DISSERTATION

PRODUCTIVITY VARIATION AND GROUP OPTIMIZATION

Submitted by

Jay Parsons

Department of Agricultural and Resource Economics

In partial fulfillment of the requirements

for the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Fall 2003

UMI Number: 3114689

COLORADO STATE UNIVERSITY

November 12, 2003

WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY JAY PARSONS ENTITLED PRODUCTIVITY VARIATION AND GROUP OPTIMIZATION BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

Committee on Graduate Work

____

Stephen R Koontz
____

____

Dana L Hoag
____
Advisor

____
Department Head

ABSTRACT OF DISSERTATION

PRODUCTIVITY VARIATION AND GROUP OPTIMIZATION

Uncertain productivity exists in many empirical settings, settings that are often characterized by nonlinear relationships. This interaction between nonlinear relationships and uncertainty provides an abundance of opportunities to apply Jensen's inequality and the concepts surrounding it. Past research has acknowledged Jensen's inequality but never formalized its treatment in a production economics setting. In group production, individuals enter the production process and exit the production process at the same time. In this dissertation, I show that the effect of Jensen's inequality must be formally accounted for in a group production setting to ensure that maximum profit conditions are being met. I introduce the concept of the Jensen effect and an aggregation premium to aid in the discussion of how an adjustment needs to be made in group optimization to account for the fact that the marginal productivity of an average individual in a group is not equivalent to the average marginal productivity of all individuals in the group.

Examples from swine production and beef production are used to show that due to the aggregation premium, the marginal productivity of the average weight individual in the group overestimates the average marginal productivity of the group. Therefore, for group optimization to occur, the animals should be marketed earlier than the date

indicated by the average weight of the group. The magnitudes of the differences are modest but direction is consistent and it's this consistent direction that when applied to a large operation could have a significant economic impact.

Most importantly, the concepts and terminology presented in this dissertation provide a formalized treatment of Jensen's inequality that previous work has failed to accomplish. It is a formalization that will aid future research discussions into the many areas where Jensen's inequality can play a pivotal role.

Jay Parsons
Department of Agricultural
and Resource Economics
Colorado State University
Fort Collins, CO 80523
Fall 2003

TABLE OF CONTENTS

Chapter 1

Introduction

The notion that output from a production process can vary is nothing new. This is especially true in agriculture where numerous biological and environmental factors can heavily influence the final outcome. Previous research addressing the issue of yield variation has tended to focus on one of two areas.

The first area might best be described as a modeling focus. This is the task of formulating a production response function that explains the expected yield while adequately capturing the yield variation through residuals representing random and/or mixed effects. This is a positive economic approach to the problem using econometric and statistical methods. The focus is on explaining a single outcome from among a distribution of outcomes.

The second area of research on variable productivity has focused on risk and uncertainty. Variation is viewed as a risk to the producer. The source of the variation has been characterized in a number of different ways but, ultimately, its effect is the same. The producer's risk preferences will combine with the variation to influence the allocation of resources toward the production process.

I take a slightly different approach to the problem in this dissertation by accounting for the entire distribution as output from production. This approach seems appropriate in a number of empirical settings where production is done in groups. For instance, in agriculture, livestock are typically fed and slaughtered in groups of several animals. These groups represent a distribution of output. Optimizing the time on feed based upon the productivity of the expected output ignores the productivity information contained in the distribution itself. I will show that there is an economic penalty for ignoring this information. That penalty may exceed the cost associated with gathering and using higher levels of information in the optimization process.

The focus of this dissertation is to address the issue of variable productivity from the perspective of group production. Specifically, my objective is to explore and explain the microeconomic principles surrounding the optimization of production from a group in the presence of a distribution of output. At the center of the theory developed in this dissertation are Jensen's inequality and nonlinear marginal value functions. Jensen's inequality applied to any nonlinear function with respect to a random variable simply states that the expected functional value will differ from the functional value of the expected value of the random variable drawn from any distribution with a nonzero variance. For a concave marginal value function, this implies that the average marginal value for an output group distributed along the marginal value curve will be less than the marginal value for the output average of the group.

Before demonstrating the impact of Jensen's inequality on classical economic optimization, I review the literature in two areas. First, I look in-depth at Jensen's inequality. Jensen's inequality can be both easy to understand and difficult to explain. In chapter 2, I formally define Jensen's inequality and many of the theoretical variations that emanate from it. Several examples of past research using Jensen's inequality are provided for review.

The second area of literature I review is animal growth functions. Livestock production provides the setting for the empirical examples developed in this dissertation. It is a natural setting where group production occurs with little need for additional explanation or story telling. It also provides a nice controlled environment from which to build the concepts and theory developed in this dissertation. Chapter 3 is a review of the animal growth functions that make up the core of livestock production theory. These functions provide the production functions and the resulting nonlinear marginal value curves used throughout the livestock applications developed in this dissertation. Justification is provided in chapter 3 for the particular growth functions employed in the examples.

Chapters 4-6 are intended to be three stand-alone papers that form the core of this dissertation. As such, there is some repetition of core knowledge and theoretical concepts in the introductory remarks and explanations contained in each of these chapters. However, I tried to keep this repetition to a minimum. Chapter 4 is the core chapter of this entire dissertation. It provides a theoretical treatise outlining the basic concept of Jensen's inequality applied to production economics and optimization of group production. It introduces the cattle and swine livestock examples as a way to show the effect of Jensen's inequality on marginal productivity for a group. It extends this effect into the optimization conditions for group profit maximization. Most importantly, it provides a formalized foundation of definitions and concepts pertaining to the effect of Jensen's inequality on aggregated groups.

Chapters 5 and 6 are extensions of the livestock examples introduced in chapter 4. Chapter 5 extends the beef example to include 24 pens of data and a more thorough introduction to the empirical setting. Chapter 6 extends the swine application into a more thorough economic exposition of the concepts and a sensitivity analysis. Finally, I wrap the dissertation up in chapter 7 with some concluding comments and conclusions.

There is a lack of past research addressing the issue of variable productivity from the perspective taken in this dissertation. Therefore, there is an opportunity for this work to make an important contribution to the literature. This contribution is a formalization of important principles and concepts that apply to a common empirical setting. By formalizing these concepts, the goal of this dissertation is to provide the tools necessary to aid in the advancement of future research into the effect of Jensen's inequality on situations characterized by variability.

Chapter 2

On Jensen's Inequality

Jensen's inequality was first described at the end of the 19th century (Hölder; Jensen) and is generally credited to the mathematician Johan L. Jensen (1859-1925). However, its origins of thought can be traced as far back as Daniel Bernoulli's concept of concave-down, risk-averse utility, first published in 1738. In its simplest form, Jensen's inequality can be described as dealing with convex (concave) functions whose second derivatives are nonnegative (nonpositive).

Definition. A scalar function $f(x)$ of a scalar argument x is *convex* if for every λ, $0 \leq \lambda \leq 1$, x_1 and x_2,

$$\lambda f(x_1) + (1-\lambda) f(x_2) \geq f(\lambda x_1 + (1-\lambda) x_2).$$

Conversely, the function is *concave* if

$$\lambda f(x_1) + (1-\lambda) f(x_2) \leq f(\lambda x_1 + (1-\lambda) x_2)$$

for every $0 \leq \lambda \leq 1$, x_1 and x_2.

Strict inequalities applied to the above definition result in *strictly convex* and *strictly concave* functions, respectively.

Lemma 2.1(a). The function $f(x)$ with a continuous second derivative over the interval $[a,b]$ is convex (strictly convex) over that interval if and only if its second derivative is nonnegative (positive) for all $x \in [a,b]$.

Lemma 2.1(b). The function $f(x)$ with a continuous second derivative over the interval $[a,b]$ is concave (strictly concave) over that interval if and only if its second derivative is nonpositive (negative) for all $x \in [a,b]$.

If the definition is extended to averages over more than two points and probabilities play the role of λ, the result is Jensen's classical inequality.

Theorem 2.1 (Jensen's inequality). Let $f:[a,b] \to \Re$ be a convex function defined on a real interval $[a,b]$ and let n be any natural number. The inequality

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

is valid for any set of nonnegative real numbers $\lambda_1, \ldots, \lambda_n$ with sum of one and all points $x_1, \ldots, x_n \in [a,b]$.

Proof. The proof is by induction. For $n = 2$, Jensen's inequality reduces to the definition of convexity. Assume Jensen's inequality is true for any distribution with k points. Then,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left(\lambda_{k+1} x_{k+1} + \sum_{i=1}^{k} \lambda_i x_i\right).$$

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right)=f\left[\lambda_{k+1}x_{k+1}+\left(1-\lambda_{k+1}\right)\left(\frac{1}{1-\lambda_{k+1}}\right)\sum_{i=1}^{k}\lambda_i x_i\right]$$

and, therefore,

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right)=f\left[\lambda_{k+1}x_{k+1}+\left(1-\lambda_{k+1}\right)\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i x_i\right]. \tag{2.1}$$

Note, x_{k+1} and $\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i x_i$ are two points in $[a,b]$. By definition, since f is convex over the interval $[a,b]$, equation (2.1) implies

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right)\le\lambda_{k+1}f\left(x_{k+1}\right)+\left(1-\lambda_{k+1}\right)f\left(\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i x_i\right). \tag{2.2}$$

Note that if $\sum_{i=1}^{k+1}\lambda_i=1$, then $\sum_{i=1}^{k}\lambda_i=1-\lambda_{k+1}$ and $\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i=1$. Since Jensen's inequality is assumed to hold for $n=k$,

$$f\left(\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i x_i\right)\le\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i f\left(x_i\right). \tag{2.3}$$

Substituting relationship (2.3) into the last term on the right-hand side of relationship (2.2) yields

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right)\le\lambda_{k+1}f\left(x_{k+1}\right)+\left(1-\lambda_{k+1}\right)\sum_{i=1}^{k}\frac{1}{1-\lambda_{k+1}}\lambda_i f\left(x_i\right)$$

or

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right)\le\lambda_{k+1}f\left(x_{k+1}\right)+\sum_{i=1}^{k}\lambda_i f\left(x_i\right).$$

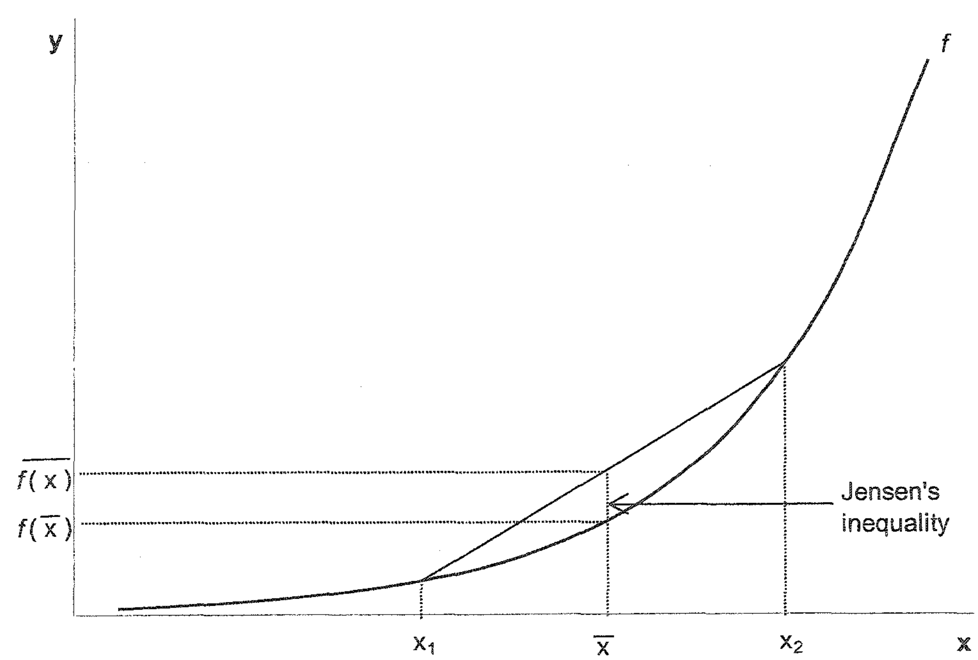
Therefore,

$$f\left(\sum_{i=1}^{k+1}\lambda_i x_i\right) \leq \sum_{i=1}^{k+1}\lambda_i f(x_i).$$

Thus, the assumption that Jensen's inequality holds for $n = k$ implies that it holds for $n = k+1$. Since it is known to hold for $n = 2$, by the principle of mathematical induction, it holds for all natural numbers greater than or equal to two.

Graphically, Jensen's inequality deals with the vertical difference between the average functional value $\overline{f(x)}$ and the functional value $f(\bar{x})$ associated with the average input argument. Figure 2.1 illustrates this point for a convex function as stated in the definition.

Figure 2.1: A convex function and Jensen's inequality.



In a continuous setting, Jensen's inequality can be written in terms of integrals where, for a convex function f,

(2.4)
$$f\left[\int_a^b x \cdot dx\right] \leq \int_a^b f(x) \cdot dx.$$

In terms of expected values, Jensen's inequality is expressed as

(2.5)
$$f[E(x)] \leq E[f(x)].$$

If $f : [a,b] \to \Re$ is a concave function defined on a real interval $[a,b]$, Jensen's inequality is stated as

(2.6)
$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i f(x_i).$$

Likewise, for concave functions in a continuous setting, inequality (2.4) and inequality (2.5) are stated as

(2.7)
$$f\left[\int_a^b x \cdot dx\right] \geq \int_a^b f(x) \cdot dx$$

and

(2.8)
$$f[E(x)] \geq E[f(x)],$$

respectively.

The area of risk as used by economists is one way to illustrate the concept of an interval $[a,b]$. For example, Rothschild and Stiglitz explored a mean preserving spread for increasing risk as follows.

Definition. Let x_i be a random variable with mean μ_x and variance $\sigma_x{}^2$. Let u_i be a random variable stochastically independent of x_i with mean zero and variance $\sigma_u{}^2 > 0$.

The random variable $y_i = x_i + u_i$ is a *mean preserving spread* of the distribution of the variable x_i with the mean $\mu_y = \mu_x$ and the variance $\sigma_y^2 > \sigma_x^2$.

Lemma 2.2(a). The expected value of a real-valued, convex function of a random variable increases or remains unchanged when the distribution of the random variable undergoes a mean preserving spread.

Proof. Let $f(x)$ be a real-valued, convex function of a random variable x. Let the random variable $y = x + u$ be a mean preserving spread of the distribution of x. Then, by definition, $E[u|x] = 0$ and $\sigma_u^2 > 0$. For any given x, Jensen's inequality states that

$$E[f(x+u)] \geq f[E(x+u)] = f[E(x)] = f(x).$$

Taking the expectations of both sides with respect to the random variable x,

$$E_x[E[f(x+u)]] \geq E_x[f(x)]$$

$$\Rightarrow E[f(y)] \geq E[f(x)].$$

For completeness, a corresponding lemma for concave functions is stated next.

Lemma 2.2(b). The expected value of a real-valued, concave function of a random variable decreases or remains unchanged when the distribution of the random variable undergoes a mean preserving spread.

Jensen realized the mathematical importance of his inequality as a way to collect a number of known, but seemingly unrelated inequalities, under one umbrella (Hansen). For example, the inequality

(2.9) $$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

for real numbers $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ can be derived from Jensen's inequality and the convex function $f(x)=x^2$ (Jensen, p. 181). This inequality is usually referred to as *Cauchy's inequality.*

Proof of Cauchy's inequality. Let $f(x)=x^2$, a convex function. Then, Jensen's inequality states

$$\left(\sum_{i=1}^{n} \lambda_i x_i\right)^2 \le \sum_{i=1}^{n} \lambda_i x_i^2$$

for any set of nonnegative real numbers $\lambda_1,\ldots,\lambda_n$ with sum of one.

Let $a_i = \sqrt{\lambda_i}$ and $b_i = a_i x_i$. Then, replacing λ_i with a_i^2 and x_i with $\frac{b_i}{a_i}$, the following results.

$$\left(\sum_{i=1}^{n} a_i^2 \cdot \frac{b_i}{a_i}\right)^2 \le \sum_{i=1}^{n} a_i^2 \left(\frac{b_i}{a_i}\right)^2$$

Therefore,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} b_i^2 = \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

where $\sum_{i=1}^{n} a_i^2 = \sum_{i=1}^{n} \lambda_i = 1$, by definition.

For inequality (2.9), the corresponding inequality in a continuous setting with integrals is usually called Schwarz's inequality even though it seems to have been first stated by Buniakowsky in 1859 (Hardy, Littlewood, and Polya). In general, inequality (2.9) can be stated in terms of expected values

(2.10) $$[E(xy)]^2 \leq E(x^2)E(y^2)$$

and is alternately referred to as the *Cauchy-Schwarz inequality* (Mittlehammer, p. 149) or simply the *Schwarz inequality* (Savage, p. 269).

An example involving a concave function and Jensen's inequality is perhaps the most famous theorem on the subject (Hardy, Littlewood, and Polya), the arithmetic mean – geometric mean inequality.

Theorem 2.2 (AM-GM inequality). If $x_1,\ldots,x_n \geq 0$, then

$$\frac{x_1+\ldots+x_n}{n} \geq \sqrt[n]{x_1\ldots x_n}.$$

Proof. Begin with the concave function $f(x)=\ln x$. Then, by Jensen's inequality, if $x_1,\ldots,x_n > 0$,

$$\ln\left(\frac{x_1+\ldots+x_n}{n}\right) \geq \frac{\ln x_1+\ldots+\ln x_n}{n}.$$

Since

$$\frac{\ln x_1+\ldots+\ln x_n}{n} = \frac{1}{n}\ln(x_1\ldots x_n) = \ln(x_1\ldots x_n)^{1/n} = \ln\left(\sqrt[n]{x_1\ldots x_n}\right),$$

it follows that

$$\ln\left(\frac{x_1+\ldots+x_n}{n}\right) \geq \ln\left(\sqrt[n]{x_1\ldots x_n}\right).$$

The theorem follows directly since $f(x) = \ln x$ is a monotonically increasing function for all $x > 0$. Note that, by inspection, it is easily seen that the theorem holds for the trivial cases when some or all of the x_i's are equal to zero. It can also easily be shown that a strict inequality will hold unless all of the x_i's are equal (Hardy, Littlewood, and Polya, p. 17).

Of course, the expression on the left-hand side of the AM-GM inequality is the arithmetic mean of a finite set of nonnegative real numbers. The expression on the right-hand side is the geometric mean. The AM-GM inequality states, in general terms, that the arithmetic mean will always be greater than or equal to the geometric mean and will be strictly greater than whenever the variance is greater than zero. The reason for this is that the geometric mean decreases relative to the arithmetic mean as variance increases. That is, the geometric mean penalizes variance (Stearns).

To illustrate, consider the two sets of numbers A = $\{5, 5, 5\}$ and B = $\{1, 5, 9\}$. Both A and B have an arithmetic mean equal to 5.0. The variance of B $\left(\sigma_b^{\ 2} = 10.67\right)$ is obviously higher than the variance of A $\left(\sigma_a^{\ 2} = 0.0\right)$. It follows that when we calculate the geometric mean of B $(G_b = 3.56)$, we get a value less than the geometric mean of A $(G_a = 5.0)$. The multiplication carried out in the calculation of the geometric mean seems to result in a greater weight being given to low values versus what occurs during the calculation of the arithmetic mean (Stearns).

An Illustration: The Optimal Growth Investment Strategy

The properties of the geometric mean have been exploited in economic studies of investment strategies under risk. The geometric mean investment strategy, also called the optimal growth strategy or the growth optimal model, was introduced into the finance and economics literature by Henry Latané in 1959. This strategy uses the objective of maximizing the geometric mean of returns as a guide for making choices among competing portfolios. It has been justified in two ways as a criteria for portfolio selection (Elton and Gruber). First, if utility functions are of the log form, it leads to the selection of portfolios that maximize the expected utility of terminal wealth. Second, it leads to the selection of portfolios with several appealing characteristics, which are described next in detail.

An investment strategy that seeks to maximize the geometric mean of returns each period has the appealing characteristic that it produces the maximum expected growth rate in wealth among all strategies for a multi-period time horizon (Elton and Gruber). Let

W_t = the investor's wealth at the end of year $t = 1,\ldots,T$.
W_0 = the investor's initial wealth.
R_{it} = a random variable equal to return plus one where t is a time period index and i denotes a particular retun.
P_{it} = the probability of R_{it} occurring.

The investor's terminal wealth is equal to

$$W_T = W_0\left(R_{i1}R_{i2}\ldots R_{iT}\right).$$

From this,

$$\ln W_T = \ln W_0 + \ln R_{i1} + \ln R_{i2} + \ldots + \ln R_{iT}$$

$$\ln \frac{W_T}{W_0} = \ln R_{i1} + \ln R_{i2} + \ldots + \ln R_{iT}$$

$$E\left(\ln \frac{W_T}{W_0}\right) = E(\ln R_{i1}) + E(\ln R_{i2}) + \ldots + E(\ln R_{iT})$$

$$(2.11) \qquad E\left(\ln \frac{W_T}{W_0}\right) = \sum_i \ln R_{i1}{}^{P_{i1}} + \sum_i \ln R_{i2}{}^{P_{i2}} + \ldots + \sum_i \ln R_{iT}{}^{P_{iT}}$$

Now, consider that for a fixed number of trials, n, the return R_{it} will occur a finite number of times, n_i. Then, consider

$$\lim_{n \to \infty} \left(\frac{n_i}{n}\right) = P_{it} = \text{the probability of } R_{it} \text{ occurring.}$$

The geometric mean of returns in period t is given by

$$G_t = \sqrt[n]{R_{1t}{}^{n_1} \cdot R_{2t}{}^{n_2} \cdot \ldots} = R_{1t}{}^{n_1/n} \cdot R_{2t}{}^{n_2/n} \cdot \ldots$$

and

$$\lim_{n \to \infty} G_t = R_{1t}{}^{P_{1t}} \cdot R_{2t}{}^{P_{2t}} \cdot \ldots = \prod_i R_{it}{}^{P_{it}} .$$

To maximize the geometric mean of returns in each period, we maximize the expression $\prod_i R_{it}{}^{P_{it}}$. Since the logarithm is a monotonic transformation, we can take the logarithm of this expression without affecting the maximization. Then, maximizing the geometric mean of returns each period is equivalent to maximizing the expected value of the logarithm of return each period (Samuelson; Latané, 1959).

$$(2.12) \qquad \ln\left(\prod_i R_{it}{}^{P_{it}}\right) = \sum_i \ln R_{it}{}^{P_{it}} = \sum_i P_{it} \ln R_{it} .$$

Let X be the growth rate (assuming continuous compounding) in capital wealth over the investor's time horizon, T. Then, by definition,

$$W_T = W_0 \cdot e^X$$

or

$$X = \ln\left(\frac{W_T}{W_0}\right). \tag{2.13}$$

From equations (2.12) and (2.11), it can easily be seen that the right-hand side of (2.11) is the summation of the geometric means of returns in each period. A strategy that maximizes the geometric mean of returns in each period will maximize equation (2.11). By inspection of the left-hand side of (2.11), this strategy will maximize the expected utility of the final wealth assuming a logarithmic utility function. From equation (2.13), this also means the strategy will maximize the expected value of the growth rate. For this reason, the geometric mean investment strategy is often called the optimal growth strategy.

Other desirable characteristics of the optimal growth strategy were developed by Latané, Breiman, and Jean. Latané (1959; 1960) showed that the portfolio having a probability distribution of returns with the highest geometric mean would maximize the probability of having the most valuable portfolio in the long run. Breiman (1960; 1961) showed two important characteristics to be true. First, a strategy that maximizes the geometric mean of returns has the highest probability of exceeding any given level of wealth over any given period of time. Second, a strategy that maximizes the geometric mean of returns has the highest probability of reaching or exceeding any given level of wealth in the shortest period of time. Jean (1980) showed that a higher geometric mean of returns is a necessary condition for any portfolio to exhibit any degree of stochastic dominance over any other portfolio.

Finally, there is the contrast between the optimal growth strategy and the mean-variance approach pioneered by Markowitz (1952). Hakansson (1971) and Grauer (1981) are among those who have studied this issue. The results indicate that the optimal growth investment strategy and strategies employing a mean-variance approach exhibit pronounced differences. Hakansson showed that the optimal growth portfolio is not even close to being mean-variance efficient. However, the set of mean-variance efficient portfolios has problems in that portfolios from this set are the only feasible portfolios that can lead to ruin in the long run, where "ruin" is defined as wealth equal to zero. In fact, the optimal growth strategy is the policy that will, in the long run, minimize the probability of ruin (Mossin, pp. 40-41).

The Geometric Mean and Daniel Bernoulli

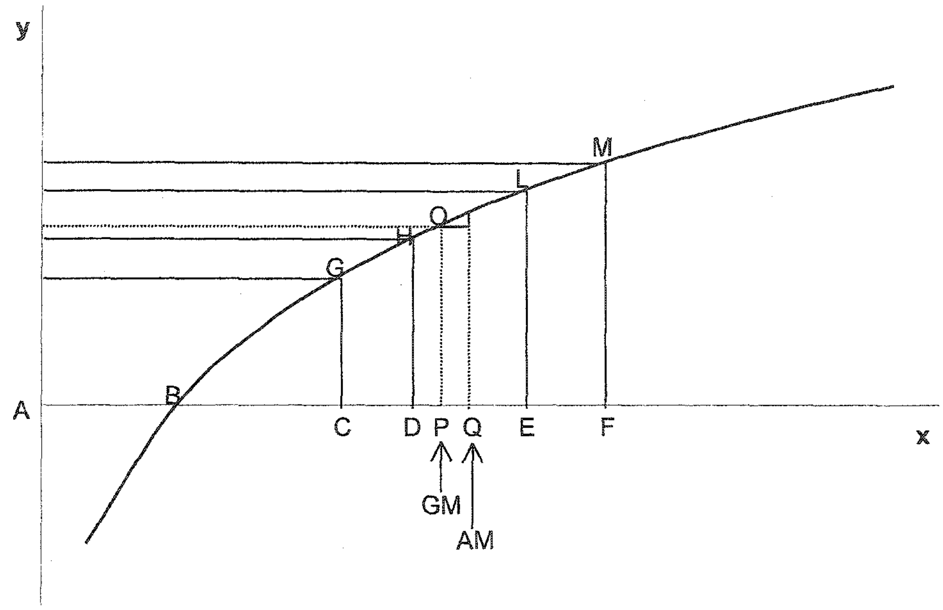
Perhaps the most famous application of the geometric mean, and the earliest use I have found, is Bernoulli's use in the context of the relationship of utility to wealth. Bernoulli employed the following utility function, which is graphed in figure 2.2,

$$y = b \cdot \mathbf{log}\left(\frac{x}{\alpha}\right)$$

where y is utility, x is wealth, $\alpha = \mathrm{AB}$ is initial wealth, and b is a positive constant that Bernoulli described as the subtangent.

[Bernoulli's subtangent is described as the length of the projection onto the Y-axis of the line tangent to the curve at point x and consisting of the segment of this tangent line between its intersection with the Y-axis and the point of contact with the curve. This

Figure 2.2: Bernoulli's logarithmic utility function with wealth (x) and utility (y).



is different from today's most common definition of subtangent, which involves a projection onto the X-axis (Stearns).]

Suppose there are m ways of gaining the wealth associated with the segment BC, n ways of gaining the wealth associated with BD, p ways of gaining the wealth associated with BE, and q ways of gaining the wealth associated with BF. Then, the mean utility of associated with these $m+n+p+q$ possible outcomes is given by

$$\mathrm{PO} = \frac{m \cdot \mathrm{CG} + n \cdot \mathrm{DH} + p \cdot \mathrm{EL} + q \cdot \mathrm{FM}}{m+n+p+q},$$

where P is the level of wealth associated with the mean utility PO.

