DISSERTATION

The Mean Meridional Circulation:
A New Potential-Vorticity, Potential-Temperature Perspective

Submitted by
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In partial fulfillment of the requirements
for the Degree of Doctor of Philosophy
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ABSTRACT OF DISSERTATION

THE MEAN MERIDIONAL CIRCULATION: A NEW POTENTIAL-VORTICITY, POTENTIAL-TEMPERATURE PERSPECTIVE

The dynamical equations governing atmospheric flow are transformed into a system of coordinates that consists of potential vorticity as meridional coordinate and potential temperature as vertical coordinate. Within this system of coordinates the atmosphere is divided into undulating tubes bounded by isentropic and constant potential vorticity surfaces, and the air moves through the tubes without penetrating through the walls, in adiabatic and frictionless conditions. A model that uses this system of coordinates has the advantage of incorporating as built-in the dry convective adjustment and baroclinic adjustment processes, which prevent the folding of isentropic and potential vorticity surfaces but simulate the effects of interaction of waves with the flow.

When applied to the study of the mean meridional circulation this system of coordinates reveals a residual circulation driven by the Lagrangian time rate of change of potential vorticity and diabatic heating, and allows to interpret the momentum exchange that affects the zonal mean flow as the zonal component of the pressure forces exerted by the eddies on a thin zonal tube bounded by surfaces of constant potential vorticity as lateral sides and undulating bottom and top isentropes.

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The biggest thanks go to Gheorghe who kept our real life going and wonderful.

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To Sonia,

I look forward to the day when I can tell you

what this dissertation shows.

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Chapter 1: Introduction

The intricate nature of the Earth's atmosphere makes the study of the atmospheric circulation an exciting topic that has been challenging many generations of atmospheric scientists. Its roots can be tracked back into the pioneering work of Hadley (1753), Maury (1855) and Ferrel (1856).

During the long history of this subject numerous diagnostic and modeling studies of the mean meridional circulation have evolved, but relatively few works sought an answer to the basic question: what fundamental processes determine its structure and strength? The main objective of the present work is to study the general circulation of the atmosphere in a new system of coordinates that consists of potential vorticity as merdional coordinate and potential temperature as vertical coordinate, and to give a new interpretation of the eddy momentum transport, one of the processes that determines the structure of the mean meridional circulation in the midlatitudes, in terms of the form drag.

The conceptual model of the mean meridional circulation is a direct product of the analysis of observational data, and it was evolving simultaneously with the development of mathematical models based on the natural physical laws.

In a very simplified theoretical framework the mean meridional circulation is divided into three segments, each segment being characterized by different mechanisms.

The Hadley cell resides in the tropical latitudes, the Ferrel cell is located in the middle latitudes and the Polar cell covers the high latitudes. The mechanisms that determine and maintain the Hadley and Polar cells were clarified by the golden generation of atmospheric scientists. This generation includes Rossby, Charney and Lorenz, nowadays we refer to as the parents of the modern theories describing the general circulation of the atmosphere. In the case of the Ferrel cell, things are more complicated, and as a consequence there is a lot of research ongoing on this topic (e.g., Barry et al., 2002; Koh and Plumb, 2004; Schneider, 2004). Although it was established a long time ago that the main factors determining the observed structure of the Ferrel cell are the eddies, the mechanisms proposed to explain the observed structure of this cell assume a lot of approximations. The major difficulties have two main causes: a) the atmosphere is able to move along on its own without any external forcing and b) the interactions of the eddies with the mean flow are nonlinear processes so in order to understand their causes it is necessary to employ numerical models that are either able to represent the eddies or simulate their effects.

In order to accomplish the objective stated above we have to answer two questions:

1.) What are the mechanisms through which the eddies interact with the mean flow?

2.) What is a simple way to represent the eddy effects on the mean flow in a numerical model?

It is obvious that throughout time there has been a tremendous amount of work devoted to address these questions, as the next two chapters attest, but each time the proposed theories were not fully satisfactory and left enough room for improvements.

In particular, this thesis aims to propose a new theory of the forcing of the zonal mean flow that arises naturally if the mean meridional circulation is studied in a system of coordinates (PVPT) which consists of potential vorticity (PV) as meridional coordinate and potential temperature (PT) as vertical coordinate.

The general definition of potential vorticity as introduced by Ertel (1942) is

$$q = \frac{1}{\rho}\boldsymbol{\zeta}_a \bullet \nabla\theta \quad , \tag{1.1}$$

where ρ is the fluid density, $\boldsymbol{\zeta}_a = 2\boldsymbol{\Omega} + \nabla\times\mathbf{V}$ is the absolute vorticity vector, $\mathbf{V} = (u, v, w)$ is the three-dimensional wind vector.When projected on the Cartesian system of coordinates (1.1) becomes

$$q = \frac{1}{\rho}\left\{\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\frac{\partial \theta}{\partial x} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\frac{\partial \theta}{\partial y} + \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial \theta}{\partial z}\right\}, \tag{1.2}$$

where $f = 2\Omega\sin\varphi$ is the Coriolis parameter and (1.2) neglects the $2\Omega\cos\varphi$ part of the horizontal vorticity. The vertical coordinate used in (1.2) is the geometrical height. θ is the potential temperature given by

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}. \tag{1.3}$$

Here T is the temperature, p pressure, p_0 a reference pressure (usually 1000 mb), R the gas constant for air and c_p the specific heat at constant pressure. Under the hydrostatic approximation[1], which characterizes most of the midlatitude large-scale processes, the definition of potential vorticity in (1.2) simplifies to

$$q = \frac{1}{\rho}\left\{-\frac{\partial v}{\partial z}\left(\frac{\partial\theta}{\partial x}\right)_z + \frac{\partial u}{\partial z}\left(\frac{\partial\theta}{\partial y}\right)_z + \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial\theta}{\partial z}\right\}. \tag{1.4}$$

When expressed in pressure coordinates (1.4) becomes

$$q = g\left\{\frac{\partial v}{\partial p}\left(\frac{\partial\theta}{\partial x}\right)_p - \frac{\partial u}{\partial p}\left(\frac{\partial\theta}{\partial y}\right)_p + (f + \zeta_p)\left(-\frac{\partial\theta}{\partial p}\right)\right\}, \tag{1.5}$$

where ζ_p is the vertical component of the relative vorticity normal on the isobaric surfaces. When expressed in isentropic coordinates (1.4) reduces to

$$q = g(f + \zeta_\theta)/\left(-\frac{\partial p}{\partial\theta}\right), \tag{1.6}$$

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[1] The vertical scale of processes is much smaller compared to the horizontal scale and the vertical component of the velocity vector is negligible when compared to the horizontal components.

where ζ_θ is the vertical component of the relative vortcity normal on the isentropic surfaces. As described by Eliassen (1987), the last term in the brackets in (1.4) or (1.5) is the barotropic contribution. The first two terms are due to the vertical shear of the wind, which characterizes most of the baroclinic flows. In isentropic coordinates, the definition of potential vorticity, (1.6), incorporates the barotropic and baroclinic contributions into one term. Because the vorticity vector of the gravity waves is tangential to the isentropic surfaces they do not possess potential vorticity, another nice property of the potential vorticity when calculated on an f-plane. This characteristic is useful to filter out the gravity waves without invoking the quasi-geostrophic approximation.

The role of the eddies in the forcing of the flow is much better defined in the PVPT framework without the presence in isobaric coordinates of mean temperature advection by the geostrophic circulation. We will show that the zonally averaged mass-weighted angular momentum changes only in response to the net form drag that arises from the undulation of potential vorticity and isentropic surfaces.

A synopsis of the thesis is as follows: Chapters 2 and 3 are reviews of literature covering observational and theoretical aspects of the mean meridional circulation, in order to place the contributions of this thesis in perspective. Chapter 2 takes a focused look at the mean meridional circulation as it is understood today from observational and theoretical studies. A selection of observational features is presented, to provide a realistic point of view from which we can appreciate subsequent modeling results. The observations were also chosen to reflect the relevance of this thesis to a wide range of important aspects of climate dynamics. We have also emphasized the accomplishments achieved in prior research by using different systems of coordinates. Chapter 3 discusses the history of the parameterization of eddy fluxes. This includes a description of various parameterizations used for the eddy heat fluxes and for the eddy momentum fluxes, and a summary of their strengths and weaknesses as applied to the zonally averaged models.

Chapters 4 and 5 are the most important chapters in this thesis, as a completely new framework is described. First Chapter 4 introduces the use of potential vorticity as a meridional coordinate, and presents in detail the derivation of the primitive equations on the sphere in PVPT coordinates. We also discuss the strengths and weaknesses of the potential vorticity and potential temperature as coordinates. Chapter 5 shows applications of the theory presented in Chapter 4 to the zonally averaged models. The characteristics of the residual circulation in PVPT coordinates are discussed, showing similarities with the residual circulations in other systems of coordinates. We propose a new interpretation of the Eliassen-Palm flux as representing the net pressure force pushing against the walls of a tube bounded by the potential vorticity and isentropic surfaces. We settle the framework of a new parameterization necessary to close the system of equations describing the mean meridional circulation in PVPT system of coordinates.

Finally, the thesis is wrapped up in Chapter 6. The conclusions reiterate the principal theoretical results. Then, some further remarks are made in the wider context of climate dynamics.

Chapter 2: Theories of the Mean Meridional Circulation: Past, Present and Future

First was the weather. Once people began to travel they understood that the weather has a local character, and its pattern is a small piece of a more general ensemble called the general circulation of the atmosphere. The complexity implied by the tridimensional structure of the general circulation was reduced when the zonal characteristics were averaged, and a new picture, called the mean meridional circulation, has emerged. Using observations atmospheric scientists have proposed theories aimed to explain and predict the encountered features for more than two centuries now. Despite this recognition, we still have to improve our understanding on the manner and extent to which the eddies affect the mean meridional circulation in the midlatitudes. Therefore, it quickly becomes apparent that one has to modify and extend the currently existing theories.

2.1: Brief background review

In the monograph, "*The nature and theory of the general circulation of the atmosphere*", Lorenz (1967) provides a complete description of conceptual models proposed to explain the mean meridional circulation of the atmosphere. Because this book represents the most comprehensive historical review on this topic, a brief summary of it is given here.

In early the 1700's, Hadley described the mean meridional circulation of the atmosphere as a single convection cell in each hemisphere. Hadley understood that the spherical shape of the Earth will make the surface to be exposed to different insolation, which will yield to warmer tropics and colder high latitudes. This would create a global circulation similar to the sea breeze, with the warm air rising and moving northward aloft. By the time the air reaches the Poles, it is very cold and subsides. It then returns to the Equator, completing the circuit. As the air moves across the surface, it is deflected by the rotation of the Earth, so it arrives at the Equator as northeasterly/southeasterly winds. In Hadley's day, the conservation principles for angular momentum and mass were very vague, and he was not concerned if the hypothesized mechanism obeys these principles. Of course, the mean meridional circulation envisioned by Hadley was based on the limited data and knowledge available at that time.

An obstacle in the way of progress was the ignorance of the role of midlatitude cyclones and anticylones in the global picture of the general circulation of the atmosphere. If the solar radiation was the only source responsible for the observed circulation of the atmosphere then we should observe higher temperatures at the Equator and much lower ones at the poles, with a much larger equator-to-pole temperature gradient than we observe. This state does not occur in the atmosphere because of the action of eddies in midlatitudes, which derive their energy from the available potential energy of the mean flow and act to reduce the equator-to-pole temperature gradient.

As the nineteenth century progressed and new observations became available, the westerlies occurring in the midlatitudes did not fit into the circulation envisioned by Hadley. Dove (1837) realized that Hadley's scheme is suitable only for low latitudes, and found it plausible to hypothesize that midlatitude storms are driven by the interaction between the polar and tropical air masses. As a result, in Dove's picture, in midlatitudes there were longitudes dominated by south-easterly winds and longitudes dominated by north-easterly winds, at the same level.

In 1855, Maury introduced a conceptual model of the mean meridional circulation in which the convective cell proposed by Hadley does not extend all the way from the Equator to the poles. Air rises at the Equator and subsides in the subtropics at about 30° N. Subsiding air produces the subtropical highs. He realized that if there were a single convection cell there would be a continuous subtropical region of high pressure associated with the subsiding side of the cell. In fact there are several highs separated by regions of low pressure. Maury proposed the existence of a midlatitude cell in which the warm air sinks in midlatitudes and the cold air rises in high latitudes. Maury understood that the water balance must be included in the big picture of the mean meridional circulation, even though his arguments were not sustained by observations. He concurred with Hadley's explanation on the cause of the trade winds, and was unable to provide a mechanism to explain the westerlies and the hypothesized indirect cell in the midlatitudes.

In the following year (1856), Ferrel completed the description of the general circulation of the atmosphere in the vertical plane. His model is known as the three-cell model. Ferrel proposed the existence of a midlatitude cell in which the vertical movement of the air is driven by the Hadley cell on the equatorial side and by the polar cell on the polar side. He understood that the large-scale atmospheric circulation pattern results from the thermal and pressure gradients that occur on the same scales. Ferrel also pointed out that the real reason for the deflection of the air moving in the cells is the conservation of angular momentum in the moving air and not the conservation of angular velocity as wrongly assumed by Hadley. He realized that the zonal flow is close to being in hydrostatic and geostrophic balance, and gave an explanation of the role of surface friction in determining the direction and strength of the surface winds. The work of Ferrel brings us very close to the modern view of the mean meridional circulation.

In the early years of 20th century, Defant (1921) and Jeffreys (1926) filled in many of the details of the very general outline available at that time. Defant introduced the idea that motion in middle latitudes was simply turbulence of a very large scale, which we now call eddies. He concluded that the heat transport in the midlatitudes is mainly by the cyclones and anticyclones occurring in this region. By considering the momentum budget Jeffreys showed that the large-scale eddies must maintain the surface winds.

In 1930, Rossby explained that since the Earth's surface is not flat, the air encounters barriers and is forced to ascend (by changing surface level), then descends (under the gravitational influence), and the resultant "squashing" and "stretching" respectively of the air columns induce modifications in the local rotation of the air flow, also known as the vorticity. These variations in the vorticity must be balanced on a rotating earth for the system to remain stable. When considering the Northern Hemisphere, air that is forced to ascend turns to the left, and as it descends, it tends to turn to the right, inducing a ridge/trough pattern to the midlatitude westerlies. These long waves are known as stationary planetary waves or Rossby waves.

Lorenz himself contributed to the understanding of the atmospheric circulation, especially regarding the partitioning of the energy between different states of the atmosphere. In his book, Lorenz (1967) concludes that, by the late 60's, scientists had identified the main forcing of the general circulation of the atmosphere: unequal heating of the Earth's surface. Also, we understood that the observed features of the mean meridional circulation are the result of the interaction between this forcing and the motion that the atmosphere undergoes, but we still had yet to explain the mechanisms through which the eddies transport momentum and heat in midlatitudes.

2.2: The modern view of the mean meridional circulation

In the current view, the mean meridional circulation consists of three main cells: the Hadley cell in the tropics, the Ferrel cell in midlatidudes and the polar cell at high latitudes, as schematically depicted in Fig. 2.1.

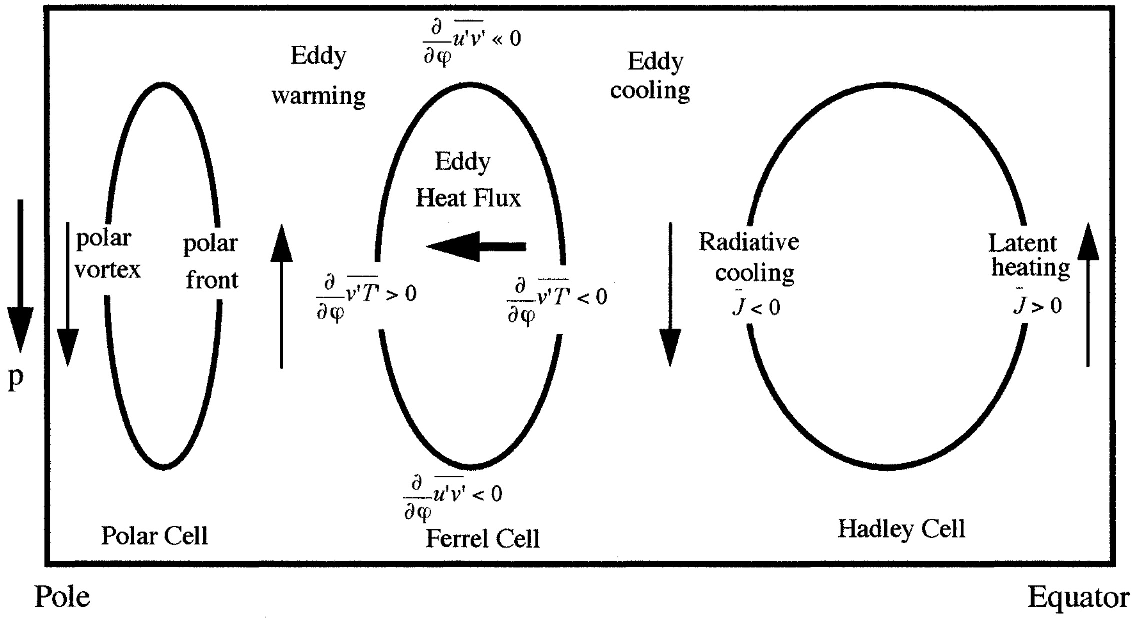

Figure 2.1: Idealized depiction of the mean meridional circulation. The Hadley cell is driven by the diabatic processes; the Ferrel cell is driven by the eddy fluxes of momentum and heat; the polar cell is driven by the polar vortex and the polar front. Note that the convergence of momentum is much stronger in the upper branch of the Ferrel cell than in the lower branch.

The Hadley cell is a closed circulation loop, which begins at the Equator with warm, moist air lifted in equatorial low pressure areas to the tropopause and carried poleward. At about 30° N latitude, it descends in a cooler high pressure area. Some of the air travels equatorward along the surface, diverted westward by the Coriolis acceleration, closing the loop of the Hadley cell and creating the tradewinds. Since in the Hadley cell warm air is rising and cool air is sinking, it is called a thermally direct circulation. The Ferrel cell, located at roughly 30° to 60° north of the Equator, is referred to as a thermally indirect circulation since warm air is sinking and cool is rising. The polar cell, located roughly 60° to 90° north of the Equator is a simple system. Air masses circulate meridionally within the troposphere, limited vertically by the tropopause, with poleward motion aloft and equatorial motion closer to the surface. When the air reaches the polar areas it descends, twisting westward as it flows equatorward at the lower levels, as a result of the Coriolis effect, to produce the polar easterlies. The polar cell and the Hadley cell are similar in that they are thermally direct; in other words, the warmer air is rising and the colder air is sinking.

Note also in Fig. 2.1 the significant differences in the mechanisms associated with these three cells: diabatic processes, eddy transports, polar vortex and polar front.

1.) Diabatic processes dominate in the tropics: the radiative heating in the tropics is partially balanced by the evaporation but the air remains warmer than the air to the north and south. The warm moist air is conditionally unstable and as soon as a small lifting occurs the air saturates and becomes unstable. Large amounts of latent heat are released in the cumulus clouds, providing further driving for the upward motion. At the top of the tropical troposphere (~10-12 km) the atmosphere is stable and the upward motion has to stop. This forces the air to diverge poleward on both sides of the Equator.

2.) Baroclinic eddies dominate in the midlatitudes, where particle accelerations are much less important (as pointed out by Charney, 1963, in his simple explanation for the differences in the dynamical balance between the tropics and midlatitudes). They are trains of alternating high and low pressure systems moving in circular motions in the westerly flow. Baroclinic eddies push the warm air poleward and the cool air southward, warming the high latitudes and cooling the subtropical latitudes. We can see the strength of the eddy fluxes in the midlatitudes by examining the latitude-potential temperature cross-section of zonally averaged pressure, which is shown in Fig. 2.2. In this figure we see that the isobars slope most strongly towards the surface in the midlatitudes, indicating strong pressure gradients and temperature gradients respectively, which are usually associated with baroclinic instability conditions that further generate eddies. One of the most important properties of the eddies is to transport heat and momentum from low to high latitudes. Observational studies show in the extratropical Northern Hemisphere (Fig. 2.3) a maximum heat flux in the lower troposphere at about 50° N due to the poleward transport of heat by the transient synoptic-scale eddies and stationary planetary waves. Thus the eddy heat flux acts to drive and maintain a mean meridional cell centered in the lower troposphere at midlatitudes with an indirect meridional circulation.

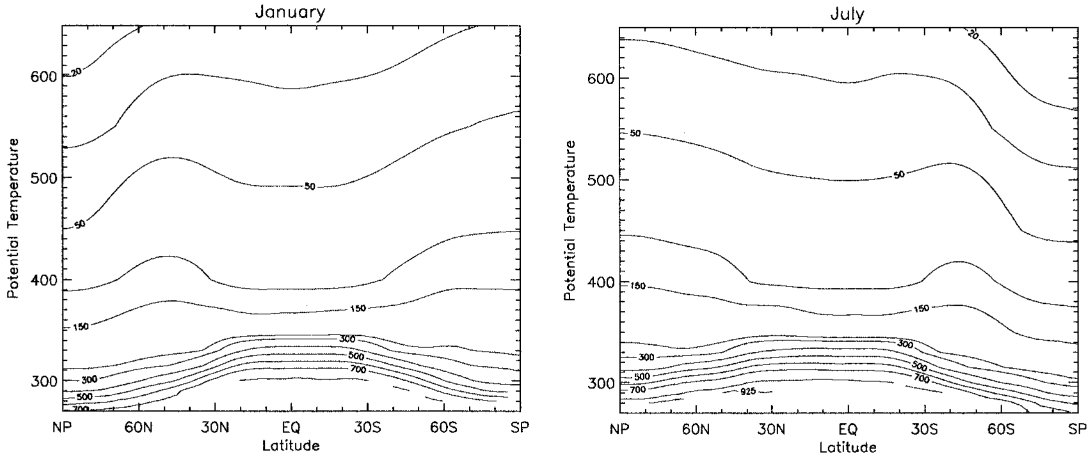

Figure 2.2: Latitude-potential temperature cross sections of the zonally averaged pressure (mb) in the NCEP-NCAR reanalysis.

As explained by Holton (1992), the mechanisms responsible for the existence of this indirect meridional circulation can be understood in terms of geostrophic[2] and hydrostatic balance, specific to the midlatitude region. The convergence of eddy heat flux north of the 50° latitude and the divergence of the eddy heat flux on the equatorward side of this latitude induce an eddy heat transport that acts to reduce the pole-to-equator mean temperature gradient. The zonal mean flow will remain in geostrophic balance if the change with height of zonal wind induced by the variation of the horizontal temperature gradient, also known as the thermal wind, decreases. Under the geostrophic approximation friction is neglected, and the decrease in the thermal wind can be accomplished by the eddy momentum transport and the "Coriolis torque" resulting from the mean meridional wind acting on the Coriolis force.

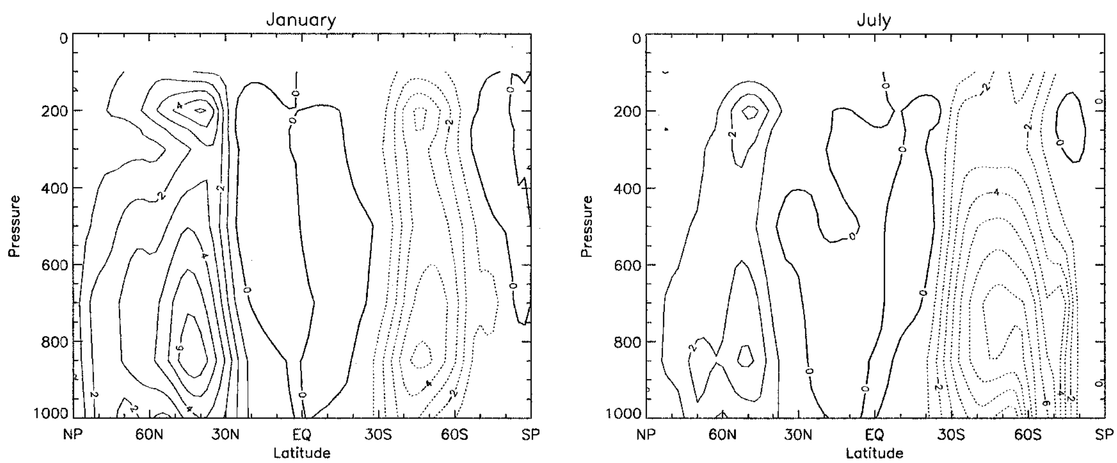

Figure 2.3: Pressure-latitude cross sections of the eddy heat flux in the NCEP-NCAR reanalysis for January and July. The contour interval is 1Km/s.

In the Northern Hemisphere the eddy momentum flux (Fig. 2.4) has a maximum near the tropopause at about 30°N (at the core of the mean jet stream). The effect of convergence of the eddy momentum flux is an acceleration of the mean flow in this region

[2] Horizontal motion is in balance with the pressure gradient force.

that will build up strong westerlies and will destroy the thermal wind balance. Therefore the "Coriolis torque" has to counteract the effects of the eddy momentum flux and restore the thermal wind balance. As a consequence the balance between the eddy momentum flux and "Coriolis torque" drives an indirect meridional cell centered in midlatitudes.

The eddy heat flux plays a major role in transporting heat from low to high latitudes and maintaining the energy balance of the atmosphere, and the eddy momentum flux is of importance in determining the surface winds and the Ferrel cell. From the point of view of the zonally averaged temperature, the midlatitude convergence of the eddy transports of sensible heat acts as a source, additional to the heating by radiation and small-scale turbulent transport. The convergence of the eddy transport of momentum acts as a mechanical force, in addition to surface friction and small-scale turbulent viscosity. Without these transports, the three-cell structure of the meridional circulation cannot be explained.

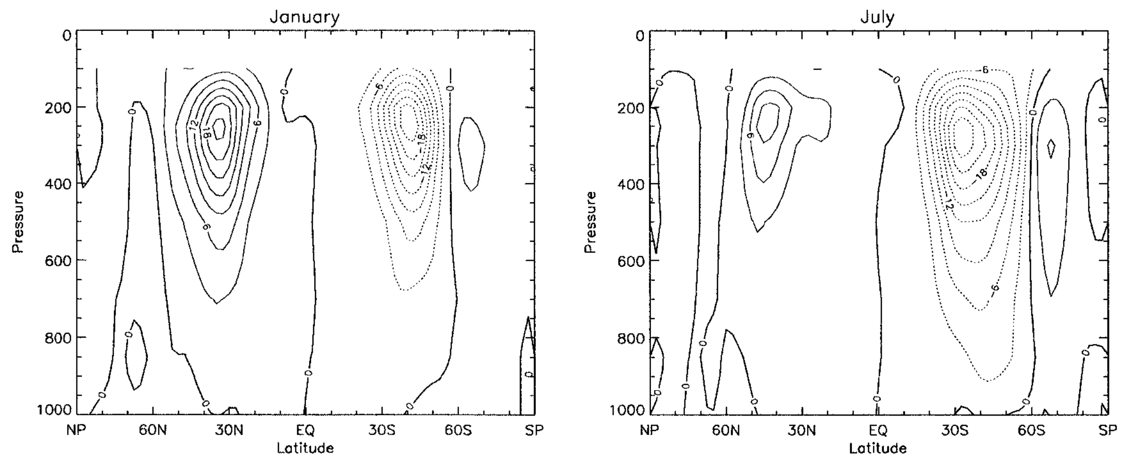

Figure 2.4: Pressure-latitude cross sections of the eddy momentum flux in NCEP-NCAR reanalysis for January and July. The contour interval is 3m^2 s^{-2}.

The eddy heat flux and the eddy momentum flux do not act independently to affect the zonal mean angular momentum, a result first recognized by Charney and Drazin (1961) and Dickinson (1969). They combined the effects of these eddies into a vectorial quantity known as the Eliassen-Palm (EP) flux. The EP flux is a vector with components in the latitude-height plane, the direction and magnitude of which determine the centers of action of the eddy heat flux and eddy momentum flux; regions where the EP flux vector points in the meridional direction, the meridional flux of the eddy momentum flux dominates, and regions where the EP flux vector points on the vertical are dominated by the meridional eddy flux of heat. The total eddy-induced forcing is the divergence of the EP flux, which under the quasi-geostrophic approximation or in the isentropic framework is equivalent to the eddy potential vorticity flux. Fig. 2.5 shows the cross-section of eddy potential vorticity flux for January and July.

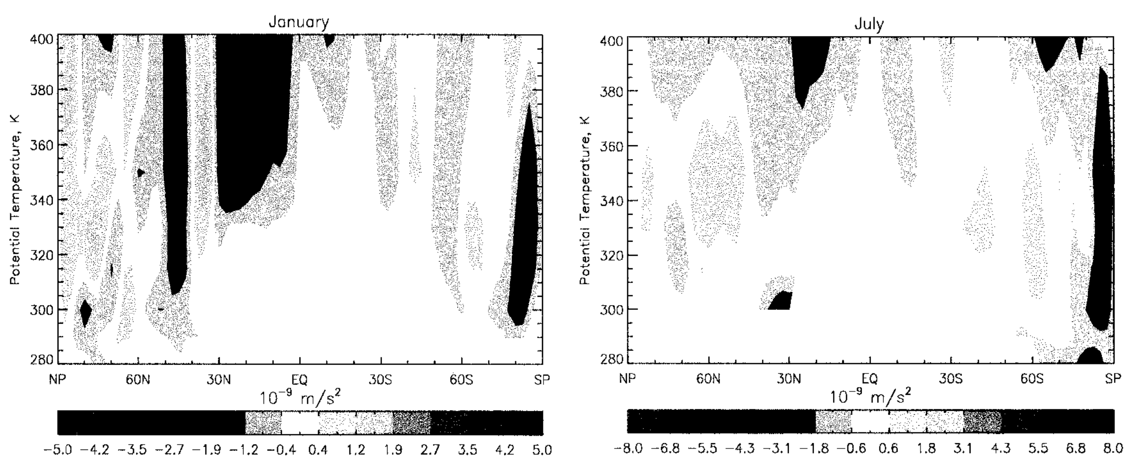

Figure 2.5: Potential temperature -latitude cross sections of the northward potential vorticity flux in NCEP-NCAR reanalysis.

We note during the Northern Hemisphere winter a convergence of the EP flux in the extratropical regions consistent with an eastward acceleration of the zonal mean flow. During the summer the eddy activity in the Northern Hemisphere is weaker than in the winter, as can be observed by comparing the two panels.

3.) Polar latitudes are dominated by the polar vortex, a low pressure system located in the middle and upper troposphere and in the stratosphere. The cold and dry air descends to the Earth's surface and moves equatorward. At about $60°$, this air encounters the warm and relatively moist air and forms the polar front.

The complexity of the general circulation of the atmosphere did not restrain researchers from imagining prototypes of the atmospheric system. An example is the classic "dishpan" experiment described in many textbooks (e.g., Holton, 1992; Chapter 10). In this experiment the atmosphere is represented by a liquid contained in a rotating vessel. A heater around the edge mimics the source of heat similar to the equatorial heating, whereas a cooling system in the middle of the vessel simulates the heat sink associated with the polar cooling. As the vessel spins counterclockwise, reproduces general feature of the atmospheric circulation. The temperature gradient induces a direct circulation that transports heat from the edge to the center of the vessel, and the flow develops troughs and ridges that nest eddies and meander around the vessel.

2.3: Characteristics of the mean meridional circulation

The atmospheric features described above strengthen and weaken regionally or seasonally, as becomes evident in Fig. 2.6. We can observe that it is only in spring and autumn when there are two sets of cells arranged symmetrically on each side of the Equator. This is partially because the equatorial trough[3], which is the center of the convergence and rising air, moves north and south with the seasons. In northern winter, Hadley circulation is dominated by a single cell with ascending motion south of the Equator and descending motion in the northern subtropics. This pattern reverses during southern winter. Another cause is the polar vortex that is strongest in winter and weakens during the summer, and also changes its location.

Another interesting characteristic of the mean meridional circulation was revealed by a series of numerical experiments performed by Williams (1988), who investigated how the mean meridional circulation would look if the Earth's rotation rate changed. He found that if the rotation rate increased the Hadley cell would move towards the Equator, and additional cells would appear in middle and high latitudes. If the rotation rate decreased the Hadley cell would extend and cover the entire hemisphere in the limit of no rotation. The streamfunctions of the mean meridional circulations obtained for different values of the rotation rates are presented in Fig. 2.7.

Although most of the time the mean meridional circulation is viewed as a secondary process forced by the large-scale eddies, diabatic heating and frictional processes (Kuo, 1956), there is evidence that the mean meridional circulation does influence the eddies by affecting the location of the eddy momentum and heat flux.

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[3]. the equatorial belt of low pressure

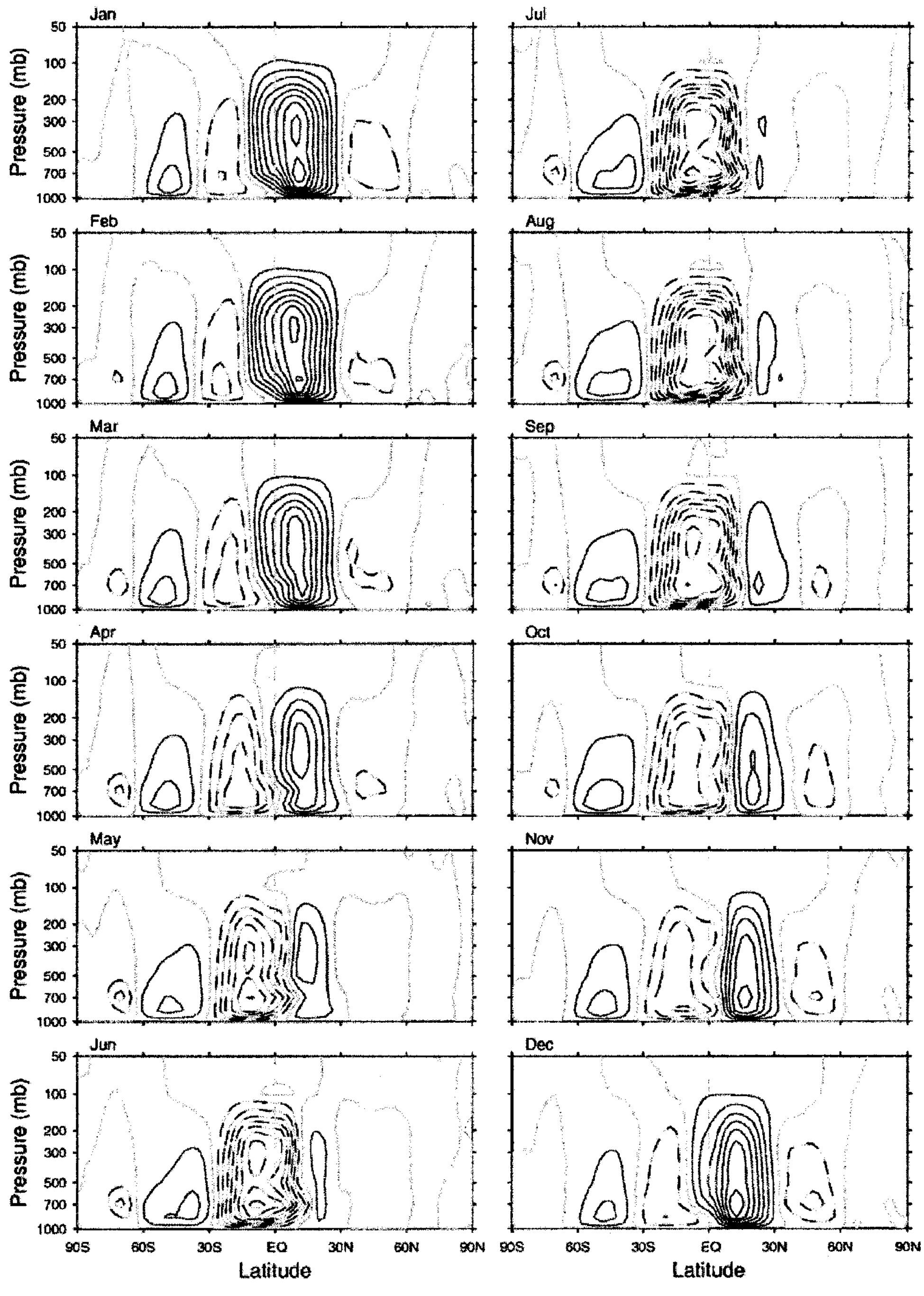

Figure 2.6: Cross-sections of monthly streamfunction in the NCAR-NCEP reanalysis. Contour interval is $2\text{x}10^{10}$ kg s^{-1}Solid contours are positive, dashed contours are negative and the zero contour is gray. From Dima and Wallace, 2003.

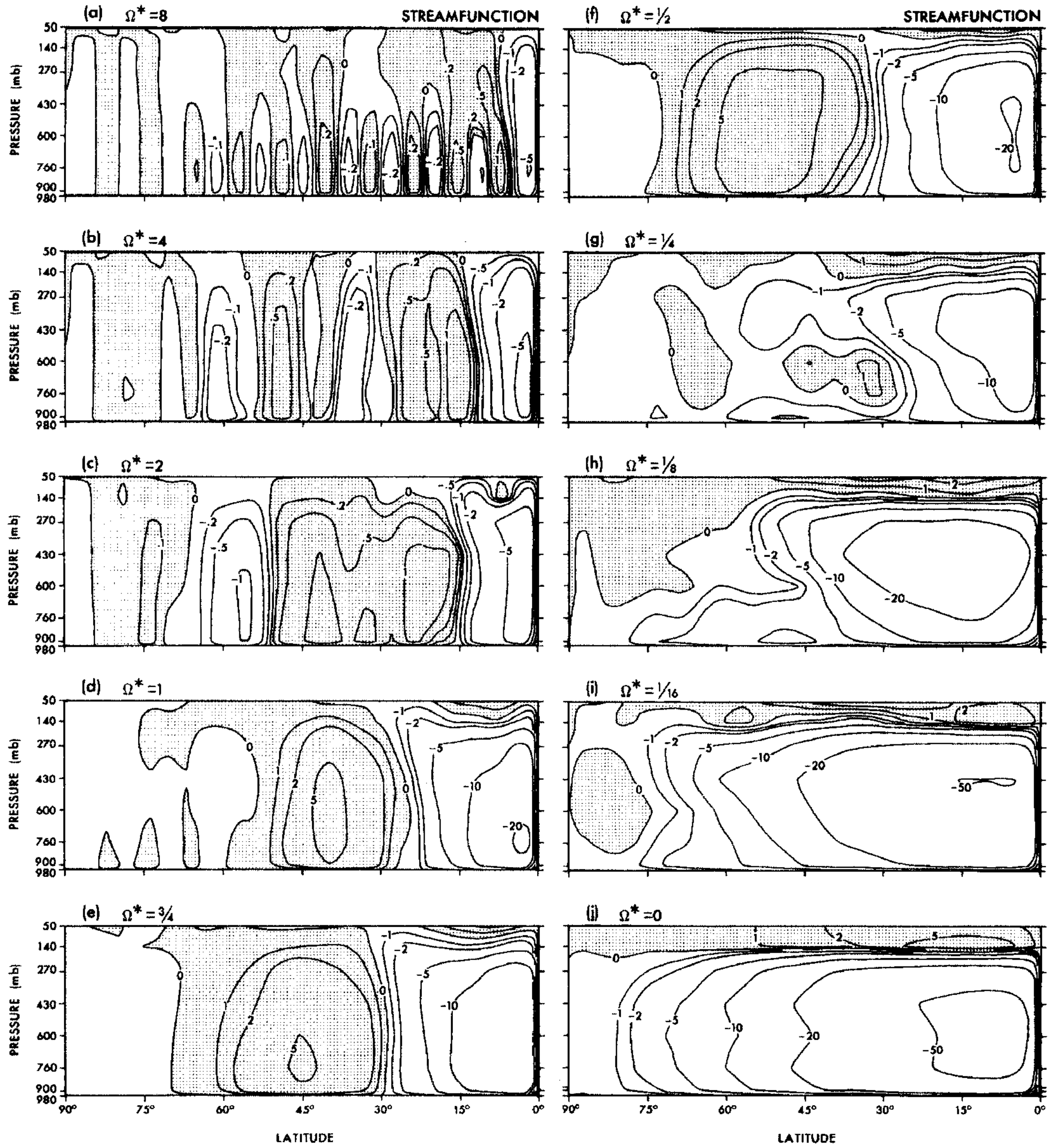

Figure 2.7: The stream function of the mean meridional circulation in 10^{10}kg $s^{-1}$for different values of the rotation rates. Ω* denotes the ratio between the specified rotation rate and the Earth's rotation rate. From Williams, 1998.

One example is the zonal flow vacillation which occurs as variations in the strength of the zonal circulation over several latitudinal bands, and more evident in the middle latitudes of the Southern Hemisphere.

Wallace and Hsu (1985) defined the vacillation as "a statistical residue of dynamically unrelated events at different longitudes".

Other examples of the variability in the atmospheric circulations are the annular modes such as the Northern Annular Mode (NAM), which has been also called by Thompson and Wallace (1998) the Arctic Oscillation, and the Southern Annular Mode (SAM). The annular modes represent northward and southward shifting of the midlatitude jet from its time-mean location. The eddy structures change to reflect the vacillation in the westerlies and the corresponding eddy momentum forcings support the shifts in jet position.

2.4: Methods of investigation of the mean meridional circulation

Most of the patterns of the atmospheric circulation presented above were inferred from observations. Because in the observations it is not always possible to separate the phenomena that contributed to the measured quantity, scientists invent different methods to test the contributions of these phenomena. The next sections review the most popular techniques: the Eulerian mean, the transformed Eulerian mean, and the isentropic zonally average.

