Source Material
for
a course in
TRANSIENT GROUND WATER HYDRAULICS

Compiled by

R.E. Glover. Civil Engineer

and

M.W. Bittinger. Assistant Engineer

Civil Engineering Section
Colorado State University

April 1959.

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FIGURES

Introduction

The ground water conditions which must be dealt with in the field are generally of a transient nature. This is to say that the conditions are changing with time. A steady state might be established if the conditions which initiated the change were maintained for a sufficiently long time but more often than not the conditions are changed before the steady state can be established. Some examples will illustrate. A well is put down and water is pumped from it and a condition of change is initiated. Before the ground water conditions can reach a new stable configuration the well is shut down and a new series of changes begins. A farmer irrigates during the summer and his deep percolation losses build up the ground water levels under his fields. Drains installed under these fields flow at increasing rates during the irrigation season. After the last irrigation, however, their flow diminishes continually until the first irrigation of the next growing season starts the cycle over again. An almost endless variety of such examples could be quoted.

Efforts have been made to apply steady state formulas to these cases because these formulas are often simpler than those for the transient cases but this only leads to the vexations which arise when attempts are made to force the application of formulas to cases to which they do not apply. It is much more satisfactory to deal with them as transient cases even though some simplifying assumptions must generally be made to reduce them to mathematically tractable form. Once the solutions have been found and graphed or tabulated applications to the actual transient condition is readily made.

The simplifications introduced are shown explicitly in each derivation but it may be stated here that they generally conform to the Dupuit-Forschheimer idealization when water table conditions are present. This idealization is based upon the assumption that the surface gradient applies throughout the depth of the saturated portion of the aquifer. It is also generally expedient to assume that the area available for the flow of ground water is not altered by the drawdowns produced. Some harsh criticisms have been leveled at the formulas developed on this basis but it is well to recognize that very many of the formulas in common use for engineering purposes are based upon similar simplifying assumptions but serve very well in spite of the shortcomings thus introduced. It is recognized that formulas having such bases must be used with judgment based upon a knowledge of their limitations. If the formulas described herein are used with a similar exercise of judgment it may be expected that they too will serve quite as well as do the formulas of the older disciplines.

Many of the formulas described herein were developed in the Chief Engineers office of the U.S. Bureau of Reclamation. In most cases they are contained in informal memoranda on which the name of the Senior author appears as one of the authors. Other names appearing on these memoranda are Wade H. Taylor, E.D. Rainville, William N. Tapp, Quentin L. Florey, William T. Moody and E.W. Kramer.

These developments profited from the support and encouragement of Charles R. Maierhofer and Waldron H. Yarger. Due to the similarity of the ground water and heat flow differential equations, solutions for ground water transient cases can ordinarily be adapted diredtly from developments found in the older and much better developed field of heat flow. This was commonly the source of the formulas described in these memos. The authors take this opportunity to thank the Bureau of Reclamation for permission to use these developments.

Notation

The notation used in this text is collected here for ready reference.

a	represents the effective radius of a well	(feet)
b,	an outer radius	(feet)
D	The original saturated thickness of an aquifer	(feet)
F	A flow as a function of r or x	
$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	a function relating to the flow of an artesian well (dimensionless)	
h	a remaining drainable depth	(feet)
H	an original drainable depth	(feet)
I_B	$= (1 - P(x))$ (See table II)	(feet)
J_o	a zero order Bessel Function (dimensionless)	
K	the permeability	(ft/sec)
L	the distance between parallel drains	
m	the thickness of a semi-permeable confining bed (Fig 6)	(feet)
n	an integral number (dimensionless)	
$P(x) = \frac{2}{\sqrt{\pi}} \int_o^x e^{-u^2} du$	The probability integral	(Table I)
p	the permeability of a semi-permeable confining bed	(ft/sec)
p_1	a pressure expressed in equivalent feet of water. In an aquifer it is the departure of the pressure from hydrostatic	(feet)
p_r	a part remaining (dimensionless)	
Q	a flow from a well	(ft^3/sec)
q	the flow which a well takes from a river	(ft^3/sec)
q_1	a flow per unit length of a line source	(ft^2/sec)
q_o	the return flow from bank storage	$(feet)^2$

r	radius	(feet)
t	time	(sec)
u	a variable of integration (dimensionless)	
V	the voids ratio. It is the ratio of the drainable or fillable voids in a unit volume to the gross volume (dimensionless)	
y	a drawdown (See fig 1)	(feet)
Y_o	a zero order Bessel Function (dimensionless)	
x, y, z,	coordinate distances	(feet)
$\alpha = \frac{K\ D}{V}$	the diffusion constant	$\left(\frac{ft^2}{sec}\right)$
ξ	a supplementary variable, defined where used.	

Note: Although dimensions have been given in English units in the above table all formulas in this text are given in consistent units. They may therefore be used in any consistent unit system without change.

Definitions

The following terms are used in this text in the sense indicated below:

Aquifer A water bearing formation.

Ground Water The water filling the saturated portion of a permeable bed.

Water Table The surface of atmospheric pressure in an aquifer. Water will stand in an observation well at the water table.

Permeability A quantity defining the ability of an aquifer to transmit water. It is represented in this text by the symbol K. It is the flow through a unit area produced by a unit gradient. The actual flow through a unit area is equal to the product of the permeability and the gradient.

Voids Ratio The ratio of the drainable or fillable voids in a unit volume to the gross volume. It is a dimensionless quantity greater than zero and less than unity.

Bessel Function A solution of Bessel's equation. The notation used here conforms to that of the British Association tables of reference. 5

Chapter 1 Well Pumped at a Constant Rate.

A cross section through a well is shown in figure 1.

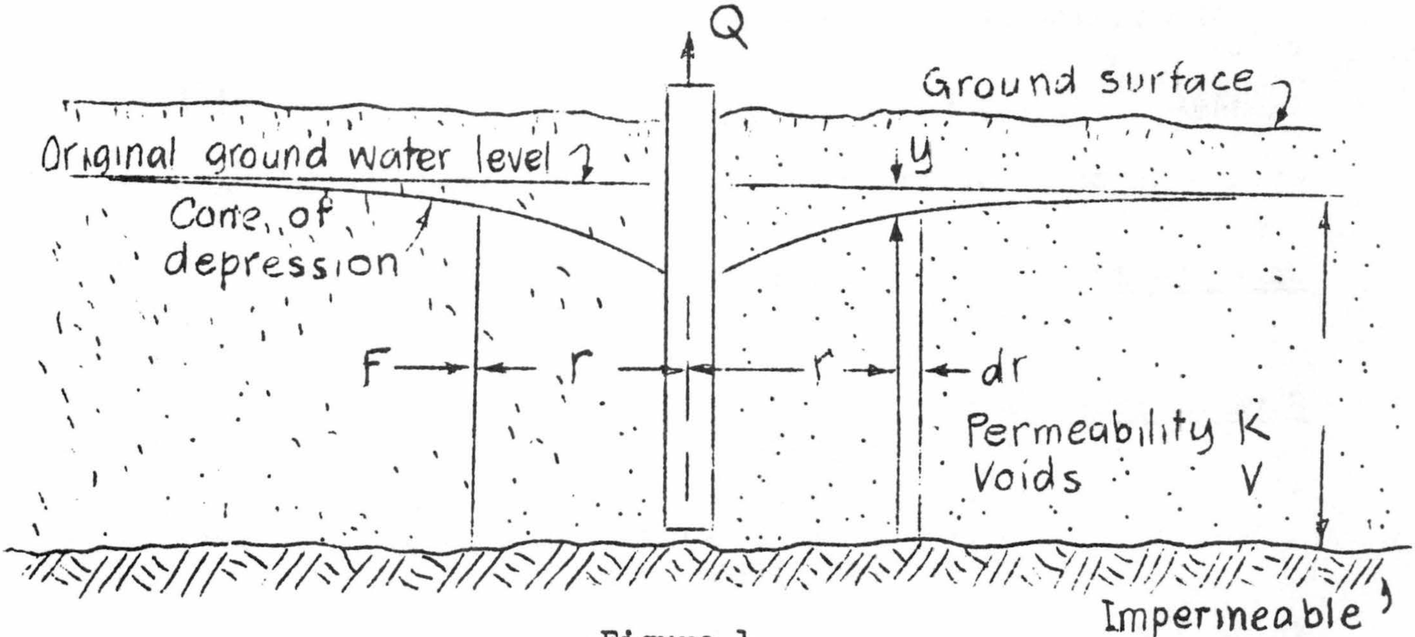


Figure 1

Pumping of the well at the constant rate Q for the time t produces a cone of depression centered at the well. At the radius r the drawdown is y.

The flow toward the well at the radius r is

$$F = -2\pi r (D-y) K \frac{\partial y}{\partial r} \tag{1}$$

If y is small compared to D this can be written

$$F = -2\pi KD r \frac{\partial y}{\partial r} \quad \text{if} \quad y << D \tag{2}$$

Under unsteady conditions, when the flow F varies with the radius r, water accumulates within the annulus of thickness dr at the rate

$$\frac{\partial F}{\partial r} = -2\pi KD \frac{\partial}{\partial r}\left(r \frac{\partial y}{\partial r}\right) \tag{3}$$

This accumulation results in a storage of water in the annulus at the rate

$$-2\pi r\, dr\ V \frac{\partial y}{\partial t} \tag{4}$$

The condition of continuity, which expresses the relationship between the flow variation across the annulus and the storage of water within it is:

$$\frac{\partial F}{\partial r} drdt = -2\pi r V \frac{\partial y}{\partial r} drdt$$

or

$$KD \frac{\partial}{\partial r} \left(r \frac{\partial y}{\partial r} \right) = rV \frac{\partial y}{\partial t} \tag{5}$$

If

$$\alpha = \frac{KD}{V} \tag{6}$$

This expression can be put in the form

$$\alpha \left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r} \frac{\partial y}{\partial r} \right) = \frac{\partial y}{\partial t} \tag{7}$$

To obtain the desired solution it will be necessary to find a solution satisfying the continuity condition (7) and meeting the boundary and initial conditions:

$$\begin{array}{llll} \text{When } t \rightarrow 0 & y \rightarrow 0 & \text{for} & r > 0 \\ \text{When } r \rightarrow 0 & F \rightarrow Q & \text{for} & t > 0 \end{array} \tag{8}$$

Consider the expression

$$y = \frac{Q}{2 \pi KD} \int_{\frac{r}{\sqrt{4 \alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du \tag{9}$$

Where u represents a variable of integration.
To show that this expression satisfies the differential equation form the derivatives

$$\frac{\partial y}{\partial t} = \frac{Q}{2 \pi KD} \frac{e^{-\frac{r^2}{4 \alpha t}}}{2t} \tag{10}$$

$$\frac{\partial y}{\partial r} = \frac{-Q}{2 \pi KD} \frac{e^{-\frac{r^2}{4 \alpha t}}}{r} \tag{11}$$

$$\frac{\partial^2 y}{\partial r^2} = \frac{Q}{2 \pi KD} \frac{2e^{-\frac{r^2}{4 \alpha t}}}{4 \alpha t} + \frac{Q}{2 \pi KD} \frac{e^{-\frac{r^2}{4 \alpha t}}}{r^2} \tag{12}$$

A substitution of these expressions into the differential equation (7) will show that equation (9) is a solution.

It remains to determine whether this solution satisfies the desired boundary conditions. If r is greater than zero and t approaches zero the lower limit of the integral approaches the upper limit and we can conclude that the initial condition is satisfied.

The flow across the cylinder of radius r and length D is, to our approximation; from equations (2) and (11).

$$F = Q \; e^{-\frac{r^2}{4\alpha t}} \tag{13}$$

When t is greater than zero and r approaches zero then F approaches Q and the second of equations (8) is satisfied.

We conclude that equation (9) is the solution which satisfies our requirements.

The application of this formula can be illustrated by the following examples:

Example

Water is pumped from a well at the rate of 350 gallons per minute. The well penetrates 170 feet of saturated thickness of an aquifer having a permeability of .0005 ft/sec and a void ratio of 0.20. Compute (a) The drawdown in an observation well at a radius r = 5 feet as a function of time.
(b) The drawdown as a function of the radius after the well has been pumped one day.

Solution

Since one cubic foot per second is equivalent to 448.8 gallons per minute, the pumping rate of 350 gallons per minute is

$$\frac{350}{448.8} = 0.780 \text{ cubic feet per second}$$

$$\alpha = \frac{KD}{V} = \frac{(.0005)(170.)}{0.2} \quad 0.425 \text{ ft}^2/\text{sec}$$

The computation of drawdown at the radius 5 feet is shown in the following table:

Values of $\displaystyle\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$ are obtained from figure (3).

Table 1

Computations of drawdowns at $r = 5$ feet in a well for which
$K = .0005$ ft/sec $D = 170$ feet $V = 0.20$
$Q = 0.780\ ft^3/sec$ (350 G.P.M.) $\alpha = 0.425\ ft^2/sec$

$\sqrt{4\alpha} = 1.342\ ft^2/sec \qquad \frac{r}{\sqrt{4\alpha}} = 3.725\ \sqrt{sec}$

Time		Time (sec)	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	y
1	minute	60	0.480	0.5543	0.810
15	minutes	900	0.124	1.5236	2.228
30	minutes	1800	0.0877	2.0779	3.040
1	hour	3600	0.0620	2.4939	3.64
2	hours	7200	0.0438	2.8405	4.10
4	hours	14400	0.0310	3.1856	4.66
8	hours	28800	0.0219	3.5330	5.17
1	day	86400	0.01268	4.0799	5.97
2	days	172800	0.00896	4.4264	6.47
4	days	345600	0.00633	4.7738	6.98
15	days	1296000	0.00327	5.4343	7.94
1	month	2628000	0.00230	5.78622	8.46

$$\frac{Q}{2\pi KD} = \frac{0.780}{(6.2832)(.0005)(170)} = \frac{0.780}{0.534} = 1.460$$

Table 2

Computation of drawdowns at various distances from the well at the end of one day of pumping.

$K = .0005$ ft/sec $\qquad D = 170$ feet $\qquad V = 0.20$

$Q = 0.780$ ft^3/sec $\qquad = 0.425$ ft^2/sec $\qquad t = 86400$ sec

$\sqrt{4\alpha t} = 384.$

radius feet	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	y	$\frac{r}{\sqrt{t}}$
1	.00262	5.65	8.26	.0034
5	.01306	4.05	5.92	.0170
10	.0261	3.37	4.93	.0340
25	.0653	2.44	3.56	.0850
50	.1306	1.75	2.65	.170
100	.261	1.088	1.59	.340
500	1.306	0.0370	0.054	1.70
1000	2.61	0.000072	0.00010	3.40

$$\frac{Q}{2\pi KD} = \frac{0.780}{(6.2832)(.0005)(170)} = 1.460$$

The drawdowns of table 1 have been plotted on figure 2 to bring out the point that this case has no ultimate steady state. Reports of pumping tests made for determination of aquifer properties often contain statements to the effect that the well was pumped to stability before the required drawdown observations were taken. The fact is that the drawdowns never come to stability. So long as all of the water taken from the well comes from storage in the aquifer, stability is impossible. The nature of the drawdown pattern for points near the well only gives an appearance of approach to stability.

Since most of the water comes from storage at a distance from the well it is possible, however, to identify the conditions under which a valid determination of aquifer properties can be made by considering the drawdowns as a part of a steady state flow system. Such a computation will always involve a certain amount of approximation but natural aquifers are generally non-uniform anyway so that approximations are permissible.