It follows that

$$b \cdot \log\left(\frac{\mathrm{AP}}{\alpha}\right) = \frac{m \cdot b \cdot \log\left(\frac{\mathrm{AC}}{\alpha}\right) + n \cdot b \cdot \log\left(\frac{\mathrm{AD}}{\alpha}\right) + p \cdot b \cdot \log\left(\frac{\mathrm{AE}}{\alpha}\right) + q \cdot b \cdot \log\left(\frac{\mathrm{AF}}{\alpha}\right)}{m+n+p+q}$$

$$\log\left(\frac{\text{AP}}{\alpha}\right)=\frac{m\cdot\log\left(\frac{\text{AC}}{\alpha}\right)+n\cdot\log\left(\frac{\text{AD}}{\alpha}\right)+p\cdot\log\left(\frac{\text{AE}}{\alpha}\right)+q\cdot\log\left(\frac{\text{AF}}{\alpha}\right)}{m+n+p+q}$$

$$\log\left(\frac{\text{AP}}{\alpha}\right)=\frac{\log\left(\frac{\text{AC}^m}{\alpha^m}\right)+\log\left(\frac{\text{AD}^n}{\alpha^n}\right)+\log\left(\frac{\text{AE}^p}{\alpha^p}\right)+\log\left(\frac{\text{AF}^q}{\alpha^q}\right)}{m+n+p+q}$$

$$\log\left(\frac{\text{AP}}{\alpha}\right)=\frac{\log\left(\frac{\text{AC}^m\cdot\text{AD}^n\cdot\text{AE}^p\cdot\text{AF}^q}{\alpha^{m+n+p+q}}\right)}{m+n+p+q}$$

$$\log\left(\frac{\text{AP}}{\alpha}\right)=\log\left(\frac{\text{AC}^m\cdot\text{AD}^n\cdot\text{AE}^p\cdot\text{AF}^q}{\alpha^{m+n+p+q}}\right)^{\frac{1}{m+n+p+q}}$$

$$\log\left(\frac{\text{AP}}{\alpha}\right)=\log\left(\frac{\left(\text{AC}^m\cdot\text{AD}^n\cdot\text{AE}^p\cdot\text{AF}^q\right)^{\frac{1}{m+n+p+q}}}{\alpha}\right)$$

$$\log(\text{AP})-\log\alpha=\log\left(\text{AC}^m\cdot\text{AD}^n\cdot\text{AE}^p\cdot\text{AF}^q\right)^{\frac{1}{m+n+p+q}}-\log\alpha$$

$$\log(\text{AP})=\log\left(\text{AC}^m\cdot\text{AD}^n\cdot\text{AE}^p\cdot\text{AF}^q\right)^{\frac{1}{m+n+p+q}}$$

$$\text{AP}=\left(\text{AC}^m\cdot\text{AD}^n\cdot\text{AE}^p\cdot\text{AF}^q\right)^{\frac{1}{m+n+p+q}}.$$

So, the wealth associated with the mean utility is the geometric mean of the possible wealth outcomes.

Let segment AQ represent the arithmetic mean of the four possible levels of wealth: AC, AD, AE, and AF. Then, by the AM-GM inequality, the geometric mean AP will always be less than the arithmetic mean AQ of the possible wealth outcomes. In modern economics literature, the difference between these two values, segment PQ, is commonly called the *risk premium*. The wealth value determined by the geometric mean, segment AP, is commonly called the certainty equivalent. The *certainty equivalent* is defined as the value of wealth such that the decision-maker is indifferent between receiving it for certain and the prospect of facing a risky outcome (Hardaker, Huirne, and Anderson).

Bernoulli illustrated this relationship with his famous example of a Petersburg merchant wondering whether he should purchase insurance on goods he is about to ship to Amsterdam. The merchant knows the market value of the goods in Amsterdam is 10,000 rubles. He also knows that out of one hundred ships sailing from Petersburg to Amsterdam, five are usually lost. If x represents the merchant's current wealth, then the merchant's certainty equivalent (CE) is given by the geometric mean

$$\mathrm{CE} = \left[(x+10000)^{95} \cdot x^{5}\right]^{\frac{1}{100}}.$$

To insure the shipment would cost the merchant 800 rubles. With insurance, the merchant would have a certain wealth of $x+9200$. The question of whether or not to purchase insurance is determined by his current level of wealth. Finding the level of wealth, x, for which the merchant is indifferent between the certain level of wealth associated with the purchase of insurance and the certainty equivalent level of wealth associated with facing the risky prospect of shipping without insurance will determine the minimum level of wealth the merchant should have before foregoing the purchase of insurance.

Equating the two values

$$\left[(x+10000)^{95} \cdot x^{5}\right]^{\frac{1}{100}} = x+9200$$

and solving numerically for x yields the value $x \approx 5043$ rubles. If the merchant possesses wealth greater than this amount, he will be right in not buying the insurance. The certainty equivalent associated with the risky prospect is greater than the certain wealth associated with the purchase of insurance. On the contrary, if he possesses wealth less than this amount, he should purchase the insurance.

In this case, the certainty equivalent associated with the initial wealth of 5043 rubles is the geometric mean of 14,243 rubles. This is exactly 300 rubles less than the arithmetic mean (AM) associated with the risk of not having the insurance.

$$\begin{aligned}\text{AM} &= \frac{95\cdot(x+10000)+5\cdot x}{100}\\ &= \frac{95\cdot(15043)+5\cdot(5043)}{100}\\ &= 14{,}543.\end{aligned}$$

This difference of 300 rubles is the merchant's risk premium. It is the maximum amount above the actuarially fair premium of 500 rubles (expected loss = (0.95)(10000) = 500 rubles) that the merchant would be willing to pay for the insurance.

Jensen's inequality applied to concave functions is equivalent to the existence of a risk premium greater than zero for risk averse decision-makers. To illustrate, let u represent a risk averse decision-maker's concave utility function. Let $\lambda_1,\ldots,\lambda_n$ sum to one and represent the probabilities of attaining levels of wealth $x_1,\ldots,x_n$, respectively. Then, recall that the certainty equivalent is the value of wealth associated with the mean utility, or

$$\text{CE} = u^{-1}\left(\sum_{i=1}^{n}\lambda_i\cdot u(x_i)\right).$$

Jensen's inequality for strictly concave functions states

$$u\left(\sum_{i=1}^{n}\lambda_i\cdot x_i\right) > \sum_{i=1}^{n}\lambda_i\cdot u(x_i)$$

or

$$u^{-1}\left(u\left(\sum_{i=1}^{n}\lambda_i \cdot x_i\right)\right) > u^{-1}\left(\sum_{i=1}^{n}\lambda_i \cdot u(x_i)\right)$$

$$\sum_{i=1}^{n}\lambda_i \cdot x_i > \text{CE}$$

$$E(x) > \text{CE}$$

where $E(x)$ is the expected value of x.

Then, it follows that

$$\text{Risk Premium} = E(x) - \text{CE} > 0\,.$$

Jensen's Inequality and Investment under Uncertainty

The microeconomic analysis of investment has been an active area of research where Jensen's inequality plays a significant role. Hartman (1972) and Abel (1983) have both established a result that implies, since profit functions are convex in prices and operating costs, uncertainty in these parameters will increase expected profits, raising the rate of return and increasing the value of investment. For comparison, refer to figure 2.1 with the input x serving in the role of the uncertain price or cost parameter and output y representing profits. The effect of uncertainty in this case was determined to increase the scale of investment. Later, Small (1999) extended this line of inquiry by connecting it to the investment timing decision, which is concerned with the irreversibility of investment and the ability to delay an investment. I will only address the issue of the scale of investment here with uncertainty limited to output price.

Consider a firm that produces output Q using capital K and labor L as input. In any period t, these variables are related by a production function

(2.14) $$Q_t = f(K_t, L_t)$$

that is assumed homogeneous of degree one, concave, and nondecreasing in each input.

The capital stock for each period $t + 1$ is determined by the equation

(2.15) $$K_{t+1} = (1-\delta)K_t + I_t$$

where I_t is the gross investment for period t and δ is the rate of depreciation.

The cost of the firm's gross investment for period t is given by an investment cost function $C(I_t)$. Assume $C(I)$ is a strictly convex, increasing function in I, with $C'(I) > 0$ and $C''(I) > 0$ for all possible I.

Let p_t and w_t denote the firm's output price and the wage rate, respectively, for period t. Assume the firm does not know p_t and w_t until the beginning of period t. All future output prices and wage rates are treated as random variables with some probability distribution.

At the beginning of period t, when p_t and w_t become known and K_t has been determined via equation (2.15), the firm's objective is to maximize short-run profits. Define

(2.16) $$h(K_t, p_t, w_t) = \max_{L_t}\left[p_t \cdot f(K_t, L_t) - w_t \cdot L_t\right]$$

as the optimal profit function for each period $t = 0, 1, 2, \ldots$

Lemma 2.2. A firm's profit function is convex with respect to output price.

Proof. (This proof follows that of Horowitz, pp. 230-231.) Let Q be a firm's output, P is the firm's output price, and $c(Q)$ represent the firm's total cost curve. Then, let

$\mathrm{MC} = {d(c(Q))}\big/{dQ}$ represent the marginal cost function. Denote the optimum output as Q^* and note that at $P = \mathrm{MC}$, the firm's profit maximum is given by

$$\pi = P \cdot Q^* - c(Q^*)$$

For a small change in price, profit will change by

$$\begin{aligned}
\frac{d\pi}{dP} &= Q^* + \frac{dQ^*}{dP} \cdot P - \frac{d(c(Q^*))}{dQ^*} \cdot \frac{dQ^*}{dP} \\
&= Q^* + \frac{dQ^*}{dP} \cdot \left[P - \frac{d(c(Q^*))}{dQ^*} \right] \\
&= Q^*
\end{aligned}$$

since the firm is at optimum output level Q^* and $P = \mathrm{MC} = {d(c(Q^*))}\big/{dQ^*}$. Therefore, with an increasing marginal cost curve, it follows that

$$\frac{d^2\pi}{dP^2} = \frac{dQ^*}{dP} > 0.$$

Since the production function (2.14) is assumed homogeneous, Euler's theorem applies. Euler's theorem states that if a function $f(x_1, \ldots, x_n)$ is homogeneous of degree r, then $r \cdot f(x_1, \ldots, x_n) = f_1 \cdot x_1 + \ldots + f_n \cdot x_n$ where $f_i = \partial f / \partial x_i$ is the usual partial derivative. In this case, the production function f is homogeneous of degree one in the inputs K and L. It follows that

$$\begin{aligned}
Q_t &= \frac{\partial f(K_t, L_t)}{\partial K_t} \cdot K_t + \frac{\partial f(K_t, L_t)}{\partial L_t} \cdot L_t \\
&= \mathrm{MPP}_k \cdot K_t + \mathrm{MPP}_l \cdot L_t
\end{aligned}$$

Therefore,

$$p_t \cdot Q_t = \mathrm{MVP}_k \cdot K_t + \mathrm{MVP}_l \cdot L_t \tag{2.17}$$

where $\mathrm{MVP}_k = p_t \cdot \frac{\partial f}{\partial K}$ and $\mathrm{MVP}_l = p_t \cdot \frac{\partial f}{\partial L}$ are the marginal value products for the respective inputs.

Equation (2.17) combined with equation (2.16) results in another way to describe the optimal profit function.

$$\begin{aligned} h(K_t, p_t, w_t) &= \max_{L_t}\left[p_t \cdot f(K_t, L_t) - w_t \cdot L_t\right] \\ &= \max_{L_t}\left[\mathrm{MVP}_k \cdot K_t + \mathrm{MVP}_l \cdot L_t - w_t \cdot L_t\right] \\ &= \mathrm{MVP}_k \cdot K_t . \end{aligned}$$

Therefore, there exists a function $g(p_t, w_t)$ such that

$$h(K_t, p_t, w_t) = K_t \cdot g(p_t, w_t). \tag{2.18}$$

Since the profit function h is convex, the function g is clearly convex also in p and w. Furthermore, the function g can be described as the marginal profit value to the firm of another unit of capital in time period t.

Example:

The result in equation (2.18) is perhaps easiest to see through a simple example. Consider the Cobb-Douglas production function $Q_t = L_t^{\alpha} K_t^{1-\alpha}$. With output price p_t, the marginal value products for L and K can easily be derived as

$\mathrm{MVP}_l = p_t \cdot \alpha \cdot L_t^{\alpha-1} K_t^{1-\alpha}$ and $\mathrm{MVP}_k = p_t \cdot (1-\alpha) \cdot L_t^{\alpha} K_t^{-\alpha}$. With wage rate w_t, the first order condition for L_t is

$$\begin{aligned} & p_t \cdot \alpha \cdot L_t^{\alpha-1} K_t^{1-\alpha} = w_t \\ \Rightarrow\ & L_t^{1-\alpha} = \left(\frac{p_t \cdot \alpha}{w_t}\right) \cdot K_t^{1-\alpha} \end{aligned}$$

$$\Rightarrow L_t = \left(\frac{p_t \cdot \alpha}{w_t}\right)^{\frac{1}{1-\alpha}} \cdot K_t.$$

Therefore, at the optimal level of L_t,

$$\text{MVP}_k = p_t \cdot (1-\alpha) \cdot \left(\left(\frac{p_t \cdot \alpha}{w_t}\right)^{\frac{1}{1-\alpha}} \cdot K_t\right)^{\alpha} K_t^{-\alpha}$$

$$= (p_t)^{\frac{1}{1-\alpha}} \cdot \left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} \cdot (1-\alpha).$$

Finally, the optimal profit function can be written as

$$h(K_t, p_t, w_t) = \max_{L_t} \left[p_t \cdot L_t^{\alpha} K_t^{1-\alpha} - w_t \cdot L_t \right]$$

or, by substitution,

$$h(K_t, p_t, w_t) = p_t \cdot \left(\left(\frac{p_t \cdot \alpha}{w_t}\right)^{\frac{1}{1-\alpha}} \cdot K_t\right)^{\alpha} \cdot K_t^{1-\alpha} - w_t \cdot \left(\frac{p_t \cdot \alpha}{w_t}\right)^{\frac{1}{1-\alpha}} \cdot K_t$$

$$= (p_t)^{\frac{1}{1-\alpha}} \cdot \left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} \cdot (1-\alpha) \cdot K_t$$

$$= \text{MVP}_k \cdot K_t.$$

Therefore, as desired,

$$h(K_t, p_t, w_t) = g(p_t, w_t) \cdot K_t$$

where $g(p_t, w_t)$ is the marginal profit value to the firm of another unit of capital.

Assume the firm is risk neutral. The firm's objective in selecting I_t is to maximize the expected value of the sum of discounted future cash flows, given by

(2.19) $$E\left(\sum_{s=t}^{\infty} R^{s-t}\left[p_s \cdot f(K_s, L_s) - w_s \cdot L_s - C(I_s)\right]\right)$$

where R is a discount factor ($0 < R < 1$) equal to the inverse of one plus the discount rate and capital stock K_t has already been determined from previous periods.

Given the firm's optimal profit function (2.18), the firm's objective function (2.19) can be reformulated as selecting I_t to maximize

(2.20) $$E\left(\sum_{s=t}^{\infty} R^{s-t}\left[K_s \cdot g(p_s, w_s) - C(I_s)\right]\right)$$

subject to the current capital stock K_t.

At the beginning of period t, the investment decision for period t is made. The marginal cost function for this investment is given by $C'(I_t)$. By equation (2.15), each marginal unit of investment undertaken in period t will contribute $(1-\delta)^{s-t-1}$ units of capital stock to periods $s = t+1, t+2, \ldots$. Hence, by equation (2.18), each marginal unit of capital investment undertaken in period t will add $E\left[(1-\delta)^{s-t-1} \cdot g(p_s, w_s)\right]$ to the expected profits in periods $s = t+1, t+2, \ldots$. The first-order optimality condition for decision I_t is that the discounted sum of the marginal contributions to expected profits must be equal to the marginal cost for another unit of investment.

(2.21) $$C'(I_t) = \sum_{s=t+1}^{\infty} R^{s-t}(1-\delta)^{s-t-1} \cdot E\left[g(p_s, w_s)\right]$$

Now consider the uncertainty in the output price p_s for periods $s = t+1, t+2, \ldots$. Since $g(p_s, w_s)$ is a convex function with respect to p_s, Jensen's inequality as applied through Lemma 2.2(a) implies that any mean preserving spread in some or all of the future output prices p_s will increase some or all of the terms contained in the summation on the right-hand side of equation (2.21). Moreover, a mean preserving spread in p_s will never decrease any of the summation terms. Since $C''(I_t) > 0$ for all t implies that $C'(t)$ is monotonically increasing, any increase in the right-hand side of equation (2.21) must be balanced by an increase in I_t in order to maintain the optimality condition. Therefore, increasing uncertainty in the future output price for a firm increases the scale of the firm's optimal investment level for the present time period by increasing the discounted marginal value of expected future returns associated with investment.

Jensen's Inequality and Life Insurance

Life insurance and annuities are another area where Jensen's inequality plays a significant role. For life insurance, formulas for compounding level premium payments made at fixed intervals are convex with respect to time for any positive interest rate. Therefore, in the case of a whole-life policy, where value is accumulated within the policy for an uncertain number of years, basing premiums on life expectancy would not be accurate. Jensen's inequality and a convex compounding interest formula combine to guarantee that policy value at life expectancy will be less than the expected policy value at death. For this reason, actuaries will generally base their policy valuations on the appropriate mortality rates for the insurance applicant rather than the applicant's life expectancy (Behan).

For annuities, the relationship is the opposite. Discounting formulas for fixed annuity payments for an uncertain number of years are concave with respect to time for any positive interest rate. Fixed annuity payments based on life expectancy will underpay the annuity recipients. The present value of level, continuous annuity payments certain for the life expectancy is greater than the actual expected present value of the annuity payments. The expected present value of the annuity payments will lie below a concave curve representing the discounted present value of total annuity payments with respect to time.

Jensen's Inequality and Ecology

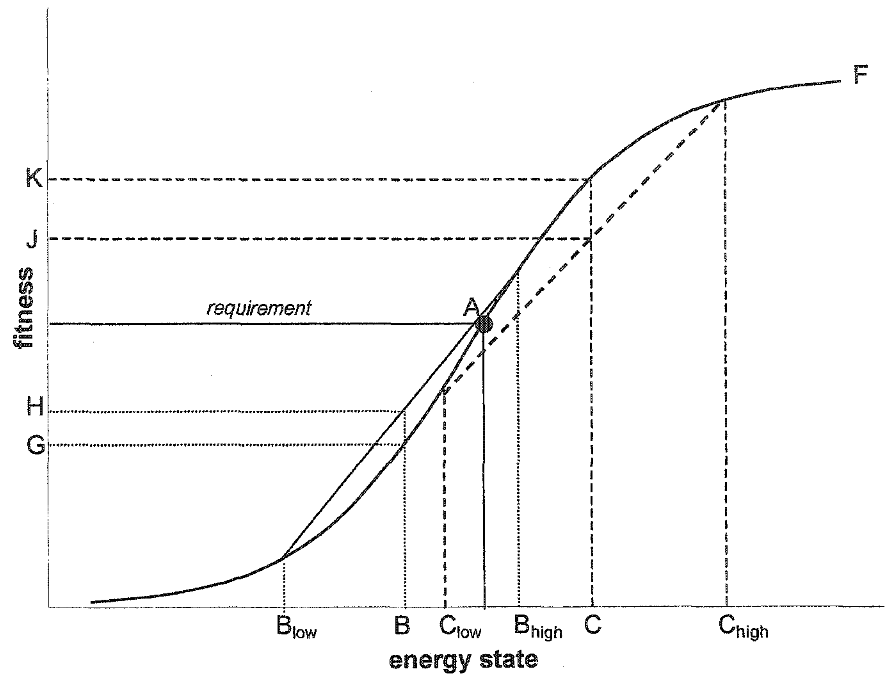
Jensen's inequality has proved to be a useful tool in ecology. I will address two applications of it here. The first is an application in explaining risk-sensitive foraging behavior. The second is an application in plant ecology and the processing of sunlight. Ruel and Ayres (1999) provide an excellent explanation of Jensen's inequality and its applications in ecology. Their paper provides the basis for the discussion of plant ecology. Smallwood's paper (1996) on risk-sensitivity provides the basis for the discussion on risk-sensitive foraging behavior.

Risk-sensitive Foraging Behavior

Studies of risk-sensitive foraging behavior originated with the independent works of Caraco and Real both done in 1980. Prior to that, animal foraging behavior was explained through models using average reward conditions (Smallwood). Risk-sensitive foraging theory takes into account that animals are sensitive to not only the average foraging reward but also the variability in that reward when making their foraging strategy decision. Caraco, Martindale, and Whittam carried out some striking experiments to illustrate this sensitivity using wild birds. In the experiments, wild-caught birds were trained to forage for seeds from covered dishes in a laboratory setting. The birds learned to associate different colors of covers with the foraging reward contained in each dish. The experiments consisted of offering dishes with two different colors of lids to the birds. Under one color, a constant number of seeds were found while under the other color, the number of seeds was variable. In both cases, the average number of seeds was the same. When the average number of seeds was above the minimum number required to maintain a positive net energy level, the birds preferred the dishes with a constant number of seeds. When the average was below the minimum requirements, the birds preferred the dishes with a variable number of seeds. This forms the basis for the logic of Caraco's energy-budget rule. According to the rule, an animal expecting to exceed its daily energy requirements will exhibit risk-averse behavior. The theoretical explanation of Caraco's energy-budget rule is best done through Jensen's inequality.

At the base of foraging decision theory is the animal's fitness function (figure 2.3). A fitness function describes the animal's level of fitness as a function of its energy state. Its energy state is a direct result of foraging rewards. Risk-sensitive foraging models assume the animal's fitness function is nonlinear. This appears to be a reasonable assumption in a majority of cases. Furthermore, it seems reasonable to assume that the fitness function will have a sigmoidal shape to it. Adding marginal increments of energy to an animal at a very low level of fitness will have increasing returns to scale up to a point. At the other end of the spectrum, adding marginal increments of energy are likely to add very small amounts to an animal's already very high level of fitness. The point of inflection (A) on the sigmoidal curve is taken by Caraco to be the best approximation of the animal's "requirement." Above this level, the animal will exhibit risk-averse foraging behavior and below this level, the behavior will be more risk prone.

Figure 2.3: Risk sensitive foraging behavior



For example, consider a situation where an animal can receive energy state B for certain or choose a different, more risk-prone foraging strategy that returns energy state B_{low} and B_{high} with equal probability. In either case, the average energy reward is at B. If the animal were to choose the risk-averse strategy, it would for certain be at a fitness level G that would be below their "requirement." Given these choices, the animal can maximize the probability of meeting or exceeding their requirement level by choosing the more risk-prone strategy. Furthermore, by Jensen's inequality, $E(F(B)) > F(E(B))$. That is, the risk-prone strategy will result in an average fitness level of H that is higher than the certain fitness level of G associated with the risk-averse strategy.

Consider a second situation where an animal can receive energy state C for certain or choose the risk-prone foraging strategy that returns energy state C_{low} and C_{high}

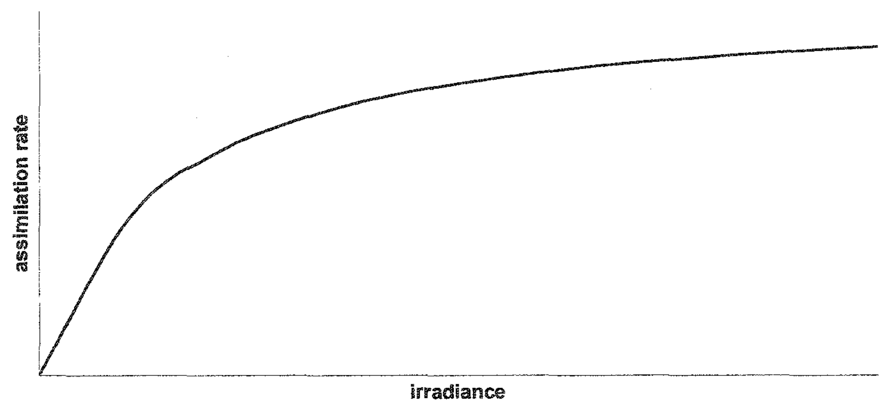
with equal probability. The average energy reward is C in both cases. Now, the animal can maximize the probability of exceeding the "requirement" by choosing the more risk-averse strategy. Thus, the animal will choose to receive C for certain. By Jensen's inequality, the risk-averse strategy will maximize the average fitness level for the animal since $E(F(C))<F(E(C))$. That is, the risk-averse strategy will result in a certain fitness level of K that is higher than the average fitness level of J associated with the risk-prone strategy.

In the experiments of Caraco, Martindale, and Whittam, they were able to duplicate this behavior with wild-caught birds. When the dishes with a constant number of seeds contained a number that was below the minimum requirement, the birds exhibited risk-prone behavior by foraging the dishes associated with variable rewards. When the constant reward was above the requirement level, the behavior became risk-averse as birds foraged the dishes associated with a constant reward. One can argue that this animal behavior is not that far detached from some behavior that humans exhibit regularly when faced with risky decisions. That is what makes these all the more fascinating and Jensen's inequality provides a nice tool to succinctly tie all of it together.

Plant Ecology

The second ecological application I will discuss is a plant ecology application as described by Ruel and Ayres. For most photosynthetic organisms, carbon assimilation as a function of irradiance (light) is a decelerating function (figure 2.4). Furthermore, because of environmental conditions and seasonal cycles, the light regime in every natural habitat is inherently variable. Jensen's inequality predicts that this variability will

Figure 2.4: Plant photosynthesis relationship



depress production relative to the production expectation based upon average irradiance. This is true across both time and space.

Light measurements will vary across time. If the measurements are aggregated across too large of a time period, it produces a systematic bias in the calculated net assimilation. For example, in an unpublished study by D. S. Canny, et. al., using a daily average light level overestimated net daily assimilation by more than 100% compared to estimates calculated using five minute intervals (Ruel and Ayres). This type of overestimation produces severe bias in any subsequent estimates of expected production.

Averaging across too big of a space produces similarly biased results. Aggregating measurements from three light sensors spaced ten meters apart produced a 14% over estimation of carbon assimilation (Ruel and Ayres). Jensen's inequality is telling us that sometimes average environmental conditions are not a good measurement. The spatial and temporal scales used for aggregation of data can have a pronounced effect on the accuracy of the subsequent estimates. If the scale is too large and the resulting variability within the aggregated unit is too high, then Jensen's inequality combined with a nonlinear response function can severely bias the results.

Summary and Conclusions

In this chapter, I have introduced Jensen's inequality, its many variants, and some of the important applications of it. On the surface, Jensen's inequality is a pretty basic idea. It provides a succinct way a progressing from the definition of concavity and convexity into a world of continuous distributions. As a tool, it has become invaluable in explaining behavior in both a normative and a positive sense. I have outlined how two very important inequalities, Cauchy's inequality and the arithmetic mean-geometric mean inequality, that can be explained under the umbrella of Jensen's inequality. The arithmetic mean-geometric mean inequality, in particular, has played an important role in economic history. Bernoulli's logarithmic utility function and Latané's optimal growth investment strategy provided two important examples where this is illustrated in this chapter.

Incentives for new capital investment when faced with uncertain future prices were also explained using Jensen's inequality along with a profit function convex in output price. Methods employed by the life insurance industry were rationalized through Jensen's inequality and nonlinear compounding/discounting functions. Finally, I provided two striking examples from ecology where Jensen's inequality is the central piece that ties together the theory. The first, Caraco and Real's risk sensitive foraging behavior is an important contribution to the science of animal behavior and provides some interesting insights into human behavior. The second, a discussion of the photosynthetic process and plant ecology, provides a nice transition into the production economics application focus of this dissertation.

Production economics often employs nonlinear curves in making theoretical explanations. In chapters 4-6, I will exploit this along with Jensen's inequality to provide new insight into the optimization of group production. Livestock production provides a natural setting for this type of application and I will exploit that in the examples throughout the remainder of this dissertation. With that in mind, I will digress for a moment in the next chapter and provide an overview of animal growth functions.

Chapter 3

Animal Growth Functions

The economic importance of the rate of maturing, the rate of gain, mature size and related characteristics leads to an interest by animal scientists and producers in weight-age relationships (Brown, Fitzhugh, and Cartwright). A data series containing weight-age observations may yield some intuitive insights into the growth characteristics of the animals. In general, however, a more systematic approach is needed to condense the data into a more usable format. One way of condensing the information is to employ a nonlinear model containing a few biologically interpretable parameters.

I consider five such models for this study: the Brody (Brody, 1945), the von Bertalanffy (von Bertalanffy, 1957), the Richards (Richards, 1959), the Gompertz (Gompertz, 1825; Winsor, 1932), and the logistic. In addition, I considered two generalizations of the logistic curve, the Nelder generalization (Nelder, 1961) and the Oliver generalization (Oliver, 1969), and one modification of the Gompertz curve (Parks, 1982). These models and their derived traits are summarized in table 3.1.

