_i has integral entries and is rationally conjugate to γ_0 by $D_i \in \mathrm{GL}_r(K_{\mathfrak{l}})$, so that $D_i g_i = \gamma_0 D_i$. Clearing the denominators of D_i results in a matrix $X_i \in M(A_{\mathfrak{l}})$ such that $X_i g_i = \gamma_0 X_i$. Choose $e(\gamma_0) = \max_{i\in\{1,\cdots,s\}}\{|\operatorname{ord}(\det(X_i)|\}$. Let $v_{\mathfrak{l}}(\gamma_0)$ be the valuation of the Weyl discriminant of γ_0 (see (4.14)). Suppose $n > 2v_{\mathfrak{l}}(\gamma_0)$, choose $d > e(\gamma_0)$, and suppose also $n \gg d$. Now it suffices to prove that $\gamma \in \mathrm{GL}_r(A_{\mathfrak{l}})$ satisfies $\gamma \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d} \pi_n(\gamma_0)$ if and only if there is a lift $\widetilde{\gamma} \in \mathrm{GL}_r(A_{\mathfrak{l}})$ such that $\pi_n(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma} \sim_{\mathrm{GL}_r(K_{\mathfrak{l}})} \gamma_0$.

The backward direction is straightforward. Suppose there is a $\widetilde{\gamma} \in \mathrm{GL}_r(A_{\mathfrak{l}})$ which is congruent to γ modulo $\mathfrak{l}^n$ and is rationally conjugate by D to γ_0, so that $D\gamma D^{-1} = \gamma_0$. Multiplying on the right by D and clearing denominators gives $X\widetilde{\gamma} = \gamma_0 X$ for some $X \in M(A_{\mathfrak{l}})$. The projection map gives $Z := \pi_n(X) \in M(A_{\mathfrak{l}}/\mathfrak{l}^n)_{|\operatorname{ord}(\det(X))|}$, and thus $Z\gamma \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_{|\operatorname{ord}(\det(X))|}} \gamma_0 \mod \mathfrak{l}^n$. In fact, $\widetilde{\gamma}$ must be in the same integral orbit as one of the g_i above, so $Z \in M(A_{\mathfrak{l}}/\mathfrak{l}^n)_e$, and we have proven the claim.

The forward direction is a consequence of Hensel's Lemma (B.0.5 and B.0.7). Suppose that $\gamma \sim_{M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d} \pi_n(\gamma_0)$. This means that there is a $D \in M(A_{\mathfrak{l}}/\mathfrak{l}^n)$ such that $D\gamma = \pi_n(\gamma_0)D$. For each n, define a subset $R_{\gamma_0}(A_{\mathfrak{l}}/\mathfrak{l}^n)$ of $M(A_{\mathfrak{l}}/\mathfrak{l}^n) \times \mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n)$ by

$$R_{\gamma_0}(A_{\mathfrak{l}}/\mathfrak{l}^n) = \{(D,\gamma) : D \in M(A_{\mathfrak{l}}/\mathfrak{l}^n), \gamma \in \mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n), D\gamma = \pi_n(\gamma_0)D\}.$$

This determines a system of r^2 equations in $2r^2$ variables; the variables are entries in the matrices D and γ, and the equations come from equating the entries in the products $D\gamma$ and $\pi_n(\gamma_0)D$. Let $n(\gamma_0)$ be the valuation of the minor formed by the first r^2 columns of the Jacobian matrix of the system at γ_0. Then if $n > 2n(\gamma_0)$, and $(D,\gamma) \in R_{\gamma_0}(A_{\mathfrak{l}}/\mathfrak{l}^n)$, by Hensel's Lemma (see B.0.7 for the statement as it appears in [5, Corollary III.4.5.3]), there is a solution to the system, which means there is a $\widetilde{\gamma} \in \mathrm{GL}_r(A_{\mathfrak{l}})$ with $\pi_n(\widetilde{\gamma}) = \gamma$ and $\widetilde{\gamma} \sim_{M(A_{\mathfrak{l}})} \gamma_0$. The authors remark that $n(\gamma_0)$ is equal to the valuation of the Weyl discriminant of γ_0, observed by Kottwitz in [22]. Lastly, the proof relies on being able to solve a system of equations over $A_{\mathfrak{l}}$, and not on γ having been an element of $\mathrm{GL}_r(A_{\mathfrak{l}})$. The argument remains the same when $\mathrm{GL}_r(A_{\mathfrak{l}})$ is replaced by $M(A_{\mathfrak{l}})$. □

Recall that $C(K_{\mathfrak{l}}) \cong \mathbb{A}^{r-1} \times \mathbb{G}_m$, and the geometric measure on an orbit is defined by the characteristic polynomial map $c : \mathrm{GL}_r(K_{\mathfrak{l}}) \to C(K_{\mathfrak{l}})$. For each n, define the nth $\mathfrak{l}$-adic neighborhood $\widetilde{U}_n(\gamma_0)$ of $c(\gamma_0)$ by

$$\begin{aligned}\widetilde{U}_n(\gamma_0) &= \pi_n^{-1}(\pi_n(c(\gamma_0))) \\ &= \{(a_{r-1}, \cdots, a_0) \in C(A_{\mathfrak{l}}) : a_i \equiv a_i(P_{\gamma_0}) \mod \mathfrak{l}^n\}.\end{aligned}$$

We may now describe the intersection of $M(A_{\mathfrak{l}})$ (or $\mathrm{GL}_r(A_{\mathfrak{l}})$) with these neighborhoods in terms of quasi-conjugacy.

Proposition 6.2.4. *Let $\mathfrak{l}$ be any prime, including $\mathfrak{p}$. Suppose $\gamma_0 \in M(A_{\mathfrak{l}})$ is regular semisimple. Then for sufficiently large n and d,*

$$M(A_{\mathfrak{l}}) \cap c^{-1}(\widetilde{U}_n(\gamma_0)) = \widetilde{Q}_{(d,n)}(\gamma_0).$$

Note that if $\gamma_0 \in GL_r(A_{\mathfrak{l}}) \subset M(A_{\mathfrak{l}})$, then $\widetilde{Q}_{(d,n)}(\gamma_0) \subset GL_r(A_{\mathfrak{l}})$ and the statement becomes $GL_r(A_{\mathfrak{l}}) \cap \widetilde{U}_n(\gamma_0) = \widetilde{Q}_{(d,n)}(\gamma_0)$.

Proof. First suppose γ is an element of $M(A_{\mathfrak{l}}) \cap c^{-1}(\widetilde{U}_n(\gamma_0))$. By definition, $P_\gamma \equiv P_{\gamma_0} \mod \mathfrak{l}^n$. The set of matrices in $M(A_{\mathfrak{l}})$ having the same characteristic polynomial in γ_0 is the set of solutions to a system of r equations in r^2 variables. Further, $\pi_n(\gamma)$ is an approximate solution in the sense that it satisfies the system modulo $\mathfrak{l}^n$. Hensel's Lemma for non-smooth systems (B.0.7) says that we can lift points for systems that are not necessarily smooth, provided that the power of $\mathfrak{l}$ dividing the system evaluated at the approximate root is more than twice the power of $\mathfrak{l}$ which divides the determinant of the $r \times r$ minor of the Jacobian matrix of the system. We can choose n arbitrarily large, so this version of Hensel's lemma applies; there is an actual solution $\gamma' \in M(A_{\mathfrak{l}})$ for which $P_{\gamma'} = P_{\gamma_0}$ (i.e., γ' is rationally conjugate to γ_0) and $\pi_n(\gamma') = \pi_n(\gamma) \mod \mathfrak{l}^n$. By Lemma 6.2.3, this is equivalent to saying that for sufficiently large n and d, $\pi_n(\gamma)$ is $M(A_{\mathfrak{l}}/\mathfrak{l}^n)_d$-conjugate to $\pi_n(\gamma_0)$, which by definition implies $\gamma \in \widetilde{Q}_{(d,n)}(\gamma_0)$.

Alternatively, for large enough n and d, if γ is in $\widetilde{Q}_{(d,n)}(\gamma_0)$, there exists an element g in $\mathrm{GL}_r(A_{\mathfrak{l}})$ with $\pi_n(g) = \pi_n(\gamma) \mod \mathfrak{l}^n$ and $g \sim_{\mathrm{GL}_r(K_{\mathfrak{l}})} \gamma_0$. The first condition implies that $P_g \equiv P_\gamma \mod \mathfrak{l}^n$ while the second condition implies that $P_g = P_{\gamma_0}$. By transitivity, we have $P_\gamma \equiv P_{\gamma_0} \mod \mathfrak{l}^n$, which means $\gamma \in M(A_{\mathfrak{l}}) \cap c^{-1}(\widetilde{U}_n(\gamma_0))$. □

6.2.2 Gekeler Ratios and Volume

We may now define Gekeler ratios and relate them to geometric orbital integrals.

Definition 6.2.5. For each finite prime $\mathfrak{l}$ of A (including $\mathfrak{p}$) define

$$v_{\mathfrak{l}}(\gamma_0) := \lim_{d\to\infty} \lim_{n\to\infty} \frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n)/|\mathfrak{l}|^{n(r-1)}}. \tag{6.4}$$

Remark 6.2.6. Suppose the characteristic polynomial of $\gamma_0 \mod \mathfrak{l}$ is square-free. Then the numerator stabilizes at $d = 0$. This is the usual conjugacy class

$$Q_{(0,n)} = \{\gamma \in M(A_{\mathfrak{l}}/\mathfrak{l}^n) : \pi_n(\gamma) \sim \pi_n(\gamma_0)\}.$$

An element $\gamma \in M(A_{\mathfrak{l}}/\mathfrak{l}^n)$ is conjugate to $\gamma_0 \mod \mathfrak{l}^n$ if and only if they have the same characteristic polynomials $\mod \mathfrak{l}^n$. Therefore,

$$Q_{(0,n)}(\gamma_0) = \{\gamma \in \mathrm{Mat}_r(A/\mathfrak{l}^n) : P_\gamma = P_{\gamma_0} \mod \mathfrak{l}^n\}.$$

This means the numerator in our Gekeler ratio (6.4) is consistent with the numerator of Gekeler's ratio at $\mathfrak{l}$ for rank two Drinfeld modules (3.1). A quick calculation shows this is also consistent with the Gekeler ratios defined for abelian varieties in [1, Definition 4.1].

Remark 6.2.7. Suppose the characteristic polynomial of $\gamma_0 \mod \mathfrak{l}$ is not square-free. We have shown that for sufficiently large n and d,

$$\widetilde{Q}_{(d,n)}(\gamma_0) = M(A_{\mathfrak{l}}) \cap c^{-1}(\widetilde{U}_n(\gamma_0)).$$

However, the set on the right-hand side is the set of matrices $\gamma \in M(A_{\mathfrak{l}})$ such that $P_\gamma \equiv P_{\gamma_0} \mod \mathfrak{l}^n$. Since $Q_{(d,n)}(\gamma_0) = \pi_n(\widetilde{Q}_{(d,n)}(\gamma_0))$, for sufficiently large n and d, the numerator of (6.4) is consistent with Gekeler's ratio at $\mathfrak{l}$ for rank two Drinfeld modules (3.1).

Lemma 6.2.8. *Let γ_0 be a regular semisimple element of $GL_r(A_{\mathfrak{l}})$, for some prime $\mathfrak{l}$ other than $\mathfrak{p}$. Then the geometric orbital integral of the characteristic function $\mathbb{1}_{GL_r(A_{\mathfrak{l}})}$ can be related to the quasi-conjugacy class $\widetilde{Q}_{(d,n)}$ by*

$$O_{\gamma_0}^{geom}(\mathbb{1}_{GL_r(A_{\mathfrak{l}})}) = \lim_{d\to\infty} \lim_{n\to\infty} \frac{\mathrm{vol}_{\mu_G^{so}}(\widetilde{Q}_{(d,n)}(\gamma_0))}{|\mathfrak{l}|^{-rn}}.$$

Proof. By definition, the geometric orbital integral is the volume of the integral points of the rational orbit of γ_0, denoted $\mathrm{GL}_r(A_{\mathfrak{l}}) \cap {}^{\mathrm{GL}_r(K_{\mathfrak{l}})}\gamma_0$, i.e.

$$O^{geom}_{\gamma_0}(\mathbb{1}_{\mathrm{GL}_r(A_\mathfrak{l})}) = \mathrm{vol}_{|\mu^{\mathrm{geom}}_{\gamma_0}|}(\mathrm{GL}_r(A_\mathfrak{l}) \cap {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0).$$

Because the underlying group is GL_r, and γ_0 is regular and semisimple, the rational orbit is equal to the stable orbit, given by the Steinberg map $c^{-1}(c(\gamma_0))$. Thus we have

$$O^{geom}_{\gamma_0}(\mathbb{1}_{\mathrm{GL}_r(A_\mathfrak{l})}) = \mathrm{vol}_{|\mu^{\mathrm{geom}}_{\gamma_0}|}(\mathrm{GL}_r(A_\mathfrak{l}) \cap c^{-1}(c(\gamma_0))).$$

By definition of the geometric measure ((4.13)) (and that the set of integral points in $\mathrm{GL}_r(K_\mathfrak{l})$ is open) we have

$$O^{geom}_{\gamma_0}(\mathbb{1}_{\mathrm{GL}_r(A_\mathfrak{l})}) = \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(\mathrm{GL}_r(A_\mathfrak{l}) \cap c^{-1}(\widetilde{U}_n(\gamma_0)))}{\mathrm{vol}_{|d\omega_C|}(\widetilde{U}_n(\gamma_0))}. \tag{6.5}$$