The integral of formula (9) can be represented by the series

$$\int_{\chi}^{\infty} \frac{e^{-u^2}}{u} = -0.288608 - \log_e \chi + \frac{\chi^2}{1|2} - \frac{\chi^4}{2|4} + \frac{\chi^6}{3|6} - \qquad (14)$$

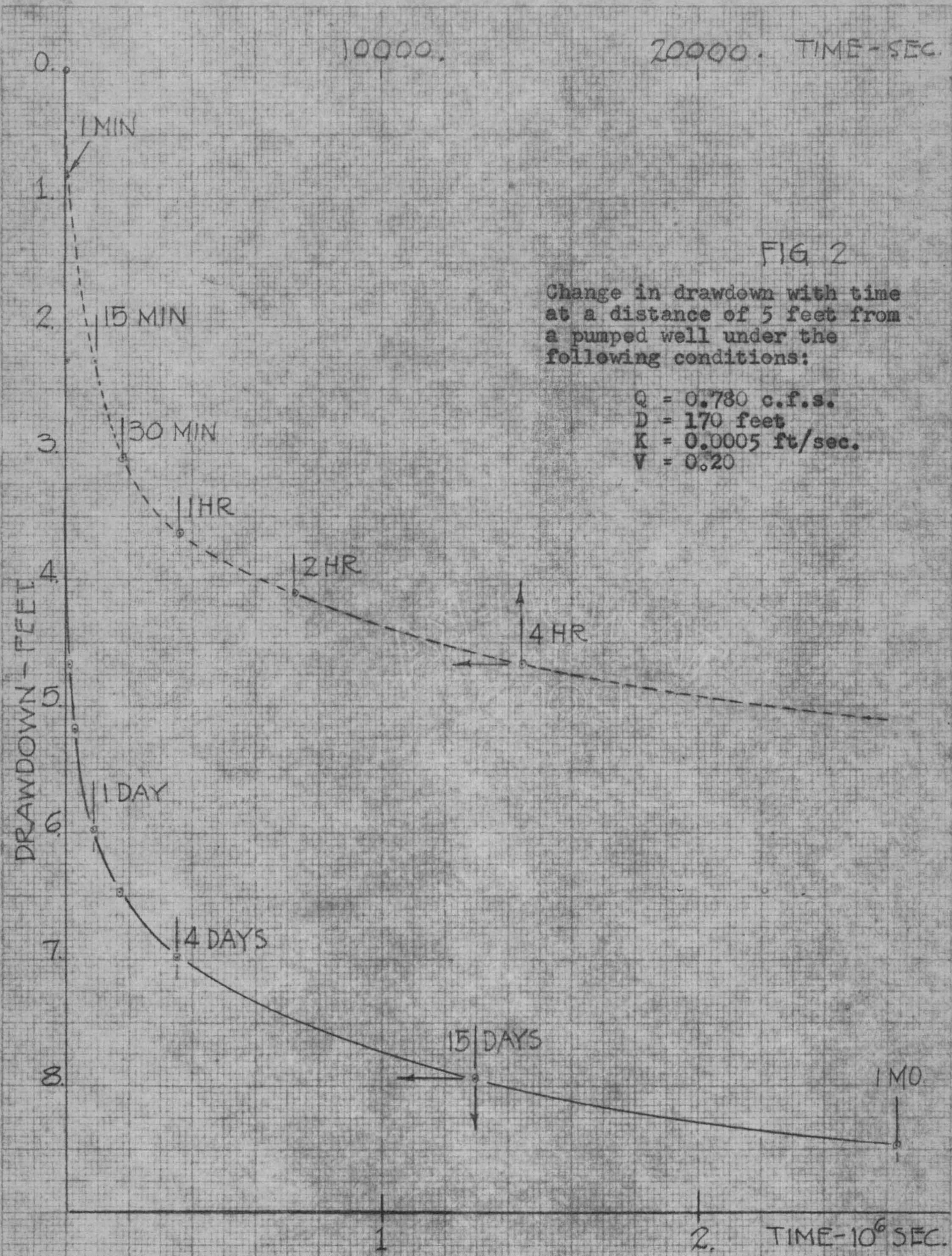


FIG 2

Change in drawdown with time at a distance of 5 feet from a pumped well under the following conditions:

Q = 0.780 c.f.s.
D = 170 feet
K = 0.0005 ft/sec.
V = 0.20

6

In our case $x = r/\sqrt{4\alpha t}$.
Suppose two observation wells are located at the radii r_1 and r_2 respectively then the difference in drawdown at the two wells will be

$$y_1 - y_2 = \frac{Q}{2\pi KD} \int_{\frac{r_1}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\, du - \frac{Q}{2\pi KD} \int_{\frac{r_2}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\, du$$

or (15)

$$y_1 - y_2 = \frac{Q}{2\pi KD}\left[-0.288608 - l\,g_e \frac{r_1}{\sqrt{4\alpha t}} + \frac{r_1^2}{2(4\alpha t)} - \right]$$
$$- \frac{Q}{2\pi KD}\left[-0.288608 - \log_e \frac{r_2}{\sqrt{4\alpha t}} + \frac{r_2^2}{2(4\alpha t)} - \right]$$

as t grows greater the value of $\frac{r}{\sqrt{4\alpha t}}$ grows smaller. Then after a certain time the term $\frac{r^4}{2 \cdot 4(4\alpha t)^2}$ and all higher power terms will become negligibly small. Then if we carry out the subtraction indicated above the result will be, approximately:

$$y_1 - y_2 = \frac{Q}{2\pi KD}\left[\log_e \frac{r_2}{r_1} + \frac{(r_1^2 - r_2^2)}{2(4\alpha t)}\right] \tag{16}$$

the formula

$$y_1 - y_2 = \frac{Q}{2\pi KD} \log_e \left(\frac{r_2}{r_1}\right) \tag{17}$$

may be recognized as a steady state formula. It can be readily obtained independently by integrating the relation:

$$Q = -KD\, 2\pi r \frac{dy}{dr} \tag{18}$$

This relation expresses the requirement that the flow Q passes through all the cylindrical surfaces of area $2\pi rD$ under the action of the gradient $-dy/dr$. It implies that the drawdowns y are small compared to D. A scrutiny of formula (16) will show that this will be a valid relationship if the numerical value of

$$\frac{r_2^2 - r_1^2}{2(4\alpha t)} \text{ is small compared to } \log_e \left(\frac{r_2}{r_1}\right) \tag{19}$$

This is the Taylor-Rainville criterion.* The question as to how small the power term must be as compared to the logarithm term must be resolved by judgement of the computer when he applies the criterion in any given case. Generally, a study of the consistency of the observation well data will provide clues as to the accuracy the basic data will permit.

The transmissibility KD is obtainable from formula (17)′ if it is rewritten in the form

$$KD = \frac{Q \log_e \frac{r_2}{r_1}}{2\pi (y_1 - y_2)} \tag{20′}$$

If the original saturated depth D is known then the permeability K can be determined.

To use the Taylor-Rainville criterion formula (19) is applied on a trial basis first. Then a value for the transmissibility KD is available and the quantity α can be computed. For these purposes only a reasonable value for V is needed.

More accurate values for the permeability can be obtained if the reduction of saturated thickness by drawdowns is accounted for. To do this we may write the steady state relation in the form:

$$Q = -K(D-y)\, 2\pi r \frac{dy}{dr} \tag{21}$$

or

$$-2(D-y)\frac{dy}{dr} = \frac{Q}{2\pi Kr} \tag{22}$$

By integration

$$(D-y)^2 = \frac{Q}{2\pi K} \log_e r + c \tag{23}$$

Where C is a constant of integration. If we let $h = (D - y)$ represent the saturated thickness and take the difference of two such expressions applying to the radii r_1 and r_2 the result is:

$$h_2^2 - h_1^2 = \frac{Q}{\pi K} \log_e \left(\frac{r_2}{r_1}\right) \tag{24′}$$

* See Bureau of Reclamation Technical Memo

This is known as the Theim formula. The permeability can be obtained from it by rewriting expression (24) in the form:

$$K = \frac{Q \log_e \left(\frac{r_2}{r_1}\right)}{\pi (h_2^2 - h_1^2)} \qquad (25)$$

To use this formula the original saturated depth of the aquifer must be known. It may be noted that while this value of K is probably a more accurate one than is obtained from formula (20) its use in formula (9) may not give as good results as the value obtained from formula (20). This is because formula (20) has the same theoretical basis as formula (9).

Use of these steady state formulas for the determination of aquifer properties confers some advantages because the time variable does not appear in them and because formula (9) is not easy to solve for the transmissibility KD by algebraic means.

A value for the voids ratio V can be obtained by estimating the volume of sediments unwatered, from observation well data, and comparing this volume with the volume of water removed by the pumps.

By using graphical methods all of the aquifer properties can be determined from observation well data. To do this a plot of

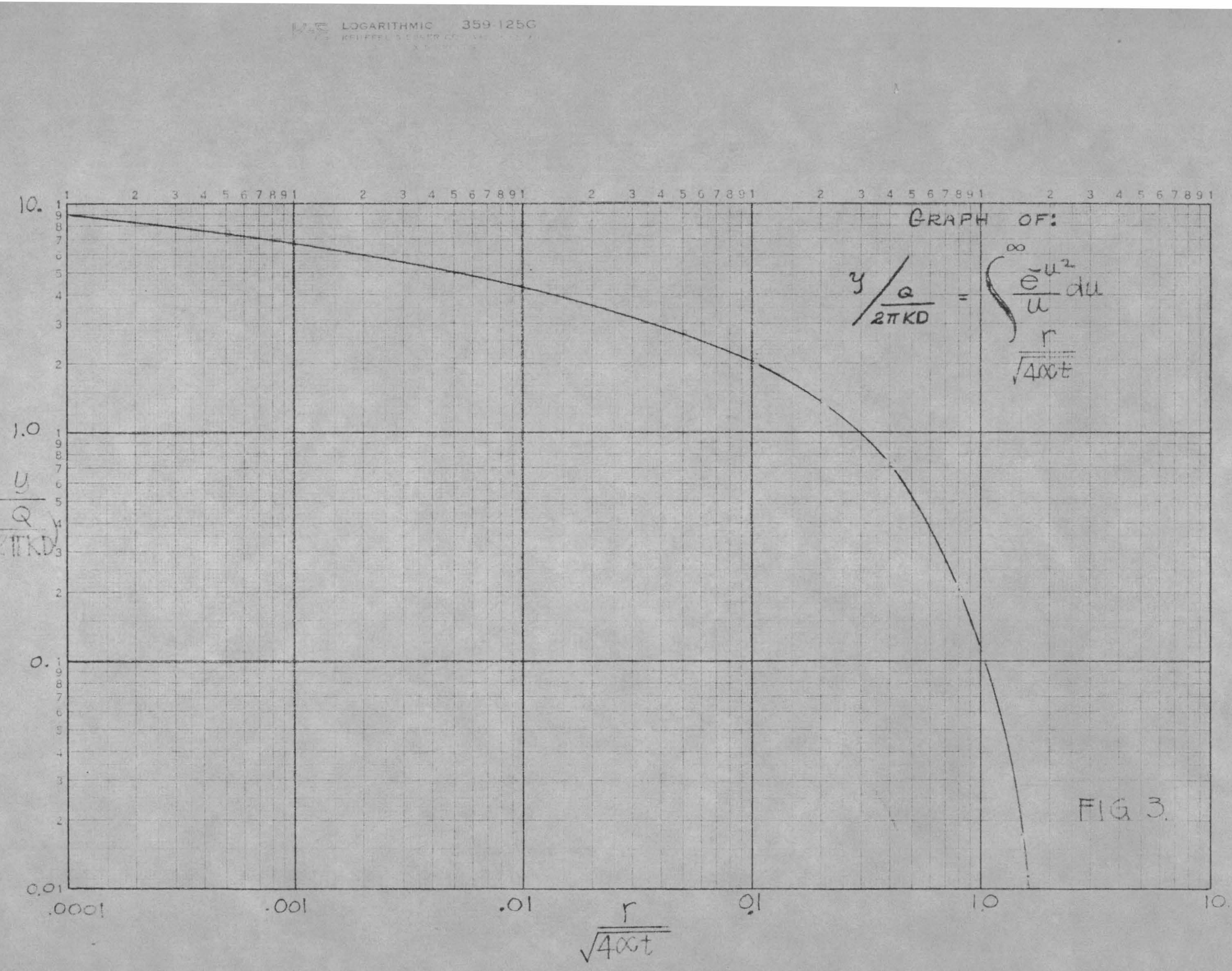
$y \Big/ \left(\frac{Q}{2\pi KD}\right)$ versus $r / \sqrt{4 \alpha t}$ is made on log paper, as

shown in figure (3). We will refer to this plot as the master chart. On a sheet of the same kind of log paper plot the drawdowns y from the observation wells versus the parameter $r/\sqrt{t}$. Then keeping the axes of the two sheets parallel adjust the plotted drawdowns to fit the curve of the master chart. This process is facilitated if one of the sheets is transparent. Select a point on the master chart for use as an index. To fix ideas we will assume that the index point is to be chosen at the intersection of the unity lines on the master chart. Then at this point we have

$$\frac{y}{\left(\frac{Q}{2\pi KD}\right)} = 1 \quad \text{and} \quad \frac{r}{\sqrt{4\alpha t}} = 1 \qquad (26)$$

When the index is read on the adjusted drawdown chart we obtain the corresponding values of y and $r / \sqrt{t}$. A substitution of these values into the above relations will permit an evaluation, first, of the transmissibility KD and then of the constant α . Since $\alpha = KD/V$ the value of V is readily found. This graphical procedure provides a means for solving equation (9) for the aquifer constants when test data are available.

A careful reader will have noted that the plot of the drawdown data is in terms of dimensioned quantities while the master chart is



LOGARITHMIC 359-125G
GRAPH OF:
$$y \Big/ \frac{Q}{2\pi KD} = \int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$$
10.
1.0
0.1
0.01
$\frac{y}{\left(\frac{Q}{2\pi KD}\right)}$
.0001
.001
.01
0.1
1.0
10.
$\frac{r}{\sqrt{4\alpha t}}$

FIG 3.

plotted in terms of dimensionless ratios. The propriety of such a procedure may well be questioned. The explanation lies in the use of logarithmic scales on both axes. With such scales a shift represents a multiplication. This principle is used in the construction of slide rules. When we fit the dimensioned plot to the master chart we produce the shifts which introduce the missing factors. These will always have the physical dimensions which were used in the drawdown plot.

We may now illustrate the use of these various methods by working out an example. For this purpose we will use the drawdowns and times of table 2 treating them as though they were observed data.

When the drawdowns of table 2 have been plotted on log paper against the values of $r/\sqrt{t}$ provided in the last column of this table and these plotted points have been fitted to the curve of figure 3 while keeping the axes parallel it will be found that the index of the master chart falls on the point:

$$y = 1.43 \qquad\qquad \frac{r}{\sqrt{t}} = 1.32$$

These values can now be substituted into equations (26). From the first of these:

$$\frac{y}{\left(\dfrac{Q}{2\,\pi\,KD}\right)} = \frac{1.43}{\left(\dfrac{0.780}{6.2832\,KD}\right)} = 1.$$

We obtain

$$KD = \frac{(0.780)(1)}{(6.2832)(1.43)} = 0.0867\ ft^2/sec$$

The original value was, as we know, KD = (0005)(170) = 0.085
Then we have recovered the original value within about 2 percent.
From the second of equations (26):

$$\frac{r}{\sqrt{4\,\alpha\,t}} = \frac{1.32}{\sqrt{4\,\alpha}} = 1 \quad \text{or} \quad \sqrt{4\alpha} = 1.32$$

$4\,\alpha = 1.74$ then $\alpha = 0.434\ ft^2/sec$
The original value, as we know, was 0.420 and, again, we have recovered the original value within 2 percent.

The value of V is obtained from the relation

$$\alpha = \frac{KD}{V}$$

By substitution:

$$0.434 = \frac{.0867}{V}$$

or

$$V = \frac{0.0867}{0.434} = 0.2$$

This agrees with the original value. We have now recovered all of the original aquifer properties by this graphical procedure.

The reader will find that he can also use the points of table 1. All of the points given in these two tables will plot along a single line regardless of the values of radius or time used. A perfectly consistent set of field data should do the same thing. All of the observed data can therefore be used in the graphical procedure for finding the aquifer properties.