I compared the goodness of fit, computational ease, and biological interpretability of each of these models as they applied to my data sets for swine and cattle on feed. In the end, I selected the modified Gompertz function suggested by Parks to model the swine growth and the logistic function to model the cattle growth. In the pages that

Table 3.1 Growth Models and Derived Traits

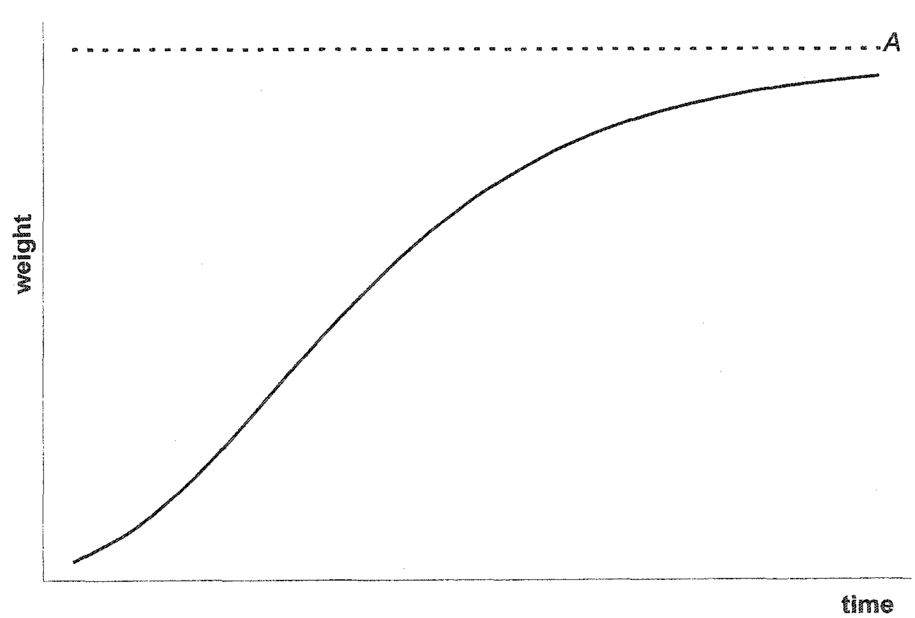
Model	Equation, $W_t =$	Initial Weight	Asymptotic Maximum Weight	Rate of Maturing	Growth rate $(dW_t/dt) =$	Point of Inflection $W_t =$
von Bertalanffy	$A(1-Be^{-Kt})^3$	$A(1-B)^3$	A	K	$3W_tBKe^{-Kt}(1-Be^{-Kt})^{-1}$	$0.296A$
Brody	$A(1-Be^{-Kt})$	$A(1-B)$	A	K	$W_tBKe^{-Kt}(1-Be^{-Kt})^{-1}$	...
Gompertz	$Ae^{-Be^{-Kt}}$	Ae^{-B}	A	K	W_tBKe^{-Kt}	$0.368A$
Gompertz (Parks modification)	$A\left(\frac{W_0}{A}\right)^{e^{-Kt}}$	W_0	A	K	$-W_tK\cdot\ln\left(\frac{W_t}{A}\right)$	$0.368A$
Logistic	$A(1+e^{-Kt})^{-1}$	$0.5A$	A	K	$W_tKe^{-Kt}(1+e^{-Kt})^{-1}$	$0.5A$
Logistic (Nelder generalization)	$A(1+e^{-Kt})^{-M}$	$A(2)^{-M}$	A	K	$W_tKMe^{-Kt}(1+e^{-Kt})^{-1}$	$A\left(\frac{M}{M+1}\right)^M$
Logistic (Oliver generalization)	$\gamma+A(1+e^{-Kt})^{-1}$	$\gamma+0.5A$	$\gamma+A$	K	$W_tKe^{-Kt}(1+e^{-Kt})^{-1}$	$\gamma+0.5A$
Richards	$A(1-Be^{-Kt})^M$	$A(1-B)^M$	A	K	$W_tBKMe^{-Kt}(1-Be^{-Kt})^{-1}$	$A\left(\frac{M-1}{M}\right)^M$

W_t = weight at time t; A, B, K, M, and γ are fitted parameters.

follow, I will briefly describe the characteristics of each of these models and justify the models selected.

Animal growth often exhibits a standard sigmoidal shape (figure 3.1). The animals grow at an increasing marginal rate up to a point of inflection. Thereafter, the animals' growth is characterized by a saturation effect. That is, each animal approaches a maximum mature weight at a diminishing but always-positive marginal rate. With the exception of the Oliver generalization of the logistic function, the fitted parameter A in each model represents an estimate of the asymptotic maximum mature weight. Craig and Schinckel (2001) showed for a sample of pig data that allowing the parameter A to vary provided an adequate explanation for between pig variation. Capturing between animal variation is a very complicated process. Therefore, Craig and Schinckel's finding provides an important simplification.

Figure 3.1 Sigmoidal Growth Curve



For each of the models, the fitted parameter K is an estimate for the rate of maturing. The rate of maturing is best described as the growth rate relative to the mature weight. Large K values indicate early maturing animals and small K values indicate late maturing animals. Genetic and environmental factors that influence the growth rate and the mature weight will influence the estimates of K and, obviously, K will always be greater than or equal to zero.

Brody

The Brody growth function,

(3.1) $$W_t = A\left(1 - Be^{-Kt}\right),$$

has three parameters to be estimated: the asymptotic mean mature weight A, the rate of maturing K, and a fitted parameter B. Consider the initial conditions

(3.2) $$W_o = A\left(1 - Be^{0}\right) = A(1 - B).$$

Equation (3.2) implies

(3.3) $$B = 1 - \frac{W_o}{A}.$$

Describe an animal's maturity level at any time t as the ratio of its current weight to its asymptotic mean mature weight. Then, equation (3.3) implies that the fitted parameter B describes an initial maturity condition. Specifically, the parameter B describes the maturity (or growth) remaining to be realized. The parameter B will be smaller (larger) the closer (farther away) the animal is to (from) its final mature weight. The value of B will always be greater than or equal to zero and less than one.

To calculate the marginal growth rate, or the rate of growth, it is necessary to take the derivative of equation (3.1) with respect to time.

$$\text{(3.4)} \qquad \mathbf{MPP}_t = \frac{dW_t}{dt} = A\left(-Be^{-Kt}\right)\left(-K\right) = ABKe^{-Kt}.$$

The Brody growth function, like all growth functions with an asymptotic mature weight, has a marginal product curve that is always positive.

The second derivative of equation (3.1),

$$\text{(3.5)} \qquad \frac{d^2W_t}{dt^2} = ABKe^{-Kt}\left(-K\right) = -ABK^2e^{-Kt},$$

is used to describe the concavity of the Brody growth function or the slope of its marginal curve. Equation (3.5) is negative for all values of t greater than or equal to zero. Therefore, the Brody growth function is characterized by diminishing marginal returns for all values of t greater than or equal to zero. There is no point of inflection. The Brody growth curve is concave everywhere and its marginal growth curve is always declining.

The third derivative of equation (3.1),

$$\text{(3.6)} \qquad \frac{d^3W_t}{dt^3} = -ABK^2e^{-Kt}\left(-K\right) = ABK^3e^{-Kt},$$

is always greater than or equal zero. Therefore, the marginal curve for the Brody growth function is always convex.

von Bertalanffy

The von Bertalanffy growth function,

$$\text{(3.7)} \qquad W_t = A\left(1 - Be^{-Kt}\right)^3,$$

also has three parameters to be estimated: the asymptotic mean mature weight A, the rate of maturing K, and a fitted parameter B. Here, the initial conditions, (3.8) $$W_o = A\left(1 - Be^{0}\right)^3 = A(1-B)^3,$$

imply

(3.9) $$B = 1 - \left(\frac{W_o}{A}\right)^{1/3}.$$

The interpretation of equation (3.9) is same as the interpretation of equation (3.3) was for the Brody growth function. The fitted parameter B describes an initial maturity condition. It will be smaller or larger depending upon initially how far the animal is away from its final mature weight. The value of B will always be greater than or equal to zero and less than one.

To calculate the marginal growth rate, or the rate of growth, it is necessary to take the derivative of equation (3.7) with respect to time.

(3.10) $$\mathbf{MPP}_t = \frac{dW_t}{dt} = 3A\left(1 - Be^{-Kt}\right)^2\left(-Be^{-Kt}\right)(-K) = 3ABKe^{-Kt}\left(1 - Be^{-Kt}\right)^2.$$

Again, like all growth functions with an asymptotic mature weight, the von Bertalanffy has a marginal product curve that is always positive.

The second derivative of equation (3.7),

$$\frac{d^2W_t}{dt^2} = 3ABKe^{-Kt} \cdot 2\left(1 - Be^{-Kt}\right)\left(-Be^{-Kt}\right)(-K) + 3ABKe^{-Kt}(-K)\left(1 - Be^{-Kt}\right)^2$$

or

(3.11) $$\frac{d^2W_t}{dt^2} = 3ABK^2e^{-Kt}\left(1 - Be^{-Kt}\right)\left(3Be^{-Kt} - 1\right),$$

is used to calculate the point of inflection for the von Bertalanffy growth function. This is where the marginal product curve (3.10) achieves its maximum and where the production curve (3.7) changes from a convex function to a concave function. This occurs when $\frac{d^2W_t}{dt^2}=0$ which implies that either $Be^{-Kt}=1$ or $3Be^{-Kt}=1$. If

$Be^{-Kt}=1$, then $t=-\frac{1}{K}\ln\left(\frac{1}{B}\right)$ which, when plugged into (3.7), implies $W_t=0$.

Therefore, the possibility that $Be^{-Kt}=1$ is ignored. If $3Be^{-Kt}=1$, then

$t=-\frac{1}{K}\ln\left(\frac{1}{3B}\right)$. Substituting $t=-\frac{1}{K}\ln\left(\frac{1}{3B}\right)$ into equation (3.7) yields

(3.12) $$W_t=\frac{8}{27}A\cong 0.296A.$$

Equation (3.11) and the result (3.12) imply that the von Bertalanffy has a fixed point of inflection relative to mature size. At a weight roughly 30% of its mature size, the animal transitions from increasing marginal growth to a pattern of decreasing marginal growth.

The third derivative of equation (3.7), which checks for the concavity of the marginal product curve (3.10), is very complicated.

$$\frac{d^3W_t}{dt^3}=3ABK^2e^{-Kt}\left(1-Be^{-Kt}\right)\left(3Be^{-Kt}\right)(-K)$$
$$+\left(3Be^{-Kt}-1\right)\cdot\left[3ABK^2e^{-Kt}\left(-Be^{-Kt}\right)(-K)+\left(1-Be^{-Kt}\right)\left(3ABK^2e^{-Kt}\right)(-K)\right]$$

or

(3.13) $$\frac{d^3W_t}{dt^3}=-3ABK^3e^{-Kt}\left(-9B^2e^{-2Kt}+8Be^{-Kt}-1\right).$$

Setting equation (3.13) equal to zero and solving yields

$$Be^{-Kt}=\frac{4+\sqrt{7}}{9}\quad\text{or}\quad Be^{-Kt}=\frac{4-\sqrt{7}}{9}$$

which occurs when

$$t = -\frac{1}{K}\ln\left(\frac{4+\sqrt{7}}{9B}\right) \quad \text{or} \quad t = -\frac{1}{K}\ln\left(\frac{4-\sqrt{7}}{9B}\right)$$

or

(3.14) $$W_t \cong 0.0179A \quad \text{or} \quad W_t \cong 0.613A.$$

Equation (3.13) and the result (3.14) imply that the marginal product curve for the von Bertalanffy growth function is convex up to $W_t \cong 0.0179A$, concave for $0.0179A < W_t < 0.613A$, and convex for $W_t > 0.613A$.

Richards

The Richards growth function,

(3.15) $$W_t = A\left(1 - Be^{-Kt}\right)^M,$$

is generalization of the Brody and the von Bertalanffy growth functions. It has four parameters to be estimated: the asymptotic mean mature weight A, the rate of maturing K, and two fitted parameters B and M. Now, the initial conditions,

(3.16) $$W_o = A\left(1 - Be^0\right)^M = A(1-B)^M,$$

imply

(3.17) $$B = 1 - \left(\frac{W_o}{A}\right)^{1/M}.$$

The initial level of maturity $\left(\frac{W_o}{A}\right)$ will influence the value of both of the fitted parameters B and M. The fitted parameter B can still be described as an initial maturity condition. It will still be smaller or larger depending upon initially how far the animal is away from its final mature weight and its value will always be greater than or equal to zero and less than one.

Now, for the rate of growth, the derivative of equation (3.15) with respect to time is

$$\mathbf{MPP}_t = \frac{dW_t}{dt} = AM\left(1 - Be^{-Kt}\right)^{M-1}\left(-Be^{-Kt}\right)\left(-K\right)$$

or

$$\mathbf{MPP}_t = \frac{dW_t}{dt} = ABKMe^{-Kt}\left(1 - Be^{-Kt}\right)^{M-1}. \tag{3.18}$$

Under the assumption that $M > 0$, it is easy to verify that equation (3.18) is always greater than or equal to zero for all $t \geq 0$. Hence, a marginal product curve that is always positive is verified for the Richards growth function.

The second derivative of equation (3.15) is

$$\frac{d^2W_t}{dt^2} = ABKMe^{-Kt} \cdot (M-1)\left(1 - Be^{-Kt}\right)^{M-2}\left(-Be^{-Kt}\right)\left(-K\right) + ABKMe^{-Kt}\left(-K\right)\left(1 - Be^{-Kt}\right)^{M-1}$$

or

$$\frac{d^2W_t}{dt^2} = MABK^2e^{-Kt}\left(1 - Be^{-Kt}\right)^{M-2}\left(MBe^{-Kt} - 1\right). \tag{3.19}$$

Since $Be^{-Kt} = 1$ only occurs when $W_t = 0$, the Richards function's only point of inflection occurs when $MBe^{-Kt} = 1$. If $MBe^{-Kt} = 1$, then $t = -\frac{1}{K}\ln\left(\frac{1}{MB}\right)$ which, when plugged into (3.15), implies

$$W_t = A\left(1 - \frac{1}{M}\right)^M = A\left(\frac{M-1}{M}\right)^M. \tag{3.20}$$

Thus, the Richards function has a variable point of inflection that depends upon the value of M. For the von Bertalanffy function, $M = 3$ resulted in a point of inflection at $W_t = (2/3)^3 A \cong 0.296A$.

The third derivative of equation (3.15) is

$$\frac{d^3W_t}{dt^3} = MABK^2e^{-Kt}\left(1 - Be^{-Kt}\right)^{M-2}\left(MBe^{-Kt}\right)\left(-K\right) + \left(MBe^{-Kt} - 1\right) \cdot \left[MABK^2e^{-Kt}(M-2)\left(1 - Be^{-Kt}\right)^{M-3}\left(-Be^{-Kt}\right)\left(-K\right) + \left(1 - Be^{-Kt}\right)^{M-2}\left(MABK^2e^{-Kt}\right)\left(-K\right)\right]$$

or

$$\frac{d^3W_t}{dt^3} = -MABK^3e^{-Kt}\left(1 - Be^{-Kt}\right)^{M-3}\left(-M^2B^2e^{-2Kt} + (3M-1)Be^{-Kt} - 1\right) \tag{3.21}$$

Again, since $Be^{-Kt} = 1$ only occurs when $W_t = 0$, I only need to examine when

$$-M^2B^2e^{-2Kt} + (3M-1)Be^{-Kt} - 1 = 0 \tag{3.22}$$

to determine the concavity of the marginal product curve.

If I let $x = Be^{-Kt}$, then I can rewrite equation (3.22) as

$$-M^2x^2 + (3M-1)x - 1 = 0. \tag{3.23}$$

Setting $a = -M^2$, $b = 3M - 1$, and $c = -1$, I can apply the quadratic formula to equation (3.23) and solve for

$$x = \frac{-(3M-1) \pm \sqrt{(-5M+1)(-M+1)}}{-2M^2}.$$

Since $x = Be^{-Kt}$ implies that $t = -\frac{1}{K}\ln\left(\frac{x}{B}\right)$, I can plug this value of t into the original Richards function (3.15) to write the weight as a function of x,

$$W_t = A(1-x)^M .$$

Thus, the marginal product curve for the Richards function has two points of inflection. The first point of inflection, where the marginal product curve changes from convexity to concavity, occurs at a lower weight of

$$W_t = A \cdot x_L ,$$

where

$$x_L = \left(1 + \frac{-3M + 1 - \sqrt{(-5M+1)(-M+1)}}{2M^2}\right)^M .$$

The second point of inflection, where the marginal product curve changes from concavity back to convexity, occurs at an upper weight of

$$W_t = A \cdot x_U ,$$

where

$$x_U = \left(1 + \frac{-3M + 1 + \sqrt{(-5M+1)(-M+1)}}{2M^2}\right)^M .$$

It is easy to verify that in the case of the von Bertalanffy growth function and $M = 3$, $x_L \cong 0.0179$ and $x_U \cong 0.613$.

Gompertz

The Gompertz growth function,

(3.24) $$W_t = Ae^{-Be^{-Kt}} ,$$

was first offered by Benjamin Gompertz in 1825 in the form $L_x = kg^{c^x}$ where L_x was the number of people living at age x. The Gompertz curve was at first only of interest to actuaries. In 1932, Winsor formalized its application as a growth curve. It has three parameters to be estimated: the asymptotic mean mature weight A, the rate of maturing K, and the fitted parameter B. The initial conditions,

$$W_o = Ae^{-Be^0} = Ae^{-B}, \tag{3.25}$$

imply

$$B = -\ln\left(\frac{W_o}{A}\right). \tag{3.26}$$

The fitted parameter B will depend upon the initial level of maturity $\left(\frac{W_o}{A}\right)$. The more mature the animal is initially, the closer $\left(\frac{W_o}{A}\right)$ is to one and the smaller B will be. This is not unlike the initial maturity condition as described for the Brody, von Bertalanffy, and Richards functions. However, unlike these earlier functions, there is nothing by the relationship (3.26) that restricts the parameter B to be less than one but it will always be greater than or equal to zero.

Now, for the rate of growth, the derivative of equation (3.24) with respect to time is

$$\text{MPP}_t = \frac{dW_t}{dt} = Ae^{-Be^{-Kt}}\left(-Be^{-Kt}\right)\left(-K\right) = ABKe^{-Kt-Be^{-Kt}}. \tag{3.27}$$

It is trivial to show that equation (3.27) is always greater than or equal to zero for all $t \geq 0$. Hence, a marginal product curve for the Gompertz growth function is always positive.

The second derivative of equation (3.24) is

$$\frac{d^2W_t}{dt^2} = ABKe^{-Kt-Be^{-Kt}}\left(-K - Be^{-Kt}(-K)\right)$$

or

(3.28) $$\frac{d^2W_t}{dt^2} = -ABK^2e^{-Kt-Be^{-Kt}}\left(1 - Be^{-Kt}\right).$$

Equation (3.28) equals zero when $Be^{-Kt} = 1$ which occurs when $t = -\frac{1}{K}\ln\left(\frac{1}{B}\right)$.

Plugging this value of t into equation (3.24) yields

(3.29) $$W_t = Ae^{-1} \cong 0.368A.$$

This is the point at which the marginal product of the Gompertz function achieves its maximum and where the production curve changes from a convex to a concave function.

The third derivative of equation (3.24) is

$$\frac{d^3W_t}{dt^3} = -ABK^2e^{-Kt-Be^{-Kt}}\left[-Be^{-Kt}(-K)\right] + \left(1 - Be^{-Kt}\right)\left(-ABK^2e^{-Kt-Be^{-Kt}}\right)\left[-K - Be^{-Kt}(-K)\right]$$

or

(3.30) $$\frac{d^3W_t}{dt^3} = -ABK^3e^{-Kt-Be^{-Kt}}\left[Be^{-Kt} - \left(1 - Be^{-Kt}\right)^2\right].$$

I only need to examine when

(3.31) $$Be^{-Kt} - \left(1 - Be^{-Kt}\right)^2 = 0$$

to determine the inflection point(s) of the marginal product curve.

Equation (3.31) can be rewritten as

(3.32) $$\left(Be^{-Kt}\right)^2 - 3Be^{-Kt} + 1 = 0 .$$

If I let $x = Be^{-Kt}$, then I can rewrite equation (3.32) as $x^2 - 3x + 1 = 0$ and apply the quadratic formula to get

(3.33) $$Be^{-Kt} = \frac{3 \pm \sqrt{5}}{2} .$$

If $Be^{-Kt} = \dfrac{3+\sqrt{5}}{2}$, then $t = -\dfrac{1}{K}\ln\left(\dfrac{3+\sqrt{5}}{2B}\right)$ and

(3.34) $$W_t = Ae^{-\left(\frac{3+\sqrt{5}}{2}\right)} \cong 0.0729A .$$

If $Be^{-Kt} = \dfrac{3-\sqrt{5}}{2}$, then $t = -\dfrac{1}{K}\ln\left(\dfrac{3-\sqrt{5}}{2B}\right)$ and

(3.35) $$W_t = Ae^{-\left(\frac{3-\sqrt{5}}{2}\right)} \cong 0.683A .$$

Equations (3.34) and (3.35) represent the two points at which the marginal product curve for the Gompertz production function changes concavity. For weights prior to 0.0729*A*, the marginal product curve is convex. It is concave for weights from 0.0729*A* to 0.683*A* and convex thereafter.

Parks' Modification

In his 1982 book, John R. Parks commented that the initial condition relationship indicated in equation (3.26) leads to estimation bias for the parameters *A* and *K*. He suggested that equation (3.26) be substituted into equation (3.25) to eliminate this bias. Having done so, the result is the following modification of the Gompertz growth function.

(3.36)
$$W_t = A\left(\frac{W_0}{A}\right)^{e^{-Kt}}$$

The impact this has on the previous computations is largely inconvenience. For the rate of growth with respect to time, the marginal equation (3.27) with Parks' modification becomes

(3.37)
$$\mathbf{MPP}_t = \frac{dW_t}{dt} = -AKe^{-Kt}\left(\frac{W_0}{A}\right)^{e^{-Kt}} \ln\left(\frac{W_0}{A}\right).$$

Equation (3.28) of the second derivative becomes

(3.38)
$$\frac{d^2W_t}{dt^2} = AK^2e^{-Kt}\left(\frac{W_0}{A}\right)^{e^{-Kt}}\left[\ln\left(\frac{W_0}{A}\right)\right]\cdot\left[e^{-Kt}\ln\left(\frac{W_0}{A}\right)+1\right]$$

and equation (3.30) of the third derivative becomes

(3.39)
$$\frac{d^3W_t}{dt^3} = -AK^3e^{-Kt}\left(\frac{W_0}{A}\right)^{e^{-Kt}}\left[\ln\left(\frac{W_0}{A}\right)\right]\cdot\left[\left(e^{-Kt}\ln\left(\frac{W_0}{A}\right)\right)^2 + 3e^{-Kt}\ln\left(\frac{W_0}{A}\right)+1\right].$$

While the equations (3.37)-(3.39) appear much more complex than their predecessors, the results that are derived from them corresponding to equations (3.29), (3.34), and (3.35) remain the same.

Logistic

The advantage to the logistic growth function,

(3.40)
$$W_t = A\left(1+e^{-Kt}\right)^{-1},$$

is that there are only two parameters to be estimated: the asymptotic mean mature weight A and the rate of maturing K. However, there are major restrictions to the model, starting with the initial conditions

(3.41) $$W_o = A\left(1+e^0\right)^{-1} = A(2)^{-1} = \frac{A}{2}.$$

Equation (3.41) implies that the initial weight, as estimated by the model, is restricted to be equal to exactly half of the estimate for the asymptotic mean mature weight. This is one of the issues that the Nelder and the Oliver generalizations are specifically designed to address.

A second major issue involves the rate of growth

(3.42) $$\mathbf{MPP}_t = \frac{dW_t}{dt} = AKe^{-Kt}\left(1+e^{-Kt}\right)^{-2}$$

and its derivative,

(3.43) $$\frac{d^2W_t}{dt^2} = -AK^2e^{-Kt}\left(1+e^{-Kt}\right)^{-3}\left(1-e^{-Kt}\right),$$

used to calculate its maximum, which is also the point of inflection for the logistic curve. From equation (3.43), the relationship $\frac{d^2W_t}{dt^2} = 0$ only occurs when $e^{-Kt} = 1$. However, if $e^{-Kt} = 1$, then $t = 0$. So, the initial conditions (3.41) define the inflection point. A quick check of equation (3.43) will reveal that its value is less than or equal to zero for all positive t. Therefore, the logistic growth curve (3.40) is everywhere concave.

The third derivative of equation (3.40), which checks for the concavity of the marginal product curve (3.42), is less of a problem.

(3.44) $$\frac{d^3W_t}{dt^3} = AK^3e^{-Kt}\left(1+e^{-Kt}\right)\left(e^{-2Kt} - 4e^{-Kt} + 1\right).$$

Setting equation (3.44) equal to zero and solving yields

$$e^{-Kt} = 2+\sqrt{3} \quad \text{or} \quad e^{-Kt} = 2-\sqrt{3}$$

which occurs when

$$t = -\frac{1}{K}\ln\left(2+\sqrt{3}\right) \text{ or } t = -\frac{1}{K}\ln\left(2-\sqrt{3}\right)$$

or

$$W_t \cong 0.211A \text{ or } W_t \cong 0.789A. \tag{3.45}$$

Equation (3.44) and the result (3.45) imply that the marginal product curve for the logistic growth function is convex up to $W_t \cong 0.211A$, concave for $0.211A < W_t < 0.789A$, and convex for $W_t > 0.789A$. Note that the initial condition (3.41) is $W_o = 0.5A$. Therefore, the marginal product curve for the logistic growth function starts out as a concave function at $t = 0$. It remains a concave function with respect to t until W_t exceeds 0.789*A*, at which time it becomes convex.

Nelder Generalization

In 1961, Nelder offered a generalization of the logistic function to address the two major restrictions on the initial weight and the point of inflection. The Nelder generalization of the logistic growth function

$$W_t = A\left(1+e^{-Kt}\right)^{-M}, \tag{3.46}$$

adds a third parameter, *M*, that he describes as a form parameter. The form parameter, along with the asymptotic mean mature weight *A*, will jointly determine the point of inflection. Nelder argues that this adds biological interpretation to the point of inflection. This is in contrast to the fixed points of inflection relative to mature size that were observed for both the von Bertalanffy and the Gompertz functions.

The initial conditions

(3.47) $$W_o = A\left(1+e^0\right)^{-M} = A(2)^{-M},$$

imply that

(3.48) $$M = -\frac{\ln\left(\frac{W_0}{A}\right)}{\ln 2}.$$

So, the estimated parameter M depends upon the initial level of maturity $\left(\frac{W_o}{A}\right)$.

Equation (3.48) works similarly to the initial condition equations for parameter B in the earlier models. The more mature the animal is initially (the closer $\left(\frac{W_o}{A}\right)$ is to one), the smaller M will be and M will always be greater than or equal to zero.

The first derivative,

(3.49) $$\mathbf{MPP}_t = \frac{dW_t}{dt} = AKMe^{-Kt}\left(1+e^{-Kt}\right)^{-M-1},$$

and the second derivative,

(3.50) $$\frac{d^2W_t}{dt^2} = -AK^2Me^{-Kt}\left(1+e^{-Kt}\right)^{-M-2}\left(1-Me^{-Kt}\right),$$

of equation (3.46) indicate that the marginal product of the Nelder generalization of logistic growth achieves its maximum when $Me^{-Kt} = 1$. Therefore, the point of inflection occurs at $t = -\frac{1}{K}\ln\left(\frac{1}{M}\right)$ or when

(3.51) $$W_t = A\left(\frac{M}{M+1}\right)^M.$$

Thus, like the Richards function, flexibility is built into the inflection point.