Because $\gamma_0 \in \mathrm{GL}_r(A_\mathfrak{l})$, Proposition 6.2.4 allows us to replace $\mathrm{GL}_r(A_\mathfrak{l}) \cap c^{-1}(\widetilde{U}_n(\gamma_0))$ with $\widetilde{Q}_{(d,n)}(\gamma_0) \subset \mathrm{GL}_r(A_\mathfrak{l})$ when d and n are sufficiently large, yielding

$$\begin{aligned} O^{geom}_{\gamma_0}(\mathbb{1}_{\mathrm{GL}_r(A_\mathfrak{l})}) &= \lim_{d\to\infty}\lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(\widetilde{Q}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{|d\omega_C|}(\widetilde{U}_n(\gamma_0))} \\ &= \lim_{d\to\infty}\lim_{n\to\infty} \frac{\mathrm{vol}_{\mu^{\mathrm{SO}}_G}(\widetilde{Q}_{(d,n)}(\gamma_0))}{|\mathfrak{l}|^{-rn}}, \end{aligned} \tag{6.6}$$

as desired. □

Corollary 6.2.9. *For* $\mathfrak{l}$ *other than* $\mathfrak{p}$,

$$O^{geom}_{\gamma_0}(\mathbb{1}_{GL_r(A_\mathfrak{l})}) = \lim_{d\to\infty}\lim_{n\to\infty} \frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{l}|^{r^2n-rn}}.$$

Proof. By the previous lemma, we know that the geometric orbital integral can be expressed in terms of the Serre-Oesterlé volume of $\widetilde{Q}_{(d,n)}(\gamma_0)$, which we now compute explicitly. Since $\pi_n : M(A_\mathfrak{l}) \to M(A_\mathfrak{l}/\mathfrak{l}^n)$ is surjective, so the set of lifts, $\widetilde{Q}_{(d,n)}(\gamma_0)$, of $Q_{(d,n)}(\gamma_0)$ is a disjoint union of fibers of π_n. Since M is smooth over $A_{\mathfrak{l}}$, each fiber has volume $|\mathfrak{l}|^{-n\dim(G)}$ under the Serre-Oesterlé measure (4.1). Therefore, we have

$$\operatorname{vol}_{\mu_G^{\mathrm{so}}}(\widetilde{Q}_{(d,n)}(\gamma_0)) = \frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{l}|^{n\dim(G)}} \tag{6.7}$$

$$= \frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{l}|^{r^2 n}}. \tag{6.8}$$

Combining this with Lemma 6.6 gives the result.

□

Lemma 6.2.10. *For primes $\mathfrak{l}$ away from $\mathfrak{p}$, the Gekeler ratio* (6.4) *is related to the geometric orbital integral $O_{\gamma_0}^{geom}(\mathbb{1}_{GL_r(A_{\mathfrak{l}})})$ by*

$$v_{\mathfrak{l}}(\gamma_0) = \frac{|\mathfrak{l}|^{r^2-1}}{\#SL_r(A_{\mathfrak{l}}/\mathfrak{l})} O_{\gamma_0}^{geom}(\mathbb{1}_{GL_r(A_{\mathfrak{l}})}). \tag{6.9}$$

Proof. By Corollary 6.2.9, the right-hand side of (6.9) is

$$\frac{|\mathfrak{l}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})} O_{\gamma_0}^{geom}(\mathbb{1}_{\mathrm{GL}_r(A_{\mathfrak{l}})}) = \frac{|\mathfrak{l}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})} \lim_{d\to\infty} \lim_{n\to\infty} \frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{l}|^{r^2 n - rn}}.$$

At sufficiently large n and d, we have

$$\begin{aligned}
\frac{|\mathfrak{l}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})} \frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{l}|^{r^2 n - rn}} &= \frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})|\mathfrak{l}|^{r^2 n - rn - r^2 + 1}} \\
&= \frac{|\mathfrak{l}|^{rn-n}\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})|\mathfrak{l}|^{r^2 n - rn - r^2 + 1 + rn - n}} \\
&= \frac{|\mathfrak{l}|^{rn-n}\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})|\mathfrak{l}|^{(n-1)(r^2-1)}}.
\end{aligned}$$

By Hensel's Lemma 4.1.2, we know that $|\mathfrak{l}|^{(n-1)(r^2-1)}\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l}) = \mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n)$. This gives

$$\frac{|\mathfrak{l}|^{r^2-1}}{\#\mathrm{SL}_r(A_\mathfrak{l}/\mathfrak{l})}\frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{l}|^{r^2n-rn}} = \frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_\mathfrak{l}/\mathfrak{l}^n)/|\mathfrak{l}|^{n(r-1)}}, \tag{6.10}$$

but this is the term at (d, n) of the Gekeler ratio $v_\mathfrak{l}(\gamma_0)$. Taking the limit of both sides of (6.10) as n and d tend to infinity yields

$$\frac{|\mathfrak{l}|^{r^2-1}}{\#\mathrm{SL}_r(A_\mathfrak{l}/\mathfrak{l})}O^{geom}_{\gamma_0}(\mathbb{1}_{\mathrm{GL}_r(A_\mathfrak{l})}) = \lim_{d\to\infty}\lim_{n\to\infty}\frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_\mathfrak{l}/\mathfrak{l}^n)/|\mathfrak{l}|^{n(r-1)}} = v_\mathfrak{l}(\gamma_0).$$

□

6.2.3 Gekeler Ratios and Volume at the "Bad Prime"

While the definitions for quasi-conjugacy in Section 6.2.1 hold at $\mathfrak{p}$, because we use the space $M(A_\mathfrak{l})$ rather than $\mathrm{GL}_r(A_\mathfrak{l})$, the methods for relating the Gekeler ratio (6.4) to a geometric orbital integral above do not apply at the prime $\mathfrak{p}$ of A. Specifically, we cannot use the orbital integral of the characteristic function of the integral points of GL_r. Instead, we will use the characteristic function of the integral points of a double coset in the Cartan decomposition of GL_r.

Let $\pi_n^M : \mathrm{Mat}_r(A_\mathfrak{p}) \to \mathrm{Mat}_r(A_\mathfrak{p}/\mathfrak{p}^n)$ be the projection map and $c : \mathrm{GL}_r(K_\mathfrak{p}) \to C(K_\mathfrak{p})$ be the characteristic polynomial map. As in (5.19), Consider the Cartan decomposition of $\mathrm{GL}_r(K_\mathfrak{p})$

$$\mathrm{GL}_r(K_\mathfrak{p}) = \bigcup_{e_i \geq e_{i+1}} \mathrm{GL}_r(A_\mathfrak{p}) \begin{pmatrix} \mathfrak{p}^{e_1} & 0 & \cdots & 0 \\ 0 & \mathfrak{p}^{e_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{p}^{e_r} \end{pmatrix} \mathrm{GL}_r(A_\mathfrak{p}).$$

It is not necessarily true that all elements in the same Cartan set have the same characteristic polynomial. However, two elements are in the same Cartan set if and only if their characteristic polynomials have the same Newton polygon. The determinant of any element in the Cartan set

$$M_{e_1,\cdots,e_r} = \mathrm{GL}_r(A_{\mathfrak{p}}) \begin{pmatrix} \mathfrak{p}^{e_1} & 0 & \cdots & 0 \\ 0 & \mathfrak{p}^{e_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{p}^{e_r} \end{pmatrix} \mathrm{GL}_r(A_{\mathfrak{p}})$$

has $\mathfrak{p}$-adic valuation $\sum_{i=1}^{r} e_i$. The next term of the characteristic polynomial has $\mathfrak{p}$-adic valuation $\sum_{i=2}^{r} e_i$. The pattern continues until finally, the trace has $\mathfrak{p}$-adic valuation e_r. The slopes of the Newton polygon are therefore $-e_1, -e_2, \cdots, -e_r$. We have already assumed that the Newton polygon of γ_0 has one root of $\mathfrak{p}$-adic valuation m and $r-1$ roots which are $\mathfrak{p}$-adic units. This means that γ_0 lies in the double coset

$$M_{m,0,\cdots,0} = \mathrm{GL}_r(A_{\mathfrak{p}})\lambda_m \mathrm{GL}_r(A_{\mathfrak{p}}),$$

where

$$\lambda_m = \begin{pmatrix} \mathfrak{p}^{m} & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{6.11}$$

The following proposition, though not groundbreaking, is useful in the application of the Fundamental Lemma to our setting.

Proposition 6.2.11. *Let $M(A_{\mathfrak{p}}) \cap {}^{GL_r(K_{\mathfrak{l}})}\gamma_0$ be the integral points of the orbit of γ_0. There is an equality of sets*

$$M(A_{\mathfrak{p}}) \cap {}^{GL_r(K_{\mathfrak{l}})}\gamma_0 = M_{m,0,\cdots,0} \cap {}^{GL_r(K_{\mathfrak{l}})}\gamma_0. \tag{6.12}$$

Proof. Suppose $\gamma \in M(A_\mathfrak{p}) \cap {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0$. Then the characteristic of γ is *exactly equal* to P_{γ_0} and therefore has the exact Newton polygon as γ_0, which means it is an element of $M_{m,0\cdots,0}$. Alternatively, $M_{m,0,\cdots,0}$ is integral because each of the powers on $\mathfrak{p}$ in λ_m are non-negative. □

In [18], Iwahori and Matsumoto show that $M_{m,0,\cdots,0}$ can be further decomposed as the disjoint union of finitely many left (or right) cosets

$$M_{m,0,\cdots,0} = \bigsqcup_{i=0}^{s} g_i \mathrm{GL}_r(A_\mathfrak{p}).$$

Define $\mathbb{1}_M : \mathrm{GL}_r(K_\mathfrak{p}) \to \{0,1\}$ to be the characteristic function of $M_{m,0,\cdots,0}$, i.e.

$$\mathbb{1}_M(\gamma) = \begin{cases} 1 & \gamma \in M_{m,0,\cdots,0}, \\ 0 & \gamma \notin M_{m,0,\cdots,0}. \end{cases} \tag{6.13}$$

The Gekeler ratio at $\mathfrak{p}$ is related to the geometric orbital integral $O^{geom}_{\gamma_0}(\mathbb{1}_M)$. In Lemma 6.2.13, we will show this relationship explicitly. First, it is convenient to elaborate on Proposition 6.2.4 in terms of the set $M_{m,0,\cdots,0}$ by relaxing Proposition 6.12 to a neighborhood of the orbit.

Proposition 6.2.12. *Let $\gamma_0 \in M(A_\mathfrak{p})$ be chosen as above; it corresponds to an ordinary Drinfeld module and thus its characteristic polynomial has $r-1$ roots which are $\mathfrak{p}$-adic units and one root with $\mathfrak{p}$-adic valuation m. Then for a neighborhood $\widetilde{U}_n(\gamma_0)$ of the characteristic polynomial, we have the equality*

$$M_{m,0,\cdots,0} \cap c^{-1}(\widetilde{U}_n(\gamma_0)) = M(A_\mathfrak{p}) \cap c^{-1}(\widetilde{U}_n(\gamma_0)). \tag{6.14}$$

Proof. Clearly, $M_{m,0,\cdots,0} \cap c^{-1}(\widetilde{U}_n(\gamma_0)) \subseteq M(A_\mathfrak{p}) \cap c^{-1}(\widetilde{U}_n(\gamma_0))$ since $M_{m,0,\cdots,0}$ is integral. To show the other subset inclusion, we take an element $\gamma \in M(A_\mathfrak{p}) \cap c^{-1}(\widetilde{U}_n(\gamma_0))$. By definition, $P_\gamma \equiv P_{\gamma_0} \mod \mathfrak{p}^n$. Since $P_{\gamma_0} \mod \mathfrak{p}^n$ has $r-1$ roots that are not divisible by $\mathfrak{p}$, $P_\gamma \mod \mathfrak{p}$ has $r-1$ roots not divisible by $\mathfrak{p}$. Further, provided that $n > m$, $P_{\gamma_0} \mod \mathfrak{p}^n$ has one root divisible by $\mathfrak{p}^m$, which implies $P_\gamma \mod \mathfrak{p}^n$ does too. This means that $r-1$ of the roots of P_γ over $A_\mathfrak{p}$ have valuation zero, and one root has valuation m. Therefore $\gamma \in M_{m,0,\cdots,0} \cap c^{-1}(\widetilde{U}_n(\gamma_0))$. □

Lemma 6.2.13.

$$O^{geom}_{\gamma_0}(\mathbb{1}_M) = \lim_{d\to\infty}\lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_{GL_r}|}(\widetilde{Q}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{|d\omega_C|}(\widetilde{U}_n(\gamma_0))}. \tag{6.15}$$

Proof. By definition, $O^{geom}_{\gamma_0}(\mathbb{1}_M)$ is the geometric volume of the set of points in both $M_{m.0,\cdots,0}$ and the orbit of γ_0. That is,

$$O^{geom}_{\gamma_0}(\mathbb{1}_M) = \mathrm{vol}_{|\mu^{geom}_{\gamma_0}|}(M_{m,0,\cdots,0} \cap c^{-1}(c(\gamma_0))).$$

By (4.13), we may write this as the limit

$$O^{geom}_{\gamma_0}(\mathbb{1}_M) = \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(M_{m,0,\cdots,0} \cap c^{-1}(\widetilde{U}_n(\gamma_0)))}{\mathrm{vol}_{|d\omega_C|}(\widetilde{U}_n(\gamma_0))}.$$

Propositions 6.2.12 and 6.2.4 together imply the result.