These statements hold if the drawdowns are everywhere small compared to the original saturated depth of the aquifer.

Chapter 2 The Artesian Well

If a permeable strata in a formation is enclosed between impermeable beds, and has an outcrop which permits water to enter it, a condition may exist as shown in figure (4).

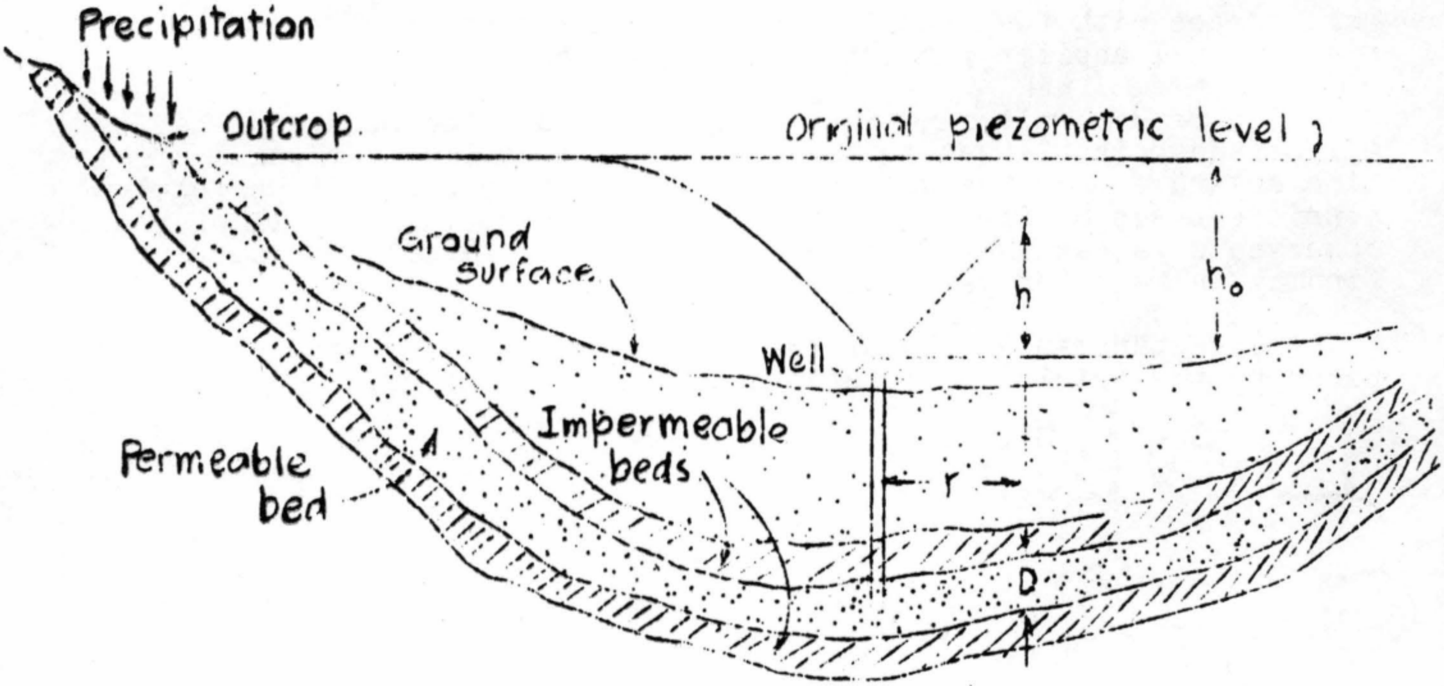


Figure 4 Artesian conditions

If the outcrop is above the top of the well casing, as is often the case, a flowing well will result. The well lowers the original pressure by the amount h_o at its location and, as time goes on, the zone of lowered pressure widens. This widening is accompanied by a reduction in the flow of the well. It will be our purpose to determine the rates at which the influence of the well spreads and the flow diminishes.

The equation of continuity, for this case, resembles closely that for the pumped well. The flow F at the radius r is now given by an expression of the type

$$F = K\,2\pi r \quad D\,\frac{\partial h}{\partial r} \qquad (27)$$

and the continuity equation becomes

$$\frac{\partial F}{\partial r}\,dr\,dt = V\,2\pi r\,dr\,\frac{\partial h}{\partial t}\,dt$$

or, if

$$\alpha = \frac{KD}{V} \quad \text{as before}$$

$$\alpha\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\,\frac{\partial h}{\partial r}\right) = \frac{\partial h}{\partial t} \qquad (28)$$

It may be noted that in this case there is no approximation since the permeable bed remains completely saturated at all times. However, because the water yield is obtained through a compression of the aquifer, instead of drainage, the values of V for these conditions will be much smaller than those generally found in unconfined aquifers.

The initial and boundary conditions for a well of radius a and an impermeable boundary at the radius b are

$$h = h_o \text{ when } t = 0 \text{ for } a < r < b$$

$$h = 0 \text{ when } r = a \text{ for } t > 0 \qquad \ldots (29)$$

$$\frac{\partial h}{\partial r} = 0 \text{ when } r = b \text{ for all values of } t.$$

Stated in this way the solution applies to an aquifer of finite extent but it will be shown later how this solution may be extended to an aquifer with an infinitely remote outer boundary.

A solution satisfying the differential equation (28) and the conditions (29) is

$$h = h_o \sum_{n=1}^{n=\infty} A_n \, U_o \left(x_n \frac{r}{b}\right) e^{-\frac{\alpha x_n^2 t}{b^2}} \qquad \ldots . (30)$$

Where U_o is a zero order Bessel Function of the type

$$U_o\left(\frac{x_n r}{b}\right) = J_o\left(\frac{x_n r}{b}\right) Y_o\left(\frac{x_n a}{b}\right) - J_o\left(\frac{x_n a}{b}\right) Y_o\left(\frac{x_n r}{b}\right). \qquad (31)$$

and

$$A_n = \frac{\frac{a}{b} U_o^1 \left(\frac{x_n a}{b}\right)}{\frac{x_n}{2}\left[(U_o(x_n))^2 - \left(\frac{a}{b} U_o^1 \left(\frac{x_n a}{b}\right)\right)^2\right]} \qquad \ldots . (32)$$

The symbol x_n represents the roots of the equation

$$U_o \; (x_n) = 0. \qquad \ldots (33)$$

The form of U_o insures that $h = 0$ when $r = a$.
The flow of the well Q is

$$Q = 2 \pi KDa \left(\frac{\partial h}{\partial r}\right)_{r=a} \qquad \ldots . (34)$$

This can be put in the form:

$$Q = 2\pi KDh_o \quad G\left(\frac{\sqrt{4\alpha t}}{a}\right)^{*} \tag{35}$$

where

$$G\left(\frac{\sqrt{4\alpha t}}{a}\right) = \sum_{n=1}^{n=\infty} A_n \left(\frac{x_n a}{b}\right) U_o^1 \left(\frac{x_n a}{b}\right) e^{-\frac{\alpha x_n^2 t}{b^2}} \tag{36}$$

The G function is here expressed in terms of a parameter of the type used throughout this text. If it is not immediately apparent that the G function can properly be expressed in terms of this parameter the question can be clarified by making the substitution

$$2m = \frac{a}{b}$$

then the exponents will be transformed in the following manner:

$$\frac{x_n^2 t}{b^2} = \frac{x_n^2 m^2 4\alpha t}{a^2}$$

It was stated previously that this solution applies to a well of radius a located at the center of a circular aquifer whose outer boundary is a circular impermeable barrier of radius b. The question will naturally arise as to how such a solution can be used when the aquifer is of infinite extent. The answer to this question may be found in a specialized computation procedure which is based upon the behavior of expressions of the type of equation (30). If we were to select a ratio $b/a = 10000$ we would find that a very large number of terms would be required to calculate the h/h_o values as a function of r for the early times. As the times increase the terms of significant magnitude decrease in number and the computation becomes much easier, Eventually the disturbance will reach the assumed outer boundary and drawdown patterns would be obtained which would not be appropriate for the case of an infinite outer boundary. We can avoid many of these difficulties by the following ruse. Suppose we begin with a choice $b/a = 10$. restrict ourselves to a few terms and determine the time when the first term discarded becomes negligible. As a representation of an infinite outer boundary case this solution is valid from this time until the disturbance reaches the outer boundary. When this happens we discard this series and take another based upon $b/a = 100$. Again we take only enough terms in it to permit us to take up the computation where we left off before. We compute with this ratio until the outer boundary is again reached. We then continue with a solution based upon $b/a = 1000$. By following such a procedure we can progress in steps, using in each step only a few terms of the series,

* A function equivalent to $G\left(\frac{\sqrt{4\alpha t}}{a}\right)$ is tabulated for a wide range of values in the paper by Jacob and Lohman.

and we can extend the outer boundary to any radius we choose. The chart of figure (5) was prepared, in part, by this method. The x_n and A_n values computed for these uses can be used also in the expression (36)′ to compute the G function. The values of the G function given in figure (6) were computed in this way.

An approximate formula which agrees well with these results is

$$\frac{h}{h_o} = 1 - \frac{\displaystyle\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du}{\displaystyle\int_{\frac{a}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du} \qquad \ldots(37)'$$

Approximate if $\frac{a}{\sqrt{4\alpha t}} < 0.02$

Example:

A one foot diameter casing penetrates a permeable bed of 170 feet total thickness lying between upper and lower confining beds of shale. The permeability of the permeable strata is K = 0.0007 ft/sec and specific yield V = .0005. If the initial pressure at the well was equivalent to 60 feet of water at the top of the casing find

(a) the pressure pattern 2 weeks after the well started to flow and

(b) the yield of the well as a function of time.

By use of the chart of figure (5) with

$$\alpha = \frac{KD}{V} = \frac{(.0007)(9170)}{.0005} = 238 \ \frac{ft^2}{sec}$$

Table 3 Pressure changes due to an artesian well

a = 0.5 feet   t = 1209600 seconds $\sqrt{\frac{4\alpha t}{a^2}} = 67800$

feet	$\frac{r}{a}$	$\frac{h}{h_o}$ *	h
0.5	1.0	0.00	0.00
1.0	2.0	.061	3.66
10.0	20	.270	16.2
1000	200	.481	28.9
1000	2000	.694	41.6
10000	20000	.905	54.9

* From figure (5)

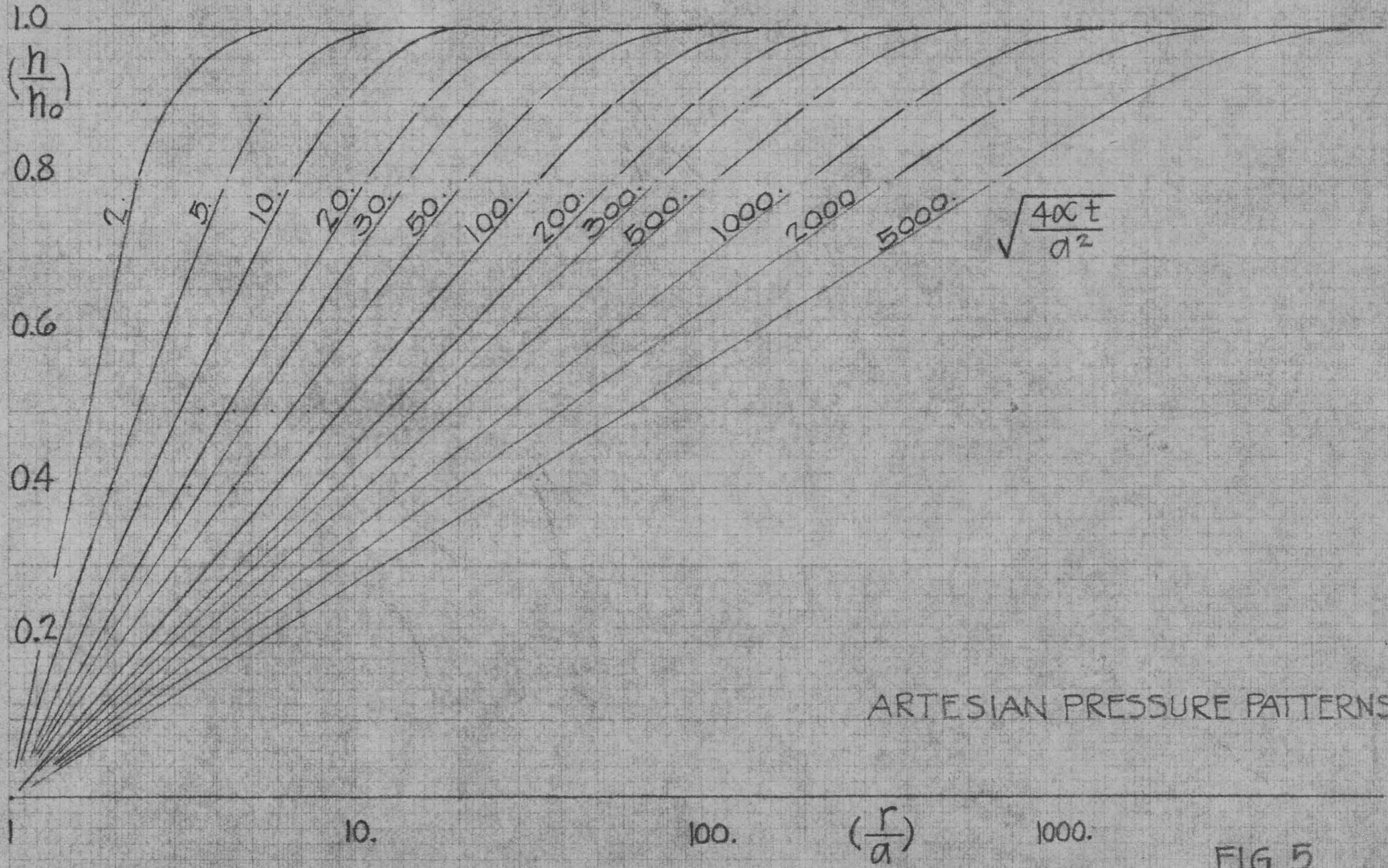
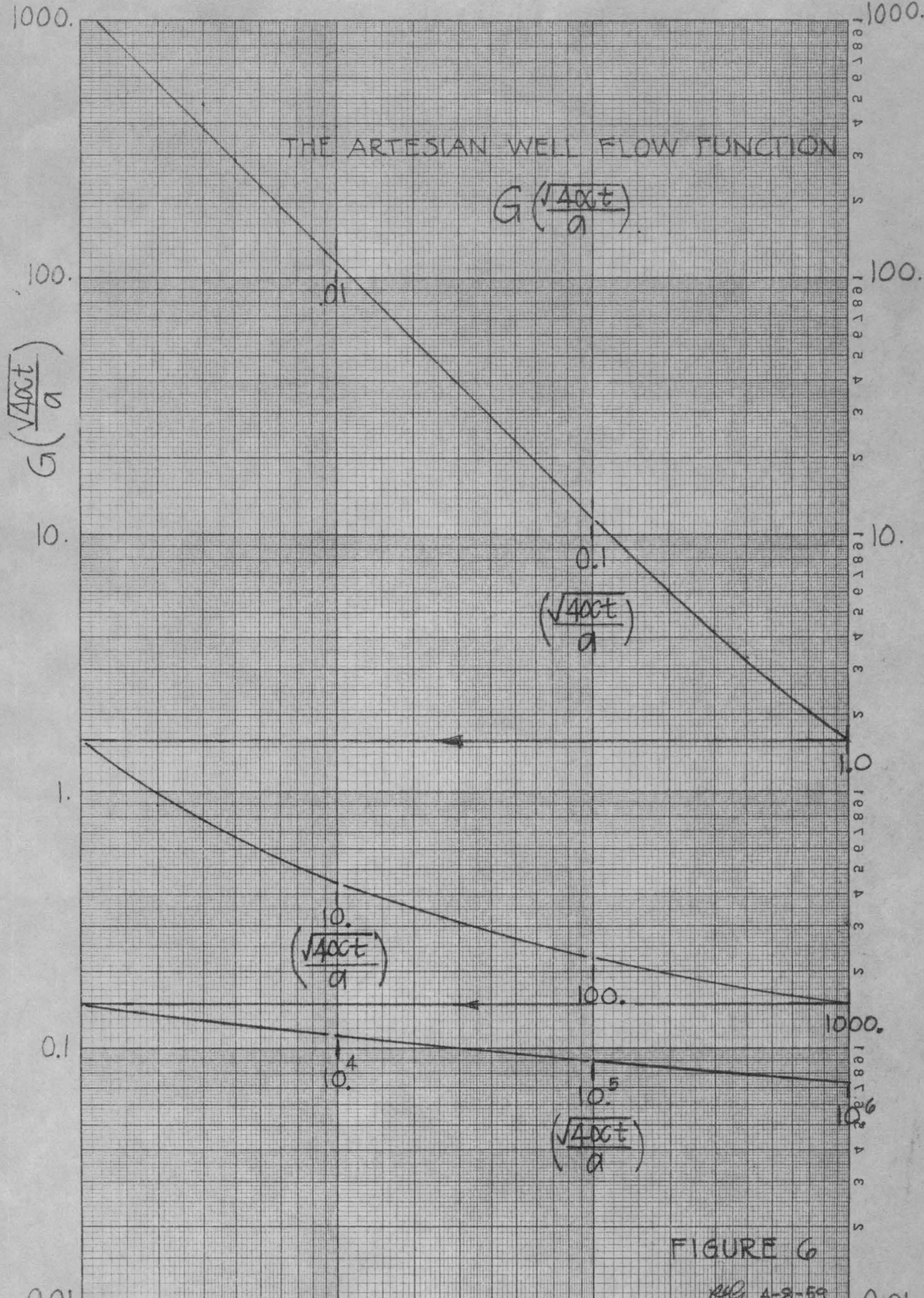

1.0
$\left(\frac{h}{h_o}\right)$
0.8
0.6
0.4
0.2
2.
5.
10.
20.
30.
50.
100.
200.
300.
500.
1000.
2000
5000.
$\sqrt{\frac{4\alpha t}{a^2}}$
ARTESIAN PRESSURE PATTERNS.
1
10.
100.
$\left(\frac{r}{a}\right)$
1000.