Also like the Richards function, the third derivative of equation (3.46) is really ugly. Equation (3.44) incorporating the Nelder generalization becomes

$$(3.52) \qquad \frac{d^3W_t}{dt^3} = AK^3Me^{-Kt}\left(1+e^{-Kt}\right)^{-M-3}\left(M^2e^{-2Kt} - 3Me^{-Kt} - e^{-Kt} + 1\right).$$

So, given estimates for the parameters K and M, equation (3.52) indicates that the concavity of the marginal product curve will be determined by the sign associated with the factor

$$(3.53) \qquad \left(M^2e^{-2Kt} - 3Me^{-Kt} - e^{-Kt} + 1\right).$$

If expression (3.53) is positive, then the marginal product curve (3.49) is convex. If expression (3.53) is negative, then the marginal product curve (3.49) is concave. Again, like the Richards function, it is easy to see how Nelder's generalization of the logistic has built flexibility into the model.

Oliver Generalization

In 1969, Oliver offered another generalization of the logistic function. The Oliver generalization of the logistic growth function

$$(3.54) \qquad W_t = \gamma + A\left(1+e^{-Kt}\right)^{-1},$$

adds a shifting parameter γ. This shifting parameter gives the Oliver generalization function an asymptotic mean mature weight of $\gamma + A$, instead of just A. This characteristic is unique to this function.

The initial conditions

$$(3.55) \qquad W_0 = \gamma + \left(1+e^{0}\right)^{-1} = \gamma + \frac{A}{2},$$

imply that

$$(3.56) \qquad \gamma = W_0 - \frac{A}{2}.$$

So, the estimated parameter γ depends upon the difference between the initial weight W_0 and the restricted estimate of the initial weight $\left(\frac{A}{2}\right)$ in the regular logistic model.

The derivatives of the Oliver generalization are the same as they were for the regular logistic function. So, the results derived in equations (3.42), (3.43), and (3.44) still hold. The Oliver generalization is everywhere concave. Its point of inflection occurs at $t = 0$ which is now defined by equation (3.55). And, finally, the marginal product curve is both convex and concave with points of inflection defined by the result (3.45).

Model Selection

In the animal growth applications that follow later in this dissertation, I use data sets for two different species. One is a data set for swine from fourteen days of age up to market weight. Another is a data set for cattle on feed from around 600 pounds up to market weight. In this section, I will detail the process I used to select a growth model to use for each data set. These models were chosen from among the eight growth models summarized in table 3.1.

The initial step in the process was to obtain fitted parameter estimates for each of the eight models. To fit the parameters, I minimized the sum of the squared deviations from the actually observed weights to the fitted weights predicted by the model. For example, if there were n weight observations for s animals over time, then the data set can be summarized as an s x n matrix $\{x_{ij}\}_{s \times n}$ where each x_{ij} is the weight of animal i at observation j. If the model predicts a fitted weight observation of y_j, then the objective of minimizing the sum of the squared deviations can be stated as

$$Min \sum_j \sum_i \left(x_{ij} - y_j\right)^2 . \tag{3.57}$$

A second set of fitted parameters was also calculated for each model to fit the curve of the average. For each weight observation *j*, an average weight for the *n* animals can be calculated as

$$\bar{x}_j = \frac{1}{n} \sum_i x_{ij} . \tag{3.58}$$

Then, the process of fitting the model to the curve of the average involves minimizing the following sum of the squared deviations.

$$Min \sum_j \left(\bar{x}_j - y_j\right)^2 \tag{3.59}$$

The model fitted to the curve of the average using criteria (3.59) was compared to the model fitted to the observed data using criteria (3.57). In most cases, I found the models consistent with one another in terms of the fitted parameters.

The fitting process was done using two quantitative tools. The first tool was a least-squares model I developed in Microsoft Excel that incorporated the use of Premium Solver, an Excel add-in marketed by Frontline Systems. The second tool was EViews version 3.1 by Quantitative Micro Software. I used Excel to come up with an initial estimate of the parameters. I then used EViews and iterative least squares to verify these estimates. In a couple of cases, EViews improved on the estimates slightly in terms of reducing the sum of the squared deviations. When this occurred, I returned to Excel to verify the results. Because these are nonlinear functions, initial parameter values were important in terms of convergence. Being able to iterate between Excel and EViews allowed me to shorten this process and feel comfortable with my results.

After the models were fitted, I then compared their goodness of fit as measured by the sum of the squared deviations, their computational ease, and their overall biological interpretation. Using these criteria, I chose the best model for each data set. For swine, I chose the Gompertz function incorporating Parks' modification. For cattle, I chose the logistic function. The details of each selection follows.

Swine Data

The swine data set consisted of twelve weight observations of individually identified hogs every one to three weeks from 14 days of age up to 171 days of age. Table 3.2 contains the sample of data used for the model selection process. The data was obtained from a Purdue University study on antibiotic treatments courtesy of Dr. Allan Schinckel via personal communication.

The plot of the average weights is shown in figure 3.2. From this plot, it is easy to see that a production function exhibiting both increasing and decreasing marginal productivity seems appropriate. The three models that exhibited only decreasing

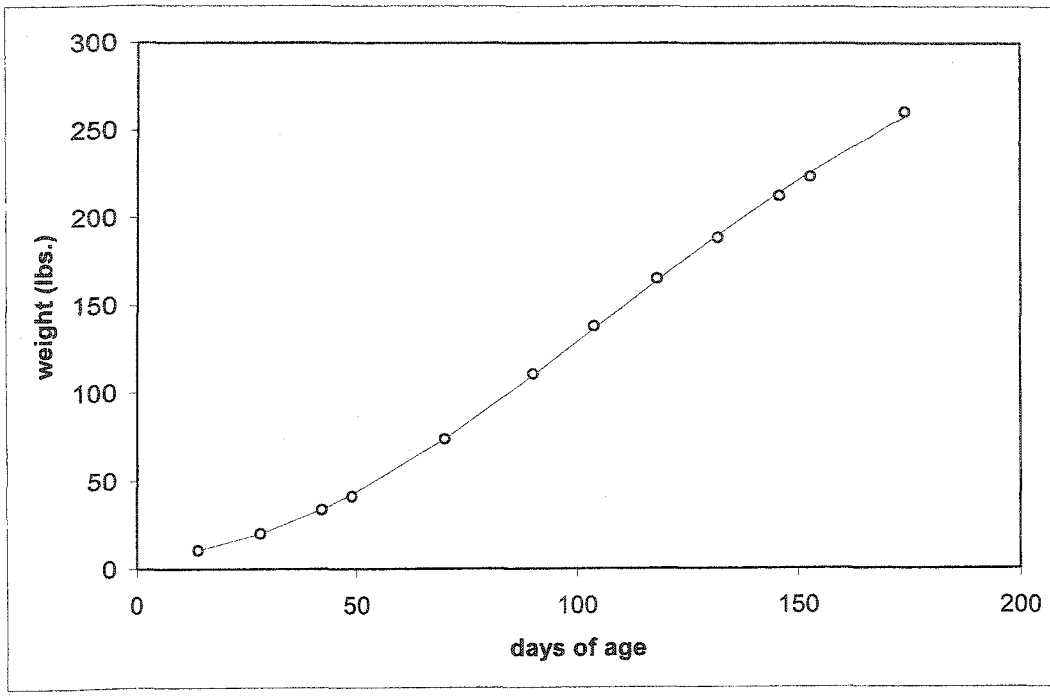
Table 3.2 Observed liveweights (in lbs.) for pigs from 14 days of age to 174 days of age

Pig Weights at *t* days of age

	Days of age											
Tag #	14	28	42	49	70	90	104	118	132	146	153	174
259	11.3	20.9	33.8	38	75	118	151	187	198	207	224	281
266	11	23.2	34.6	40	70	107	134	167	196	224	238	274
275	11.7	19.3	33.6	41	70	105	129	156	186	218	230	267
299	11.2	19.9	35.3	43	75	115	141	160	183	216	227	265
308	11.2	22	35.6	44	75	110	135	159	180	211	222	253
380	10.2	16.7	30.2	38.5	73.5	105	134	161	190	207.5	221.5	256
425	11.1	18.5	30.1	36.5	74.5	112	142	167.5	187.5	211.5	224.5	254.5
450	10.8	19.9	32.8	40	67	95	121.5	149.5	175.5	197.5	198.5	238.5
451	8.8	20.6	36.3	45.5	81	124	154	176	197.5	222	226.5	263.5
461	9.4	18.6	34.1	44	78	116	140	168.5	191.5	210	222	249
Average	10.7	20.0	33.6	41.1	73.9	110.7	138.2	165.2	188.5	212.5	223.4	260.2

Data courtesy of Allan Schinckel, Purdue University

Figure 3.2 Fitted growth path for the average weight in the swine sample data.



o - Plotted average weights through time.
— - Fitted Gompertz curve with Parks' modification

marginal productivity were the Brody, the logistic, and the Oliver generalization of the logistic. When I fit each of these models to the data, the result was very poor compared to the other five models in terms of the value of the sum of the squared deviations. Also, the estimated parameter values were biologically questionable, such as an asymptotic mature weight as low as 172 pounds and as high as 2000 pounds. Therefore, these three functions were eliminated from further consideration as models for the swine data. The estimated parameters and the sum of the squared residuals for each of the other five models under consideration are summarized in table 3.3. The sums of the squared residuals are reported for the entire data set using criteria (3.57) and for the curve of the average using criteria (3.59). As expected, the four parameter Richards model provides the best fit. However, this model also proved to be the trickiest to estimate requiring

Table 3.3 Model parameter estimates and goodness of fit for swine.

Model	Parameter estimates A	B	K	M	Sum of Squared Residuals for the panel of data (Equation 3.57)	Sum of Squared Residuals for the average animal (Equation 3.59)
von Bertalanffy	486	0.7478	0.0086	...	6478	31.00
Gompertz	378	3.5617	0.01392	...	6434.4	26.69
Gompertz (Parks modification)	377	...	0.01395	...	6434.7	26.72
Logistic (Nelder generalization)	345	...	0.0172	4.8181	6654	48.62
Richards	413	0.4285	0.0116	6.7431	6383	21.55

several iterations between Excel and EViews. Estimates for the parameters B and M had high standard errors on the Richards model as reported by EViews and are of questionable statistical reliability.

I chose the Gompertz function with Parks' modification as the best model for the swine data. Whittemore states that the use of a Gompertz curve to describe swine growth has proved useful. After the Richards function, the Gompertz functions provide the best fit for our sample data. Parks' modification, which addresses a possible biasedness in the estimates of A and K, eliminates the need for the parameter B. It does so with virtually no loss in terms of goodness of fit. It also lowers the standard errors for both A and K improving an already high statistical reliability. The estimated Gompertz function with Parks' modification is graphed in figure 3.2 with the plot of the average weights.

Cattle Data

The cattle data set consisted of seven weight observations of individually identified cattle on feed in southeastern Colorado. The observations all took place four weeks apart from one another with the exception of the final two observations which are only two weeks apart. Table 3.4 provides the sample of data used for this model selection process. Dr. Clinton Parsons of Texas Christian University provided the data via personal communication.

Figure 3.3 contains a plot of the average cattle weights at each of the seven observations. The estimated parameters and the sum of the squared residuals for each model under consideration are summarized in table 3.5. The plot in figure 3.3 suggests that, unlike the swine data, it is not clear if the production function needs to exhibit both increasing and decreasing marginal returns. A look at the sum of the square residuals in table 3.5 shows that the logistic functions appear to provide the best fit. Furthermore, the Nelder and Oliver generalizations of the logistic function do not appear to add much in terms of providing a better fit than the basic logistic. For completeness, I conducted a Wald test on the parameter estimate for *M* in the Nelder logistic function. This test revealed that the estimated value of 1.0262 was not significantly different from the assumed value of 1.0 in the logistic function (p-value on the Wald test was 0.83 implying

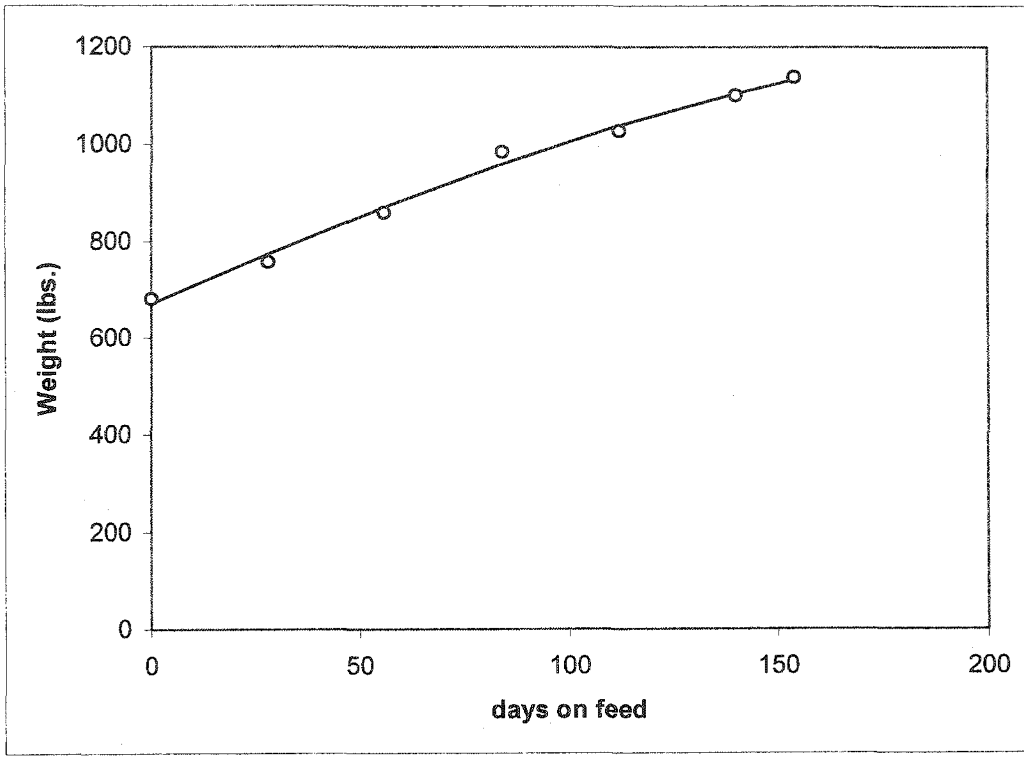
Table 3.4 Observed live body weights for a pen of five Brangus X Angus steers on feed for 154 days in southeastern Colorado.

Cattle Weights in lbs.

		Days on feed						
		0	28	56	84	112	140	154
ID #	470	714	778	882	991	1014	1088	1118
	484	708	762	866	1010	1059	1122	1162
	508	631	732	806	913	937	984	1018
	510	668	758	860	985	1049	1157	1206
	562	682	757	880	1020	1075	1148	1185
	Average	680.6	757.4	858.8	983.8	1026.8	1099.8	1137.8

Data courtesy of Clinton Parsons, Texas Christian University.

Figure 3.3 Fitted growth path for the average weight steer.



o - Plotted average cattle weights through time

— - Fitted logistic curve

Table 3.5 Model parameter estimates and goodness of fit for cattle.

Model	Parameter estimates					Sum of Squared Residuals for the panel of data (Equation 3.57)	Sum of Squared Residuals for the average animal (Equation 3.59)
	A	*B*	*K*	*M*	γ		
von Bertalanffy	1564	0.2458	0.00578	...	...	76988	1318
Brody	1811	0.6298	0.00342	...	...	77337	1388
Gompertz	1494	0.8004	0.00697	...	...	76830	1287
Gompertz (Parks modification)	1570	...	0.00618	...	...	77435	1408
Logistic	1341	...	0.01098	...	...	76524	1226
Logistic (Nelder generalization)	1367	...	0.01047	1.0262	...	76440	1209
Logistic (Oliver generalization)	1390	...	0.01045	...	-24	76465	1214
Richards	1531	0.1190	0.00635	6.5082	...	76903	1301

not to reject the null hypothesis of $M = 1$). I also conducted a t-test on the significance of the parameter estimate for γ in the Oliver generalization. That test revealed that the coefficient estimate of –24 was not significantly different from the assumed value of zero in the logistic function (p-value on the t-test was 0.86 implying not to reject the null hypothesis of $\gamma = 0$). In summary, I chose the logistic function as the best model for the cattle data. The estimated logistic function is graphed in figure 3.3 with the plots of the average weights.

Summary and Conclusions

In this chapter, I provided a thorough derivation of the characteristics of each of the eight growth functions that I considered for modeling the animal growth data used in this dissertation. I then provided a summary of how well these functions did in modeling in the data as well as the thought processes I used in selecting one function to use for each data set. For the swine data, which ran from 14 days of age up through market weight, I chose the Parks modified Gompertz function as the best model. For the cattle data, I chose the logistic function as the best model. The cattle data, which ran from about 600 pounds up to market weight, really only captured about the last half of the growth cycle. This distinguished it from the swine data and allowed me to use a function that only exhibited decreasing marginal growth.

In the next chapter, I will provide a general theoretical discussion of Jensen's inequality applied to marginal analysis and group optimization. In the examples that follow, I will incorporate the use of the Parks modified Gompertz and the logistic functions selected in the above analysis.

Chapter 4

Production Uncertainty and the Theory of Group Production

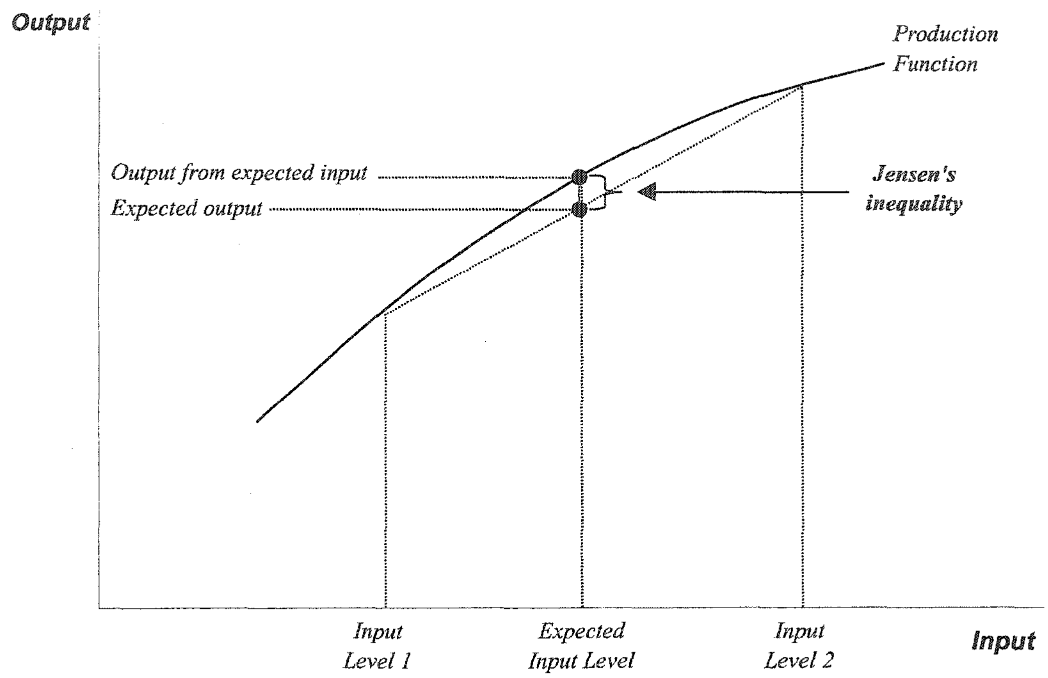
Anytime uncertainty enters the theoretical construct of production economics, the decision-maker's problem becomes more complex. In the classical theory of the firm, the decision-maker purchases inputs, places the inputs into the production process, and converts them into output. Under the assumption of certainty, the decision-maker maximizes profits for the firm by purchasing and using more of each input until the value of the marginal productivity from each of the inputs no longer exceeds their respective marginal cost.

It is empirically common for production functions and marginal product functions to be nonlinear with respect to inputs. It is also common for the production environment to be characterized by some degree of uncertainty. Anytime a function is nonlinear with respect to an uncertain variable, Jensen's inequality can be applied. Jensen's inequality states that for any function *f(x)* that is convex (concave) with respect to a random variable x, $E(f(x)) \geq (\leq) f(E(x))$.

As described in chapter 2, Jensen's inequality has played an important role in a number of different areas in the literature. In mathematics, it has been used to prove Cauchy's inequality (Jensen) and the arithmetic mean-geometric mean (AM-GM) inequality (Hansen). The AM-GM inequality, in particular, has proved useful in discussions on risk and logarithmic utility dating back to Bernoulli (1738). It can be shown that the existence of a nonzero risk premium for risk averse decision-makers is a direct application of Jensen's inequality. Jensen's inequality has also been used to show that, for a firm faced with uncertainty in future output prices, a wider distribution of possible prices leads to an increased incentive to make new capital investments (Hartman; Abel; Small). In the field of actuarial science, Jensen's inequality has been used to justify the use of mortality rates rather than life expectancies when calculating life insurance premiums or annuity payments (Behan). In ecology, it has been used to explain risk-sensitive foraging behavior (Smallwood) and biased estimation of plant response functions (Ruel and Ayres). However, in production economics, the use of Jensen's inequality has been limited. It has been largely confined to use as an explanation why, if input is uncertain and the production response function is concave, the expected output is less than the output from the expected input (Horowitz; Robison and Barry). Furthermore, these applications have been accompanied by statements like, "Nevertheless, the expected value of the output depends critically on the (expected) value of (input) L." (Robison and Barry, p. 119) Figure 4.1 provides graphical representation of this situation.

For the theory of the firm, uncertainty on any level can have a profound impact on decision-making. The risk literature has documented the theoretical arguments that management strategies based on average but variable conditions may not achieve a desired objective (Pope and Shumway). However, the risk literature has usually cast their arguments based on the interaction between the uncertainty and the utility maximizing producer with known risk preferences. In doing so, the focus has been on

Figure 4.1 Jensen's inequality, input uncertainty, and a concave production function



the variance of possible profit outcomes and utility maximization. In this setting, the utility maximizing decision-maker is about to experience a single outcome drawn from a distribution of possible outcomes. What if the decision-maker was about to experience a distribution of outcomes? This is the case for a producer engaged in group production. For example, a livestock producer usually feeds a group of animals and then takes the entire group to market. Jensen's inequality can be applied to the theory of the firm on the total input/output level and on the marginal level. The marginal level is where profit-maximizing decisions are made for the firm. In a group optimization problem, this is precisely where Jensen's inequality needs to be applied.

In the discussion that follows, I will outline the different models that can be used to explain an observed variation in production. I will then present two simple models for production function uncertainty and quality of input uncertainty that capture the important impacts of variable productivity on the group optimization conditions. These models will be used to show how producers need to pay attention to the effect of Jensen's inequality in order to calculate the correct average marginal productivity for a group.

Uncertainty to the Firm

Uncertainty to the firm can be introduced in a number of different ways including output price uncertainty, input price uncertainty, production function uncertainty, and quality of input uncertainty (Robison and Barry). Of these, price uncertainty has received the most extensive treatment in the literature (Batra and Ullah; Sandmo). However, there are many sectors of the economy where production uncertainty may have a greater impact than market uncertainty (Pope and Kramer). Furthermore, price is simply a multiplicative factor when it comes to computing values and costs. Therefore, price uncertainty does not warrant special treatment in a setting where multiplicative production uncertainty is accounted for in the model. On the other hand, there is a difference between whether the uncertainty is introduced to the firm through the production function or through the quality of the inputs (Walters). Therefore, those two areas will receive separate treatments in the work that follows.

Robison and Barry offer two forms of production function risk models that, respectively, account for additive random effects and multiplicative mixed effects. In an additive model, q is output and $\mathbf{X}$ is a vector of inputs with $q = f(\mathbf{X}) + \varepsilon$ where $\varepsilon \sim \left(0, \sigma_\varepsilon{}^2\right)$ and $f_x > 0$, $f_{xx} < 0$ for all $x \in \mathbf{X}$. Then, for any individual within a group, the marginal productivity f_x is the same. The variation observed in the residuals ε is independent of the individual observations. Craig and Schinckel (2001) argue that this type of a model, also called a fixed effects model, is not appropriate in some empirical settings. For example, empirical evidence shows that animals that are heavier at birth and weaning have a competitive advantage and usually remain heavier throughout their stay within a production group (Le Dividich). In a multiplicative model, q_i is output from individual *i* and **X** is a vector of inputs with $q_i = \delta_i \cdot f(\mathbf{X}) + \varepsilon$ where $\varepsilon \sim \left(0, \sigma_\varepsilon^2\right)$, $\delta_i \sim \left(1, \sigma_\delta^2\right)$, and $f_x > 0$, $f_{xx} < 0$ for all $x \in \mathbf{X}$. Then, each individual has its own marginal productivity dependent upon the value of δ_i.

Robison and Barry also offer two quality of input risk models. These models are sometimes described as involving uncertainty in factor services, terminology introduced by Walters (1960). The idea is that although the firm may hire a certain quantity of input to be used in production, the quality of that input may be stochastic. For example, if *L* units of labor are hired to provide service to production, the actual effectiveness of that labor may be $L + \varepsilon$ where $\varepsilon \sim \left(0, \sigma_\varepsilon^2\right)$ and ε depends upon previous training, dependability, etc. Then, for an additive quality of input model, *q* is output and **X** is a vector of known inputs with $q = f(L + \varepsilon | \mathbf{X})$ where $\varepsilon \sim \left(0, \sigma_\varepsilon^2\right)$ and $f_L > 0$, $f_{LL} < 0$. For a multiplicative model, *q* is output and **X** is a vector of known inputs with $q = f\left(\delta L | \mathbf{X}\right)$ where $\delta \sim \left(1, \sigma_\delta^2\right)$ and $f_L > 0$, $f_{LL} < 0$. In either case, Jensen's inequality can be applied directly to any nonlinear production function *f* and any nonlinear marginal product function f_L.

A Model for Production Function Uncertainty

Let us suppose a firm is producing output Q from input factors $F_1, F_2, \ldots, F_n$ according to the production function $Q = q(F_1, F_2, \ldots, F_n)$. Introduce into the firm, the random variable u described by the density function $\gamma(u)$. Production function uncertainty for the firm is then described by a new production function

(4.1) $$Q = q(u, F_1, F_2, \ldots, F_n),$$

with inputs $u, F_1, F_2, \ldots, F_n$.

By Jensen's inequality, the following lemma is true in describing an inequality between the expected output and the output from the expected input.

Lemma 4.1. Let $Q = q(u, F_1, F_2, \ldots, F_n)$ represent a production function with known input factors $F_1, F_2, \ldots, F_n$ and random variable u. Then,

(a) $E(Q) \le q(E(u), F_1, F_2, \ldots, F_n)$ if Q is concave with respect to u; and

(b) $E(Q) \ge q(E(u), F_1, F_2, \ldots, F_n)$ if Q is convex with respect to u.

Let

(4.2) $$E(MP_i) = \int_u \frac{\partial Q(u)}{\partial F_i} \gamma(u) du$$

designate the expected marginal product of input F_i. Note, that equation (4.2) is an integration over the entire distribution of the random variable u of the marginal productivity with respect to input F_i. By Jensen's inequality, the follow lemma is true in describing an inequality between the expected marginal product (EMP) of input F_i and the certain marginal product of F_i based on the expected input $E(u)$.

Lemma 4.2. Let the marginal product function for input F_i be represented by

$MP_i = \frac{\partial Q(u)}{\partial F_i}$, where u is a random variable. Then,

a) $E(MP_i) \leq \frac{\partial Q(E(u))}{\partial F_i}$ if MP_i is concave with respect to u; and

b) $E(MP_i) \geq \frac{\partial Q(E(u))}{\partial F_i}$ if MP_i, is convex with respect to u.

From lemma 4.2, it is clear that if the firm ignores the distribution $\gamma(u)$, it may be costly in terms of optimizing profits. The presence of uncertainty in the production function, through the random variable u, can change the EMP for some or all of the inputs. The result is the possibility of an input's EMP not being a point on the input's marginal product curve associated with the input quantities $F_1, F_2, \ldots, F_n$.

A Model for Quality of Input Uncertainty

Again, let the random variable u be described by the density function $\gamma(u)$. Quality of input uncertainty to the firm can be described by the production function

(4.3) $$Q = q(F_1, \ldots, F_i(u), \ldots, F_n).$$

Lemma 4.1 still holds but with the following modifications.