□

Unlike in the case when $\mathfrak{l} \neq \mathfrak{p}$, $\widetilde{Q}_{(d,n)}(\gamma_0)$ is not a subset of $\mathrm{GL}_r(A_\mathfrak{p})$, so the Serre-Oesterlé measure does not coincide with the canonical measure. The next lemma allows us to transfer between the two measures on $\widetilde{Q}_{(d,n)}(\gamma_0)$, which as above, is a subset of $M_{m,0,\cdots,0}$.

Lemma 6.2.14. *On subsets of $M_{m,0,\cdots,0}$, the Serre-Oesterlé measure is related to the canonical measure by*

$$|\mu^{SO}_G| = |\det(\gamma_0)|^r_\mathfrak{p} \cdot |d\omega_{GL_r}|. \tag{6.16}$$

Proof. Since $M_{m,0,\cdots,0} = \bigsqcup_{i=0}^s g_i\mathrm{GL}_r(A_\mathfrak{p})$ is the *disjoint union* of $\mathrm{GL}_r(A_\mathfrak{p})$-cosets, and the Serre-Oesterlé measure is left (and right) invariant on $\mathrm{GL}_r(A_\mathfrak{p})$-cosets, it suffices to compare the two measures on just one of these cosets. In particular, since $\mathrm{vol}_{\mu_G^{\mathrm{so}}}(g_i \mathrm{GL}_r(A_{\mathfrak{p}}))$ is independent of g_i, we may as well compare $\mathrm{vol}_{\mu_G^{\mathrm{so}}}(\lambda_m \mathrm{GL}_r(A_{\mathfrak{p}}))$ and $\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(\lambda_m \mathrm{GL}_r(A_{\mathfrak{p}}))$. By definition of the Serre-Oesterlé measure, we have

$$\mathrm{vol}_{\mu_G^{\mathrm{so}}}(\lambda_m \mathrm{GL}_r(A_{\mathfrak{p}})) = \lim_{n\to\infty} \frac{\#(\pi_n(\lambda_m \mathrm{GL}_r(A_{\mathfrak{p}})))}{|\mathfrak{p}|^{n\dim(G)}}. \tag{6.17}$$

The canonical measure is normalized to give each $\mathrm{GL}_r(A_{\mathfrak{p}})$-coset volume $\frac{\#\mathrm{GL}_r(A_{\mathfrak{p}}/\mathfrak{p})}{|\mathfrak{p}|^{\dim(G)}}$. To compare the two measures, we first simplify (6.17). The number $\#(\pi_n(\lambda_m \mathrm{GL}_r(A_{\mathfrak{p}})))$ is the number of matrices of the form

$$\begin{pmatrix} \mathfrak{p}^m \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_r} \end{pmatrix}, \text{ where } \begin{pmatrix} \vec{a_1} \\ \vdots \\ \vec{a_r} \end{pmatrix} \in \mathrm{GL}_r(A_{\mathfrak{p}}/\mathfrak{p}^n)$$

provided $n \gg m$. (It doesn't matter if we reduce modulo $\mathfrak{p}^n$ and then multiply by $\mathfrak{p}^m$ or if we multiply then reduce.) First observe that there are

$$(|\mathfrak{p}|^r - 1)(|\mathfrak{p}|^r - |\mathfrak{p}|)\cdots(|\mathfrak{p}|^r - |\mathfrak{p}|^{r-1})$$

elements in $\mathrm{GL}_r(A_{\mathfrak{p}}/\mathfrak{p})$. For each choice in $\mathrm{GL}_r(A_{\mathfrak{p}}/\mathfrak{p})$, we obtain an element of the desired form by counting the number of lifts of the last $r-1$ rows to $A_{\mathfrak{p}}/\mathfrak{p}^{\mathrm{n}}$ and the number of ways to lift the first row to $\mathfrak{p}^m A_{\mathfrak{p}}/\mathfrak{p}^n$. Each of the last $r-1$ rows has $(|p|^{n-1})$ choices of lifts to $A_{\mathfrak{p}}/\mathfrak{p}^n$. The number of lifts of the first row $\vec{a_1}$ to $\mathfrak{p}^m A_{\mathfrak{p}}/\mathfrak{p}^n$ is equal to the number of vectors $\vec{b} \in (\mathfrak{p}^m A_{\mathfrak{p}}/\mathfrak{p}^n)^{\oplus r}$ such that $\frac{1}{\mathfrak{p}^m}\vec{b} = \vec{a_1} \mod \mathfrak{p}$. The total number of vectors $\vec{b}$ in $(\mathfrak{p}^m A_{\mathfrak{p}}/\mathfrak{p}^n)^{\oplus r}$ is $(|\mathfrak{p}|^{n-m})^r$. For each entry, the proportion of choices which project to the corresponding entry of $\vec{a_1} \mod \mathfrak{p}$ is $\frac{1}{\mathfrak{p}}$. This gives that the number of lifts of $\vec{a_1}$ to $(\mathfrak{p}^m A_{\mathfrak{p}}/\mathfrak{p}^n)^{\oplus r}$ is $\frac{(|\mathfrak{p}|^{n-m})^r}{(|\mathfrak{p}|)^r} = (|\mathfrak{p}|^{n-m-1})^r$. All together,

$$\#(\pi_n(\lambda_m \mathrm{GL}_r(A_\mathfrak{p}))) = ((|\mathfrak{p}|^{n-1})^r)^{r-1}(|\mathfrak{p}|^{n-m-1})^r(|\mathfrak{p}|^r - 1)(|\mathfrak{p}|^r - |\mathfrak{p}|)\cdots(|\mathfrak{p}|^r - |\mathfrak{p}|^{r-1})$$
$$= ((|\mathfrak{p}|^{n-1})^r)^{r-1}(|\mathfrak{p}|^{n-m-1})^r \#\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}).$$

We may now simplify the Serre-Oesterlé volume of Equation (6.17) to obtain

$$\begin{aligned}
\mathrm{vol}_{\mu_G^{\mathrm{SO}}}(\pi_n(\lambda_m \mathrm{GL}_r(A_\mathfrak{p}))) &= \lim_{n\to\infty} \frac{((|\mathfrak{p}|^{n-1})^r)^{r-1}(|\mathfrak{p}|^{n-m-1})^r \#\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p})}{|\mathfrak{p}|^{nr^2}} \\
&= \lim_{n\to\infty} \frac{|\mathfrak{p}|^{nr^2 - r^2 - nr + r + nr - mr - r} \#\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p})}{|\mathfrak{p}|^{nr^2}} \\
&= \lim_{n\to\infty} |\mathfrak{p}|^{-(r^2+mr)} \#\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p}).
\end{aligned}$$

We have thus shown that if $n \gg m$, the limit stabilizes. We must now compare the quantities

$$\mathrm{vol}_{\mu_G^{\mathrm{SO}}}(\lambda_m \mathrm{GL}_r(A_\mathfrak{p})) = \frac{\#\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p})}{|\mathfrak{p}|^{r^2+mr}} \tag{6.18}$$

and

$$\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(\lambda_m \mathrm{GL}_r(A_\mathfrak{p})) = \frac{\#\mathrm{GL}_r(A_\mathfrak{p}/\mathfrak{p})}{|\mathfrak{p}|^{r^2}}. \tag{6.19}$$

Clearly, (6.18) differs from (6.19) by a factor of $|\mathfrak{p}|^{-mr}$. However, $|\mathfrak{p}|^{-mr} = |\det(\gamma_0)|_\mathfrak{p}^r$, so

$$|\mu_G^{\mathrm{SO}}| = |\det(\gamma_0)|_\mathfrak{p}^r \cdot |d\omega_{\mathrm{GL}_r}|.$$

□

Corollary 6.2.15.

$$O_{\gamma_0}^{geom}(\mathbb{1}_M) = \lim_{d\to\infty}\lim_{n\to\infty} \frac{|\det(\gamma_0)|_\mathfrak{p}^{-r}\,\mathrm{vol}_{\mu_G^{so}}(\widetilde{Q}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{\mu_C^{so}}(\widetilde{U}_n(\gamma_0))} \tag{6.20}$$

Proof. By Lemma 6.2.14, we may replace $\mathrm{vol}_{|d\omega_{\mathrm{GL}_r}|}(\widetilde{Q}_{(d,n)}(\gamma_0))$ in Lemma 6.2.13 with

$$|\det(\gamma_0)|_{\mathfrak{p}}^{-r}\,\mathrm{vol}_{\mu_G^{\mathrm{so}}}(\widetilde{Q}_{(d,n)}(\gamma_0)).$$

□

Lemma 6.2.16.

$$v_{\mathfrak{p}}(\gamma_0) = |\det(\gamma_0)|_{\mathfrak{p}}^{r-1}\frac{|\mathfrak{p}|^{(r^2-1)}}{\#SL_r(A_{\mathfrak{p}}/\mathfrak{p})}O_{\gamma_0}^{geom}(\mathbb{1}_M). \tag{6.21}$$

Proof. By Corollary 6.2.15, the right-hand side of Equation (6.21) is

$$|\det(\gamma_0)|_{\mathfrak{p}}^{r-1}\frac{|\mathfrak{p}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{p}}/\mathfrak{p})}\lim_{d\to\infty}\lim_{n\to\infty}\frac{|\det(\gamma_0)|_{\mathfrak{p}}^{-r}\,\mathrm{vol}_{\mu_G^{\mathrm{so}}}(\widetilde{Q}_{(d,n)}(\gamma_0))}{\mathrm{vol}_{\mu_C^{\mathrm{so}}}(\widetilde{U}_n(\gamma_0))}. \tag{6.22}$$

The Serre-Oesterlé volume of $\widetilde{U}_n(\gamma_0)$ is $\frac{|\mathfrak{p}|^{-rn}}{|\det(\gamma_0)|_{\mathfrak{p}}}$, while as in Corollary 6.2.9, the Serre-Oesterlé volume of $\widetilde{Q}_{(d,n)}(\gamma_0)$ is $\frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{p}|^{nr^2}}$. Substituting these quantities into (6.22) and simplifying gives the equality between the right-hand side of (6.21) and $\lim_{d\to\infty}\lim_{n\to\infty}\frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{p}}/\mathfrak{p})|\mathfrak{p}|^{nr^2-rn-r^2+1}}$:

$$\begin{aligned}
&|\det(\gamma_0)|_{\mathfrak{p}}^{r-1}\frac{|\mathfrak{p}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{p}}/\mathfrak{p})}\lim_{d\to\infty}\lim_{n\to\infty}\frac{|\det(\gamma_0)|_{\mathfrak{p}}^{-r}\frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{p}|^{nr^2}}}{\frac{|\mathfrak{p}|^{-rn}}{|\det(\gamma_0)|_{\mathfrak{p}}}}\\
&\qquad= |\det(\gamma_0)|_{\mathfrak{p}}^{r-1}\frac{|\mathfrak{p}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{p}}/\mathfrak{p})}\lim_{d\to\infty}\lim_{n\to\infty}\frac{|\det(\gamma_0)|_{\mathfrak{p}}^{-r+1}\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{p}|^{nr^2-rn}}\\
&\qquad= \frac{|\mathfrak{p}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{p}}/\mathfrak{p})}\lim_{d\to\infty}\lim_{n\to\infty}\frac{\#Q_{(d,n)}(\gamma_0)}{|\mathfrak{p}|^{nr^2-rn}}\\
&\qquad= \lim_{d\to\infty}\lim_{n\to\infty}\frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{p}}/\mathfrak{p})|\mathfrak{p}|^{nr^2-rn-r^2+1}}.
\end{aligned}$$

As in the proof of Lemma 6.2.10, we multiply and divide by $|\mathfrak{p}|^{rn-n}$ to obtain that the right-hand side of (6.21) is equal to $\lim_{d\to\infty}\lim_{n\to\infty}\frac{|\mathfrak{p}|^{rn-n}\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_\mathfrak{p}/\mathfrak{p})|\mathfrak{p}|^{(r^2-1)(n-1)}}$. Hensel's Lemma implies that the right-hand side of (6.21) is

$$\lim_{d\to\infty}\lim_{n\to\infty}\frac{|\mathfrak{p}|^{rn-n}\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_\mathfrak{p}/\mathfrak{p})|\mathfrak{p}|^{(r^2-1)(n-1)}}=\lim_{d\to\infty}\lim_{n\to\infty}\frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_\mathfrak{p}/\mathfrak{p}^n)/|\mathfrak{p}|^{n(r-1)}}=v_\mathfrak{p}(\gamma_0).$$

□

We have now shown that the Gekeler ratio at $\mathfrak{p}$ can be written in terms of an ordinary orbital integral on the orbit of γ_0 in $\mathrm{GL}_r(K_\mathfrak{p})$, not a twisted orbital integral as in (5.18). Luckily, the Fundamental Lemma allows us to translate between ordinary orbital integrals for $M_{m,0,\cdots,0}$ and twisted orbital integrals for $\mathrm{GL}_r(A_L)\epsilon_p\mathrm{GL}_r(A_L)$, provided the measure used on the ordinary orbital integral has been normalized to give $\mathrm{GL}_r(A_\mathfrak{p})$ volume one. Since the geometric measure has not been normalized in this way, we must first relate the geometric measure to a measure consistent with Laumon's formula and the Fundamental Lemma.