FIG 5.


THE ARTESIAN WELL FLOW FUNCTION
$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$
$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$
$\left(\frac{\sqrt{4\alpha t}}{a}\right)$
.01
0.1
1.0
10.
100.
1000.
10^4
10^5
10^6
1000.
100.
10.
1.
0.1
0.01
FIGURE 6
4-8-59
K+E LOGARITHMIC 359-125G
KEUFFEL & ESSER CO.
3 X 5 CYCLES

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A check by formula (37) yields the following results:

Table 4.

$\frac{a}{\sqrt{4\alpha t}}$		$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{a}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du$	$\frac{\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du}{\int_{\frac{a}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du}$	$\frac{h}{h_o}$	
0.5	.000,014,74	.000,014,74	10.836	10.836	1.0000	0	0
1.0		.000,029,5		10.143	0.935	.065	3.9
10		.000295		7.8406	0.724	.276	16.5
100		.00295		5.5380	0.511	.489	29.3
1000		.0295		3.2358	0.298	.702	42.1
10000		.295		0.9763	0.0902	.9098	54.5

This formula could be used for all later times.

The yield of the well can be computed as follows:

Table 5. Computation of yield of an artesian well.

Time	Time (seconds)	$\sqrt{\frac{4 \alpha t}{a^2}}$	$G\left(\sqrt{\frac{4 \alpha t}{a^2}}\right)$	Q (ft^3/sec)
One day	86,400	18,100	0.106	4.75
One week	604,800	48,000	0.095	4.26
One month	2,628,000	99,800	0.090	4.04
Six months	15,768,000	244,000	0.083	3.72
One year	31,536,000	346,000	0.080	3.59

$$\sqrt{\frac{4\alpha}{a^2}} = \sqrt{\frac{(4)(238)}{0.25}} = \sqrt{3800} = 61.6$$

$$2\pi KDh_o = (6.2832)(.0007)(170)(60) = 44.9$$

Chapter 3. The Aquifer With A Semi-Permeable Confining Bed.*

A case which occurs frequently is illustrated in figure (6) below. In this case a permeable stratum of thickness D is overlain with a stratum of low permeability of thickness m. The water table lies above this stratum. We consider the case where a well is drilled into the permeable stratum and pumped at the rate Q.

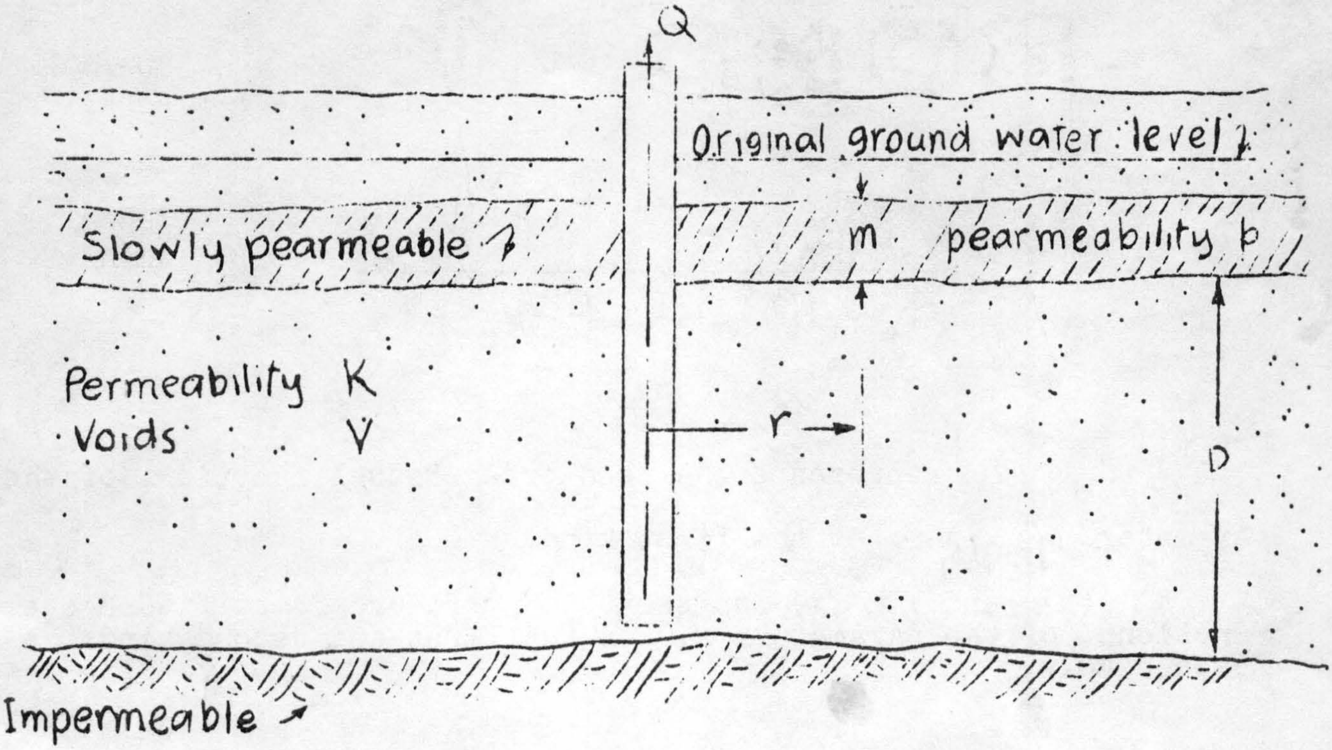


Figure (7) Aquifer with a semi-permeable confining bed.

When pumping begins the pressure in the permeable stratum is lowered. This has two effects. The first is to draw water from storage in the permeable bed which, in this case, behaves as an artesian aquifer in the sense that the water released from storage comes from a compression of the bed. The second effect is to cause water to seep downward through the semi-permeable confining bed. If the reduction of pressure at the radius r is represented by y then the condition of continuity is given by the differential equation

$$\frac{\partial y}{\partial t} = \frac{KD}{V}\left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r}\right) - \frac{p}{mV}y \tag{38}$$

A comparison will show that this equation differs from that of the artesian case only by the last term which accounts for the downward seepage through the slowly permeable confining bed. It is assumed, as a basis for formulating this term, that the downward flow per unit of horizontal area is proportional to the reduction of pressure

*This case has been given an elegant treatment by Jacob and Hantush in terms of the integral

$$W(u,r,B) = \int_u^\infty \frac{e^{-(y + r^2/4B^2y)}}{y}\,dy$$

y and the permeability p and inversely proportional to the thickness m.

For the finite case where a supply maintains the ground water level unchanged at an outer radius b a solution is:

$$\mu = \left[K_0(\rho) - \frac{K_0(\rho_b)}{I_0(\rho_b)} I_0(\rho) \right] - \sum_{n=1}^{n=\infty} \frac{2 J_0(\beta_n \rho)\, e^{-(1+\beta_n^2)\eta}}{\rho_b^2 J_1^2(\beta_n \rho_b)(1+\beta_n^2)} \qquad (39)$$

where:

$J_0(\beta_n \rho)$ represents the zero order Bessel function, of the parameter $(\beta_n \rho)$, of the first kind

K_0 (x) and I_0(x) represent the zero order modified Bessel functions, of the parameter x, of the first and second kinds respectively,

$$\rho = r\sqrt{\frac{p}{mKD}} \qquad \rho_b = b\sqrt{\frac{p}{mKD}}$$

$$\mu = y \Big/ \left(\frac{Q}{2\pi KD}\right) \qquad \eta = t\left(\frac{p}{mV}\right) \qquad (40)$$

and

β n are the roots of $J_0(\beta_n \rho_b) = 0$

The first term in the right hand member of the solution (39) represents the ultimate steady state and it will be helpful to note that there will always be an ultimate steady state when the upper confining permits seepage through it. In the case of the infinitely remote outer boundary the steady state is reached when all of the flow of the well is supplied by seepage through the slowly permeable confining bed. The ultimate steady state is then

$$\mu = K(\rho) \qquad \ldots\ldots\ldots\ldots\ldots \qquad (41)$$

The case of the infinitely remote outer boundary can be treated by the computation procedure previously used for the artesian case. A chart prepared in this way is shown in figure (8). An alternative arrangement is shown in figure (9) which has advantages where

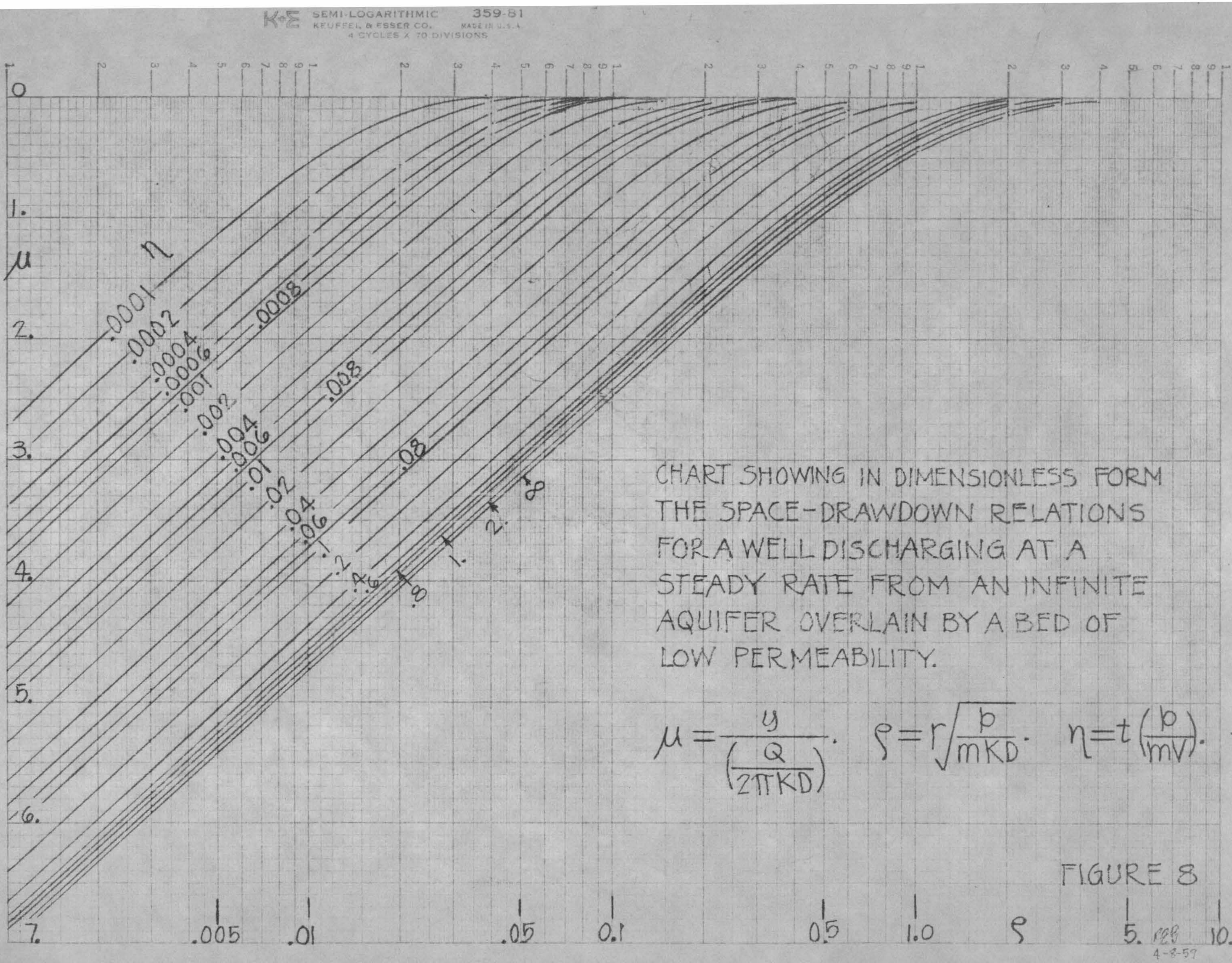
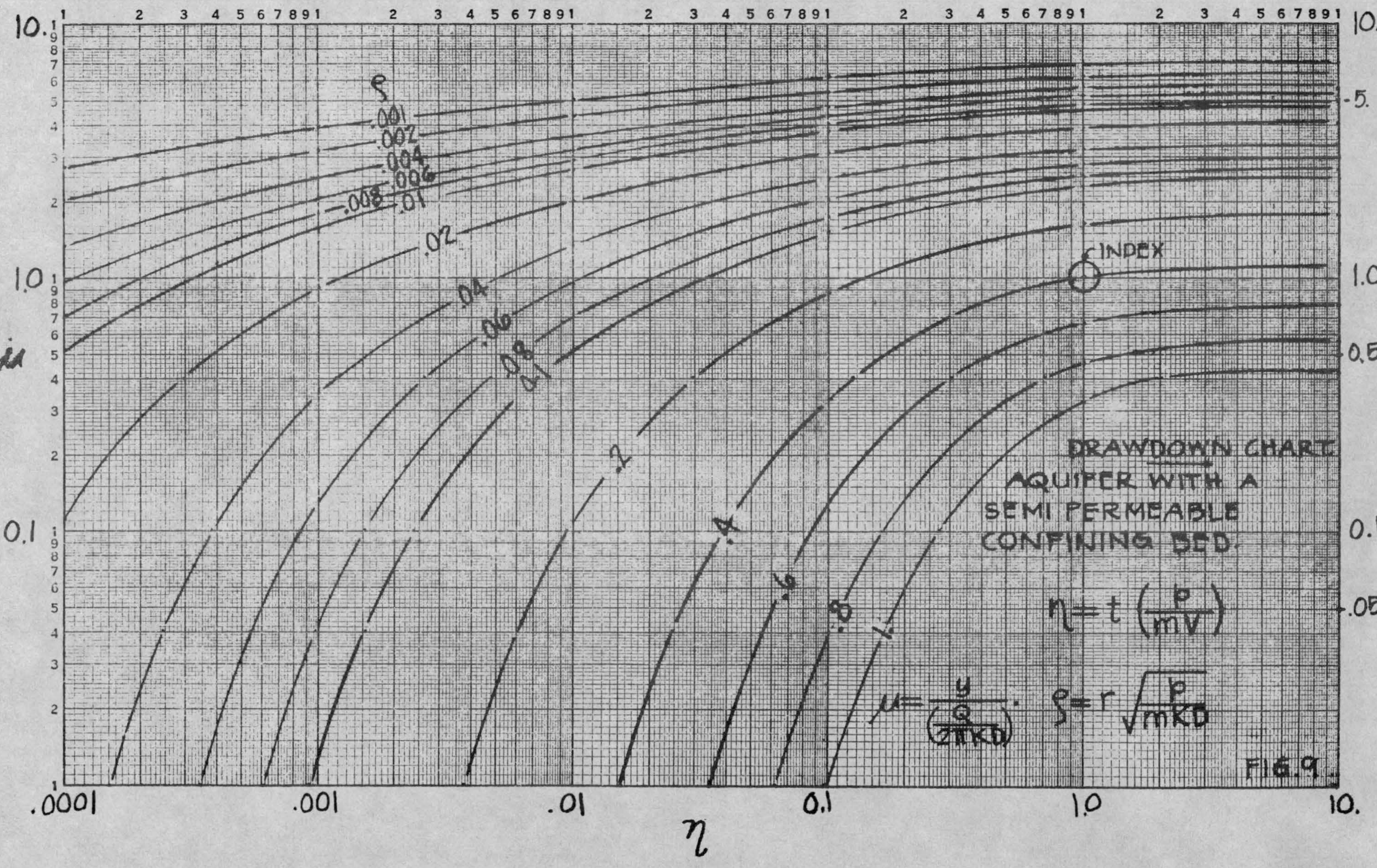