Lemma 4.3. Let $Q = q(F_1, \ldots, F_i(u), \ldots, F_n)$ represent a production function with input factors $F_1, F_2, \ldots, F_n$ and random variable u. Then,

a) $E(Q) \leq q(F_1, \ldots, E[F_i(u)], \ldots, F_n)$ if Q is concave with respect to u; and

b) $E(Q) \geq q(F_1, \ldots, E[F_i(u)], \ldots, F_n)$ if Q is convex with respect to u.

Lemma 4.2 still holds as stated. However, a new and more specific lemma can now be stated for the particular input F_i that introduces the input uncertainty to the firm.

Lemma 4.4. Let the marginal product function for input F_i be represented by $MP_i = \frac{\partial Q(u)}{\partial F_i}$, where u is a random variable. Then,

a) $E(MP_i) \leq \frac{\partial Q(E(u))}{\partial F_i}$ if MP_i is concave with respect to $F_i(u)$; and

b) $E(MP_i) \geq \frac{\partial Q(E(u))}{\partial F_i}$ if MP_i, is convex with respect to $F_i(u)$.

From lemma 4.2, in general, and lemma 4.4, in particular, it is clear that if the firm ignores the uncertainty in the quality of input F_i, then it may be costly. The true EMP is not necessarily equal to the marginal product associated with the expected value of the input. Therefore, using the expected value of the input and the marginal product curve to optimize profits may lead to a suboptimal result.

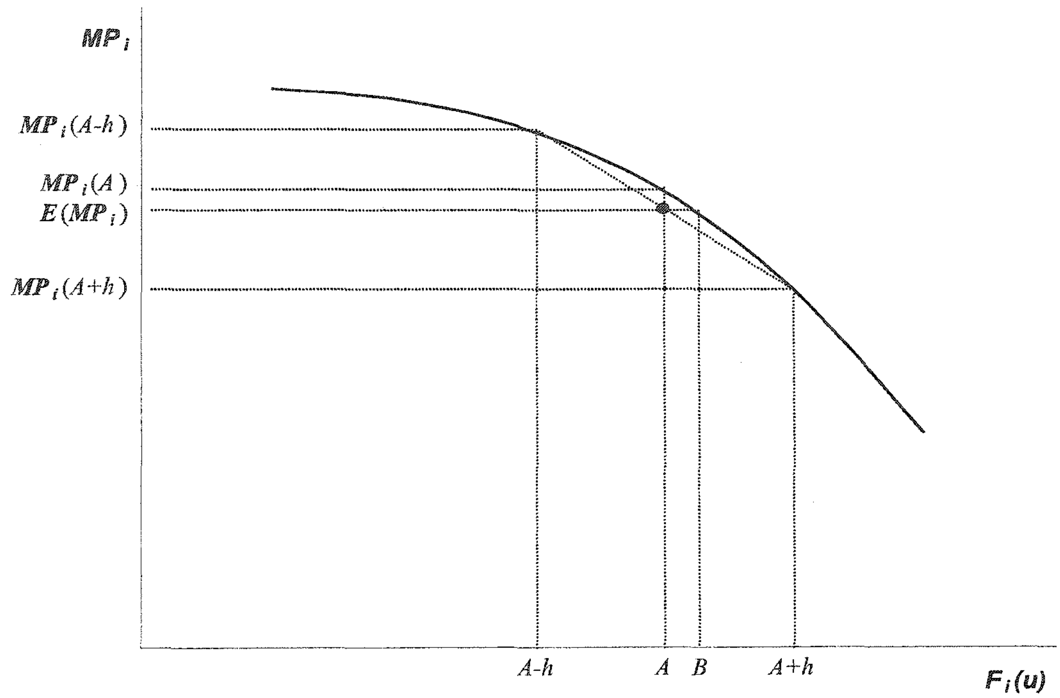
Group Optimization and the Jensen Effect

It is best to demonstrate these concepts with an illustration. Consider figure 4.2 and the quality of input model (4.3). Let the random variable u be such that it has an equal probability of being equal to $-h$ or $+h$. Then, let

$$F_i(u) = A + u. \tag{4.4}$$

According to the function (4.4), the uncertain input F_i has an equal probability of having the equivalent quality of $A+h$ or of $A-h$. Then, the expected quality for input F_i is equivalent to the input quantity A. Now consider the marginal product associated with

Figure 4.2 Marginal productivity with quality of input uncertainty.



each of level of input for F_i. According to the distribution associated with the random variable u, there is an equal probability of having a marginal product equal to $MP_i(A+h)$ or $MP_i(A-h)$. The expected marginal product is $E(MP_i)$. Since the depicted marginal product curve in figure 4.2 is concave, lemma 4.4(a) applies and $E(MP_i) \leq MP_i(A)$. That is, the expected marginal product for the random input F_i as established by u is less than the marginal product of the expected input. The following set of three definitions will help clarify the inequalities that are illustrated in figure 4.2.

Definition. Let $F(x)$ be either a concave or a convex function with respect to a random variable x. Define the variable J as the *Jensen effect* where $J = E(F(x)) - F(E(x))$.

Definition. Let $F(x)$ be a function with respect to a random variable x. Define the *aggregate equivalent* as the value of x equal to B such that $F(B) = E(F(x))$.

Definition. Let $F(x)$ be a function with respect to a random variable x. Define the *aggregation premium* as being equal to $B - E(x)$ where B is the aggregate equivalent.

In figure 4.2, the marginal product curve for input F_i is concave and monotonically decreasing with respect to $F_i(u)$. Therefore, the Jensen effect, equal to $E(MP_i) - MP_i(A)$, is less than or equal to zero. It follows that the aggregation premium, equal to $B - A$, is positive. A production group governed by this relationship will have an average marginal product that is less than the marginal product associated with the average individual in the group. Hence, the aggregation premium is positive. Therefore, the group behaves *as if* the individual representing the group's average marginal productivity is at the aggregate equivalent level of input (B) lying somewhere to the right of the average level of input (A). The sign on the Jensen effect and the aggregation premium can be generalized for all increasing and decreasing functions that can be classified as either concave or convex. These generalizations are made in a summary fashion in table 4.1.

A decision-maker for a firm engaged in group production needs to account for the Jensen effect on marginal productivity and marginal costs in order to optimize profits. Using an average individual may misrepresent the marginal characteristics for the aggregate group. Instead, an aggregate equivalent individual equal to the average individual plus an aggregation premium is a better representative of the marginal characteristics that should be used in the optimization process. The following two

Table 4.1 Let $F(x)$ be either a concave or a convex function and monotonically increasing or decreasing with respect to a random variable x. The following table summarizes the signed values of the Jensen effect (J) and the aggregation premium (AP) for production groups.

Type of Function	Increasing	Decreasing
Concave	$J \leq 0$ $AP \leq 0$	$J \leq 0$ $AP \geq 0$
Convex	$J \geq 0$ $AP \geq 0$	$J \geq 0$ $AP \leq 0$

empirical examples are provided to help illustrate the marginal productivity relationships represented in the present discussion.

An Example for Quality of Input Uncertainty

Consider a situation where a feedlot owner accepts deliver of a group of five cattle and places them on feed for t days. Over time the animals grow according to a logistic growth function

(4.5) $$W_t = 1341\left(1 + e^{-0.01098t}\right)^{-1}$$

where W_t is their weight in pounds on day t.

Now, the animals do not weigh the same identical amount when they are purchased nor do they weigh the same throughout the feeding process. Suppose on day 154, the feedlot owner weighs each of the five animals and obtains the weights shown in table 4.2. There are a number of potential explanations that could be given for the differences in weight recorded in table 4.2. One plausible explanation might be that the animals are at slightly different points of biological maturity. Suppose the feedlot owner

Table 4.2 Individual weight observations for five steers on day 154 along with calculated input t_i and calculated marginal productivity MP_i based on each animal's individual weight.

	Animal ID#						
	470	484	508	510	562	$\overline{W}$	$\overline{t}$
W_i (lbs.)	1118	1162	1018	1206	1185	1137.8	1145.8
t_i	146.8	170.4	104.5	199.4	184.7	156.9	161.2
MP_i	2.04	1.70	2.69	1.33	1.51	1.89	1.83

$t_i = -\frac{1}{0.01098}\ln\left[1341W_i^{-1} - 1\right]$ where W_i is the weight each animal i on day 154. The inverse of equation (4.5).

$\overline{W}$ is based on the average output weight of 1137.8 lbs. for the five animals.

$\overline{t}$ is based on the average input, $\overline{t} = \frac{1}{n}\sum_i t_i = 161.2$ days

does not know exactly how biologically mature the animals are when they are purchased. Throughout the feeding process, the animals travel along the growth path defined by equation (4.5). However, each animal's individual level of biological maturity is associated with a t_i unique to that animal. That is, input t in equation (4.5) is actually input $t_i = t + u$ for each animal i where u is a random variable with a zero mean and constant variance. The second row of table 4.2 contains the calculated t_i's associated with each animal according to the weights observed on day 154 and the inverse of equation (4.5).

The marginal product function is calculated by taking the first derivative of equation (4.5).

$$MP_t = 14.72418e^{-0.01098t}\left(1 + e^{-0.01098t}\right)^{-2} \tag{4.6}$$

It is easy to show that equation (4.6) is positive for all values of t. Therefore, the production function (4.5) is monotonically increasing and the marginal product curve is always positive.

To analyze the concavity of the production function (4.5), it is necessary to take its the second derivative. This is calculated as

$$\text{(4.7)} \qquad \frac{d^2W}{dt^2} = -0.1617e^{-0.01098t}\left(1 + e^{-0.01098t}\right)^{-3}\left(1 - e^{-0.01098t}\right).$$

From the last factor in equation (4.7), it is easy to see that equation (4.7) equals zero when $t = 0$ and that equation (4.7) is negative for all $t > 0$. Therefore, the production function (4.5) is concave and the marginal product function (4.6) is decreasing for all $t > 0$. According to table 4.1, a production function that is concave and monotonically increasing will have a negative Jensen effect and a negative aggregation premium. To see this, define $\bar{t}$ as the arithmetic mean of the five individual t_i's on day 154 displayed in table 4.2 and calculate $\bar{t} \cong 161.2$. According to production function (4.5), $W_{\bar{t}} \cong 1145.8$. So, the Jensen effect reduced the weight associated with the average quality of input $\bar{t}$ from 1145.8 down to the average weight of 1137.8 observed for the group. Furthermore, the aggregation premium of –4.3 reduced the average input $\bar{t}$ from 161.2 down to the aggregate equivalent of 156.9.

Thus far, everything has been confined to an analysis of the observed weights. To move forward with the analysis of the group from a marginal productivity standpoint, a third derivative is required. The third derivative of equation (4.5) is calculated as

$$\text{(4.8)} \qquad \frac{d^3W}{dt^3} = 0.001775e^{-0.01098t}\left(1 + e^{-0.01098t}\right)\left(e^{-0.02196t} - 4e^{-0.01098t} + 1\right).$$

Equation (4.8) is positive or negative depending upon the sign on the last factor $\left(e^{-0.02196t} - 4e^{-0.01098t} + 1\right)$. For $t = 0$, equation (4.8) is negative. Also, equation (4.8) is equal to zero when $t \cong 119.9$ and is greater than zero for all $t > 119.9$. It follows that the marginal product curve (4.6) is a decreasing, convex function with respect to t for all $t > 119.9$. So, according to table 4.1, a distribution in this portion of the curve will result in a positive Jensen effect and a negative aggregation premium.

In the third line of table 4.2, the marginal product is calculated according to equation (4.6) for each of the individual animals. Note that, with the exception of animal 508, all of the individual animal t_i's are much greater than 119.9, putting them well within the convex portion of the marginal product curve. Define $\overline{MP}$ as the arithmetic mean of the five individual MP_i's and calculate $\overline{MP} \cong 1.85$. So, the Jensen effect increased the average marginal product for the group by the quantity 0.02 from the 1.83 associated with the average quality of input $\bar{t}$. Furthermore, an aggregation premium of -1.3 reduced the average input $\bar{t}$ down to an aggregate equivalent of $t = 159.9$ that, by equation (4.6), is associated with $\overline{MP} \cong 1.85$. The natural question is why does the aggregate equivalent of 159.9 obtained in marginal product space not agree with the aggregate equivalent of 156.9 obtained in production function space? The answer to this question is essential in understanding why using averages may lead to misleading results.

First, in production function space, the aggregate equivalent of 156.9 was viewed in terms of a relationship to average weight. To derive corresponding results in marginal product space, the marginal product function (4.6) must be rewritten as a function of weight instead of time. To do this, consider that, from equation (4.5), the following two results can be derived.

$$e^{-0.01098t} = \frac{1341}{W_t} - 1 \text{ and } \left(1 + e^{-0.01098t}\right)^{-2} = \left(\frac{W_t}{1341}\right)^2$$

Substituting each of these into equation (4.6) and simplifying yields an equation for the marginal product as a function of weight.

$$MP_w = 0.01098 W_t \left(1 - \frac{W_t}{1341}\right) \tag{4.9}$$

To discuss the concavity of equation (4.9), two derivatives need to be calculated with respect W_t to correspond to equations (4.7) and (4.8). The results are as follows.

$$\frac{dMP_w}{dW_t} = 0.01098\left(1 - \frac{2W_t}{1341}\right) \tag{4.10}$$

$$\frac{d^2 MP_w}{dW_t^2} = -\frac{2(0.01098)}{1341} \cong -0.0000164 . \tag{4.11}$$

Equation (4.10) is negative for all $W_t > 670.5$. Together with equation (4.11), this implies that the marginal product function (4.9) is a decreasing, concave function with respect to weight. According to table 4.1, the Jensen effect for this function should be negative and the aggregation premium should be positive. Indeed, the marginal product associated with the average weight is 1.89 as displayed in the last row of table 4.2 and calculated using equation (4.9). The negative Jensen effect of $J = -0.04$ lowers this value to the average marginal product of $\overline{MP} \cong 1.85$. Likewise, a positive aggregation premium of +5.6 pounds explains the difference between the average weight value of 1137.8 pounds associated with $t = 156.9$ and the aggregate equivalent weight of 1143.4 associated with $t = 159.9$.

Figures 4.3 and 4.4 summarize these results. In figure 4.3, the average input value $\bar{t}$ misrepresents the average marginal product for the group. The amount of the

Figure 4.3 The Jensen effect on marginal productivity as a function of input (time).

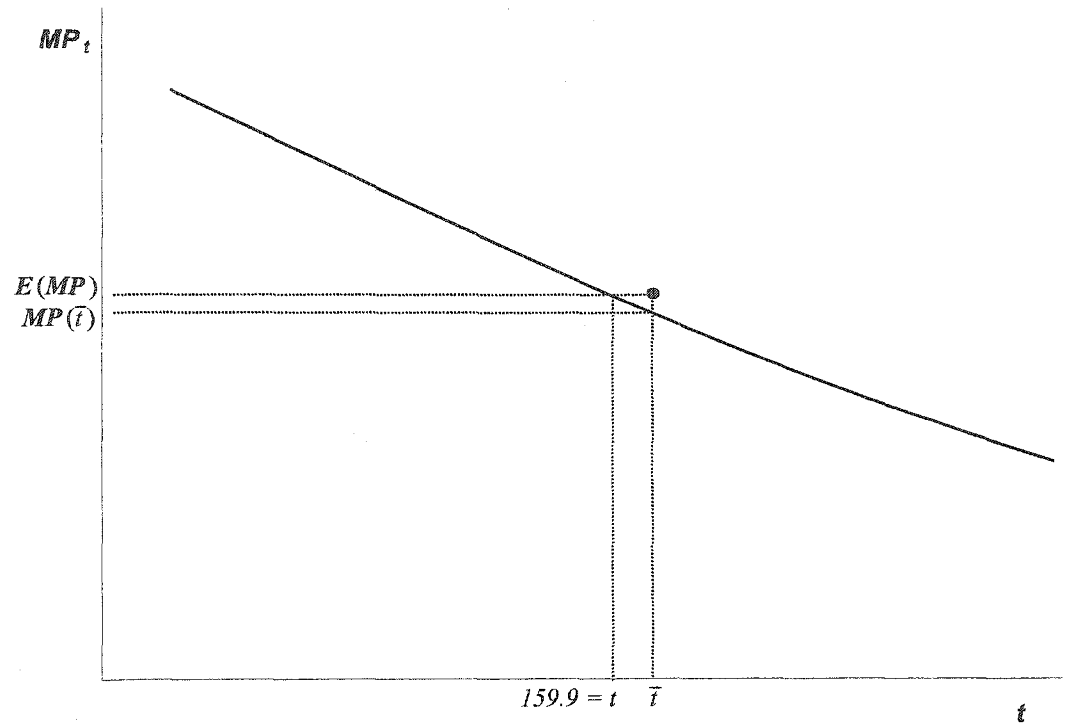
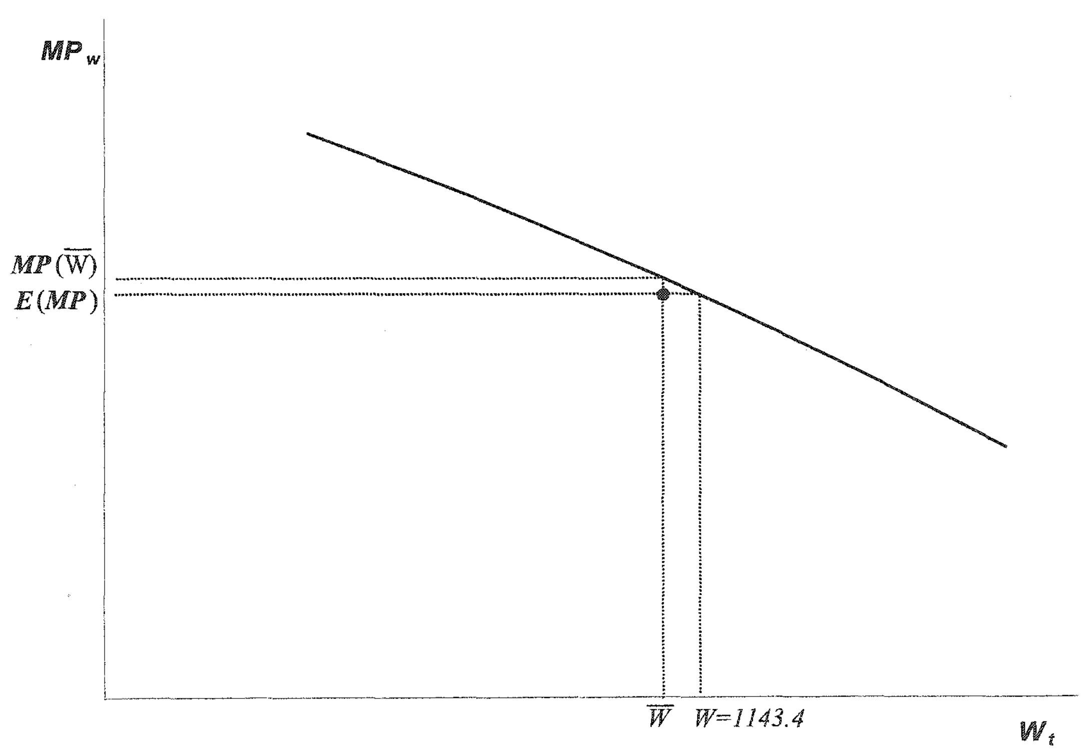


Figure 4.4 The Jensen effect on marginal productivity as a function of output (weight).



misrepresentation is called the Jensen effect and is calculated as a positive number $J = E(MP) - MP(\bar{t})$. A negative aggregation premium added to $\bar{t}$ leads to the aggregate equivalent level of $t = 159.9$. In figure 4.4, the average weight $\overline{W}$ misrepresents the average marginal product for the group. The amount of the misrepresentation is a negative Jensen effect and is calculated as $J = E(MP) - MP(\overline{W})$. A positive aggregation premium added to $\overline{W}$ leads to the aggregate equivalent level of $W = 1143.4$. In either case, using an average input level or an average output level leads to a misrepresentation of the average marginal productivity for the group.

By accounting for a Jensen effect, the average marginal productivity for a group can be derived from the marginal product associated with the average input level or the marginal product associated with the average output level. Furthermore, an aggregation premium can be added to the average output level to derive an aggregate equivalent level of output and an aggregation premium can be added to the average input level to derive an aggregate equivalent level of input. These aggregate equivalent levels of input and output ($t = 159.9$ and $W = 1143.4$, respectively, in the example) indicate the true individual that represents the average marginal productivity of the group. It is this aggregate equivalent individual, not the individual at $\bar{t} = 161.2$ nor the individual at $\overline{W} = 1137.8$, which should be used in any subsequent optimization process to represent a measure for average marginal productivity.

An Example for Production Function Uncertainty

Consider a situation where a group of ten hogs is raised together from 14 days up to 174 days of age. Every two to three weeks individual weight observations are made and the data is recorded as in table 4.3. A Gompertz curve has proved useful in modeling swine growth (Whittemore). Here, a modified Gompertz function (Parks) is employed as a production function for this group of animals.

$$W_t = A\left(\frac{W_o}{A}\right)^{e^{-kt}} \tag{4.12}$$

In production function (4.12), an animal's weight at time t is given as a function of the animals initial weight W_o, a kinetic growth parameter k, and an asymptotic maximum mature weight A. Consider the observation in table 4.3 at 174 days of age, which is equivalent to $t = 160$. There are a number of reasons that could be sighted for the distribution of weights observed for the ten animals. One plausible explanation is that animals are headed toward their own individual asymptotic maximum mature weight. That is, there is uncertainty in the production function (4.12) in the form of a variable parameter A. The production function can be written as

$$W_{it} = A_i\left(\frac{W_o}{A_i}\right)^{e^{-kt}} \tag{4.13}$$

where W_{it} is animal i's weight at time t given animal i's asymptotic maximum mature

Table 4.3 Observed liveweights (in lbs.) for pigs from 14 days of age to 174 days of age.

Pig Weights (in lbs.) at t days of age

	Days of age												
Tag #	14	28	42	49	70	90	104	118	132	146	153	174	A_i
259	11.3	20.9	33.8	38	75	118	151	187	198	207	224	281	398
266	11	23.2	34.6	40	70	107	134	167	196	224	238	274	395
275	11.7	19.3	33.6	41	70	105	129	156	186	218	230	267	379
299	11.2	19.9	35.3	43	75	115	141	160	183	216	227	265	381
308	11.2	22	35.6	44	75	110	135	159	180	211	222	253	368
380	10.2	16.7	30.2	38.5	73.5	105	134	161	190	207.5	221.5	256	370
425	11.1	18.5	30.1	36.5	74.5	112	142	167.5	187.5	211.5	224.5	254.5	376
450	10.8	19.9	32.8	40	67	95	121.5	149.5	175.5	197.5	198.5	238.5	336
451	8.8	20.6	36.3	45.5	81	124	154	176	197.5	222	226.5	263.5	396
461	9.4	18.6	34.1	44	78	116	140	168.5	191.5	210	222	249	374

Data courtesy of Allan Schinckel, Purdue University.

weight A_i. Although all of the parameters could theoretically be random, scientists have found that simply allowing the mature body weight in a growth model to vary across pigs has adequately modeled the empirically observed variations in weight as a function of time (Craig and Schinckel). An estimated A_i is listed for each of the individuals in the last column of table 4.3 under the assumption that $W_o = 10.7$ and $k = 0.01395$.

The marginal product function is calculated by taking the first derivative of equation (4.13) with respect to *t*.

$$MP_i = \frac{dW_{it}}{dt} = -A_i k e^{-kt} \left(\frac{W_o}{A_i}\right)^{e^{-kt}} \cdot \ln\left(\frac{W_o}{A_i}\right) \qquad (4.14)$$

Equation (4.14) can be used to calculate an individual's marginal product at any point in time. Table 4.4 contains the individual marginal products for $t = 160$. The production function uncertainty is contained in the parameter *A*, which translates into a distribution of marginal products as shown in the last row of table 4.4.

The Jensen effect on the marginal productivity for the group will be determined by the concavity of equation (4.14) with respect to the random parameter *A*. The

Table 4.4 Observed marginal products for ten swine at 174 days of age (t = 160 days).

	Animal ID#										
	259	266	275	299	308	380	425	450	451	461	$\bar{A}$
Weight (lbs.)	281	274	267	265	253	256	254.5	238.5	263.5	249	257.3
A_i	398	395	379	381	368	370	376	336	396	374	377.18
MP_i	1.46	1.45	1.38	1.39	1.33	1.34	1.37	1.20	1.45	1.36	1.3725

Assumptions: t = 160, k = 0.01395, and W_o = 10.7.

following two derivatives of equation (4.14) with respect to A are needed to complete the analysis.

$$\frac{dMP_i}{dA} = ke^{-kt}\left(\frac{W_o}{A_i}\right)^{e^{-kt}} \cdot \left[1 + \ln\left(\frac{W_o}{A_i}\right)\left(e^{-kt} - 1\right)\right] \tag{4.15}$$

$$\frac{d^2MP_i}{dA^2} = -ke^{-kt}\left(\frac{W_o}{A_i}\right)^{e^{-kt}} A^{-1} \cdot \left[\left(e^{-kt} - 1\right) + e^{-kt}\left[1 + \ln\left(\frac{W_o}{A_i}\right)\left(e^{-kt} - 1\right)\right]\right] \tag{4.16}$$

The sign on equation (4.15) will depend upon the sign on the factor $\left[1 + \ln\left(\frac{W_o}{A_i}\right)\left(e^{-kt} - 1\right)\right]$. The sign on this factor will be positive for all $t > 0$. The derivation of this result proceeds as follows. The value of e^{-kt} will range from a positive one to zero as t goes from zero to infinity. Therefore, the value of $\left(e^{-kt} - 1\right)$ will be negative for all $t > 0$. Since $W_o < A_i$, the value of $\ln\left(\frac{W_o}{A_i}\right)$ will also be negative. It follows that the sign on the factor $\left[1 + \ln\left(\frac{W_o}{A_i}\right)\left(e^{-kt} - 1\right)\right]$ will be positive for all $t > 0$. Therefore, by equation (4.15), the marginal product is an increasing function with respect to the parameter A.

The sign on equation (4.16) is more difficult to analyze but it also depends upon the sign on a factor. In this case, that factor is $\left[\left(e^{-kt} - 1\right) + e^{-kt}\left[1 + \ln\left(\frac{W_o}{A_i}\right)\left(e^{-kt} - 1\right)\right]\right]$. The sign on this factor depends jointly on A_i and t. Given the empirical setting, it will suffice to say that if $t \geq 132$, the sign on this factor is negative for all $A_i < 1784$. It follows that equation (4.16) is positive if $t \geq 132$ and $A_i < 1784$. Therefore, if $t \geq 132$, the marginal product curve is a convex function with respect to the parameter A for all $A < 1784$.

The conclusion is that the marginal product (4.14) is an increasing, convex function with respect to the parameter A for all $t \geq 132$. According to table 4.1, the Jensen effect and the aggregation premium should both be positive for this function with respect to the uncertain parameter A. Define $\overline{A}$ as the arithmetic mean of the A_i's displayed in table 4.3 and calculate $\overline{A} \cong 377.18$. Using this value in equation (4.14) results in a marginal product of 1.3725 as displayed in the lower right-hand corner of table 4.4. Define $\overline{MP}$ as the arithmetic mean of the ten individual MP_i's displayed in the bottom row of table 4.4 and calculate $\overline{MP} \cong 1.3727$. So, the Jensen effect increased the average marginal product for the group by the quantity 0.0002 from the 1.3725 associated with the average parameter $\overline{A}$ to the actual average of 1.3727. The aggregate equivalent level of A that would yield a marginal product of 1.3727 can be calculated from equation (4.14) as $A \cong 377.24$. An aggregation premium of +0.06 explains the difference between the average parameter $\overline{A}$ and the aggregate equivalent of $A \cong 377.24$.