6.3 Transfer of Measures

The task at hand is to relate the geometric measures at each place to the measures $\frac{dh_\mathfrak{l}}{d\omega^{Tama}_{\mathbf{T},\mathfrak{l}}}$ chosen in Section 6.1.

Proposition 6.3.1. *For all places $\mathfrak{l}\neq\infty$ of K,*

$$d\omega^{geom}_{\gamma_0,\mathfrak{l}}=|D(\gamma_0)|_\mathfrak{l}^{1/2}|\det(\gamma_0)|_\mathfrak{l}^{\frac{-(r-1)}{2}}\frac{\#GL_r(A_\mathfrak{l}/\mathfrak{l})}{|\mathfrak{l}|^{r^2}}\cdot L_\mathfrak{l}(1,\sigma_T)\cdot\frac{dh_\mathfrak{l}}{d\omega^{Tama}_{\mathbf{T},\mathfrak{l}}}. \tag{6.23}$$

Proof. Starting with Equation (4.17), we know that

$$d\omega^{geom}_{\gamma_0,\mathfrak{l}}=|D(\gamma_0)|_\mathfrak{l}^{1/2}|\det(\gamma_0)|_\mathfrak{l}^{-\frac{r-1}{2}}\cdot d\omega_{\mathbf{T}\backslash\mathrm{GL}_r,\mathfrak{l}}. \tag{6.24}$$

We then observe that the measure $d\omega_{\mathrm{GL}_r}$ used to define the quotient measure $d\omega_{\mathbf{T}\backslash \mathrm{GL}_r,\mathfrak{l}}$ has not been normalized to give volume one to the integral points while $dh_{\mathfrak{l}}$ has been, i.e.,

$$\frac{d\omega_{\mathrm{GL}_r,\mathfrak{l}}}{\#\mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l})/|\mathfrak{l}|^{r^2}} = dh_{\mathfrak{l}}. \tag{6.25}$$

Finally, the measure $d\omega_{\mathbf{T},\mathfrak{l}}$ used in $d\omega_{\mathbf{T}\backslash \mathrm{GL}_r}$ differs from $d\omega^{Tama}_{\mathbf{T},\mathfrak{l}}$ by a factor of $L_{\mathfrak{l}}(1,\sigma_{\mathbf{T}})^{-1}$. This implies that

$$d\omega_{\mathbf{T}\backslash \mathrm{GL}_r,\mathfrak{l}} = \frac{\#\mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l})/|\mathfrak{l}|^{r^2} dh_{\mathfrak{l}}}{L_{\mathfrak{l}}(1,\sigma_{\mathbf{T}})^{-1} d\omega^{Tama}_{\mathbf{T},\mathfrak{l}}}. \tag{6.26}$$

Substituting this into Equation (6.24) shows the desired relationship. □

We can now write the Gekeler ratios in terms of orbital integrals with respect to the local measures $d\omega^{Tama}_{\mathbf{T},\mathfrak{l}}$ consistent with Laumon's formula.

Corollary 6.3.2. *For $\mathfrak{l} \neq \mathfrak{p}$, the Gekeler ratio and local ordinary orbital integrals are related by*

$$v_{\mathfrak{l}}(\gamma_0) = |D(\gamma_0)|_{\mathfrak{l}}^{1/2} L_{\mathfrak{l}}(1,\sigma_{\mathbf{T}})(1-1/|\mathfrak{l}|) \cdot O^{Tama}_{\gamma_0,\mathfrak{l}}(\mathbb{1}_{GL_r(A_{\mathfrak{l}})}). \tag{6.27}$$

Proof. We start by combining Proposition 6.3.1 and Lemma 6.2.10 to obtain

$$v_{\mathfrak{l}}(\gamma_0) = \frac{|\mathfrak{l}|^{r^2-1}}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l})} \cdot |D(\gamma_0)|_{\mathfrak{l}}^{1/2} |\det(\gamma_0)|_{\mathfrak{l}}^{\frac{-(r-1)}{2}} \frac{\#\mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l})}{|\mathfrak{l}|^{r^2}} \cdot L_{\mathfrak{l}}(1,\sigma_T) \cdot O^{Tama}_{\gamma_0}(\mathbb{1}_{\mathrm{GL}_r(A_{\mathfrak{l}})}). \tag{6.28}$$

We simplify by observing $|\det(\gamma_0)|_{\mathfrak{l}} = 1$ for $\mathfrak{l} \neq \mathfrak{p}$, and $\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l}) = \frac{\#\mathrm{GL}_r(A_{\mathfrak{l}}/\mathfrak{l})}{|\mathfrak{l}|(1-\frac{1}{|\mathfrak{l}|})}$.

□

Corollary 6.3.3. *At $\mathfrak{p}$, the Gekeler ratio is related to an ordinary orbital integral by*

$$v_{\mathfrak{p}}(\gamma_0) = |\mathfrak{p}|^{\frac{-m(r-1)}{2}} |D(\gamma_o)|_{\mathfrak{p}}^{1/2} L_{\mathfrak{p}}(1,\sigma_{\mathbf{T}})(1-1/|\mathfrak{p}|) O^{Tama}_{\gamma_0}(\mathbb{1}_M). \tag{6.29}$$

Proof. For $\mathfrak{p}$, the only differences are that there is an additional factor of $|\det(\gamma_0)|_{\mathfrak{p}}^{r-1}$ at the beginning and $|\det(\gamma_0)|_{\mathfrak{p}} = |\mathfrak{p}|^{-m}$, so we have

$$v_{\mathfrak{p}}(\gamma_0) = |\mathfrak{p}|^{-m(r-1)}|D(\gamma_o)|_{\mathfrak{p}}^{1/2} \cdot |\mathfrak{p}|^{\frac{m(r-1)}{2}} L_{\mathfrak{p}}(1, \sigma_{\mathbf{T}})(1 - 1/|\mathfrak{p}|)O_{\gamma_0}^{Tama}(\mathbb{1}_M) \tag{6.30}$$

$$= |\mathfrak{p}|^{\frac{-m(r-1)}{2}}|D(\gamma_o)|_{\mathfrak{p}}^{1/2} L_{\mathfrak{p}}(1, \sigma_{\mathbf{T}})(1 - 1/|\mathfrak{p}|)O_{\gamma_0}^{Tama}(\mathbb{1}_M). \tag{6.31}$$

□

6.3.1 The product of Gekeler Ratios

So far, we have shown that Gekeler ratios can be reformulated as local orbital integrals. Taking the product over all primes of Gekeler ratios yields an expression involving the adèlic orbital integral for the finite and prime-to-$\mathfrak{p}$ places and an *ordinary* orbital integral at $\mathfrak{p}$.

Corollary 6.3.4. *The product over finite places of Gekeler ratios is related to the ordinary adèlic orbital integrals of Laumon's formula by*

$$\prod_{\mathfrak{l} \neq \infty} v_{\mathfrak{l}}(\gamma_0) = |\mathfrak{p}|^{\frac{-m(r-1)}{2}}|D(\gamma_0)|_{\infty}^{-1/2} \cdot \prod_{\mathfrak{l} \neq \infty}(1 - 1/|\mathfrak{l}|) \cdot \prod_{\mathfrak{l} \neq \infty} L_{\mathfrak{l}}(1, \sigma_{\mathbf{T}}) \cdot O_{\gamma_0, f}^{Tama}(\mathbb{1}_{GL_r(\hat{\mathbf{A}}^{\mathfrak{p}})}) \cdot O_{\gamma_0, \mathfrak{p}}^{Tama}(\mathbb{1}_M). \tag{6.32}$$

Proof. In combining the previous two corollaries and simplifying, we find

$$\begin{aligned}
\prod_{\mathfrak{l} \neq \infty} v_{\mathfrak{l}}(\gamma_0) = &\prod_{\mathfrak{l} \neq \mathfrak{p}, \infty} |D(\gamma_0)|_{\mathfrak{l}}^{1/2} L_{\mathfrak{l}}(1, \sigma_{\mathbf{T}})(1 - 1/|\mathfrak{l}|) \cdot O_{\gamma_0, \mathfrak{l}}^{Tama}(\mathbb{1}_{\mathrm{GL}_r(A_{\mathfrak{l}})}) \\
&\cdot |\mathfrak{p}|^{\frac{-m(r-1)}{2}}|D(\gamma_0)|_{\mathfrak{p}}^{1/2} L_{\mathfrak{p}}(1, \sigma_{\mathbf{T}})(1 - 1/|\mathfrak{p}|) \cdot O_{\gamma_0, \mathfrak{p}}^{Tama}(\mathbb{1}_M) \\
&= |\mathfrak{p}|^{\frac{-m(r-1)}{2}}|D(\gamma_0)|_{\infty}^{-1/2} \cdot \prod_{\mathfrak{l} \neq \infty}(1 - 1/|\mathfrak{l}|) \cdot \prod_{\mathfrak{l} \neq \infty} L_{\mathfrak{l}}(1, \sigma_{\mathbf{T}}) \\
&\cdot O_{\gamma_0, f}^{Tama}(\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}) \cdot O_{\gamma_0, \mathfrak{p}}^{Tama}(\mathbb{1}_M).
\end{aligned} \tag{6.33}$$

□

6.4 The Fundamental Lemma

In this section, we will prove that the Fundamental Lemma applies to relate Laumon's twisted orbital integral with our ordinary one. We start with the twisted orbital integral. To verify that Theorem 5.3.7 applies, we compute the Satake transform of $\mathbb{1}_{S_m}$ and its base change explicitly. After this, it suffices to show that the resulting function agrees with $\mathbb{1}_M$ on the sets of interest.

By assumption on the Newton polygon of γ_0, we know that it is a closed m-admissible element γ_0 of GL_r. Further, we know that the sign function is just one. Therefore, we have the equality

$$O_{\gamma_0}^{Tama}(f) = TO_{\gamma_0}^{Tama}(\mathbb{1}_{S_m}) \tag{6.34}$$

between ordinary and twisted orbital integrals, where f is the base change of $\mathbb{1}_{S_m}$ from the Hecke algebra $\mathcal{H}_m$ to the Hecke algebra $\mathcal{H}$. Recall that we have completely characterized f by Theorem 5.3.7 and the subsequent remarks. Indeed, $f = b(\mathbb{1}_{S_m})$ is the function $f : \mathrm{GL}_r(K_\mathfrak{p}) \to \mathbb{Z}$ defined by

$$f(\gamma) = \begin{cases} \prod_{i=1}^{\varrho-1}(1-q^i) & \gamma \in M(A_\mathfrak{p}) \cap \mathrm{GL}_r(K_\mathfrak{p}) \text{ and } \mathrm{val}_\mathfrak{p}(\det(\gamma_0)) = m, \\ 0 & \text{otherwise,} \end{cases} \tag{6.35}$$

where ϱ is the nullity of $\gamma \mod \mathfrak{p}$. In this section, we show that our ordinary orbital integral $O_{\gamma_0}^{Tama}(\mathbb{1}_M)$ can be replaced by Laumon's twisted orbital integral $TO_{\gamma_0}^{Tama}(\mathbb{1}_{S_m})$. It suffices to show

$$O_{\gamma_0}^{Tama}(\mathbb{1}_M) = O_{\gamma_0}^{Tama}(f) \tag{6.36}$$

by showing the functions agree. Clearly, f and $\mathbb{1}_M$ *do not agree* in general since f can take on values other than zero and one. Thankfully, since we are integrating these functions over the orbit of γ_0, we only need that $f(\gamma) = \mathbb{1}_M(\gamma)$ for elements γ in the orbit ${}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0$. By the fact that γ_0 has a square-free characteristic polynomial, all elements in the orbit satisfy have the same characteristic polynomial as that of γ_0 and therefore satisfy the condition that the valuation on the determinant is m. Likewise, the Newton polygon of the characteristic polynomial implies that all elements in the orbit have exactly one eigenvalue divisible by $\mathfrak{p}$. Therefore, for *every* $\gamma \in {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0$ has nullity one modulo $\mathfrak{p}$. Hence on the orbit ${}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0$, it is always the case that $\varrho - 1 = 0$. Together, this means that on ${}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0$, we have

$$f(\gamma) = \begin{cases} 1 & \gamma \in M(A_\mathfrak{p}) \cap {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0, \\ 0 & \gamma \notin M(A_\mathfrak{p}) \cap {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0. \end{cases} \tag{6.37}$$

Consequently, for $\gamma \in {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0$, $\mathbb{1}_M(\gamma) = f(\gamma)$ if and only if

$$M(A_\mathfrak{p}) \cap {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0 = M_{m,0,\cdots,0} \cap {}^{\mathrm{GL}_r(K_\mathfrak{l})}\gamma_0.$$

However, this is exactly Proposition 6.12, so we have proven the next lemma and subsequent corollary.