CHART SHOWING IN DIMENSIONLESS FORM THE SPACE-DRAWDOWN RELATIONS FOR A WELL DISCHARGING AT A STEADY RATE FROM AN INFINITE AQUIFER OVERLAIN BY A BED OF LOW PERMEABILITY.

$$\mu = \frac{y}{\left(\frac{Q}{2\pi KD}\right)}. \quad \rho = r\sqrt{\frac{p}{mKD}}. \quad \eta = t\left(\frac{p}{mV}\right).$$

FIGURE 8

REB
4-8-57


DRAWDOWN CHART
AQUIFER WITH A SEMI PERMEABLE CONFINING BED
$\eta = t\left(\frac{p}{mV}\right)$
$\mu = \frac{y}{\left(\frac{Q}{2\pi KD}\right)}$
$\rho = r\sqrt{\frac{p}{mKD}}$
INDEX
ρ
.001
.002
.004
.006
.008
.01
.02
.04
.06
.08
0.1
.2
.4
.6
.8
1.
μ
η
.0001
.001
.01
0.1
1.0
10.
0.1
1.0
10.
0.5
.05
5.
FIG. 9.

aquifer properties are to be determined from test data.

Example:

An aquifer of 125 feet thickness having a permeability 0.00040 ft/sec and a specific yield of .0009 is overlain by a bed of glacial till having a thickness of 32 feet and a permeability of $(35)(10)^{-9}$ ft/sec. A well penetrating the full thickness of the aquifer is pumped at the rate of 0.25 ft /second.

Required.

(a) The drawdown as a function of the radius at the end of 24 hours of pumping.

(b) The drawdowns representing the final steady state condition and an estimate of the time required to establish this steady state.

Solutions

With

$$K = .00040 \text{ ft/sec}$$
$$D = 125 \text{ ft} \qquad KD = .050 \text{ ft}^2\text{/sec}$$
$$V = 0.0009 \text{ (dimensionless)}$$
$$p = (35.0)(10)^{-9} \text{ ft/sec}$$
$$m = 32 \text{ ft}$$
$$Q = 0.250 \text{ ft}^3\text{/sec}$$
$$t = 86400 \text{ seconds (24 hours)}$$

$$\frac{Q}{2\pi KD} = \frac{0.250}{(6.2832)(050)} = 0.795$$

$$\sqrt{\frac{p}{m\,KD}} = \sqrt{\frac{(35.)(10)}{(32)(.050)}} = \sqrt{(2.19)(10)} = (1.480)(10)$$

$$\left(\frac{p}{mV}\right) = \frac{(35)(10)}{(32)(.0009)} = (1.213)(10)$$

$$\eta = t\left(\frac{p}{mV}\right) = (86400)(1.213)(10) = .1048$$

Table 6. Computed drawdowns for a well with a semi-permeable confining bed.

radius feet	ρ	μ	y feet
10.0	.00148	5.73	4.56
50.0	.0074	4.12	3.28
100.0	.0148	3.43	2.73
500	.074	1.83	1.46
1000	.148	1.44	1.15
000	0.74	0.10	0.080
10000	1.48	0.00	0.00

The ultimate steady state is given by

$$\mu = K_o(\rho)$$

From tables or from figure 8

Table 7. Ultimate steady state

radius feet	ρ	$K_o(\rho)$	y_m
1.0	.000148	8.9342	7.17
5.0	.000740	7.3248	5.84
10.0	.00148	6.6316	5.02
50.0	.00740	5.0222	3.95
100.0	0.0148	4.3290	3.44
500.0	.0740	2.7248	2.16
1000.0	.148	2.0412	1.62
5000	.740	.6202	0.493
10000	1.480	.2194	0.174

For small values of ρ, approximately:

$$K_o(\rho) = -\left(\gamma + \log_e\left(\frac{\rho}{2}\right)\right) + \left(\frac{\rho}{2}\right)^2. \quad (\text{for } \rho < 0.1)*$$

where γ = 0.57721 is Euler's constant.
A computation for the small values of $K_o(\rho)$ is given below:

* For ρ = 0.1 This expression gives K_o (0.1) = 2.4235. The true value is K_o (0.1) $\doteq$ 2.42706 +. In this expression $I_o(\rho)$ has been replaced by unity. The power series is

$$K_o(\rho) = -\left(\gamma + \log_e\left(\frac{\rho}{2}\right)\right) I_o(\rho) + \left(\frac{\rho}{2}\right)^2 + \frac{3}{8}\left(\frac{\rho}{2}\right)^4 +$$

A computation for the small values of $K_0(\rho)$ is given below:

Table 8. Computation of $K_0(\rho)$ for small values of ρ

r feet	ρ	$\frac{\rho}{2}$	$\left(\frac{\rho}{2}\right)^2$	$-(\gamma + \log_e(\frac{\rho}{2}))$	$K_0(\rho)$	Comparisons from tables
1.0	.000148	000074	----	8.9342	8.9342	
5.0	.000740	00037	-----	7.3248	7.9348	
10.0	.00148	00074	-----	6.6316	6.6316	
50	.00740	0037	-----	5.0222	5.0222	
100	.0148	0074	-----	4.3290	4.3290	
500	.0740	037	.0014	2.7196	2.7248	
1000	.148	074	.0055	2.0246	2.0412	2.0434

A glance at figure (8) will show that the ultimate steady state will be nearly attained when $\eta = 4$. Then the corresponding time is

$$t = \frac{\eta}{\left(\frac{p}{m\,V}\right)} = \frac{4}{(1.213)(10)^{-6}} = 3,300,000 \text{ seconds}$$

This is equivalent to about 38 days.

Chapter 4. Bank Storage

When a reservoir has been filled for some time and is then drawn down a condition such as is shown on figure (10) exists.

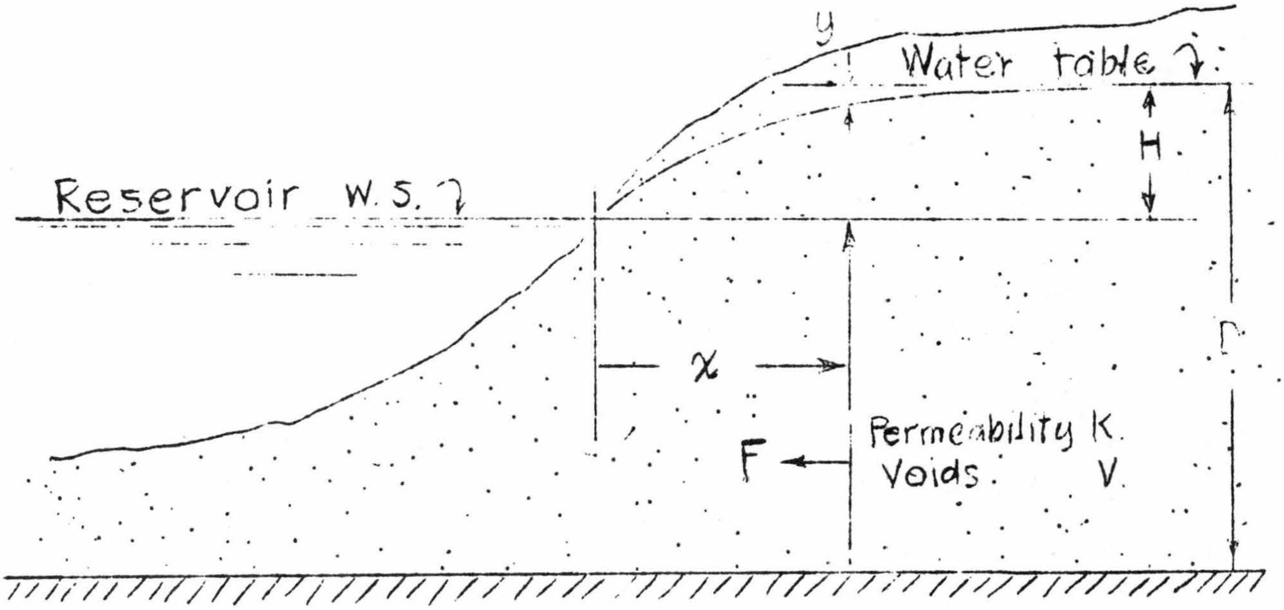


Figure (10). Conditions following a reservoir drawdown.

The flow F per unit length of bank is:

$$F = -K\,(D-y)\,\frac{\partial y}{\partial x} \quad \ldots\ldots \quad (42)$$

If y is small compared to D it can be neglected. It will be assumed that this is done.
The continuity condition is:

$$\frac{\partial F}{\partial x}\,dx\,dt = -V\,\frac{\partial y}{\partial t}\,dt\,dx$$

or, by making use of (42)

$$KD\,\frac{\partial^2 y}{\partial x^2} = V\,\frac{\partial y}{\partial t} \quad \ldots\ldots \quad (43)$$

If

$$\alpha = \frac{KD}{V} \qquad (44)$$

This equation becomes

$$\alpha\,\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t} \qquad (45)$$

A solution satisfying the conditions:

$$\begin{aligned} &\text{when} \quad x = 0 \quad y = H \quad \text{for} \quad t > 0 \\ &\text{when} \quad t = 0 \quad y = 0 \quad \text{for} \quad x > 0 \end{aligned} \qquad (46)$$

is:

$$y = H\left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4\alpha t}}} e^{-u^2} du\right] \quad \ldots\ldots \quad (47)$$

The integral which appears here is the "probability integral." It has been extensively tabulated. Values are given in table I. Values of one minus this integral are given in table II.

The flow F at x is given by the expression

$$F = H \frac{2\,KD}{\sqrt{\pi}} \frac{e^{-\frac{x^2}{4\alpha t}}}{\sqrt{4\alpha t}} \quad \ldots\ldots \quad (48)$$

The flow F_o at $x = 0$ is

$$F_o = \frac{2\,HKD}{\sqrt{4\pi\alpha t}} \quad \ldots\ldots \quad (49)$$

The total amount of water which has flowed out of the bank up to the time t is obtained by integrating F_o with respect to t. It is:

$$q_o = HV \sqrt{\frac{4\alpha t}{\pi}} \quad \ldots\ldots \quad (50)$$

The use of these formulas may be illustrated by the following example:

Example

A reservoir with a shore line of 21 miles has been filled for a considerable time and the reservoir level is then dropped 8 feet. If D = 280 feet K = 0.0001 ft/sec V = 0.08 Compute the rate of return flow and the total return after the lake level has been drawn down for one month.

Solution:

$$\alpha = \frac{KD}{V} = \frac{(.0001)(280)}{0.08} = 0.35 \text{ ft}^2/\text{sec}$$

One month is 2,628,000 seconds.

The rate of return flow is :

$$F_o = \frac{2\ HKD}{\sqrt{4\pi\alpha t}} = \frac{(2)(8)(.0001)(280)}{\sqrt{4\pi\,(0.35)(2628000)}} = \frac{0.448}{3400} = .0001318$$

cubic feet per foot of bank per second

The total perimeter is (21)(5280) = 110,800 feet
Then the return flow is (110800)(.0001318) = 14.6 cubic feet per second.
The total amount of return flow is:

$$q_o = HV\sqrt{\frac{4\alpha t}{\pi}} = (8)(0.08)\sqrt{\frac{(4)(0.35)(2628000)}{\pi}}$$

or

q_o = (0.64)(1057) = 677 cubic feet per foot of bank. Since the total perimeter is 110,800 feet the total return flow in the first month is:

(110800)(677) = 74,800,000 cubic feet. Since an acre foot is 43,560 cubic feet the total return flow in the first month is:

$$\frac{74\ 800\ 000}{43\ 560} = 1{,}720 \text{ acre feet}$$

Chapter 5. The Line Source or Sink

The solution described in chapter 4 applies when the drawdown is held constant at $x = 0$ and the flow varies with time. The alternative case where the flow is held constant at $x = 0$ and the drawdown varies with time will now be presented. The physical conditions are shown in figure (11) below

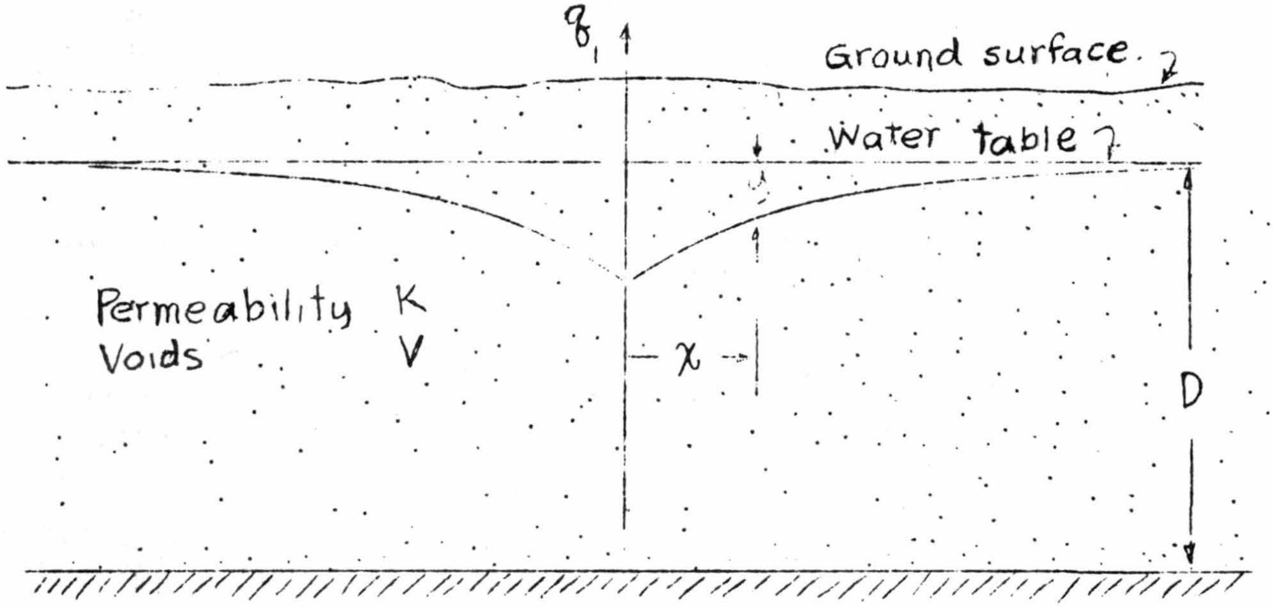


Figure (11). The line source or sink

Of the total flow q_1 half comes from the direction of positive x and the other half comes from the direction of negative x.