2.4.1: The Eulerian mean

The Eulerian mean (denoted by []) represents an averaging over a latitudinal circle, and for any arbitrary quantity A is defined by (e.g., Andrews et al., 1987)

$$[A] = \frac{1}{2\pi}\int_0^{2\pi} A(\lambda, \varphi, z, t)d\lambda. \quad (2.1)$$

The complete system of equations describing the quasi-geostrophic motion in the log-pressure system of coordinates consists of the zonal momentum equation, thermodynamic equation, continuity equation, and the thermal wind equation (e.g., Holton, 1992):

$$\frac{\partial}{\partial t}[u] - f_0[v] = -\frac{\partial}{\partial y}[u^*v^*] + [X] \quad (2.2)$$

$$\frac{\partial}{\partial t}[T] + \underbrace{\frac{N^2H}{R}[w]}_{I} = -\underbrace{\frac{\partial}{\partial y}[v^*T^*]}_{II} + \frac{[J]}{c_p}, \quad (2.3)$$

$$\frac{\partial}{\partial y}[v] + \frac{1}{\rho_0}\frac{\partial}{\partial z}\rho_0[w] = 0 \quad (2.4)$$

$$f_0\frac{\partial}{\partial z}[u] + \frac{R}{H}\frac{\partial}{\partial y}[T] = 0. \quad (2.5)$$

Equations (2.2)- (2.5) use the log-pressure coordinate system $z = -H\log(p/p_s)$, with H and p_s being the constant reference scale height and surface pressure respectively. u, v and w are the velocity components, T represents the temperature, f_0 denotes the Coriolis parameter on the midlatitude beta-plane, N is the buoyancy frequency, R is the gas constant for dry air, X and J designate the zonal component of turbulent drag force and the diabatic heating rate, respectively. c_p is the specific heat of dry air at constant pressure. The mass density in log-pressure coordinate is $\rho_0 = \rho_s e^{-z/H}$, where $\rho_s = p_s/RT_s$. The star quantities represent deviations from the zonal/Eulerian mean.

For time-average conditions, the horizontal branches of the Eulerian mean circulation in log-pressure coordinates are maintained by the imbalance between the friction and the eddy transport of angular momentum, whereas the vertical branches are maintained by the imbalances between the diabatic heating and the convergence of eddy heat flux (Eliassen, 1951).

Holton (1992) has shown that using the thermal wind relation (2.5), the zonal momentum equation and the thermodynamic equation can be combined as

$$N^2\frac{\partial}{\partial y}[w]-f_0^2\frac{\partial}{\partial z}[v] = \frac{R}{H}\frac{\partial}{\partial y}\left(\frac{[J]}{c_p}-\frac{\partial}{\partial y}[v^*T^*]\right)-f_0\left(\frac{\partial^2}{\partial y\partial z}[u^*v^*]-\frac{\partial}{\partial z}[X]\right). \qquad (2.6)$$

Using the continuity equation (2.4) we can describe the mean meridional circulation $([v],[w])$ using the stream function ψ, such that

$$([v],[w]) = \left(-\frac{1}{\rho_0}\frac{\partial\psi}{\partial z}, \frac{1}{\rho_0}\frac{\partial\psi}{\partial y}\right). \qquad (2.7)$$

The stream function ψ provides a topological view of the mean meridional circulation, and can be diagnosed when (2.6) is simplified to

$$\left\{\frac{\partial^2}{\partial y^2}+\frac{f_0^2}{N^2}\rho_0\frac{\partial}{\partial z}\left(\frac{1}{\rho_0}\frac{\partial}{\partial z}\right)\right\}\psi=\frac{\rho_0 R}{N^2 H}\frac{\partial}{\partial y}\left(\frac{[J]}{c_p}-\frac{\partial}{\partial y}[v^*T^*]\right)$$
$$-\frac{f_0\rho_0}{N^2}\left(\frac{\partial^2}{\partial y\partial z}[u^*v^*]-\frac{\partial}{\partial z}[X]\right) \tag{2.8}$$

Fig. 2.8 shows the stream function of the mean meridional circulation obtained with pressure as vertical coordinate. There is a seasonal variation in the position of the cells. A strong Hadley cell dominates the winter hemisphere with its rising branch in the summer hemisphere.

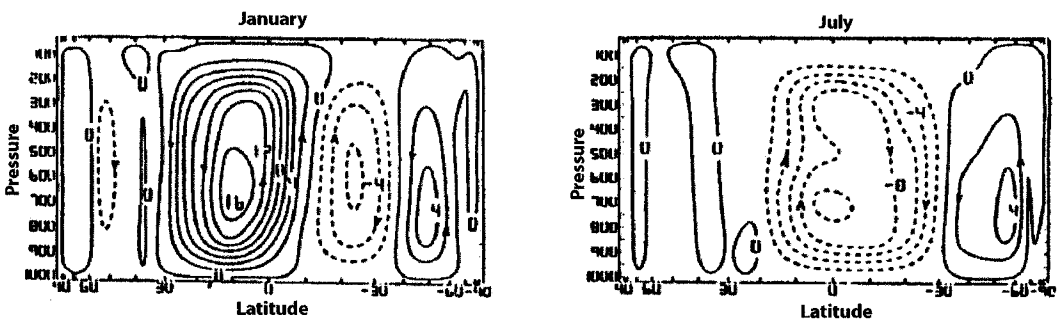

Figure 2.8: The stream function of the mean meridional circulation in pressure coordinates, for January and July. Units are 10^{10}kgs^{-1}. From Townsend and Johnson (1985).

The Eulerian-mean formalism is a convenient mathematical framework, but has some significant weaknesses when used as a tool for investigating the atmospheric transport processes. In this frame of reference the mean state of the atmosphere (given by $[u]$, $[T]$) changes in response to small imbalances between the forcing terms (momentum and heat eddy fluxes) and mean meridional circulation ($[v]$, $[w]$). Observational studies show that adiabatic cooling (term I in (2.3)) tends to cancel the convergence of the heat flux (term *II* in (2.3)). As a result the mean temperature would change only in response to variations in the diabatic heating. Observations show that the diabatic heating term is small and cannot explain the observed variations in the mean temperature field.

Some of the problems with Eulerian-mean methods can be avoided by defining a residual-mean circulation in the transformed Eulerian-mean (TEM) framework or by the use of isentropic coordinates (e.g., Andrews 1983, Andrews et al. 1987; Stone et al. 1999, Held and Schneider 1999).

2.4.2: The transformed Eulerian mean

The Transformed Eulerian Mean (TEM) equations can be obtained by defining the residual circulation $(v^{\dagger}, w^{\dagger})$ as follows (e.g., Holton, 1992),

$$v^{\dagger} = [v] - \frac{R}{\rho_0 H}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}[v^*T^*]\right) \tag{2.9}$$

$$w^{\dagger} = [w] - \frac{R}{H}\frac{\partial}{\partial y}\left(\frac{1}{N^2}[v^*T^*]\right) \tag{2.10}$$

By including the divergence of the eddy heat flux into the definition of the residual vertical velocity $w^{\dagger}$, the temperature does not change only in response to the diabatic heating.

The transformed Eulerian mean version of (2.2)-(2.5) (e.g., Holton, 1992) is

$$\frac{\partial}{\partial t}[u] - f_0 v^{\dagger} = \frac{1}{\rho_0}\nabla \bullet \mathbf{F} + [X], \tag{2.11}$$

$$\frac{\partial}{\partial t}[T] + \frac{N^2 H}{R} w^{\dagger} = \frac{[J]}{c_p}, \tag{2.12}$$

$$\frac{\partial}{\partial y} v^{\dagger} + \frac{1}{\rho_0} \frac{\partial}{\partial z}(\rho_0 w^{\dagger}) = 0 \tag{2.13}$$

$$f_0 \frac{\partial}{\partial z}[u] + \frac{R}{H} \frac{\partial}{\partial y}[T] = 0, \tag{2.14}$$

where $\mathbf{F} \equiv \mathbf{j} F_y + \mathbf{k} F_z$, the EP flux, is a vector in the meridional (y, z) plane, which for large-scale quasi-geostrophic eddies has the components

$$F_y = -\rho_0 [u^* v^*], \qquad F_z = \frac{\rho_0 f_0}{N^2 H}[v^* T^*]. \tag{2.15}$$

By inspecting (2.11)- (2.14) it is obvious that in the TEM formalism the cancellation between the eddy heat flux convergence and adiabatic cooling is removed. The form of (2.11) shows that the net effect of the eddy forcing is a redistribution of the zonal mean momentum (Tung, 1986).

Haynes et al. (1991) have derived a diagnostic equation for the stream function of the residual circulation $(v^{\dagger}, w^{\dagger})$, similar to (2.8):

$$\left\{ \frac{\partial^2}{\partial y^2} + \frac{f_0^2}{N^2} \rho_0 \frac{\partial}{\partial z}\left(\frac{1}{\rho_0} \frac{\partial}{\partial z} \right) \right\} \psi^{\dagger} = \frac{f_0}{N^2} \rho_0 \frac{\partial}{\partial z}\left(\frac{1}{\rho_0} \nabla \bullet \mathbf{F} \right) + \frac{R}{H c_p} \frac{\partial}{\partial y}[J] + \frac{\partial}{\partial z}[X], \tag{2.16}$$

where the streamfunction $\psi^{\dagger}$ is given by

$$(v^{\dagger}, w^{\dagger}) = \left(-\frac{1}{\rho_0}\frac{\partial}{\partial z}\psi^{\dagger}, \frac{1}{\rho_0}\frac{\partial}{\partial y}\psi^{\dagger}\right). \tag{2.17}$$

Thus, according to equation (2.16) the mean meridional circulation is determined by three forcings: eddy potential vorticity flux, zonal-mean diabatic heating and zonal-mean drag force. Therefore, the mean state $([u], [T])$ evolves totally in response to forcing terms.

Fig. 2.9 shows the stream function of the residual circulation in pressure coordinates. Unlike the Eulerian mean meridional circulation, the residual circulation consists of a single Hadley cell in each hemisphere extending all the way to the pole.

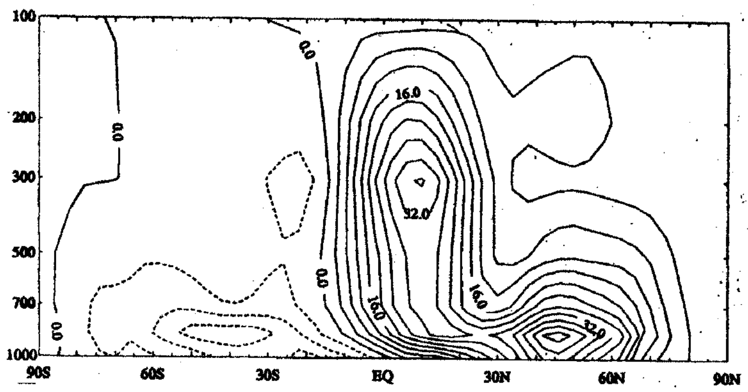

Figure 2.9: The stream function of the residual circulation in pressure coordinates, for January. Units are hPa m s^{-1}. From Holton (1992).

Referring to the differences between the Eulerian mean and the TEM, Held (personal communication) considers that the TEM formalism offers a semi-Lagrangian perspective of the flow. Fig. 2.10 provides a comparison of the volume control as appears in the two formalisms. The atmosphere is highly idealized and consists of a single layer with a rigid flat top and an undulating bottom following the topography.

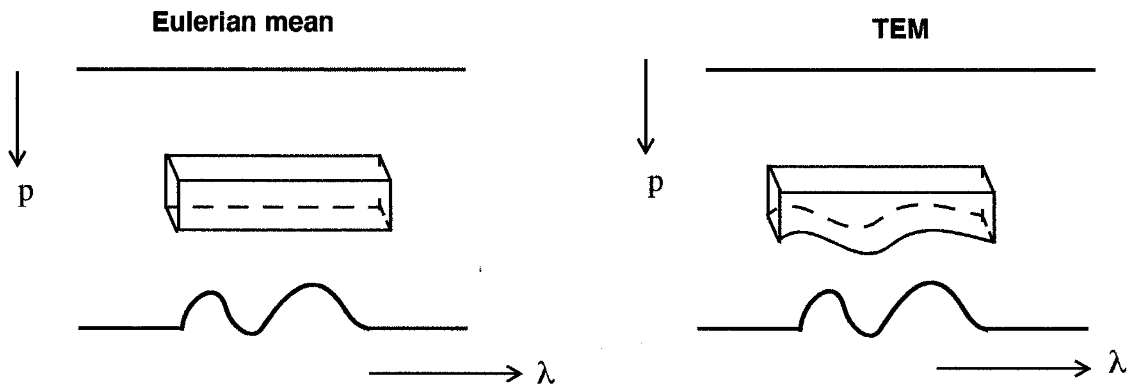

Figure 2.10: Schematic representation of the volume control as appears in the Eulerian mean formalism and the TEM formalism. Adapted from Held (personal communication).

2.4.3: The isentropic coordinate

Andrews (1983), Tung (1986), Haynes and McIntyre (1987), Hoskins (1991) and very recently Schneider (2004, 2005) have pointed out the usefulness of isentropic coordinates to study the interaction between the mean flow and baroclinic eddies since the potential vorticity flux has only components along isentropes, but not across isentropes. Under adiabatic inviscid conditions, isentropic surfaces act as material surfaces thus the air flows along isentropic surfaces.

The zonally averaged hydrostatic primitive equations with an isentropic coordinate (e.g., Andrews et al, 1987) are:

$$\begin{aligned} &\frac{\partial}{\partial t}[u] + \frac{\hat{v}}{a\cos\varphi}\frac{\partial}{\partial\varphi}([u]\cos\varphi) - f\hat{v} + \hat{\dot{\theta}}\frac{\partial}{\partial\theta}[u] = \\ &-\frac{1}{[\sigma]}\frac{\partial}{\partial t}[\sigma^* u^*] + \frac{\nabla\bullet\mathbf{F}}{[\sigma]a\cos\varphi} + [X] \end{aligned}, \qquad (2.18)$$

$$\frac{\partial}{\partial t}[\sigma] + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([\sigma][v]\cos\varphi) + \frac{\partial}{\partial\theta}[\sigma\dot{\theta}] = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([\sigma^* v^*]\cos\varphi) \qquad (2.19)$$

$$\left(f + \frac{2[u]\tan\varphi}{a}\right)\frac{\partial}{\partial\theta}[u] + \frac{1}{a}\frac{\partial}{\partial\varphi}\Pi([p]) \approx 0\,, \qquad (2.20)$$

$$\hat{\dot{\theta}} = \hat{Q}, \qquad (2.21)$$

where $\hat{Q}$ is the mean diabatic heating and the function Π on the right-hand side of equation (2.20) is given by

$$\Pi(p) = c_p\left(\frac{p}{p_0}\right)^{R/c_p}.$$

The EP flux divergence appearing in the forcing term of the zonal mean flow is

$$\nabla\bullet\mathbf{F} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([(\sigma v)^* u^*]\cos\varphi) + \frac{\partial}{\partial\theta}\left(-\cos\varphi\left[(\sigma u)^*\dot{\theta}^* - p^*\frac{\partial}{\partial\lambda}z^*\right]\right). \qquad (2.22)$$

The mass-density in isentropic coordinates is

$$\sigma = -\frac{1}{g}\left(\frac{\partial p}{\partial\theta}\right). \qquad (2.23)$$

The mass-weighted zonal mean is defined as

$$[\hat{A}] = \frac{[\sigma A]}{[\sigma]}. \qquad (2.24)$$

A similar mass-weighted zonal mean could be defined in pressure coordinate, but because the mass-density (given by $1/g$) in this system of coordinates is constant at each level, the mass-weighted zonal mean is identical to the zonal mean.

The first term in the vertical component of the EP flux expressed in isentropic coordinates represents the flux of momentum across the isentropic surfaces, while the second term is the pressure force exerted by the surface on the atmospheric flow. This representation makes the isentropic coordinates to offer a semi-Lagrangian description of the fluid when compared to the pure Eulerian mean formalism. Another advantage of the isentropic coordinates is that the Coriolis force acts on the meridional motion of the center of the mass of the fluid, which most of the time is a conserved quantity. In the Eulerian mean formalism the Coriolis force acts on the mean meridional motion that is not a conserved quantity.

In θ-coordinate, unlike the TEM frame of reference, the analytical expression of the continuity equation makes difficult to eliminate the time derivatives from the momentum and continuity equations using the thermal wind equation. To avoid this inconvenience Townsend and Johnson (1985) defined the isentropic mass stream function ψ_θ, analogous to the pressure coordinate, that satisfies

$$(\hat{v}, \hat{\dot{\theta}}) = \left(-\frac{1}{2\pi a \cos\varphi[\sigma]}\frac{\partial\psi_\theta}{\partial\theta}, \frac{1}{2\pi a^2 \cos\varphi[\sigma]}\frac{\partial\psi_\theta}{\partial\varphi}\right). \qquad (2.25)$$

Fig. 2.11 shows the stream function of the mean meridional circulation using potential temperature as vertical coordinate. In the isentropic coordinates the mean meridional circulation in each hemisphere is dominated by a single Hadley cell extending all the way to the pole, similar to the residual circulation in the TEM formalism. This feature appears because, as in the TEM formalism, the motion in the vertical branches is proportional only to the diabatic heating rate, the eddies affecting only the motion in the horizontal branches. We notice again a seasonal variation in the position of the cell, with a weaker cell in the summer hemisphere.

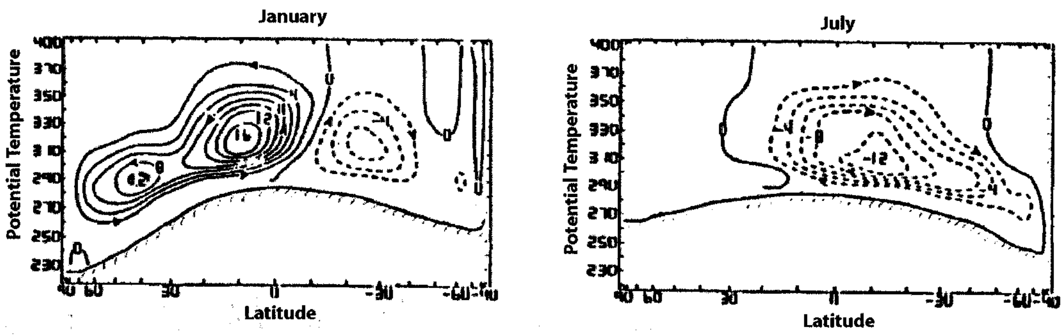

Figure 2.11: The stream function of the mean meridional circulation in θ-coordinates, for January and July. Units are $10^{10}kgs^{-1}$. From Townsend and Johnson (1985).

Townsend and Johnson (1985) found that in isentropic coordinates the pattern of the mean meridional circulation is similar to that when only the geostrophic part of the wind ($\hat{v} \approx \hat{v}_g = [\sigma^* v^*_g]/[\sigma]$) is considered. When only the ageostrophic part of the wind ($\hat{v} \approx \hat{v}_{ag} = [\sigma^* v^*_{ag}]/[\sigma]$) was considered the structure of the mean meridional circulation resembled that of the residual circulation in the log-pressure coordinate.

The presented methods, used for the theoretical investigation of the mean meridional circulation, have lead to the development of numerical models of different complexities determined primarily by the approach used to treat the eddies in these models.

2.5: Where do we go from here?

To tackle the problem of understanding and modeling the mean meridional circulation we need first to understand the mechanisms through which dynamical fluxes represented by the eddies influence the mean state. However, direct simulation of the eddies is a difficult and time-consuming process when dealing with the behavior of turbulent systems over long period of time. It is well known from theory and analysis of observations that a) the development and behavior of the eddies are influenced by the actual state of the mean field such us baroclinicity; b) the eddies play an active role in the formation, maintenance and variation of the mean field; and, c) the correlation between the mean field and statistics of eddies represents the energetic exchange between the mean and eddy fields. Therefore, relationships should exist between the eddy fluxes and the zonally averaged temperature and zonal wind fields.

In this thesis we propose a new approach to study the mean meridional circulation of the atmosphere that we envision as a framework for future understanding, especially the mechanisms that generate and maintain the eddy transports. The proposed PVPT system of coordinates divides the atmosphere into invariant flux tubes along which air parcels travel in the absence of friction, during adiabatic processes. These tubes are bounded by the potential vorticity surfaces as undulating lateral-sides, and undulating bottom and top isentropes. Unlike theories based on potential vorticity dynamics, in which the zonal mean flow is driven by the eddy transports, in the PVPT system of coordinates the main forcing of the mean meridional circulation is the form drag exerted by the walls of the tube on the air that flows through it. The form drag concept is analogous to the form drag exerted by mountains on the atmosphere (James, Chapter 8). The shape of the walls is modulated by the baroclinic eddies.

The manner and extent to which the eddies affect the mean meridional circulation in the middle latitudes is still an open issue. Most of the attempts describing the eddy - mean flow interactions are focused on the effects of the eddy heat fluxes and eddy momentum fluxes on the mean flow, but do not provide explanations on the mechanisms of interaction. The PVPT system of coordinates reveals that the mechanism of interaction is a deceleration that the mean flow experiences due to the form drag. In other words, in the invariant tubes the eddies have a similar effect on the flow as a mountain has when the mean meridional circulation is studied in pressure or height coordinates.

Chapter 3: The Early History of Eddy Flux Parameterizations

The atmosphere as a continuous system can be characterized by an infinite number of degrees of freedom associated with all processes that occur on different spatial and temporal scales (Phillips, 1951). The eclectic nature of phenomena describing the general circulation of the atmosphere loads a very difficult task on the shoulders of the numerical modeling community in its attempt to simulate the general circulation of the atmosphere. Even though such a model will never be built, it is the responsibility of modelers to reduce the complexity of the fundamental laws to simple mathematical forms - known as parameterizations, without losing the essence of the mechanisms involved. The simple model may not be a qualitatively realistic representation of the atmosphere, but the elimination of the complex details allows us to make the basic physical principles clear.

In the American Meteorological Society-Glossary of Meteorology, parameterization is defined as "*the representation in a dynamical model of physical effects in terms of admittedly oversimplified parameters, rather than realistically requiring such effects to be consequences of the dynamics of the system.*" Thus, the main goal of using parameterizations is to test different physical mechanisms, that we hypothesize as being responsible for driving the atmospheric circulation, not through an explicit representation but implicitly by reproducing their statistical effects.

The observed properties of the atmospheric flow described in the previous chapter suggest that effects of eddies can be indeed parameterized in climate models when their explicit simulation has no direct relevance to the scales on which the studied phenomena occur. The design of eddy parameterizations is of paramount importance since it largely determines the quality of modeling results. A tremendous amount of work has been consecrated to this subject, but much remains to be done.

Since 1921, most of the theoretical work dedicated to the study of the general circulation has been devoted to overcoming the problem of how to treat transports due to large-scale eddies. Forty years later, Lorenz (1967), referring to the nonlinear interaction of eddies with the mean zonal flow, noted that "the structure of eddies constitutes one of the outstanding aspects of the general circulation not yet theoretically explained. This complexity arises from the fact that while attempting to reach any equilibrium configuration, eddies will produce a new zonally averaged circulation which will turn to demand a new equilibrium configuration for the eddies."

Because it is extremely difficult to measure the eddy transports, the numerical simulations of the general circulation of the atmosphere have emerged as a critical tool for understanding the development and evolution of large-scale atmospheric turbulence. The advent of computers fueled interest in the development of eddy flux parameterizations, specifically for numerical models. In the next section we will discuss the most important methods proposed for the eddy flux parameterizations and their application to modeling the mean meridional circulation.

3.1: Eddy transport parameterization

In 1921 Defant, a pioneer among the theoretician peers looking at the latitudinal transport of heat, used diffusion as a method of parameterizing eddy transports. This approach works very well to describing the eddy heat transport, but is not suitable in circumstances when the flux is directed against the gradient, like in the case of the eddy momentum transport (Saltzman, 1978).

The weakness of the parameterization proposed by Defant challenged Gambo and Arakawa (1958) to derive a prognostic equation for the eddy momentum flux, in order to close the system of equations used for predicting the zonal mean flow. In this parameterization the time evolution of the eddy momentum flux is assumed to be proportional to the zonal mean flow and variances of the zonal mean flow. The coefficients are considered constants and determined by assuming that the disturbance affecting the mean flow is simple harmonic.

One year later, Charney (1959), based on observational studies (Starr et. al, 1950), envisioned the dynamical necessity and importance of eddy transports for the mean meridional circulation, and constructed a model of the general circulation of the atmosphere in which the zonal flow is modified by the joint action of the eddy momentum flux and the eddy heat flux. The eddy fluxes are parameterized as being driven by the amplitude of the most unstable disturbances. He hypothesized that baroclinic waves will develop and affect the mean flow until the rate at which they receive energy from the mean flow is balanced by the rate at which they dissipate the available potential energy by destroying the temperature gradient, and dissipate kinetic energy by surface and internal friction. The results of Charney's study show an alteration of the equator-to-pole temperature gradient caused by the action of eddies, and agree with some features of the mean atmospheric circulation, such as the zonal wind structure in middle latitudes. The deviations from the observed circulation in midlatitudes are not significant because if no standing waves are available the eddy momentum is transported by the transient waves, a result obtained almost twenty years later by Held and Suarez (1978). The major discrepancies appear in the tropical latitudes. The differences in the tropical mean meridional circulation obtained by Charney and that observed could be due to the heating of the atmosphere by the release of latent heat in deep convection, a mechanism which occurs primarily in the tropics, and has been neglected. He also noticed that the geostrophic approximation equations used for the study of the tropical region must be replaced by the primitive equations. The weakness of Charney's parameterization is the assumption that only the most unstable wave can interact with the mean flow, whereas observations show that a large amount of meridional transport is by standing waves.

In 1960 the pioneering work of Eliassen and Palm finally proved mathematically that the eddy fluxes of heat and momentum do not act independently to each other in modifying the mean state of the atmosphere. In the following year, Charney and Drazin

(1961), derived the nonacceleration theorem which states that if the divergence of the EP flux is zero then the disturbances cannot affect the zonal mean flow. Therefore the divergence of the EP flux represents one of the main driving mechanisms of the zonal circulation in the meridional plane. This result was later generalized by Andrews and McIntyre (1976, 1978, 1983) and Boyd (1976).

In 1964 Smagorinsky, in an attempt to study the general circulation of the atmosphere, proposed a two-level primitive-equation model in which the eddy heat flux is diagnosed by assuming that at a given latitude the total heat flux is balanced by the external diabatic heating. The eddy momentum flux is parameterized in terms of surface wind stress by assuming an empirical relationship according to which the processes responsible for the momentum balance of a zonal ring in a layer above the surface are the Coriolis force and the surface boundary stress, as schematically shown in Fig. 3.1.

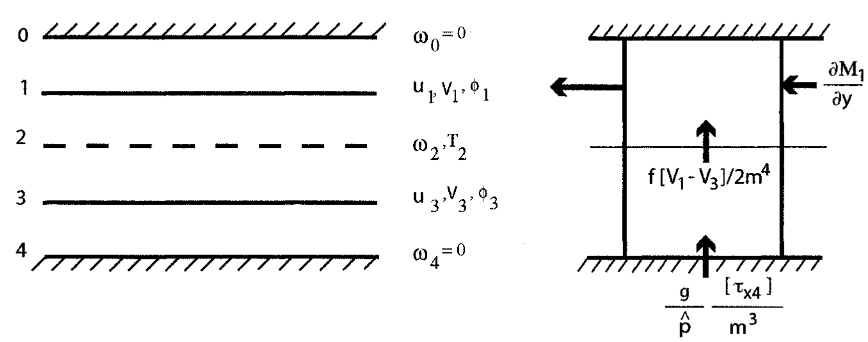

Figure 3.1: Schematic illustration of the processes assumed to maintain the momentum balance of a latitude belt. M **represents the eddy flux of momentum,** $-2\hat{p} = p_4 = 1000mb$, τ_{x4} **denotes the zonal component of the boundary stress tensor,** $m = 1/\cos\varphi$, **where** φ **is the longitude.** ω **is the vertical velocity in pressure coordinates and** ϕ **is the geopotential height. All the other notations are standard. From Smagorinsky 1964.**

The weakness of the eddy momentum flux parameterization is that, according to this parameterization, the global integral of the surface wind stress should vanish, a requirement that is not necessary true. The results obtained with this model show that to some degree of approximation the eddy heat flux can be calculated from the radiative heating without using the zonal mean temperature distribution, and the eddy momentum flux can be calculated from the surface stress without employing the zonal wind distribution or assuming a dependence of the surface stress on the surface wind. This approach is based on different assumptions than the diffusive theories.

In the same year, Leovy (1964) parameterized the eddy heat transport by assuming that the divergence of the eddy heat flux is proportional to the difference between the local temperature and a radiatively equilibrium temperature. The equilibrium temperature is assumed a function of height only, and the coefficient of proportionality is a constant. The eddy momentum flux parameterization uses the same concept and thus, the divergence of the eddy momentum flux is proportional to the zonal mean wind, and the coefficient of proportionality is chosen constant. These parameterizations are based on very simple assumptions and therefore have many deficiencies such as: *i)* in the case of the eddy heat this parameterization may give rise to significant errors especially when the difference between the local temperature and the radiatively determined temperature is large. Other misrepresentations are made by omitting the latitudinal structure of the mean state temperature and the variation of the coefficient of proportionality with the local temperature; *ii)* in the case of the eddy momentum flux this parameterization is similar to the diffusive representation proposed by Defant (1921), thus it manifests the same lacking in representing the countergradient fluxes.

Saltzman and Vernekar (1968), closed the system of equations describing the time and zonal mean state of the atmosphere by assuming that the eddy momentum flux is transported by the transient waves excited by the baroclinic instability, and that shortly after the development they start to comport as barotropic waves. In this parameterization "the eddy momentum flux is proportional to the amplitude of the waves as measured by the variance of the meridional wind, and to the shear of the angular phase velocity which causes the tilting of the wave". A good point of this parameterization is the dependence on the wave number, which allows all baroclinic waves to be taken into account, rather than just most unstable wave, as in the case of Charney's parameterization. The model of Saltzman and Vernekar can be subject to criticisms because it does not include any heat transport.

In 1970, Green presumed that an accurately representation of the momentum transport can be obtained if the parameters describing the mean state of the atmosphere are functions of both latitude and height. For example if the zonal mean zonal wind is independent of latitude, the eddy momentum transport is independent of latitude. This uniform distribution is not compatible with the observed counterpart. If the zonal mean flow does not varies with height then by the thermal wind relationship the temperature field is uniform and the potential energy is not available for conversion. In this case the baroclinic waves would extract their kinetic energy from the kinetic energy of the mean flow. The net effect of the eddies would be to transfer momentum countergradient, a result that does not support the westerlies observed at the Earth's surface in midlatitudes. Considering the classic diffusive theory as inappropriate to parameterize the eddy transports, Green represented the interaction between the baroclinic waves and the mean flow in terms of a non-isotropic diffusive theory, which he calls the "transfer theory". The main characteristic of this theory is that the eddy fluxes are related to the quantities conserved during the eddy-lifetime cycles through the transfer coefficients that depend on the trajectory of the conserved quantity. The parameterization of eddy momentum flux is based on the equation derived by Charney and Stern (1962) which relates the transport of potential vorticity, the meridional divergence of the eddy momentum flux, and the vertical convergence of eddy heat flux. Since potential temperature and potential vorticity are approximately conserved during baroclinic development they can be treated in terms of the transfer theory. To parameterize the eddy heat flux, Green applied the same theory based on the conservation of entropy in large-scale eddy motion, and the fact that the entropy flux is nearly proportional to the heat flux. Green's work is referred to as a milestone in the development of models ability to allow some feedback in the climatic system. A major problem of this method was the calculation of the transfer coefficients, which represent the components of the diffusion coefficients tensor. For motion of small amplitude Green found that the horizontal and vertical transfer coefficients are proportional to the amplification rate of the baroclinic wave and are positive. In the case of a more complex motion Green was able to provide only a qualitative description of the transfer coefficients and left the door open for further investigations

In the early 1970s the fever of climate change was likely one of the reason that determined Stone (1972, 1973, 1974) to develop simple models to study the interaction between the baroclinic waves and men flow in order to assess the impact of dynamical feedback on the climate change. In these models the effects of the eddies are parameterized in terms of the Richardson number. The eddy heat parameterization relies on the assumption that radiative fluxes are in balance with the fluxes of heat and potential energy due to large-scale eddy motion. In his studies, Stone omitted some significant contributions to the energy balance of the Earth's atmosphere such as transports of latent heat and sensible heat due to small-scale convection, the mean large-scale motions, and the effects oceanic circulations on the atmosphere. Despite the neglected components the solutions are surprisingly good. The reason for model's success is the strong feedback which exists in the dynamical fluxes arising from baroclinic instability. This feedback is also evident in numerical results of Smagorinsky (1964). In fact both parameterizations are based on the same balance assumption with regard to representing the eddy heat transport. The eddy momentum flux is parameterized in a similar manner in terms of the Richardson number.

In a review of eddy flux parameterizations Brascome (1983) provides a nice a comparison between the Stone and Green works. He noticed that both parameterizations are based on dynamical assumptions rather than empirical assumptions as in most of the previous theories, and they have some common characteristics such us: *i*) in both methods eddies transporting the heat flux have their energy partitioned equally between the kinetic energy and available potential energy. This is because the baroclinic waves have the horizontal scales on the order of the deformation radius. *ii*) the eddy available potential energy is proportional to zonal available potential energy. There also differences such as Green assumes the meridional scale is determined by the width of latitudes where the baroclinic waves are active, whereas Stone assumes that the meridional scale is the deformation radius of the Eady model, which is much smaller than the meridional scale of the zonal mean flow.

In 1975 Held started a series of long dedicated studies of baroclinic eddy fluxes, using what he called "the simplest model of a turbulent flow generated by baroclinic instability." In the context of the two-layer quasi-geostrophic model, Held showed that in a viscous fluid the meridional transport of the eddy momentum flux is proportional to the source/sink of the mean square eddy potential vorticity. For the parameterization of the eddy heat flux, Held found that the characteristic height to which the eddy potential vorticity flux extends above the surface is an important parameter in establishing a relationship between the poleward eddy heat flux and the horizontal temperature gradient. When the characteristic height is much bigger than the scale height of the atmosphere, using scale arguments, Held proved that the poleward eddy heat flux displays a squared dependence on the horizontal temperature gradient. This result was also obtained by

Green (1970) and Stone (1972). When the characteristic height is much less than the scale height of the atmosphere, the poleward heat flux is proportional to the fifth power of the horizontal temperature gradient.

While the general trend was to design parameterizations capable to simulate the statistical effects of the eddies on the zonally averaged circulation and incorporate them into zonally-averaged statistical-dynamics models, there were studies that tried to minimize the effects of eddy transports in climate models as in the following example.

In 1977, Schneider and Lindzen studied the general circulation of the atmosphere with a steady-state model that includes parameterization of physical processes such as cumulus convection, radiation, and friction, but does not represent any effect of large scale eddies. Transports of heat and momentum by large-scale eddies are suppressed on the assumption that eddies are not the main factors responsible for the observed mean meridional circulation. The structure of the mean meridional circulation obtained within this model resembles the observations in the tropical latitudes, but structural differences appear in midlatitudes, namely the magnitude of the zonal wind in the core of the subtropical jet is very big. This is not a surprising result because as observations show the diabatic heating is maximum in the tropical latitudes and is very small away from this region. The authors speculate that the large errors obtained for the amplitude of the zonal mean in midlatitudes are not necessarily induced by the lack of representation of the eddy effects, but don't provide any other explanation. If they would have been included parameterization of physical processes along with parameterization of eddy fluxes of heat and momentum, then their results could be significantly improved. The role of eddy transports is to avoid the built up of excessively strong westerlies at the latitude of the subtropical jet stream.

Up to this point many modelers sought a new method of parameterization for eddy transports. Since the 1980's, the advent of computers put the early parameterizations to new tests and the sophistication of eddy parameterizations have grown in response to recent technology.

Quasi-geostrophic flow has been extensively studied both for the investigation of the dynamical balance of zonal flows as well as for the development of closure schemes. Taylor (1980) developed a model to study the zonally averaged circulations in which for the eddy momentum flux adopted the Saltzamn and Vernekar's (1968) parameterization, and the eddy heat flux is driven by the temperature gradient. The proportionality coefficient depends on the magnitude of the temperature gradient, as in Green (1970).

In 1981, Gallimore and Johnson formulated a numerical diagnostic model of the zonally averaged circulation in isentropic coordinates. This model was used to investigate the constrains applied on the mean meridional circulation in this frame of reference and the influence of the angular momentum torques[4] on the circumpolar vortex. In the

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[4]. The angular momentum torques includes the zonal pressure torque, friction torque and the divergence of the eddy momentum transport.

isentropic zonal model the parameterization of processes affecting the mean state of the atmosphere is based on different mechanisms than those used so far in pressure and Cartesian frame of references. There are no eddy heat and momentum fluxes. The surrogate of the thermal forcing is the diabatic heating (radiative heating, latent heat and sensible heating), and the eddy momentum transport is substituted by the zonal pressure torque. In the isentropic framework, "the zonal pressure torques represent the vertical transfer of angular momentum by pressure stresses within large-scale waves, and, thereby dynamically force a meridional mass circulation through the development of mean geostrophic poleward and equatorward flows". The results show an isentropic Hadley circulation extending from tropical to polar latitudes. Tropical diabatic heating maintains the upward branch in low latitudes, whereas polar cooling drives the downward branch in high latitudes. Despite their success in numerical modeling the isentropic coordinates were not used much with zonally averaged models.

In 1983 Andrews extended Gallimore and Johnson's idea and derived the Eliassen-Palm theorem in isentropic coordinates. The mathematical framework provided new insight on the forcing of the zonal mean flow. The vertical eddy transport of energy in isentropic coordinate is by the pressure force exerted on the isentropic surfaces by the fluid above.

Branscome (1983) designed a parameterization for the eddy heat flux using the baroclinic instability theory on a beta plane. Using the beta plane approximation

Branscome included a realistic representation of the location where the baroclinic waves are active. He assumes that the mean flow adjusts to an equilibrium state as the baroclinic instability depletes the available potential energy existing in the mean flow, and therefore the eddy heat flux can by expressed in terms of the quasi-geostrophic eddy streamfunction. Although Branscome acknowledges the importance of the whole spectrum of waves in transporting the eddy heat flux he chose to represent the total flux by the most unstable wave, as Charney (1959). Using scale arguments the eddy heat flux is parameterized in terms of the static stability. The magnitude of the parameterized eddy heat flux is similar to the observed flux, but there is a discrepancy in the location of the maximum eddy heat transport relative to the atmospheric counterpart.

In 1986, Tung generalized Andrew's (1983) expression of the eddy forcing to enclose the diabatic contribution. He also noted that the exact relationship between the quasi-geostrophic potential vorticity and the divergence of the quasi-geostrophic EP flux is maintained in the isentropic coordinates, without quasi-geostrophic balance requirements, if the pseudodivergence[5] of the EP flux is considered. Then the forcing of the zonal mean flow, given by the EP is parameterized using the transfer theory (Green, 1970). Results obtained by Tung represent an important generalization of the previous works that were focused on particular states of the atmospheric system. Another major achievement of Tung's work is to point out the ability of isentropic coordinates to represent the

5. The pseudodivergence operator differs from the pure divergence operator due to the addition a derivative with respect to time.

atmospheric flow in different regimes. When the eddy forcing is strong the mean meridional circulation in midlatitudes is in geostrophic balance, and when the eddy forcing is weak the circulation is "nearly inviscid" (Held and Hou, 1980).

In 1987 and 1990, Stone and Yao conducted complex investigation of the mean meridional circulation of the atmosphere using a zonally-averaged statistical-dynamical model that included both a dynamical core and a complete description of physical processes. The experiments performed with this model were compared with the 3D version of the zonally averaged model that was running in parallel as the "truth". They consider this method more useful than the direct comparison with the observations because the latter have embedded the effects of stationary waves. The parameterizations used for representing the effects of eddy momentum and heat fluxes are a combination and generalization of previous results obtained by Green (1970), Branscome (1983). First they derive an expression for the eddy momentum transport suitable for a "dry" atmosphere using as starting point the transfer theory proposed by Green (1970). Remember the major problem in the approach proposed by Green (1970) was how to calculate the transfer coefficients. Neglecting the vertical variations in the eddy momentum transports, Stone and Yao (1987) assume the transfer coefficient equal to the transfer coefficient calculated by Branscome (1983) plus a correction term that is non zero in the mature-stage of the baroclinic wave. If only the Branscome's transfer coefficient was considered the effects of the induced eddy momentum transport would be a uniform acceleration of the zonal mean flow that would violate the angular momentum conservation. This parameterization gives rise to big errors when the eddy momentum transport is compared against the structure obtained from the 3D simulation. Because the largest discrepancies were observed in the summer hemisphere they concluded that effects of moisture associated with baroclinic eddies are important and have to be included in the parameterization of eddy momentum transport. The presence of condensation processes raises the question of the validity of the transfer theory, since potential temperature and potential vorticity are no longer conserved in a "moist" atmosphere. To overcome this impediment they defined corresponding variables that are presumably conserved in a "moist" atmosphere, and allow the transfer theory to be applied. The potential temperature is replaced by the equivalent potential temperature, and for the quasi-geostrophic potential vorticity they chose as alternate variable the equivalent potential vorticity, which is not in general a conserved quantity (Schubert, 2004). They also derive a diagnostic relationship for the "moist" EP flux similar to that for "dry" EP flux. The transfer coefficient in the "moist" atmosphere case is the same for the "dry" atmosphere because the conservation of angular momentum is independent of condensation processes. They tested their parameterization for the meridional eddy momentum transport by using it in a perpetual January 2D-simulation. The meridional circulations produced by the parameterization simulate the observed structures fairly well. In the following paper Stone and Yao (1990) improved Branscome's (1983) parameterization of the eddy heat flux by damping the total eddy heat flux in the planetary boundary layer. When compared with the eddy fluxes produced by the 3D model the results show similar characteristics but also large errors such as the underestimation of the eddy heat transport in the summer hemisphere because is not capable to reproduce the maximum observed near the tropopause. When compared to the Branscome's (1983) parameterization the new one does not show significant improvements.

Although the slantwise moist convection concept proposed by Emanuel (1988) has no direct relevance to the present discussion, is helpful in understanding the results obtained by Stone and Yao. Slantwise convection is a form of convection that results from the imbalance between gravitational and centrifugal forces, and has space and time scales intermediate between those of moist convection and baroclinic instability. Observations show the occurence of instability of the flow to slantwise convection in regions characterized by baroclinic instability.

In 1990, Yang et al., in a diagnostic study of the EP flux pseudodivergence, used the same parameterization proposed by Tung (1986), but instead of making any assumption about the isentropic mixing coefficient, they calculated the mixing coefficient from observational data. They also identified the importance of eddy flux parameterization for the study of tracer transports.