Lemma 6.4.1. *On the orbit of γ_0, we have the equality of integrals*

$$TO_{\gamma_0}^{Tama}(\mathbb{1}_{S_m}) = O_{\gamma_0}^{Tama}(\mathbb{1}_M). \tag{6.38}$$

Corollary 6.4.2. *The product over finite places of Gekeler ratios is*

$$\prod_{\mathfrak{l}\neq\infty} v_\mathfrak{l}(\gamma_0) = |\mathfrak{p}|^{\frac{-m(r-1)}{2}} |D(\gamma_0)|_\infty^{-1/2} \cdot \prod_{\mathfrak{l}\neq\infty}(1-1/|\mathfrak{l}|) \cdot \prod_{\mathfrak{l}\neq\infty} L_\mathfrak{l}(1,\sigma_\mathbf{T}) \cdot O_{\gamma_0,f}^{Tama}(\mathbb{1}_{GL_r(\hat{\mathbf{A}}^\mathfrak{p})}) \cdot TO_{\gamma_0,\mathfrak{p}}^{Tama}(\mathbb{1}_{S_m}). \tag{6.39}$$

Remark 6.4.3. In preparation for the final theorem, we rewrite the previous corollary as follows.

$$O_{\gamma_0,f}^{Tama}(\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}) \cdot TO_{\gamma_0,\mathfrak{p}}^{Tama}(\mathbb{1}_{S_m}) = |\mathfrak{p}|^{\frac{m(r-1)}{2}} |D(\gamma_0)|_\infty^{1/2} \frac{\prod_{\mathfrak{l}\neq\infty}(1-1/|\mathfrak{l}|)^{-1}}{\prod_{\mathfrak{l}\neq\infty} L_{\mathfrak{l}}(1,\sigma_{\mathbf{T}})} \prod_{\mathfrak{l}\neq\infty} v_{\mathfrak{l}}(\gamma_0). \quad (6.40)$$

6.5 The Global Factor

The purpose of this section is to explicitly compute the global factor

$$\mathrm{vol}_{d\omega_{\mathbf{T},f}^{Tama}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)). \quad (6.41)$$

At the beginning of this chapter, we chose to use the measure $d\omega_{\mathbf{T},f}^{Tama}$ on this space so that we could leverage known results about the Tamagawa number of the torus for our calculation.

The Tamagawa measure is a measure on the set of norm one adèlic points of the torus defined conventionally in the following way. To begin, one may fix *any* differential form on $\mathbf{T}(K)$ and associated measure. Recall that the differential form $\omega_{\mathbf{T}}$ and associated differential form $d\omega_{\mathbf{T}}$ as defined in Chapter 4 are not defined over K unless $\mathbf{T}$ splits over K. However, there is some $\Delta \in K$ such that the measure $\frac{d\omega_{\mathbf{T}}}{\sqrt{|\Delta|}}$ is defined over K (see [14, Section 5.1.1] for details). Then let

$$\rho_{\mathbf{T}} := \lim_{s\to 1}(s-1)^{r_{\mathbf{T}}} L(s,\sigma_{\mathbf{T}}) \quad (6.42)$$

be the residue of the Artin L-function associated with the representation $\sigma_{\mathbf{T}}$ at $s=1$, as seen in Chapter 4. By $r_{\mathbf{T}}$, we mean the number of characters defined over K. In particular, since $\mathbf{T}$ is the restriction of scalars of the multiplicative group of E, the only character is the field norm map $N_{E/K}$, and thus $r_{\mathbf{T}} = 1$. Let $d_{\mathbf{T}}$ be the dimension of $\mathbf{T}$ over K, in this case, $d_{\mathbf{T}} = r$. We additionally let g be the genus of the *base field* field K, so $g = 0$. Then $\mu_{\mathbf{T}(\mathbb{A})}$ is the measure on the adèlic points of $\mathbf{T}$ defined by

$$\begin{aligned} d\mu_{\mathbf{T}(\mathbb{A})} &:= q^{-d_{\mathbf{T}}(g-1)}\rho_{\mathbf{T}}^{-1}\prod_{\mathfrak{v}} L_{\mathfrak{v}}(1,\sigma_{\mathbf{T}})\frac{d\omega_{\mathbf{T},\mathfrak{v}}}{\sqrt{|\Delta|_{\mathfrak{v}}}} \\ &= q^{-d_{\mathbf{T}}(g-1)}\rho_{\mathbf{T}}^{-1}\prod_{\mathfrak{v}} L_{\mathfrak{v}}(1,\sigma_{\mathbf{T}})d\omega_{\mathbf{T},\mathfrak{v}}. \end{aligned} \tag{6.43}$$

The simplification is due to the product formula – indeed, $\prod_{\mathfrak{v}} \frac{1}{\sqrt{|\Delta|_{\mathfrak{v}}}} = 1$. In our specific setting, we have

$$d\mu_{\mathbf{T}(\mathbb{A})} = q^{r}\rho_{\mathbf{T}}^{-1}\prod_{\mathfrak{v}} L_{\mathfrak{v}}(1,\sigma_{\mathbf{T}})d\omega_{\mathbf{T},\mathfrak{v}}. \tag{6.44}$$

The *Tamagawa number of* $\mathbf{T}$ is by definition the volume of $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1$, where $\mathbf{T}(\mathbb{A})^1$ is defined as follows.

Definition 6.5.1. Define the adèlic norm one torus to be

$$\mathbf{T}(\mathbb{A})^1 := \{(x_{\mathfrak{v}}) \in \mathbf{T}(\mathbb{A}) : \prod_{\mathfrak{v}} |\chi(x_{\mathfrak{v}})|_{\mathfrak{v}} = 1 \text{ for all } \chi \in X^*(\mathbf{T}) \text{ defined over } K\}. \tag{6.45}$$

For $\mathbf{T} = \mathrm{Res}_{E/K}(\mathbb{G}_m)$, the only rational character is the field norm for E/K. Thus

$$\mathbf{T}(\mathbb{A})^1 := \{(x_{\mathfrak{v}}) \in \mathbf{T}(\mathbb{A}) : \prod_{\mathfrak{v}} |N_{E_{\mathfrak{v}}/K_{\mathfrak{v}}}(x_{\mathfrak{v}})|_{\mathfrak{v}} = 1\}. \tag{6.46}$$

The measure $d\mu_{\mathbf{T}(\mathbb{A})}$ naturally induces *the Tamagawa measure* $d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}$ on $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1$ since $\mathbf{T}(\mathbb{A})^1$ includes into $\mathbf{T}(\mathbb{A})$.

Definition 6.5.2. The *Tamagawa number of* $\mathbf{T}$ *over* K is the measure one has to give to the compact subgroup $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1$ in order that $d\mu_{\mathbf{T}(\mathbb{A})}$ decomposes as

$$d\mu_{\mathbf{T}(\mathbb{A})} = d\mu_{\mathbf{T}(\mathbb{A})^1\backslash\mathbf{T}(\mathbb{A})} \cdot d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1} \cdot d\mu_K \tag{6.47}$$

where $d\mu_{\mathbf{T}(\mathbb{A})^1\backslash\mathbf{T}(\mathbb{A})}$ comes from giving measure $\ln(q)$ to each point and $d\mu_K$ is the canonical discrete measure.[5]

The *Tamagawa measure* can be explained via the homomorphism

$$\begin{aligned} \alpha : \mathbf{T}(\mathbb{A}) &\to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{T})_K, q^{\mathbb{Z}}), \\ (x_{\mathfrak{v}}) &\mapsto \prod_{\mathfrak{v}} |N_{E_{\mathfrak{v}}}/K_{\mathfrak{v}}(x)_{\mathfrak{v}}|_{\mathfrak{v}}. \end{aligned} \tag{6.48}$$

By definition, the kernel of α is $\mathbf{T}(\mathbb{A})^1$, and to this kernel, we give each point measure $\ln(q)$. (Note, the power on $\ln(q)$ is equal to the number of characters.) If α is surjective, then $d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}$ is simply the measure

$$d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1} := \frac{d\mu_{\mathbf{T}(\mathbb{A}))}}{(\ln(q))^{r_{\mathbf{T}}}}.$$

Unfortunately, as Oesterle points out, α fails to be surjective when the field of constants in E/K is larger than $\mathbb{F}_q$ [25, Remarque 5.7] Gekeler refers to this case in [12] as the exceptional case (E1). However, the image of α is always of finite index in $\mathbf{T}(\mathbb{A})$, so $|\operatorname{coker}(\alpha)|$ is finite, and divides r. In the greatest generality, we have

$$d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1} := \frac{d\mu_{\mathbf{T}(\mathbb{A})|_{\mathbf{T}(\mathbb{A})^1}}}{\ln(q)|\operatorname{coker}(\alpha)|}. \tag{6.49}$$

It is well known that the Tamagawa number of the restriction of scalars of $\mathbb{G}_m$ is one (see [25,26,33] for an in-depth treatment of this calculation), which by definition means that

$$\mathrm{vol}_{d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1) = 1. \tag{6.50}$$

It turns out that the volume of the compact subgroup $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)$ is related to the volume of $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1$. This relationship comes from the natural homomorphism

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[5]There is a slight abuse of notation; $d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}$ refers to both the measure on $\mathbf{T}(\mathbb{A})$ and the quotient of this measure by the discrete measure on $\mathbf{T}(K)$.

$$\beta : \mathbf{T}(\mathbb{A})^1 \to \mathbf{T}(\mathbb{A}_f), \tag{6.51}$$

which truncates the adèlic tuple $(x_{\mathfrak{v}}) = ((x_{\mathfrak{l}})_{\mathfrak{l}\neq\infty}, x_\infty)$ by deleting the term at infinity. The kernel of β is the set of tuples with value one at each finite place and *norm one* at the infinite place, i.e.,

$$\ker(\beta) = \{((1_{\mathfrak{l}})_{\mathfrak{l}\neq\infty}, x_\infty) : |N_{E_{\tilde{\infty}}/K_\infty}(x_\infty)| = 1\}.$$

This kernel is isomorphic to the compact subgroup

$$\mathbf{T}(K_\infty)^c = \{t \in \mathbf{T}(K_\infty) : |\chi(t)|_{\mathfrak{l}} = 1 \text{ for } \chi \in X^*(\mathbf{T})_{K_\infty}\} \subset \mathbf{T}(K_\infty)$$

described in detail in 4, as per [14]. Again, the only character is $N_{E_{\tilde{\infty}}/K_\infty}$ (by Theorem 2.3.5, there is only one prime above ∞). Next, observe that an element $(x_{\mathfrak{l}})$ of $\mathbf{T}(\mathbb{A}_f)$ is in the image of β if and only if there is an element of the form $((x_{\mathfrak{l}}), x_\infty)$ in $\mathbf{T}(\mathbb{A})^1$. Equivalently, $(x_{\mathfrak{l}}) \in \mathrm{im}(\beta)$ if and only if there is an element $x_\infty \in E_{\tilde{\infty}}^\times$ such that

$$\prod_{\mathfrak{l}\neq\infty} |N_{E_{\mathfrak{l}}/K_{\mathfrak{l}}}(x_{\mathfrak{l}})|_{\mathfrak{l}} \cdot |N_{E_{\tilde{\infty}}/K_\infty}(x_\infty)|_\infty = 1,$$

i.e.,

$$\prod_{\mathfrak{l}\neq\infty} |N_{E_{\mathfrak{l}}/K_{\mathfrak{l}}}(x_{\mathfrak{l}})|_{\mathfrak{l}} = (|N_{E_{\tilde{\infty}}/K_\infty}(x_\infty)|_\infty)^{-1}.$$

This happens exactly as often as elements in $K_\infty^\times$ are in the image of $N_{E_{\tilde{\infty}}/K_\infty}$. According to Serre ([30, Page 87]), the index of the image of the norm map in $K_\infty^\times$ is controlled by the splitting behavior of ∞ in E. Specifically, we have that $\tilde{\infty}$ is the unique prime above ∞ in E, and the image of the norm map has index $f(E_{\tilde{\infty}}/K_\infty)$, the residue degree of ∞. (Note that if ∞ is *unramified* in E, the image of the norm map is of index r. If ∞ ramifies, the norm map is surjective.) Consequently, the image of β also has index $f(E_{\tilde{\infty}}/K_\infty)$ in $\mathbf{T}(\mathbb{A}_f)$. This is equivalent to saying $|\operatorname{coker}(\beta)| = f(E_{\tilde{\infty}/K_\infty})$.

According to this description of β, we have the short exact sequence

$$1 \to \mathbf{T}(K_\infty)^c \to \mathbf{T}(\mathbb{A})^1 \xrightarrow{\beta} \mathbf{T}(\mathbb{A}_f) \to \operatorname{coker}(\beta) \to 1. \tag{6.52}$$

Since the diagonal embedding of $\mathbf{T}(K) = E^\times$ intersects the image of $\mathbf{T}(K_\infty)^c$ in $\mathbf{T}(\mathbb{A})$ trivially, (6.52) gives the exact sequence for quotients by $\mathbf{T}(K)$:

$$1 \to \mathbf{T}(K_\infty)^c \to \mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1 \xrightarrow{\beta} \mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f) \to \operatorname{coker}(\beta) \to 1. \tag{6.53}$$

With this set up – the specific choice of measures used and a relationship between the compact subgroups $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1$ and $\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)$ – we can finally compute $\operatorname{vol}_{d\omega^{Tama}_{\mathbf{T},f}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f))$ explicitly.