The solution required is:

$$y = \frac{q_1 x}{2\pi K D}\sqrt{\pi}\int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty}\frac{e^{-u^2}}{u^2}\,du \qquad .....(51)$$

This solution satisfies the differential equation

$$\alpha\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t} \qquad ....(52)$$

and the conditions

when $x = 0. \quad -2\,KD \dfrac{\partial y}{\partial x} = q_1$ for $t > 0.$ ...... (53)

when $t = 0.\ y = 0.$ for $x > 0.$

Values of the integral may be found in table III., of the Appendix.

At $x = 0$ the solution becomes indeterminate but at this point the drawdown y_o can be computed from the expression

..... (54)

$$y_o = \frac{q_1 \sqrt{4 \pi \alpha t}}{2 \pi K D}$$

The use of this formula may be illustrated by the following example:

Example.

An unlined irrigation canal has a seepage loss of 1 cubic foot per second per mile of canal. It delivers water for six months each year. If it overlies an aquifer with a depth $D = 60$ feet a permeability of 0.0004 ft/sec and a voids ratio of 0.20 estimate the height of the ground water ridge produced by its seepage losses at the end of the six months of operation.

Solution

$$\alpha = \frac{KD}{V} = \frac{(.0004)(60)}{(0.2)} = 0.120$$

Six months is 15,768,000 seconds.

$$\sqrt{4 \alpha t} = \sqrt{(4)(0.120)(15768000)} = 2750 \text{ ft}$$

$$q_1 = \frac{-1.00}{5280} = -0.000,278$$ cubic feet per second-per foot.

$$\frac{q_1}{2 \pi K D} = \frac{0.000278}{(6.2832)(.0004)(60)} = .001845$$

$$y_o = -\frac{q_1 \sqrt{4 \pi \alpha t}}{2 \pi K D} = -(001845)(2750) = -8.98 \text{ ft.}$$

Computation of the variation of the height of the ground water mound with distance is shown in the following table:

Table 9. Heights of ground water mound.

x (feet)	$\frac{x}{\sqrt{4\alpha t}}$	$\int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u^2}\,du$	y (feet)
0	0	----	-8.98
100	.0364	26.0	-8.50
200	.0727	12.1	-7.91
500	.1820	4.00	-6.55
1000	.364	1.45	-4.74
5000	1.820	0.0035	-0.06

The minus sign indicates a rise in the ground water level.

Chapter 6. Parallel Drains.

When land is to be drained for agricultural purposes a system of parallel drains as shown in figure (12) is often used.

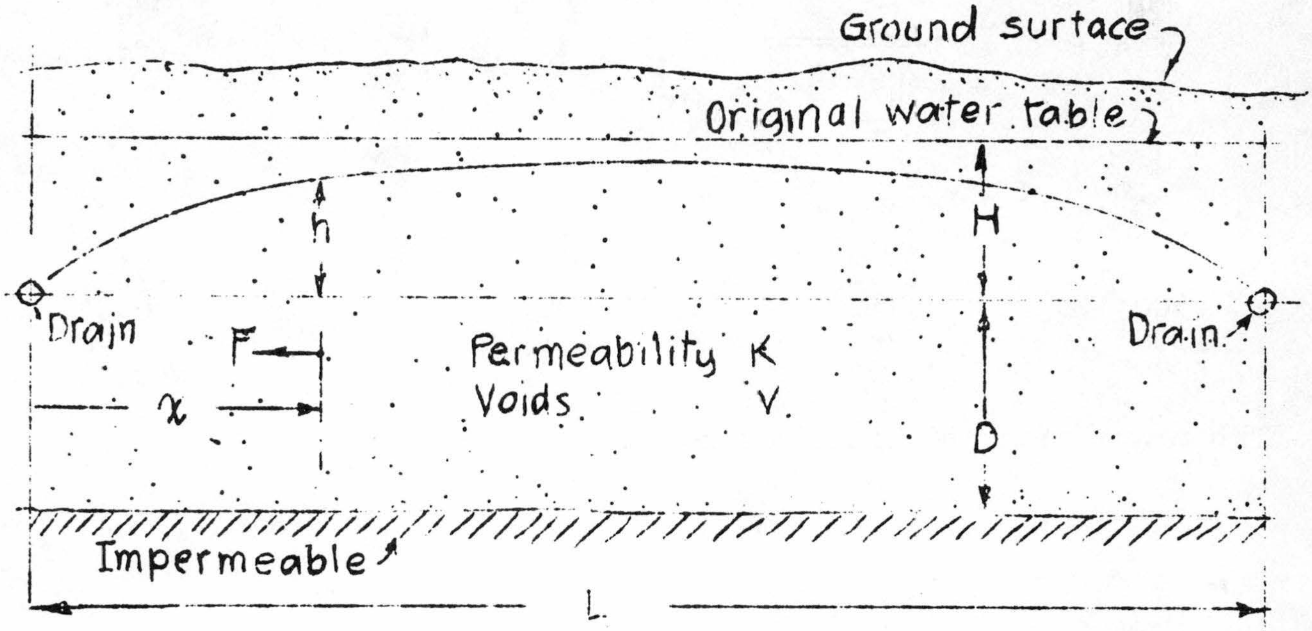


Figure (12). Parallel drains

As a basis for treating this case it will be assumed that an impermeable bed underlies the area at a depth d below the drains. Due to application of irrigation water or otherwise a uniform depth H is saturated with water. The distance between drains is L and the remaining saturated depth at the distance x from one of the drains is h. The permeability is K and the drainable voids V. The flow F per unit length of drain is

$$F = K\,(d+h)\,\frac{\partial h}{\partial x} \qquad \text{.... (55)}$$

And the condition of continuity is:

$$\frac{\partial F}{\partial x}\,dx\,dt = V\,\frac{\partial h}{\partial t}\,dt\,dx$$

In order to avoid a non-linear expression which would cause serious mathematical difficulties we will replace the quantity (d+h) by an average value.

$$D_a = (d + \frac{H}{2}) \qquad \text{.... (56)}$$

Then the differential equation becomes

$$K\,D_a\,\frac{\partial^2 h}{\partial x^2} = V\,\frac{\partial h}{\partial t}$$

or

$$\alpha_a = \frac{KD_a}{V} \qquad \text{.... (57)}$$

$$\alpha_a \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \qquad \text{.... (58)}$$

A solution which meets the initial and boundary conditions:

$$\begin{array}{llll} \text{When } x = 0 & h = 0 & \text{for} & t > 0 \\ \text{When } x = L & h = 0 & \text{for} & t > 0 \\ \text{When } t = 0 & h = H & \text{for} & 0 < x < L \end{array} \qquad (59)$$

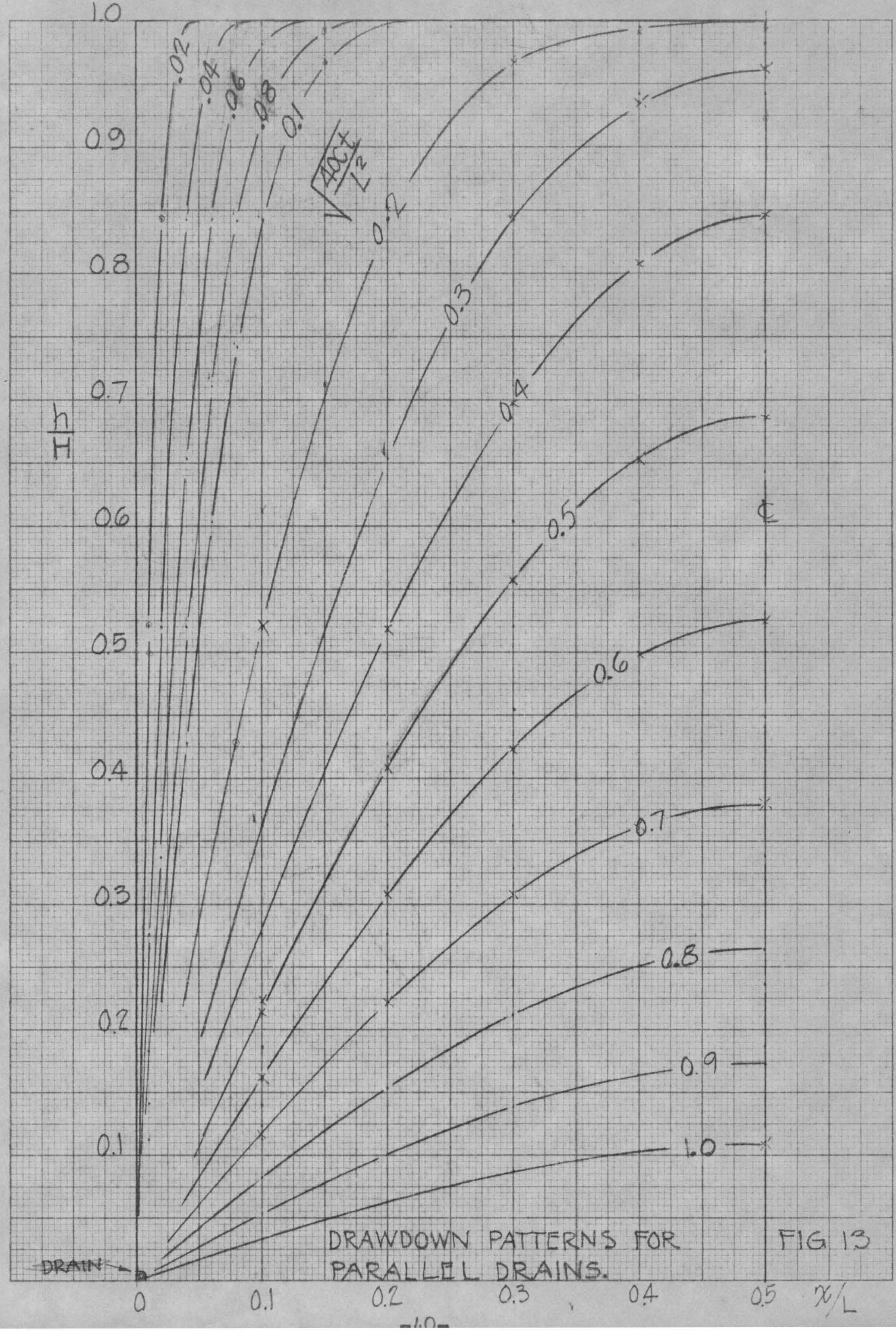
is:

$$h = H\frac{4}{\pi} \sum_{n=1,3,5 \text{ etc}}^{n=\infty} \frac{e^{-\frac{\alpha_a n^2 \pi^2 t}{L^2}}}{n} \sin \frac{n\pi}{L} x \qquad \text{.... (60)}$$

A plot of this function is shown in figure (13). For the sake of uniformity the time variable is put in the form $\frac{\sqrt{4\alpha_a t}}{L}$ while the space variable is put in the form x/L. This plot represents in a generalized form, the succession of profiles assumed by the water table during the draining period. Because of the average depth introduced to avoid a non-linear type of differential equation the solution is an approximate one and the approximation is best when H is small compared to d. However, comparison with a solution of the non-linear differential equation for the case where the drains are at the impermeable bed indicates that the choice made above for the value of D_a will permit this solution to be used without grave error even though H is not small compared to d.

Drainage is slowest at the point midway between the drains. The ratio of the remaining drainable depth at the center h_c to the original drainable depth H is shown on figure (13). The ratio (h/H) is called the part remaining. This ratio plays an important role when there is a drainage flow in more than one direction. If there is a flow in both the direction x and z equation (58) is replaced by:

$$\alpha_a \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} \right) = \frac{\partial h}{\partial t} \qquad (61)$$



DRAWDOWN PATTERNS FOR PARALLEL DRAINS.

FIG 13

If h_1 is a solution of

$$\alpha_a \frac{\partial^2 h_1}{\partial x^2} = \frac{\partial h_1}{\partial t} \tag{62}$$

If
and h_2 is a solution of

$$\alpha_a \frac{\partial^2 h_2}{\partial z^2} = \frac{\partial h_2}{\partial t} \tag{63}$$

Then the product $h = h_1 h_2$ is a solution of (61). This relation, here called the product law, permits some important cases of two dimensional flow to be solved by using one dimensional solutions. This method will be illustrated by an example later.

The rate at which water flows into the drains is:

$$F_o = KD_a \left(\frac{\partial h}{\partial x}\right) \quad x = 0 \tag{64}$$

or since, by differentiation of (60)

$$\frac{\partial h}{\partial x} = \frac{4H}{L} \sum_{n=1,3,5 \text{ etc}}^{n=\infty} e^{-\frac{n^2\pi^2\alpha_a t}{L^2}} \cos \frac{n\pi}{L} x \quad \ldots\ldots \tag{65}$$

Then

$$F_o = \frac{4KD_aH}{L} \sum_{n=1,3,5 \text{ etc}}^{n=\infty} e^{-\frac{n^2\pi^2\alpha_a t}{L^2}} \tag{66}$$

A plot derived from this relation is also shown on figure (14). It is useful for estimating the capacity of drains.

The use of these formulas may now be illustrated by means of examples.