In 1999, Zou and Chen proposed a modified version of Green's (1970) and Branscome's (1983) scheme to parameterize the eddy fluxes in the atmosphere. Zou and Chen aimed their work to determine the transfer coefficient that describes the vertical variations of the eddy heat flux, variations that were considered unimportant by previous investigators. They attributed the underestimation of the eddy heat flux, obtained by Stone and You (1990), to the neglected contribution. The magnitude of the transfer coefficient is obtained from observations using the least square technique. For the eddy momentum flux, Zou and Chen used the same method as Green (1970, and the transfer coefficient is determined using the observed eddy heat flux. Contrary to Stone and Yao (1978) results, the transfer coefficients from the eddy flux parameterization used by Zou and Chen yield to good representation of the vertically integrated eddy momentum flux for a dry atmosphere. Since only the most unstable wave is used, the maximum eddy heat flux in the vicinity of the tropopause cannot be reproduced.

In 2000, Petoukhov et al., in a complex climate model, chose to incorporate in the dynamical core a parameterization of the eddy momentum heat transport in order to simulate the effects of large-scale eddies in the tropopause.The eddy heat transport is represented in terms of the classical diffusion theory. The diffusion coefficient is proportional to the eddy kinetic energy, and is designed to include effects of the static stability and turbulence in the planetary boundary layer. Because of the complexity of the model, it is not suitable to discuss the role of the eddy fluxes parameterization. We just mentioned it as an example of the usefulness of eddy flux parameterizations.

In 2002, Barry et al. noted that "the atmospheric heat transport on Earth from the equator-to-poles is largely carried by the mid-latitude storms. However, there is no satisfactory theory to describe this fundamental feature of the Earth's climate." They proposed to describe the midlatitudes as a "heat engine" that works between a warm pool, represented by the tropics and a cold pool, represented by the pole and produces eddy kinetic energy. They exploited this concept in deriving a new form of the diffusive coefficient used by the diffusive approximation.

In 2003, Lapeyre and Held bring again on the spot the diffusive eddy closure theory for the parameterization of eddy heat flux, and point out the significance of the relationship between the eddy diffusivity and energy generation rate.

This historical review organized around the various views and controversies in the development of the eddy flux parameterizations can be summarized as in Fig. 3.2. The complexity implied by processes that contribute to these transports and their effects on the mean flow does not bring about sturdy theories. The subject is eclectic and this makes the problem of parameterization so exciting.

We have just outlined the milestones in the development of the eddy flux parameterizations, and in the next two tables a comprehensive list of all work done is presented. Table 3.1 contains the parameterizations of the horizontal eddy flux and Table 3.2 the parameterizations of the meridional momentum flux. These tables include the original contributions that introduced each type of parameterization as well as other papers that either use these parameterizations or have improved them over the time. Each type of parameterization in accompanied by a brief description which emphasizes its strengths and pinpoints the weaknesses.

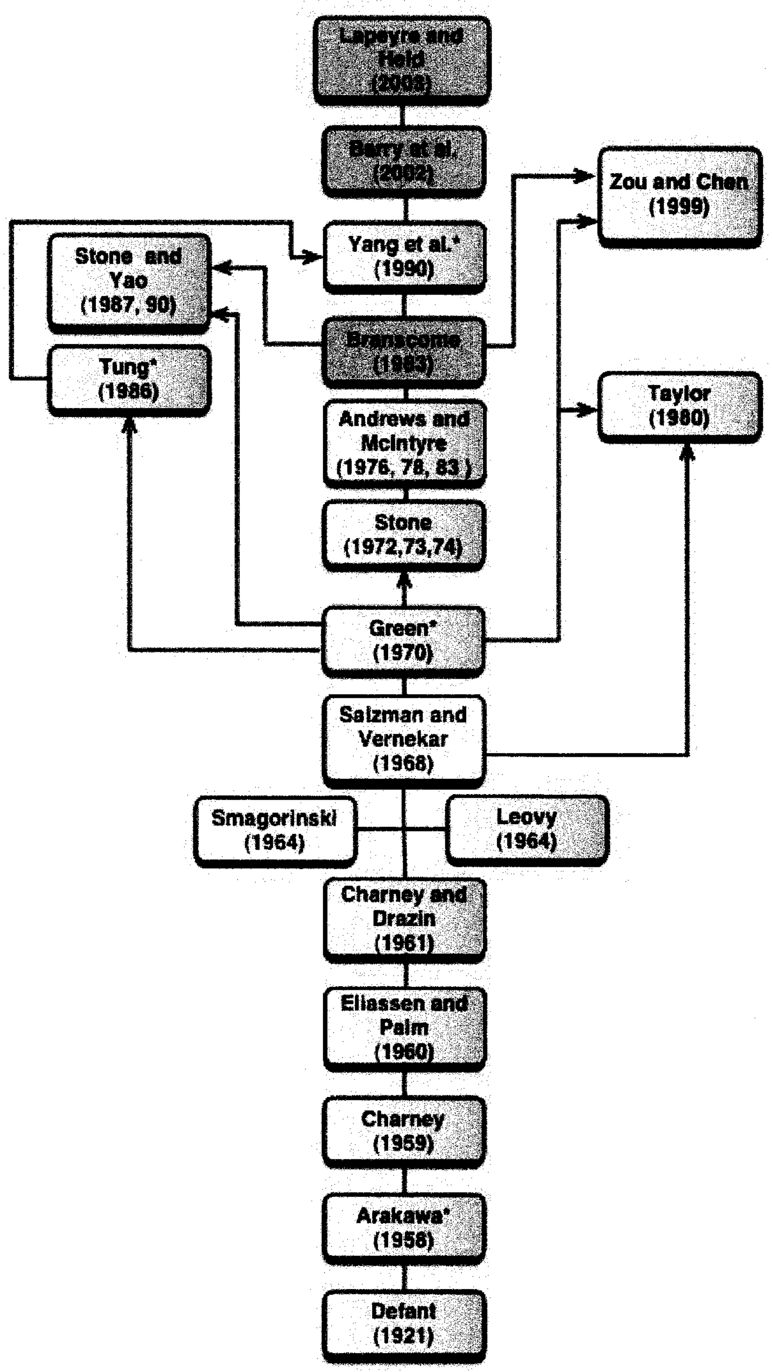

Figure 3.2: The diagram of investigators who contributed to the development of parameterizations of the eddy fluxes of momentum (in yellow) and heat (in red). Boxes filled with both colors denote work that involves parameterization of both type of fluxes. Arrows connect works that use previous ideas as bases for developing new parameterizations. Names accompanied by a star mark contributors that used potential vorticity in formulating the parameterization.

Table 3.1: Horizontal eddy heat flux

Parameterization Type	Used by	Comments
$\overline{\mathbf{V}'T'} = -K\nabla\bar{T}$	1921, '50, Defant 1926, Angstrom 1953, Hess and Frank 1960, Fritz 1962, '63, '64, '65, 71', '73, '75, Adem 1965, Williams and Davis 1968, '71, Shaw and Donn 1969, '71, Dolzhanskiy 1972, Faegre 1970, Wiin-Nielsen 1973, Schneider and Gal-Chen 1975, North (a,b) 1975, Paltrige 1976, Ghil 1976, Suarez and Held 1977, Drazin and Grieffel 1977, Lee and Snell 1977, Lindzen and Farrel 1971, '72, Pike 1973, MacCracken 1975, Lahif 1976, Petukhov 1977, Ohring and Gyoeri 1978, Ohring and Adler	The linear diffusive heat flux with a prescribed constant eddy diffusivity K is one of the simplest possible assumption relating the eddy heat flux to the zonal mean temperature gradient. The transfer is thus assumed to be downgradient in direction and diffusive in character. Analysis of observations reveals that the meridional eddy fluxes of heat depend on latitude, and height. In the stratosphere they are countergradient in many regions. This type of parameterization is not used.

Parameterization Type	Used by	Comments
$\overline{\mathbf{V}'T'} = -K(\varphi)\nabla\bar{T}$	1969, Sellers 1973, Dwyer and Peterson 1976, Temkin and Snell	Linear diffusive heat flux with K prescribed as function of latitude did not enhance the accuracy of the results.
$\overline{\mathbf{V}'T'} = -K(T)\nabla\bar{T}$	1970, Green 1968, '71(a,b), '72, '75, Saltzman and Vernekar 1972(a,b), '73, '74, Stone 1973, '76, Sellers 1974, Held and Suarez 1974, Mac Craken and Luther 1975, Wiin-Nielsen and Fuenzalinda 1976, Gal-Chen and Schneider 1977, Webster and Lau 1980, Haidvogel and Held 1980,Taylor 1983, Brascome 1990, Stone and Yao 1996, Held and Larichev 1998, Haine and Marshall 2000, Petoukhov et al. 2002, Barry et al.	Diffusive heat with K as a dependent variable from baroclinic wave theory either as a constant leading to linear a formula or a function of the temperature field (e.g., the temperature gradient) leading to a nonlinear formula.

Parameterization Type	Used by	Comments
$\nabla\bullet[\mathbf{V}'T'] = \beta(\bar{T} - \tilde{T})$	1964, Leovy 1969, '70, '72 Budyko 1974, Gordon and Davis 1974, Held and Suarez 1975, Cheylek and Coakley 1976, Frederiksen 1976, Gal-Chen and Schneider 1976, Su and Hsieh 1977, Lian and Cess 1977, Lindzen and Farrel 1978, Schroebel and Strobel 1980, Held and Hou 1986, James and Gray 1987, Garcia 1988, Lindzen and Hou 1991, Haynes et al. 1992, Chen and Robinson 1994, Held and Suarez 1995, Holton et al. 1998, Sankey 1999, Fang and Tung 2002, Walker and Magnusdottir 2002, Semeniuk and Shepherd	In the Newtonian type parameterization the zonal mean of the heat flux is proportional to the difference between the zonal mean temperature and a radiatively equilibrium temperature. Most of the models consider the variation of the radiative damping rate β with latitude, but do not address the influence of temporal variations. This parameterization does not guarantee that the global integral of the right -hand side vanishes as the left-hand side. It gives good results for the extratropical middle atmosphere since one of the properties of this region is the strongly relaxation nature of the diabatic heating.

Parameterization Type	Used by	Comments
HOC	1952, Kuo 1959, Charney 1961, Arakawa 1964, Smagorinski 1970, '73, Kurihara 1975, Egger 2003, Lapeyre and Held	Equation for heat flux with closure at higher order.

Table 3.2: Meridional eddy momentum flux

Parameterization Type	Used by	Comments
$\overline{u'v'} = -K_u \frac{\partial}{\partial \varphi} \bar{u}$	1964, Leovy 1968, Starr 1971, '72, Pike 1973, MacCracken 1986, James and Gray 1988, Vallis 1992, Chen and Robinson 1994, Held and Suarez 2002, Franzke	In the diffusive approximation the eddy momentum flux is proportional to the meridional gradient of the zonal mean wind.

Parameterization Type	Used by	Comments
$\frac{\partial}{\partial\varphi}\overline{u'v'} = C\bar{u}$	1975, Lahiff 1985, Hendon and Hartman 1988, Panetta and Held 1989, James and James 1992, Suarez and Duffy 1993, Yu and Hartman 2000, Magnusdottir and Walker	A simpler version of the diffusive approximation. The both have the same deficiency that cannot represent the downgradient flux characteristic of large scale atmospheric behavior.
$\overline{u'v'} = -K\frac{\partial}{\partial\varphi}\bar{T}$	1965, Wiliam and Davis 1969, '71, Dolzhanskiy 1976, Petukhov	
$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\overline{u'v'} = -K\overline{\tau}_e$	1964, Smagorinsky 1970, '73, Kurihara	Diagnostic relations to surface wind stress in a two-layer model.
$\overline{u'v'} = T_c\left[v'^2\right]\cos\varphi\frac{\partial\mu}{\partial\varphi}$	1968, '72, Saltzman and Vernekar 1980, Taylor	
$[v'q'] \sim \nabla\bullet\mathbf{F}$	1970, Green 1971, Wiin-Nielsen and Sela 1975, Wiin-Nielsen and Fuezalinda 1977, Webster and Lau 1977, White 1978, Ohring and Adler 1983, Branscome 1984, White and Green 1986, Wu and White 1987, Stone and Yao 1990, Yang et al. 1999, Zou and Gal-Chen 2004, Zurita and Lindzen	Relations based on combined diffusive approximation for potential vorticity and sensible heat.

Parameterization Type	Used by	Comments
HOC	1959, Charney 1961, Arakawa 1982, Hoyer and Sadourny	Equation for momentum flux with closure at higher order.

3.2: Discussion

After a century of research, one of the major difficulties to be overcome in the zonally averaged models continues to be how to treat meridional and vertical transports due to large-scale eddies. While these transports can be fairly well simulated by atmospheric general circulation models, there is still no widely accepted quantitative theory to explain these transports. The main problem, as pointed by the tremendous amount of work spent on this issue, is that both momentum and heat are not conservative properties of the atmospheric flow. Another major difficulty arises from the fact that the parameterization of eddies requires two parts: a theory to explain the mechanisms that produce the eddies and another one to explain how they interact with the zonal mean flow. The most convenient mathematical formulation of the zonal mean flow may represent the eddy terms in such a way that their physical significance is not evident, and they are not easy to parameterize.

In the conventional Eulerian-mean framework, the effects of eddies on the mean flow can be parameterized only in terms of the eddy momentum transport. The eddy heat transport term does not appear in the equation of the zonal mean zonal wind.

Within the TEM equations the EP flux is the main eddy forcing of the zonal mean flow. This makes the TEM equations an appropriate investigation tool for the mean meridional circulation. Unfortunately, the mathematical form of the eddy terms, as derived in different systems of coordinates, does not provide much physical insight on how to design the parameterization of these transports. Equation (3.1) gives the mathematical expression for the EP flux when the vertical coordinate is the geometrical height.

$$\nabla\bullet\mathbf{F} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([u^*v^*]\cos\varphi) + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left\{\frac{\rho_0 f_0}{N^2 H}[v^*T^*]\right\}. \tag{3.1}$$

Here $\mathbf{F}$ is the EP flux vector, (u^*, v^*) are the horizontal components of the wind (departure from the zonal mean), N^2 is the Brunt-Vaisala frequency, ρ_0 is the density of the basic state, H is the equivalent depth of the atmosphere and T^* is the deviation of the temperature from the zonal mean. [ ] denotes zonal averaging defined in Chapter 2.

From this equation we can infer that the forcing of the zonal mean flow is the eddy momentum flux and eddy heat flux, but nothing more about the way that these fluxes interact with the zonal mean flow. Similar arguments apply for the case of pressure as vertical coordinate.

A somewhat more restrictive result can be obtained under the quasi-geostrophic approximation, when the divergence of the EP flux is proportional to the meridional flux of eddy potential vorticity,

$$\nabla \bullet \mathbf{F} = \rho_0[v^* q^*], \tag{3.2}$$

where q^* is the eddy potential vorticity. This representation allowed the development of some of the parameterizations presented above.

In contrast to height and pressure coordinates, with the isentropic coordinates the vertical component of the EP flux (eq. 3.3) can be interpreted as form drag on undulating isentropic surface, in the absence of diabatic sources:

$$\nabla \bullet \mathbf{F} = -\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}([(\sigma v)^* u^*]\cos\varphi) + \frac{\partial}{\partial\theta}\left[p^*\frac{\partial}{\partial\lambda}z^*\right], \tag{3.3}$$

where $\sigma = -\frac{1}{g}\frac{\partial p}{\partial\theta}$ is the pseudo-density in isentropic coordinates, p^* is the departure of pressure from the zonal mean and z^* is the departure of the geometric height (the height of the isentropic surface) from the zonal mean. This result was first derived by Andrews (1983) and shows that the vertical component of the EP flux equals the zonal component of the pressure torque, as introduced in Chapter 2, exerted by the fluid between two isentropes.

Despite the fact that the isentropic coordinate reveals the physical meaning of one of the terms of the eddy forcing of the zonal mean flow, the interpretation of the eddy momentum flux is still unclear.

Chapter 4: Potential-Vorticity, Potential -Temperature Coordinates

In Chapters 2 and 3 we reviewed the existing literature which covers the studies of the mean meridional circulation of the atmosphere, and identified two issues:

1. The mathematical representation of different aspects of the mean meridional circulation of the atmosphere depends to a large extent on the system of coordinates adopted in the analysis. Clearly, a zonal average system that avoids large cancellations between eddy and mean transport is most physically meaningful.

2. The greater insight into the mechanisms driving the mean meridional circulation of the atmosphere is achieved by using zonally averaged models that must include parameterizations of the eddy transports.

This thesis is aimed at answering the questions posed by the two issues in two parallel but inter-connected lines: one on finding a more useful system of coordinates, and the other to develop a model in the PVPT coordinates that allows a better interpretation of the forcing.

From previous studies it is obvious that the isentropic zonal average is the framework which satisfies most of the requirements addressed by the first issue, but the parameterization of the eddy transports leaves enough room for improvements.

When we asked what is the property of the potential temperature that makes it very distinct from the other quantities used as vertical coordinates, the answer is: its material conservativeness in the absence of heating. It quickly became apparent that one has to transform the isentropic system of coordinates into a system in which most of the coordinates have this nice property. Having decided on the potential vorticity as the meridional coordinate, one of the objectives of this thesis was born.

In the process of building up this thesis we learned about the qualitative work carried out by Obukhov (1964) who introduced the concept of "invariant" (potential temperature, Ertel potential vorticity) tubes which divide the atmosphere in regions into which the air masses have the same potential temperature and potential vorticity. For adiabatic and frictionless processes the walls of the tubes cannot be crossed by the air and the flow can be studied by following the vertical and horizontal displacements of the tubes.

To picture these tubes, it is useful to look at the cross-sections of potential temperature and potential vorticity. One example is shown in Fig. 4.1. Here we notice some regions where the potential vorticity is not a monotonic function with respect to latitude. We show in a later section how we treat such situations, which are usually associated with baroclinic instability or blocking events.

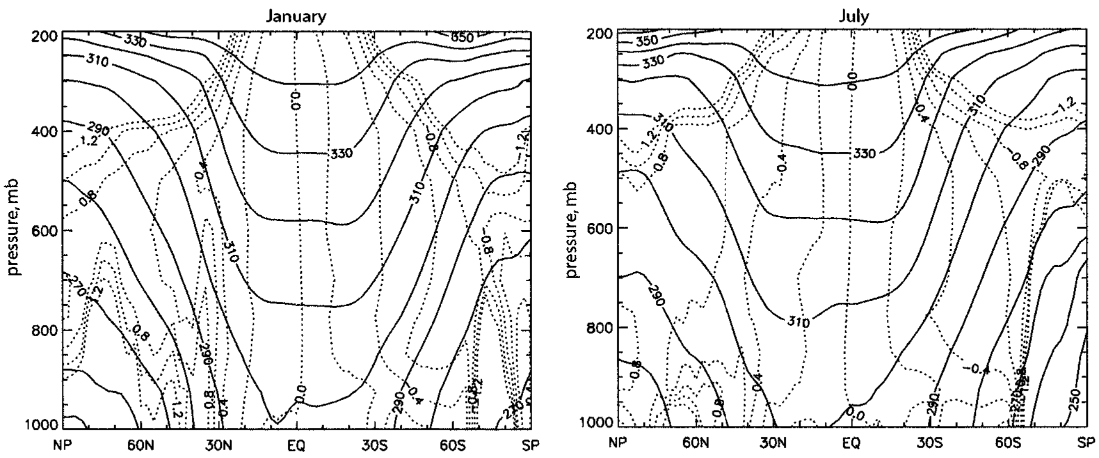

Figure 4.1: Distribution of zonal mean potential temperature and Ertel potential vorticity in the ECMWF ERA-40 reanalysis, for January and July. Solid lines represent the potential temperature with the contour interval 10K and the dotted lines represent the Ertel potential vorticity with the contour interval 0.2 PVU (1 PVU = $10^{-6}m^{-2}Ks^{-1}kg^{-1}$).

The work of McIntyre and Palmer (1983) and Hoskins et al. (1985) has revived interest in PV on isentropic surfaces, and PV is becoming a powerful diagnostic tool for studies of the general circulation and even for weather forecasting. Since the ingredients incorporated in the definition of PV carry most of the dynamical information of the flow, the PV has the ability to capture many aspects of the flow in a single scalar field (Schneider et al., 2003). Rhines and Young (1982) and Tung (1986) discuss some of the advantages of PV dynamics on isentropes over the quasi-geostrophic PV dynamics.

There are many properties of the potential vorticity and potential temperature fields that make them a useful combination for the study of the geophysical fluid dynamics. The most important is the fact that PV and PT are both conserved following an air parcel in adiabatic and frictionless flows, and therefore they can be used to define material surfaces. The advantage of using material surfaces was emphasized by Eliassen

(1962). He pointed out that material surfaces are convenient to use in modeling of the atmospheric flow since there is no mass exchange between model layers. The result of Eliassen was extended by Haynes and McIntyre (1987), whose results are known as the impermeability theorem. According to this theorem, "*whether or not diabatic heating and frictional or other forces are acting: i) there can be no net transport of Rossby-Ertel potential vorticity (PV) across any isentropic surface; ii) PV can neither be created nor destroyed, within a layer bounded by two isentropic surfaces*". The first statement above stems from the fact that the PV is not affected by the vertical advection, and by the sound waves (Danielsen, 1990), a very attractive property especially from the point of view of modeling.

Another advantage of using potential temperature as vertical coordinate is that in regions where the heating is small or zero, parcels move along isentropic surfaces that slope upwards towards the poles. The vertical motion is not zero in regions where diabatic physical processes are at work. The distribution of vertical velocity on an isentropic map shows the intensity of such physical processes.

Hoskins (1991) proposed a PV-θ view of the general circulation of the atmosphere. He replaced the classic layers of the atmosphere (e.g., troposphere, stratosphere, mesosphere, thermosphere) by three regions: the "Overworld", where $\theta > 380K$, the "Middleworld", where $300K < \theta < 380K$ and the "Underworld", where $\theta < 300K$. In the Overworld the isentropes reside above the tropopause, in the Middleworld the isentropes cross the tropopause and in the Underworld the isentropes intersect the ground. A schematic picture of this separation of the atmosphere is shown in Fig. 4.2.

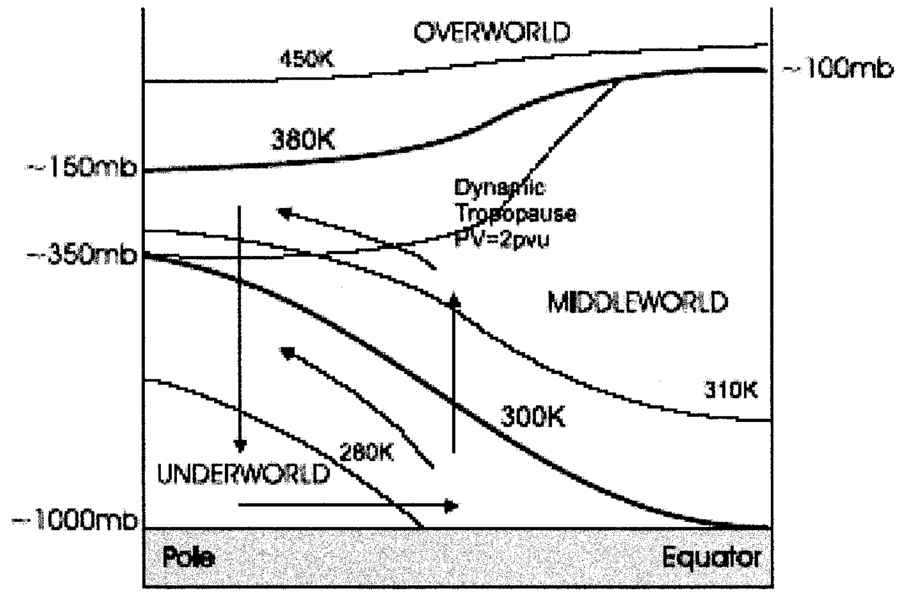

Figure 4.2: A schematic view of the atmosphere divided into the three regions: the Overworld, the Middleworld and the Underworld. Isentropes are shown as thick and thin solid lines. The dynamic tropopause is represented by a dashed line. Arrows represent the isentropic zonal average circulation after removing the Hadly cell. From Hoskins (1991).

Despite the fact that a PV-θ perspective of the general circulation of the atmosphere is envisioned as an adequate framework, and isentropic analysis of the general circulation has been extensively carried out, none of the previous investigators actually explored the possibility of using PV as a real coordinate. It is our main goal in this chapter to derive the primitive equations in a system of coordinates that consists of longitude, potential vorticity as meridional coordinate, and potential temperature as vertical coordinate.

Besides all the theoretical arguments, summarized above, that sustain PV as a good candidate for a meridional coordinate, we should also consider some observational viewpoints. For example in the latitude-potential temperature cross-section we can observe that the PV is generally positive in the Northern Hemisphere, negative in the Southern Hemisphere and vanishes near the Equator. The isolines of potential vorticity in a meridional cross-section are nearly vertical close to the Equator and throughout tropics (Fig. 4.1).

The 50-year climatology of potential vorticity on the isentropic surface $\theta = 350K$ presented in Fig. 4.3 for winter and summer shows that over long periods of time isolines of potential temperature are parallel to the latitudinal circles.

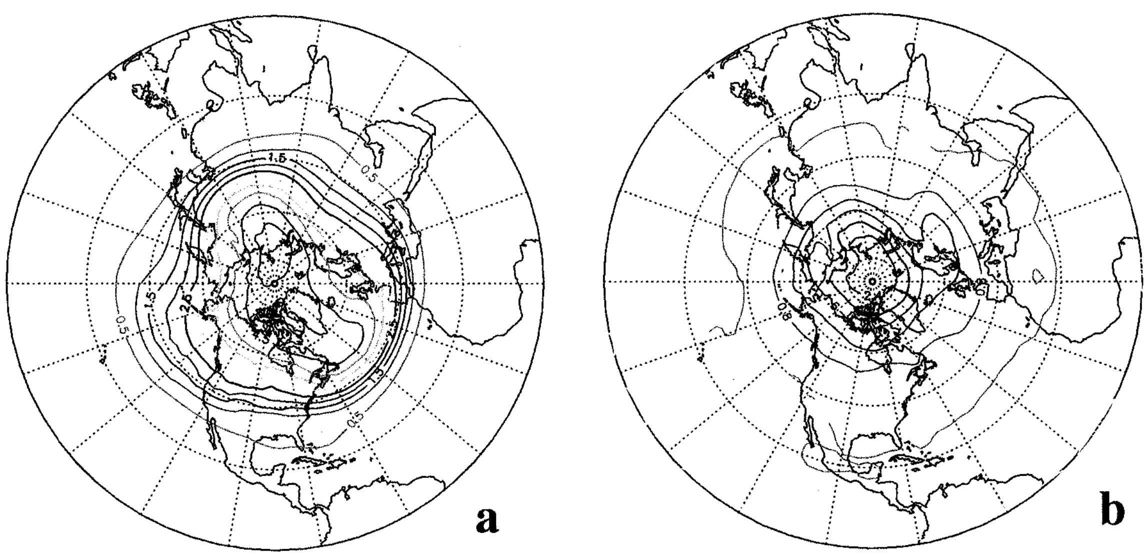

Figure 4.3: 50-year climatology of potential vorticity on the isentropic surface θ=315 K in the NCEP-NCAR reanalysis for a) winter and b) summer. The contour interval is 0.5 PVU (1 PVU = $10^{-6}m^2Ks^{-1}kg^{-1}$) for winter and 0.4 PVU for the summer.

In Fig. 4.3 we notice a discontinuity in the outermost contour corresponding to the summer. In these regions the isentropic surface $\theta = 350K$ intersects the ground, and the PV is not defined in the reanalysis data. We show later how to treat these situation in a model that uses PV as meridional coordinate.

Before deriving the governing equations in the PVPT system of coordinates, we review the basic equations in isentropic coordinates, which are the starting point for deriving the new ones. The primitive equations in isentropic coordinate will be presented in the next section, with an emphasis on the properties that make them useful for our purpose.

4.1: Starting equations

The primitive equations in isentropic coordinates, expressing momentum balance in the zonal and meridional directions, continuity of mass, thermodynamic relation between the Lagrangian change of potential vorticity and diabatic heating, $\dot{\theta}$, and hydrostatic balance in the vertical can be written as

$$\frac{DU}{Dt} - 2\Omega\mu V + \frac{1}{a}\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{\mu,\,\theta} = X_\lambda, \tag{4.1}$$

$$\frac{DV}{Dt} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{(1-\mu^2)}{a}\left(\frac{\partial \mathcal{M}}{\partial \mu}\right)_{\lambda,\,\theta} = X_\mu, \tag{4.2}$$

where X_λ and X_φ are the horizontal components of friction, or other non conservative mechanical forcing, and $K \equiv \frac{1}{2}\left(\frac{U^2 + V^2}{1 - \mu^2}\right)$ represents the horizontal kinetic energy per unit mass. Note that (U, V) are the modified components of the horizontal velocity

$$U = u\cos\varphi \text{ and } V = v\cos\varphi, \tag{4.3}$$

$$\mu \equiv \sin\varphi, \text{ where } -1 \le \mu \le 1. \tag{4.4}$$

The advantage of using the modified components of the horizontal velocity consists in removing the discontinuities in (u, v) over the poles and some of the metric terms implied by the spherical coordinates disappear.

$$\left(\frac{\partial\sigma}{\partial t}\right)_{\mu,\,\theta} + \frac{1}{a(1-\mu^2)}\left\{\frac{\partial}{\partial\lambda}(\sigma U)\right\}_{\mu,\,\theta} + \frac{1}{a}\left\{\frac{\partial(\sigma V)}{\partial\mu}\right\}_{\lambda,\,\theta} + \left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{\lambda,\,\mu} = 0, \tag{4.5}$$

$$\frac{D\theta}{Dt} = \dot{\theta} \tag{4.6}$$

$$\left(\frac{\partial\mathcal{M}}{\partial\theta}\right)_{\lambda,\,\mu} = \Pi, \tag{4.7}$$

where the quantity $\mathcal{M}$ is called the Montgomery stream function, defined by

$$\mathcal{M} = \theta\Pi(p) + gz, \tag{4.8}$$

and the quantity σ is the "density" in (λ, μ, θ) space and is defined by

$$\sigma = -\frac{1}{g}\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,\,\mu}. \tag{4.9}$$

The Exner function is

$$\Pi \equiv c_p\left(\frac{p}{p_0}\right)^{\kappa}. \tag{4.10}$$

The Lagrangian derivative following the components of the flow on an isentropes can be expressed by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\mu,\,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_{\mu,\,\theta} + \frac{V}{a}\left(\frac{\partial}{\partial \mu}\right)_{\lambda,\,\theta} + \dot{\theta}\left(\frac{\partial}{\partial \theta}\right)_{\lambda,\,\varphi}. \tag{4.11}$$

The absolute angular momentum per unit mass is

$$M \equiv aU + \Omega a^2(1-\mu^2). \tag{4.12}$$

Differentiation of the absolute angular momentum, as given by (4.12), yields

$$\begin{aligned}\frac{DM}{Dt} &= a\frac{DU}{Dt} - 2\Omega a^2\mu\frac{D\mu}{Dt} \\ &= a\frac{DU}{Dt} - 2\Omega a\mu V\end{aligned} \tag{4.13}$$

Using (4.13), the zonal momentum equation can be written as

$$\frac{DM}{Dt} + \left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{\mu} = X_{\lambda} \tag{4.14}$$

and represents the angular momentum principle according to which in the absence of friction the total angular momentum per unit mass is conserved.

Finally, the potential vorticity is

$$q \equiv \frac{1}{\sigma}\left\{2\Omega\mu + \frac{1}{a(1-\mu^2)}\left(\frac{\partial V}{\partial\lambda}\right)_{\mu,\,\theta} - \frac{1}{a}\left(\frac{\partial U}{\partial\mu}\right)_{\lambda,\,\theta}\right\} = \frac{1}{\sigma}(2\Omega\mu + \zeta_\theta). \tag{4.15}$$

The isentropic relative vorticity ζ_θ is the same as the component of the relative vorticity vector normal to the isentropic surface.

Using equations (4.1)-(4.2) we can derive a prognostic equation for the potential vorticity, similar to Haynes and McIntyre (1987)

$$\sigma\left\{\left(\frac{\partial q}{\partial t}\right)_{\mu,\,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial q}{\partial\lambda}\right)_{\mu,\,\theta} + \frac{V}{a}\left(\frac{\partial q}{\partial\mu}\right)_{\lambda,\,\theta}\right\} = q\frac{\partial}{\partial\theta}(\sigma\dot{\theta}) - \nabla\bullet\mathbf{J}_{\dot{\theta}} - \nabla\bullet\mathbf{J}_X, \tag{4.16}$$

where

$$\mathbf{J}_{\dot{\theta}} = \left\{\dot{\theta}\frac{\partial V}{\partial\theta}, -\dot{\theta}\frac{\partial U}{\partial\theta}, 0\right\}, \tag{4.17}$$

$$\mathbf{J}_X = \{-X_\lambda, X_\mu, 0\} \tag{4.18}$$

are the fluxes due to diabatic heating and friction in the momentum equation, respectively.

Equation (4.16) is the mathematical expression of the impermeability theorem discussed earlier in this chapter. The form of the conservation equation makes possible a distinct separation between the dynamical processes on one side and the diabatic and frictional processes on the other side.

If friction is in the form of molecular viscosity, then the last term on the right-hand side of (4.16) can be written as

$$-\nabla \bullet \mathbf{J}_X = \nabla^2 \zeta_\theta + \frac{2}{a^2} \zeta_\theta . \tag{4.19}$$

For a detailed derivation of (4.19) see Appendix A.

Hoskins et al. (1985) have shown that for large-scale processes (4.16) simplifies to

$$\left(\frac{\partial q}{\partial t}\right)_{\mu,\,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial q}{\partial \lambda}\right)_{\mu,\,\theta} + \frac{V}{a}\left(\frac{\partial q}{\partial \mu}\right)_{\lambda,\,\theta} + \dot{\theta}\left(\frac{\partial q}{\partial \theta}\right)_{\lambda,\,\mu} \approx q\left(\frac{\partial \dot{\theta}}{\partial \theta}\right)_{\lambda,\,\mu} . \tag{4.20}$$

In flux form (4.16) becomes

$$\frac{\partial}{\partial t}(\sigma q) + \nabla \bullet \mathbf{J} = 0 , \tag{4.21}$$

which shows that the "PV-substance" is transported as any chemical substance on an isentropic surface. $\mathbf{J}$ is the flux of "PV-substance" on isentropic surfaces, given by

$$\mathbf{J} = \zeta_\theta \mathbf{V} + \mathbf{J}_{\dot{\theta}} + \mathbf{J}_X , \tag{4.22}$$

where $\mathbf{V} = (U, V, 0)$ and $\mathbf{J}_{\dot{\theta}}$, $\mathbf{J}_X$ were defined above. The concept of "PV-substance" was introduced by Haynes and McIntyre (1990), who interpreted PV as the mixing ratio of the "PV-substance".

We prefer to rewrite (4.16) in a more compact form, for reasons that will become evident starting in the next section

$$\left(\frac{\partial q}{\partial t}\right)_{\mu} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial q}{\partial \lambda}\right)_{\mu,\,\theta} + \frac{V}{a}\left(\frac{\partial q}{\partial \mu}\right)_{\lambda,\,\theta} + \dot{\theta}\left(\frac{\partial q}{\partial \theta}\right)_{\lambda,\,\mu} = \dot{q}, \tag{4.23}$$

where

$$\dot{q} = \frac{1}{\sigma}\left\{\frac{\partial}{\partial \theta}(\dot{\theta}\sigma q) - \nabla \bullet \mathbf{J}_{\dot{\theta}} - \nabla \bullet \mathbf{J}_X\right\}. \tag{4.24}$$

It is obvious that, in the absence of diabatic and frictional sources $\dot{q} = 0$, a result known as the conservation of PV. In an adiabatic and inviscid flow the PV following a parcel of fluid is materially conserved.

To describe the state of the atmosphere in isentropic coordinates only three prognostic equations are necessary: (4.1), (4.2) and (4.5). In these equations the horizontal velocity components (U, V) have to be initialized for each isentropic surface. Diabatic heating $\dot{\theta}$ and friction terms are specified through parameterizations. The horizontal components of the pressure gradient force in the momentum equations are not initialized variables. The Montgomery stream function is calculated by integrating the hydrostatic equation (4.7), therefore the pressure on each isentropic surface has to be determined in order to calculate the value of the Exner function given by (4.10). By knowing the pressure, the pseudo-density is known from (4.9), too. The continuity equation predicts the pseudo-density of each layer and then by integrating (4.9) the value of pressure corresponding to each isentropic layer is calculated. Any other quantity that describes other properties of the flow can be diagnosed using these predicted variables.

4.2: Transforming to PVPT coordinates

4.2.1: General Considerations

In Chapter 2 we introduced the property of the PVPT system of coordinates to divide the atmosphere into invariant flux tubes, along which air parcels travel during adiabatic processes. These tubes are bounded by the potential vorticity surfaces as undulating lateral-sides, and undulating bottom and top isentropes. Kurgansky (2002) in his book "*Adiabatic invariants in large-scale atmospheric dynamics*" has shown that in order to characterize the orientation of the line resulting from the intersection of the isentropic surfaces with the surfaces of constant PV it is useful to define a vector $\mathbf{B} = \nabla q \times \nabla \theta$, which is tangent to each point along the intersection line. This section reviews Chapter 4 of Kurgansky's book which serves as a primary reference for the physical understanding of the PVPT framework. We shall repeatedly refer to this book as K02 in this section.

For an adiabatic and frictionless flow, the time variation of vector $\mathbf{B}$ is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{V}\times\mathbf{B}), \tag{4.25}$$

and using the vector identity $\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\bullet\mathbf{b}) - \mathbf{b}(\nabla\bullet\mathbf{a}) + (\mathbf{b}\bullet\nabla)\mathbf{a} - (\mathbf{a}\bullet\nabla)\mathbf{b}$, equation (4.25) becomes,

$$\frac{D\mathbf{B}}{Dt} - (\mathbf{B}\bullet\nabla)\mathbf{V} + \mathbf{B}(\nabla\bullet\mathbf{V}) = 0. \tag{4.26}$$

A derivation of (4.25) is reproduced from K02 in Appendix B.

Equation (4.26) represents the analytical expression of the Friedmann theorem (K02) according to which the necessary and sufficient condition for the conservation of both vector lines and vector tubes of an arbitrary field $\mathbf{B}(\mathbf{x}, t)$ is that (4.26) be satisfied.

Using the transformation relationship derived by Ertel (1960, Appendix B), that allows to express the material derivative as a spatial derivative, K02 found another governing equation for the vector $\mathbf{B}$:

$$\frac{D\mathbf{B}}{Dt} - (\mathbf{B}\bullet\nabla)\mathbf{V} + \mathbf{B}(\nabla\bullet\mathbf{V}) = \nabla\times(\dot{q}\nabla\theta - \dot{\theta}\nabla q). \tag{4.27}$$

The derivation of (4.27) is given in Appendix B. In order to satisfy the Friedmann theorem the right-hand side of (4.26) must be the gradient of a scalar,

$$\dot{q}\nabla\theta - \dot{\theta}\nabla q = \nabla\psi. \tag{4.28}$$

This implies that the tubes bounded by the potential vorticity and isentropic surfaces remain invariant under non-adiabatic and friction processes. We will use this relationship to define the streamfunction of the mean meridional circulation in PVPT coordinates later in Chapter 5.

4.2.2: *Basic equations*

Considering q as a meridional coordinate, we adopt the Kushner and Held (1999) definition of the "potential-vorticity thickness"

$$h = \left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\,\theta}, \tag{4.29}$$

and assume that h is positive and finite. For an arbitrary A, we can write

$$\left(\frac{\partial A}{\partial t}\right)_{\mu,\,\theta} = \left(\frac{\partial A}{\partial t}\right)_{q,\,\theta} - \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{t}\left(\frac{\partial \mu}{\partial t}\right)_{q,\,\theta}, \tag{4.30}$$

$$\left(\frac{\partial A}{\partial \lambda}\right)_{\mu,\,\theta} = \left(\frac{\partial A}{\partial \lambda}\right)_{q,\,\theta} - \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\lambda,\,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\,\theta}, \tag{4.31}$$

$$\left(\frac{\partial A}{\partial \theta}\right)_{\lambda,\,\mu} = \left(\frac{\partial A}{\partial \theta}\right)_{\lambda,\,q} - \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\lambda,\,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,\,q}, \tag{4.32}$$

$$\left(\frac{\partial A}{\partial \mu}\right)_{\lambda,\,\theta} = \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\lambda,\,\theta}. \tag{4.33}$$

Note that $\left(\frac{\partial A}{\partial \theta}\right)_{\lambda,\, q}$, which appears in (4.32), is not a "true" vertical derivative, because the isosurfaces of q can be tilted with respect to the true vertical.

In order to remain orthogonal we choose the unit vectors of the new system of coordinates to remain parallel to the units vectors of the $(\lambda, \varphi, \theta)$ system of coordinates, as schematically shown in Fig. 4.4.

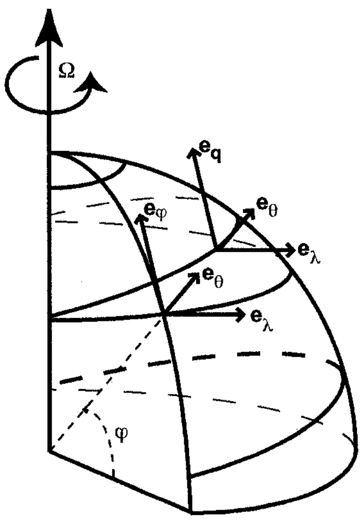

Figure 4.4: Illustration of the unit vectors of the spherical system of coordinates $(e_\lambda, e_\varphi, e_\theta)$ (in black) and of the PVPT system of coordinates $(e_\lambda, e_q, e_\theta)$ (in blue). The blue circles represents contours of constant PV.