Using (6.49), we observe that the Tamagawa number of $\mathbf{T}$ over K can be written in terms of the measure $d\mu_{\mathbf{T}(\mathbb{A})}$ as follows

$$1 = \operatorname{vol}_{d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1) = \frac{\operatorname{vol}_{d\mu_{\mathbf{T}(\mathbb{A})}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1)}{\ln(q)|\operatorname{coker}(\alpha)|}. \tag{6.54}$$

Next, the short exact sequence (6.53) implies that

$$\operatorname{vol}_{d\mu_{\mathbf{T}(\mathbb{A})}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1) = \frac{\operatorname{vol}_{d\omega^{Tama}_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)}{f(E_{\tilde{\infty}}/K_\infty)} \cdot \operatorname{vol}_{d\mu_{\mathbf{T}(\mathbb{A})}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)). \tag{6.55}$$

Note that the measures here are all chosen consistently; the measure $d\omega_{\mathbf{T},\infty}$ is the measure at infinity used in the definition of $d\mu_{\mathbf{T}(\mathbb{A})}$. Implicitly, the measure $d\mu_{\mathbf{T}(\mathbb{A})}$ has been restricted to the finite places in the expression $\operatorname{vol}_{d\mu_{\mathbf{T}(\mathbb{A})}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f))$. Finally, we want the volume in terms of the measure $d\omega^{Tama}_{\mathbf{T},f}$, not $d\mu_{\mathbf{T}(\mathbb{A})}$. By the definitions of these two measures, in our setting where $d_{\mathbf{T}} = r, g = 0$, they are related by

$$d\mu_{\mathbf{T}(\mathbb{A})} = q^r \rho_{\mathbf{T}}^{-1} d\omega_{\mathbf{T},f}^{Tama}. \tag{6.56}$$

Putting together Equations (6.54) and (6.55) yields

$$1 = \mathrm{vol}_{d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1) = \frac{\mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c) \cdot \mathrm{vol}_{d\mu_{\mathbf{T}(\mathbb{A})}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f))}{\ln(q) \cdot |\operatorname{coker}(\alpha)| \cdot f(E_{\tilde{\infty}}/K_\infty)}. \tag{6.57}$$

Finally, we transfer measures using Equation (6.56) to obtain

$$1 = \mathrm{vol}_{d\mu^{Tama}_{\mathbf{T}(\mathbb{A})^1}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A})^1) = \frac{\mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c) \cdot q^r \rho_{\mathbf{T}}^{-1}\, \mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},f}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f))}{\ln(q) \cdot |\operatorname{coker}(\alpha)| \cdot f(E_{\tilde{\infty}}/K_\infty)}. \tag{6.58}$$

In solving for $\mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},f}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f))$, and observing that

$$\mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c) = L_\infty(1, \sigma_{\mathbf{T}}) \cdot \mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c),$$

we have proven the next lemma.

Lemma 6.5.3. *The global volume term of Laumon's formula is explicitly given by*

$$\mathrm{vol}_{d\omega^{Tama}_{\mathbf{T},f}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)) = \frac{\ln(q) \cdot \rho_{\mathbf{T}} \cdot |\operatorname{coker}(\alpha)| \cdot f(E_{\tilde{\infty}}/K_\infty)}{q^r \cdot L_\infty(1, \sigma_{\mathbf{T}}) \cdot \mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)}. \tag{6.59}$$

6.6 Main Theorem

Equipped with this specific computation of the global volume, and the interpretation of the adèlic orbital integrals in terms of Gekeler ratios, we can now express the size of an isogeny class of ordinary rank r Drinfeld modules as a Gekeler-style product formula. Recall that for all finite places, we define the Gekeler ratios as follows.

Definition 6.6.1. For each finite prime $\mathfrak{l}$ of A (including $\mathfrak{p}$) define

$$v_{\mathfrak{l}}(\gamma_0) := \lim_{d\to\infty}\lim_{n\to\infty} \frac{\#Q_{(d,n)}(\gamma_0)}{\#\mathrm{SL}_r(A_{\mathfrak{l}}/\mathfrak{l}^n)/|\mathfrak{l}|^{n(r-1)}}. \tag{6.4}$$

At the infinite place, we introduce the following definition.

Definition 6.6.2. At the infinite place, define the number

$$v_\infty(\gamma_0) := |D(\gamma_0)|_\infty^{1/2} \frac{\rho_{\mathbf{T}} \cdot |\operatorname{coker}(\alpha)| \cdot f(E_{\tilde{\infty}}/K_\infty)}{q^r \cdot L(1,\sigma_{\mathbf{T}}) \cdot \mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)} \tag{6.60}$$

Equipped with these definitions, we are finally able to prove Theorem A.

Theorem A. *Let ϕ be an ordinary rank r Drinfeld module over the finite field L of degree m over $A/\mathfrak{p}$. Let $\gamma_0 \in GL_r(\mathbb{F}_q[T])$ be a matrix with the same characteristic polynomial as ϕ. The weighted size of the isogeny class of ϕ is*

$$h_\phi^*(L) = |\mathfrak{p}|^{\frac{m(r-1)}{2}} v_\infty(\gamma_0) \prod_{\mathfrak{l}} v_{\mathfrak{l}}(\gamma_0).$$

Proof. Starting with Laumon's formula, the weighted size of the isogeny class is given by

$$h_\phi^*(L) = \mathrm{vol}_{d\omega_{\mathbf{T},f}^{Tama}}(\mathbf{T}(K)\backslash\mathbf{T}(\mathbb{A}_f)) \cdot O_{\gamma_0,f}^{Tama}(\mathbb{1}_{\mathrm{GL}_r(\hat{\mathbf{A}}^{\mathfrak{p}})}) \cdot TO_{\gamma_0,\mathfrak{p}}^{Tama}(\mathbb{1}_{S_m}).$$

Into this formula, we directly substitute the values of the global volume term and the product of the two orbital integrals from Equations (6.59) and (6.40) respectively to obtain the size of the isogeny class as the product

$$h_\phi^*(L) = \frac{\ln(q)\cdot\rho_{\mathbf{T}} \cdot |\operatorname{coker}(\alpha)| \cdot f(E_{\tilde{\infty}}/K_\infty)}{q^r \cdot L_\infty(1,\sigma_{\mathbf{T}}) \cdot \mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)} \cdot |\mathfrak{p}|^{\frac{m(r-1)}{2}} |D(\gamma_0)|_\infty^{1/2} \frac{\prod_{\mathfrak{l}\neq\infty}(1-1/|\mathfrak{l}|)^{-1}}{\prod_{\mathfrak{l}\neq\infty} L_{\mathfrak{l}}(1,\sigma_{\mathbf{T}})} \prod_{\mathfrak{l}\neq\infty} v_{\mathfrak{l}}(\gamma_0). \tag{6.61}$$

Observe that the product

$$\prod_{\mathfrak{l}\neq\infty}(1-|\mathfrak{l}|^{-1})^{-1}$$

conditionally converges to the zeta function of A evaluated at one, which has residue $1/\ln(q)$, so this factor cancels with $\ln(q)$ [28, Chapter 5]. The product of the remaining factors is exactly $v_\infty(\gamma_0)$ as in Equation (6.60). Therefore, we have finally shown the desired result,

$$h_\phi^*(L) = |\mathfrak{p}|^{\frac{m(r-1)}{2}} v_\infty(\gamma_0) \prod_{\mathfrak{l}} v_{\mathfrak{l}}(\gamma_0). \tag{6.62}$$

□

6.7 Comparison with Gekeler's Formula

We now verify that our formula for the size of an isogeny class (6.62) of a rank two Drinfeld module ϕ over the "prime field" $\mathbb{F}_p[T]/\mathfrak{p}$ is equal to Gekeler's formula. The first step in comparing Formula (6.62) with Gekeler's (Equation (3.3)) is noting that the terms $v_{\mathfrak{l}}(\gamma_0)$ coincide with $v_{\mathfrak{l}}(a,b)$ as defined in (3.1) for $\mathfrak{l} \neq \infty$ (see Remark 6.2.7). Next, we both have a factor of $|\mathfrak{p}|^{1/2}$ at the beginning. Lastly, it suffices to show that our definition for $v_\infty(\gamma_0)$ (see (6.60)) is equal to Gekeler's definition for $v_\infty(a,b)$ (see (3.2)).

To do so, we will assume that E/K is a regular extension, and that the characteristic of K is not two. This implies that $|\operatorname{coker}(\alpha)| = 1$ in $v_\infty(\gamma_0)$. This gives

$$v_\infty(\gamma_0) = |D(\gamma_0)|_\infty^{1/2} \frac{\rho_{\mathbf{T}} \cdot f(E_{\tilde{\infty}}/K_\infty)}{q^2 \cdot L(1,\sigma_{\mathbf{T}}) \cdot \operatorname{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)}.$$

Next, the Artin L-function $L(1,\sigma_{\mathbf{T}})$ converges (conditionally) to $\rho_{\mathbf{T}}$. Additionally, Gekeler's $v_\infty(a,b)$ is a function of the polynomial discriminant $\Delta := a^2 - 4b$, while our $v_\infty(\gamma_0)$ involves the $D(\gamma_0)$, the Weyl discriminant of γ_0. By Equation (4.16), we have

$$|D(\gamma_0)|_\infty = \left| \frac{\Delta}{\det(\gamma_0)} \right|_\infty.$$

Even further, $\det(\gamma_0) = b$, which implies

$$|D(\gamma_0)|_\infty^{1/2} = \left| \frac{\Delta}{b} \right|_\infty^{1/2}.$$

Substituting this into our definition of v_∞, we now have

$$v_\infty(\gamma_0) = \left|\frac{\Delta}{b}\right|_\infty^{1/2} \cdot \frac{f(E_{\tilde{\infty}}/K_\infty)}{q^2 \cdot \mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c)}.$$

We can simplify further using the fact that there is only one prime above ∞ in E, which when combined with Equation (4.5) implies

$$\mathrm{vol}_{d\omega_{\mathbf{T},\infty}}(\mathbf{T}(K_\infty)^c) = \begin{cases} \frac{(q-1)(q+1)}{q^2} \text{ if } \infty \text{ is unramified in } E, \\ \frac{(q-1)}{q^{3/2}} \text{ if } \infty \text{ ramifies in } E. \end{cases}$$

We also know that $f(E_{\tilde{\infty}}/K_\infty) = 2$ (resp. 1) if the extension is unramified (resp. ramified). The extension is unramified if (and only if) the degree of Δ is even, and ramified when the degree of Δ is odd. Therefore,

$$v_\infty(\gamma_0) = \left|\frac{\Delta}{b}\right|^{1/2} \left\{ \begin{matrix} \dfrac{2}{(q+1)(q-1)} \\ \dfrac{q^{-1/2}}{q-1} \end{matrix} \right\} \text{ if } \infty \left\{ \begin{matrix} \text{is unramified in } E \\ \text{ramifies in } E \end{matrix} \right\},$$

which coincides exactly with Gekeler's definition as stated in Equation (3.2).

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Appendix A

Computing the Characteristic Polynomial

Continue to view L as a finite extension of $A/\mathfrak{p}$, viewed as an A -algebra with structure map γ. The method for determining the characteristic polynomial is as outlined in [12, Sections 2,3].

A.1 General Strategies (for Rank Two)

As stated in the Section 2.4, the characteristic polynomial of ϕ is the characteristic polynomial $i_{\mathfrak{l}}(F)$. We determine the coefficients using a system of equations. For an arbitrary rank Drinfeld module, such a system is daunting, but for rank two, the situation is not so bad. We demonstrate how to compute $P_\phi(X)$ for a rank two Drinfeld module ϕ given ϕ_T. First, we describe this process when L is any finite extension of $A/\mathfrak{p}$. Then, in the next section, we will restrict to the much easier case $L = A/\mathfrak{p}$. Following Gekeler's notation in [12], each rank two Drinfeld module is given by ϕ_T of the form $\phi_T = \gamma(T) + g\tau + \Delta\tau^2$ and the characteristic polynomial will be of the form $P_\phi(X) = X^2 - aX + b$. Define the invariant $j(\phi) := g^{q+1}/\Delta$. Because L is a finite extension of a finite field, it has Galois group Gal$(L : \mathbb{F}_q)$, with $n = d \cdot m$ elements. The evaluation of the norm $N^L_{\mathbb{F}_q}$ at α for any α in L is given by

$$N^L_{\mathbb{F}_q}(\alpha) = \prod_{\sigma \in \text{Gal}(L:\mathbb{F}_q)} \sigma(\alpha).$$

Similarly, the trace of the field extension is given by

$$\text{Tr}^L_{\mathbb{F}_q}(\alpha) = \sum_{\sigma \in \text{Gal}(L:\mathbb{F}_q)} \sigma(\alpha).$$

If a is an element of $\mathbb{F}_q$, then $N^L_{\mathbb{F}_q}(a) = a^n$.

Let ϕ be a rank two Drinfeld module over L, the degree m extension of $A/\mathfrak{p}$ defined by $\phi_T = \gamma(T) + g\tau + \Delta\tau^2$. Then $P_\phi(X) = X^2 - aX + b$. We saw in Theorem 2.4.3 that b is a generator of $(\mathfrak{p}^m)$. Explicitly, in [17, Equation 7], we see that $b = \epsilon(\phi)\mathfrak{p}^m(T)$, where

$$\epsilon(\phi) = N^L_{\mathbb{F}_q}(-\Delta)^{-1}.$$

See [10, 17] for more details.