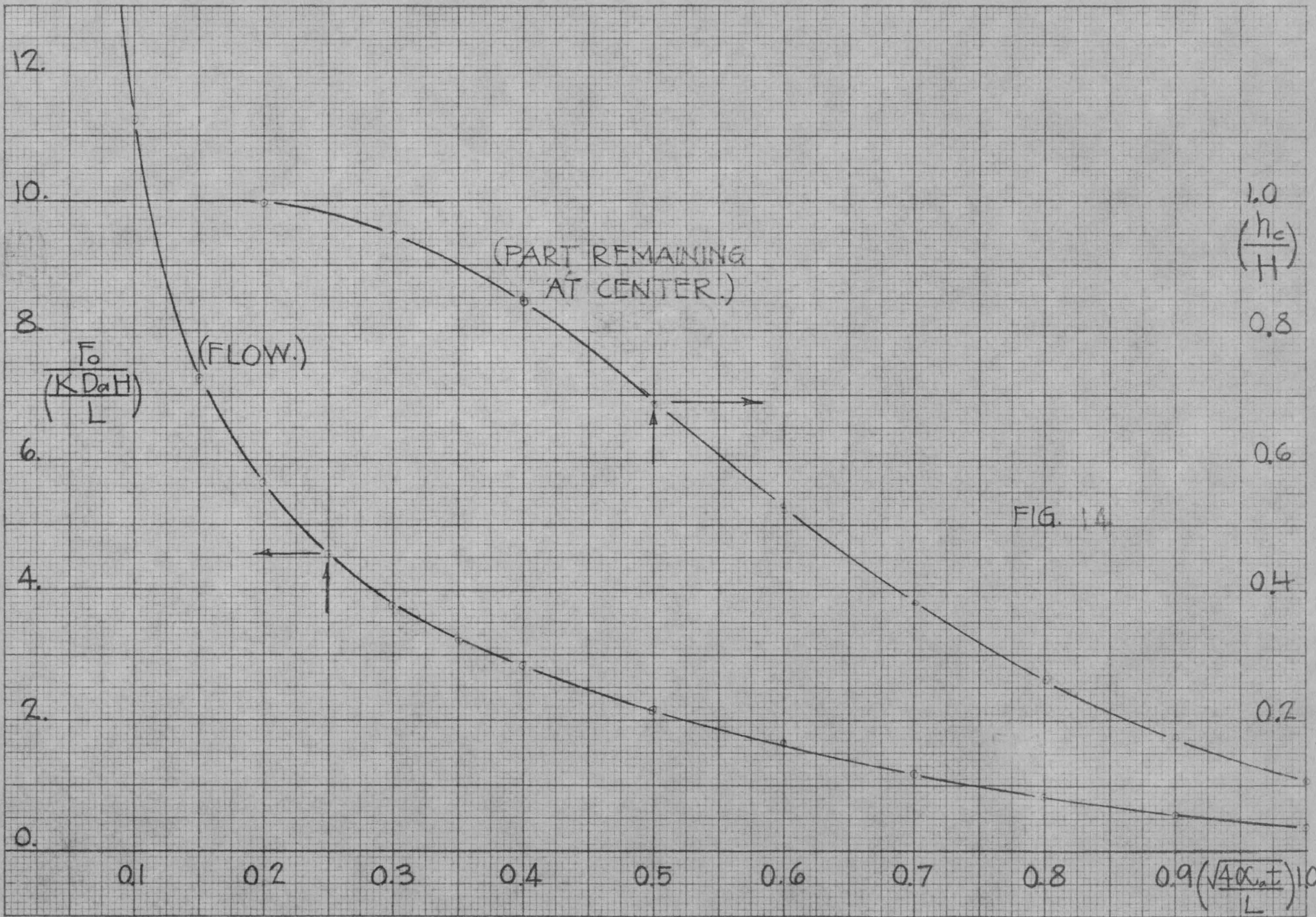

12.
10.
8.
6.
4.
2.
0.
$\frac{F_0}{\left(\frac{K D_a H}{L}\right)}$
(FLOW.)
(PART REMAINING AT CENTER.)
1.0
$\left(\frac{h_c}{H}\right)$
0.8
0.6
0.4
0.2
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
$\left(\frac{\sqrt{4\alpha_a t}}{L}\right)$

FIG. 14

Example:

An irrigated area receives 2.5 feet of water each year, of which it is estimated that about 1 foot is lost by deep percolation. With this amount of water loss the land is in need of drainage. The sediments have a depth of 80 feet below the proposed level of the drains, a permeability of .0002 ft/sec and a drainable voids ratio of 0.15. The ground surface is 14 feet above the drain level. What drain spacing would be needed to keep the water table at least 5 feet below the ground surface at all times.

Solution

If all of the loss occurred at one time the rise of the water table would be $1.00 / 0.15 = 6.67$ feet. Then if the water table is to be kept 5 feet below the surface, at all times, the elevation of the water table above, the level of the drains, at the point midway between them, could be as much as:

$$14 - 5 - 6.67 = 2.33 \text{ feet.}$$

On this basis a year would be available to drain the excess away. Then approximately $H = 6.67 + 2.33 = 9.00$ feet $h = 2.33$ feet $(h / H) = (2.33 / 9) = 0.259$
From the part remaining curve of figure (14) with $(h / H) = 0.259$ read

$$\sqrt{4 \alpha t} / L = 0.802$$

with $D = 80 + \frac{10}{2} = 85.$

$$\alpha_a = \frac{KD_a}{V} = \frac{(.0002)(85.)}{0.15} = 0.1133$$

One year is 31,536,000 seconds

Then $4 \alpha_a t = (4)(0.1133)(31536000) = 14300000$

Since

$$\frac{\sqrt{4 \alpha_a t}}{L} = 0.802 \qquad L = \frac{\sqrt{4 \alpha_a t}}{0.802}$$

or

$$L = \frac{3780}{0.802} = 4720 \text{ feet}$$

This figure could be refined somewhat by a trial process which would account for the fact that, when the last irrigation is made, the previous imcrements have had some opportunity to drain away. We will assume that there will be four irrigations applied one month apart beginning June 1 and we will try a 5280 foot drain spacing. Then the increment of ground water depth due to each irrigation would be $\frac{100}{(4)(0.15)}$ = 1.667 feet

The critical period would come at the time of the last irrigation on September 1.

One month is 2,628,000 seconds. The computation is made in the manner indicated in the table below.

Table 10. Remaining drainable depths.

Time of irrigation	Increment of depth (ft)	Time to Sept 1 (sec)	$\frac{\sqrt{4\alpha_a t}}{L}$	$\left(\frac{h_c}{H}\right)$	h
June 1	1.667	7,884,000	.358	0.892	1.484
July 1	1.667	5,256,000	.292	0.953	1.587
Aug 1	1.667	2,628,000	.207	0.995	1.658
Sept 1	1.667	0	0	1.000	1.667
Totals	6.668				6.396

The point that must now be reached in a year of drainage is approximately 14.00 - 5 - 6.396 = 2.604

Then, as before, take h_c = 2.604 H = 9.00 ft

$$(h_c / H) = 0.289$$

From the chart of figure $\sqrt{4\alpha_a t} / L = 0.776$

$$L = \frac{\sqrt{4\alpha_a t}}{0.776} = \frac{(4)(.1133)(31536000)}{0.776} = \frac{3780}{0.776} = 4870$$

By repeated trials we could bring these figures close together. The spacing we would ultimately find would be between 4870 and 5280 feet. It would be close to the lower figure. The final figure for this case is approximately 4900 feet.

In the process we have been using a small concession has been made to obtain a more expeditious computation procedure. When computing the drain spacing by use of a chart prepared for a drainable increment of uniform depth a small error has been committed because the part carried over from the previous year is not of uniform depth. An exact computation could be made by superimposing drainable increments as they arise over a period of years. Such a computation would be much more cumbersome than the one described.

To illustrate the use of the product law we may assume that the above drains, spaced 4900 feet apart, terminate in a collecting drain, so that drainage can move in two directions. We will compute the part remaining at a point midway between the drains and 2640 feet away from the collecting drain. The parallel drains will be assumed to be much longer than the distance between them. For the flow toward the collecting drain the idealization of chapter 4 may be considered appropriate.

This is:

$$\frac{h_z}{H} = \frac{2}{\sqrt{\pi}}\int_0^{\frac{z}{\sqrt{4\alpha_a t}}} e^{-u^2}\,du$$

For $z = 2640$ ft $t = 31,536,000$ sec

$$\frac{z}{\sqrt{4\alpha_a t}} = \frac{2640}{3780} = 0.698$$

From tables of the probability integral $\frac{h_z}{H} = 0.676$

For the point midway between drains

$$\frac{\sqrt{4\alpha_a t}}{L} = \frac{3780}{4900} = 0.770$$ and from the chart of figure $\frac{h_c}{H} = 0.296$ then $\frac{h}{H} = \left(\frac{h_z}{H}\right)\left(\frac{h_c}{H}\right) = (0.676)(0.296)$ $= 0.200$

Then with the parallel drains only, the original 9.00 foot increment would be drained away to a depth of (9.00)(0.296) = 2.66 feet. With the effect of the collecting drain included the corresponding depth is (9.00)(0.200) = 1.80 feet.

To estimate the drain capacity required we can use the flow curve of figure (14). For our case, with L = 4900 ft.

$$\frac{KD_a\ H}{L} = \frac{(.0002)(85)(9)}{4900} = .000,031,2\ \text{ft}^2/\text{sec}$$

A reference to this figure will show that the flow to the drain is not constant but decreases with time as the drainage progresses. This is the behavior we should expect but it requires the exercise of a little judgement in the selection of drain capacity. If we arbitrarily select the point $\left(\sqrt{4\alpha_a t}/L\right) = 0.1$ we read

$$F_c \Big/ \left(\frac{KD_a}{L}\right) = 11.25$$

and the flow to the drain from <u>one side</u> is, at this time, (.000,0312)(11.25) = .000351 cubic feet per second per foot of drain. Then 1000 feet of this drain would collect (1000)(000351) = 0.351 second feet from each side, or 0.702 cubic feet per second from both sides. A reference to figure (14) will show that the time

$\left(\sqrt{4\alpha_a t}/L = 0.1\right.$ is very early in the draining cycle and, as a matter of judgement, it can be concluded that a drain of this capacity would be satisfactory. The worst that could happen would be that the drainage would be retarded slightly in the early part of the drainage cycle.

Chapter 7. The Use of Images.

The solution of chapter 1 is appropriate when the aquifer, from which the well draws water, extends to an infinite distance from it in all directions. Such conditions are rarely met in nature and it is then important to know how the solution can be adapted to represent the conditions as they are. Many of these conditions aan be met by the use of images. Such uses are illustrated by the following cases:

Case 1. A well near a river.

Many irrigation wells draw water from sediments in a river valley. When pumping begins water is, at first, drawn from storage but as the cone of depression deepens and widens it finally makes contact with the river. Because the river will maintain the water table elevation along its bank no drawdown will occur there. If the stream bank is idealized as a straight line it is possible to account for this condition by use of an image well. Suppose we return to the infinitely extended aquifer and lay out upon it a line representing the idealized position of the river bank. At right angles to this line another line is drawn which passes through the pumped well. If a recharge well, having an inflow equal to the outflow of the pumped well, is now located on this second line on the opposite side of the idealized river bank line, and the same distance from it, the drawdown of the pumped well will be neutralized, by the rise due to the recharge well, with the result that the original ground water level will be maintained along the river bank line. Then on the pumped well side the position of the water table will be given by the sum of these two solutions.

Such a situation leads to an ultimate steady state even though the original solutions do not have one. A series representation for the integral of formula (9) is:

$$\int_{\xi}^{\infty} \frac{e^{-u^2}}{u}\,du = -0.288607 + \log_e\left(\frac{1}{\xi}\right) + \frac{\xi^2}{1!2} - \frac{\xi^4}{2!4} \quad \ldots(67)$$

In our case $\xi = \dfrac{r}{\sqrt{4\alpha t}}$. As t grows large ξ grows small and finally the terms containing powers of ξ grow small compared to $\log_e\left(\frac{1}{\xi}\right)$. Ultimately, because the flow of the recharge well is negative, the algebraic sum of the two solutions reduces to

$$y = \frac{Q}{2\pi KD} \log_e \frac{\sqrt{(2x_1 - x)^2 + z^2}}{\sqrt{x^2 + z^2}} \quad \text{for } x < x_1 \qquad (68)$$

In this expression x and z represent rectangular coordinate distances measured from the pumped well as origin. The coordinate x is measured toward the river and the coordinate z along it. The distance from the pumped well to the river is x_1 . In the region $x < x_1$., The radius from the pumped well to the point x, z is:

$$r = \sqrt{x^2 + z^2} \tag{69}$$

and the radius from the recharge well to the same point is:

$$r = \sqrt{(2x_1 - x)^2 + z^2} . \tag{70}$$

If the water table was originally level the stream would now supply the entire flow of the well.

Example:

A well penetrates 90 feet of saturated sediments of permeability K = 0.0004 ft/sec and is developed to a diameter of 3.0 feet by means of a gravel pack. If the well is 700 feet from a river and is pumped at the rate Q = 0.45 /sec compute the ultimate steady state drawdown along a line from the well to the nearest point of the river.

Solution

$$\frac{Q}{2\pi KD} = \frac{0.45}{(6.2832)(.0004)(90)} = 1.99$$

$$x = 700 \text{ ft} \qquad 2x = 1400 \text{ ft} \qquad z = 0$$

Table 11. Ultimate steady state drawdown.

x_1	$(2x_1 - x)$	$\sqrt{\frac{(2x_1 - x)^2}{x^2}}$	$\log\sqrt{\frac{(2x_1 - x)^2}{x^2}}$	y (feet)
700	700	1.000	0	0
600	800	1.333	.2874	.572
500	900	1.800	.5878	1.17
400	1000	2.500	.9163	1.82
300	1100	3.667	1.2994	2.58
200	1200	6.000	1.7918	3.56
100	1300	13.000	2.5649	5.08
25	1375	27.5	3.3142	6.60
1.5	1398.5	932.	6.8373	13.6

At a point 50 feet downstream of the well where x = 0, z = 50

$$\sqrt{\frac{(2x_1 - x)^2 + z^2}{z^2}} = \sqrt{\frac{1,962,500}{2500}} = \frac{1401}{50} = 28.02, \log 28.02 =$$

3.3329

$$y = (1.99)(3.3329) = 6.63 \text{ feet.}$$

For comparative purposes we will compute the drawdown along the lines $z = 0$ at the end of three months of pumping. Assume $V = 0.20$. Since this is before the permanent steady state has been established it is to be expected that these drawdowns will be somewhat less than those of the ultimate steady state.
Then with

$$\alpha = \frac{KD}{V} = \frac{(.0004)(90)}{0.20} = 0.180$$

$$t = 7{,}884{,}000 \text{ seconds} \qquad x_1 = 700 \text{ ft.}$$

$$r = x \qquad r_1 = (2x_1 - x)$$

Table 12.
Drawdowns computed along a line from the well to the river.

r	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du$	y	r_1	$\frac{r_1}{\sqrt{4\alpha t}}$	$\int_{\frac{r_1}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du$	y_1	$(y+y_1)$
700	0.294	0.99	1.97	700	0.294	0.99	-1.97	0
600	0.252	1.11	2.22	800	0.336	0.86	-1.72	0.45
500	0.210	1.30	2.59	900	0.378	0.73	-1.45	1.14
400	0.168	1.50	2.99	1000	0.420	0.66	-1.31	1.68
300	0.125	1.80	3.59	1100	0.462	0.59	-1.17	2.42
200	0.084	2.20	4.38	1200	0.504	0.51	-1.01	3.37
100	0.042	2.85	5.68	1300	0.546	0.45	-0.89	4.79
50	0.021	3.59	7.14	1350	0.567	0.42	-0.84	6.30
1.5	0.0063	4.88	9.72	1398	0.588	0.40	-0.80	8.92

Note:
The quantity y represents the drawdown due to the real well.
The quantity y_1 represents the drawdown due to the image well.
The quantity $(y + y_1)$ represents the drawdown with the river present.

River depletion due to pumping a well.

It can be shown, by considerations based upon the use of the two images described, that if q represents the depletion of river flow due to pumping a well at the rate Q the part of the flow taken from the river is:

$$\frac{q}{Q} = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{x_1}{\sqrt{4\alpha t}}} e^{-u^2} du \qquad (71)$$

As t grows large this ratio approaches unity. Then, ultimately, the well will deplete the stream by the amount Q.

The river depletion caused by pumping the well of the above example, for three months, would be, if $V = 0.2$

$$\alpha = \frac{KD}{V} = \frac{(0004)(90)}{0.2} = 0.180$$

$$t = 7884000 \text{ seconds} \qquad x_1 = 700 \text{ ft.}$$

$$\frac{x_1}{\sqrt{4 \alpha t}} = \frac{700}{(4)(0.180)(7884000)} = \frac{0.294}{2380} = 0.294$$

From tables of the probability integral

$$\frac{2}{\sqrt{\pi}} \int_0^{0.294} e^{-u^2} du = 0.3224 \qquad 1 - \frac{2}{\sqrt{\pi}} \int_0^{0.294} e^{-u^2} du = 0.6776$$

Then

$$\frac{q}{Q} = 0.6776$$

A little over two thirds of the flow of the well will then be taken from the river at the end of the three months pumping period.

Case 2. A Well Near an Impermeable Boundary.

If the sediments from which a well takes water terminate at an impermeable boundary it will be possible for water to move along the boundary but no flow will cross it. This condition can be reproduced in the idealized infinitely extended aquifer if a line is drawn at the position of the impermeable boundary and an image well is used. The term image stems from the relationships which would exist if a mirror were erected on the line representing the boundary. The image well occupies the position which the pumped well appears to have if viewed in this mirror. The line between the pumped well and the image well crosses the boundary line at right angles and the pumped well and the image well are at equal distances from it. To impose the condition of no flow across the boundary the image well, in this case, must be a pumped well having the same flow as the real well.