We can use (4.30)-(4.33) to express the pseudo-density as

$$\sigma = -\frac{1}{g}\left\{\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,\, q} - \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\, \theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,\, q}\right\}, \tag{4.34}$$

and the potential vorticity as

$$ah\sigma q = 2\Omega a\mu h + \frac{1}{1-\mu^2}\left\{h\left(\frac{\partial V}{\partial\lambda}\right)_{q,\theta} - \left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda,\theta}$$

$$= a2\Omega\mu\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta} + \frac{1}{1-\mu^2}\left\{\left(\frac{\partial V}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta} - \left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda,\theta}. \quad (4.35)$$

$$= a\Omega\left(\frac{\partial\mu^2}{\partial q}\right)_{\lambda,\theta} + \frac{1}{1-\mu^2}\left\{\left(\frac{\partial V}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta} - \left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda,\theta}$$

Applying (4.30)-(4.33) to the potential vorticity equation, (4.23), we obtain:

$$\left\{\left(\frac{\partial q}{\partial t}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial q}{\partial q}\right)_t\left(\frac{\partial\mu}{\partial t}\right)_{q,\theta}\right\} + \frac{U}{a(1-\mu^2)}\left\{\left(\frac{\partial q}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial q}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}$$
$$+ \frac{V}{a}\left(\frac{\partial q}{\partial\mu}\right)_{\lambda,\theta} + \dot\theta\left\{\left(\frac{\partial q}{\partial\theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial q}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\} = \dot q \quad (4.36)$$

$$\left\{-\frac{1}{h}\left(\frac{\partial\mu}{\partial t}\right)_{q,\theta}\right\} + \frac{U}{a(1-\mu^2)}\left\{-\frac{1}{h}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + \frac{V1}{ah} + \dot\theta\left\{-\frac{1}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\} = \dot q, \quad (4.37)$$

Multiplying (4.37) by h and rearranging terms we can write

$$\left(\frac{\partial\mu}{\partial t}\right)_{q,\theta} + \frac{1}{a}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + \dot q\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta} + \dot\theta\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q} = \frac{V}{a}, \quad (4.38)$$

which represents the equation of the geographical latitude of a PV contour, that is related to the PV through the potential-vorticity thickness

$$\mu = \int h\,dq. \quad (4.39)$$

From (4.38) we can observe that in the PVPT system of coordinates V is the velocity of a fluid parcel moving along a surface of constant PV, or also can be interpreted as the velocity of a fluid parcel that conserves its PV moving on an isentropic surface. A similar formula would apply for any conservative variable. In this derivation we also obtained the Lagrangian derivative in PVPT system of coordinates as

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_q + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_q + \dot{q}\left(\frac{\partial}{\partial q}\right) + \dot{\theta}\left(\frac{\partial}{\partial \theta}\right)_q . \tag{4.40}$$

Next, we transform the hydrostatic equation

$$\left(\frac{\partial \mathcal{M}}{\partial \theta}\right)_{\lambda,\varphi} = \Pi = \left(\frac{\partial \mathcal{M}}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} \tag{4.41}$$

and use (4.8), so that we can write

$$\begin{aligned} \Pi &= \Pi + \theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} - \frac{g}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} \\ 0 &= \theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} - \frac{g}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} \end{aligned} \tag{4.42}$$

We can arrive at the same form of the hydrostatic equation if we start from

$$\theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,\mu} + g\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,\mu} = 0 \tag{4.43}$$

and transform it to PVPT coordinates using (4.32).

Next, we transform the continuity equation:

$$\left(\frac{\partial\sigma}{\partial t}\right)_{\mu,\,\theta} + \frac{1}{a(1-\mu^2)}\left\{\frac{\partial}{\partial\lambda}(\sigma U)\right\}_{\mu,\,\theta} + \frac{1}{a}\frac{\partial(\sigma V)}{\partial\mu} + \frac{\partial}{\partial\theta}(\sigma\dot{\theta}) = 0 \tag{4.44}$$

In (λ, μ, θ) the latitude μ and longitude λ are independent coordinates, therefore we can rearrange the second term as

$$\left(\frac{\partial\sigma}{\partial t}\right)_{\mu,\,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_{\mu,\,\theta} + \frac{1}{a}\frac{\partial(\sigma V)}{\partial\mu} + \frac{\partial}{\partial\theta}(\sigma\dot{\theta}) = 0 \tag{4.45}$$

Using (4.30)-(4.33) in (4.45) this transforms to PVPT coordinates

$$\begin{aligned} &\left\{\left(\frac{\partial\sigma}{\partial t}\right)_{q,\,\theta} - \frac{1}{h}\left(\frac{\partial\sigma}{\partial q}\right)\left(\frac{\partial\mu}{\partial t}\right)_{q,\,\theta}\right\} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_{q,\,\theta} - \frac{1}{ah}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}\left\{\frac{\partial}{\partial q}\left(\frac{\sigma U}{1-\mu^2}\right)\right\} \\ &+ \frac{1}{ah}\frac{\partial(\sigma V)}{\partial q} + \left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{q,\,\theta} - \frac{1}{ah}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\left\{\frac{\partial}{\partial q}(\sigma\dot{\theta})\right\} = 0 \end{aligned} \quad . \tag{4.46}$$

Use (4.38) to eliminate $\left(\frac{\partial\mu}{\partial t}\right)_{q,\,\theta}$ in (4.46)

$$
\begin{aligned}
&\left(\frac{\partial\sigma}{\partial t}\right)_{q,\,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_{q,\,\theta} \underline{- \frac{1}{ah}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}\left\{\frac{\partial}{\partial q}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}} + \overline{\frac{1}{ah}\frac{\partial(\sigma V)}{\partial q}} \\
&\quad + \left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{q,\,\theta} \underbrace{- \frac{1}{ah}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\left\{\frac{\partial}{\partial q}(\sigma\dot{\theta})\right\}} \\
&\quad + \frac{1}{h}\left(\frac{\partial\sigma}{\partial q}\right)\left\{\underline{\frac{1}{a}\frac{U}{(1-\mu^2)}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}} + \dot{q}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta} + \underbrace{\dot{\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}} - \overline{\frac{V}{a}}\right\} = 0
\end{aligned}
\tag{4.47}
$$

Combine the underlined terms as follows,

$$
\begin{aligned}
&- \frac{1}{ah}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}\left\{\frac{\partial}{\partial q}\left(\frac{\sigma U}{1-\mu^2}\right)\right\} + \frac{1}{h}\left(\frac{\partial\sigma}{\partial q}\right)\frac{1}{a}\frac{U}{(1-\mu^2)}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta} \\
&\quad = -\frac{\sigma}{ah}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}\frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right)
\end{aligned}
\tag{4.48}
$$

$$
\frac{1}{ah}\frac{\partial(\sigma V)}{\partial q} - \frac{1}{h}\left(\frac{\partial\sigma}{\partial q}\right)\frac{V}{a} = \frac{\sigma}{ah}\left(\frac{\partial V}{\partial q}\right)_{\lambda,\,\theta}
\tag{4.49}
$$

$$
-\frac{1}{ah}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\left\{\frac{\partial}{\partial q}(\sigma\dot{\theta})\right\} + \frac{1}{h}\left(\frac{\partial\sigma}{\partial q}\right)\dot{\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q} = -\frac{\sigma}{ah}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\left(\frac{\partial}{\partial q}\dot{\theta}\right)_{\lambda,\,\theta}
\tag{4.50}
$$

Using (4.48)-(4.50) we can rewrite (4.47) as

$$
\begin{aligned}
&\left(\frac{\partial\sigma}{\partial t}\right)_{q,\,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_{q,\,\theta} + \dot{q}\left(\frac{\partial\sigma}{\partial q}\right)_{\lambda,\,\theta} + \left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{q,\,\theta} \\
&\quad - \frac{\sigma}{ah}\left\{\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}\frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right) + \left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\left(\frac{\partial}{\partial q}\dot{\theta}\right)_{\lambda,\,\theta} - \left(\frac{\partial V}{\partial q}\right)_{\theta}\right\}.
\end{aligned}
\tag{4.51}
$$

By differentiating (4.38) with respect to q, we find that

$$\left(\frac{\partial h}{\partial t}\right)_{q,\theta} + \frac{1}{a}\frac{\partial}{\partial q}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} + \dot{q}h + \dot{\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\} = \frac{1}{a}\left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta} \quad . \tag{4.52}$$

Now we eliminate $\left(\frac{\partial V}{\partial q}\right)_{\theta}$ in (4.51) by substituting from (4.52):

$$\begin{aligned}
&\left(\frac{\partial \sigma}{\partial t}\right)_q + \underline{\frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_q} + \dot{q}\left(\frac{\partial \sigma}{\partial q}\right)_{\lambda,\theta} + \overline{\left\{\frac{\partial}{\partial \theta}(\sigma\dot{\theta})\right\}_{q,\theta}} \\
&\quad \underline{-\frac{\sigma}{ah}\left(\frac{\partial \mu}{\partial \lambda}\right)_q \frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right)} \overline{-\frac{\sigma}{ah}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial}{\partial q}\dot{\theta}\right)_{\lambda,\theta}} \\
&\quad + \frac{\sigma}{h}\left(\left(\frac{\partial h}{\partial t}\right)_q + \frac{1}{a}\frac{\partial}{\partial q}\left[\underline{\left(\frac{\partial \mu}{\partial \lambda}\right)_q \frac{U}{(1-\mu^2)}} + \dot{q}h + \overline{\dot{\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}}\right]\right) = 0
\end{aligned} \tag{4.53}$$

Again combine the underlined terms to obtain

$$\begin{aligned}
&\frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_q - \frac{\sigma}{ah}\left(\frac{\partial \mu}{\partial \lambda}\right)_q \frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right) + \frac{\sigma}{ah}\frac{\partial}{\partial q}\left\{\left(\frac{\partial \mu}{\partial \lambda}\right)_q \frac{U}{(1-\mu^2)}\right\} \\
&\quad = \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_q + \frac{\sigma}{ah}\frac{U}{1-\mu^2}\left(\frac{\partial h}{\partial \lambda}\right)_{q,\theta}
\end{aligned} \tag{4.54}$$

$$\left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{q,\theta} - \frac{\sigma}{ah}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial}{\partial q}\dot{\theta}\right)_{\lambda,\theta} + \frac{\sigma}{ah}\frac{\partial}{\partial q}\left\{\dot{\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\}$$
$$= \left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{q,\theta} + \frac{\sigma\dot{\theta}}{ah}\left(\frac{\partial h}{\partial\theta}\right)_{\lambda,q} . \tag{4.55}$$

Using (4.54) and (4.55) we can write (4.53) as

$$\left(\frac{\partial\sigma}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma U}{1-\mu^2}\right)\right\}_q + \frac{\sigma}{ah}\frac{U}{1-\mu^2}\left(\frac{\partial h}{\partial\lambda}\right)_{q,\theta} + \dot{q}\left(\frac{\partial\sigma}{\partial q}\right)_{\lambda,\theta}$$
$$\frac{\sigma}{h}\frac{\partial}{\partial q}(\dot{q}h) + \left\{\frac{\partial}{\partial\theta}(\sigma\dot{\theta})\right\}_{q,\theta} + \frac{\sigma\dot{\theta}}{ah}\left(\frac{\partial h}{\partial\theta}\right)_{\lambda,q} + \frac{\sigma}{h}\left(\frac{\partial h}{\partial t}\right)_q = 0 . \tag{4.56}$$

We finally obtain the continuity equation after multiplying (4.56) by h and combining terms using the chain rule:

$$\left\{\frac{\partial}{\partial t}(\sigma h)\right\}_{q,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma h U}{1-\mu^2}\right)\right\}_{q,\theta} + \left\{\frac{\partial}{\partial q}(\sigma h\dot{q})\right\}_{\lambda,\theta} + \left\{\frac{\partial}{\partial\theta}(\sigma h\dot{\theta})\right\}_{\lambda,q} = 0. \tag{4.57}$$

Equation (4.57) represents the mass conservation principle in the PVPT system of coordinates. Despite its similarity to the continuity equation in the (λ, μ, θ) system of coordinates, equation (4.57) reduces to a simpler form in the case of a diabatic and frictionless flow, when compared to the continuity equation for an adiabatic and frictionless flow in the (λ, μ, θ) system of coordinates. The physical interpretation of the mass conservation for an adiabatic and inviscid fluid is easy to see in the schematic representation in Fig. 4.5. The air flows only in the longitudinal direction through a volume bounded by the potential vorticity surfaces as lateral sides, and bottom and top isentropes.

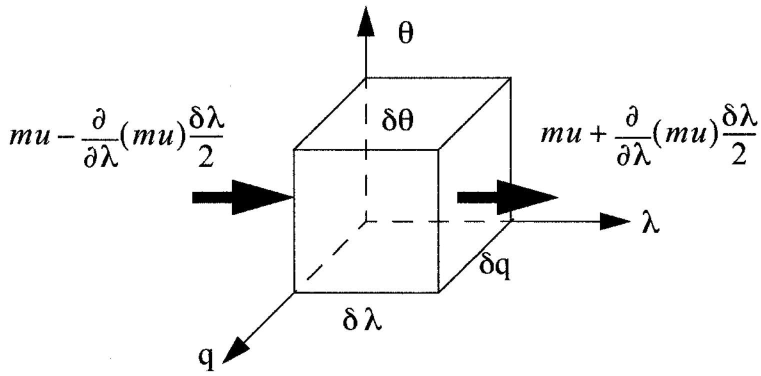

Figure 4.5: Mass inflow into a fixed (Eulerian) control volume; $m = \sigma h$

Examples of the σh-distribution are presented in Fig. 4.6. and Fig. 4.7 where the vertical axis is the potential temperature and the horizontal axis is the normalized PV. The effect of normalization is to eliminate the strong dependence of PV on altitude, which arises mainly due to the pseudo-density in the denominator. As we expect the air is concentrated in the lower part of the atmosphere and seasonal changes are clearly seen. More air resides at lower potential temperatures in the winter hemisphere compared to the summer hemisphere. The maximum air quantity corresponds to a warmer isentropic surface in the Northern Hemisphere compared to the Southern Hemisphere.

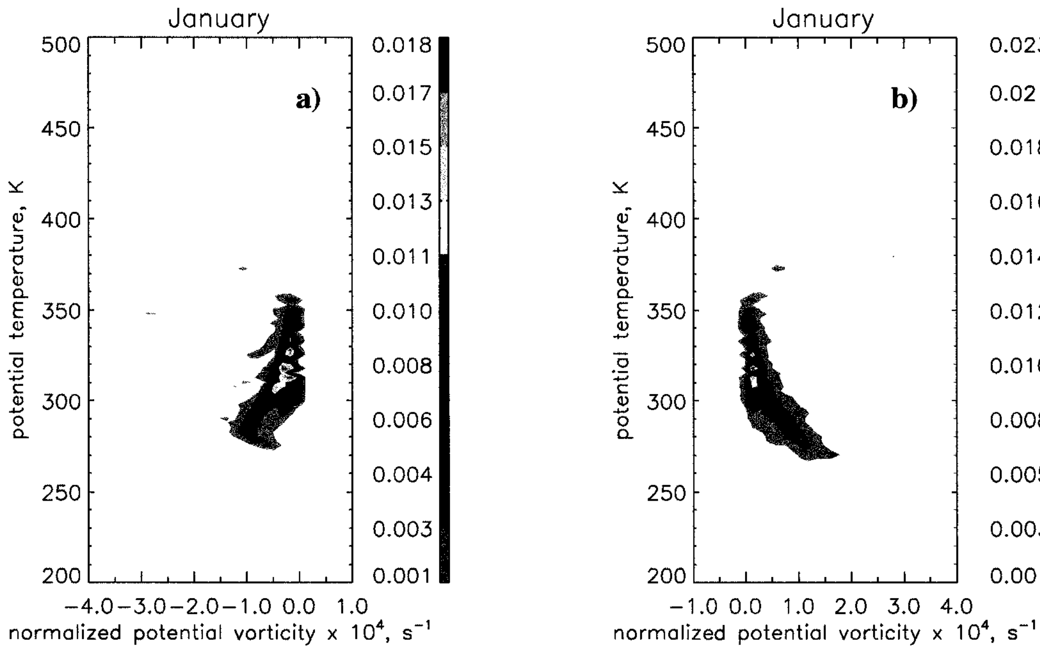

Figure 4.6: Mass distribution in PVPT coordinates. Southern Hemisphere (a) and Northern Hemisphere (b) in NCEP-NCAR reanalysis, for January.

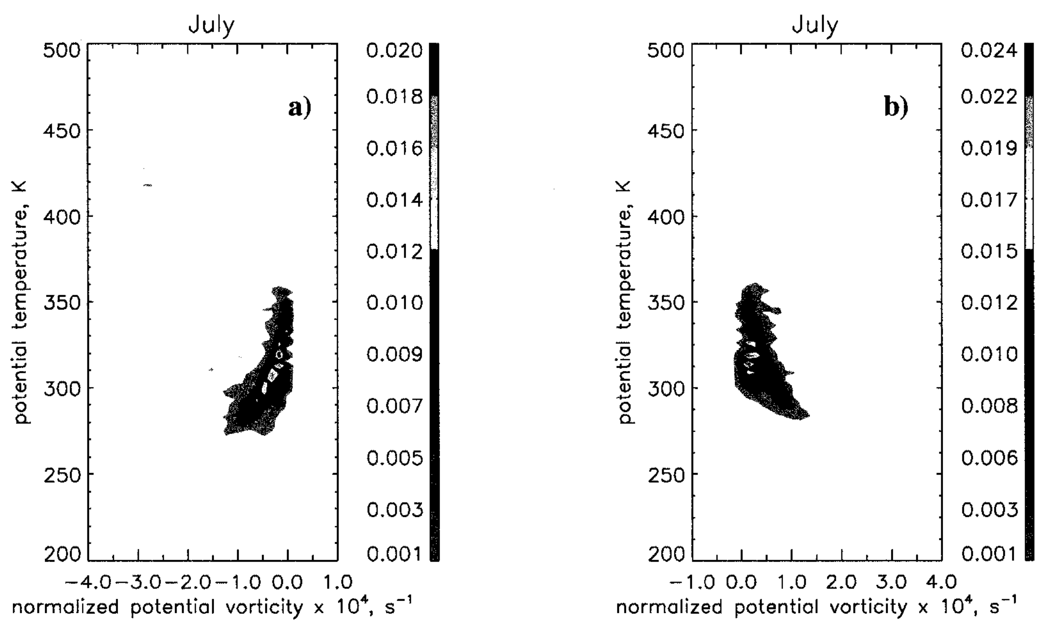

Figure 4.7: Mass distribution in PVPT coordinates. Southern Hemisphere (a) and Northern Hemisphere (b) in NCEP-NCAR reanalysis, for July.

The angular momentum equation can be transformed to

$$\left(\frac{\partial M}{\partial t}\right)_{\lambda, q}+\frac{U}{a(1-\mu^2)}\left(\frac{\partial M}{\partial \lambda}\right)_{q, \theta}+\dot{q}\left(\frac{\partial M}{\partial q}\right)_{\lambda, \theta}+\dot{\theta}\left(\frac{\partial M}{\partial \theta}\right)_{\lambda, q}$$
$$+\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q, \theta}-\frac{1}{h}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda, \theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q, \theta}=X_{\lambda} \quad . \qquad (4.58)$$

Convert (4.58) to flux form, using the continuity equation (4.57)

$$\left\{\frac{\partial}{\partial t}(\sigma h M)\right\}_{q, \theta}+\frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma h U M}{1-\mu^2}\right)\right\}_{q, \theta}+\left\{\frac{\partial}{\partial q}(\sigma h \dot{q} M)\right\}_{\lambda, \theta}+\left\{\frac{\partial}{\partial \theta}(\sigma h \dot{\theta} M)\right\}_{\lambda, q}$$
$$+\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q, \theta}-\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda, \theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q, \theta}=\sigma h X_{\lambda} \quad . \qquad (4.59)$$

The advective form of the zonal momentum equation can be transformed to

$$\left(\frac{\partial U}{\partial t}\right)_{\lambda, q}+\frac{U}{a(1-\mu^2)}\left(\frac{\partial U}{\partial \lambda}\right)_{q, \theta}+\dot{q}\left(\frac{\partial U}{\partial q}\right)_{\lambda, \theta}+\dot{\theta}\left(\frac{\partial U}{\partial \theta}\right)_{\lambda, q}-2\Omega\mu V$$
$$+\frac{1}{a}\left\{\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q, \theta}-\frac{1}{h}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda, \theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q, \theta}\right\}=X_{\lambda} \quad . \qquad (4.60)$$

The flux form is

$$\left\{\frac{\partial}{\partial t}(\sigma h U)\right\}_{q, \theta}+\frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma h U U}{1-\mu^2}\right)\right\}_{q, \theta}+\left\{\frac{\partial}{\partial q}(\sigma h \dot{q} U)\right\}_{\lambda, \theta}+\left\{\frac{\partial}{\partial \theta}(\sigma h \dot{\theta} U)\right\}_{\lambda, q}$$
$$-2\Omega\mu V+\frac{1}{a}\left\{\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q, \theta}-\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda, \theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q, \theta}\right\}=\sigma h X_{\lambda} \; . \qquad (4.61)$$

Similarly, the meridional momentum equation can be transformed to

$$\left(\frac{\partial V}{\partial t}\right)_{\lambda, q, \theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_{q, \theta} + \dot{q}\left(\frac{\partial V}{\partial q}\right)_{\lambda, \theta} + \dot{\theta}\left(\frac{\partial V}{\partial \theta}\right)_{\lambda, q} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{g(1-\mu^2)}{ah}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda, \theta} = X_q, \tag{4.62}$$

Convert (4.62) to flux form, using the continuity equation (4.57):

$$\left\{\frac{\partial}{\partial t}(\sigma h V)\right\}_{q, \theta} + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma h U V}{1-\mu^2}\right)\right\}_{q, \theta} + \left\{\frac{\partial}{\partial q}(\sigma h \dot{q} V)\right\}_{\lambda, \theta} + \left\{\frac{\partial}{\partial \theta}(\sigma h \dot{\theta} U)\right\}_{\lambda, q} + 2\mu\sigma h\left(\Omega U + \frac{K}{a}\right) + \frac{(1-\mu^2)\sigma}{a}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda, \theta} = \sigma h X F_q \tag{4.63}$$

The governing equations derived in this section were obtained working under the assumption that PV varies monotonically with the latitude and potential temperature is a monotonic function of height. In nature folding of a PV surface or an isentropic surface may exist, but in the model these are not allowed and these aspects are treated in the next two sections.

4.2.3: Convective instability

A necessary condition for any quantity to be used as a surrogate coordinate is that it varies monotonically with the "original" coordinate. For example, in the case of hydrostatic models in pressure coordinates, $\left(\frac{\partial p}{\partial z}\right)_{\lambda, \mu} < 0$ is always guaranteed by the hydrostatic approximation $\left(\frac{\partial p}{\partial z}\right)_{\lambda,\,\mu} = -\rho g$. Another condition that a hydrostatic model has to satisfy is to produce statically stable solutions, i.e. $\frac{\partial \theta}{\partial z} > 0$. In pressure coordinates, the thermodynamic equation predicts the potential temperature,

$$\frac{\partial \theta}{\partial t} = -\omega \frac{\partial \theta}{\partial p}. \tag{4.64}$$

In equation (4.64) we neglect the horizontal advection of potential temperature and heating as irrelevant for this discussion, and $\omega = Dp/Dt$ is the vertical p-velocity. In a hydrostatic atmosphere, the static stability is positive ($-(\partial\theta/\partial p) > 0$), and according to equation (4.64) rising air ($\omega < 0$) is cooling and sinking air ($\omega > 0$) is warming, and so the atmosphere is statically stable. If $\partial\theta/\partial p$ becomes positive, then sinking air is cooling and rising air is warming, and so the atmosphere is unstable, and also the numerical model might become unstable. In order to prevent the occurrence of such a convective instability, a mixing process is assumed to adjust the potential temperature such that the mass-weighted temperature is unchanged, and $\partial\theta/\partial p$ becomes zero. This assumed mixing process is also known as dry convective adjustment (Manabe et al., 1965). Requiring the condition $\partial\theta/\partial p < 0$ also means that the pseudodensity in θ-coordinates remains non-negative, a necessary physical condition.

In θ-coordinates, the dry convective adjustment is a built-in process (Hsu and Arakawa, 1990) because the thermodynamic equation is also the mass continuity equation.

Although we still call the process dry convective adjustment, in isentropic coordinates we cannot assume a mixing of potential temperature, because parcels with different potential temperatures cannot mix in this framework. In the isentropic coordinates the mass is predicted, and we have to guarantee that our model produces positive values for this quantity. The definition of pseudo-density in the isentropic coordinates can be also interpreted as the inverse of the static stability, and therefore the static stability of the model is automatically guaranteed by the this constraint imposed on the mass. In this case pressure is diagnosed as

$$p(\theta + \Delta\theta) = p(\theta) + \sigma\Delta\theta. \tag{4.65}$$

To illustrate how the dry convective adjustment restores the static stability in both pressure and isentropic coordinates, we performed two simplified numerical experiments. For the pressure coordinates we consider that the only source term in the thermodynamic equation is the prescribed diabatic heating shown in Fig. 4.8 a),

$$\left(\frac{\partial\theta}{\partial t}\right)_p = \dot{\theta}, \tag{4.66}$$

and for the isentropic coordinates the only forcing term in the thermodynamic equation is the vertical advection of mass by the diabatic motion,

$$\left(\frac{\partial\sigma}{\partial t}\right)_\theta = -\frac{\partial}{\partial\theta}(\sigma\dot{\theta}). \tag{4.67}$$

In both cases we assume the initial profile $\theta(p)$ is stable, and integrate the models for few time steps: a) without imposing any adjustment and b) with the dry convective adjustment imposed on the potential temperature predicted by eq. (4.66) and preventing the mass predicted by eq. (4.67) from becoming negative.

In Fig. 4.9 we present the results obtained using pressure and potential temperature as vertical coordinates corresponding to numerical experiment a). In both cases the potential temperature profile becomes unstable in response to the action of the diabatic heating. Although in both models at the initial time the right-hand sides have the same pattern, the effect of diabatic heating in isentropic coordinates is more dramatic than in isobaric coordinates. This is a consequence of the form of the governing equations that we integrate in each case. In pressure coordinates we integrate a first-order ordinary differential equation, whereas in isentropic coordinates we integrate an advection equation. In the first case the potential temperature θ changes only in response to changes in the source/sink term. In the advection equation, the pseudodensity σ changes in response to mass fluxes that flow into the isentropic layer.

The oscillating pattern observed in the pressure profile obtained from integration using isentropic coordinates (Fig. 4.9 b) is due to a violation of physical laws. The given profile of the diabatic heating yields to formation of negative mass, which further makes the pressure to oscillate and, therefore the layer where this occurs experiences convective instability.

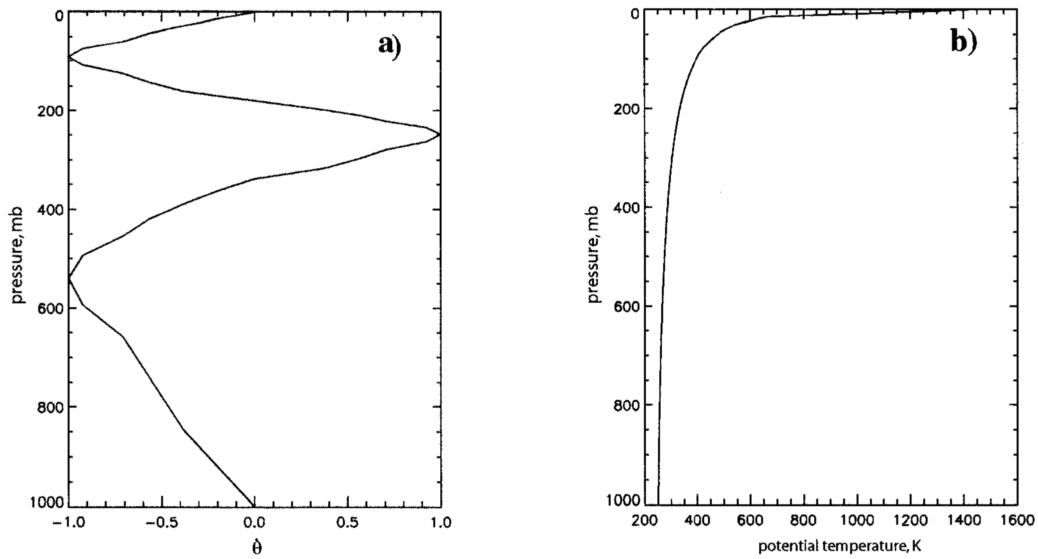

Figure 4.8: a) The vertical profile of the prescribed diabatic heating. b) The vertical profile of the potential temperature at t=0.

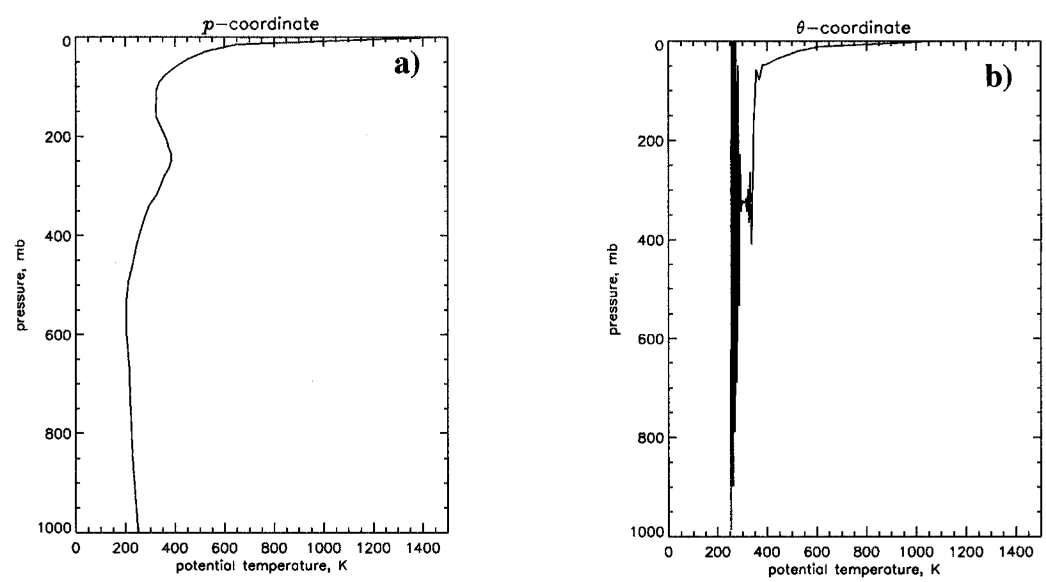

Figure 4.9: The vertical profile of potential temperature as obtained using pressure a) and potential temperature b) as vertical coordinate, without the adjustment

To observe the onset of convective instability we plot in Fig. 4.10 time series of the static stability in an unstable layer, corresponding the two systems of coordinates. In pressure coordinates $-(\partial\theta/\partial p)$ becomes negative which is equivalent to occurrence of convective instability in the model. In isentropic coordinates $-(\partial p/(\partial\theta))$ increases and eventually becomes infinity.

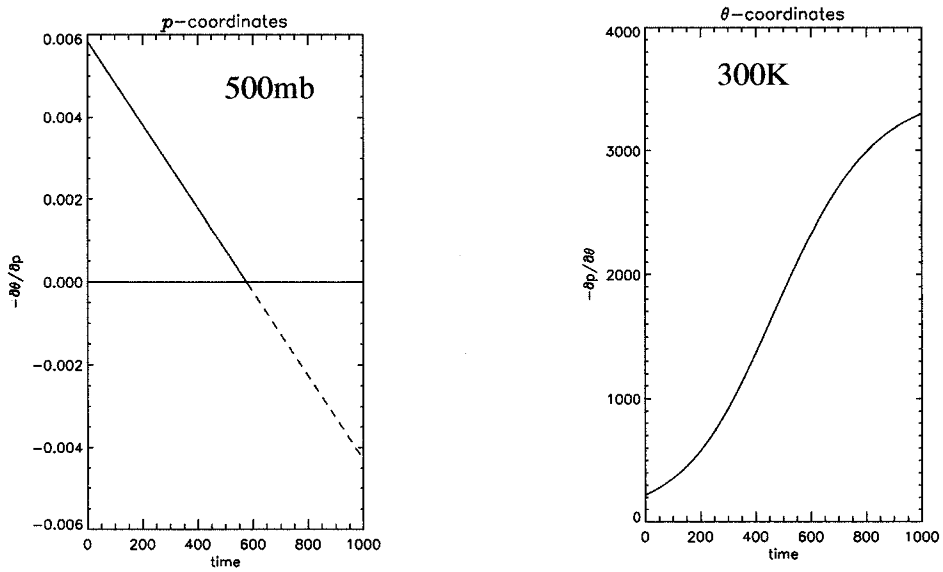

Figure 4.10: Time series of the static stability parameter as obtained using pressure a) and potential temperature b) as vertical coordinate.

In Fig. 4.11 we present the results obtained using pressure and potential temperature as vertical coordinate corresponding to the numerical experiment b). In both cases the profiles are statically stable and they look similar. In p-coordinates we artificially created a layer with neutral stability by mixing the mass-weighted temperature, but in θ-coordinates this layer appears just by preventing the model from producing negative mass. The method used to accomplish this requirement is the so called hole-filling scheme, which is based on redistributing mass from all the points in the domain to the points where the numerical scheme produces negative mass. This is the demonstration that the dry convective adjustment acts indirectly in an isentropic model.

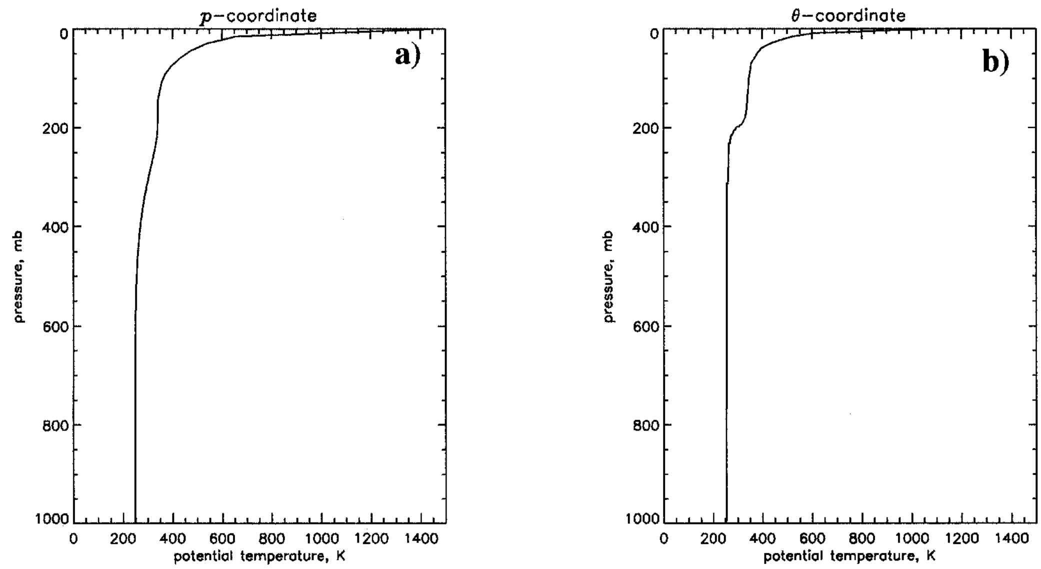

Figure 4.11: The vertical profile of potential temperature as obtained using pressure a) and potential temperature b) as vertical coordinate, with the adjustment

4.2.4: Baroclinic instability

Charney and Stern (1962) showed that small amplitude disturbances superimposed upon a horizontally uniform current cannot grow by extracting available potential energy from the mean flow if the meridional component of the potential vorticity gradient has the same sign everywhere in the fluid. This statement is known as the Charney-Stern theorem (e.g., Pedlosky, 1989 Chapter 4, pg. 490), and represents the necessary condition for the combined barotropic-baroclinic instability. The reader is referred to Eliassen (1983) for a discussion review of the Charney-Stern theorem in isentropic coordinates.

Using potential vorticity as a substitute for latitude, the former has to vary monotonically with respect to latitude. In nature there are situations when this condition is not satisfied and the potential vorticity gradient changes sign. An example of the folding of PV surfaces is presented in Fig. 4.12, which shows maps of the PV on the isentropic surface $\theta = 315K$, for three consecutive days in January, 2000. The domain chosen extends from 20N to 80N and from 140W to 140E. To see the evolution of the PV gradient during these days, in Fig. 4.13 the time series of the PV gradient at a point centered at 45N and 180 is shown. By inspecting Fig. 4.13 we observe an event, marked with the blue oval, that persists for about 3 days. During this event the PV gradient starts from a positive value, that is considered a "normal" pattern, and becomes negative for about 3 days, after which it returns back to its "normal" sign.

Based on the Charney-Stern theorem we assume that the violations of the monotonicity of PV are accompanied by development of baroclinic waves, which homogenize the PV, and in a similar manner to the dry convective adjustment a baroclinic adjustment process prevents the folding of PV surfaces.

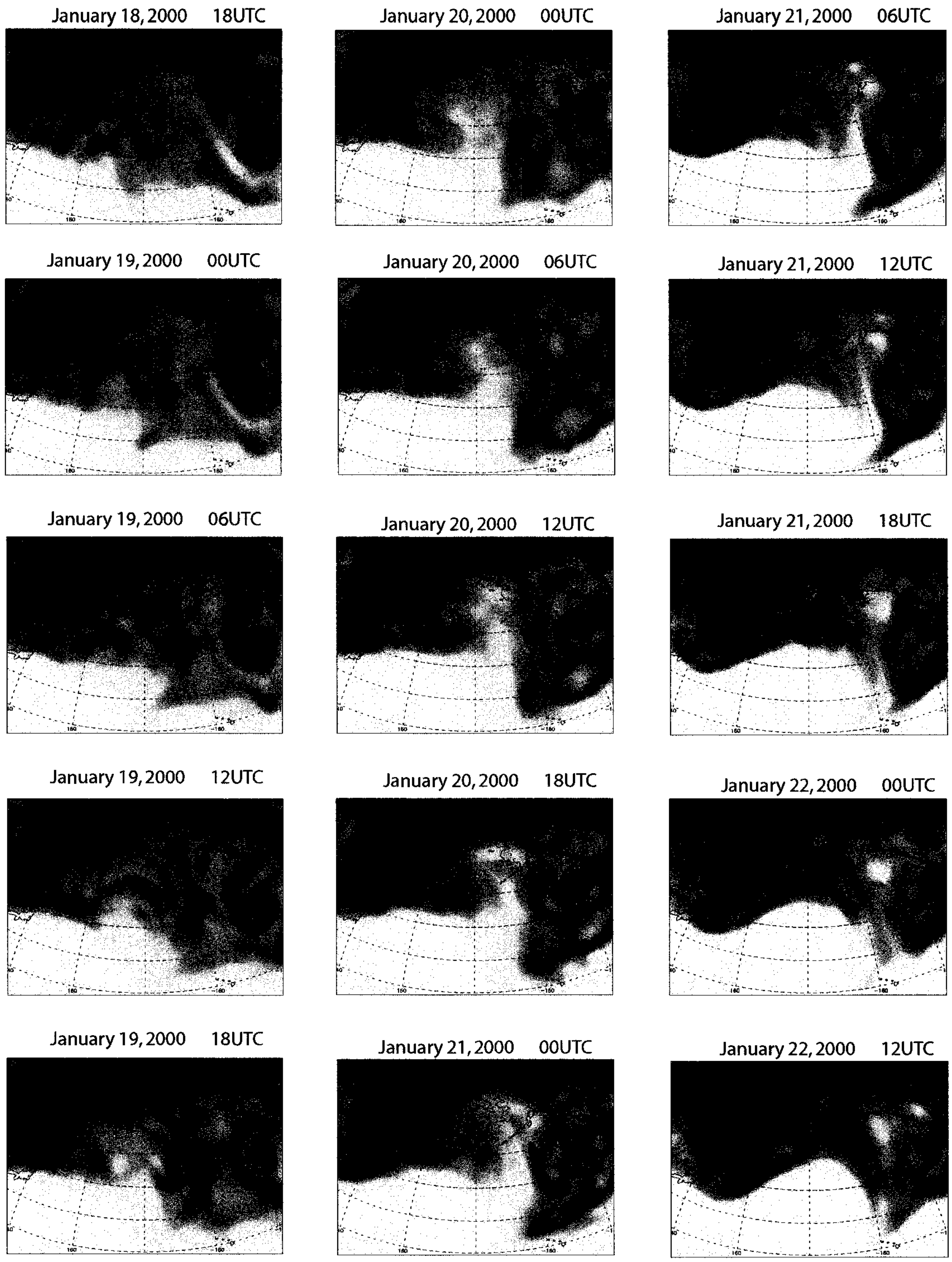

Figure 4.12: PV on the isentropic surface $\theta = 350K$ in the ERA-40 reanalysis. The range of values expands between 0PVU (white) and 8 PVU (dark red).

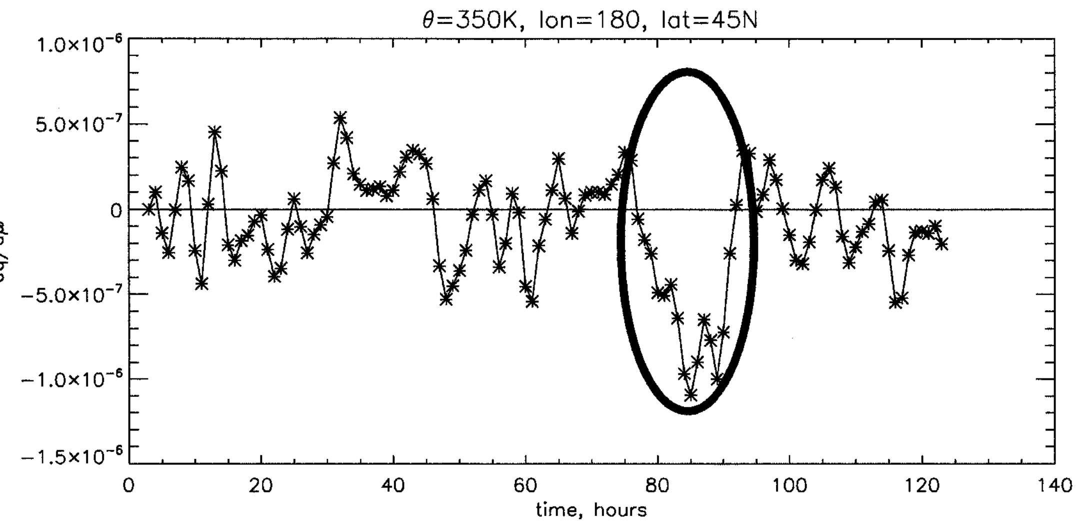

Figure 4.13: Time series of the PV gradient in 6 hours daily ERA-40 reanalysis for January.