The Effects on Group Optimization

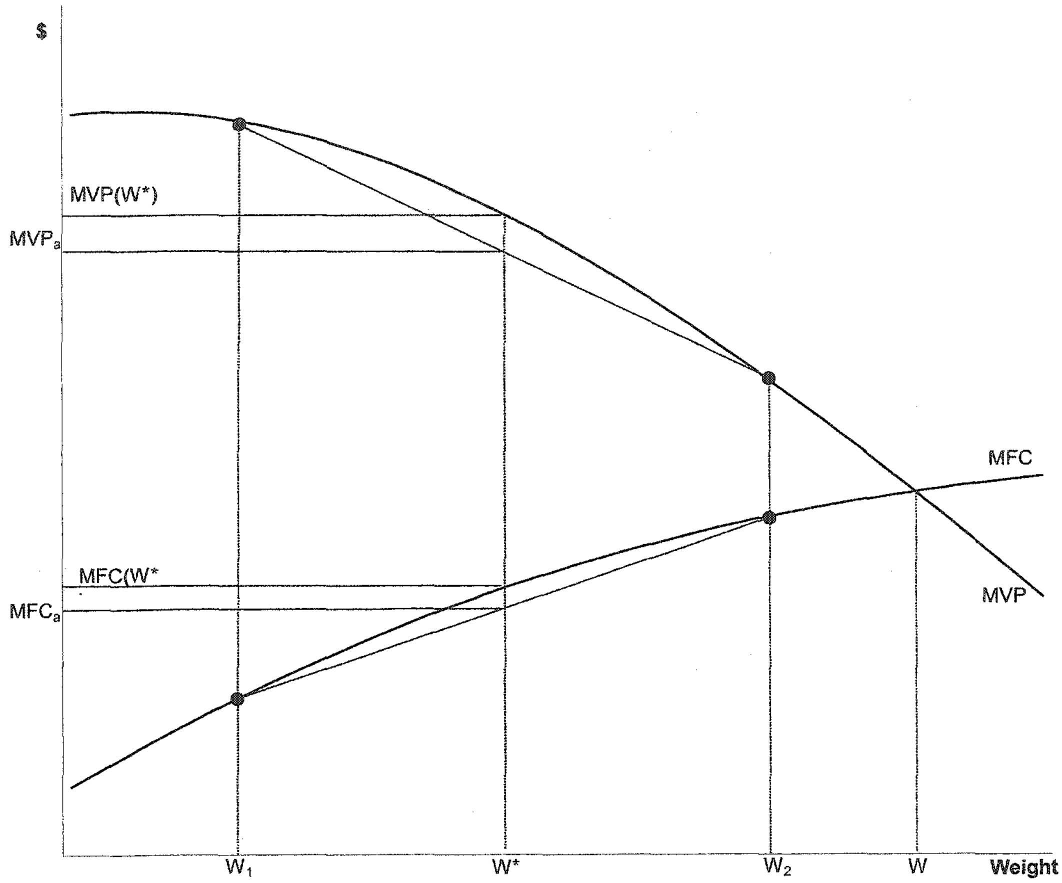
The conclusions at this point are that Jensen's effect means the average marginal productivity of a group of individuals may not equal the marginal productivity of an average individual in the group and that there exists an aggregate equivalent individual that does represent the average marginal productivity of the group. Furthermore, the aggregate equivalent individual can be derived from the average individual by adding an aggregation premium.

In group optimization, the objective is to maximize profits from the group. General principles from microeconomics state that to do so, the firm should continue to apply inputs to the group until the value of marginal productivity associated with those inputs equals their marginal cost. As such, the marginal productivity of a group is the average marginal productivity multiplied times the number of individuals in the group. What the work leading up to this point indicates is that the average marginal productivity should not be derived from an average individual based on input or output quantities. Rather, it should be derived from an aggregate equivalent individual where the Jensen effect has been accounted for in the process.

The misallocation of inputs that can occur when the Jensen effect is ignored can be demonstrated through further use of the livestock examples presented in the two preceding sections of this chapter. In these examples, the production functions (4.12) and (4.5) are used to indicate animal growth. You will notice that in each of these production functions the input is time. Therefore, the profit-maximizing producer is deciding how many days to feed the animals. The profit maximizing point is the point in time where the marginal cost of feeding the animals another day is equal to the value of the marginal productivity that results.

The marginal factor cost (MFC) function representing the marginal cost of feeding the animals another day is likely to be an increasing concave function as shown in figure 4.5. This is due to the fact that feeding costs are directly related to an animal's

Figure 4.5 Example marginal function curves for profit maximization.



weight. In this case, each animal's weight is increasing but at a decreasing marginal rate. Therefore, the marginal cost of feeding those animals can be expected to do the same.

The result is a marginal function picture as shown in figure 4.5. The function for the value of marginal productivity (MVP) is a decreasing concave function and the function for the marginal factor cost (MFC) is an increasing concave function. Therefore, according to table 4.1, the Jensen effect for both curves is negative. Since the marginal profit is given by the difference between the marginal value product and the marginal factor cost, the two Jensen effects will counter-act one another; they have the same sign. Perhaps an easier way to identify the existence of this counter-balancing effect is the fact that the aggregation premium for the MVP curve is positive, while the aggregation premium for the MFC curve is negative. Opposite signs on the aggregation premium for each curve will always signify a counter-balancing effect. The net effect on the point of group profit maximization is determined by the relative magnitudes of the Jensen effects and the aggregation premiums.

For example, the point of intersection for the MVP and MFC curves in figure 4.5 is at a weight of W. However, due to the Jensen effect, when the average weight of the individuals in the group is W,

$$MVP_a < MVP(W)$$

and

$$MFC_a < MFC(W)$$

where MVP_a and MFC_a represent the average marginal value product and the average marginal factor cost, respectively, for the group. Unless the Jensen effect for each curve is identical, the situation exists where $MVP_a \neq MFC_a$. Therefore, when the average weight in the group is W, the group is not at a point of profit maximization. However, the economic importance of the error is diminished due to the counter-acting effects of the two Jensen effects.

Let W* represent the average weight of the animals at the true point of group profit maximization. If the negative Jensen effect associated with the MVP curve is larger in magnitude than the negative Jensen effect associated with the MFC curve, then W* will lie to the left of W. Otherwise, if the negative Jensen effect associated with the MFC curve is more dominant, then W* would lie to the right of W. The primary concern for the group optimization decision-maker is the magnitude and direction of the shift from W to W* along with its subsequent economic impact. There are three primary factors that determine the magnitude and direction of the shift from W to W*. The first factor is the relative curvature of the respective marginal functions. This can affect both the direction and the magnitude of the shift. The second factor is the slope of each of the marginal functions. The slope of the functions can only affect the magnitude of the shift. Finally, the third factor is the shape of the underlying distribution describing the individuals in the group. This adds an interesting dimension to the problem that can have a complicated affect the magnitude of the shift. The first two factors, involving the curvature and the slope of the marginal curves, will be discussed together in the examples and illustrations that immediately follow. The impact of the distribution is addressed afterwards in a general discussion.

Consider figure 4.6, where the point of intersection between the MFC and MVP curves is determined to occur at W = 245.59 lbs. Note that, at the point of intersection, MVP(W) = MFC(W) = 0.7101. If a normal distribution with a mean of 245.59 lbs. and a standard deviation of 30 lbs. is numerically imposed onto the graphical model, a weighted average for the MVP and MFC can be calculated for a fictional group of individuals. The results are $MVP_a = 0.6972$ and $MFC_a = 0.7090$, respectively. Because these are not equal, W = 245.59 is not the point of group profit maximization.

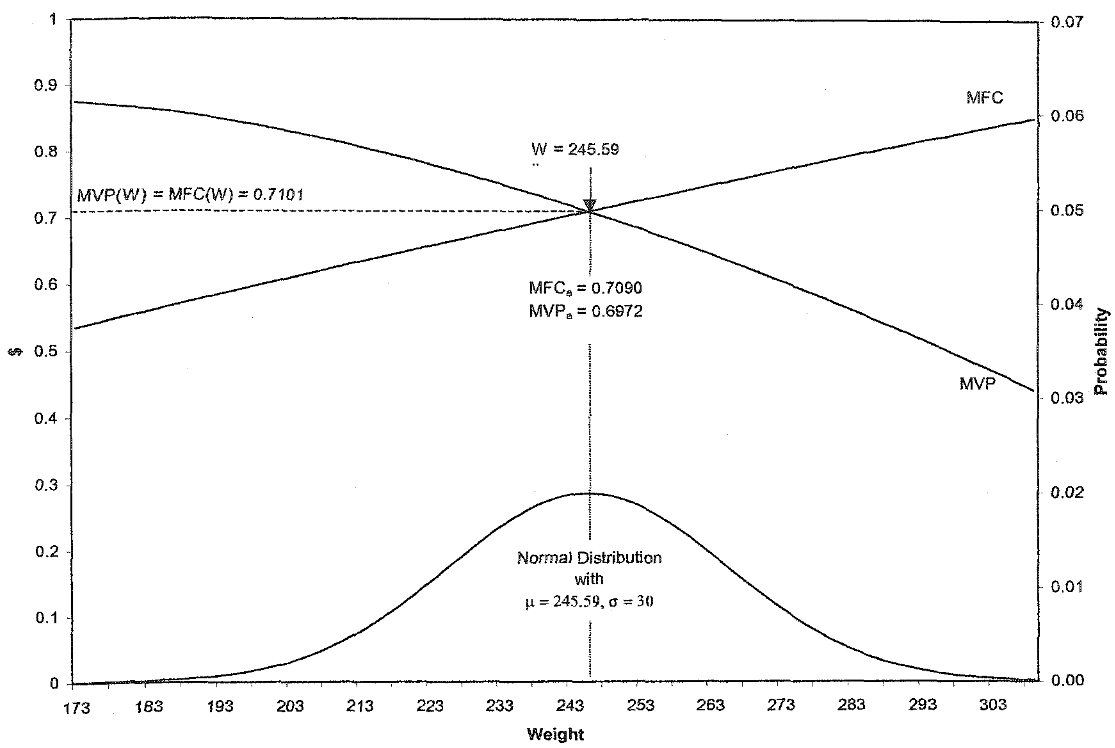
The Jensen effect for the MVP curve at W = 245.59 can be calculated as

$$J_{mvp} = MVP_a - MVP(W) = 0.6972 - 0.7101 = -0.0129,$$

and the Jensen effect for the MFC curve at W = 245.59 can be calculated as

$$J_{mfc} = MFC_a - MFC(W) = 0.7090 - 0.7101 = -0.0011.$$

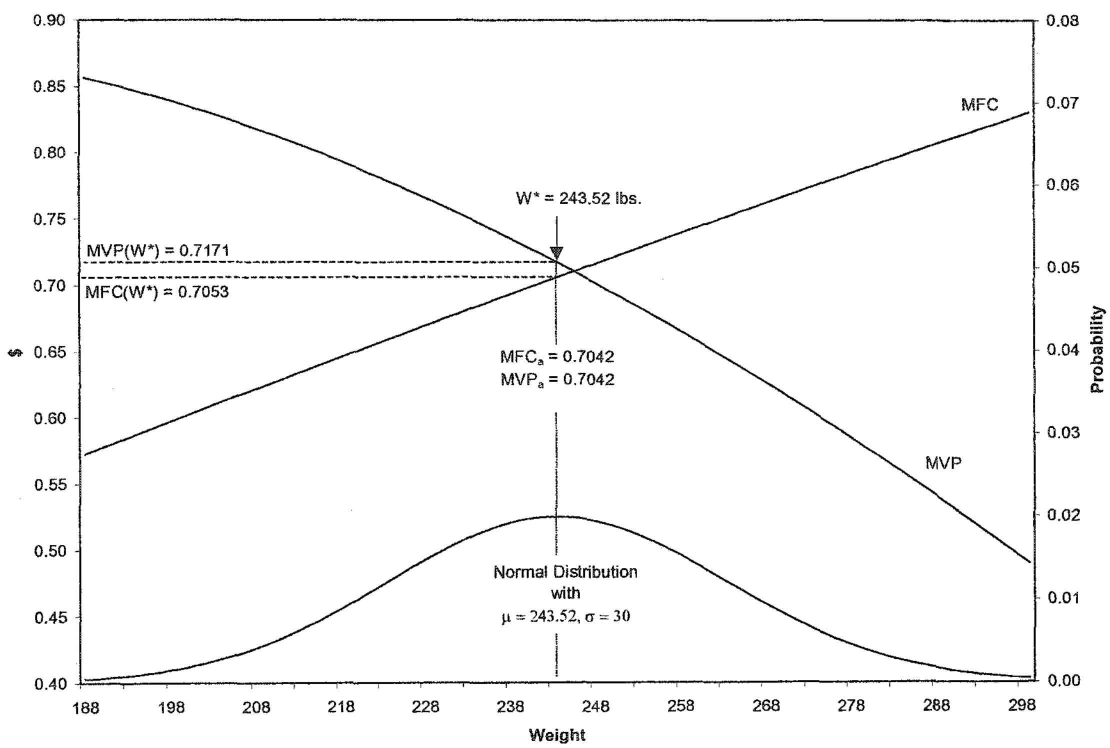
Figure 4.6 A graphical representation of a "false" group optimization condition.



Because the Jensen effect associated with the MVP curve is larger in magnitude than the Jensen effect associated with MFC curve, the result is a situation where $MVP_a < MFC_a$ at W = 245.59. Therefore, W*, the point of true group profit maximization, is somewhere to the left of W. W* is shown graphically in figure 4.7.

Next, consider a shift to the left in the normal distribution function depicted in figure 4.6 from W = 245.59 to W* = 243.52 as depicted in figure 4.7. This indicates the mean of the group is now at W* rather than at W. Note that at W* = 243.52 lbs., MVP(W*) = 0.7171 and MFC(W*) = 0.7053 indicating that the marginal value of feeding an average weight individual from the group another day is greater than the marginal cost. However, taking into account the distribution, the average value of

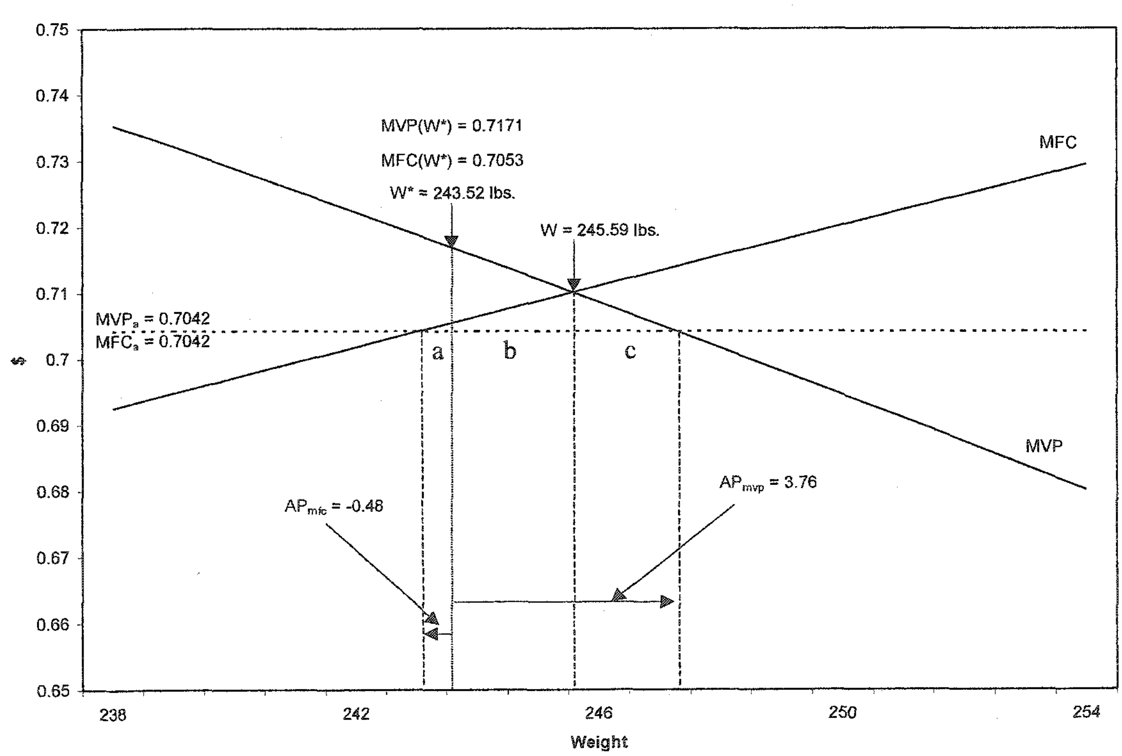
Figure 4.7 A graphical representation of a group profit maximization condition.



marginal productivity for the group is $MVP_a = 0.7042$ and the average marginal factor cost for the group is $MFC_a = 0.7042$ indicating that W* = 243.52 is the true point of group profit maximization. That is, when the average weight in the group is equal to W* = 243.52 lbs., the marginal value of feeding the group another day is equal to the marginal cost and it is the optimal time to market the group of animals. The resulting equality of $MVP_a = MFC_a$, despite the fact that MVP(W*) ≠ MFC(W*), is due to the Jensen effect on each curve ($J_{mvp} = -0.0129$ and $J_{mfc} = -0.0011$, respectively).

Figure 4.8 shows a magnified version of figure 4.7, zeroing in on the profit maximization conditions. A horizontal line indicating where $MVP_a = MFC_a = 0.7042$ is drawn into the graph for reference. Note two things that diminish the magnitude of the shift necessary to move from the point of intersection at W = 245.59 to the point of group profit maximization at W* = 243.52. First, both curves have a negative Jensen effect due to concavity. Therefore, their Jensen effects counter-act one another. Another way of describing this is that at W* = 243.52, the negative aggregation premium of –0.48 associated with the MFC curve somewhat counter acts the positive aggregation premium of +3.76 associated with the MVP curve. Second, the magnitude of the shift is even more

Figure 4.8 A graphical look at the aggregation premiums at group optimal conditions with counter-acting impacts from the Jensen effects.



a – indicates a counter-acting curvature effect in diminishing the magnitude of the shift from W to W*.
b – indicates the remaining magnitude of the shift from W to W*.
c – indicates the counter-acting slope effect in diminishing the magnitude of the shift from W to W*.

affected by the slope effect resulting from the increasing marginal cost curve. Because it is increasing, the aggregate equivalent individual on the MVP curve will lie to the right of the point of intersection at W = 245.59. Therefore, a good part of the aggregation premium of +3.76 associated with the MVP curve is accounted for by the upward sloping MFC curve.

Generalized Cases for Curvature and Slope Effects

It follows that there are several cases that can exist where the MFC curve has a counter-acting, neutral, or compounding effect on the magnitude of the shift from W to W* necessary to reach conditions of group profit maximization. Four of these cases are presented in figure 4.9 under the assumption that the MVP curve is a decreasing concave function. If the MFC curve is a horizontal line representing constant marginal factor costs (figure 4.9a), then the entire Jensen effect associated with the MVP curve at W* accounts for the difference between the MVP curve and the MFC curve at W*. Also, the entire aggregation premium associated with the MVP curve at W* must be accounted for in the shift from W to W*.

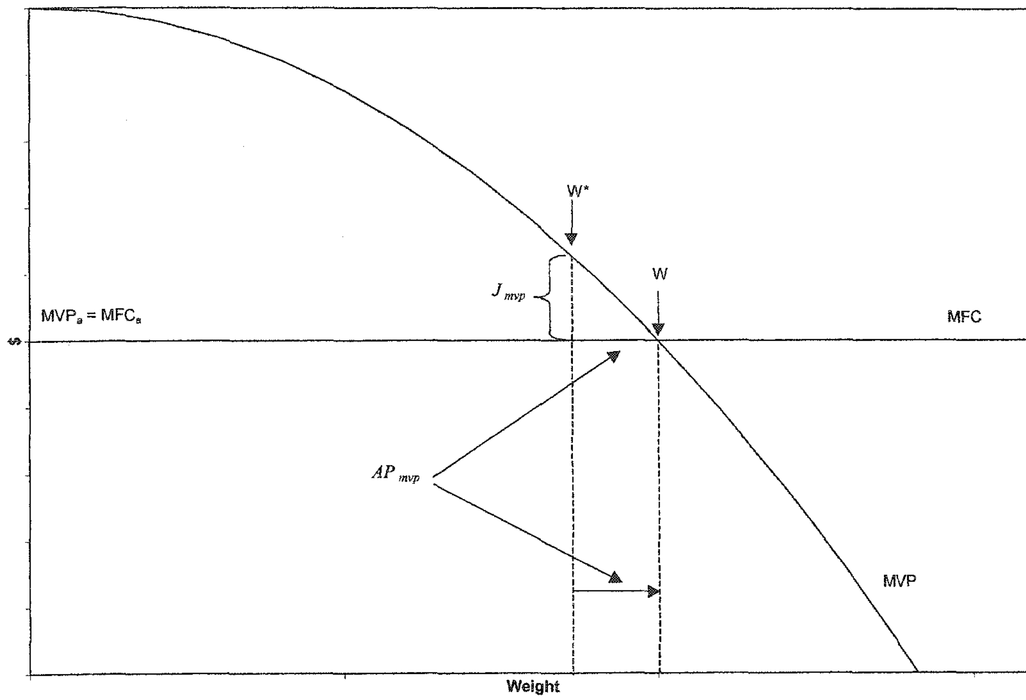
If the MFC curve is an increasing linear function (figure 4.9b), then the entire Jensen effect associated with the MVP curve at W* would still account for the difference between the MVP curve and the MFC curve at W*. However, part of the aggregation premium associated with the MVP curve at W* will be accounted for by the upward slope of the MFC curve. That is, because of the upward slope of the MFC curve, part of the aggregation premium associated with the MVP curve at W* will lie to the right of W. Therefore, the shift from W to W* must no longer account for the entire aggregation premium. The conclusion is that the fact that the MFC curve is increasing diminishes the magnitude of the shift necessary to move from W to W*.

If the MFC curve is an increasing concave function (figure 4.9c), then the Jensen effect associated with the MVP curve will be counter-acted by a Jensen effect associated with the MFC curve. It is theoretically possible that the Jensen effect associated with the MFC curve could be greater in magnitude than the Jensen effect associated with the MVP curve. If that were the case, then W* would be to the right of W. However, it is more likely that the Jensen effect associated with the MVP curve has the greater magnitude. Then, as depicted in figure 4.9c, the Jensen effect associated with the MVP curve is partially counter-acted by the Jensen effect associated with the MFC curve. The net difference in the magnitudes of the two Jensen effects will account for the difference between the MVP curve and the MFC curve at W*. This counter-acting effect along with the effects of the upward sloping MFC curve will diminish the magnitude of the shift from W to W*.

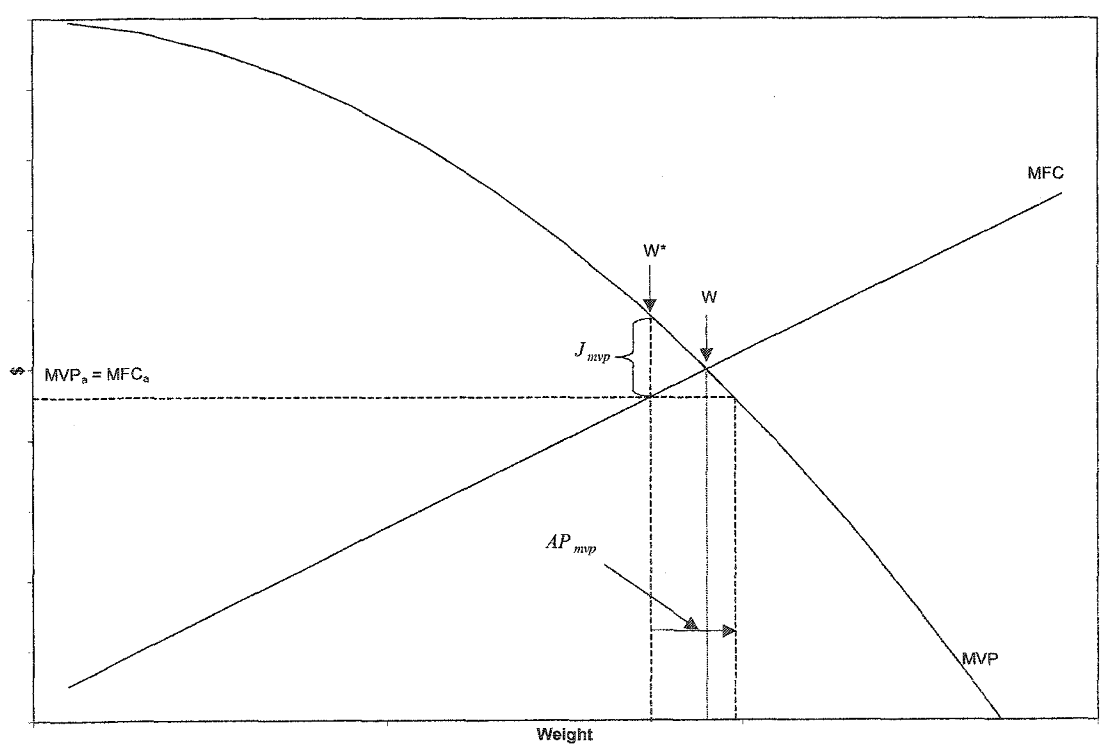
Finally, if the MFC curve is an increasing convex function (figure 4.9d), then the Jensen effect associated with the MVP curve at W* will compound with the Jensen effect associated with the MFC curve at W*. The sum of the magnitudes of these two Jensen effects will account for the difference between the MVP curve and the MFC curve at W*. Because of this compounding effect, the magnitude of the shift from W to W* would be increased. However, the upward slope of the MFC curve will continue to account for some of the aggregation premium associated with the MVP curve at W*. This will somewhat offset the effect of the convexity of the MFC curve in increasing the magnitude of the shift from W to W*.

Figure 4.9 Four cases of accounting for the optimum shift from W to W* under the assumption of a decreasing concave MVP curve.

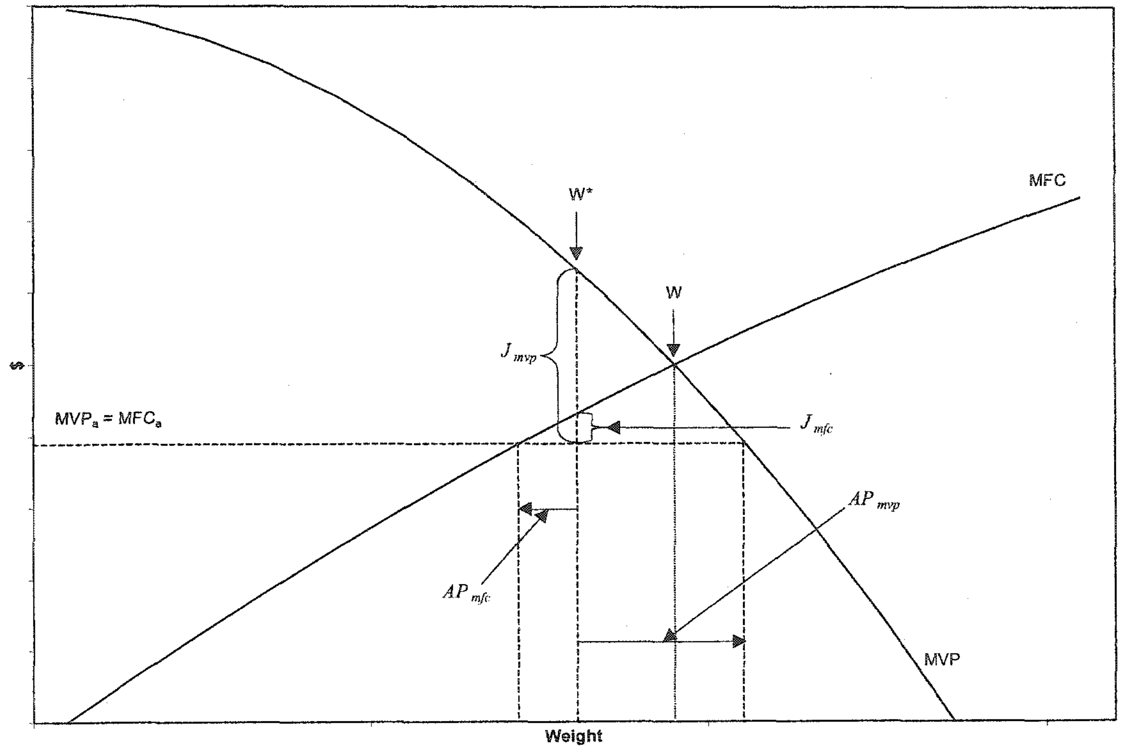
(a) The case with constant marginal factor costs.