Determining a is more difficult; a is the Frobenius trace of ϕ over L. It can be written as the polynomial in T

$$a = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_i T^i. \tag{A.1}$$

In Proposition 2.14 of [12], we learn that

$$a_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} \mathrm{Tr}^{\mathbb{F}_{q^2}}_{\mathbb{F}_q}(N^L_{\mathbb{F}_{q^2}}(\Delta)^{-1}) \text{ for } n \text{ even}, \\ -N^L_{\mathbb{F}_q}(g)^{-1}\mathrm{Tr}^L_{\mathbb{F}_q}(j(\phi)^{(q^n-q)/(q^2-1)+1}) \text{ for } n \text{ odd and } g \neq 0, \\ 0 \text{ for } n \text{ odd and } g = 0. \end{cases} \tag{A.2}$$

To solve for the remaining coefficients of a, we must solve the system of equations given by

$$-\sum_{0 \le i \le \lfloor \frac{n}{2} \rfloor} a_i f_{i,j-n} + \epsilon(\phi) \sum_{j/2 \le i \le n} p_i f_{i,j} = \begin{cases} -1 \text{ if } j = 2n, \\ 0 \text{ if } aj < 2n. \end{cases} \tag{A.3}$$

Here, the $f_{i,j}$ are the coefficients in L of $\phi_{T^i} = \sum_{0 \le j \le 2i}$. They are determined by recursion, with $f_{i,0} = \gamma(T^i)$, $f_{1,1} = g$, and $f_{1,2} = \Delta$. To obtain subsequent terms, use that ϕ is a ring homomorphism – see [12, (2.12)] for the details. The p_i's are the coefficients of the monic generator $\mathfrak{p}^m(T)$.

Several of the above equations are redundant; the j-th equations can be deleted for $j = n + 1, n + 2, \ldots 2n$, if n is odd, or $j = n+1, n+2, \ldots, 2n-1$, if n is even. This reduces the system to $\lfloor \frac{n}{2} \rfloor + 1$ equations. The remaining system is triangular [12, (3.3)].

A.2 A Specific Case for Rank Two

Calculating the characteristic polynomial when L is $A/\mathfrak{p}$, or equivalently when $m = 1$ and $d = n$, is much simpler. We calculate b as before. To compute a, we need the image of $\mathfrak{p}(T)$ under ϕ. Since $\mathfrak{p}(T)$ is a degree d polynomial in T, we know $\phi_{\mathfrak{p}(T)}$ is a degree $2d$ polynomial in τ (in general it is a degree rd polynomial in τ). Write

$$\phi_{\mathfrak{p}(T)} = \sum_{i=0}^{2d} h_i(\phi)\tau^i.$$

The first $d-1$ coefficients vanish. The d-th coefficient $h_d(\phi)$ is called the *Hasse invariant*, also denoted $H(\phi)$. We compute $H(\phi)$ using $\phi_{\mathfrak{p}(T)}$ in Example A.3.1. Another way to compute $H(\phi)$ is via the sequence $\{g_k\}_{k\geq 0}$ defined by recursion. Let $g_0 = 1$, and $g_1 = g$ (the coefficient of τ in ϕ_T). Put $[k] := T^{q^k} - T$, and regard it as an element of $A/\mathfrak{p} = L$. Then

$$g_k = -[k-1]g_{k-2}\Delta^{q^{k-2}} + g_{k-1}g^{q^{k-1}},$$

for $k \geq 2$. Proposition 3.7 in [12] gives that the Hasse invariant $H(\phi)$ is the d-th term in the sequence; $H(\phi) = g_d$. We use this method in Examples A.3.2, A.3.3 and A.3.4. The Hasse satisfies

$$\gamma(a) = \epsilon(\phi)N^L_{\mathbb{F}_q}(H(\phi)). \tag{A.4}$$

By (A.1), the degree of a is less than or equal to $\lfloor \frac{n}{2} \rfloor$. When $n = d$, a has degree less than that of $\mathfrak{p}$. This implies that a is determined by its residue class $\gamma(a)$ modulo $\mathfrak{p}$.

A.3 Examples

We compute the characteristic polynomials for a few different rank two Drinfeld modules, all with $A = \mathbb{F}_2[T]$ with $L = A/\mathfrak{p}$ for varying primes $\mathfrak{p}$.

Example A.3.1. This example is also in [12, Example 3.8]. Let $\mathfrak{p}(T) = T^3 + T + 1$ and $\mathfrak{p}$ be the ideal generated by $\mathfrak{p}(T)$. Then $L = A/\mathfrak{p} \cong \mathbb{F}_8$ can be viewed as an A-field via $\gamma : A \to L$, the $\mod \mathfrak{p}$ map. We have that $A/\mathfrak{p}$ is a degree $d = 3$ extension of A, and L is a degree $m = 1$ extension of $A/\mathfrak{p}$, so $n = d \cdot m = 3$. Let ϕ be the rank two Drinfeld module defined by

$$\phi_T = \gamma(T) + T\tau + \tau^2 = T + T\tau + \tau^2 \mod \mathfrak{p}.$$

Here $g = T$ and $\Delta = 1$. We aim to compute the characteristic polynomial $P_\phi(X) = X^2 - aX + b$, where $b = \epsilon(\phi) \cdot \mathfrak{p}^m(T)$. We begin by computing the sign function $\epsilon(\phi)$ via the field norm map $N^L_{\mathbb{F}_2}(\Delta)$ for L over $\mathbb{F}_2$. This is defined in terms of the Galois group of L of $\mathbb{F}_2$, which is cyclic and of degree three, Gal$(L : \mathbb{F}_2) \cong C_3 = \{\text{Id}, \sigma, \sigma^2\}$. By definition, we have

$$N^L_{\mathbb{F}_2}(\Delta) = \text{Id}(1) \cdot \sigma(1) \cdot \sigma^2(1) = 1$$

so $\epsilon(\phi) = (-1)^n N^L_{\mathbb{F}_2}(\Delta)^{-1} = 1$. We may now explicitly write b as $\epsilon(\phi) \cdot \mathfrak{p}^m(T)$, which simplifies to $b = \mathfrak{p}(T) = T^3 + T + 1$.

To compute a, we begin finding the Hasse invariant $H(\phi)$ by finding the coefficient on the degree d term of $\phi_{\mathfrak{p}(T)}$. Using the homomorphism properties, we have

$$\begin{aligned}
\phi_{\mathfrak{p}(T)} &= \phi_{T^3} + \phi_T + \phi_1, \\
&= (\phi_T)^3 + \phi_T + \phi_1, \\
&= (T + T\tau + \tau^2)^3 + (T + T\tau + \tau^2) + 1 \mod \mathfrak{p}.
\end{aligned}$$

Simplifying according to the definition of multiplication in $L\{\tau\}$ gives

$$\phi_{\mathfrak{p}(T)} = (T+1)\tau^5 + \tau^6.$$

The coefficient on the third term is zero, so $H(\phi) = 0$ and $\gamma(a) = 1 \cdot N_{\mathbb{F}_q}^{A/\mathfrak{p}}(H(\phi)) = 0 \mod \mathfrak{p}$. This implies that a is an element of the ideal $\mathfrak{p}$, but since $\deg_T(a) \leq \lfloor d/2 \rfloor$, $a = 0$. Therefore, the characteristic polynomial of ϕ is

$$P_\phi(X) = X^2 + aX + b = X^2 + (T^3 + T + 1).$$

Example A.3.2. Now let $\mathfrak{p}(T) = T^2+T+1$, $\mathfrak{p}$ be the ideal generated by $\mathfrak{p}(T)$, and $L = \mathbb{F}_2[T]/\mathfrak{p} \cong \mathbb{F}_4$. Then L is a degree $n = d \cdot m = 2$ extension of $\mathbb{F}_2$ where $m = 1$ and $d = 2$. Define a rank two Drinfeld module $\psi : A \to L\{\tau\}$ by

$$\psi_T = \gamma(T) + T\tau + T\tau^2 = T + T\tau + T\tau^2.$$

The Galois group Gal$(L : \mathbb{F}_2)$ has order two, and is generated by $\sigma : x \mapsto x^2$. Observe

$$\begin{aligned} N_{\mathbb{F}_2}^L(\Delta) &= N_{\mathbb{F}_2}^L(T) \mod \mathfrak{p} \\ &= T \cdot T^2 \mod \mathfrak{p} \\ &= 1, \end{aligned}$$

so $\epsilon(\psi) = 1$ and $b = \mathfrak{p}(T) = T^2 + T + 1$. To compute a, we use the recursive formula for $H(\psi) = g_2$. First, we have $g_0 = 1$ and $g_1 = T$. By the recursive definition,

$$\begin{aligned} g_2 &= -[2-1]g_0\Delta^{2^0} + g_1 g^{2^1} \\ &= -(T^2 - T)(1)(T)^1 + (T)(T)^2 \mod \mathfrak{p} \\ &= T + 1 \mod \mathfrak{p}. \end{aligned}$$

The Hasse invariant is $H(\psi) = T + 1$ and satisfies

$$\gamma(a) = \epsilon(\psi) N^L_{A/\mathfrak{p}}(H(\psi)) = (1) N^L_{A/\mathfrak{p}}(T + 1).$$

Since $L = A/\mathfrak{p}$, we have $\gamma(a) = T+1 \mod \mathfrak{p}$. Since $\deg_T(a) \leq \lfloor d/2 \rfloor$, $a = T+1$. All together, we have

$$P_\psi(X) = X^2 + (T+1)X + \mathfrak{p}(T) = X^2 + (T+1)X + T^2 + T + 1.$$

Example A.3.3. As in the previous example let $L = \mathbb{F}_2[T]/\langle T^2 + T + 1 \rangle$. Define a rank two Drinfeld A-module ζ over L by $\zeta_T = \gamma(T) + T\tau + \tau^2$. Again, we compute b by first computing $\epsilon(\zeta)$.

$$\begin{aligned} \epsilon(\zeta) &= (-1)^2 (N^L_{\mathbb{F}^2}(\Delta))^{-1} \\ &= (1)(N^L_{\mathbb{F}_2}(1))^{-1} \\ &= 1. \end{aligned}$$

Thus $b = \mathfrak{p}(T) = T^2 + T + 1$. As in the previous example, we compute $H(\zeta) = g_2$ using the recursive formula with $g_0 = 1$ and $g_1 = T$.

$$
\begin{aligned}
g_2 &= -[2-1]g_0\Delta^{2^0} + g_1 g^{2^1} \\
&= -(T^2 - T)(1)(T+1)^1 + (T)(T)^2 \mod \mathfrak{p} \\
&= T^3 \mod \mathfrak{p} \\
&= 1 + 1 \mod \mathfrak{p} = 0 \mod \mathfrak{p}.
\end{aligned}
$$

Since the Hasse invariant is zero, $\gamma(a) = 0 \mod \mathfrak{p}$, but again $\deg_T(a) \leq \lfloor d/2 \rfloor$, so $a = 0$. Then

$$
P_\zeta(X) = X^2 + \mathfrak{p}(T) = X^2 + T^2 + T + 1.
$$

Example A.3.4. Again take $L = \mathbb{F}_2/\langle T^2 + T + 1\rangle$. Define a rank two Drinfeld module θ by $\theta_T = T + (T+1)\tau + (T+1)\tau^2$. Then $g = (T+1)$ and $\Delta = (T+1)$. We again start by computing $\epsilon(\theta)$:

$$
\begin{aligned}
\epsilon(\theta) &= (-1)^2 (N^L_{\mathbb{F}^2}(\Delta))^{-1} \mod \mathfrak{p} \\
&= (1)(N^L_{\mathbb{F}_2}(T+1))^{-1} \mod \mathfrak{p} \\
&= (1)((T+1)^3)^{-1} = (1)(T^3 + T^2 + T + 1)^{-1} \mod \mathfrak{p} \\
&= (1)(1)^{-1} = 1.
\end{aligned}
$$

This means that $\epsilon(\theta) = 1$ and $b = T^2 + T + 1$. Now, $H(\theta) = g_2$ is found by the recursion formula with $g_0 = 1$ and $g_1 = T + 1$:

$$\begin{aligned} g_2 &= -[2-1]g_0\Delta^{2^0} + g_1 g^{2^1} \\ &= -(T^2 - T)(1)(T+1)^1 + (T)(T+1)^2 \mod \mathfrak{p} \\ &= T + 1 \mod \mathfrak{p}. \end{aligned}$$

Then $\gamma(a) = \epsilon(\theta) N^L_{A/\mathfrak{p}}(T+1) = T+1$. Since $\deg_T(a) \leq \lfloor d/2 \rfloor$, we know $a = T+1$. Thus

$$P_\theta(X) = X^2 + (T+1)X + \mathfrak{p}(T) = X^2 + (T+1)X + T^2 + T + 1.$$

A.4 Isogenies

By Theorem 2.6.1, we now explain which of the above Drinfeld modules are isogenous. There is an isogeny between the Drinfeld modules ψ and θ from Examples A.3.2 and A.3.4 respectively because both have characteristic polynomial $X^2 + (T+1)X + T^2 + T + 1$. Neither is isogenous over L to ζ in Example A.3.3 because $P_\zeta(X) = X^2 + T^2 + T + 1$. We also know that ϕ in Example A.3.1 is not isogenous to any of the others over the base field $L = A/\langle T^2 + T + 1 \rangle$ because it is defined over a different field entirely. However, all four examples are defined over fields with the same algebraic closure. We may thus ask if there is a common field of definition over which the Drinfeld modules become isogenous. The first observation is that Examples A.3.1 and A.3.3 are *supersingular* because the Hasse invariant vanishes. Even over an extension, these examples cannot become isogenous to Examples A.3.2 and A.3.4, though they may become isogenous to one another.