As an evidence that foldings of the PV surfaces are not persistent events and they are removed through the baroclinic adjustments in Fig. 4.14 and Fig. 4.15 we show regions where the reversal of the PV gradient persists for more than three consecutive days. The maps correspond to the isentropic surface $350K$ during the winter and summer for a period spanning five years, 1996-2000. To quantify the occurrence of these reversals for each season, we also plot the fraction of this events over the latitude belts. The fractions are defined as the the total number of the points in a latitude belt where the PV gradient is reversed for three consecutive days during a month divided by the total number of points in that latitude belt, multiplied by the total number of days in that month. The pictures show a distinct minimum in midlatitudes for both hemispheres, except for January 1996 in the Norther Hemisphere.

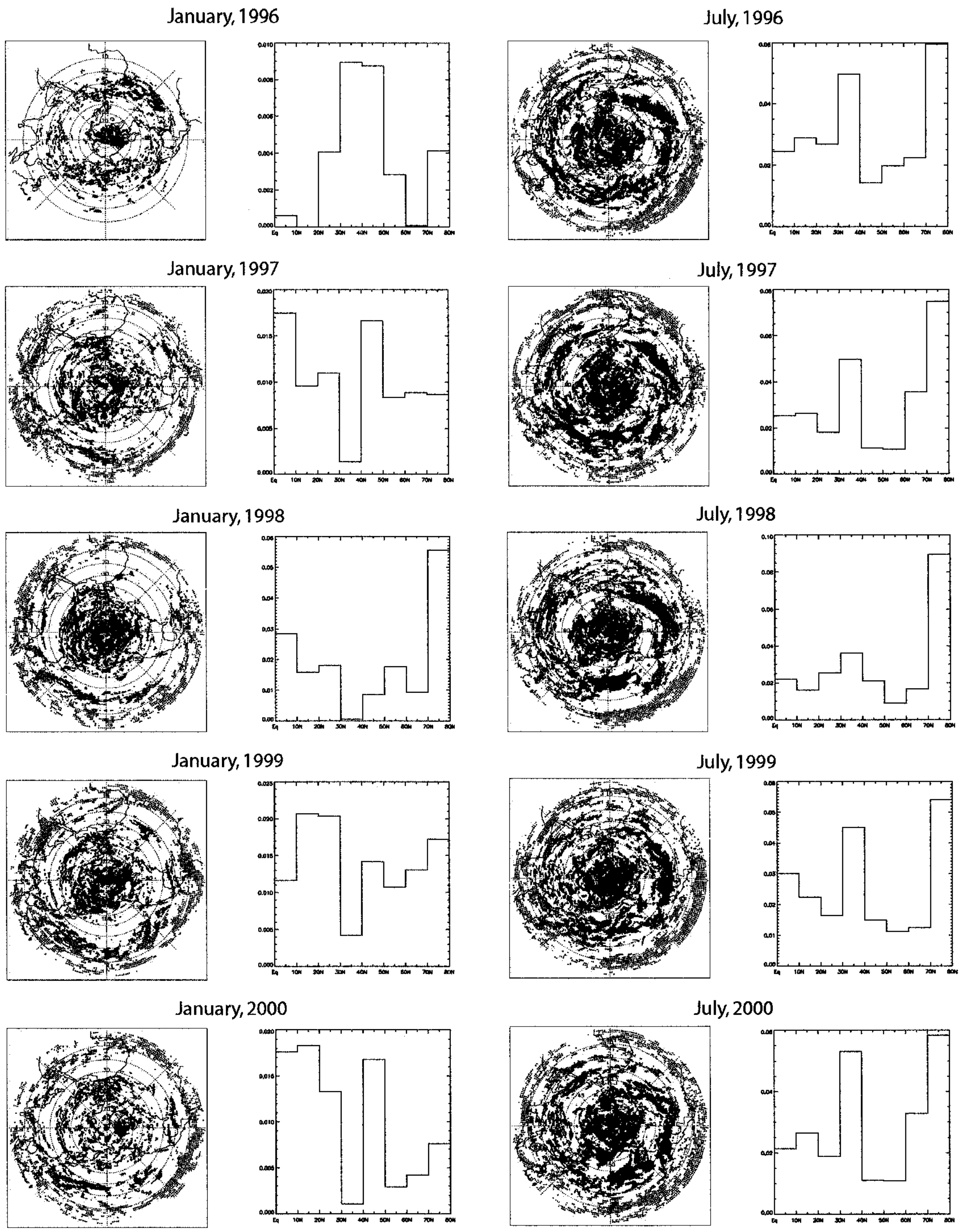

Figure 4.14: Norther Hemisphere statistical distribution of regions where the PV gradient on the isentropic surface $\theta = 350K$ in the ERA-40 reanalysis is negative for more than three consecutive days.

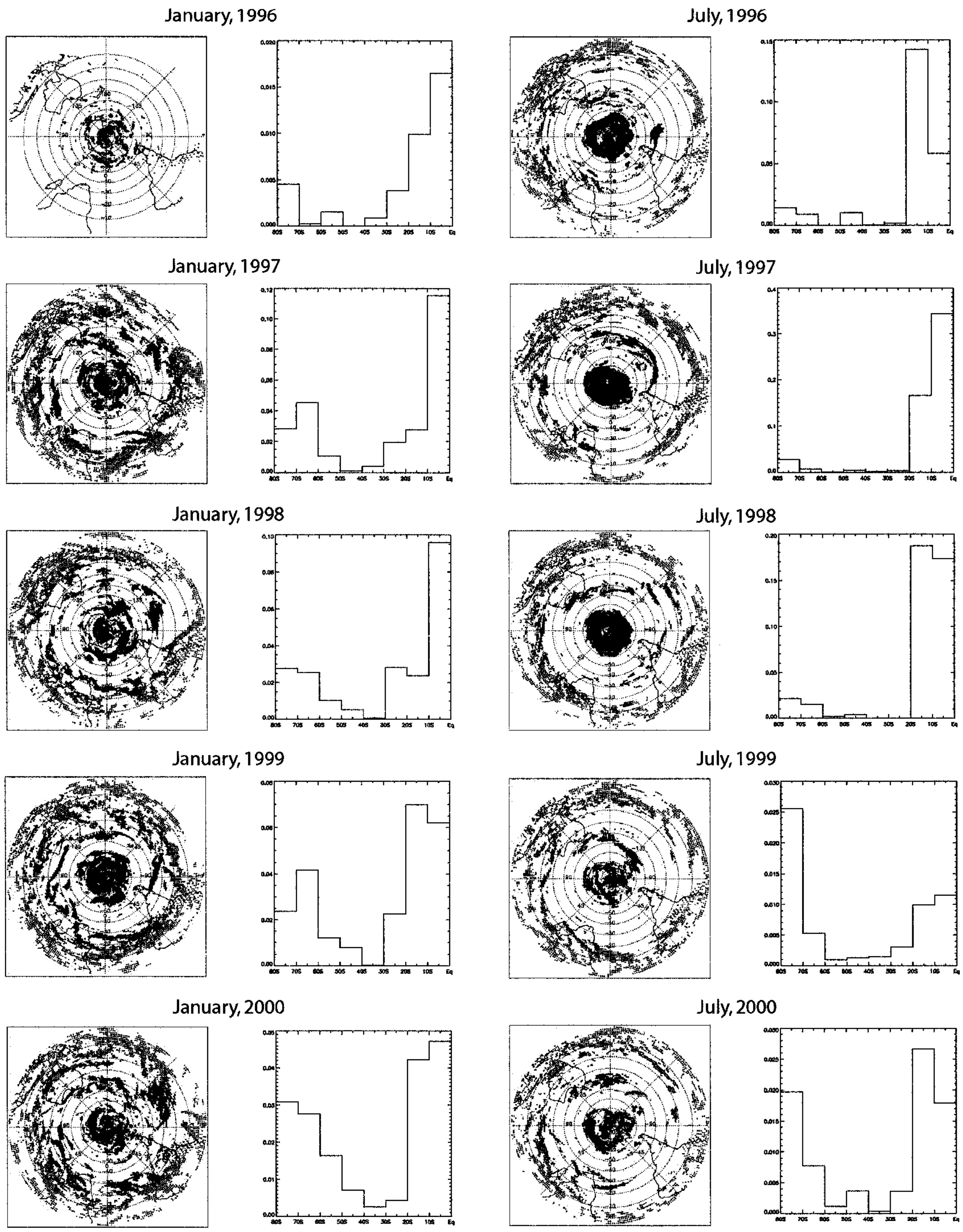

Figure 4.15: Southern Hemisphere statistical distribution of regions where the PV gradient on the isentropic surface $\theta = 350K$ in the ERA-40 reanalysis is negative for more than three consecutive days.

In a model that uses PV as meridional coordinate the instability of the model is suppressed by the fact that the continuity equation in PVPT coordinates predicts the inverse of gradient of PV,

$$\left(\frac{\partial h}{\partial t}\right)_{q,\,\theta} = -\left\{\frac{\partial}{\partial q}(h\dot{q})\right\}_{\lambda,\,\theta}. \tag{4.68}$$

Eq. (4.68) represents a modified version of the continuity equation (4.57), in which σ was omitted on purpose, since the built in dry convective adjustment process prevents formation of static instability.

By preventing h from becoming negative, for the simple reason that it represents mass in the PVPT coordinates, we guarantee that PV is a monotonic function of latitude. This constraint on the PV gradient removes the baroclinic instabilities but incorporates its effects through $\dot{q}$ which acts as a source/sink responsible for this process. Note the similarity between equation (4.68) and the continuity equation in θ-coordinates.

For a better understanding of the similarities between dry convective adjustment and baroclinic adjustment we show in Fig. 4.16a schematic representation of the evolution of static stability and PV gradient, in a longitude, pressure coordinates model and in an PV, isentropic coordinates model, respectively. From this sketch we infer that baroclinic adjustment occurs "automatically" in a model that uses PV as a meridional coordinate, just as convective adjustment occurs automatically in a model that uses potential temperature as vertical coordinate.

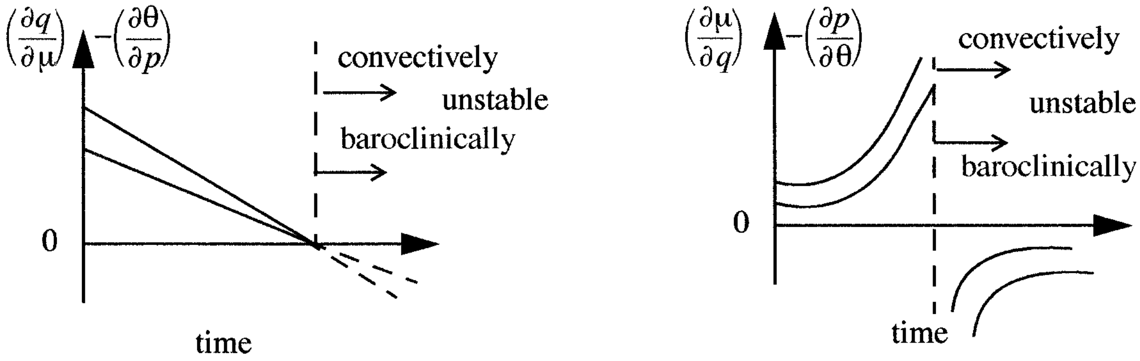

Figure 4.16: Schematic representation of convective /baroclinic instability in a latitude, pressure coordinates model and in a PVPT coordinates model.

Although there are some analogies between the dry convective and baroclinic adjustments, they differ in the space and time scales on which they act. The dry convective adjustment is parameterized as a very rapid adjustment process in which the static instability is removed by vertical mixing on scales of ~15 km (Lilly and Kennedy, 1973). The space and time scale of the baroclic adjustment are on the order thousand of kilometers and 3-5 days (e.g., Simmons and Hoskins, 1978). The baroclinic problem is also more complex and no instability condition is known that is both necessary and sufficient.

The baroclinic adjustment has been described by Stone (1978) as a process in which baroclinic eddies tend to adjust the atmosphere to a state that is neutral to the further growth of eddies. The baroclinic waves can grow in time when the temperature gradient is large, which means the eddy flux is weak because the waves have just started to form. As the waves grow, they produce strong eddy fluxes which in turn reduce the temperature gradient and the waves do not have the available potential energy necessary for further growth.

Gutowski (1985) has shown that the adjustment could also be accomplished by vertical heat fluxes that arise from baroclinic instability. These fluxes tend to increase the static stability, and thus raise the critical vertical shear needed for instability.

James and Gray (1986) have proposed an alternative adjustment called “the barotropic governor mechanism”, in which baroclinic waves stabilize the mean state not by adjustment of the mean temperature field by eddy heat fluxes, but by eddy momentum flux which creates strong meridional shears. The strong meridional shears confine baroclinic waves to narrow meridional bands and substantially reduce linear growth rates. Surface friction can also interfere with this adjustment and reduce wind speeds near the surface, so barotropic stabilization may be affected by the addition of surface drag.

4.3: Summary and discussion

The combined use of isentropic and potential vorticity coordinates is essential to make both the vertical and meridional advection implicit in a diabatic frictionless fluid, but there are also some disadvantages implied by both potential temperature and potential vorticity.

At the Earth’s surface θ is a function of latitude and this makes difficult to define the bottom boundary condition. As proposed by Lorenz (1955) we can make the lower boundary isentropic by adding at the lower boundary an infinite thin layer which has no mass and infinite potential vorticity, the so called mass-less layers.

One disadvantage of the potential vorticity coordinates is that along the surface layers the potential vorticity fluctuates due to boundary effects, and so is difficult to satisfy the necessary conditions for PV to be used as a coordinate. In order to avoid this inconvenience we adopt the definition of the surface potential vorticity proposed by Schneider et al. (2003)

$$q_s = \frac{f+\zeta_s}{\sigma}\theta\delta(\theta-\theta_s), \tag{4.69}$$

where ζ_s denotes the relative vorticity along the surface. Under no-slip boundary conditions $\zeta_s = 0$ and the potential vorticity along the surface layer reduces to

$$q_s = \frac{f}{\sigma}\theta\delta(\theta-\theta_s). \tag{4.70}$$

According to (4.70) the potential vorticity at the surface can be approximated by the potential temperature at the surface. Equation (4.70) is used to calculate the PV at the surface in the model we employ to study the mean meridional circulation in PVPT coordinates, and it is described in Chapter 5.

To summarize, the transformed prognostic equations in PVPT coordinates are

$$\left\{\frac{\partial}{\partial t}(\sigma h)\right\}_{q,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma h U}{1-\mu^2}\right)\right\}_{q,\theta} + \left\{\frac{\partial}{\partial q}(\sigma h\dot{q})\right\}_{\lambda,\theta} + \left\{\frac{\partial}{\partial\theta}(\sigma h\dot{\theta})\right\}_{\lambda,q} = 0, \tag{4.71}$$

$$\left(\frac{\partial U}{\partial t}\right)_{\lambda,q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial U}{\partial \lambda}\right)_{q,\theta} + \dot{q}\left(\frac{\partial U}{\partial q}\right)_{\lambda,\theta} + \dot{\theta}\left(\frac{\partial U}{\partial \theta}\right)_{\lambda,q} - 2\Omega\mu V + \frac{1}{a}\left\{\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right\} = X_\lambda \tag{4.72}$$

$$\left(\frac{\partial V}{\partial t}\right)_{\lambda,q,\theta} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_{q,\theta} + \dot{q}\left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta} + \dot{\theta}\left(\frac{\partial V}{\partial \theta}\right)_{\lambda,q} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{g(1-\mu^2)}{ah}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta} = X_q, \tag{4.73}$$

$$\left\{\frac{\partial}{\partial t}(\sigma h M)\right\}_{q,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{\sigma h U M}{1-\mu^2}\right)\right\}_{q,\theta} + \left\{\frac{\partial}{\partial q}(\sigma h \dot{q} M)\right\}_{\lambda,\theta} + \left\{\frac{\partial}{\partial \theta}(\sigma h \dot{\theta} M)\right\}_{\lambda,q} + \sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} - \sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} = \sigma h X_\lambda \tag{4.74}$$

$$\left(\frac{\partial \mu}{\partial t}\right)_{q,\theta} + \frac{1}{a}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right\} + \dot{q}\left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\theta} + \dot{\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} = \frac{V}{a}, \tag{4.75}$$

$$\left(\frac{\partial h}{\partial t}\right)_{q,\theta} + \frac{1}{a}\frac{\partial}{\partial q}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta} + \dot{q}\left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\theta} + \dot{\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\} = \frac{1}{a}\left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta}. \tag{4.76}$$

We also have the diagnostic relationships

$$0 = \theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} - \frac{g}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}, \tag{4.77}$$

or the equivalent formulation in terms of the Montgomery streamfunction

$$\left(\frac{\partial \mathcal{M}}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} = \Pi, \tag{4.78}$$

the definition of the angular momentum

$$M \equiv aU + \Omega a^2(1-\mu^2), \tag{4.79}$$

the expression of the PV using PV as meridional coordinate

$$a\sigma hq = a\Omega\left(\frac{\partial \mu^2}{\partial q}\right)_{\lambda,\theta} + \frac{1}{(1-\mu^2)}\left\{\left(\frac{\partial V}{\partial \lambda}\right)_{q,\theta}\left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\theta} - \left(\frac{\partial V}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\theta}\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda,\theta}, \tag{4.80}$$

and the definition of the Lagrangian time rate of change of the PV

$$\dot{q} = \frac{1}{\sigma}\left\{\frac{\partial}{\partial \theta}(\dot{\theta}\sigma q) - \nabla\bullet\mathbf{J}_{\dot{\theta}} - \nabla\bullet\mathbf{J}_X\right\}. \tag{4.81}$$

In this set of equations the vertical advection is determined by the diabatic heating and the meridional advection by the Lagrangian time rate of change in PV. To describe the state of the atmosphere the necessary prognostic equations are: the continuity equation (4.71), the horizontal momentum equations (4.72)-(4.73) and the latitude of a PV contour equation (4.75). The variables that must to be initialized are: the horizontal components of the velocity, (U, V), the geographical latitude μ and the pressure. These variables must be specified for each isentropic layer on a longitude, PV grid. The diabatic heating $\dot{\theta}$ and the friction terms (X_λ, X_q) can be specified through parameterizations. Since geographical latitude is known, the PV thickness h given by (4.29) is known. The next step is to calculate the pseudo-density σ as given in (4.34) and the Exner function Π. The Montgomery streamfunction necessary to calculate the horizontal components of the pressure gradient force is calculated by integrating the hydrostatic equation (4.78). The $\dot{q}$ can be calculated according to (4.81) if friction and heating are specified. At the next time step the continuity equation predicts σh, which further gives σ, and by integrating (4.34) we obtain the value of the pressure necessary to continue the integration.

We have derived the shallow water model using PV as meridional coordinate and obtained similar results. The shallow water model is presented in Appendix C.

Chapter 5: Interactions between Eddies and the Mean Flow as Seen in PVPT Coordinates

5.1: Zonally averaged equations

In the preceding chapter, a primitive system of equations in the PVPT framework has emerged, which is able to predict the mass-weighted circulation in a rigorous manner. At this point we put the theory to test and use it to investigate the mean meridional circulation in the PVPT formalism.

We introduce the zonal average on an isentropic surface similar to the Eulerian mean described in Chapter 2, as

$$[A] = \frac{1}{2\pi}\int_0^{2\pi} A d\lambda. \tag{5.1}$$

The departure from the zonal mean will be denoted by a star

$$A^* = A - [A], \tag{5.2}$$

and the following identities will be used herein

$$\begin{aligned} \left[\frac{\partial A}{\partial \mu}\right] &= \frac{\partial}{\partial \mu}[A] & \left[\frac{\partial A}{\partial \theta}\right] &= \frac{\partial}{\partial \theta}[A] \\ \left[\frac{\partial A}{\partial t}\right] &= \frac{\partial}{\partial t}[A] & \frac{\partial}{\partial \lambda}[A] &= 0 \end{aligned} \quad . \tag{5.3}$$

We shall repeatedly make use of the mass-weighted average ($\hat{}$) defined in Chapter 2.

5.1.1: The residual circulation

The residual circulation $\left(\hat{\dot{q}}, \hat{\dot{\theta}}\right)$ in the PVPT coordinates becomes very simple:

1) the meridional mean flow is given by

$$\hat{\dot{q}} = \frac{[\sigma\dot{q}]}{[\sigma]}, \tag{5.4}$$

where the expression for $\dot{q}$ has been already introduced in Section 4.1. and is repeated below in an extended form,

$$\dot{q} = \frac{1}{\sigma}\left\{\frac{\partial}{\partial\theta}(\dot{\theta}\sigma q) - \frac{1}{a(1-\mu^2)}\frac{\partial}{\partial\lambda}\left(\dot{\theta}\frac{\partial V}{\partial\theta}\right) + \frac{1}{a}\frac{\partial}{\partial\mu}\left(\dot{\theta}\frac{\partial U}{\partial\theta}\right) + \frac{1}{a(1-\mu^2)}\frac{\partial X_\lambda}{\partial\lambda} - \frac{1}{a}\frac{\partial X_\mu}{\partial\mu}\right\}. \tag{5.5}$$

Zonally averaging (5.5) gives us

$$[\sigma\dot{q}] = \frac{1}{a}\frac{\partial}{\partial\mu}\left[\dot{\theta}\frac{\partial U}{\partial\theta} - X_\lambda\right] + \frac{\partial}{\partial\theta}[\dot{\theta}\sigma q] = \nabla \bullet \mathbf{J}_q, \tag{5.6}$$

where $\mathbf{J}_q = \left(0, \left[\dot{\theta}\frac{\partial U}{\partial \theta} - X_\lambda\right], [\sigma\dot{\theta}q]\right)$ is the eddy PV flux across isentropes. According to Haynes and McIntyre's formulation (1987), the meridional component of $\mathbf{J}_q$ represents the diabatic and frictional PV flux along isentropic surfaces whereas the vertical component represents the cross-isentropic advective flux of PV.

Combining (5.4) and (5.6), the meridional mean flow becomes,

$$\hat{\dot{q}} = \frac{\nabla \bullet \mathbf{J}_q}{[\sigma]}. \tag{5.7}$$

2) the vertical mean flow is given by the mean diabatic heating,

$$\hat{\dot{\theta}} = \hat{Q}. \tag{5.8}$$

Collecting the results of 1) and 2) we have the proof that in the PVPT coordinates the mean meridional circulation is directly driven by forcings in the form of mean diabatic heating and the divergence of eddy PV flux, which arises in part from the mean turbulent drag. This result agrees with the classic result of TEM described in Chapter 2, but was obtained without using the quasi-geostrophic approximation.

These two equations (5.4) and (5.8) determine the residual circulation in the meridional plane, which predicts the mean pseudo-density through the the zonally averaged mass continuity equation,

$$\left\{\frac{\partial}{\partial t}[\sigma h]\right\}_{q,\,\theta} + \left\{\frac{\partial}{\partial q}\hat{\dot{q}}[\sigma h]\right\}_{\lambda,\,\theta} + \left\{\frac{\partial}{\partial \theta}\hat{\dot{\theta}}[\sigma h]\right\}_{\lambda,\,q} = 0. \tag{5.9}$$

Note in (5.9) $\hat{\dot{q}}$ and $\hat{\dot{\theta}}$ are defined as zonally averaged quantities along a contour of constant PV and constant θ which makes necessary to include the PV-thickness h.

We should be able to find a streamfunction ψ analogous to the streamfunctions corresponding to different frameworks presented in Chapter 2. For this we use the relationship between $\dot{q}$ and $\dot{\theta}$, derived by K02, which has already been introduced in section 4.2.1,

$$\dot{q}\nabla\theta - \dot{\theta}\nabla q = \nabla\psi, \tag{5.10}$$

and that yields

$$\dot{q} = \left(\frac{\partial\psi}{\partial\theta}\right)_q. \tag{5.11}$$

$$\dot{\theta} = -\left(\frac{\partial\psi}{\partial q}\right)_\theta. \tag{5.12}$$

Notice that equation (5.11) gives the mean meridional circulation in terms of the streamfunction ψ, which can be determined up to a constant as

$$\psi = \left. \int_{\theta_s}^{\theta} \dot{q} d\theta \right|_{q = const} . \tag{5.13}$$

While it is tempting to use available global-scale datasets for the current investigations, it is easy to anticipate the number of complications that will arise. For example the data contain information from all physical processes that influence the meteorological variables and that part of the signal that we are interested in may appear as a small noise. Therefore, the best approach to address the questions we asked in Chapter 1 is to use a simple numerical model. The model used is acquired from Heikes (2002). It is an isentropic model that has simple parameterization for the physics of radiation and surface friction, based on Held-Suarez (1994) forcing. Radiative processes are parameterized through the Newtonian relaxation to a fixed equilibrium temperature distribution specific for the winter months. The dissipation processes are parameterized as Rayleigh damping of low-level winds. In the horizontal plane the model uses a geodesic grid with 2562 cells, which has the advantages of eliminating the pole problems and providing an approximately homogeneous resolution on the sphere. For the sake of simplicity, the model does not contain any topography or land-sea contrast, even though these factors influence the large-scale circulation.

Using this model we calculated $\left(\hat{\dot{q}}, \hat{\dot{\theta}}\right)$ as given by (5.7) and (5.8). The results are shown in Fig. 5.1. The meridional component $\hat{\dot{q}}$ presented in Fig. 5.1 does not contain the friction term X_λ that appears in its definition. An equatorial flow at lower levels of magnitude ~ 0.2 PVU/day and a poleward flow at upper levels of magnitude ~ 0.35 PVU/day dominates in each hemisphere. The dipolar structure with convergence of the eddy PV flux near the surface and divergence of the eddy PV flux aloft extends all the way from the Equator to pole. Because the meridional flow is driven by the diabatic heating we can also interpret this pattern as cooled air is transported equatorward at lower levels and heated air is transported poleward aloft. The distribution of the mass-weighted zonal-mean diabatic heating shows that at the surface flows are almost adiabatic, because the radiative forcing is weak.

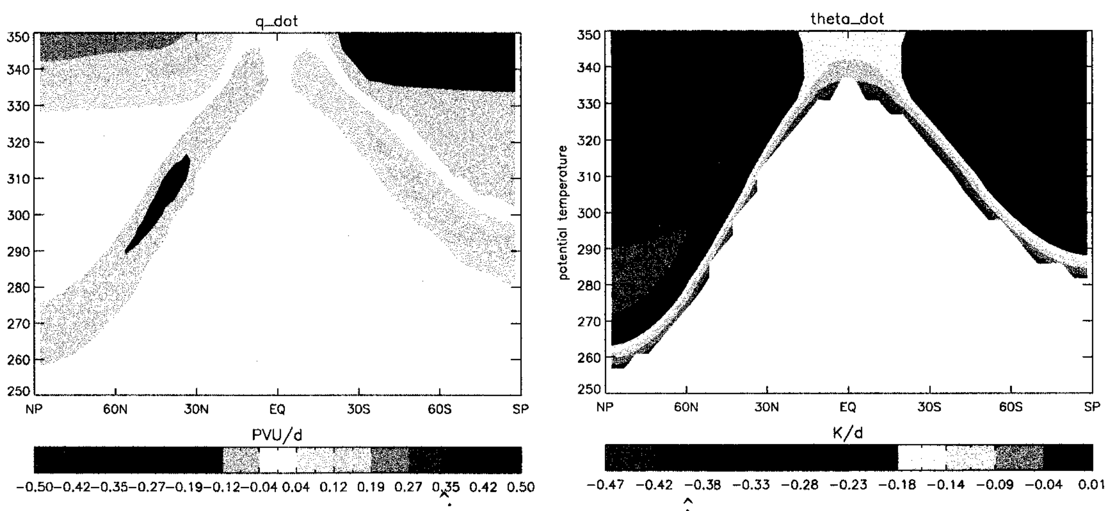

Figure 5.1: The residual circulation ($\dot{q}$ on the left panel, $\dot{\theta}$ on the right panel) in PVPT coordinates, for January month.

5.1.2: Horizontal pressure gradient force

To study the evolution of the mean state in PVPT coordinates we apply the zonal average (5.1) to the primitive equations obtained in section 4.3.

Taking the zonal averages of the continuity equation (4.71), we find that

$$\left\{\frac{\partial}{\partial t}[\sigma h]\right\}_{q,\,\theta} + \left\{\frac{\partial}{\partial q}[\sigma h\dot{q}]\right\}_{\lambda,\,\theta} + \left\{\frac{\partial}{\partial \theta}[\sigma h\dot{\theta}]\right\}_{\lambda,\,q} = 0. \tag{5.14}$$

Here the square brackets denote a zonal average at fixed q. From (5.14), we see that in the absence of diabatic and frictional sources, $[\sigma h]$ remains at its initial value forever, along each q contour. The amount of air on the $dqd\theta$ surface can be modified only by diabatic processes and friction.

We want to write the pressure gradient term of (4.73) in a convenient form, so that we can see what we expect when we take the zonal average. The horizontal pressure-gradient force (as it appears in flux form of the angular momentum equation) is given by

$$HPGF = \frac{1}{a}\left\{\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\,\theta} - \sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\,\theta}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\,\theta}\right\}. \tag{5.15}$$

We use the flux form of the equation only for convenience. As Haynes and McIntyre (1990) noticed the advective form and the flux form describe the same physical processes and are based on the same concepts.

It can be shown (see Appendix D for the derivation) that the $HPGF$ can be rearranged as,

$$HPGF = \frac{1}{a}\frac{\partial}{\partial \lambda}\left\{F\left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\,\theta}\right\} - \frac{1}{a}\frac{\partial}{\partial \theta}\left\{G\left(\frac{\partial \mu}{\partial q}\right)_{\lambda,\,\theta}\right\} + \frac{1}{a}\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial \mu}{\partial \lambda}\right)_{q,\,\theta} + G\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,\,q}\right\}. \tag{5.16}$$

where $F(p)$ is function of pressure only, and satisfies

$$\frac{p}{g}\frac{\partial\Pi}{\partial\lambda} \equiv \frac{\partial F}{\partial\lambda},\ \frac{p}{g}\frac{\partial\Pi}{\partial q} \equiv \frac{\partial F}{\partial q}\ \text{and}\ \frac{p}{g}\frac{\partial\Pi}{\partial\theta} \equiv \frac{\partial F}{\partial\theta}. \tag{5.17}$$

The function $G(\lambda, q, \theta, t)$ is defined as,

$$G(\lambda, q, \theta) \equiv \theta\left\{\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + p\left\{\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}. \tag{5.18}$$

Upon close inspection of (5.18) we notice that the equivalent expression in $(\lambda, \varphi, \theta)$ coordinates is

$$G(\lambda, q, \theta) = \frac{p}{g}\left(\frac{\partial \mathcal{M}}{\partial\lambda}\right)_{\mu,\theta}. \tag{5.19}$$

Substituting from (5.16) into (4.73), we find that

$$\begin{aligned} &\left\{\frac{\partial}{\partial t}(\sigma h M)\right\}_{q,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma h U M}{1-\mu^2}\right) + F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right\}_{q,\theta} \\ &= \frac{1}{a}\frac{\partial}{\partial\theta}\left\{G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right\} - \frac{1}{a}\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} + G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\} \end{aligned} \tag{5.20}$$

Note that when compare (5.20) with (4.73) the meridional advection, vertical advection and friction terms have been dropped because they are not relevant for the present discussion. These terms will not be included in this chapter, except where their presence is needed for the discussion.

Taking the zonal average of (5.20) gives

$$\left\{\frac{\partial}{\partial t}[\sigma h M]\right\}_{q,\,\theta} = \frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right]\right\}_{\lambda,\,q} - \frac{1}{a}\left\{\frac{\partial}{\partial q}\left(\left[-F^*\left(\frac{\partial\mu^*}{\partial\lambda}\right)_{q,\,\theta}\right] + \left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right]\right)\right\}_{\lambda,\,\theta} \tag{5.21}$$

Note that in equation (5.21) the zonal average is taken along of a constant PV contour on an isentropic surface.

Substituting from (5.16) into (4.61), we find that

$$\begin{gathered}\left\{\frac{\partial}{\partial t}(\sigma h U)\right\}_{q,\,\theta} + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma h U U}{1-\mu^2} + F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right)\right\}_{q,\,\theta} - 2\Omega\mu\sigma h V = \\ \frac{1}{a}\frac{\partial}{\partial\theta}\left\{G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} - \frac{1}{a}\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta} + G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right\}\end{gathered} \quad . \tag{5.22}$$

Taking the zonal average of (5.22) gives

$$\begin{gathered}\left\{\frac{\partial}{\partial t}[\sigma h U]\right\}_{q,\,\theta} - 2\Omega[\mu\sigma h V] = \\ \frac{1}{a}\left\{\frac{\partial}{\partial\theta}\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right]\right\}_{\lambda,\,q} - \frac{1}{a}\left\{\frac{\partial}{\partial q}\left(\left[-F^*\left(\frac{\partial\mu^*}{\partial\lambda}\right)_{q,\,\theta}\right] + \left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right]\right)\right\}_{\lambda,\,\theta}\end{gathered} \tag{5.23}$$

Equation (5.23) shows that the total zonal momentum can be changed only by external forces.

We can prove that when the flow is steady and $\dot{q} = 0$ and $\dot{\theta} = 0$, $[\mu\sigma h V] = 0$.

For this we multiply (4.71) by $\mu^2/2$ and (4.75) by $\mu\sigma h$ and then add both together to obtain

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\sigma h\mu^2\right)_{q,\theta} + \frac{1}{2a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sigma h U\mu^2}{(1-\mu^2)}\right)\right\}_{q,\theta} - \sigma h\mu V = 0. \tag{5.24}$$

Taking the zonal average of (5.24) gives,

$$\frac{1}{2}\left(\frac{\partial}{\partial t}[\sigma h\mu^2]\right)_{q,\theta} - [\sigma h\mu V] = 0, \tag{5.25}$$

and for steady flow (5.25) reduces to

$$[\sigma h\mu V] = 0. \tag{5.26}$$

Under this condition, (5.23) becomes

$$\left\{\frac{\partial}{\partial\theta}\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right]\right\}_{\lambda,q} + \left\{\frac{\partial}{\partial q}\left(\left[-F^*\left(\frac{\partial\mu^*}{\partial\lambda}\right)_{q,\theta}\right] + \left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right]\right)\right\}_{\lambda,\theta} = 0. \tag{5.27}$$

Note that Equation (5.27) can be written as

$$\nabla\bullet\boldsymbol{E} = 0. \tag{5.28}$$

The vector $\boldsymbol{E} \equiv (0, \boldsymbol{E}^{(q)}, \boldsymbol{E}^{(\theta)})$ is the Eliassen-Palm flux (EP flux); its components are given by

$$
\begin{aligned}
E^{(q)} &= \left[-F^*\left(\frac{\partial\mu^*}{\partial\lambda}\right)_{q,\,\theta}\right] + \left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right] \\
E^{(\theta)} &= \left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right]
\end{aligned}
\quad . \tag{5.29}
$$

Dynamically speaking equation (5.27) or (5.29) shows that in the absence of an EP flux, eddies do not accelerate the zonal mean flow, a result first recognized by Charney and Drazin (1961) and Dickinson (1969).

A physical interpretation of $\nabla \bullet E$ can be given, following Andrews (1983): $\nabla \bullet E$ can be viewed as the λ-component of the forces exerted by eddies on a thin tube with its axis oriented along the λ-direction bounded by undulating lateral sides which are located at q and $q + dq$, and undulating bottom and top isentropes θ and $\theta + d\theta$. The first term in the horizontal component of the EP flux, $E^{(q)}$, represents the net pressure force pushing against the slanted potential vorticity surface (the so called form drag).

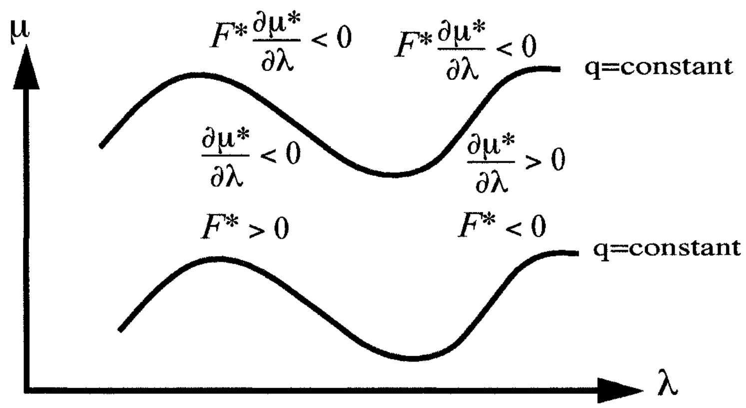

Figure 5.2: The schematic illustration of the form drag. μ^* represents the geographical latitude of the undulating PV contour. F being a function of pressure only bulged regions are associated with high pressure, $F^* > 0$ and valleys are associated with low pressure, $F^* < 0$.

To picture the physical representation of the form drag consider the air mass bounded by two surfaces of constant PV as illustrated in Fig. 5.2. Because F is only a function of pressure we can interpret regions where $\frac{\partial\mu^*}{\partial\lambda} < 0$ dominated by positive F^* and regions where $\frac{\partial\mu^*}{\partial\lambda} > 0$ by negative F^*, respectively. Taking the zonal average over the whole domain, $\left[F^*\frac{\partial\mu^*}{\partial\lambda}\right]$ will be negative everywhere and the net result is a drag on the flow moving towards the east.

The form drag nature of the other two terms, namely $\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right]$ and $\left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right]$ is more apparent if we think at G given by (5.19). We can write them as

$$\left[G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right] = \left[\frac{p}{g}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\left(\frac{\partial\mathcal{M}}{\partial\lambda}\right)_{\mu,\,\theta}\right] = \left[p_q{}^*\left(\frac{\partial\mathcal{M}^*}{\partial\lambda}\right)_{\mu,\,\theta}\right] \tag{5.30}$$

$$\left[G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right] = \left[\frac{p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\left(\frac{\partial\mathcal{M}}{\partial\lambda}\right)_{\mu,\,\theta}\right] = \left[p_\theta{}^*\left(\frac{\partial\mathcal{M}^*}{\partial\lambda}\right)_{\mu,\,\theta}\right]. \tag{5.31}$$

If we assume in (5.23) $G = G(\lambda)$ then the zonal momentum on a PV surface changes only in response to the action of the Coriolis force and meridional variations in the form drag. If we further consider the case when F^* does not depend on λ then the form drag is zero and in consequence the zonal momentum is affected only by the Coriolis force, a result in accordance with the physical laws.

The use of PVPT coordinates does not simplify the zonally averaged equations. It does, however, provide a basis for assessing the physical significance of various terms in these equations. The PVPT system of coordinates is self-consistent and provides an unifying theory for the forcing of the mean meridional circulation.

The general theoretical framework presented here shows that the main forcing of the non-geostrophic mean flow is the form drag. This result have implications for the accurate parameterization of eddy fluxes for use in climate models which do not include an explicit representation of baroclinic disturbances. We present in Appendix E an example of potential parameterization suitable for the this purpose. The effects of eddies on the mean flow are parameterized, based on the form drag concept, in terms of the ratio between the length scale over which the eddy equivalent potential enstrophy varies relative to the gradient of mean equivalent potential vorticity and the length scale over which the eddy equivalent potential enstrophy varies relative to the gradient of eddy equivalent potential enstrophy.

Chapter 6: Summary, Conclusions and Prospects

6.1: Summary and conclusions

The present study has improved our understanding of the mechanisms sustaining the mean meridional circulation of the atmosphere, especially in midlatitude regions. The major achievements of this thesis have been to shed some light on two main topics of research interests in the atmospheric dynamics community: 1) the mechanisms through which the eddies affect the mean flow in midlatitudes, 2) the nature of eddy parameterizations. In carrying out the main objectives of this thesis two important theoretical contributions were made. The first is the introduction of potential vorticity as meridional coordinate. The second is to develop a framework adequate for parameterization of the eddy momentum transport.

This is not the first time that an alternative for the meridional coordinate has been proposed. For example Hoskins (1975) replaced the geographical latitude and longitude with the semigeostrophic coordinates that specify the position of particles as they would move with the geostrophic velocity. Shutts (1979) used as meridional coordinate the angular momentum and investigated its usefulness for the study of the mean meridional circulation of the atmosphere. Schubert and Magnusdottir (1994) generalized the semigostrophic coordinates to the so called vorticity coordinates. The vorticity coordinates have the property that their Jacobian with respect to the geographical latitude and longitude is the vertical component of absolute vorticity vector. The vorticity coordinates are an appropriate frame of reference to investigate the zonally symmetric balanced flows.

With the introduction of the TEM formalism it became obvious that a Lagrangian description of the flow provides more insight into the mechanisms through which the eddies interact with the mean flow. Because the TEM formalism offers only a semi-Lagrangian perspective of the flow, McIntyre (1980) introduces the generalized Lagrangian mean (GLM) formalism, which offers a Lagrangian description of the flow. In the GLM theory the averaging is applied along the trajectory followed by a particle. In a schematic representation similar to Fig. 2.10 in Chapter 2, the control volume in the GLM is shown in Fig. 6.1. Despite the fact that the GLM offers a Lagrangian description of the flow the position is not a conserved quantity, and this makes the formalism very complicated.

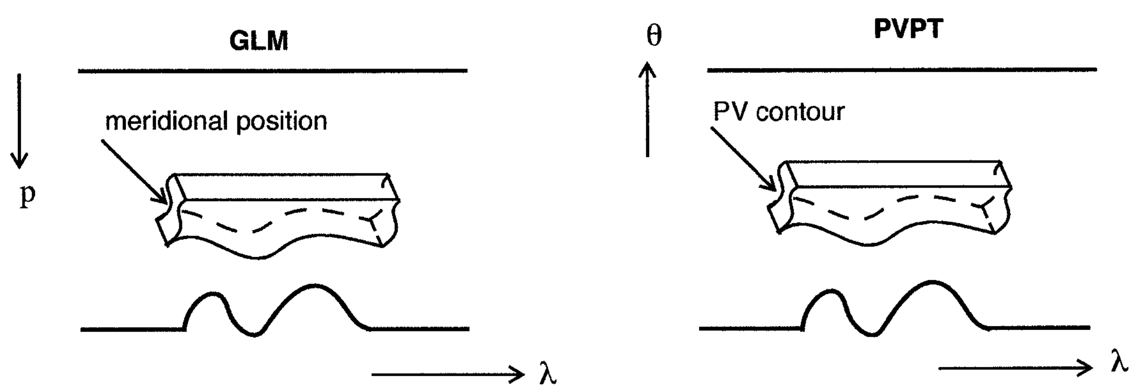

Figure 6.1: Schematic representation of the volume control as appears in the GLM formalism and the PVPT formalism and PVPT system of coordinates.