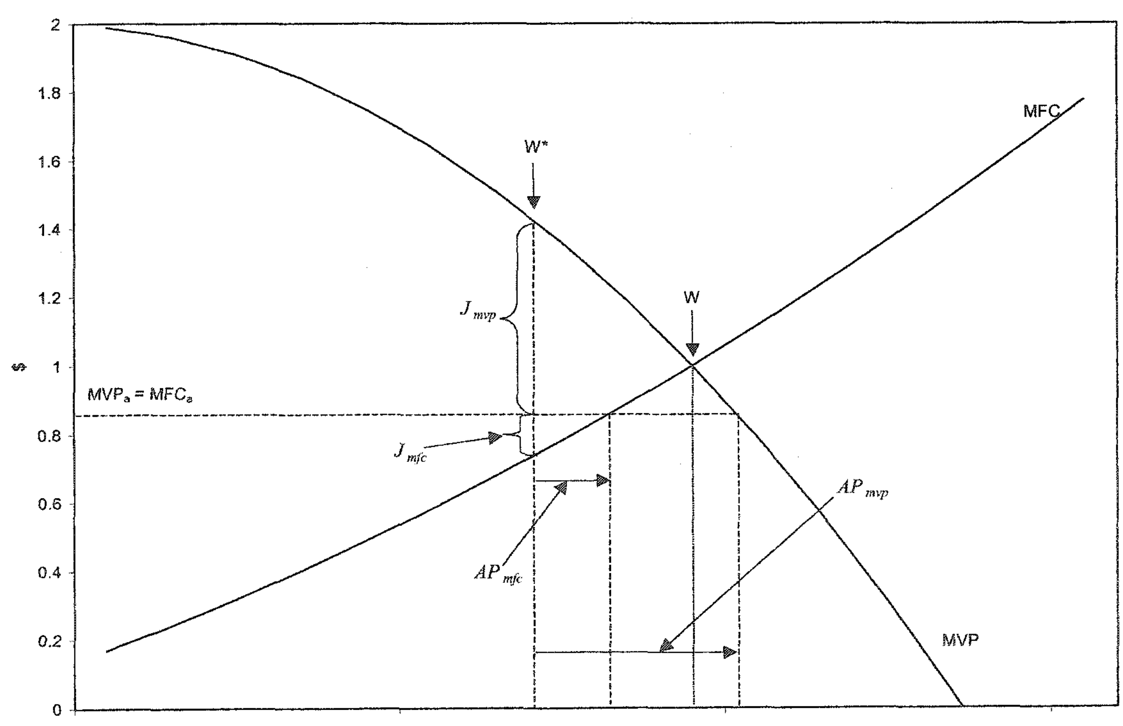
(b) The case with an increasing linear marginal factor cost curve.



(c) The counter-acting case with an increasing concave marginal factor cost curve.



(d) The compounding case with an increasing convex marginal factor cost curve.



In summary, an increasing MFC curve, as opposed to a constant marginal factor cost, will diminish the magnitude of the shift from W to a smaller W*. A convex MFC curve coupled with a decreasing concave MVP curve will have a compounding effect that increases the magnitude of the shift from W to a smaller W*. A concave MFC curve would have a counter-acting effect that would diminish the magnitude of the shift from W to W* and leave the direction of the shift to be determined by the relative magnitudes of the Jensen effect for each curve.

For completeness, an analogous set of statements can be made for an MVP curve that is a decreasing convex function or a decreasing linear function. Table 4.5 contains a summary of the relationships that exist for various combinations of linear and nonlinear MVP and MFC curves. The assumption is made in table 4.5 that the MVP curve is a decreasing function and the MFC curve is an increasing function. However, if one of the curves is a constant horizontal linear function, then the aggregation premium of the other curve will equal the shift from W to W*. This relationship was shown earlier using constant marginal factor costs in figure 4.9(a). As shown in contrast in figure 4.9(b), an increasing MFC curve will decrease the magnitude of the shift from W to W* relative to the magnitude of the shift associated with constant marginal factor costs. In general, the steeper the slope of either the MFC or the MVP curve, the lower in magnitude the shift from W to W* needs to be to compensate for the Jensen effect.

As table 4.5 summarizes, if the Jensen effect of the MFC curve has the opposite sign as the Jensen effect of the MVP curve, then there is a compounding effect that will increase the magnitude of the shift from W to W*. Alternatively, if the MFC curve and the MVP curve are either both concave or both convex then the Jensen effect for each

Table 4.5 A summary of the impact of the Jensen effect on optimal conditions.

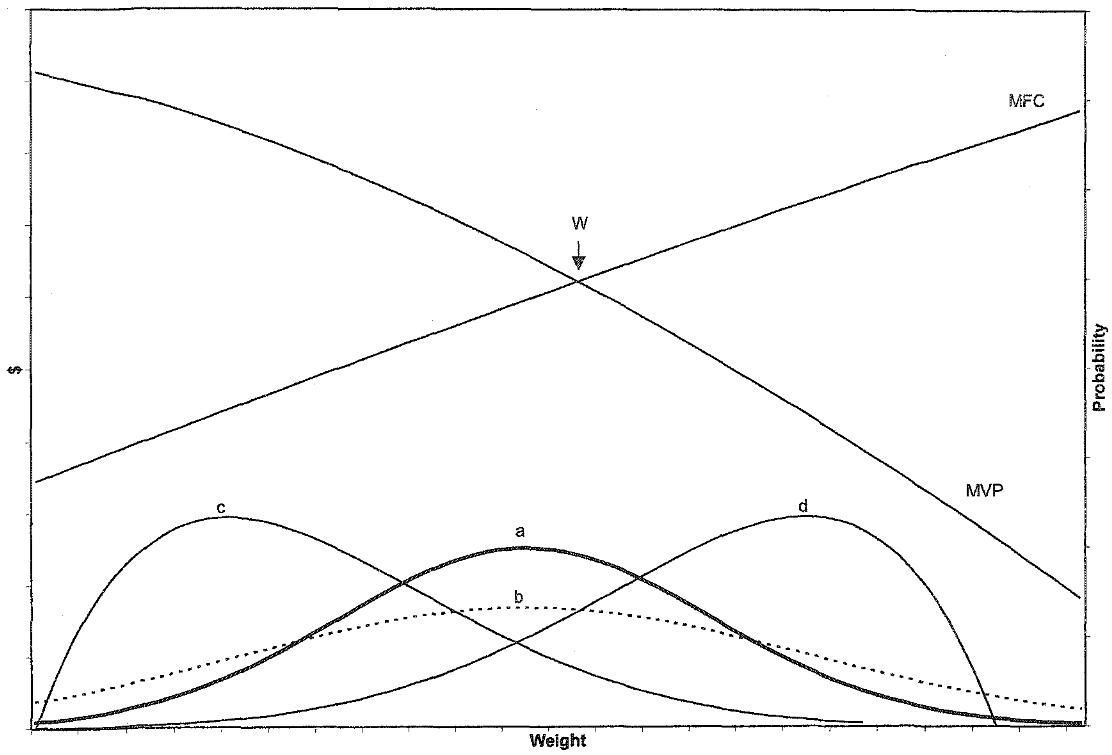
		Increasing MFC Curve		
		Linear	Concave	Convex
Decreasing MVP Curve	Linear	$J_{mvp} = 0$ $J_{mfc} = 0$ *No Jensen effect* $W^* = W$	$J_{mvp} = 0$ $J_{mfc} < 0$ $AP_{mfc} < W - W^* < 0$ $W^* > W$	$J_{mvp} = 0$ $J_{mfc} > 0$ $AP_{mfc} > W - W^* > 0$ $W^* < W$
	Concave	$J_{mvp} < 0$ $J_{mfc} = 0$ $AP_{mvp} > W - W^* > 0$ $W^* < W$	$J_{mvp} < 0$ $J_{mfc} < 0$ *Counter-acting effects*	$J_{mvp} < 0$ $J_{mfc} > 0$ *Compounding effects* $W^* < W$
	Convex	$J_{mvp} > 0$ $J_{mfc} = 0$ $AP_{mvp} < W - W^* < 0$ $W^* > W$	$J_{mvp} > 0$ $J_{mfc} < 0$ *Compounding effects* $W^* > W$	$J_{mvp} > 0$ $J_{mfc} > 0$ *Counter-acting effects*

curve will have the same sign. When the Jensen effects have the same sign, there is a counter-acting effect that will decrease the magnitude of the shift from W to W* and leave the direction of the shift to be determined by the relative magnitudes of the Jensen effect for each curve.

The Distribution Effect

The distribution effect can manifest itself in a number of complicated ways. The shape of the underlying distribution of individuals in the group will either magnify or diminish the Jensen effect and its impact on the magnitude of the shift from W to W*. Where the distribution mass is relative to the curvature of the two curves will determine the impact that the distribution has on the magnitude of the shift. Figure 4.10 shows four

Figure 4.10 Four example underlying distributions.



example distributions super-imposed on a graph of marginal value and marginal cost curves. Distribution (a) is a normal distribution as assumed in the previous example. Distribution (b) is also a normal distribution but with a mean preserving spread imposed. A mean preserving spread will magnify the Jensen effect as long as the distribution remains strictly within a concave (or convex) portion of the curve. This would increase the magnitude of the shift for W to W*. Distribution (c) is skewed to the right and distribution (d) is skewed to the left. How these skewed distributions interact with the curvature will determine their impact relative to a symmetric distribution. If the more of the probability mass interacts with a portion of the curves that has more curvature, then the Jensen effect can be expected to be magnified. Otherwise, if the bulk of the probability mass interacts with a more flat portion of the curve, then the Jensen effect can be expected to be diminished.

The final comment on the impact of the distribution function is that sometimes the underlying distribution may be unknown. Then, the problem of group optimization becomes very interesting but, unfortunately, beyond the scope of this dissertation. In general, it moves the decision-maker back into the realm of decision-making under uncertainty rather than assuming a certain degree of variability exists as I have done in the present work.

Summary and Concluding Remarks

Jensen's inequality can be applied anytime a function is nonlinear with respect to an uncertain variable. In this chapter, I have demonstrated how this can have an effect on marginal values in a production economics setting. This can be important when production is done in groups and uncertainty exists that results in variable productivity. I characterized production uncertainty as either a random variable that affects the production function or a random variable that affects the input that goes into it. In either case, if the marginal product curve is nonlinear with respect to the random variable, an aggregate equivalent individual that represents the average marginal productivity for the group can be calculated. This individual will differ from individuals representing the average level of input and from individuals representing the average level of output. These differences can be accounted for through Jensen's effect and an aggregation premium applied to the nonlinear marginal curves.

The effect these differences have on the optimal conditions associated with group profit maximization will be determined by the curvatures of the marginal product curve and the marginal factor cost curve relative to one another. If the curvatures result in aggregation premiums associated with each curve having the same sign, then the Jensen effects of each curve will be opposite in sign and will compound with one another to increase the magnitude of the difference between the point of intersection between the two curves and optimal point pertaining to profit maximization. If the curvatures result in aggregation premiums associated with each curve having opposite signs, then the Jensen effects will have the same sign and will counter-act one another. As a result, the magnitude of the difference between the point of intersection between the two curves and the optimal point pertaining to profit maximization will be diminished. Furthermore, the direction of the shift from the point of intersection to the optimum point will be determined by the relative magnitudes of the Jensen effect associated with each curve.

In this chapter, I have introduced new terminology in an effort to formalize the treatment of Jensen's inequality in production economics. In a group setting with production uncertainty that leads to variable productivity, this is an important foundation of information to have available. I hope that it will help stimulate future research into empirical applications.

Chapter 5

Variable Cattle Growth and Group Optimization

Under the assumption of certainty, the world of microeconomic analysis is a comforting arena in which to operate. The hypothetical producer, with perfect knowledge of costs and productivity, employs an increasing amount of each input until the marginal cost for each input is equivalent to the value of its marginal productivity. Unfortunately (or fortunately, depending upon your viewpoint), we live in a world that is predominantly uncertain. For producers of goods and services, this can have a profound effect on the decisions that they make to maximize benefits received from operating a business. Economists have analyzed the effects of uncertainty on the theory of the firm from a number of different angles but the angle involving optimized group production has been largely ignored. This chapter steps forward in addressing that issue by analyzing the marginal characteristics of the production economics decision for a hypothetical cattle producer feeding a group of 600-weight animals for 154 days up to a slaughter weight. The application is chosen because it provides a natural, convenient setting from which to build the concepts and theories developed. The most important contribution of the work that follows is in the theoretical foundation and model that is presented for this type of analysis. The concepts presented here add to the toolkit for future microeconomic analysis of the firm.

Analysis of uncertainty in the theory of the firm has largely been confined to two areas: price uncertainty or production uncertainty. Of these, price uncertainty has received the most extensive treatment in the literature (e.g. Batra and Ullah; Sandmo). However, there are many sectors of the economy where production uncertainty may have a greater impact than market uncertainty (Pope and Kramer). Agriculture is one of the settings where this is often true. Factors such as weather and genetics can combine to have profound effects on producer profitability.

Production uncertainty can be modeled in a couple of different ways. Walters (1960) introduced the idea of uncertainty in factor services. Other authors (Robison and Barry) term this concept as uncertainty in the quality of input. The idea is that although a firm may hire a certain quantity of input, the quality of that input in the production process may be stochastic. For example, although L units of labor are hired, the actual effectiveness of that labor in production may be $L+\varepsilon$ where ε is a random variable with zero mean and constant variance.

A second way to model production uncertainty is through uncertainty in the production function. When analyzing empirical data, econometricians invariably include stochastic terms in their model. This is born out of necessity dictated by hard empirical facts. However, the stochastic component is often viewed as an inconvenience. One that could be eliminated by a properly formed deterministic model (McCall). Despite the best efforts of highly qualified individuals, the perfect deterministic model has yet to be formed that models empirical data. A stochastic component may be an inconvenience but it is also a reality. There are a number of ways to incorporate it into a production function model. The easiest way is to do it additively. However, this is an unsatisfactory approach in many empirical settings. For example, empirical evidence shows that animals in a production group that are heavier at birth and weaning have a competitive advantage. Thus, they will usually remain heavier throughout their stay within the group (Le Dividich). An independent additive random effect is therefore an unsatisfactory explanation for why one animal is bigger or smaller than another is. A mixed effects model may be a better tool for addressing such a situation. Craig and Schinckel (2001) have developed such a model for swine in which the stochastic component is multiplicatively incorporated into a biological parameter in the model.

The theory of the firm in the presence of production uncertainty has also been addressed in the operations research literature. However, the context there has involved the determination of optimal production quantities or run time when the possibility exists for some of the output to be defective (Agnihothri, Lee, and Kim; Yao; Rosenblatt and Lee). While related to the work carried out here, these particular works do not involve an optimization procedure that analyzes a distribution of marginal productivity. Either the production process worked and produced a non-defective product or it didn't. Variable productivity created the situation but understanding the entire distribution of variability isn't necessary to resolve it. Optimization of profits from a group of producing animals is a bit more complex. Disregarding death, there are no defective individual animals in the sense of being worthless to the firm. Nor are all non-defective individuals of the same value. Therefore, the theory of group production presented here is a more general approach to the problem of variable productivity than that presented in the operations research literature.

Cattle producers are particularly concerned about variable productivity. This concern has manifested itself in a number of studies done that attempt to quantify the economic returns related to sorting (Brethour; Basarab, et al., 1997; Basarab, et al., 1999; Walker; Wang and Roe). The results have been mixed but benefits do appear to exist. These benefits are realized by producing a more uniform product within a group. Or, perhaps a better way to say it is that benefits are realized by a more uniform production of the product. That is, sorting puts individuals into groups sharing similar characteristics. These individuals then progress through the production process together and, theoretically, produce output exhibiting a tightly bunched set of characteristic measurements. The trouble is that, even with a well thought out sorting regime, there will still be some variability in the final product. The first step toward understanding the economic value in reducing this variability is understanding the impact variability has on the microeconomic principles of group optimization.

Economists approach group optimizations much the same way they approach many things. They look for a representative individual, the "rational man" so to speak. The natural candidate for this representative is some average individual. However, if a function is nonlinear, Jensen's inequality states that the functional value of the average is not equivalent to the average functional value. This impacts the situation in a number of ways. One way is that the average input will not produce the average output if the production function is nonlinear. Therefore, choosing the representative average individual is not a trivial matter. If it is based on the average output level, then it will not represent the average input level. If it is based on the average input level, then it will not represent the average output. Furthermore, when it comes to the optimization of profits from group production, the problem can be even more complex. In optimization, the chosen individual should represent the average marginal productivity for the group. In some cases, with nonlinear marginal product curves, neither the individual representing average input nor the individual representing average output is representative of the group's average marginal productivity. Fortunately, through Jensen's inequality, these differences can be quantified and analyzed.

The objective of this chapter is to lay a foundation of microeconomic theory for group production as it is applied to a pen of cattle fed for slaughter. The optimization results will be analyzed in order to quantify differences observed as a result of Jensen's inequality. These differences will be used to explain why sometimes using average output levels in the microeconomic analysis can lead to suboptimal results.

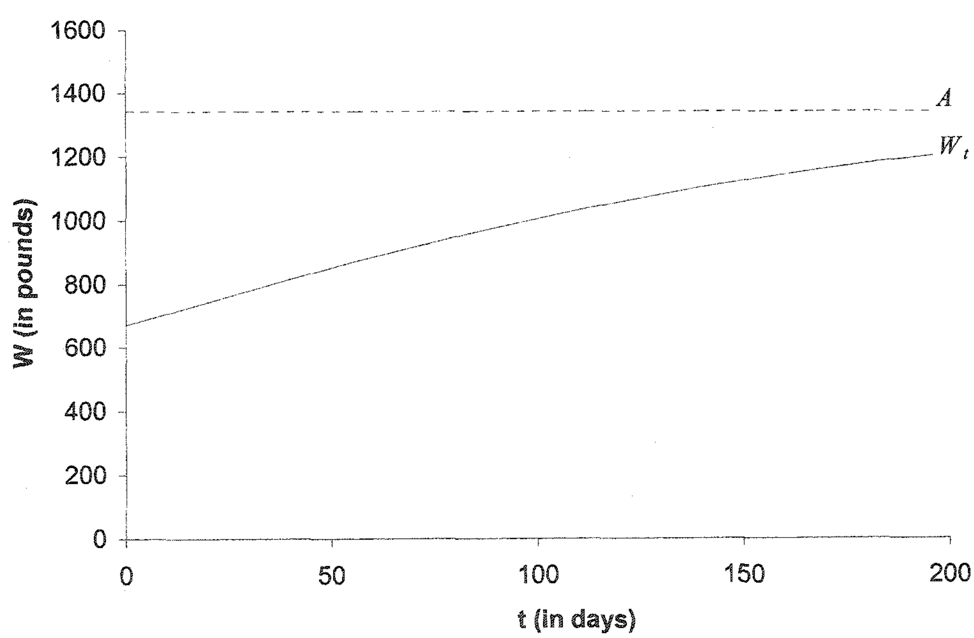
The Problem

In group production, individuals enter the production process and exit the production process at the same time. Whenever a group of cattle is brought into a feedlot and fed for slaughter there is bound to be some degree of variability within the group. The goal of sorting them into groups is to reduce this variability as much as possible. The effect this variability has on profits for the firm will be determined by how pronounced it is and by the market implications that result.

Determining the time that should elapse while the animals are on feed requires an understanding of the value of the marginal productivity of the group and the associated marginal cost. In a perfect world, complete knowledge of each individual animal's microeconomic information would be available. However, this level of information may be impossible to obtain and, at the very least, extremely costly to obtain. Therefore, an animal that is representative of the group is sought for purposes of determining the optimum time on feed.

The natural procedure used to select a representative for the group might be to use an average animal. For example, simply taking the average weight of the animals throughout the feeding process to create a fictitious representative individual for the group seems reasonable. The problem is that animals do not grow in a linear fashion. The typical animal growth path is a nonlinear concave function with respect to time as it asymptotically approaches a maximum mature weight (see figure 5.1). Therefore, by Jensen's inequality, the average weight is not the functional value of the average input. I will clarify the details of this relationship in the model that follows as it extends into the all-important optimization space of marginal productivity.

Figure 5.1: A logistic animal growth path as weight W_t asymptotically approaches a maximum mature weight A.



The Model

Consider the logistic growth function

$$W_t = A\left(1 + e^{-kt}\right)^{-1} \tag{5.1}$$

where W_t represents an animal's weight at time t. The parameter A represents the animal's asymptotic maximum mature weight. The parameter k is a kinetic growth constant representing the rate of maturing. Function (5.1) has a severe restriction at $t = 0$ in that $W_o = \frac{A}{2}$. That is, this model restricts the estimated initial weight to be equal to half of the estimated maximum mature weight. This eliminates use of this model in many animal growth applications. However, cattle are often placed into feedlots at a time that they are approximately half grown. For the data used in the analysis that follows, the logistic growth function turns out to be a very good model choice.

A sample pen of data is contained in table 5.1. As observed earlier, there are a number of plausible explanations that one could give for the variable weights observed at any one point in time. However, since the animals are the same breed and presumed to be of very similar composition (Parsons), one plausible explanation would be that the animals are at slightly different stages of biological maturity. Under this assumption, the

Table 5.1: Observed live body weights for a pen of five Brangus X Angus steers on feed for 154 days in southeastern Colorado January-May 2000.

	Body Weight, lbs						
ID	day 0	day 28	day 56	day 84	day 112	day 140	day 154
470	714	778	882	991	1014	1088	1118
484	708	762	866	1010	1059	1122	1162
508	631	732	806	913	937	984	1018
510	668	758	860	985	1049	1157	1206
562	682	757	880	1020	1075	1148	1185
Average	680.6	757.4	858.8	983.8	1026.8	1099.8	1137.8

Data courtesy of Clinton Parsons, Texas Christian University.

animals are all growing according to the same growth function (5.1). However, at any given point in time, each animal is at a unique point along the growth path. This is classified as a model with uncertainty in the quality of the input. That is, at any time t, each animal i has a unique input level equal to $t + \varepsilon_i$ that explains its observed weight. Then, in figure 5.1, one can think of the distribution of weights observed in table 5.1 at any point in time as producing a distribution of points along the growth curve. Each point represents an individual animal's unique weight and input level $t + \varepsilon_i$ at time t. Furthermore, each point would represent a unique marginal product for each animal. It is this resulting distribution of marginal productivity that the firm is concerned about incorporating into the profit maximization process.

If the production function (5.1) results in a marginal product function that is nonlinear with respect to output W then, by Jensen's inequality, the marginal product of the average weight will not equal the average marginal product for the group. Similarly, if the marginal product function is nonlinear with respect to t then the marginal product of the average input will not equal the average marginal product for the group.

For production function (5.1), the marginal product with respect to another day on feed is given by

$$MP_t = \frac{dW_t}{dt} = Ake^{-kt}\left(1 + e^{-kt}\right)^{-2}. \tag{5.2}$$

It can be shown that equation (5.2) is a decreasing convex function with respect to t, for all $t > -\frac{1}{k}\ln\left(2 - \sqrt{3}\right)$ and a decreasing concave function for $0 \le t < -\frac{1}{k}\ln\left(2 - \sqrt{3}\right)$. Therefore, in general, the marginal product of the average input will not equal the average marginal product for the group.

In terms of output, the marginal product equation (5.2) can be written as

(5.3) $$MP_{W_t} = \frac{dW_t}{dt} = W_t k - W_t^2 k A^{-1}$$

It can be shown that the marginal product equation (5.3) is a decreasing concave function with respect to W_t for all $W_t > W_o$. Therefore, the marginal product of the average output, also, will not equal the average marginal product for the group.

An Illustration

Using the data presented in table 5.1, a production function equation of the form in equation (5.1) is estimated as

(5.4) $$W_t = 1341\left(1 + e^{-0.01098t}\right)^{-1}$$

where the parameters A and k are estimated such that the sums of the squared deviations are minimized. For the panel data set of five individual animals represented in table 5.1, the R-squared associated with equation (5.4) is 0.92. For the one average weight animal represented in the last row of table 5.1, the R-squared associated with equation (5.4) is 0.99. Both of these indicate a reasonably good fit of equation (5.4) to the data contained in table 5.1.

Using equation (5.4), a marginal product equation corresponding to equation (5.3) can be derived. The results are as follows.

(5.5) $$MP_{W_t} = 0.01098W_t - \left(8.1879 \times 10^{-6}\right)W_t^2.$$

Consider the observation made on day 154 in table 5.1. Using the distribution of weights observed and equation (5.5), the marginal product for each individual can be calculated. These calculations are contained in the second row of table 5.2.

Table 5.2: Individual weight observations and marginal productivity of another day on feed based on the observation made on day 154.

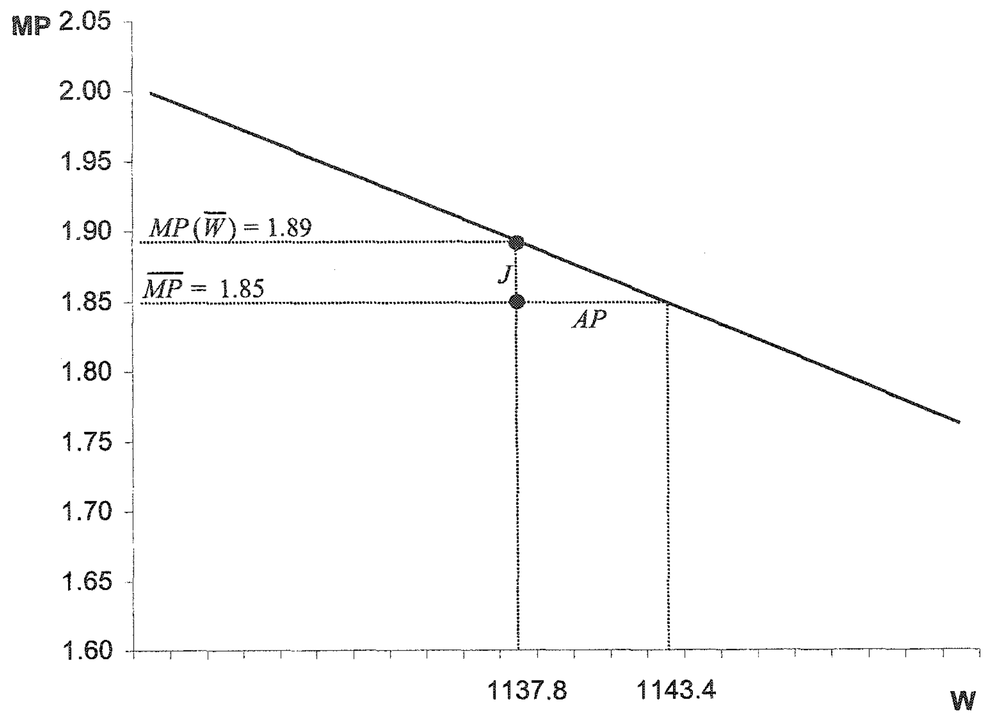
	Animal ID#					
	470	484	508	510	562	Average
W_t (lbs.)	1118	1162	1018	1206	1185	1137.8
MP_{Wt}	2.04	1.70	2.69	1.33	1.51	1.85

Note, in table 5.2, that the average of the five marginal products is 1.85. This is the expected weight gain from another day on feed. Also, notice that the average weight on day 154 is 1,137.8 pounds. Plugging this value into equation (5.5) for W_t yields a marginal product of $MP_{1137.8} \cong 1.89$. Therefore, as shown in figure 5.2, the average marginal product of 1.85 is less than the marginal product of 1.89 for the average weight. This is due to Jensen's inequality and the fact that the marginal product curve (5.5) is concave with respect to weight.

The vertical distance between the average marginal product $\left(\overline{MP} = 1.85\right)$ and the marginal product of the average weight $\left(MP\left(\overline{W}\right) = 1.89\right)$ is called the *Jensen effect*. In figure 5.2, this is labeled as J and it has a negative value (J = 1.85-1.89 = - 0.04) signifying that the average marginal productivity for the group lies below the curve associated with the marginal productivity of the average weight. An *aggregation premium* (AP) of 5.6 pounds added to the average weight results in an aggregate equivalent weight of 1,143.4 pounds. The *aggregate equivalent* weight is the weight of the animal that is representative of the group's average marginal productivity. It is this aggregate equivalent animal, not the average weight animal, that should be used in the optimization process.