Example:

Compute the drawdown for the well of the previous problem if there is an impermeable boundary at a distance of 700 feet. We will compute the drawdown, along a line drawn from the well to the nearest point of the boundary, at the end of three months of pumping. As before, let x and z represent rectangular coordinates drawn from the well, as origin, toward and along the doundary. At the boundary $x = x_1$.

Table 13.
Computation of drawdowns along the line $z = 0$ when there is an impermeable boundary at $x = x_1 = 700$ft. $r = x$.

r	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}$	y	r_1	$\frac{r_1}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}$	y_1	$(y+y_1)$
700	0.294	0.99	1.97	700	0.294	0.99	1.97	3.94
600	0.252	1.11	2.22	800	0.336	0.86	1.72	3.94
500	0.210	1.30	2.59	900	0.378	0.73	1.45	4.04
400	0.168	1.50	2.99	1000	0.420	0.66	1.31	4.30
300	0.125	1.80	3.59	1100	0.462	0.59	1.17	4.76
200	0.084	2.20	4.38	1200	0.504	0.51	1.01	5.39
100	0.042	2.85	5.68	1300	0.546	0.45	0.89	6.57
50	0.021	3.59	7.14	1350	0.567	0.42	0.84	7.98
1.5	0.00630	4.38	9.72	1398.5	0.588	0.40	0.80	10.52

Note:

$r_1 = (2x_1 - x)$, y represents the drawdown due to the real well and y_1 represents the drawdown due to the image well. $(y + y_1)$ represents the drawdown with the impermeable boundary present.

Case 3. A Well between a Stream and impermeable boundary.

It is not uncommon for the alluvial sediments of a river valley to lie in a trench eroded in a material like shale which has a permeability that is very small as compared to that of the alluvial sediments. To treat this case we may assume the one boundary to be impermeable and idealize the river and the outer boundary as parallel straight lines. To compute the draft of the well on the river we may use the well and image of reference 3 . Then if we represent a pumped well by an open circle and a recharge well by a

Figure (15). Well between a stream and an impermeable boundary.

The first pair are those of reference 3. The draft they make on the river can be computed by the use of formula (71), but this pair will cause a flow across the impermeable boundary. To rectify this the first pair is imaged in the impermeable boundary as the second pair. The addition of the second pair has, however, upset slightly the condition that the drawdown is to be zero at the river bank. To rectify this the second pair is imaged in the river bank as the third pair. This, in turn, does a modicum of damage to the conditions along the impermeable boundary. Imaging the fourth pair in the impermeable boundary will yield a fifth pair and a sixth pair will be obtained if the fifth pair is imaged in the river bank. The process results in an infinite series, which, however, may be expected to converge rapidly.

If the draft on the river due to the first pair is considered positive the draft due to the second and third pairs, taken together, will also be positive. The draft due to the fourth and fifth pairs, taken together, will be positive. In all cases the draft can be computed by use of formula (71).

Example:

Suppose we recompute the draft on the river, as obtained in a previous example, but modified by the presence of an impermeable boundary 1400 feet from the well. The distances are as shown in figure 15 .
As before

$$\alpha = \frac{KD}{V} = \frac{(.0004)(90)}{0.2} = 0.180$$

$$t = 7{,}884{,}000 \text{ seconds} \qquad (3 \text{ months})$$

For the first pair $x = 700$ ft.

$$\frac{x_1}{\sqrt{4\alpha t}} = \frac{700}{\sqrt{(4)(0.180)(7884000)}} = 0.294$$

From tables

$$\frac{2}{\sqrt{\pi}}\int_0^{\frac{x_1}{\sqrt{4\alpha t}}} e^{-u^2}du = 0.32243 \qquad 1-\frac{2}{\sqrt{\pi}}\int_0^{0.294} e^{-u^2}du = 0.67757$$

For the inside wells of the second and third pairs

$$x = 3500 \text{ ft} \qquad \frac{x_1}{\sqrt{4\alpha t}} = (5)(0.294) = 1.470$$

For the outside wells of the second and third pair

$$x_1 = 4900 \text{ ft} \qquad \frac{x_1}{\sqrt{4 \alpha t}} = (7)(0.294 = 2.058$$

$$\frac{2}{\sqrt{\pi}} \int_0^{2.058} e^{-u^2} du = 0.99630 \qquad 1 - \frac{2}{\sqrt{\pi}} \int_0^{2.058} e^{-u^2} du = .00361$$

For the inside wells of the fourth and fifth pair

$$x_1 = 5600 \text{ ft} \qquad \frac{x_1}{\sqrt{4 \alpha t}} = (8)(0.294) = 2.352$$

$$\frac{2}{\sqrt{\pi}} \int_0^{2.352} e^{-u^2} du = 0.99912 \qquad 1 - \frac{2}{\sqrt{\pi}} \int_0^{2.352} e^{-u^2} du = .00088$$

For the outside wells of the fourth and fifth pair

$$x_1 = 7000 \qquad \frac{x_1}{\sqrt{4 \alpha t}} = (10)(0.294) = 2.94$$

$$\frac{2}{\sqrt{\pi}} \int_0^{2.94} e^{-u^2} du = 0.99997 \qquad 1 - 0.99997 = .00003$$

Then with due regard to sign the resulting value for q/Q will be:

$$\begin{array}{r} +\ 0.67757 \\ +\ 0.03763 \\ -\ 0.00361 \\ -\ 0.00088 \\ +\ 0.00003 \\ \hline +\ 0.71074 \end{array}$$

Then $q/Q = 0.71074$
And the stream depletion is:

$$q = (0.71074)(0.45) = 0.320 \text{ ft } /\text{sec}$$

A comparison with the previous result will show that the impermeable boundary causes the depletion q to approach the ultimate value Q more rapidly.

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18 Non Steady Flow to a Well of Constant Drawdown in an Extensive Aquifer, by C.E. Jacob and S.W. Lohman in Trans A.G.U. 33- Pages 559-569. - 1952.

19 Bureau of Reclamation Technical Memorandum, in preparation. (A compilation of informal Bureau memoranda on ground-water subjects which have been prepared over a period of years.)

Table I. Probability Integral: $P(x) = \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4\alpha t}} e^{-u^2}\,du$

$x/\sqrt{4\alpha t}$	P (x)
0.0000	0.00000
0.0005	0.00056
0.0010	0.00113
0.0020	0.00226
0.0030	0.00339
0.0040	0.00451
0.0050	0.00564
0.0060	0.00677
0.0070	0.00790
0.0080	0.00903
0.0090	0.01016
0.0100	0.01128
0.0200	0.02256
0.0300	0.03384
0.0400	0.04511
0.0500	0.05637
0.0600	0.06762
0.0700	0.07886
0.0800	0.09008
0.0900	0.10128
0.1000	0.11246
0.1100	0.12362
0.1200	0.13476
0.1300	0.14587
0.1400	0.15695
0.1500	0.16800
0.1600	0.17901
0.1700	0.18999
0.1800	0.20094
0.1900	0.21184
0.2000	0.22270
0.2100	0.23352
0.2200	0.24430
0.2300	0.25502
0.2400	0.26570
0.2500	0.27633
0.2600	0.28690
0.2700	0.29742
0.2800	0.30788
0.2900	0.31828
0.3000	0.32863
0.31	0.33891
0.32	0.34913

$x/\sqrt{4\alpha t}$	P (x)
0.33	0.35928
0.34	0.36936
0.35	0.37938
0.36	0.38933
0.37	0.39921
0.38	0.40901
0.39	0.41874
0.40	0.42839
0.41	0.43797
0.42	0.44747
0.43	0.45689
0.44	0.46623
0.45	0.47548
0.46	0.48466
0.47	0.49375
0.48	0.50275
0.49	0.51167
0.50	0.52050
0.51	0.52924
0.52	0.53790
0.53	0.54646
0.54	0.55494
0.55	0.56332
0.56	0.57162
0.57	0.57982
0.58	0.58972
0.59	0.59594
0.60	0.60386
0.60	0.60386
0.61	0.61168
0.62	0.61941
0.63	0.62705
0.64	0.63459
0.65	0.64203
0.66	0.64938
0.67	0.65663
0.68	0.66378
0.69	0.67084
0.70	0.67780
0.71	0.68467
0.72	0.69143
0.73	0.69810
0.74	0.70468
0.75	0.71116

$x/\sqrt{4\alpha t}$	P (x)
0.76	0.71754
0.77	0.72382
0.78	0.73001
0.79	0.73610
0.80	0.74210
0.81	0.74800
0.82	0.75381
0.83	0.75952
0.84	0.76514
0.85	0.77067
0.86	0.77610
0.87	0.78144
0.88	0.78669
0.89	0.79184
0.90	0.79691
0.91	0.80188
0.92	0.80677
0.93	0.81156
0.94	0.81627
0.95	0.82089
0.96	0.82542
0.97	0.82987
0.98	0.83423
0.99	0.83851
1.00	0.84270
1.1	0.88021
1.2	0.91031
1.3	0.93401
1.4	0.95229
1.5	0.96611
1.6	0.97635
1.7	0.98379
1.8	0.98909
1.9	0.99279
2.0	0.99532
2.1	0.99702
2.2	0.99814
2.3	0.99886
2.4	0.99931
2.5	0.99959
2.6	0.99976
2.7	0.99987
2.8	0.99992
2.9	0.99996
3.0	0.99998
∞	1.00000

Source: National Bureau of Standards, Tables of Probability Functions, Vol.I, MT8, U.S. Government Printing Office, 1941.

Table II. One Minus The Probability Integral: $I_B = (1.0 - P(x))$

$x/\sqrt{4\alpha t}$	I_B	$x/\sqrt{4\alpha t}$	I_B	$x/\sqrt{4\alpha t}$	I_B
0.0000	1.00000	0.34	0.63064	0.78	0.26999
0.0005	0.99944	0.35	0.62062	0.79	0.26390
0.001	0.99887	0.36	0.61067	0.80	0.25790
0.002	0.99774	0.37	0.60079		
0.003	0.99661	0.38	0.59099	0.81	0.25200
0.004	0.99549	0.39	0.58126	0.82	0.24619
0.005	0.99436	0.40	0.57161	0.83	0.24048
0.006	0.99323			0.84	0.23486
0.007	0.99210	0.41	0.56202	0.85	0.22933
0.008	0.99097	0.42	0.55253	0.86	0.22390
0.009	0.98984	0.43	0.54311	0.87	0.21856
0.01	0.98872	0.44	0.53377	0.88	0.21331
		0.45	0.52452	0.89	0.20816
0.02	0.97744	0.46	0.51534	0.90	0.20309
0.03	0.96616	0.47	0.50625		
0.04	0.95489	0.48	0.49725	0.91	0.19812
0.05	0.94363	0.49	0.48833	0.92	0.19323
0.06	0.93238	0.50	0.47950	0.93	0.18844
0.07	0.92114			0.94	0.18373
0.08	0.90992	0.51	0.47076	0.95	0.17911
0.09	0.89872	0.52	0.46210	0.96	0.17458
0.10	0.88754	0.53	0.45354	0.97	0.17013
		0.54	0.44506	0.98	0.16577
0.11	0.87638	0.55	0.43668	0.99	0.16149
0.12	0.86524	0.56	0.42838	1.00	0.15730
0.13	0.85413	0.57	0.42018		
0.14	0.84305	0.58	0.41208	1.1	0.11979
0.15	0.83200	0.59	0.40406	1.2	0.08969
0.16	0.82099	0.60	0.39614	1.3	0.06599
0.17	0.81001			1.4	0.04771
0.18	0.79906	0.61	0.38832	1.5	0.03389
0.19	0.78816	0.62	0.38059	1.6	0.02364
0.20	0.77730	0.63	0.37295	1.7	0.01621
		0.64	0.36541	1.8	0.01091
0.21	0.76643	0.65	0.35797	1.9	0.00721
0.22	0.75570	0.66	0.35062	2.0	0.00467
0.23	0.74498	0.67	0.34337		
0.24	0.73430	0.68	0.33622	2.1	0.00298
0.25	0.72367	0.69	0.32916	2.2	0.00186
0.26	0.71310	0.70	0.32220	2.3	0.00114
0.27	0.70258			2.4	0.00069
0.28	0.69212	0.71	0.31533	2.5	0.00041
0.29	0.68172	0.72	0.30857	2.6	0.00024
0.30	0.67137	0.73	0.30190	2.7	0.00013
		0.74	0.29532	2.8	0.00008
0.31	0.66109	0.75	0.28884	2.9	0.00004
0.32	0.65087	0.76	0.28246	3.0	0.00002
0.33	0.64072	0.77	0.27618		

Computed from National Bureau of Standards, Tables of Probability Functions, Vol. I, MT8, U.S. Government Printing Office, 1941.

Table III. Line Source Integral: $I_x = \sqrt{\pi}\int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u^2}\,du$

$x/\sqrt{4\alpha t}$	I_x	$x/\sqrt{4\alpha t}$	I_x	$x/\sqrt{4\alpha t}$	I_x
0.0000	∞	0.34	2.6628	0.78	0.38848
0.0005	3541.8	0.35	2.5306	0.79	0.37294
0.001	1769.3	0.36	2.4065	0.80	0.35804
0.002	883.07	0.37	2.2901		
0.003	587.68	0.38	2.1805	0.81	0.34373
0.004	439.98	0.39	2.0774	0.82	0.33000
0.005	351.36	0.40	1.9802	0.83	0.31681
0.006	292.28			0.84	0.30415
0.007	250.08	0.41	1.8885	0.85	0.29199
0.008	213.43	0.42	1.8018	0.86	0.28032
0.009	193.81	0.43	1.7199	0.87	0.26911
0.01	174.12	0.44	1.6424	0.88	0.25834
		0.45	1.5689	0.89	0.24800
0.02	85.516	0.46	1.4993	0.90	0.23807
0.03	55.993	0.47	1.4333		
0.04	41.241	0.48	1.3706	0.91	0.22853
0.05	32.396	0.49	1.3110	0.92	0.21936
0.06	26.506	0.50	1.2544	0.93	0.21056
0.07	22.303			0.94	0.20210
0.08	19.156	0.51	1.2005	0.95	0.19397
0.09	16.712	0.52	1.1493	0.96	0.18616
0.10	14.760	0.53	1.1004	0.97	0.17866
		0.54	1.0539	0.98	0.17146
0.11	13.166	0.55	1.0096	0.99	0.16453
0.12	11.841	0.56	0.96728	1.00	0.15788
0.13	10.722	0.57	0.92692		
0.14	9.7661	0.58	0.88840	1.1	0.10414
0.15	8.9397	0.59	0.85162	1.2	0.06820
0.16	8.2186	0.60	0.81647	1.3	0.04426
0.17	7.5845			1.4	0.02843
0.18	7.0227	0.61	0.78289	1.5	0.01806
0.19	6.5219	0.62	0.75078	1.6	0.01133
0.20	6.0728	0.63	0.72008	1.7	0.00702
		0.64	0.69070	1.8	0.00429
0.21	5.6682	0.65	0.66260	1.9	0.00259
0.22	5.3018	0.66	0.63570	2.0	0.00154
0.23	4.9688	0.67	0.60994		
0.24	4.6650	0.68	0.58527	2.1	0.00090
0.25	4.3868	0.69	0.56164	2.2	0.00052
0.26	4.1313	0.70	0.53900	2.3	0.00029
0.27	3.8959			2.4	0.00016
0.28	3.6785	0.71	0.51730	2.5	0.00009
0.29	3.4772	0.72	0.49651	2.6	0.00005
0.30	3.2905	0.73	0.47657	2.7	0.00003
		0.74	0.45745	2.8	0.00001
0.31	3.1168	0.75	0.43912	2.9	0.00001
0.32	2.9550	0.76	0.42153	3.0	0.00000
0.33	2.8040	0.77	0.40466		

Computed from National Bureau of Standards, Tables of Probability Functions, Vol.I, MT8, U.S. Government Printing Office, 1941.