Another formalism, the modified Lagrangian-mean (MLM) was proposed by Nakamura (1995, 1996) to study the properties of the zonal mean flow. In the MLM theory the meridional coordinate is defined as the mass/area enclosed by the contour of a conservative variable (tracer, in this particular case) on an isentropic surface.

By using the potential vorticity as meridional coordinate the meridional advection is zero for an adiabatic frictionless fluid, thus the air flows in only one direction, parallel to the latitudinal circle on a surface of constant PV. In connection with the potential temperature as vertical coordinate, the newly created system of coordinates allows us to divide the atmosphere into undulating (both in zonal and meridional directions) tubes bounded by isentropic and constant PV surfaces, and the air moves through these tubes without penetrating through the walls, again in adiabatic frictionless conditions. In Fig. 6.1 we notice the similarity between the volume control in GLM and PVPT. This resemblance appears because the PV coordinate represents in fact the trajectory of the particle, which allows this system of coordinates to describe the flow from a Lagrangian perspective. The advantage of the PVPT coordinates is that unlike the position, the PV surfaces act as material surfaces in adiabatic frictionless conditions.

A model that uses this system of coordinates has the advantage of incorporating as built-in the dry convective adjustment and baroclinic adjustment processes, which prevent the folding of isentropic and potential vorticity surfaces but simulate the effects of interaction of waves with the flow. Dry convective instability occurs when the pressure increases with increases potential temperature, but in an isentropic model this is equivalent with the production of negative “isentropic-mass”. Since the numerical scheme is designed not to allow formation of negative mass, an isentropic model cannot experience dry convective instability. This is a very well known result (Hsu and Arakawa, 1990) that we mention just to point the reader towards an analogy when thinking about the built-in baroclinic adjustment. A necessary, but not sufficient, condition for the baroclinic instability to occur in a certain region of the domain is that meridional gradient of potential vorticity have both signs in that region. In the PVPT coordinates the potential vorticity gradient represents a part of the “PVPT-mass”, but the numerical scheme of the model in PVPT coordinates does not allow the formation of negative mass. As a result a model in PVPT coordinates cannot experience baroclinic instability. Another argument that sustains this mechanism is a mass conservation argument that occurs in the absence of diabatic and friction processes as schematically represented in Fig. 6.2. The left panel shows a hypothetical distribution of the PV field in a balanced state that at a later time experiences baroclinic instability as shown in the right panel. Under the assumption that diabatic contributions are negligible the mass distribution against the PV (bottom panel), calculated for the appropriate region in the balanced state and during the baroclinic event, has the same shape. The mass is calculated as the mass enclosed by the area delimited by a contour of constant PV and a contour of constant potential temperature.

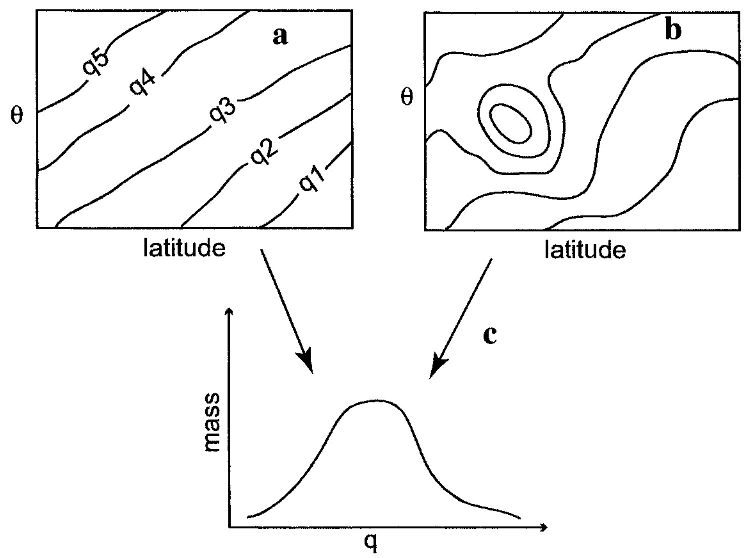

Figure 6.2: Distribution of PV in a balanced state (a), and during a baroclinic event (b) produces the same mass distribution when represented in PV PT coordinates (c).

The atmospheric counterpart of this mechanism is the mixing process by baroclinic waves, which develop in connection with the baroclinic instability and adjust the flow to a balanced state. Therefore the atmospheric flow simulated by a model that uses PVPT coordinates can capture the characteristics of the flow before and after baroclinic instability occurs but cannot be used to study the life-cycle of baroclinic waves.

The mass distribution in PVPT coordinates shows a novel pattern. When we track the seasonal variations of the mass distribution we can observe that the mass contained in PVPT tubes is almost conserved. For example in the winter season the synoptic-scale baroclinic waves show more variability than in the summer and therefore we would expect the diabatic processes associated with midlatitude storms to transport mass across the PVPT walls, and the mass distribution to be significantly different. From the PV-inversion theory we can state that the distribution of PV is equivalent to the distribution of temperature and momentum, therefore a change in any variable with respect to PV can be seen as a change with respect the mean state of the atmosphere as well.

It is important to stress that all the considerations developed in PVPT coordinates are general in the sense that no quasi-geostrophic equilibrium is assumed and there is no dependence on a particular choice of the state of the atmosphere. As is well known (Hoskins, 1975) in the quasi-geostrophic frame of reference the eddy available potential energy is generated as a result of poleward heat transport, whereas in the primitive equation system the available potential energy is released as the heat is transported upward. Therefore, the PVPT system of coordinate allows to include the vertical variations that were neglected in the quasi-geostrophic approximation.

When applied to the study of the mean circulation in the meridional plane, we obtained the following results:

1). In the PVPT coordinates the residual circulation is driven by the two forcings: mean diabatic heating and the divergence of the eddy PV flux across isentropes.

2). The zonal mean pseudo-density changes only when there is divergence/ convergence in the residual circulation. For an adiabatic, frictionless atmosphere it remains at its initial value forever, along each PV contour.

3). The forcing of the mean angular momentum is the form drag. In the "meridional" direction there are two terms. One represents the net pressure force pushing against the slanted PV surface with respect to the zonal direction, whereas the second one represents the net pressure force pushing against the slanted PV surface with respect to the vertical direction. In the vertical direction the form drag is given by the net pressure pushing against the slanted isentropic surface.

6.2: Prospects

We conclude this thesis by discussing future research directions through the open question: *What could be the implications of the new system of coordinates and the new eddy parameterization in predictions of climate?*

The theory developed using the PVPT framework emerges at the same time with the ERA-40 reanalysis products, which provide direct model outputs on the PV surface 2 PVU. Since there are no other theoretical views of the atmospheric circulation using this coordinate, the present work will provide the physical background for the diagnostic studies of the observed circulation using PVPT coordinates.

How large-scale eddy activity is maintained in the wintertime storm-tracks is still an open debate with large implication for climate simulations. As computational power increases exponentially, it is now possible to run models that produce realistic features on the smallest scale, but it is unknown whether the models are simulating the dynamics in any realistic manner. Thus, as model resolution changes, the need for parameterization does not simply go away. Simple zonally averaged climate models with parameterization of eddies provide the test beds for understanding hypothesized mechanisms.

APPENDIX A

The viscous stress-tensor on the sphere

Here we discuss the form of the viscous stress tensor in spherical coordinates, and its effects on linear momentum, angular momentum, and vorticity.

Consider a Cartesian system of coordinates, (x, y, z), with its origin at the Earth's center, and also a spherical system of coordinates (λ, φ, r) with its origin at the same point, as illustrated in Fig A.1.

$$x = r\cos\varphi\cos\lambda \qquad \frac{\partial}{\partial x} = \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}$$

$$y = r\cos\varphi\sin\lambda \qquad \frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial\varphi}$$

$$z = r\sin\varphi \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial r}$$

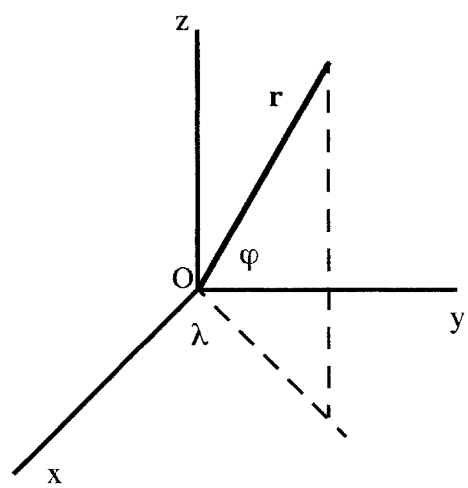

Figure A.1: The relationship between Cartesian and spherical system of coordinates.

Assume that the normals of the three surfaces $r = r(x, y, z)$, $\varphi = \varphi(x, y, z)$ and $\lambda = \lambda(x, y, z)$ intersect at the point M, and form an orthogonal system of coordinates (x', y', z'). Attached to every point on the sphere we have a Cartesian frame. The directions of axes of the frames vary from point to point and are given by the direction cosines in Table A.1.

Table A1: Direction cosines

	x	y	z	
x'	$-\sin\lambda$	$\cos\lambda$	0	u'
y'	$-\sin\varphi\cos\lambda$	$-\sin\varphi\sin\lambda$	$\cos\varphi$	v'
z'	$\cos\varphi\cos\lambda$	$\cos\varphi\sin\lambda$	$\sin\varphi$	w'
	u	v	w	

In Table A.1 (u, v, w) and (u', v', w') represent the projections of displacement of point M on the older system coordinates (x, y, z) and on (x', y', z'), respectively. Table A.1 allows us to express a vector with components (u, v, w) in terms of (u', v', w') and vice versa, as follows:

$$\begin{aligned} u &= -u'\sin\lambda - v'\sin\varphi\cos\lambda + w'\cos\varphi\cos\lambda \\ v &= u'\cos\lambda - v'\sin\varphi\sin\lambda + w'\cos\varphi\sin\lambda \\ w &= v'\cos\varphi + w'\sin\varphi \end{aligned}, \tag{A.1}$$

$$\begin{aligned} u' &= -u\sin\lambda + v\cos\lambda \\ v' &= -u\sin\varphi\cos\lambda - v\sin\varphi\sin\lambda + w\cos\varphi \\ w' &= u\cos\varphi + v\cos\varphi\sin\lambda + w\sin\varphi \end{aligned}. \tag{A.2}$$

The components of the stress tensor are:

$$
\begin{aligned}
\widehat{rr} &= -p + \varepsilon\delta + 2\mu\frac{\partial u'}{\partial x'} & \widehat{r\varphi} &= \mu\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) \\
\widehat{\varphi\varphi} &= -p + \varepsilon\delta + 2\mu\frac{\partial v'}{\partial y'}, & \widehat{\lambda\varphi} &= \mu\left(\frac{\partial w'}{\partial y'} + \frac{\partial v'}{\partial z'}\right) \\
\widehat{\lambda\lambda} &= -p + \varepsilon\delta + 2\mu\frac{\partial w'}{\partial z'} & \widehat{\lambda r} &= \mu\left(\frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'}\right)
\end{aligned}
\tag{A.3}
$$

where the divergence is $\delta = \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'}$. The coefficients (ε, μ) are assumed to satisfy Stokes' hypothesis (Lamb, 1945),

$$
3\varepsilon + 2\mu = 0 . \tag{A.4}
$$

The first three relations in (A.3) give the normal components of the stress tensor, and the next three give the tangential components.

Using (A.1) we can write

$$
\frac{\partial u'}{\partial x'} = -\sin\lambda\frac{\partial u}{\partial x'} + \cos\lambda\frac{\partial v}{\partial x'}, \tag{A.5}
$$

which in spherical coordinates becomes

$$
\frac{\partial u'}{\partial x'} = \frac{\sin\lambda}{r\cos\varphi}\frac{\partial u}{\partial\lambda} + \frac{\cos\lambda}{r\cos\varphi}\frac{\partial u}{\partial\lambda}. \tag{A.6}
$$

In (A.6), u' is expressed in terms of the projections onto the older system of coordinates, but we want it expressed in terms of the projections onto the new system of coordinates. We can obtain this by using (A.2), as follows

$$\frac{\partial u'}{\partial x'} = \frac{1}{r\cos\varphi}\frac{\partial u'}{\partial\lambda} - \frac{v'\sin\varphi}{r\cos\varphi} + \frac{w'}{r} \tag{A.7}$$

Using the same method, we obtain the other components of the stress tensor:

$$\begin{aligned}
\widehat{\lambda\lambda} &= -p + \varepsilon\delta + 2\mu\left(\frac{1}{r\cos\varphi}\frac{\partial u'}{\partial\lambda} - \frac{\sin\varphi}{\cos\varphi}\frac{v'}{r} + \frac{w'}{r}\right)\\
\widehat{\varphi\varphi} &= -p + \varepsilon\delta + 2\mu\left(\frac{1}{r}\frac{\partial v'}{\partial\varphi} + \frac{w'}{r}\right)\\
\widehat{rr} &= -p + \varepsilon\delta + 2\mu\frac{\partial w'}{\partial r}\\
\widehat{\lambda\varphi} &= \mu\left(\frac{1}{r}\frac{\partial u'}{\partial\varphi} + \frac{1}{r\cos\varphi}\frac{\partial v'}{\partial\lambda} + \frac{\sin\varphi}{\cos\varphi}\frac{u'}{r}\right)\\
\widehat{\lambda r} &= \mu\left(\frac{\partial u'}{\partial r} - \frac{u'}{r} + \frac{1}{r\cos\varphi}\frac{\partial w'}{\partial\lambda}\right)\\
\widehat{r\varphi} &= \mu\left(\frac{\partial v'}{\partial r} - \frac{v'}{r} + \frac{1}{r}\frac{\partial w'}{\partial\varphi}\right)
\end{aligned} \tag{A.8}$$

Jeffery (1915) generalized (A.8) for any curvilinear coordinates (α, β, γ)

$$\begin{aligned}
\widehat{\alpha\alpha} &= -p + \varepsilon\delta + 2\mu h_1\left[\frac{\partial u}{\partial\alpha} + h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right)v + h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)w\right]\\
\widehat{\beta\beta} &= -p + \varepsilon\delta + 2\mu h_2\left[\frac{\partial v}{\partial\beta} + h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right)v + h_3\frac{\partial}{\partial\gamma}\left(\frac{1}{h_2}\right)w\right],\\
\widehat{\gamma\gamma} &= -p + \varepsilon\delta + 2\mu h_3\left[\frac{\partial w}{\partial\gamma} + h_2\frac{\partial}{\partial\beta}\left(\frac{1}{h_3}\right)v + h_1\frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)u\right]
\end{aligned} \tag{A.9}$$

$$\widehat{\alpha\beta} = \mu\left[\frac{h_1}{h_2}\frac{\partial}{\partial\alpha}(h_2 v) + \frac{h_2}{h_1}\frac{\partial}{\partial\beta}(h_1 u)\right]$$

$$\widehat{\alpha\gamma} = \mu\left[\frac{h_1}{h_3}\frac{\partial}{\partial\alpha}(h_3 w) + \frac{h_3}{h_1}\frac{\partial}{\partial\gamma}(h_1 u)\right], \qquad \text{(A.10)}$$

$$\widehat{\beta\gamma} = \mu\left[\frac{h_2}{h_3}\frac{\partial}{\partial\beta}(h_3 w) + \frac{h_3}{h_2}\frac{\partial}{\partial\gamma}(h_2 v)\right]$$

where h_i are the metric terms given by

$$h_i = \left[\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2\right]^{-\frac{1}{2}}, \qquad \text{(A.11)}$$

with $q_i = (\alpha, \beta, \gamma)$.

Ames and Murnaghan (1958) have shown that the divergence of a tensor $\boldsymbol{A}$, which is a vector, can be written as

$$\begin{aligned}\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{xy}}{\partial y} + \frac{\partial A_{xz}}{\partial z} &= h_1 h_2 h_3\left[\frac{\partial}{\partial\alpha}\left(\frac{\widehat{\alpha\alpha}}{h_2 h_3}\right) + \frac{\partial}{\partial\beta}\left(\frac{\widehat{\alpha\beta}}{h_3 h_1}\right) + \frac{\partial}{\partial\gamma}\left(\frac{\widehat{\alpha\gamma}}{h_1 h_2}\right)\right]\\ &+ \widehat{\beta\alpha} h_1 h_2 \frac{\partial}{\partial\beta}\left(\frac{1}{h_1}\right) + \widehat{\gamma\alpha} h_1 h_3 \frac{\partial}{\partial\gamma}\left(\frac{1}{h_1}\right)\\ &- \widehat{\beta\beta} h_1 h_2 \frac{\partial}{\partial\alpha}\left(\frac{1}{h_2}\right) - \widehat{\gamma\gamma} h_1 h_3 \frac{\partial}{\partial\alpha}\left(\frac{1}{h_3}\right)\end{aligned} \qquad \text{(A.12)}$$

$$
\begin{aligned}
\frac{\partial A_{yx}}{\partial x} + \frac{\partial A_{yy}}{\partial y} + \frac{\partial A_{yz}}{\partial z} = {} & h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha}\left(\frac{\overparen{\beta\alpha}}{h_2 h_3} \right) + \frac{\partial}{\partial \beta}\left(\frac{\overparen{\beta\beta}}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma}\left(\frac{\overparen{\beta\gamma}}{h_1 h_2} \right) \right] \\
& + \overparen{\alpha\beta} h_1 h_2 \frac{\partial}{\partial \alpha}\left(\frac{1}{h_2} \right) + \overparen{\gamma\beta} h_2 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_2} \right) \\
& - \overparen{\alpha\alpha} h_1 h_2 \frac{\partial}{\partial \beta}\left(\frac{1}{h_1} \right) - \overparen{\gamma\gamma} h_2 h_3 \frac{\partial}{\partial \beta}\left(\frac{1}{h_3} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A_{zx}}{\partial x} + \frac{\partial A_{zy}}{\partial y} + \frac{\partial A_{zz}}{\partial z} = {} & h_1 h_2 h_3 \left[\frac{\partial}{\partial \alpha}\left(\frac{\overparen{\gamma\alpha}}{h_2 h_3} \right) + \frac{\partial}{\partial \beta}\left(\frac{\overparen{\gamma\beta}}{h_3 h_1} \right) + \frac{\partial}{\partial \gamma}\left(\frac{\overparen{\gamma\gamma}}{h_1 h_2} \right) \right] \\
& + \overparen{\alpha\gamma} h_1 h_3 \frac{\partial}{\partial \alpha}\left(\frac{1}{h_3} \right) + \overparen{\beta\gamma} h_2 h_3 \frac{\partial}{\partial \beta}\left(\frac{1}{h_3} \right) \\
& - \overparen{\alpha\alpha} h_1 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_1} \right) - \overparen{\beta\beta} h_2 h_3 \frac{\partial}{\partial \gamma}\left(\frac{1}{h_2} \right)
\end{aligned}
$$

For the case of spherical coordinates $\alpha = \lambda$, $\beta = \varphi$ and $\gamma = r$, the metric terms (A.11) become

$$
\begin{aligned}
h_1 &= \frac{1}{r\cos\varphi} \\
h_2 &= \frac{1}{r} \quad , \\
h_3 &= 1
\end{aligned}
\tag{A.13}
$$

and (A.12) reduces to:

$$
\begin{aligned}
& \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}\overparen{\lambda\lambda} + \frac{1}{r}\frac{\partial}{\partial\varphi}\overparen{\lambda\varphi} + \frac{\partial}{\partial r}\overparen{\lambda r} - \frac{2\sin\varphi}{r\cos\varphi}\overparen{\varphi\lambda} + \frac{3}{r}\overparen{\lambda r} \\
& \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}\overparen{\varphi\lambda} + \frac{1}{r}\frac{\partial}{\partial\varphi}\overparen{\varphi\varphi} + \frac{\partial}{\partial r}\overparen{\varphi r} + \frac{\sin\varphi}{r\cos\varphi}(\overparen{\lambda\lambda} - \overparen{\varphi\varphi}) + \frac{3}{r}\overparen{\varphi r} \\
& \frac{1}{r\cos\varphi}\frac{\partial}{\partial\lambda}\overparen{r\lambda} + \frac{1}{r}\frac{\partial}{\partial\varphi}\overparen{r\varphi} + \frac{\partial}{\partial r}\overparen{rr} - \frac{\sin\varphi}{r\cos\varphi}\overparen{r\varphi} + \frac{2}{r}\overparen{rr} - \frac{1}{r}(\overparen{\varphi\varphi} + \overparen{\lambda\lambda})
\end{aligned}
\tag{A.14}
$$

After substitution of (A.8) into (A.14) we obtain the Navier-Stokes equations in spherical coordinates:

$$
\begin{aligned}
\frac{Du}{Dt}+\frac{uw}{r}+\frac{vw\sin\varphi}{r\cos\varphi} &= -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial\lambda}+\nu\left[\frac{1}{3r\cos\varphi}\frac{\partial\delta}{\partial\lambda}+\nabla^2 u-\frac{2\sin\varphi}{r^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}+\frac{2}{r^2\cos\varphi}\frac{\partial w}{\partial\lambda}\right.\\
&\qquad\left.-\frac{u}{r^2\cos^2\varphi}\right]\\
\frac{Dv}{Dt}+\frac{uv}{r}+\frac{u^2\sin\varphi}{r\cos\varphi} &= -\frac{1}{\rho r}\frac{\partial p}{\partial\varphi}+\nu\left[\frac{1}{3r}\frac{\partial\delta}{\partial\varphi}+\nabla^2 v+\frac{2\sin\varphi}{r^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}+\frac{2}{r^2}\frac{\partial w}{\partial\varphi}-\frac{v}{r^2\cos^2\varphi}\right]\\
\frac{Dw}{Dt}-\frac{u^2+v^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\frac{1}{3}\frac{\partial\delta}{\partial r}+\nabla^2 w+\frac{2\sin\varphi}{r^2\cos\varphi}v-\frac{2w}{r}-\frac{2}{r^2}\frac{\partial v}{\partial\varphi}-\frac{2}{r^2\cos\varphi}\frac{\partial u}{\partial\lambda}\right]
\end{aligned}
\tag{A.15}
$$

Equations (A.15) represent the exact primitive equations in spherical coordinates. In order to derive (4.19) we have to consider the shallow atmosphere case, when $r = a$. In this case the metric terms reduce to

$$
\begin{aligned}
h_1 &= \frac{1}{a\cos\varphi},\\
h_2 &= \frac{1}{a}
\end{aligned}
\tag{A.16}
$$

and the components of the stress tensor become,

$$
\begin{aligned}
\widehat{\lambda\lambda} &= -p+\varepsilon\delta+2\mu\left(\frac{1}{a\cos\varphi}\frac{\partial u}{\partial\lambda}-\frac{\sin\varphi}{\cos\varphi}\frac{v}{r}\right)\\
\widehat{\varphi\varphi} &= -p+\varepsilon\delta+2\mu\left(\frac{1}{a}\frac{\partial v}{\partial\varphi}\right)\\
\widehat{\lambda\varphi} &= \mu\left(\frac{1}{a}\frac{\partial u}{\partial\varphi}+\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{\sin\varphi}{\cos\varphi}\frac{u}{a}\right)
\end{aligned}
\tag{A.17}
$$

The components of the divergence of the stress tensor (A.12) for the shallow atmosphere become

$$\begin{aligned}&\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\widehat{\lambda\lambda}+\frac{1}{a}\frac{\partial}{\partial\varphi}\widehat{\lambda\varphi}-\frac{2\sin\varphi}{r\cos\varphi}\widehat{\varphi\lambda}\\&\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\widehat{\varphi\lambda}+\frac{1}{a}\frac{\partial}{\partial\varphi}\widehat{\varphi\varphi}+\frac{\sin\varphi}{r\cos\varphi}(\widehat{\lambda\lambda}-\widehat{\varphi\varphi})\end{aligned} \qquad (A.18)$$

Now we are able to write the Navier-Stokes equations for the shallow atmosphere case

$$\begin{aligned}\frac{Du}{Dt} &= -\frac{1}{\rho a\cos\varphi}\frac{\partial p}{\partial\lambda}+\nu\left[\frac{1}{3a\cos\varphi}\frac{\partial\delta}{\partial\lambda}+\nabla^2u-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}+\frac{u}{a^2}-\frac{u\sin^2\varphi}{a^2\cos^2\varphi}\right]\\\frac{Dv}{Dt} &= -\frac{1}{\rho a}\frac{\partial p}{\partial\varphi}+\nu\left[\frac{1}{3a}\frac{\partial\delta}{\partial\varphi}+\nabla^2v+\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial u}{\partial\lambda}+\frac{v}{a^2}-\frac{v\sin^2\varphi}{a^2\cos^2\varphi}\right]\end{aligned}. \qquad (A.19)$$

When applied to the atmosphere as a solid-body rotation, for which $u = u_0\cos\varphi$ and $v = 0$, the viscous term is zero in both equations.

The relative angular momentum of the shallow atmosphere is

$$M = ua\cos\varphi. \qquad (A.20)$$

To derive an equation for the angular momentum multiply the first equation in (A.19) by $a\cos\varphi$

$$\frac{DM}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial\lambda}+\nu\left[\frac{1}{3}\frac{\partial\delta}{\partial\lambda}+a\cos\varphi\nabla^2u-\frac{2\sin\varphi}{a\cos\varphi}\frac{\partial v}{\partial\lambda}+\frac{u\cos\varphi}{a}-\frac{u\sin^2\varphi}{a\cos\varphi}\right]. \qquad (A.21)$$

To calculate the total angular momentum of the shallow atmosphere we integrate over the whole mass of the atmosphere

$$\frac{d}{dt}\int_{\Omega} M d\Omega = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi \nabla^2 u a^3 d\lambda d\varphi - 2\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial\lambda}(v\sin\varphi) a d\lambda d\varphi$$
$$+ \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\cos^2\varphi a d\lambda d\varphi - \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\sin^2\varphi a d\lambda d\varphi \,.$$

Writing the Laplacian operator on components gives,

$$\frac{d}{dt}\int_{\Omega} M d\Omega = \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial u}{\partial\varphi}\right) a d\lambda d\varphi + \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial\lambda}\left(\frac{\partial u}{\partial\lambda}\right) a d\lambda d\varphi$$
$$+ \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\cos^2\varphi a d\lambda d\varphi - 2\int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial\lambda}(v\sin\varphi) a d\lambda d\varphi - \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\sin^2\varphi a d\lambda d\varphi \,.$$

Integrating by parts the first term on the right-hand side and using the fact that the domain is periodic in the zonal direction yield,

$$\frac{d}{dt}\int_{\Omega} M d\Omega = ua\sin\varphi\cos\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\sin^2\varphi a d\lambda d\varphi - \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\cos^2\varphi a d\lambda d\varphi$$

$$- \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\sin^2\varphi a d\lambda d\varphi + \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u\cos^2\varphi a d\lambda d\varphi$$

$$= 0.$$

The total angular momentum of the shallow atmosphere is not affected by molecular viscosity.

The relative vorticity of the shallow atmosphere is defined as

$$\zeta = \frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(u\cos\varphi). \tag{A.22}$$

To derive a governing equation for the relative vorticity multiply the first equation in (A.19) by $\cos\varphi$ and then take $\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}$ from the result and subtract the result from $\frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}$ of the second equation in (A.19) to obtain

$$\begin{aligned}\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda} - \frac{\sin^2\varphi}{\cos^2\varphi}\frac{v}{a^2} + \frac{v}{a^2}\right) \\ & - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda} - \frac{\sin^2\varphi}{\cos^2\varphi}\frac{u}{a^2} + \frac{u}{a^2}\right)\right]\end{aligned} \tag{A.23}$$

in (A.23) we rewrite the last two terms as $-\frac{\sin^2\varphi}{\cos^2\varphi}\frac{u}{a^2}+\frac{u}{a^2} = -\frac{2\sin^2\varphi}{\cos^2\varphi}\frac{u}{a^2}+\frac{1}{\cos^2\varphi}\frac{u}{a^2}$

$$\begin{aligned}\frac{\partial\zeta}{\partial t} &= \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v+\frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda}-\frac{\sin^2\varphi}{\cos^2\varphi}\frac{v}{a^2}+\frac{v}{a^2}\right)\\ &-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}-\frac{2\sin^2\varphi}{\cos^2\varphi}\frac{u}{a^2}+\underline{\frac{1}{\cos^2\varphi}\frac{u}{a^2}}\right)\right]\end{aligned} \tag{A.24}$$

Apply the derivative with respect to φ to the last term in (A.24)

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{\cos\varphi}\frac{u}{a^2}\right) = \frac{1}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi}+\frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} \tag{A.25}$$

$$\begin{aligned}\frac{\partial\zeta}{\partial t} &= \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v+\frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda}-\frac{\sin^2\varphi}{\cos^2\varphi}\frac{v}{a^2}+\frac{v}{a^2}\right)\\ &-\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u-\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}-\frac{2\sin^2\varphi}{\cos^2\varphi}\frac{u}{a^2}\right)\right]\\ &-\frac{1}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi}-\frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}\end{aligned} \tag{A.26}$$

Apply the derivative with respect to φ to the $\frac{2\sin^2\varphi}{\cos^2\varphi}\frac{u}{a^2}$ term

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin^2\varphi}{\cos\varphi}\frac{u}{a^2}\right) = \frac{2\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi}+\frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}+\frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3}$$

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda} - \frac{\sin^2\varphi}{\cos^2\varphi}\frac{v}{a^2} + \frac{v}{a^2}\right) \\
& - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right)\right] + \underline{\frac{2\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi}} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} \\
& + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \underline{\frac{1}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi}} - \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}
\end{aligned}
\tag{A.27}
$$

Arrange the underlined terms in (A.27) as

$$
\begin{aligned}
\frac{2\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{1}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} & = \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{2}{a^3}\frac{\partial u}{\partial\varphi} - \left(\frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} + \frac{1}{a^3}\frac{\partial u}{\partial\varphi}\right) \\
& \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{3}{a^3}\frac{\partial u}{\partial\varphi}
\end{aligned}
$$

and substitute the result in (A.27),

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda} - \underline{\frac{\sin^2\varphi}{\cos^2\varphi}\frac{v}{a^2}} + \underline{\frac{v}{a^2}}\right) \\
& - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right)\right] + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} \quad . \\
& - \frac{\sin^2\varphi}{a^3\cos 2\varphi}\frac{\partial u}{\partial\varphi} - \frac{3}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}
\end{aligned}
\tag{A.28}
$$

Apply the derivative with respect to λ to the underlined terms in the first parenthesis

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) - \underline{\frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda}} + \underline{\frac{1}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda}} \\
& - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u - \frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}\right)\right] + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} \\
& - \frac{\sin^2\varphi}{a^3\cos 2\varphi}\frac{\partial u}{\partial\varphi} - \frac{3}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}
\end{aligned}
\quad . \qquad \text{(A.29)}
$$

Rewrite the marked terms as,

$$
\frac{1}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} = -\frac{2}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} + \frac{3}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda},
$$

and substitute this result into (A.29)

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) - \underbrace{\frac{2}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda}} \\
& + \frac{1}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} + \frac{2}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} \\
& - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\nabla^2 u - \underline{\frac{2\sin\varphi}{a^2\cos^2\varphi}\frac{\partial v}{\partial\lambda}}\right)\right] + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} \\
& - \frac{\sin^2\varphi}{a^3\cos 2\varphi}\frac{\partial u}{\partial\varphi} - \frac{3}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}
\end{aligned}
\quad . \qquad \text{(A.30)}
$$

Apply the derivative with respect to φ to the marked underlined term

$$\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{2\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial\lambda}\right) = \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{2}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda},$$

when we substitute this result back into (A.30) we notice that $\frac{2}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda}$ appears in there (the under braced term) with the opposite sign, so these two terms cancel

$$\begin{aligned}\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial\lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\lambda}\right) + \frac{1}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} \\ & + \frac{2}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi\nabla^2 u) \\ & + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{\sin^2\varphi}{a^3\cos 2\varphi}\frac{\partial u}{\partial\varphi} \\ & - \underbrace{\frac{3}{a^3}\frac{\partial u}{\partial\varphi}} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}\end{aligned} \qquad (A.31)$$

Apply the derivative with respect to φ to $\cos\varphi\nabla^2 u$,

$$\begin{aligned}\frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi\nabla^2 u) & = \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left[\cos\varphi\left(\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 u}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial^2 u}{\partial\varphi^2} - \frac{\sin\varphi}{a^2\cos\varphi}\frac{\partial u}{\partial\varphi}\right)\right] \\ & = \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{1}{a^2\cos\varphi}\frac{\partial^2 u}{\partial\lambda^2}\right) + \frac{1}{a\cos\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\cos\varphi}{a^2}\frac{\partial^2 u}{\partial\varphi^2}\right) - \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) \\ & = \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u}{\partial\lambda^2}\right) + \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2} + \frac{1}{a^3}\frac{\partial^3 u}{\partial\varphi^3} - \frac{2\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} - \frac{1}{a^3}\frac{\partial u}{\partial\varphi}\end{aligned}$$

Substituting this result into (A.31) we observe that (A.31) contains the term $\frac{1}{a^3}\frac{\partial u}{\partial \varphi}$ (the under braced term), so this two terms combine,

$$
\begin{aligned}
\frac{\partial \zeta}{\partial t} = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial \lambda}\right) + \frac{3}{a^3\cos\varphi}\frac{\partial v}{\partial \lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial \lambda} \\
& - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial \varphi}\left(\frac{\partial^2 u}{\partial \lambda^2}\right) - \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial \lambda^2} - \frac{1}{a^3}\frac{\partial^3 u}{\partial \varphi^3} \\
& + \frac{2\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{a^3}\frac{\partial u}{\partial \varphi} + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial \lambda \partial \varphi} \\
& + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial \varphi} - \frac{\sin^2\varphi}{a^3\cos 2\varphi}\frac{\partial u}{\partial \varphi} - \underbrace{\frac{3}{a^3}\frac{\partial u}{\partial \varphi}} \\
& + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}
\end{aligned}
\quad . \qquad \text{(A.32)}
$$

Apply the derivative with respect to λ to the $\nabla^2 v + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial \lambda}$

$$
\begin{aligned}
\frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}(\nabla^2 v) = {} & \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{a^2}\frac{\partial^2 v}{\partial \varphi^2} - \frac{\sin\varphi}{a^2\cos\varphi}\frac{\partial v}{\partial \varphi}\right) \\
& + \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(\frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial \lambda}\right)
\end{aligned}
$$

$$
= \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial \lambda^2}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial \lambda}\right) + \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial \lambda}\left(\frac{\partial^2 v}{\partial \varphi^2}\right) - \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial \lambda \partial \varphi} + \frac{2\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial \lambda^2}
$$

Substituting this result in (A.32) it follows

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 v}{\partial\varphi^2}\right) - \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} \\
& + \underline{\frac{2\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2}} + \frac{3}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u}{\partial\lambda^2}\right) \\
& - \underline{\frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2}} - \frac{1}{a^3}\frac{\partial^3 u}{\partial\varphi^3} + \frac{2\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{a^3}\frac{\partial u}{\partial\varphi} \\
& + \frac{2\sin\varphi}{a^3\cos2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} \\
& - \frac{2}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}
\end{aligned}
\tag{A.33}
$$

Combining the underlined terms we obtain

$$
\begin{aligned}
\frac{\partial\zeta}{\partial t} = {} & \frac{1}{a^2\cos^2\varphi}\frac{\partial^2}{\partial\lambda^2}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial\lambda}\right) + \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 v}{\partial\varphi^2}\right) + \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} \\
& + \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2} + \frac{3}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u}{\partial\lambda^2}\right) \\
& - \frac{1}{a^3}\frac{\partial^3 u}{\partial\varphi^3} + \frac{2\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} + \frac{1}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} \\
& - \frac{2}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3} - \frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}
\end{aligned}
\quad . \tag{A.34}
$$

We will show now that the right-hand side of (A.33) is $\nabla^2\zeta + \frac{2}{a^2}\zeta$.

$$
\nabla^2\zeta = \frac{1}{a^2\cos^2\varphi}\frac{\partial^2\zeta}{\partial\lambda^2} + \frac{1}{a^2}\frac{\partial^2\zeta}{\partial\varphi^2} - \frac{\sin\varphi}{a^2\cos\varphi}\frac{\partial\zeta}{\partial\varphi} \tag{A.35}
$$

$$\frac{\partial^2 \zeta}{\partial \lambda^2} = \frac{\partial}{\partial \lambda}\left(\frac{\partial \zeta}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda}\left\{\frac{\partial}{\partial \lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial \lambda} - \frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\sin\varphi}{\cos\varphi}\frac{u}{a}\right)\right\}$$

$$= \frac{\partial}{\partial \lambda}\left(\frac{1}{a\cos\varphi}\frac{\partial^2 v}{\partial \lambda^2} - \frac{1}{a}\frac{\partial}{\partial \varphi}\left(\frac{\partial u}{\partial \lambda}\right) + \frac{\sin\varphi}{a\cos\varphi}\frac{\partial u}{\partial \lambda}\right) \tag{A.36}$$

$$= \frac{1}{a\cos\varphi}\frac{\partial^3 v}{\partial \lambda^3} - \frac{1}{a}\frac{\partial}{\partial \varphi}\left(\frac{\partial^2 u}{\partial \lambda^2}\right) + \frac{\sin\varphi}{a\cos\varphi}\frac{\partial^2 u}{\partial \lambda^2}$$

$$\frac{1}{a^2\cos^2\varphi}\frac{\partial^2 \zeta}{\partial \lambda^2} = \frac{1}{a^3\cos^3\varphi}\frac{\partial^3 v}{\partial \lambda^3} - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial \varphi}\left(\frac{\partial^2 u}{\partial \lambda^2}\right) + \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial \lambda^2} \tag{A.37}$$

$$\frac{\partial^2 \zeta}{\partial \varphi^2} = \frac{\partial}{\partial \varphi}\left(\frac{\partial \zeta}{\partial \varphi}\right) = \frac{\partial}{\partial \varphi}\left\{\frac{\partial}{\partial \varphi}\left(\frac{1}{a\cos\varphi}\frac{\partial v}{\partial \lambda} - \frac{1}{a}\frac{\partial u}{\partial \varphi} + \frac{\sin\varphi}{\cos\varphi}\frac{u}{a}\right)\right\}$$

$$= \frac{\partial}{\partial \varphi}\left\{\frac{1}{a\cos\varphi}\frac{\partial^2 v}{\partial \lambda \partial \varphi} + \frac{\sin\varphi}{a\cos^2\varphi}\frac{\partial v}{\partial \lambda} - \frac{1}{a}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\sin\varphi}{a\cos\varphi}\frac{\partial u}{\partial \varphi} + \frac{1}{\cos^2\varphi}\frac{u}{a}\right\}$$

$$= \frac{1}{a\cos\varphi}\frac{\partial}{\partial \lambda}\left(\frac{\partial^2 v}{\partial \varphi^2}\right) + \frac{2\sin\varphi}{a\cos^2\varphi}\frac{\partial^2 v}{\partial \lambda \partial \varphi} + \frac{1}{a\cos\varphi}\frac{\partial v}{\partial \lambda} + \frac{2\sin^2\varphi}{a\cos^3\varphi}\frac{\partial v}{\partial \lambda} \tag{A.38}$$

$$- \frac{1}{a}\frac{\partial^3 u}{\partial \varphi^3} + \frac{\sin\varphi}{a\cos\varphi}\frac{\partial^2 u}{\partial \varphi^2} + \frac{2}{a\cos^2\varphi}\frac{\partial u}{\partial \varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a}$$

$$\frac{1}{a^2}\frac{\partial^2 \zeta}{\partial \varphi^2} = \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial \lambda}\left(\frac{\partial^2 v}{\partial \varphi^2}\right) + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial \lambda \partial \varphi} + \frac{1}{a^3\cos\varphi}\frac{\partial v}{\partial \lambda} + \frac{2\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial \lambda} \tag{A.39}$$

$$- \frac{1}{a^3}\frac{\partial^3 u}{\partial \varphi^3} + \frac{\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial \varphi^2} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial \varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}$$

$$
\begin{aligned}
\frac{\sin\varphi}{a^2\cos\varphi}\frac{\partial\zeta}{\partial\varphi} &= \frac{\sin\varphi}{a^2\cos\varphi}\left\{\frac{1}{a\cos\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{\sin\varphi}{a\ \cos^2\varphi}\frac{\partial v}{\partial\lambda} - \frac{1}{a}\frac{\partial^2 u}{\partial\varphi^2}\right. \\
&\qquad \left. + \frac{\sin\varphi}{a\cos\varphi}\frac{\partial u}{\partial\varphi} + \frac{1}{\cos 2\varphi}\frac{u}{a}\right\} \\
&= \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} - \frac{\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} \\
&\qquad + \frac{\sin 2\varphi}{a^3\cos 2\varphi}\frac{\partial u}{\partial\varphi} + \frac{\sin\varphi}{\cos 3\varphi}\frac{u}{a^3}
\end{aligned} \tag{A.40}
$$

$$
\begin{aligned}
\nabla^2\zeta &= \frac{1}{a^3\cos^3\varphi}\frac{\partial^3 v}{\partial\lambda^3} - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u}{\partial\lambda^2}\right) + \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2} \\
&+ \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 v}{\partial\varphi^2}\right) + \frac{2\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{1}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{2\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} \\
&- \frac{1}{a^3}\frac{\partial^3 u}{\partial\varphi^3} + \frac{\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} \\
&- \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} - \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} - \frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}
\end{aligned} \tag{A.41}
$$

$$
\begin{aligned}
\nabla^2\zeta &= \frac{1}{a^3\cos^3\varphi}\frac{\partial^3 v}{\partial\lambda^3} - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u}{\partial\lambda^2}\right) + \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2} \\
&+ \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 v}{\partial\varphi^2}\right) + \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{1}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda} \\
&- \frac{1}{a^3}\frac{\partial^3 u}{\partial\varphi^3} + \frac{2\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} + \frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} \\
&- \frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi}
\end{aligned} \tag{A.42}
$$

$$\nabla^2\zeta + \frac{2}{a^2}\zeta = \frac{1}{a^3\cos^3\varphi}\frac{\partial^3 v}{\partial\lambda^3} - \frac{1}{a^3\cos^2\varphi}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 u}{\partial\lambda^2}\right) + \frac{\sin\varphi}{a^3\cos^3\varphi}\frac{\partial^2 u}{\partial\lambda^2}$$

$$+ \frac{1}{a^3\cos\varphi}\frac{\partial}{\partial\lambda}\left(\frac{\partial^2 v}{\partial\varphi^2}\right) + \frac{\sin\varphi}{a^3\cos^2\varphi}\frac{\partial^2 v}{\partial\lambda\partial\varphi} + \frac{3}{a^3\cos\varphi}\frac{\partial v}{\partial\lambda} + \frac{\sin^2\varphi}{a^3\cos^3\varphi}\frac{\partial v}{\partial\lambda}$$

$$- \frac{1}{a^3}\frac{\partial^3 u}{\partial\varphi^3} + \frac{\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} + \frac{2}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos^3\varphi}\frac{u}{a^3} \tag{A.43}$$

$$+ \frac{\sin\varphi}{a^3\cos\varphi}\frac{\partial^2 u}{\partial\varphi^2} - \frac{\sin^2\varphi}{a^3\cos^2\varphi}\frac{\partial u}{\partial\varphi} - \frac{\sin\varphi}{\cos^3\varphi}\frac{u}{a^3}$$

$$- \frac{2}{a^3}\frac{\partial u}{\partial\varphi} + \frac{2\sin\varphi}{\cos\varphi}\frac{u}{a^3}$$

By comparing the right-hand side of (A.34) and (A.43) we notice that they are the same, so we conclude that in the presence of molecular viscosity the governing equation for the vorticity is given by

$$\frac{\partial\zeta}{\partial t} = \nu\left(\nabla^2\zeta + \frac{2}{a^2}\zeta\right). \tag{A.44}$$

For the case of the atmosphere as a solid-body rotation

$$\zeta = \frac{2u_0}{a}\sin\varphi, \tag{A.45}$$

$$\nabla^2\zeta = -\frac{2}{a^2}\zeta, \tag{A.46}$$

$$\nabla^2\zeta + \frac{2}{a^2}\zeta = 0. \tag{A.47}$$

APPENDIX B

Derivation of governing equation for the **B** *vector*

In order to derive (4.25) we consider the thermodynamic equation for an adiabatic frictionless fluid and the corresponding potential vorticity principle,

$$\frac{\partial \theta}{\partial t} + \mathbf{V} \bullet \nabla\theta = 0 \tag{B.1}$$

$$\frac{\partial q}{\partial t} + \mathbf{V} \bullet \nabla q = 0 . \tag{B.2}$$

Using these two principles the time rate of change of vector $\mathbf{B} = \nabla q \times \nabla\theta$ becomes

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla\left(\frac{\partial q}{\partial t}\right) \times \nabla\theta + \nabla q \times \nabla\left(\frac{\partial \theta}{\partial t}\right) = \nabla\times\left(\frac{\partial q}{\partial t}\nabla\theta\right) - \nabla\times\left(\frac{\partial \theta}{\partial t}\nabla q\right) \\ &= \nabla\times\{\nabla q(\mathbf{V} \bullet \nabla\theta) - \nabla\theta(\mathbf{V} \bullet \nabla q)\} = \nabla\times\{\mathbf{V} \times (\nabla q \times \nabla\theta)\} \\ &= \nabla\times(\mathbf{V} \times \mathbf{B}) \end{aligned} \tag{B.3}$$

In deriving (B.3) we used the vectorial identities $\nabla\times\nabla\psi = 0$, where ψ is any arbitrary scalar; and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \bullet \mathbf{c}) - \mathbf{c}(\mathbf{a} \bullet \mathbf{b})$, were $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are any three arbitrary vectors.