Appendix B

Hensel's Lemma

In this appendix, we give more details about Hensel's Lemma. The first result stated, also referred to as Hensel's lemma allows us to find roots of polynomials given approximate roots, much in the way we use Newton's method to approximate roots in calculus.

Lemma B.0.1 ([8]). *Let R be a complete ring with respect to the ideal ϖ, and let $f(x) \in R[x]$ be a polynomial. If a is an approximate root of f, in the sense that $f(a) \equiv 0 \mod f'(a)^2\varpi$, then there is a root b of f near a, in the sense that $f(b) = 0$ and $b \equiv a \mod f'(a)\varpi$. If $f'(a)$ is a non-zero-divisor in R, then b is unique.*

The proof of this lemma is a consequence of the following properties of power series rings.

Theorem B.0.2. *[8, Theorem 7.16] Let R be any ring and S an R-algebra that is complete with respect to some ideal $\mathfrak{n}$. Given $f_1, \cdots, f_n \in \mathfrak{n}$:*

(a) There is a unique R-algebra homomorphism

$$\varphi : R[[x_1 \cdots x_n]] \to S$$

sending x_i to f_i for each i. The map φ takes a power series $g(x_1, \cdots, x_n)$ to $g(f_1, \cdots, f_n) \in S$.

(b) If the induced map $R \to S/\mathfrak{n}$ is an epimorphism and $f_1, \cdots, f_n$ generate $\mathfrak{n}$, then φ is an epimorphism.

(c) If the induced map of associated graded rings

$$gr\varphi : R[x_1, \cdots, x_n] \cong gr_{(x_1, \cdots, x_n)} R[[x_1, \cdots, x_n]] \to gr_{\mathfrak{n}} S$$

is a monomorphism, then φ is a monomorphism.

The proof is given explicitly in [8, Theorem 7.16]. For (a), we combine the universal properties of the polynomial ring, the quotient, and the inverse limit to obtain the result. The hypotheses of (b) imply that the map

$$(x_1, \cdots, x_n)/(x_1, \cdots, x_n)^2 \to \mathfrak{n}/\mathfrak{n}^2$$

is surjective, further implying that the induced map grφ : gr$_{(x_1,\cdots,x_n)}R \to$ gr$_{\mathfrak{n}}S$ is surjective. Using the surjectivity of grφ, together with the completeness of S, for any $g \neq 0$ in S, we can construct an infinite sequence of $g_j \in (x_1, \cdots, x_n)^{i+j}$, where i is the largest number such that $g \in \mathfrak{n}^i$, such that $\varphi(\sum_{j=1}^{\infty} g_j) = g$. Finally, for (c), take any $g \neq 0$ in $R[[x_1, \cdots, x_n]]$, and suppose d is the greatest number such that g is in the degree d part of the associated graded ring. Then in$(g) := g \mod (x_1, \cdots, x_n)^{d+1}$ is nonzero. We also have that $\varphi(g) \equiv$ grφ(in$(g)) \mod \mathfrak{n}^{d+1}$. Since the map on associated graded rings is injective, grφ(in(g)) is also nonzero in the degree d part of gr$_{\mathfrak{n}}S$, so $\phi(g) \neq 0$ in S.

Corollary B.0.3. *[8, Corollary 7.17] Let $f \in xR[[x]]$ be a power series. If φ is the endomorphism*

$$\varphi : R[[x]] \to R[[x]],$$
$$x \mapsto f,$$

which is the identity on R and sends x to f, then φ is an isomorphism if and only if $f'(0) \in R^{\times}$.

Suppose $f'(0) = u$ is a unit in R. The associated graded ring is gr$_{(x)}R[[x]] \cong R[x]$, and the induced map grφ : $R[x] \to R[x]$ is defined by gr$\varphi(x) = ux$. Since u is a unit, this is an isomorphism. By Theorem B.0.2, this implies φ is injective. We can also write $f = ux + hx^2 = (u+hx)x$ for some power series h in $R[[x]]$. Since $u+hx$ is a unit in $R[[x]]$, f is a generator of (x). Since the induced map $R \to R[[x]]/(x)$ is surjective (b) of B.0.2 implies that φ is also surjective. Alternatively, we observe that φ preserves the set of elements of $R[[x]]$ not in (x), so because φ is an isomorphism, $\phi((x)) = (x)$. This implies f is a generator or (x), so $f+(x)^2$ generates $(x)/(x^2)$. Because $f \equiv f'(0)x(\mod (x^2))$, $f'(0)$ is a unit in R. This proof is in [8, Corollary 7.17].

We can now return to the proof of Lemma B.0.1, as in [8]. Let $f'(a) = e$. We choose $h(x)$ so

$$f(a + ex) = f(a) + f'(a)ex + h(x)(ex)^2 = f(a) + e^2(x + x^2h(x)). \tag{B.1}$$

We know $x + x^2h(x) = x(1 + h(x))$ is a generator of the ideal (x) in $R[[x]]$, so condition (a) of Theorem B.0.2 implies there is a unique homomorphism $\varphi : R[[x]] \to R[[x]]$ defined by $\varphi(x) = x + x^2h(x)$. Further, $\frac{\mathrm{d}}{\mathrm{dx}}(x + x^2h(x))|_{x=0} = 1$ is a unit, so φ is an isomorphism with inverse φ^{-1} taking $x + x^2h(x)$ to x by Corollary B.0.3. Applying the inverse map to (B.1) yields

$$f(a + e\varphi^{-1}(x)) = f(a) + e^2x. \tag{B.2}$$

We assumed that a is an approximate solution to $f(x)$, meaning $f(a) \equiv 0 \mod f'(a)^2\varpi$. Equivalently, $f(a) = c \cdot e^2$ for some $c \in \varpi$. Theorem B.0.2 again implies there is a unique algebra homomorphism ψ taking x to $-c$. This homomorphism applied to Equation (B.2) gives

$$f(a + e\psi(\varphi^{-1}(x)) = 0. \tag{B.3}$$

Then $b = a + e\psi(\varphi(x))$ is the desired root. The proof that $f'(a)$ being a nonzero divisor implies that b is unique again invokes the uniqueness of a homomorphism given by Theorem B.0.2, but the details are left to the reader. By similar proof techniques, Hensel's Lemma may be extended to a system of equations in many variables.

Lemma B.0.4 (Hensel's Lemma, [5, 8])**.** *If $(a_1, \cdots, a_n) \in R^n$ is an approximate solution to the system of equations $f_i(x) = 0$ in the sense that $f_i(a_1, \cdots, a_n) \equiv 0 \mod \varpi$ for $i = 1, \cdots, n$,* *and the determinant of the Jacobian matrix at* $(a_1, \cdots, a_n)$ *is a unit in* R*, then there is an actual solution* $(b_1, \cdots, b_n) \in R^n$ *of the equations such that each* b_i *differs from* a_i *by an element of* ϖ*.*

Bourbaki explicitly describes a similar result for systems of equations for which the Jacobian matrix is not square [5, Corollary III.4.5.2].

Corollary B.0.5 (Hensel's Lemma for non-square systems)**.** *Let* A *be a ring and* $\mathfrak{m}$ *an ideal of* A *such that* $(A, \mathfrak{m})$ *satisfies Hensel's conditions. (That is* $\mathfrak{m}$ *is closed in* A *and its elements are topologically nilpotent.) Let* r, n *be integers such that* $0 \le r < n$ *and* $\vec{f} = (f_{r+1}, \cdots, f_n)$ *is a system of* $r-1$ *elements of* $A[X_1, \cdots, X_n]$*; let* $J^{(n-r)}_{\vec{f}}(\vec{X})$ *denote the determinant of the Jacobian matrix of* $M_{\vec{f}}(\vec{X})$ *consisting of the columns of index* j *such that* $r+1 \le j \le n$*. Let* $\vec{a} \in A^n$ *be such that* $J^{(n-r)}_{\vec{f}}(\vec{a})$ *is invertible in* A *and* $\vec{f}(\vec{a}) \equiv 0 \mod \mathfrak{m}^{n-r}$*. Then there exists a unique* $\vec{x} = (x_1, \cdots, x_n) \in A^n$ *such that* $x + k = a_k$ *for* $1 \le k \le r$ *and* $\vec{x} \equiv \vec{a} \mod \mathfrak{m}^n$ *and* $\vec{f}(\vec{x}) = 0$*.*

In the simplest case of the p-adic integers, Conrad illustrates this idea in his proof of Hensel's Lemma for a single variable polynomial [6]. Given the approximate root a to $f(x) \in \mathbb{Z}_p[x]$, such that $f'(a) \not\equiv 0 \mod p$, we can construct the unique root b in $\mathbb{Z}_p$ as a limit of solutions a_n modulo p^n. The following argument uses the same ideas in Conrad's notes to lift solutions to a system of polynomials over $A_{\mathfrak{l}} \mod \mathfrak{l}^k$ to solutions $\mod \mathfrak{l}^{k+1}$ of Let $f_1, \cdots, f_t$ in $A_{\mathfrak{l}}[x_1, \cdots, x_n]$. Suppose $\overline{a} = (a_1, \cdots, a_n)$ is a solution to the system

$$\overline{f}(\overline{x}) = \overline{0}$$

modulo $\mathfrak{l}^k$ where $\overline{f}$ is the vector of functions $f_1, \cdots, f_t$. We want to lift $\overline{a}$ to a solution $\overline{b} = (b_1, \cdots, b_n)$ modulo $\mathfrak{l}^{k+1}$. That is $f_i(\overline{b}) \equiv 0 \mod \mathfrak{l}^{k+1}$ for all $1 \le i \le t$ and $b_j \equiv a_j \mod \mathfrak{l}^k$ for all $1 \le j \le n$. The second condition is true if and only if $b_j = a_j + \mathfrak{l}^k c_j$ for some c_j in the residue field $A/\mathfrak{l}$. The Taylor expansion of the system centered at $\overline{a}$ evaluated at $\overline{b}$ is

$$\overline{f}(\overline{b}) = \overline{f}(\overline{a}) + J(\overline{a})(\overline{b} - \overline{a}) + O((\overline{b} - \overline{a})^2).$$

Since $\bar{b} - \bar{a} = \bar{c}\mathfrak{l}^k$, the higher order terms all have a factor of at least $\mathfrak{l}^{k+1}$, so $\bar{b}$ is a solution $\mod \mathfrak{l}^{k+1}$ if and only if

$$\overline{f}(\bar{b}) = \overline{f}(\overline{a}) + J(\overline{a})(\mathfrak{l}^k\overline{c}) \equiv 0 \mod \mathfrak{l}^{k+1},$$

if and only if

$$\frac{1}{\mathfrak{l}^k}\overline{f}(\overline{a}) + J(\overline{a})(\overline{c}) \equiv 0 \mod \mathfrak{l}. \tag{B.4}$$

By the assumption that $\overline{f}(\overline{a}) \equiv 0 \mod \mathfrak{l}^k$, we know each entry of $\frac{1}{\mathfrak{l}^k}\overline{f}(\overline{a})$ is an $\mathfrak{l}$-adic integer. The assumption that $\mathcal{X}$ is smooth implies that the Jacobian has non-zero rank, i.e., there is a solution to the system of Equation (B.4). Since $\overline{c}$ has n-entries, there are $|\mathfrak{l}|^n$ choices for $\overline{c}$. This result can be further refined for systems which are not necessarily smooth [5, Corollary III.4.5.3].

Corollary B.0.6. *(Hensel's lemma for non-smooth systems) With the notation of B.0.5, let $\vec{a} \in A^n$; let $e = J_{\vec{f}}^{(n-r)}(\vec{a})$ be the determinant of the Jacobian matrix (not necessarily invertible in A) and suppose $\vec{f}(\vec{a}) \equiv 0 \mod e^2\mathfrak{m}^{n-r}$. Then there exist $n-r$ formal power series without constant term ϕ_i $(r+1 \leq i \leq n)$ in $A[[X_1, \cdots, X_r]]$ such that, for all $\vec{t} = (t_1, \cdots, t_r) \in \mathfrak{m}^r$,*

$$f_i(a_1 + e^2t_1, \cdots, a_r + e^2t_r, a_{r+1} + e\phi_{r+1}(\vec{t}), \cdots, a_n + e\phi_n(\vec{t})) = 0$$

for $r+1 \leq i \leq n$.

For smooth schemes, the result is stated as follows.

Lemma B.0.7 (Hensel's Lemma)**.** *Let $\mathcal{X}$ be a smooth scheme of dimension n, over a complete local ring $A_{\mathfrak{l}}$, with maximal ideal $\mathfrak{l}$. Each point modulo $\mathfrak{l}^k$ lifts to exactly $|\mathfrak{l}|^n$ solutions modulo $\mathfrak{l}^{k+1}$. That is*

$$\#\mathcal{X}(A/\mathfrak{l}^{k+1}A) = |\mathfrak{l}|^{nk}\#\mathcal{X}(A/\mathfrak{l}A). \tag{B.5}$$