Ertel (1960; in English by Schubert et al., 2004) had shown that the material derivative of any function can be expressed as a spatial divergence multiplied by the specific volume of the fluid as follows,

$$\frac{D}{Dt}\ln\{\rho\nabla\Theta_1 \bullet (\nabla\Theta_2 \times \nabla\Theta_3)\} = \frac{\partial}{\partial\Theta_j}\left(\frac{D\Theta_j}{Dt}\right) \tag{B.4}$$

$$\frac{D}{Dt}\{\nabla\Theta_1 \bullet (\nabla\Theta_2 \times \nabla\Theta_3)\} = \rho\nabla\Theta_1 \bullet (\nabla\Theta_2 \times \nabla\Theta_3)\left\{\frac{\partial\dot{\Theta}_1}{\partial\Theta_1} + \frac{\partial\dot{\Theta}_2}{\partial\Theta_2} + \frac{\partial\dot{\Theta}_3}{\partial\Theta_3}\right\} \tag{B.5}$$

$$\begin{aligned}\frac{D}{Dt}\{\rho\nabla\Theta_1 \bullet (\nabla\Theta_2 \times \nabla\Theta_3)\} = \rho\{&\nabla\dot{\Theta}_1 \bullet (\nabla\Theta_2 \times \nabla\Theta_3) + \nabla\Theta_1 \bullet (\nabla\dot{\Theta}_2 \times \nabla\Theta_3) \\ &+ \nabla\Theta_1 \bullet (\nabla\Theta_2 \times \nabla\dot{\Theta}_3)\}\end{aligned}, \tag{B.6}$$

where $\Theta_k = \Theta_k(x_1, x_2, x_{3,\,t})$ with $k = 1, 2, 3$, and ρ is the fluid density which satisfies the continuity equation

$$\frac{D\rho}{Dt} = \rho\nabla\bullet\mathbf{V}. \tag{B.7}$$

By considering

$$\Theta_1 = x, y, z \qquad \Theta_2 = q \qquad \Theta_3 = \theta, \tag{B.8}$$

(B.6) becomes

$$\frac{D}{Dt}\{\rho(\nabla q \times \nabla\theta)_x\} = \rho\left\{\frac{\partial u}{\partial x}(\nabla q \times \nabla\theta)_x + \frac{\partial u}{\partial y}(\nabla q \times \nabla\theta)_y + \frac{\partial u}{\partial z}(\nabla q \times \nabla\theta)_z + (\nabla\dot{q} \times \nabla\theta)_x + (\nabla q \times \nabla\dot{\theta})_x\right\}$$

$$\frac{D}{Dt}\{\rho(\nabla q \times \nabla\theta)_y\} = \rho\left\{\frac{\partial v}{\partial x}(\nabla q \times \nabla\theta)_x + \frac{\partial v}{\partial y}(\nabla q \times \nabla\theta)_y + \frac{\partial v}{\partial z}(\nabla q \times \nabla\theta)_z + (\nabla\dot{q} \times \nabla\theta)_y + (\nabla q \times \nabla\dot{\theta})_y\right\}, \tag{B.9}$$

$$\frac{D}{Dt}\{\rho(\nabla q \times \nabla\theta)_z\} = \rho\left\{\frac{\partial w}{\partial x}(\nabla q \times \nabla\theta)_x + \frac{\partial w}{\partial y}(\nabla q \times \nabla\theta)_y + \frac{\partial w}{\partial z}(\nabla q \times \nabla\theta)_z + (\nabla\dot{q} \times \nabla\theta)_z + (\nabla q \times \nabla\dot{\theta})_z\right\}$$

where the subscripts x, y, z denotes the Cartesian components of the vectors. Multiplying the first equation in (B.9) by the unit vector $\mathbf{i}$, the second one by $\mathbf{j}$ and the third one by $\mathbf{k}$ and then add them together we obtain

$$\frac{D}{Dt}(\rho\mathbf{B}) = \rho\{(\mathbf{B} \bullet \nabla\ )\mathbf{V} + (\nabla\dot{q} \times \nabla\theta) + (\nabla q \times \nabla\dot{\theta})\}, \tag{B.10}$$

Using the continuity equation (B.7) we can write

$$\frac{D\mathbf{B}}{Dt} + \mathbf{V}(\nabla \bullet \mathbf{B}) - (\mathbf{B}\nabla \bullet\ )\mathbf{V} = (\nabla\dot{q} \times \nabla\theta) + (\nabla q \times \nabla\dot{\theta}) \quad . \tag{B.11}$$

APPENDIX C

Shallow Water Model

A shallow water model is formulated in which the role of latitude is taken by the potential vorticity. In the absence of friction and baroclinic factors the potential vorticity is invariant.

The shallow water equations are a test bed for the horizontal spatial and temporal discretization schemes used in numerical weather prediction models and general circulation models. The golden rule seems to be that, if discretization scheme won't work on the shallow water equations, then won't work on the more complicated equations.

C.1: Starting equations

The shallow-water equations can be written as

$$\frac{DU}{Dt} - 2\Omega\mu V + \frac{g}{a}\left(\frac{\partial H}{\partial \lambda}\right)_{\mu} = 0, \tag{C.1}$$

$$\frac{DV}{Dt} + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{g(1-\mu^2)}{a}\left(\frac{\partial H}{\partial \mu}\right)_{\lambda} = 0, \tag{C.2}$$

$$\left(\frac{\partial H}{\partial t}\right)_{\mu} + \frac{1}{a(1-\mu^2)}\left\{\frac{\partial}{\partial \lambda}(HU)\right\}_{\mu} + \frac{1}{a}\frac{\partial (HV)}{\partial \mu} = 0, \tag{C.3}$$

where H represents the free surface of the fluid in the absence of topography, and the other symbols have the same meaning as the ones used in Chapter 4.

Here the Lagrangian derivative can be expressed by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_\mu + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_\mu + \frac{V}{a}\frac{\partial}{\partial \mu}. \tag{C.4}$$

The angular momentum per unit mass is

$$M \equiv aU + \Omega a^2(1-\mu^2). \tag{C.5}$$

and the potential vorticity is given by

$$q = \frac{1}{H}\left\{2\Omega\mu + \frac{1}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_\mu - \frac{1}{a}\frac{\partial U}{\partial \mu}\right\}. \tag{C.6}$$

Differentiation of the angular momentum, as given by (C.5), yields

$$\begin{aligned} \frac{DM}{Dt} &= a\frac{DU}{Dt} - 2\Omega a^2\mu\frac{D\mu}{Dt} \\ &= a\frac{DU}{Dt} - 2\Omega a\mu V \end{aligned} \tag{C.7}$$

Using (C.1), equation (C.7) can be rewritten as

$$\frac{DM}{Dt} + g\left(\frac{\partial H}{\partial \lambda}\right)_\mu = 0. \tag{C.8}$$

Conservation of potential vorticity is expressed by:

$$\left(\frac{\partial q}{\partial t}\right)_\mu + \frac{U}{a(1-\mu^2)}\left(\frac{\partial q}{\partial \lambda}\right)_\mu + \frac{V}{a}\frac{\partial q}{\partial \mu} = 0. \tag{C.9}$$

C.2: Transforming to PV coordinates

Adopting q as a meridional coordinate, we define the "potential-vorticity thickness" as

$$h = \left(\frac{\partial\mu}{\partial q}\right)_{\lambda}, \tag{C.10}$$

and assume that h is positive and finite. For an arbitrary A, we can write

$$\left(\frac{\partial A}{\partial t}\right)_{\mu} = \left(\frac{\partial A}{\partial t}\right)_{q} - \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{t}\left(\frac{\partial\mu}{\partial t}\right)_{q}, \tag{C.11}$$

$$\left(\frac{\partial A}{\partial\lambda}\right)_{\mu} = \left(\frac{\partial A}{\partial\lambda}\right)_{q} - \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\lambda}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q}, \tag{C.12}$$

$$\left(\frac{\partial A}{\partial\mu}\right)_{\lambda} = \frac{1}{h}\left(\frac{\partial A}{\partial q}\right)_{\lambda}. \tag{C.13}$$

We can use (C.11)-(C.13) to express the potential vorticity as

$$\begin{aligned} ahHq &= 2\Omega a\mu h + \frac{1}{1-\mu^2}\left\{h\left(\frac{\partial V}{\partial\lambda}\right)_q - \left(\frac{\partial V}{\partial q}\right)_{\lambda}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda} \\ &= a2\Omega\mu\left(\frac{\partial\mu}{\partial q}\right)_{\lambda} + \frac{1}{1-\mu^2}\left\{\left(\frac{\partial V}{\partial\lambda}\right)_q\left(\frac{\partial\mu}{\partial q}\right)_{\lambda} - \left(\frac{\partial V}{\partial q}\right)_{\lambda}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda}. \\ &= a\Omega\left(\frac{\partial\mu^2}{\partial q}\right)_{\lambda} + \frac{1}{1-\mu^2}\left\{\left(\frac{\partial V}{\partial\lambda}\right)_q\left(\frac{\partial\mu}{\partial q}\right)_{\lambda} - \left(\frac{\partial V}{\partial q}\right)_{\lambda}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\} - \left(\frac{\partial U}{\partial q}\right)_{\lambda} \end{aligned} \tag{C.14}$$

Applying (C.11)-(C.13) to the potential vorticity equation, (C.9), we obtain a formula that governs the latitude of a potential-vorticity contour:

$$\left\{\left(\frac{\partial q}{\partial t}\right)_q - \frac{1}{h}\left(\frac{\partial q}{\partial q}\right)_t\left(\frac{\partial \mu}{\partial t}\right)_q\right\} + \frac{U}{a(1-\mu^2)}\left\{\left(\frac{\partial q}{\partial \lambda}\right)_q - \frac{1}{h}\left(\frac{\partial q}{\partial q}\right)_\lambda\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\} + \frac{V}{a}\left(\frac{\partial q}{\partial \mu}\right)_\lambda = 0 \quad \text{(C.15)}$$

$$\left\{-\frac{1}{h}\left(\frac{\partial \mu}{\partial t}\right)_q\right\} + \frac{U}{a(1-\mu^2)}\left\{-\frac{1}{h}\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\} + \frac{V1}{ah} = 0 \qquad \text{(C.16)}$$

$$\left(\frac{\partial \mu}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_q - V\right\} = 0. \qquad \text{(C.17)}$$

A similar formula would apply for any conservative variable. By differentiating (C.17), we find that

$$\left(\frac{\partial h}{\partial t}\right)_q + \frac{1}{a}\frac{\partial}{\partial q}\left\{\left(\frac{\partial \mu}{\partial \lambda}\right)_q \frac{U}{(1-\mu^2)} - V\right\} = 0. \qquad \text{(C.18)}$$

Next, we transform the continuity equation:

$$\left(\frac{\partial H}{\partial t}\right)_\mu + \frac{1}{a(1-\mu^2)}\left\{\frac{\partial}{\partial \lambda}(HU)\right\}_\mu + \frac{1}{a}\frac{\partial (HV)}{\partial \mu} = 0 \qquad \text{(C.19)}$$

$$\left(\frac{\partial H}{\partial t}\right)_\mu + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_\mu + \frac{1}{a}\frac{\partial (HV)}{\partial \mu} = 0 \qquad \text{(C.20)}$$

$$\left\{\left(\frac{\partial H}{\partial t}\right)_q - \frac{1}{h}\left(\frac{\partial H}{\partial q}\right)\left(\frac{\partial \mu}{\partial t}\right)_q\right\} + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q$$
$$-\frac{1}{ah}\left(\frac{\partial \mu}{\partial \lambda}\right)_q\left\{\frac{\partial}{\partial q}\left(\frac{HU}{1-\mu^2}\right)\right\} + \frac{1}{ah}\frac{\partial (HV)}{\partial q} = 0 \tag{C.21}$$

Use (C.17) to eliminate $\left(\frac{\partial \mu}{\partial t}\right)_q$ in (C.21):

$$\left(\frac{\partial H}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q - \frac{1}{ah}\left(\frac{\partial \mu}{\partial \lambda}\right)_q\left\{\frac{\partial}{\partial q}\left(\frac{HU}{1-\mu^2}\right)\right\} + \frac{1}{ah}\frac{\partial (HV)}{\partial q}$$
$$-\frac{1}{h}\left(\frac{\partial H}{\partial q}\right)\left(\frac{\partial \mu}{\partial t}\right)_q = 0 \tag{C.22}$$

$$\left(\frac{\partial H}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q - \frac{1}{ah}\left(\frac{\partial \mu}{\partial \lambda}\right)_q\left\{\frac{\partial}{\partial q}\left(\frac{HU}{1-\mu^2}\right)\right\} + \frac{1}{ah}\frac{\partial (HV)}{\partial q}$$
$$+\frac{1}{h}\left(\frac{\partial H}{\partial q}\right)\left\{\frac{U}{a(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_q - \frac{V}{a}\right\} = 0 \tag{C.23}$$

$$\left(\frac{\partial H}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q - \frac{H}{ah}\left\{\left(\frac{\partial \mu}{\partial \lambda}\right)_q\frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right) - \frac{\partial V}{\partial q}\right\} = 0. \tag{C.24}$$

Now we eliminate $\frac{\partial V}{\partial q}$ in (C.24) by substituting from (C.18):

$$\left(\frac{\partial H}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q - \frac{H}{ah}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right)$$
$$+\frac{H}{h}\left(\left(\frac{\partial h}{\partial t}\right)_q + \frac{1}{a}\frac{\partial}{\partial q}\left\{\left(\frac{\partial\mu}{\partial\lambda}\right)_q\frac{U}{(1-\mu^2)}\right\}\right) = 0 \tag{C.25}$$

$$h\left(\frac{\partial H}{\partial t}\right)_q + \frac{h}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q - \frac{H}{a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right)$$
$$+H\left(\frac{\partial h}{\partial t}\right)_q + \frac{H}{a}\frac{\partial}{\partial q}\left\{\left(\frac{\partial\mu}{\partial\lambda}\right)_q\frac{U}{(1-\mu^2)}\right\} = 0 \tag{C.26}$$

$$\left\{\frac{\partial}{\partial t}(hH)\right\}_q + \frac{h}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q - \frac{H}{a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\frac{\partial}{\partial q}\left(\frac{U}{1-\mu^2}\right)$$
$$+\frac{H}{a}\frac{\partial}{\partial q}\left\{\left(\frac{\partial\mu}{\partial\lambda}\right)_q\frac{U}{(1-\mu^2)}\right\} = 0 \tag{C.27}$$

$$\left\{\frac{\partial}{\partial t}(hH)\right\}_q + \frac{h}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{HU}{1-\mu^2}\right)\right\}_q + \frac{HU}{a(1-\mu^2)}\frac{\partial}{\partial q}\left\{\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\} = 0 \tag{C.28}$$

$$\left\{\frac{\partial}{\partial t}(hH)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHU}{1-\mu^2}\right)\right\}_q = 0 \tag{C.29}$$

This is very nice and simple. The mass flows in only one direction, parallel to the latitudinal circle on a surface of constant potential vorticity.

Next, transform the Lagrangian derivative:

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{\mu} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_{\mu} + \frac{V}{a}\frac{\partial}{\partial \mu} \tag{C.30}$$

$$\frac{D}{Dt} = \left\{\left(\frac{\partial}{\partial t}\right)_{q} - \left(\frac{\partial \mu}{\partial t}\right)_{q}\frac{1}{h}\left(\frac{\partial}{\partial q}\right)_{t}\right\} + \frac{U}{a(1-\mu^2)}\left\{\left(\frac{\partial}{\partial \lambda}\right)_{q} - \left(\frac{\partial \mu}{\partial \lambda}\right)_{q}\frac{1}{h}\left(\frac{\partial}{\partial q}\right)_{\lambda}\right\} + \frac{V}{ah}\frac{\partial}{\partial q} \tag{C.31}$$

$$\begin{aligned}\frac{D}{Dt} &= \left\{\left(\frac{\partial}{\partial t}\right)_{q} - \left(\frac{\partial \mu}{\partial t}\right)_{q}\frac{1}{h}\left(\frac{\partial}{\partial q}\right)_{t}\right\} + \frac{U}{a(1-\mu^2)}\left\{\left(\frac{\partial}{\partial \lambda}\right)_{q} - \left(\frac{\partial \mu}{\partial \lambda}\right)_{q}\frac{1}{h}\left(\frac{\partial}{\partial q}\right)_{\lambda}\right\} + \frac{V}{ah}\frac{\partial}{\partial q} \\ &= \left(\frac{\partial}{\partial t}\right)_{q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_{q} - \frac{1}{h}\left\{\left(\frac{\partial \mu}{\partial t}\right)_{q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q} - \frac{V}{a}\right\}\frac{\partial}{\partial q}\end{aligned} \tag{C.32}$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t}\right)_{q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial}{\partial \lambda}\right)_{q}. \tag{C.33}$$

The angular momentum equation can be transformed as follows:

$$\frac{DM}{Dt} + g\left(\frac{\partial H}{\partial \lambda}\right)_{\mu} = 0, \tag{C.34}$$

$$\left(\frac{\partial M}{\partial t}\right)_{\lambda, q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial M}{\partial \lambda}\right)_{q} + g\left\{\left(\frac{\partial H}{\partial \lambda}\right)_{q} - \frac{1}{h}\left(\frac{\partial H}{\partial q}\right)_{\lambda}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q}\right\} = 0. \tag{C.35}$$

Convert (C.35) to flux form, using the continuity equation (C.29):

$$\left(\frac{\partial M}{\partial t}\right)_{\lambda, q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial M}{\partial \lambda}\right)_{q} + g\left(\frac{\partial H}{\partial \lambda}\right)_{q} - \frac{g}{h}\left(\frac{\partial H}{\partial q}\right)_{\lambda}\left(\frac{\partial \mu}{\partial \lambda}\right)_{q} = 0 \tag{C.36}$$

$$\left\{\frac{\partial}{\partial t}(hHM)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHUM}{1-\mu^2}\right)\right\}_q + ghH\left(\frac{\partial H}{\partial\lambda}\right)_q - \frac{ghH}{h}\left(\frac{\partial H}{\partial q}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q = 0. \tag{C.37}$$

We want to write the pressure-gradient term of (C.37) in the form

$$ghH\left(\frac{\partial H}{\partial\lambda}\right)_q - \frac{ghH}{h}\left(\frac{\partial H}{\partial q}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q = \left(\frac{\partial A}{\partial\lambda}\right)_q + \left\{\frac{\partial}{\partial q}B\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\}_\lambda. \tag{C.38}$$

For this we can write,

$$\begin{aligned}
&ghH\left(\frac{\partial H}{\partial\lambda}\right)_q - \frac{ghH}{h}\left(\frac{\partial H}{\partial q}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q \\
&= \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q + g\left\{hH\left(\frac{\partial H}{\partial\lambda}\right)_q - \left(\frac{\partial}{\partial\lambda}h\frac{H^2}{2}\right)_q\right\} - g\left(\frac{\partial}{\partial q}\frac{H^2}{2}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q \\
&= \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q + g\left\{h\left(\frac{\partial}{\partial\lambda}\frac{H^2}{2}\right)_q - \left(\frac{\partial}{\partial\lambda}h\frac{H^2}{2}\right)_q\right\} - g\left(\frac{\partial}{\partial q}\frac{H^2}{2}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q \\
&= \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q - g\frac{H^2}{2}\left(\frac{\partial h}{\partial\lambda}\right)_q - g\left(\frac{\partial}{\partial q}\frac{H^2}{2}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q \\
&= \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q - g\frac{H^2}{2}\left\{\frac{\partial\left(\frac{\partial\mu}{\partial q}\right)_\lambda}{\partial\lambda}\right\}_q - g\left(\frac{\partial}{\partial q}\frac{H^2}{2}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q \\
&= \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q - g\frac{H^2}{2}\left\{\frac{\partial\left(\frac{\partial\mu}{\partial\lambda}\right)_q}{\partial q}\right\}_q - g\left(\frac{\partial}{\partial q}\frac{H^2}{2}\right)_\lambda\left(\frac{\partial\mu}{\partial\lambda}\right)_q \\
&= \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q - g\left(\frac{\partial}{\partial q}\left\{\frac{H^2}{2}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\}\right)_\lambda
\end{aligned} \tag{C.39}$$

Substituting back, we get

$$\left\{\frac{\partial}{\partial t}(hHM)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{hHUM}{1-\mu^2}\right)\right\}_q + \left(\frac{\partial}{\partial \lambda}h\frac{gH^2}{2}\right)_q - \left(\frac{\partial}{\partial q}\left\{\frac{gH^2}{2}\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\}\right)_\lambda = 0. \quad \text{(C.40)}$$

The advective form of the zonal momentum equation can be transformed to

$$\left(\frac{\partial U}{\partial t}\right)_{\lambda, q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial U}{\partial \lambda}\right)_q - 2\Omega\mu V + \frac{g}{a}\left\{\left(\frac{\partial H}{\partial \lambda}\right)_q - \frac{1}{h}\left(\frac{\partial H}{\partial q}\right)_\lambda\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\} = 0. \quad \text{(C.41)}$$

The flux form is

$$\begin{aligned} &\left\{\frac{\partial}{\partial t}(hHU)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{hHUU}{1-\mu^2}\right)\right\}_q - 2\Omega\mu hHV \\ &+ \left(\frac{\partial}{\partial \lambda}h\frac{gH^2}{2a}\right)_q - \left(\frac{\partial}{\partial q}\left\{\frac{gH^2}{2a}\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\}\right)_\lambda = 0 \end{aligned} \quad . \quad \text{(C.42)}$$

Similarly, the meridional momentum equation can be transformed to

$$\left(\frac{\partial V}{\partial t}\right)_{\lambda, q} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_q + 2\mu\left(\Omega U + \frac{K}{a}\right) + \frac{g(1-\mu^2)}{ah}\left(\frac{\partial H}{\partial q}\right)_\lambda = 0. \quad \text{(C.43)}$$

Convert (C.43) to flux form, using the continuity equation (C.29):

$$\left\{\frac{\partial}{\partial t}(hHV)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{hHUV}{1-\mu^2}\right)\right\}_q + 2\mu hH\left(\Omega U + \frac{K}{a}\right) + (1-\mu^2)\left(\frac{\partial}{\partial q}\frac{gH^2}{2a}\right)_\lambda = 0. \quad \text{(C.44)}$$

Finally, the kinetic energy equation can be rewritten, in flux form, as

$$\left\{\frac{\partial}{\partial t}(hHK)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHUK}{1-\mu^2}\right)\right\}_q$$
$$+\frac{U}{(1-\mu^2)}\left\{\left(\frac{\partial}{\partial\lambda}h\frac{gH^2}{2a}\right)_q - \left(\frac{\partial}{\partial q}\left\{\frac{gH^2}{2a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\}\right)_\lambda\right\} + V\left\{\left(\frac{\partial}{\partial q}\frac{gH^2}{2a}\right)_\lambda\right\} = 0 \quad . \tag{C.45}$$

C.3: Summary and discussion

The transformed prognostic equations are

$$\left\{\frac{\partial}{\partial t}(hH)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHU}{1-\mu^2}\right)\right\}_q = 0\,, \tag{C.46}$$

$$\left\{\frac{\partial}{\partial t}(hHM)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHUM}{1-\mu^2}\right)\right\}_q + \left(\frac{\partial}{\partial\lambda}gh\frac{H^2}{2}\right)_q - \left(\frac{\partial}{\partial q}\left\{\frac{gH^2}{2}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\}\right)_\lambda = 0\,, \tag{C.47}$$

$$\left\{\frac{\partial}{\partial t}(hHK)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHUK}{1-\mu^2}\right)\right\}_q$$
$$+\frac{U}{(1-\mu^2)}\left\{\left(\frac{\partial}{\partial\lambda}h\frac{gH^2}{2a}\right)_q - \left(\frac{\partial}{\partial q}\left\{\frac{gH^2}{2a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\}\right)_\lambda\right\} + V\left\{\frac{1}{a}\left(\frac{\partial}{\partial q}\frac{gH^2}{2}\right)_\lambda\right\} = 0 \quad , \tag{C.48}$$

$$\left\{\frac{\partial}{\partial t}(hHU)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial\lambda}\left(\frac{hHUU}{1-\mu^2}\right)\right\}_q - 2\Omega\mu hHV$$
$$+\left(\frac{\partial}{\partial\lambda}h\frac{gH^2}{2a}\right)_q - \left(\frac{\partial}{\partial q}\left\{\frac{gH^2}{2a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right\}\right)_\lambda = 0 \quad , \tag{C.49}$$

$$\left\{\frac{\partial}{\partial t}(hHV)\right\}_q + \frac{1}{a}\left\{\frac{\partial}{\partial \lambda}\left(\frac{hHUV}{1-\mu^2}\right)\right\}_q + 2\mu hH\left(\Omega U + \frac{K}{a}\right) + \frac{g(1-\mu^2)}{a}\left(\frac{\partial}{\partial q}\frac{H^2}{2}\right)_\lambda = 0, \quad \text{(C.50)}$$

$$\left(\frac{\partial \mu}{\partial t}\right)_q + \frac{1}{a}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_q - V\right\} = 0, \quad \text{(C.51)}$$

$$\left(\frac{\partial h}{\partial t}\right)_q + \frac{1}{a}\frac{\partial}{\partial q}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_q - V\right\} = 0. \quad \text{(C.52)}$$

Only three of these prognostic equations are independent. We also have the diagnostic relationships

$$M \equiv aU + \Omega a^2(1-\mu^2), \quad \text{(C.53)}$$

$$K \equiv \frac{1}{2}\left(\frac{U^2+V^2}{1-\mu^2}\right), \quad \text{(C.54)}$$

$$ahHq = a\Omega\left(\frac{\partial \mu^2}{\partial q}\right)_\lambda + \frac{1}{(1-\mu^2)}\left\{\left(\frac{\partial V}{\partial \lambda}\right)_q\left(\frac{\partial \mu}{\partial q}\right)_\lambda - \left(\frac{\partial V}{\partial q}\right)_\lambda\left(\frac{\partial \mu}{\partial \lambda}\right)_q\right\} - \left(\frac{\partial U}{\partial q}\right)_\lambda. \quad \text{(C.55)}$$

As an exercise, we can take $\left(\frac{\partial}{\partial t}\right)_q$ of (C.55), and substitute for $\left\{\frac{\partial}{\partial t}(hH)\right\}_q$, $\left(\frac{\partial U}{\partial t}\right)_q$, and $\left(\frac{\partial V}{\partial t}\right)_q$. This should yield (C.51).

C.4: The nondivergent barotropic model

The nondivergent barotropic model is a special case of the divergent barotropic model in which the continuity equation is replaced by:

$$\frac{1}{a\cos\varphi}\left(\frac{\partial u}{\partial\lambda}\right)_{\varphi} + \frac{1}{a\cos\varphi}\left(\frac{\partial}{\partial\varphi}v\cos\varphi\right)_{\lambda} = 0 \cdot \tag{C.56}$$

Using the modified horizontal velocity components equation (C.56) to be written as

$$\frac{1}{a(1-\mu^2)}\left(\frac{\partial U}{\partial\lambda}\right)_{\mu} + \left(\frac{\partial V}{\partial\mu}\right)_{\lambda} = 0\,. \tag{C.57}$$

When the flow is horizontally nondivergent, we can express the modified horizontal velocity components U and V in terms of a single variable, the stream function ψ, i.e.

$U = -(1-\mu^2)\frac{\partial\psi}{a\partial\mu}$, $V = \frac{\partial\psi}{a\partial\lambda}$ which automatically satisfy equation (C.57).

The potential vorticity principle (C.9) reduces to the conservation of the absolute vorticity in case of the nondivergent shallow water model as follows,

$$\left(\frac{\partial\eta}{\partial t}\right)_{\lambda,\,\mu} + \frac{U}{a(1-\mu^2)}\left(\frac{\partial\eta}{\partial\lambda}\right)_{\mu} + \frac{V}{a}\left(\frac{\partial\eta}{\partial\mu}\right)_{\lambda} = 0\,, \tag{C.58}$$

where $\eta = 2\Omega\mu + \frac{1}{a(1-\mu^2)}\left(\frac{\partial V}{\partial \lambda}\right)_\mu - \frac{1}{a}\left(\frac{\partial U}{\partial \mu}\right)_\lambda = 2\Omega\mu + \nabla^2\psi$ represents the vertical component of the absolute vorticity.

We define the 'potential vorticity thickness' $h = \left(\frac{\partial \mu}{\partial \eta}\right)_\lambda$ with h positive and finite and using (C.11)-(C.13) we can transform the vorticity equation (C.58) into (λ, η, t) coordinates as follows

$$\frac{1}{h}\left(\frac{\partial \mu}{\partial t}\right)_{\lambda, \eta} + \frac{1}{ah}\left\{\frac{U}{(1-\mu^2)}\left(\frac{\partial \mu}{\partial \lambda}\right)_\eta - V\right\} = 0, \tag{C.59}$$

Which is similar to equation (C.17) and can also be written as

$$\frac{1}{h}\left\{\left(\frac{\partial \mu}{\partial t}\right)_{\lambda, \eta} - \frac{\partial \psi}{a^2 \partial \mu}\left(\frac{\partial \mu}{\partial \lambda}\right)_\eta - \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \lambda}\right)_\mu\right\} = 0, \tag{C.60}$$

and reduces to

$$\frac{1}{h}\left\{\left(\frac{\partial \mu}{\partial t}\right)_{\lambda, \eta} - \frac{1}{a^2}\left(\frac{\partial \psi}{\partial \lambda}\right)_\eta\right\} = 0, \tag{C.61}$$

since $\left(\frac{\partial \psi}{\partial \lambda}\right)_\eta = \left(\frac{\partial \psi}{\partial \lambda}\right)_\mu + \left(\frac{\partial \mu}{\partial \lambda}\right)_\eta \frac{\partial \psi}{\partial \mu}$.

Using the definition of potential vorticity we can express μ as a function of ψ:

$$\mu = \frac{1}{2\Omega}(\eta - \nabla^2\psi), \tag{C.62}$$

and substitute (C.62) into (C.61) we obtain the nondivergent shallow water model,

$$\frac{1}{h}\left\{\frac{\partial}{\partial t}\nabla^2\psi + \frac{2\Omega}{a^2}\left(\frac{\partial\psi}{\partial\lambda}\right)\right\} = 0, \tag{C.63}$$

with the following boundary conditions

$$\begin{aligned} \psi(\lambda = 0) &= \psi(\lambda = 2\pi) \\ \int_S \psi dS &= 0 \end{aligned} \tag{C.64}$$

Equation (C.63) is similar with the linearized equation of the relative vorticity in (λ, φ, t) coordinates.

Note that the analytical expression of the Laplacian operator in equation (C.63) is different from that in spherical coordinates. Equation (C.63) gives the kinetic energy principle for the inviscid fluid. To obtain the kinetic energy principle we multiply equation (C.63) by the streamfunction $-\psi$:

$$\left\{\frac{1}{h}\frac{\partial}{\partial t}\nabla^2\psi + \frac{2\Omega}{a^2 h}\left(\frac{\partial\psi}{\partial\lambda}\right) = 0\right\}\cdot(-\psi) \tag{C.65}$$

$$\frac{1}{h}\left\{-\psi\frac{\partial}{\partial t}\nabla^2\psi - \frac{2\Omega}{a^2}\psi\left(\frac{\partial\psi}{\partial\lambda}\right)\right\} = 0 \tag{C.66}$$

The first term on the left hand side of the equation can be written as,

$$-\psi\frac{\partial}{\partial t}\nabla^2\psi = -\nabla\bullet\left(\psi\frac{\partial}{\partial t}\nabla\psi\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\nabla\psi\bullet\nabla\psi\right) \tag{C.67}$$

Using equation (C.67) we can write (C.66) as,

$$\frac{1}{h}\left\{-\nabla\bullet\left(\psi\frac{\partial}{\partial t}\nabla\psi\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\nabla\psi\bullet\nabla\psi\right) - \frac{\Omega}{a^2}\left(\frac{\partial\psi^2}{\partial\lambda}\right)\right\} = 0 \tag{C.68}$$

The integration of (C.68) over the sphere then yields,

$$\iint\frac{1}{h}\left\{-\nabla\bullet\left(\psi\frac{\partial}{\partial t}\nabla\psi\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\nabla\psi\bullet\nabla\psi\right) - \frac{\Omega}{a^2}\left(\frac{\partial\psi^2}{\partial\lambda}\right)\right\}hd\lambda dq = 0\,, \tag{C.69}$$

which after the integration gives,

$$\frac{dK}{dt} = 0 \tag{C.70}$$

where,

$$K = \iint\frac{1}{2}\nabla\psi\bullet\nabla\psi d\lambda d\eta\,. \tag{C.71}$$

is the kinetic energy, per unit mass.

C.5: The equatorial β-plane approximation

First it is convenient to make a geometrical simplification, by replacing the spherical coordinates (λ, q, t) with the cartesian coordinates (x, q, t). The nonlinear primitive equations in (x, q, t) coordinates reduce to

$$\left(\frac{\partial u}{\partial t}\right)_{x,q} + u\left(\frac{\partial u}{\partial x}\right)_q - fv + g\left(\frac{\partial H}{\partial x}\right)_q - \frac{g}{h}\left(\frac{\partial H}{\partial q}\right)_x\left(\frac{\partial y}{\partial x}\right)_q = 0 \ , \tag{C.72}$$

$$\left(\frac{\partial v}{\partial t}\right)_{x,q} + u\left(\frac{\partial v}{\partial x}\right)_q + fu + \frac{g}{h}\left(\frac{\partial H}{\partial q}\right)_x = 0 \ , \tag{C.73}$$

$$\left(\frac{\partial}{\partial t}hH\right)_{x,q} + \left(\frac{\partial}{\partial x}hHu\right)_q = 0 \,. \tag{C.74}$$

These equations are very complicated because of nonlinearity and we shall consider small amplitude motions about a basic state of rest: $u = u'(x, q, t)$, $v = v'(x, q, t)$, $H = H + H'(x, q, t)$, $h = \bar{h} + h'(x, q, t)$ and $y = y + y'(x, q, t)$. We also restrict the flow domain to some neighborhood of the equatorial latitude.

In (x, y) coordinates the β-plane approximation arises from expanding the Coriolis parameter in a Taylor series about the latitude y_0 as: $f = f_0 + \beta y$ + (higher-order terms), where $\beta = \left(\frac{df}{dy}\right)_{y_0}$ and $y = 0$ at the Equator.

In the same way we can expand the Coriolis parameter in a Taylor series about the potential vorticity $q(y_0)$ as: $f = f_0 + \beta^* q$ + (higher-order terms), where $\beta^* = \left(\frac{df}{dq}\right)_{y_0}$ and $q = 0$ at the Equator.

The linearized shallow water equations governing such motions are:

$$\left(\frac{\partial u'}{\partial t}\right)_{x,q} + u'\left(\frac{\partial u'}{\partial x}\right)_q - \beta^* q v' + g\left(\frac{\partial H'}{\partial x}\right)_q - \frac{g}{\bar{h}+h'}\left(\frac{\partial H'}{\partial q}\right)_x\left(\frac{\partial y'}{\partial x}\right)_q = 0, \tag{C.75}$$

$$\left(\frac{\partial v'}{\partial t}\right)_{x,q} + u'\left(\frac{\partial v'}{\partial x}\right)_q + \beta^* q u' + \frac{g}{\bar{h}+h'}\left(\frac{\partial H'}{\partial q}\right)_x = 0, \tag{C.76}$$

$$\left\{\frac{\partial(\bar{H}+H')(\bar{h}+h')}{\partial t}\right\}_{x,q} + u'\left\{\frac{\partial(\bar{H}+H')(\bar{h}+h')}{\partial x}\right\}_q + (\bar{H}+H')(\bar{h}+h')\left(\frac{\partial u'}{\partial x}\right)_q = 0. \tag{C.77}$$

Now $\frac{1}{\bar{h}+h'} = \frac{1}{\bar{h}\left(1+\frac{h'}{\bar{h}}\right)} \approx \frac{1}{\bar{h}}\left(1-\frac{h'}{\bar{h}}\right)$, since $\frac{h'}{\bar{h}} \ll 1$. The perturbation fields must be small enough so that all terms that are products of perturbation variables can be neglected and the nonlinear equations are reduced to linear differential equations in the perturbation variables.

$$\left(\frac{\partial u'}{\partial t}\right)_{x,q} - \beta^* q v' + g\left(\frac{\partial H'}{\partial x}\right)_q = 0, \tag{C.78}$$

$$\left(\frac{\partial v'}{\partial t}\right)_{x,q} + \beta^* q u' + \frac{g}{\bar{h}}\left(\frac{\partial H'}{\partial q}\right)_x = 0 \tag{C.79}$$

$$\bar{H}\left(\frac{\partial h'}{\partial t}\right)_{x,q} + \bar{h}\left(\frac{\partial H'}{\partial t}\right)_{x,q} + \bar{H}\bar{h}\left(\frac{\partial u'}{\partial x}\right)_q = 0. \tag{C.80}$$

Before solving these equations it is convenient to put the problem in nondimensional form. We define $c = (g\bar{H})^{1/2}$ as the constant gravity wave speed based on the mean depth H. As a horizontal length scale let us choose $L = (c\bar{h}/\beta^*)^{1/2}$ and $L_q = (c/\beta^*\bar{h})^{1/2}$, similarly let us choose as unit of time $T = (\bar{h}/c\beta^*)^{1/2}$. Then, choosing c as the unit of speed and H as the unit of depth, we obtain

$$\frac{c^{3/2}\beta^{*3/2}}{\bar{h}^{1/2}}\frac{\partial u}{\partial t} - \frac{c^{3/2}\beta^{*3/2}}{\bar{h}^{1/2}}vq + \frac{c^{3/2}\beta^{*3/2}}{\bar{h}^{1/2}}\frac{\partial H}{\partial x} = 0 , \tag{C.81}$$

$$\frac{c^{3/2}\beta^{*3/2}}{\bar{h}^{1/2}}\frac{\partial v}{\partial t} + \frac{c^{3/2}\beta^{*3/2}}{\bar{h}^{1/2}}qu + \frac{c^{3/2}\beta^{*3/2}}{\bar{h}^{1/2}}\frac{\partial H}{\partial q} = 0 , \tag{C.82}$$

$$\frac{\bar{H}\bar{h}^{1/2}\beta^{*1/2}}{c^{-1/2}}\frac{\partial h}{\partial t} + \frac{\bar{H}\bar{h}^{1/2}\beta^{*1/2}}{c^{-1/2}}\frac{\partial H}{\partial t} + \frac{\bar{H}\bar{h}^{1/2}\beta^{*1/2}}{c^{-1/2}}\frac{\partial u'}{\partial x} = 0 , \tag{C.83}$$

Therefore we can write

$$\frac{\partial u}{\partial t} - vq + \frac{\partial H}{\partial x} = 0, \tag{C.84}$$

$$\frac{\partial v}{\partial t} + qu + \frac{\partial H}{\partial q} = 0, \tag{C.85}$$

$$\frac{\partial h}{\partial t} + \frac{\partial H}{\partial t} + \frac{\partial u}{\partial x} = 0. \tag{C.86}$$

where all variables are now nondimensional.

To obtain the eigenvalues and eigenfunctions we search for solutions of the form

$$\begin{bmatrix} u(x,q,t) \\ v(x,q,t) \\ H(x,q,t) \\ h(x,q,t) \end{bmatrix} = \begin{bmatrix} \mathcal{U}(k,q) \\ \mathcal{V}(k,q) \\ \mathcal{H}(k,q) \\ \mathcal{W}(k,q) \end{bmatrix} e^{i(kx+\nu t)} \tag{C.87}$$

where k is the zonal wavenumber and ν is the frequency.

Substituting (C.87) in (C.84)-(C.86) we obtain

$$i\nu\mathcal{U} - q\mathcal{V} + ik\mathcal{H} = 0, \tag{C.88}$$

$$i\nu\mathcal{V} + q\mathcal{U} + \frac{d\mathcal{H}}{dq} = 0, \tag{C.89}$$

$$i\nu\mathcal{W} + i\nu\mathcal{H} + ik\mathcal{U} = 0. \tag{C.90}$$

Because we have four unknowns and three equations, we need one more equation and this is

$$\left(\frac{\partial v}{\partial q}\right)_x = \left(\frac{\partial h}{\partial t}\right)_{x,q} + u\left(\frac{\partial h}{\partial x}\right)_q + \left(\frac{\partial u}{\partial q}\right)_x\left(\frac{\partial y}{\partial x}\right)_q, \tag{C.91}$$

which after linearization becomes

$\frac{\partial v}{\partial q} = \frac{\partial h}{\partial t}$, where all variables are dimensionless. Using again solutions (C.87) we can substitute for $\mathcal{W}$ in equation (C.90) which becomes

$$\frac{d\mathcal{V}}{dq} + i\nu\mathcal{H} + ik\mathcal{U} = 0\,. \tag{C.92}$$

Eliminating $\mathcal{U}(k, q)$ and $\mathcal{H}(k, q))$ between (C.88) and (C.90) we get the equation for $\mathcal{V}(k, q)$

$$\frac{d^2\mathcal{V}}{dq^2} + \left(\nu^2 - k^2 + \frac{k}{\nu} - q^2\right) = 0\,. \tag{C.93}$$

Note that (C.93) is similar to that obtained by Matsuno (1966) using a divergent barotropic model on a local cartesian coordinate system.

APPENDIX D

Derivation of the pressure gradient force

We first begin to work on the first term of the $HPGF$

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{h}{g}\left\{\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\}\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} \\
&= -\frac{h}{g}\left\{\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\}\left\{\frac{\partial}{\partial \lambda}(\theta\Pi + gz)\right\}_{q,\theta} \\
&= -\frac{\theta h}{g}\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} - h\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta} \\
&\quad + \frac{\theta}{g}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}
\end{aligned}
\tag{D.1}
$$

Now we want to use the hydrostatic equation, and for this we have to construct terms that appear in the hydrostatic equation,

$$
0 = \theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} - \frac{g}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}.
$$

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{\partial}{\partial \theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{p}{g}\frac{\partial}{\partial \theta}\left\{\theta h\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} - \frac{\partial}{\partial \theta}\left\{hp\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} \\
&\quad + p\frac{\partial}{\partial \theta}\left\{h\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} - \frac{p}{g}\frac{\partial}{\partial q}\left\{\theta\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} \\
&\quad + \frac{\partial}{\partial q}\left\{p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} - p\frac{\partial}{\partial q}\left\{\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\}
\end{aligned}
\tag{D.2}
$$

$$\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} = -\frac{\partial}{\partial \theta}\left\{\theta h \frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \underline{\frac{p\theta}{g}\left(\frac{\partial h}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}}$$

$$+\frac{ph\theta}{g}\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} - \frac{\partial}{\partial \theta}\left\{hp\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \underline{p\left(\frac{\partial h}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}}$$

$$+ph\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} -$$

$$\underline{-\frac{p\theta}{g}\left(\frac{\partial h}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}} - \frac{p\theta}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial q}\left\{\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\}$$

$$+\frac{\partial}{\partial q}\left\{p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} - \underline{p\left(\frac{\partial h}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}} - p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial q}\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\}$$

Some nice cancellations occur:

$$\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} = -\frac{\partial}{\partial \theta}\left\{\theta h \frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \frac{ph\theta}{g}\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\}$$

$$-\frac{\partial}{\partial \theta}\left\{hp\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + ph\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} \quad , \qquad \text{(D.3)}$$

$$-\frac{p\theta}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial q}\left\{\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{\partial}{\partial q}\left\{p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} - p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial q}\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\}$$

rearrange terms:

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{\partial}{\partial \theta}\left\{\theta h \frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} \\
&\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \frac{ph\theta}{g}\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\}, \\
&+ ph\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} - \frac{p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial q}\left\{\theta\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}\right\} - p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial q}\left\{\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\}
\end{aligned}
\tag{D.4}
$$

change $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial q}$ derivatives with $\frac{\partial}{\partial \lambda}$,

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{\partial}{\partial \theta}\left\{\theta h \frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} \\
&\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \frac{ph}{g}\frac{\partial}{\partial \lambda}\left\{\theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q}\right\}, \\
&+ ph\frac{\partial}{\partial \lambda}\left\{\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q}\right\} - \frac{p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial \lambda}\left\{\theta\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} - p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial \lambda}\left\{\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}
\end{aligned}
\tag{D.5}
$$

in the last two terms multiply and divide by h,

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{\partial}{\partial \theta}\left\{\theta h \frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} \\
&\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + \frac{ph}{g}\frac{\partial}{\partial \lambda}\left\{\theta\left(\frac{\partial \Pi}{\partial \theta}\right)_{\lambda,q}\right\}, \\
&+ ph\frac{\partial}{\partial \lambda}\left\{\left(\frac{\partial z}{\partial \theta}\right)_{\lambda,q}\right\} - \frac{ph}{gh}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial \lambda}\left\{\theta\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} - \frac{ph}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\frac{\partial}{\partial \lambda}\left\{\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}
\end{aligned}
\tag{D.6}
$$

in the last two terms push $1/h$ inside the $\frac{\partial}{\partial\lambda}$ derivative,

$$\sigma h\left(\frac{\partial\mathcal{M}}{\partial\lambda}\right)_{q,\theta} = -\frac{\partial}{\partial\theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\}$$

$$\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} + p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} + \frac{ph}{g}\frac{\partial}{\partial\lambda}\left\{\theta\left(\frac{\partial\Pi}{\partial\theta}\right)_{\lambda,q}\right\}$$

$$+ ph\frac{\partial}{\partial\lambda}\left\{\left(\frac{\partial z}{\partial\theta}\right)_{\lambda,q}\right\} - \frac{ph}{g}\frac{\partial}{\partial\lambda}\left\{\frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{ph\theta}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\frac{\partial}{\partial\lambda}\left\{\frac{1}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\}, \quad \text{(D.7)}$$

$$-\frac{ph}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\frac{\partial}{\partial\lambda}\left\{\frac{g}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + ph\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\frac{\partial}{\partial\lambda}\left\{\frac{1}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\}$$

rearrange terms,

$$\sigma h\left(\frac{\partial\mathcal{M}}{\partial\lambda}\right)_{q,\theta} = -\frac{\partial}{\partial\theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\}$$

$$\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} + p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}, \quad \text{(D.8)}$$

$$\frac{ph}{g}\frac{\partial}{\partial\lambda}\left\{\theta\left(\frac{\partial\Pi}{\partial\theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial\theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} - \frac{g}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}$$

$$+\frac{ph\theta}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\frac{\partial}{\partial\lambda}\left\{\frac{1}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\} + ph\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\frac{\partial}{\partial\lambda}\left\{\frac{1}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right\}$$

the third line is zero because of the hydrostatic equation,

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{\partial}{\partial \theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} \\
&\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta}, \\
&+\frac{ph\theta}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\frac{\partial}{\partial \lambda}\left\{\frac{1}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\} + ph\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\frac{\partial}{\partial \lambda}\left\{\frac{1}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\}
\end{aligned}
\tag{D.9}
$$

expand the last two $\frac{\partial}{\partial \lambda}$ derivatives,

$$
\begin{aligned}
\sigma h\left(\frac{\partial \mathcal{M}}{\partial \lambda}\right)_{q,\theta} &= -\frac{\partial}{\partial \theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} \\
&+\left\{\frac{\theta p}{g}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} + p\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial \lambda}\right)_{q,\theta}\right\} + \frac{ph}{g}\left(\frac{\partial \Pi}{\partial \lambda}\right)_{q,\theta} \\
&-\left\{\frac{p\theta}{gh}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \frac{p}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\left(\frac{\partial h}{\partial \lambda}\right)_{q,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q} \\
&+\left\{\frac{p\theta}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\frac{\partial^2 \mu}{\partial \lambda \partial \theta}
\end{aligned}
\tag{D.10}
$$

Apply the same steps on the second term of the *HPGF*:

$$\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta} = -\frac{1}{g}\left\{\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\}\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta}$$

$$= -\frac{1}{g}\left\{\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q} - \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\right\}\left\{\frac{\partial}{\partial q}(\theta\Pi + gz)\right\}_{\lambda,\theta}$$

$$= -\frac{\theta}{g}\left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} - \left(\frac{\partial p}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}$$

$$+\frac{\theta}{gh}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \frac{1}{h}\left(\frac{\partial p}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta} \quad , \qquad \text{(D.11)}$$

$$= -\frac{\partial}{\partial \theta}\left\{\theta\frac{p}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{p}{g}\frac{\partial}{\partial \theta}\left\{\theta\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} - \frac{\partial}{\partial \theta}\left\{p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}$$

$$+p\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} - \frac{p}{g}\frac{\partial}{\partial q}\left\{\frac{\theta}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\}$$

$$+\frac{\partial}{\partial q}\left\{\frac{p}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} - p\frac{\partial}{\partial q}\left\{\frac{1}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}$$

change $\frac{\partial}{\partial \theta}$ derivative with $\frac{\partial}{\partial q}$ in the third term on the first line,

$$\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta} = -\frac{\partial}{\partial \theta}\left\{\theta\frac{p}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{p}{g}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta} + \frac{p\theta}{g}\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\}$$

$$-\frac{\partial}{\partial \theta}\left\{p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + p\frac{\partial}{\partial \theta}\left\{\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} \qquad \text{(D.12)}$$

$$-\frac{p}{g}\frac{\partial}{\partial q}\left\{\frac{\theta}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial \Pi}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{p}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} - p\frac{\partial}{\partial q}\left\{\frac{1}{h}\left(\frac{\partial \mu}{\partial \theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}$$

$$\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta} = -\frac{\partial}{\partial\theta}\left\{\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} + \frac{p}{g}\frac{\partial}{\partial q}\left\{\theta\left(\frac{\partial\Pi}{\partial\theta}\right)_{\lambda,q}\right\}$$

$$-\frac{\partial}{\partial\theta}\left\{p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{p}{g}\frac{\partial}{\partial q}\left\{g\left(\frac{\partial z}{\partial\theta}\right)_{\lambda,q}\right\} + \frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\right\} \qquad \text{(D.13)}$$

$$-\frac{p}{g}\frac{\partial}{\partial q}\left\{\frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{\partial}{\partial q}\left\{\frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} - \frac{p}{g}\frac{\partial}{\partial q}\left\{\frac{g}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}$$

rearrange terms,

$$\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta} = -\frac{\partial}{\partial\theta}\left\{\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} + p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}$$

$$+\frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} + \frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} \quad , \qquad \text{(D.14)}$$

$$+\frac{p}{g}\frac{\partial}{\partial q}\left\{\theta\left(\frac{\partial\Pi}{\partial\theta}\right)_{\lambda,q} + g\left(\frac{\partial z}{\partial\theta}\right)_{\lambda,q} - \frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} - \frac{g}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}$$

we can observe that using the hydrostatic equation the last line becomes zero.

$$\sigma\left(\frac{\partial \mathcal{M}}{\partial q}\right)_{\lambda,\theta} = -\frac{\partial}{\partial\theta}\left\{\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} + p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} + \frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}$$

$$+\frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta} + \frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\} \qquad \text{(D.15)}$$

Now combine (D.10) with (D.15) and the pressure gradient term becomes:

$$
\begin{aligned}
HPGF = & -\frac{\partial}{\partial\theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\} \\
& +\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\}+\frac{ph}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} \\
& -\left\{\frac{p\theta}{gh}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+\frac{p}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\left(\frac{\partial h}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q} \\
& +\left\{\frac{p\theta}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\frac{\partial^2\mu}{\partial\lambda\partial\theta} \\
& +\frac{\partial}{\partial\theta}\left\{\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}-\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} \\
& -\frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+\frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}
\end{aligned}
\quad , \tag{D.16}
$$

push $\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}$ inside of $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial q}$ derivatives,

$$
\begin{aligned}
HPGF = & -\frac{\partial}{\partial\theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\} \\
& +\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\} \\
& +\frac{ph}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}\underline{-\left\{\frac{p\theta}{gh}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+\frac{p}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\left(\frac{\partial h}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}} \\
& \underline{\left\{\frac{p\theta}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\frac{\partial^2\mu}{\partial\lambda\partial\theta}}+\frac{\partial}{\partial\theta}\left\{\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}+p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}
\end{aligned}
\tag{D.17}
$$

$$-\underline{\left\{\frac{p\theta}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\frac{\partial^2\mu}{\partial\lambda\partial\theta}}-\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}$$

$$-\frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}+\frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}$$

$$+\underline{\left\{\frac{p\theta}{gh}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}+\frac{p}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\right\}\left(\frac{\partial h}{\partial\lambda}\right)_{q,\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}}$$

Some cancellations occur,

$$\begin{aligned}
HPGF = {} & -\frac{\partial}{\partial\theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\} \\
& +\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}\right\}+\frac{ph}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta} \\
& +\frac{\partial}{\partial\theta}\left\{\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}+p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}-\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} \\
& -\frac{\partial}{\partial q}\left\{\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}+\frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}
\end{aligned}, \qquad \text{(D.18)}$$

rearrange terms,

$$\begin{aligned}
HPGF = {} & -\frac{\partial}{\partial\theta}\left\{\theta h\frac{p}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}-\theta\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}-p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} \\
& +\frac{\partial}{\partial q}\left\{\frac{\theta p}{g}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}+p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta}-\frac{\theta p}{gh}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right. \\
& \left.-\frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}+\frac{ph}{g}\left(\frac{\partial\Pi}{\partial\lambda}\right)_{q,\theta}-\frac{p}{g}\left(\frac{\partial\Pi}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}
\end{aligned} \qquad \text{(D.19)}$$

Define $F(p)$ such that

$$\frac{p\partial\Pi}{g\partial\lambda} \equiv \frac{\partial F}{\partial\lambda}. \tag{D.20}$$

For future reference, note that $F(p)$ will also satisfy

$$\frac{p\partial\Pi}{g\partial q} \equiv \frac{\partial F}{\partial q} \text{ and } \frac{p\partial\Pi}{g\partial\theta} \equiv \frac{\partial F}{\partial\theta}. \tag{D.21}$$

Using (D.20) and (D.21) the pressure gradient force becomes,

$$\begin{aligned} HPGF = & -\frac{\partial}{\partial\theta}\left\{\theta h\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \theta\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} - p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} \\ & + \frac{\partial}{\partial q}\left\{\theta\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} + p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right. \\ & \left. - \frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + \left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} - \left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} \end{aligned} \tag{D.22}$$

work on the last two terms,

$$HPGF = -\frac{\partial}{\partial\theta}\left\{\theta h\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} + ph\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \theta\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} - p\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}$$

$$+\frac{\partial}{\partial q}\left\{\theta\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} + p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{\theta}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right.$$

$$\left.-\frac{p}{h}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + \frac{\partial}{\partial\lambda}\left\{F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right\} - \underline{F\left(\frac{\partial h}{\partial\lambda}\right)_{q,\theta}} \tag{D.23}$$

$$-\frac{\partial}{\partial q}\left\{F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + \underline{F\left(\frac{\partial h}{\partial\lambda}\right)_{q,\theta}}$$

cancel the underline terms,

$$HPGF = \frac{\partial}{\partial\lambda}\left\{F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\theta}\right\} - \frac{\partial}{\partial\theta}\left\{\theta h\left[\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right.$$

$$\left.+ph\left[\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right\}$$

$$+\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta} + \theta\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left[\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right. \tag{D.24}$$

$$\left.+p\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\left[\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right]\right\} .$$

We can introduce the notation

$$G(\lambda, q, \theta) \equiv \theta\left\{\left(\frac{\partial F}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial F}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\} + p\left\{\left(\frac{\partial z}{\partial\lambda}\right)_{q,\theta} - \frac{1}{h}\left(\frac{\partial z}{\partial q}\right)_{\lambda,\theta}\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\theta}\right\}, \tag{D.25}$$

and the pressure gradient force is,

$$HPGF = \frac{1}{a}\frac{\partial}{\partial\lambda}\left\{F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} - \frac{1}{a}\frac{\partial}{\partial\theta}\left\{G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} + \frac{1}{a}\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta} + G\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q}\right\} \quad \text{(D.26)}$$

As a check, suppose that isolines of q are perfectly lined up along latitude circles. Then (D.26) reduces to

$$HPGF = \frac{1}{a}\frac{\partial F}{\partial\lambda}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta} - \frac{1}{a}\frac{\partial}{\partial\theta}\left\{G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\}, \quad \text{(D.27)}$$

and (D.25) reduces to

$$G(\lambda, q, \theta) \equiv \theta\left(\frac{\partial F}{\partial\lambda}\right)_{q,\,\theta} + p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\,\theta}. \quad \text{(D.28)}$$

Substituting (D.28) into (D.27), we get

$$\begin{aligned} HPGF &= \frac{1}{a}\frac{\partial F}{\partial\lambda}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta} - \frac{1}{a}\frac{\partial}{\partial\theta}\left(\left\{\theta\left(\frac{\partial F}{\partial\lambda}\right)_{q,\,\theta} + p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\,\theta}\right\}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right) \\ &= \frac{1}{a}\frac{\partial F}{\partial\lambda}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta} - \frac{1}{a}\frac{\partial}{\partial\theta}\left\{\theta\left(\frac{\partial F}{\partial\lambda}\right)_{q,\,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} - \frac{1}{a}\frac{\partial}{\partial\theta}\left\{p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} \\ &= \frac{1}{a}\frac{\partial F}{\partial\lambda}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta} - \frac{1}{a}\frac{\partial F}{\partial\lambda}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta} - \frac{\theta}{a}\frac{\partial}{\partial\theta}\left\{\left(\frac{\partial F}{\partial\lambda}\right)_{q,\,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} \\ &\quad - \frac{1}{a}\frac{\partial}{\partial\theta}\left\{p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\}. \end{aligned} \quad \text{(D.29)}$$

This can be simplified to

$$HPGF = -\frac{\theta}{a}\frac{\partial}{\partial\theta}\left\{\left(\frac{\partial F}{\partial\lambda}\right)_{q,\,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} - \frac{1}{a}\frac{\partial}{\partial\theta}\left\{p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\,\theta}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\}. \tag{D.30}$$

If we now assume that $\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}$ is independent of θ, which it would be if there was a simple one-to-one correspondence between μ and q, we get

$$HPGF = -\frac{1}{a}\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\left(\theta\frac{\partial}{\partial\theta}\left(\frac{\partial F}{\partial\lambda}\right)_{q,\,\theta} + \frac{\partial}{\partial\theta}\left\{p\left(\frac{\partial z}{\partial\lambda}\right)_{q,\,\theta}\right\}\right), \tag{D.31}$$

which corresponds with the $HPGF$ in isentropic coordinates.

If we assume that there are no vertical variations then, $\frac{\partial}{\partial\theta}\left\{G\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} = 0$ and $\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,\,q} = 0$ and the $HPGF$ reduces to

$$HPGF = \frac{1}{a}\frac{\partial}{\partial\lambda}\left\{F\left(\frac{\partial\mu}{\partial q}\right)_{\lambda,\,\theta}\right\} + \frac{1}{a}\frac{\partial}{\partial q}\left\{-F\left(\frac{\partial\mu}{\partial\lambda}\right)_{q,\,\theta}\right\} \tag{D.32}$$

which corresponds to the $HPGF$ for the shallow water case.

APPENDIX E

Parameterization of the form drag

Upon examination of (5.23), we see that in trying to predict the zonal mean angular momentum will be useful to parameterize the right-hand side of (5.23). We will focus on the form drag term $F^*\left(\frac{\partial\mu^*}{\partial\lambda}\right)_{q,\theta}$ and find a prognostic equation for this quantity using (4.75) rearranged as

$$\left(\frac{\partial\mu}{\partial t}\right)_q + \frac{U_r}{a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q + \dot{q}\left(\frac{\partial\mu}{\partial q}\right) + \dot{\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q} = \frac{V}{a}, \tag{E.1}$$

where $U_r = \dfrac{U}{1-\mu^2}$ and the other notations are standard.

Zonally averaging (E.1) gives,

$$\frac{\partial}{\partial t}[\mu] + \left[\frac{U_r}{a}\left(\frac{\partial\mu}{\partial\lambda}\right)_q\right] + \left[\dot{q}\left(\frac{\partial\mu}{\partial q}\right)\right] + \left[\dot{\theta}\left(\frac{\partial\mu}{\partial\theta}\right)_{\lambda,q}\right] = \frac{[V]}{a}. \tag{E.2}$$

Subtracting (E.2) from (E.1) yields,

$$\frac{\partial}{\partial t}\mu^* + [U_r]\frac{\partial}{\partial\lambda}\mu^* + \frac{U_r^*}{a}\frac{\partial}{\partial\lambda}\mu^* = \frac{V^*}{a} - \left(\dot{q}\frac{\partial\mu}{\partial q}\right)^* - \left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^* + \left[\frac{U_r^*}{a}\frac{\partial}{\partial\lambda}\mu^*\right]. \tag{E.3}$$

Taking $\frac{\partial}{\partial\lambda}$(E.3), we find,

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \lambda}\mu^*\right) + [U_r]\frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial \lambda}\mu^*\right) + \frac{U_r^*}{a}\frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial \lambda}\mu^*\right) + \frac{1}{a}\frac{\partial}{\partial \lambda}U_r^*\frac{\partial}{\partial \lambda}\mu^* = \frac{1}{a}\frac{\partial}{\partial \lambda}V^*$$
$$-\frac{\partial}{\partial \lambda}\left(\dot{q}\frac{\partial \mu}{\partial q}\right)^* - \frac{\partial}{\partial \lambda}\left(\dot{\theta}\frac{\partial \mu}{\partial \theta}\right)^* \qquad \text{(E.4)}$$

Multiplying (E.4) by F^* we obtain,

$$F^*\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \lambda}\mu^*\right) + [U_r]F^*\frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial \lambda}\mu^*\right) + \frac{F^*U_r^*}{a}\frac{\partial}{\partial \lambda}\left(\frac{\partial}{\partial \lambda}\mu^*\right) + \frac{F^*}{a}\frac{\partial}{\partial \lambda}U_r^*\frac{\partial}{\partial \lambda}\mu^*$$
$$= \frac{F^*}{a}\frac{\partial}{\partial \lambda}V^* - F^*\frac{\partial}{\partial \lambda}\left(\dot{q}\frac{\partial \mu}{\partial q}\right)^* - F^*\frac{\partial}{\partial \lambda}\left(\dot{\theta}\frac{\partial \mu}{\partial \theta}\right)^* \qquad \text{(E.5)}$$

Using the chain rule rewrite the first term on the left-hand site in (E.5) as,

$$F^*\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \lambda}\mu^*\right) = \frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial \lambda}\mu^*\right) - \frac{\partial}{\partial \lambda}\mu^*\frac{\partial}{\partial t}F^*$$
$$= \frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial \lambda}\mu^*\right) - \frac{\partial}{\partial \lambda}\mu^*\frac{dF^*}{dp}\frac{\partial p}{\partial t} \quad , \qquad \text{(E.6)}$$

in (E.5) we used the fact that

$$\frac{\partial}{\partial t}F^* = \frac{dF^*}{dp}\frac{\partial p}{\partial t} \ . \qquad \text{(E.7)}$$

Substituting (E.6) into (E.5) gives,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial\lambda}\mu^*\right)-\frac{\partial}{\partial\lambda}\mu^*\frac{dF^*}{dp}\frac{\partial p}{\partial t}+[U_r]F^*\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\lambda}\mu^*\right)+\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\lambda}\mu^*\right)\\
&+\frac{F^*}{a}\frac{\partial}{\partial\lambda}U_r^*\frac{\partial}{\partial\lambda}\mu^* = \frac{F^*}{a}\frac{\partial}{\partial\lambda}V^*-F^*\frac{\partial}{\partial\lambda}\left(\dot{q}\frac{\partial\mu}{\partial q}\right)^*-F^*\frac{\partial}{\partial\lambda}\left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^*
\end{aligned}
\quad . \tag{E.8}
$$

We can also combine $\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\lambda}\mu^*\right)+\frac{F^*}{a}\frac{\partial}{\partial\lambda}U_r^*\frac{\partial}{\partial\lambda}\mu^*$ and write them as,

$$
\begin{aligned}
\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\lambda}\mu^*\right)+\frac{F^*}{a}\frac{\partial}{\partial\lambda}U_r^*\frac{\partial}{\partial\lambda}\mu^* &= \frac{\partial}{\partial\lambda}\left(\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\mu^*\right)-U_r^*\frac{\partial}{\partial\lambda}F^*\frac{\partial}{\partial\lambda}\mu^*\\
&= \frac{\partial}{\partial\lambda}\left(\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\mu^*\right)-U_r^*\frac{dF^*}{dp}\frac{\partial p}{\partial\lambda}\frac{\partial}{\partial\lambda}\mu^*
\end{aligned}
\quad , \tag{E.9}
$$

where in (E.9) we used again the fact that $\frac{\partial}{\partial\lambda}F^* = \frac{dF^*}{dp}\frac{\partial p}{\partial\lambda}$.

Substituting (E.9) into (E.8) yields,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial\lambda}\mu^*\right)\underline{-\frac{\partial}{\partial\lambda}\mu^*\frac{dF^*}{dp}\frac{\partial p}{\partial t}}+[U_r]F^*\frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\lambda}\mu^*\right)+\frac{\partial}{\partial\lambda}\left(\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\mu^*\right)\\
&\underline{-U_r^*\frac{dF^*}{dp}\frac{\partial p}{\partial\lambda}\frac{\partial}{\partial\lambda}\mu^*} = \frac{F^*}{a}\frac{\partial}{\partial\lambda}V^*-F^*\frac{\partial}{\partial\lambda}\left(\dot{q}\frac{\partial\mu}{\partial q}\right)^*-F^*\frac{\partial}{\partial\lambda}\left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^*
\end{aligned}
\quad . \tag{E.10}
$$

Note that the underlined terms in (E.10) can be combined as follows,

$$\frac{\partial}{\partial\lambda}\mu^*\frac{dF^*}{dp}\frac{\partial p}{\partial t} + U_r^*\frac{dF^*}{dp}\frac{\partial p}{\partial\lambda}\frac{\partial}{\partial\lambda}\mu^* = \left(\frac{\partial p}{\partial t} + U_r^*\frac{\partial p}{\partial\lambda}\right)\frac{dF^*}{dp}\frac{\partial}{\partial\lambda}\mu^* \\ = \omega_q\frac{dF^*}{dp}\frac{\partial}{\partial\lambda}\mu^*, \tag{E.11}$$

where $\omega_q = \frac{\partial p}{\partial t} + U_r^*\frac{\partial p}{\partial\lambda}$.

Now using the relationship existing between the function F and the Exner function Π, we can write $\frac{\partial}{\partial p}F^* = \frac{p}{g}\frac{d\Pi}{dp} = \frac{\Pi}{g}$, and (E.11) becomes,

$$\frac{\partial}{\partial\lambda}\mu^*\frac{\partial}{\partial p}F^*\frac{\partial p}{\partial t} + U_r^*\frac{dF^*}{dp}\frac{\partial p}{\partial\lambda}\frac{\partial}{\partial\lambda}\mu^* = \frac{\omega_q\Pi}{g}\frac{\partial}{\partial\lambda}\mu^*. \tag{E.12}$$

Substituting (E.12) into (E.10) gives,

$$\frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial\lambda}\mu^*\right) + [U_r]\frac{\partial}{\partial\lambda}\left(F^*\frac{\partial}{\partial\lambda}\mu^*\right) = \frac{\omega_q\Pi}{g}\frac{\partial}{\partial\lambda}\mu^* + [U_r]\frac{\partial}{\partial\lambda}F^*\frac{\partial}{\partial\lambda}\mu^* \\ -\frac{\partial}{\partial\lambda}\left(\frac{F^*U_r^*}{a}\frac{\partial}{\partial\lambda}\mu^*\right) + \frac{F^*}{a}\frac{\partial}{\partial\lambda}V^* - F^*\frac{\partial}{\partial\lambda}\left(\dot{q}\frac{\partial\mu}{\partial q}\right)^* - F^*\frac{\partial}{\partial\lambda}\left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^* \tag{E.13}$$

In (E.13) we can rearrange the remaining terms as,

$$\frac{F^*}{a}\frac{\partial}{\partial\lambda}V^* - F^*\frac{\partial}{\partial\lambda}\left(\dot{q}\frac{\partial\mu}{\partial q}\right)^* + F^*\frac{\partial}{\partial\lambda}\left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^* = \frac{\partial}{\partial\lambda}\left\{\frac{F^*V^*}{a} - F^*\left(\dot{q}\frac{\partial\mu}{\partial q}\right)^* - F^*\left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^*\right\} \\ -\left\{\frac{V^*}{a} - \left(\dot{q}\frac{\partial\mu}{\partial q}\right)^* - \left(\dot{\theta}\frac{\partial\mu}{\partial\theta}\right)^*\right\}\frac{\partial}{\partial\lambda}F^*, \tag{E.14}$$

and substituting (E.14) into (E.13) we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial \lambda}\mu^*\right) + [U_r]\frac{\partial}{\partial \lambda}\left(F^*\frac{\partial}{\partial \lambda}\mu^*\right) &= \frac{\omega_q \Pi}{g}\frac{\partial}{\partial \lambda}\mu^* \\ &+ \frac{\partial}{\partial \lambda}\left\{\frac{F^* V^*}{a} - F^*\left(\dot{q}\frac{\partial \mu}{\partial q}\right)^* - F^*\left(\dot{\theta}\frac{\partial \mu}{\partial \theta}\right)^*\right\} \\ &- \left\{\frac{V^*}{a} - \left(\dot{q}\frac{\partial \mu}{\partial q}\right)^* - \left(\dot{\theta}\frac{\partial \mu}{\partial \theta}\right)^* - [U_r]\frac{\partial}{\partial \lambda}\mu^*\right\}\frac{\partial}{\partial \lambda}F^*\end{aligned} \quad . \qquad \text{(E.15)}$$

By inspecting (E.15) we note that in a zonally average sense the terms affecting the form drag is the first one on the right-hand side and the third one, thus we can write

$$\frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial \lambda}\mu^*\right) + [U_r]\frac{\partial}{\partial \lambda}\left(F^*\frac{\partial}{\partial \lambda}\mu^*\right) \sim \frac{\omega_q \Pi}{g}\frac{\partial}{\partial \lambda}\mu^* - \gamma\frac{\partial}{\partial \lambda}F^* , \qquad \text{(E.16)}$$

where we used the notation $\gamma \equiv \frac{V^*}{a} - \left(\dot{q}\frac{\partial \mu}{\partial q}\right)^* - \left(\dot{\theta}\frac{\partial \mu}{\partial \theta}\right)^* - [U_r]\frac{\partial}{\partial \lambda}\mu^*$.

To evaluate $\frac{\partial}{\partial \lambda}\mu^*$ a scale analysis similar to Wilson and Williams (2004) is applied. This analogy is possible because μ and $(\mu^*)^2$ on PV surfaces carry the same information as the potential vorticity and eddy potential enstrophy, respectively, in PVPT coordinates, and we will call μ the **E**quivalent **P**otential **V**orticity (EPV) and $\frac{1}{2}(\mu^*)^2$ the eddy **E**quivalent **P**otential **E**nstrophy (EPE). The next step will be to investigate the main factors contributing to the variability of eddy EPE, as described in Wilson and Williams.

To derive an equation for the eddy EPE rewrite (E.3) as,

$$\frac{\partial}{\partial t}\mu^* + [U_r]\frac{\partial}{\partial \lambda}\mu^* + \frac{U_r^*}{a}\frac{\partial}{\partial \lambda}\mu^* + \dot{q}^*\frac{\partial}{\partial q}[\mu] + \dot{q}^*\frac{\partial}{\partial q}\mu^* + [\dot{q}]\frac{\partial}{\partial q}\mu^*$$
$$+ \dot{\theta}^*\frac{\partial}{\partial \theta}[\mu] + \dot{\theta}^*\frac{\partial}{\partial \theta}\mu^* + [\dot{\theta}]\frac{\partial}{\partial \theta}\mu^* = \frac{V^*}{a} + \left[\frac{U_r^*}{a}\frac{\partial}{\partial \lambda}\mu^*\right] \tag{E.17}$$

Multiply (E.17) by μ^*, and then take the zonal average,

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}[(\mu^*)^2]\right\} + [\dot{q}]\frac{\partial}{\partial q}\left\{\frac{1}{2}[(\mu^*)^2]\right\} + [\dot{\theta}]\frac{\partial}{\partial \theta}\left\{\frac{1}{2}[(\mu^*)^2]\right\}$$
$$+ \frac{1}{a}\left[U_r^*\frac{\partial}{\partial \lambda}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] + \left[\dot{q}^*\frac{\partial}{\partial q}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] + \left[\dot{\theta}^*\frac{\partial}{\partial \theta}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] = \tag{E.18}$$
$$\frac{[\mu^* V^*]}{a} - [\dot{q}^*\mu^*]\frac{\partial}{\partial q}[\mu] - [\dot{\theta}^*\mu^*]\frac{\partial}{\partial \theta}[\mu]$$

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}[(\mu^*)^2]\right\} = -[\dot{q}]\frac{\partial}{\partial q}\left\{\frac{1}{2}[(\mu^*)^2]\right\} - [\dot{\theta}]\frac{\partial}{\partial \theta}\left\{\frac{1}{2}[(\mu^*)^2]\right\}$$
$$-\frac{1}{a}\left[U_r^*\frac{\partial}{\partial \lambda}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] - \left[\dot{q}^*\frac{\partial}{\partial q}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] - \left[\dot{\theta}^*\frac{\partial}{\partial \theta}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] \tag{E.19}$$
$$-[\dot{q}^*\mu^*]\frac{\partial}{\partial q}[\mu] - [\dot{\theta}^*\mu^*]\frac{\partial}{\partial \theta}[\mu] + \frac{[\mu^* V^*]}{a}$$

The first two terms on the right-hand side of (E.19) represent the mean advection of the eddy EPE by the mean meridional circulation. The next three terms represent the mean advection of the eddy EPE by the eddies. The first two terms on the third line are the gradient production/destruction and describe the exchange with the equivalent enstrophy of the mean flow through diabatic heating and friction.

Equation (E.19) can be also written as,

$$\left[\frac{D}{Dt}\left\{\frac{1}{2}(\mu^*)^2\right\}\right] + [\mu^*\mathbf{U}^*] \bullet \nabla[\mu] = \frac{1}{a}[\mu^* V^*] \ , \tag{E.20}$$

where, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{U}{a}\frac{\partial}{\partial\lambda} + \dot{q}\frac{\partial}{\partial q} + \dot{\theta}\frac{\partial}{\partial\theta}$, $\nabla = \mathbf{e}_q\frac{\partial}{\partial q} + \mathbf{e}_\theta\frac{\partial}{\partial\theta}$, and $\mathbf{U}^* = (\dot{q}^*, \dot{\theta}^*)$. In regions where $\frac{1}{a}[\mu^* V^*]$ is negative (positive) the eddy flux of EPV ($[\mu^*\mathbf{U}^*]$) must be downgradient (upgradient).

The direction of the eddy EPV flux can be determined by considering the scalar product $(\mu^*\mathbf{U}^*) \bullet \nabla\mu$, which can be written in terms of eddy advection of eddy EPE,

$$\begin{aligned}(\mu^*\mathbf{U}^*) \bullet \nabla\mu &= (\mu^*\mathbf{U}^*) \bullet \nabla[\mu] + (\mu^*\mathbf{U}^*) \bullet \nabla\mu^* \\ &= (\mu^*\mathbf{U}^*) \bullet \nabla[\mu] + \mathbf{U}^* \bullet \nabla\frac{1}{2}(\mu^*)^2\end{aligned} \tag{E.21}$$

These contributions on the right-hand side of (E.21) can either cancel each other or have the same sign. Consider an idealized perturbation such that there is a temporal variation of the eddy EPE and in consequence a variation in the eddy EPV field as depicted in Fig. E.1.

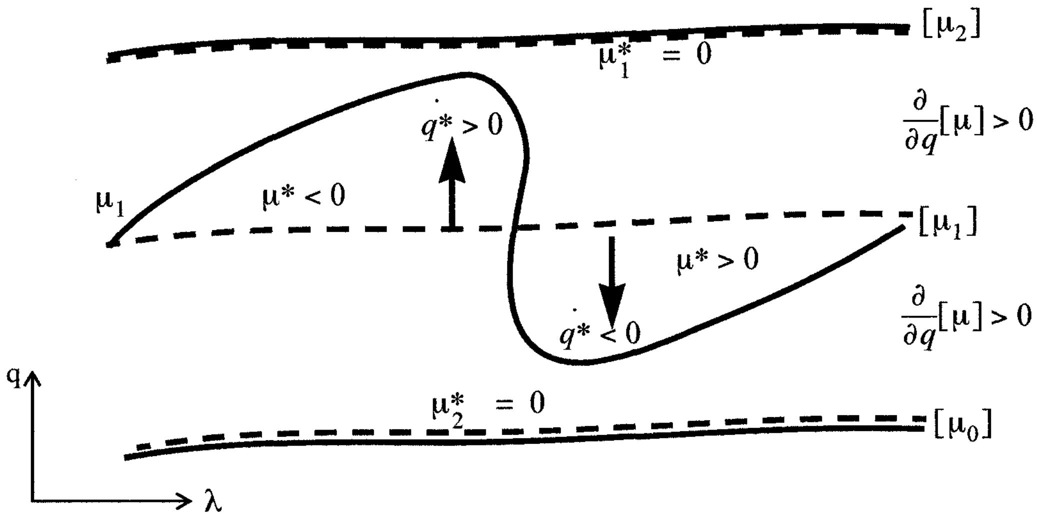

Figure E.1: .Schematic representation of the interaction between a baroclinic wave, represented by the perturbed geographical latitude and the mean flow, given by the zonal mean of the geographical latitude.

In Fig. E.1 the instantaneous μ_1 contour (solid line) is perturbed from the zonal mean contour $[\mu_1]$ (dashed line) in a channel bounded by unperturbed μ_0 and μ_2 contours. We notice $\mu^*\dot{q}^* < 0$ everywhere which can be interpreted as a downward flux of EPV relative to the mean northwards mean EPV gradient $[h]$. μ^* is negative on the left-hand side of the picture because we assume μ below μ_1 has values between $[\mu_0]$ and $[\mu_1]$ ($[\mu_0] < [\mu_1] < [\mu_2]$), and above μ_1 has values between $[\mu_1]$ and $[\mu_2]$. As a result $\mu^*\dot{q}^*\frac{\partial}{\partial q}[\mu] < 0$ everywhere in this channel. In the middle of the channel there is a maximum of eddy EPE, whereas it vanishes towards the edges of the channel giving

$\dot{q}^* \frac{\partial}{\partial q}\left\{\frac{1}{2}(q^*)^2\right\} < 0$. We can conclude that in this case the two terms reinforce each other.

From this analysis we can conclude that the variation of eddy EPE is determined by the eddy advection of eddy EPE and the scalar product of eddy EPV flux and mean EPV gradient. We can apply a scale analysis and define

$$\left|\left[\mathbf{U}^* \bullet \nabla \frac{1}{2}(\mu^*)^2\right]\right| \sim L_{[\mu]} \text{ and } |[\mu^* \mathbf{U}^*] \bullet \nabla[\mu]| \sim L_{\mu^*} \tag{E.22}$$

where $L_{[\mu]} \sim \left|\frac{\left[\frac{1}{2}(\mu^*)^2\right]^{1/2}}{\nabla[\mu]}\right|$ and reflects the length scale over which the eddy EPE varies relative to the gradient of mean EPV. $L_{\mu^*} = \left|\frac{[(\mu^*)^2]^{1/2}}{\nabla[(\mu^*)^2]^{1/2}}\right|$ and reflects the length scale over which the eddy EPE varies relative to the gradient of eddy EPE.

$$\frac{\left|\left[\mathbf{U}^* \bullet \nabla \frac{1}{2}(\mu^*)^2\right]\right|}{|[\mu^* \mathbf{U}^*] \bullet \nabla[\mu]|} \sim \frac{L_{[\mu]}}{L_{\mu^*}}. \tag{E.23}$$

In a steady state $\frac{L_{[\mu]}}{L_{\mu^*}} \sim 1$. When $\frac{L_{[\mu]}}{L_{\mu^*}} < 1$ the eddy advection of eddy EPE is the main forcing of the eddy EPE whereas $\frac{L_{[\mu]}}{L_{\mu^*}} > 1$ means the scalar product of the mean of eddy EPV flux and mean EPV gradient determine the variations in eddy EPE.

From this analysis we can say that,

$$\frac{\partial}{\partial\lambda}\mu^* \sim \frac{L_{[\mu]}}{L_{\mu^*}}. \tag{E.24}$$

and

$$\frac{\partial}{\partial t}\left(F^*\frac{\partial}{\partial\lambda}\mu^*\right) \sim \frac{\omega_q \Pi}{g}\frac{L_{[\mu]}}{L_{\mu^*}} - \gamma\frac{\partial}{\partial\lambda}F^* . \tag{E.25}$$

Assuming that the second term in (E.25) is small compared to the first one, these arguments suggest the following explanation for the zonally averaged zonal circulation: when the gradient of the zonal mean PV and the gradient of the zonal mean eddy PV vary on the same length scales, i.e., when $\frac{L_{[\mu]}}{L_{\mu^*}} \sim 1$, the form drag due to the undulation of surfaces of constant PV cannot affect the zonal mean zonal flow. When $\frac{L_{[\mu]}}{L_{\mu^*}}$ is either less or greater than one, the parameterization allows development of baroclinic waves that remove the baroclinic instabilities.

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