In the optimization process, the producer will seek to equate the total value of the marginal productivity from the group to the total marginal cost of continuing to feed the

Figure 5.2:



group one more day. For illustration purposes only, suppose an average marginal productivity of 1.85 pounds per day proves to be the optimum. Then, as illustrated in figure 5.2, the producer should terminate the production process on day 154 when the average weight is 1,137.8 pounds. If the producer were to mistakenly base the optimization process on the marginal productivity of the average weight, they would instead seek to continue the production process until the average weight has reached 1,143.4 pounds, the aggregate equivalent weight on day 154. According to the production function (5.4), this would require another 3.0 days on feed. So, the Jensen effect would indicate that termination of the production process approximately three days earlier than the date indicated if the average weight animal were used in the optimization process for the group.

Results

Using the procedure illustrated above, an analysis of twenty-four pens with five animals per pen was carried out. The results are summarized in table 5.3. The most significant Jensen effect observed was –0.14. This corresponded to an aggregation premium of 15.4 pounds and an additional time of 9.3 days. However, on average, the aggregation premium was just less than six pounds with a corresponding additional time of just over three days. So, the distributions of marginal productivity and the Jensen effect as steers approach slaughter weight appears to create about a three day differential between the observed average weight and the aggregate equivalent weight reflecting the average marginal productivity for the group.

There are a number of factors that a producer takes into account when determining an optimal time to harvest a group of animals. The marginal productivity as reflected by the pounds gained per day is only one of them. However, as these results indicate, ignoring the Jensen effect when it comes to marginal productivity can lead to an erroneous extension of the production process. Output price was purposely ignored in this presentation because it is beyond the scope of emphasis. However, as livestock reach heavier weights, it is the tendency of liveweight market prices for those animals to decline. Therefore, output market price issues could serve to exacerbate the negative impact of ignoring the Jensen effect.

Additional factors to consider include the cost of feeding the animals. As the animals grow larger, they will theoretically consume more feed. This increased

Table 5.3: An analysis of 24 pens each containing five Brangus X Angus steers on feed for 154 days in southeastern Colorado.

$\overline{W}$	$MP(\overline{W})$	$\overline{MP}$	J	AE	AP	Δt
1137.8	1.89	1.85	-0.04	1143.4	5.6	3.0
1184.8	1.66	1.61	-0.05	1190.4	5.6	3.4
1200.4	1.72	1.58	-0.14	1215.8	15.4	9.3
1190.6	1.63	1.62	-0.01	1191.6	1.0	0.6
1165.6	1.52	1.46	-0.06	1171.3	5.7	3.8
1226.0	1.36	1.29	-0.07	1231.9	5.9	4.5
1185.4	1.98	1.93	-0.05	1191.4	6.0	3.0
1176.0	1.79	1.68	-0.11	1191.0	15.0	8.6
1148.8	1.89	1.83	-0.06	1155.7	6.9	3.7
1218.6	1.61	1.60	-0.02	1220.1	1.5	0.9
1265.0	1.70	1.61	-0.08	1273.6	8.6	5.2
1250.2	1.84	1.81	-0.02	1253.1	2.9	1.6
1225.6	1.91	1.89	-0.02	1228.1	2.5	1.3
1274.4	1.65	1.62	-0.03	1278.3	3.9	2.4
1271.6	1.89	1.86	-0.03	1275.3	3.7	2.0
1292.8	1.86	1.84	-0.02	1295.0	2.2	1.2
1327.6	1.96	1.86	-0.10	1338.3	10.7	5.6
1291.0	2.01	2.00	-0.01	1292.9	1.9	0.9
1318.4	2.16	2.05	-0.10	1331.8	13.4	6.3
1283.4	2.08	2.01	-0.07	1293.6	10.2	5.0
1309.0	1.85	1.82	-0.03	1312.6	3.6	1.9
1315.2	2.20	2.18	-0.02	1319.0	3.8	1.7
1280.0	2.24	2.22	-0.02	1283.7	3.7	1.7
1332.4	2.23	2.21	-0.02	1335.5	3.1	1.4
				Averages:	5.9	3.3

The data in the table pertains to the 154th day that coincided with the day that all of the animals were slaughtered. In the columns from left to right are the average live weight, the marginal product associated with the average live weight, the average marginal product, the Jensen effect, the aggregate equivalent weight, the aggregation premium, and the additional time required for the average weight to rise to meet the aggregate equivalent weight.

consumption will increase marginal costs through time. Erroneously extending the production process will not only result in lost profit due to the declining marginal productivity but also because of increasing marginal costs.

In the results presented here, it is unlikely that three days on feed or an extra six pounds of liveweight will have a profound effect on output price or on marginal feeding costs. Therefore, qualifying these results under the assumptions of a constant output price and a constant input price seems like a valid thing to do. However, in situations where these assumptions do not hold, quality discounts for over-finished animals and increasing costs associated with extended feeding of those animals will only serve to increase the economic gains associated with accounting for the Jensen effect in the group optimization process.

Concluding Comments

Anytime a nonlinear function is influenced by a random variable, the Jensen effect should be accounted for in any group analysis. In production economics, the function of marginal productivity is sometimes nonlinear with respect to uncertain variables. Whenever this happens, the "average" animal may not be the best representative of the average marginal productivity within the group. In this chapter, I have shown how, due to Jensen's inequality, the average weight steer being fed for slaughter can lag three days behind the aggregate equivalent steer for the group. It is this aggregate equivalent steer that truly reflects the average marginal productivity of the individuals in the group. Therefore, the aggregate equivalent animal and not the average weight animal should be used in the optimization process to determine the optimal time to slaughter the group.

Chapter 6

Variable Growth Impacts on Optimal Market Timing in All-Out Production Systems

In all agricultural production systems, numerous factors such as weather and genetics jointly determine the distribution of final output. Precision agriculture has emerged as a way to address spatial yield variations within fields. And in the livestock sector, animals are monitored with video and ultrasound to take advantage of yield variation in pens. Yield variation within a management unit, such as an acre for crops or a pen for livestock, is especially pertinent in the livestock industry where we typically see entire pens of animals marketed at one time based on the average size in the pen. Ideally, to be entirely confident about these marketing decisions, the entire range of the data should be understood (Pringle). Averages can mask information; information that might return more than it costs to collect.

Most research about the optimal slaughter weight of livestock has focused on feeding strategies, genetics, and pricing systems. For instance, it has been shown that there are higher profits per hog for leaner gilts relative to the fatter barrows and that the gilts pay more marketed through a component pricing system while the barrows pay more in a live weight pricing system (Boland, Preckel, and Schinckel). Other studies have shown that feed prices and animal replacement costs are important in determining the optimal market weight (Chavas, Kliebenstein, and Crenshaw), have examined how producers might modify their feeding decisions to best respond to changes in input and output prices (Crabtree), and used gain isoquants to establish decision rules for optimal rations through various growing phases (Heady, Sonka, and Dahm).

In general, past research has focused on establishing decision rules based on a representative animal from the group, even in the case where individual animals are ultrasounded and monitored by video. This may be appropriate in industries like poultry where variability has been reduced to minimal levels in recent years. However, these same decision rules may be sub-optimal for heterogeneous animals such as cattle, where there are frequent calls to improve quality and consistency (Smith et al.). Variability in animal growth results in some animals in the pen being over-finished, while others have not yet reached their full economic potential at the time the pen is marketed.

The objective of this paper is to present a model that accounts for the distribution of the animals in the market timing decision. This model is used to evaluate the marginal value of incorporating increasing amounts of information into the profit maximization problem for an all-out production system. The impact of this information on the optimality conditions is explored through a thorough analysis of the marginal curves resulting from the production process. Using familiar microeconomic theory, we clearly show that making decisions on a representative animal, were marginal value product is equal to marginal factor cost, does not yield an optimal solution (when information is free).

Swine production provides the application focus of the present paper but the methods extend to other species. By choosing swine as our application focus, we are able to utilize extensive data sets available from university researchers to test our model. As it turns out, hogs have relatively uniform growth and other circumstances that make the improvement in market timing not worth the cost of gathering more information. Nevertheless, the value of our model is not limited by this application, which clearly demonstrates when information would be valuable based on the model we have constructed. Specifically, we show that when marginal returns decrease at anything other than a constant rate, the average output level may not be the basis from which to compute the average marginal value product for a group of animals. A model accounting for the entire distribution of output levels provides a more accurate assessment of the marginal value associated with continuing to feed a pen of animals. This may lead to situations where market timing decisions based on the average output level can be proven to be significantly less than optimal.

This paper extends previous research in two ways. First, whereas previous research has focused on decision rules as they pertain to a representative animal for a given group, we are considering the entire distribution of animals. Therefore, the decision rules developed in this paper are a better representation of the full economic potential of all-in, all-out pen marketing practices. Second, by developing this model, we present a framework to explore the impact of production variability on any production situation characterized by a simultaneous termination of the production process across multiple producing units. In doing so, we make it possible to better assess the impact of practices such as tightening the genetic line or employing a sophisticated sorting regime on the potential profits of an all-in, all-out production system.

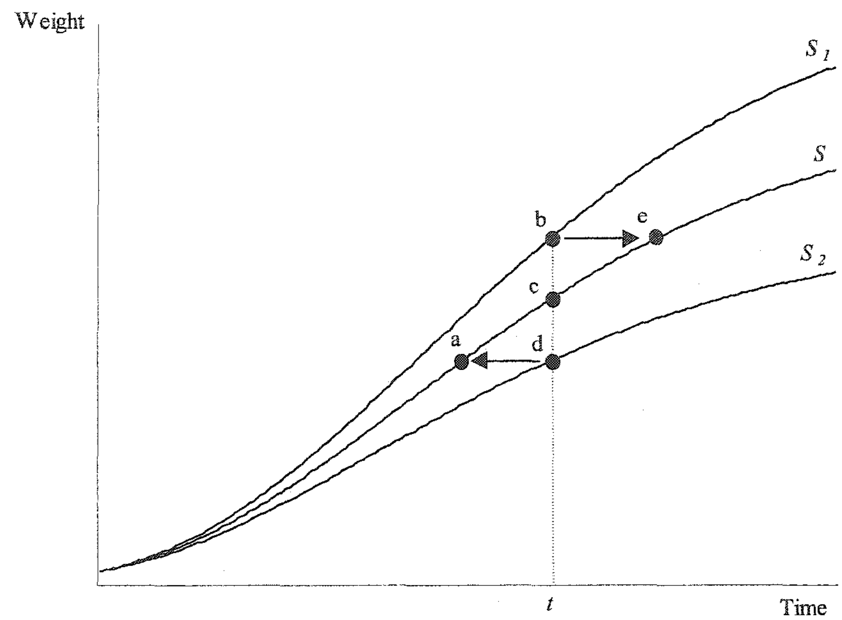
Problem Identification

In the neoclassical theory of the firm, the profit maximizing level of input is determined by equating the marginal value product of the input to its marginal factor cost. However, with variable output in an all-in, all-out production system with multiple producing units, the optimization problem becomes slightly more complicated. First of all, in an all-out production system, we are interested in the total marginal value product and the total marginal factor cost. That is, we need to account for all producing units. This would involve obtaining full information about each unit's individual production function. In some situations, this information would involve considerable cost to obtain. Whether this cost is justified would depend upon the marginal returns associated with using this "full" information in the profit maximization process versus some other level of information.

In figure 6.1, three levels of information are depicted as they apply to livestock production. The first level involves simply knowing how the average weight of the lot changes through time. This is depicted by curve S in figure 6.1. Define this as a basic level of information. Using only this knowledge in the profit maximization problem would result in the assumption that, at any point in time t, the marginal value product associated with the average animal (point c) is the marginal value product of every animal. An analogous assumption can be described for the marginal factor cost but, for now, the analysis is confined to a discussion of the effect on the marginal value product.

The second level of information combines knowledge of the curve S together with knowledge about the current distribution of weights. For simplicity, assume the group is of size n, with $n = 2$. Then, at a point in time t, knowledge of the curve S is available

Figure 6.1 Different levels of information in group production of livestock.



along with the individual animal weights at points *b* and *d* that combine to produce the average weight at point *c*. Define this as an intermediate level of information. There may be several approaches to using this level information in the profit maximization problem. The model described in this paper presents an approach that involves using knowledge of the individual weights to extrapolate backward or forward to corresponding points on curve *S* (points *a* and *e*, respectively). These points are then used together with the knowledge of curve *S* to approximate marginal value products for the individual animals. Essentially, the distribution of weights is combined with curve *S* to produce a distribution of marginal value products that can then be used in the profit maximization problem.

The third level of information involves knowledge of individual growth curves, S_1 and S_2, for each animal. Define this as the level of full information. It provides a distribution of actual marginal value products for use in the profit maximization problem.

Each of these levels of information comes with an associated cost. It is safe to assume that an increase in the level of information will involve an increase in the cost to obtain the information. On the other hand, incorporating a higher level of information in the profit maximization problem should produce a marginal benefit reflected by a higher net profit in the final outcome.

In the sections that follow, we describe a model that captures the dynamics of a growing pen of swine. Our model allows us to describe and assess the theoretical impact of moving from a basic level of information to an intermediate level of information in the profit maximization problem. We conclude the paper with an assessment of the financial impact on net profit per animal of incorporating each of the three levels of information in the profit maximization problem.

Model

The first step in developing the model is the determination of an appropriate production function. The use of a Gompertz sigmoidal curve to describe potential growth in swine has proved useful (Whittemore). The curve to give weight W_t at time t is given by $W_t = Ae^{-be^{-kt}}$ where A is the upper asymptotic weight, k is a growth constant, and b is a time scale parameter. However, Parks (1982) points out that this form makes the determinations of A and k biased. Therefore, using the suggestion of Parks, the following modification of the Gompertz function is employed as a model for potential growth.

(6.1) $$W_t = A\left(\frac{W_o}{A}\right)^{e^{-kt}}$$

where W_o is defined to be the initial weight and t is defined to be the time that has elapsed since the initial weight was observed. Then, as $t \to \infty$, $W_t \to A$ and at $t = 0$, $W_t = W_o$. The parameter $k > 0$ serves as a shape parameter that influences the slope, curvature, and point of inflection of the sigmoidal curve.

Given output as a function of time as described in equation (6.1), the marginal physical product with respect to time can be derived as a function of time

(6.2) $$\mathrm{MPP}(t) = \frac{\partial W_t}{\partial t} = -k \cdot \ln\left(\frac{W_o}{A}\right) \cdot A \cdot \left(\frac{W_o}{A}\right)^{e^{-kt}} \cdot e^{-kt}$$

or as a function of weight

(6.3) $$\mathrm{MPP}(W_t) = \frac{\partial W_t}{\partial t} = -k \cdot W_t \cdot \ln\left(\frac{W_t}{A}\right).$$

Since the late 19th century, physicists and chemists have been studying differential equations of the type in equation (6.3) (Parks). That is, the rate of change in output W with respect to the independent variable t is uniquely related to the value of W at that t. Scientists argue that this is more likely to lead to natural laws of nature than differential equations of the form expressed in equation (6.2). The argument is that more fundamental information can be gained by comparing marginal products at the same value of output W than at the same value of input t. This concept is adopted here and equation (6.3) together with a constant output price P_w is used to produce the equation for the marginal value product with respect to time as a function of weight.

(6.4) $$\mathrm{MVP}(W_t) = -k \cdot P_w \cdot W_t \cdot \ln\left(\frac{W_t}{A}\right)$$

Therefore, a point on the MVP curve in figure 6.2 represents the marginal value of feeding the animal another unit of time (one day) given its current weight.

Note that $0 < W_t < A$ for all $t < \infty$. Therefore, the MVP is always positive. Also, the second derivative

(6.5)
$$\frac{\partial^2 W_t}{\partial t^2} = k^2 \cdot P_w \cdot W_t \cdot \ln\left(\frac{W_t}{A}\right) \cdot \left[1 + \ln\left(\frac{W_t}{A}\right)\right]$$

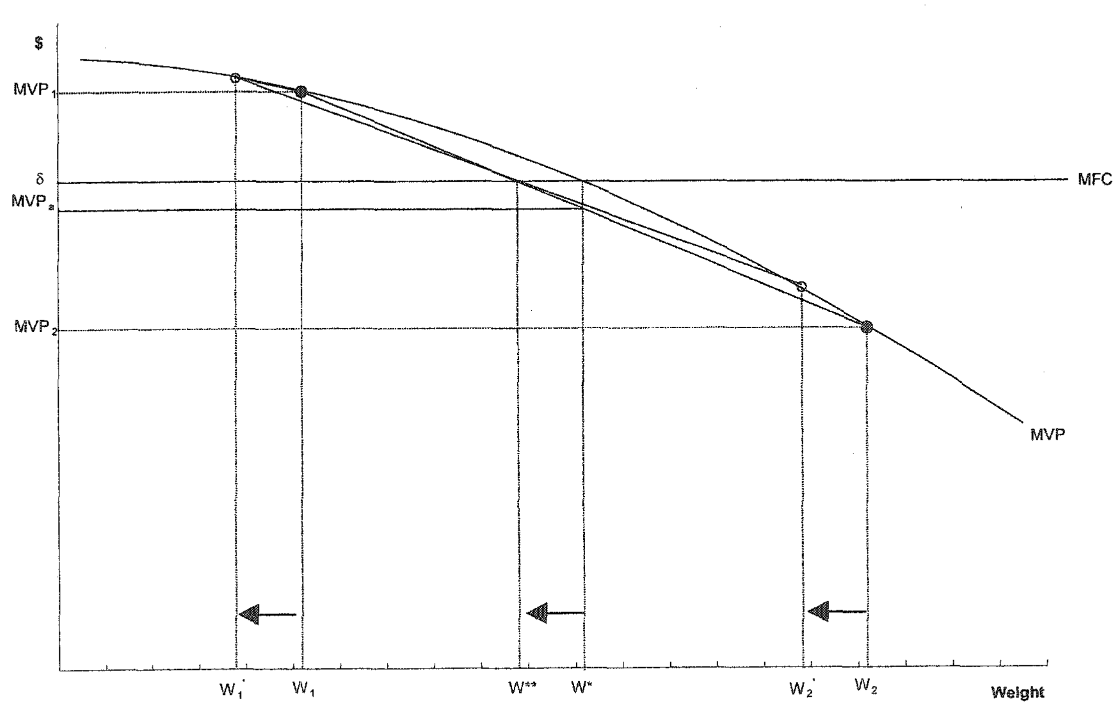
is negative precisely when

(6.6)
$$1 + \ln\left(\frac{W_t}{A}\right) > 0.$$

Therefore, the production function (6.1) is characterized by positive but diminishing marginal returns with respect to time whenever

(6.7)
$$W_t > A \cdot e^{-1}.$$

Figure 6.2 Example marginal function curves with respect to another day on feed.



Furthermore, to analyze the concavity of the MVP curve, calculate a third derivative

(6.8)
$$\frac{\partial^3 W_t}{\partial t^3} = -k^3 \cdot P_w \cdot W_t \cdot \ln\left(\frac{W_t}{A}\right) \cdot \left[1 + 3 \cdot \ln\left(\frac{W_t}{A}\right) + \left[\ln\left(\frac{W_t}{A}\right)\right]^2\right].$$

Equation (6.8) is negative when

(6.9)
$$0.0729A < W_t < 0.6825A.$$

Therefore, under the assumption of a constant output price P_w, the marginal value product curve is concave over the approximate weight region from $0.07A$ to $0.68A$, and convex otherwise.

Jensen's Inequality

To maximize profits from the production of a single animal, the animal is fed until the marginal value product equals the marginal factor cost. Let the unit of time t be days and start with a simplified assumption that the marginal factor cost is represented by a constant δ that captures the daily cost of feeding the animal.

For a single animal with the marginal curves depicted in figure 6.2, the profit maximizing weight to terminate production is represented by W^* where MVP=MFC. Now, assume there are two animals in a pen that are to be marketed together. Let their weights be represented by W_1 and W_2 with $W^* = (W_1 + W_2)/2$. By Jensen's inequality (Jensen), the average of the marginal value product (MVP_a) will be less than the marginal value product of the average weight $(\text{MVP}(W^*) = \delta)$ over any concave region of the marginal value product curve. This difference between $\text{MVP}(W^*)$ and MVP_a is called the *Jensen effect*. In the case of maximizing profits for the pen marketed together, it is the average of the marginal value products that needs to be equated to the constant δ representing the average of the marginal factor costs. Therefore, as figure 6.2 indicates, with two animals in the pen, profits are maximized by shifting the market weight to the left. The result is a lower market weight for each animal (W_1' and W_2', respectively) and a lower average weight W^{**}.

The difference between W^* and W^{**} is called an *aggregation premium*. That is, when the average weight of the animals is W^{**}, there exists an *aggregate equivalent* animal weighing W^* whose marginal value product is equivalent to the average marginal value product for the group. So, accounting for the aggregation premium will lower the optimal average market weight for the animals.

The magnitude of the shift and its subsequent effect on market timing decisions will be influenced by two things, the curvature of the marginal value product curve and the distribution of the animal weights. In (6.9), it was determined that, under the assumption of a constant output price, the MVP curve (6.4) would be concave when the weight is between $0.07A$ and $0.68A$. Whittemore (p. 6) points out that, "prime meat is found from pigs slaughtered between 30% and 60% of mature size." One can conclude that, in the case of swine, it is likely that the MVP curve will be in the latter stages of concavity around the profit maximizing market weight.

All symmetric distributions lying predominantly within the declining concave region can be expected to behave similar to the two animals depicted in figure 6.2. That is, they can be expected to have a negative Jensen effect and a positive aggregation premium. Obviously, the larger the standard deviation of the distribution, the larger the Jensen effect. Thus, the degree of dispersion will affect the magnitude of the aggregation premium and subsequently the difference between W^* and W^{**}. If the distribution is asymmetric, we can expect W^* to lie closer to either W_1 or W_2 where W_1 and W_2 represent the minimum and maximum weights, respectively, in the distribution. Thus, an asymmetric distribution will likely decrease the difference between δ and MVP_a. Finally, if the distribution is pushed further to the right with more of it lying outside of the concave region of the MVP curve, then a further decrease in the Jensen effect can be expected with the possibility existing that at W^*, δ could be less than MVP_a. This would result in a positive Jensen effect and a subsequent negative aggregation premium.

Increasing Marginal Factor Costs

Marginal factor cost has been limited to estimating the daily cost of feed. In figure 6.2, this was naively assumed constant at δ. This served its purpose as a simplifying assumption in the above exposition but, in reality, the daily cost of feeding an animal grows with the size of the animal. One of the "laws of animal science" is the long held belief that to maintain body weight, animals should be fed in proportion to their "metabolic" body size $W^{0.75}$ (Parks; Kleiber). Therefore, a function of the form $F = aW^{0.75}$ seems appropriate where F is daily feed intake, W is the weight of the animal, and a is some constant.

Whittemore (1993) points out that most empirical estimates of feed intake for pigs of various weights involve pigs growing positively. He suggests a value for a between 0.09 and 0.11 when the weight units are measured in kilograms and the pigs are being fed under commercial conditions. Adopting the lower bound and converting to English units leaves a naive but practical formula

$$(6.10) \qquad F = 0.20W^{0.75}$$

to represent pounds of daily feed intake, F, as a function of weight, W. Assuming a constant positive feed price of P_f per pound, the marginal factor cost with respect to time, representing the cost of feeding the animal another day, can be written as

$$(6.11) \qquad \mathrm{MFC}(W_t) = P_f \cdot F = 0.20 \cdot P_f \cdot W_t^{0.75}.$$

Examining the characteristics of the marginal factor cost curve (6.11), first note the obvious that it is positive for all positive values of W_t. Second, note that the marginal factor cost with respect to time is monotonically increasing since

$$(6.12) \qquad \frac{\partial[\mathrm{MFC}(W_t)]}{\partial t} = -0.15 \cdot P_f \cdot k \cdot W_t^{0.75} \cdot \ln\left(\frac{W_t}{A}\right) > 0$$

for all $0 < W_t < A$. Finally, to analyze the concavity of the marginal factor cost curve (6.11), calculate its second derivative

$$(6.13) \qquad \frac{\partial^2[\mathrm{MFC}(W_t)]}{\partial t^2} = 0.15 \cdot P_f \cdot k^2 \cdot W_t^{0.75} \cdot \ln\left(\frac{W_t}{A}\right) \cdot \left[1 + 0.75 \cdot \ln\left(\frac{W_t}{A}\right)\right].$$

Equation (6.13) is positive when

$$(6.14) \qquad 1 + 0.75 \cdot \ln\left(\frac{W_t}{A}\right) < 0$$

or

$$(6.15) \qquad W_t < A \cdot e^{-\frac{4}{3}} \cong 0.2636 \cdot A.$$

Therefore, the marginal factor cost curve (6.11) is convex whenever relationship (6.15) holds and concave otherwise. Applying Whittemore's observation from above, it is then likely that the marginal factor cost curve will be concave over the weight regions in which marketing of swine occurs. Jensen's inequality then presents a situation where the average of the marginal factor cost is expected to be less than the marginal factor cost of the average weight. That is, the Jensen effect on the marginal factor cost is negative.

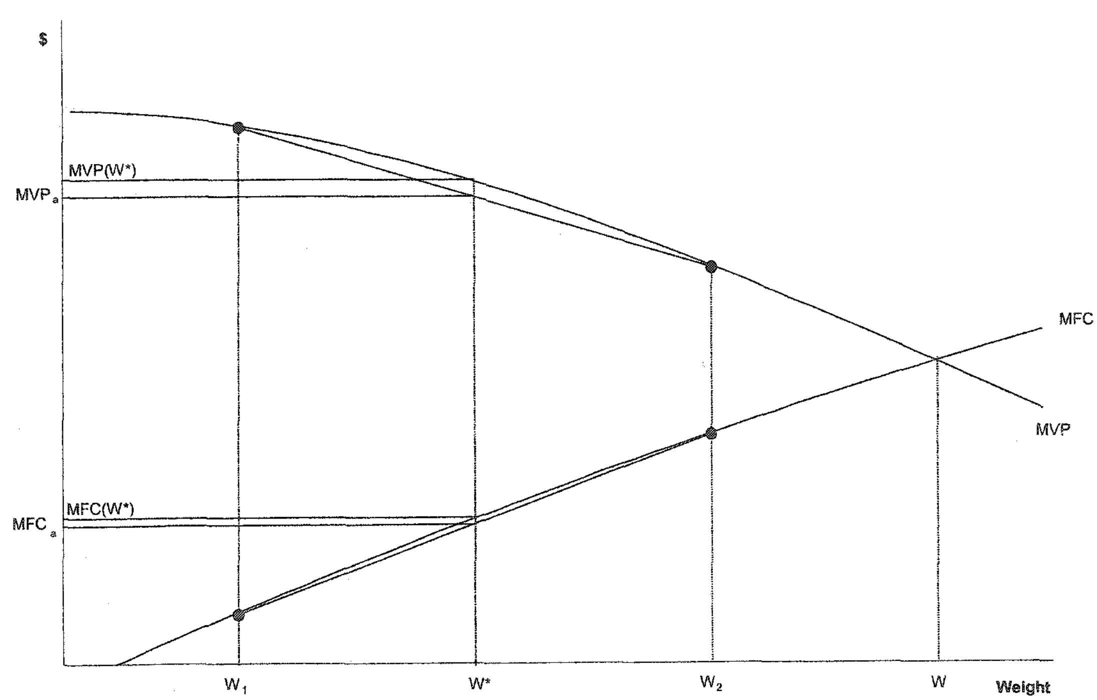
Figure 6.3 depicts the previous situation with two animals weighing W_1 and W_2, respectively. With both the MFC and MVP curves being concave over the applicable region, the negative Jensen effect presents a situation where $MVP_a < \mathrm{MVP}(W^*)$ and $MFC_a < \mathrm{MFC}(W^*)$ with

(6.16) $$MVP_a = 0.5\mathrm{MVP}(W_1) + 0.5\mathrm{MVP}(W_2)$$

and

(6.17) $$MFC_a = 0.5\mathrm{MFC}(W_1) + 0.5\mathrm{MFC}(W_2).$$

Figure 6.3 Example marginal functions with increasing marginal factor costs.



However, since MFC is an increasing function, the aggregation premium associated with its Jensen effect will be negative. That is, the average marginal factor cost is associated with an aggregate equivalent weight that is less than the average weight of the animals in the group. The net effect that all of this has on the marginal profit and the subsequent point of optimality will be determined by the relative curvature of the two curves over the applicable region.

Intuitively, one might expect the situation as it is depicted in figure 6.3 where the curvature of the MVP curve is more pronounced than that for the MFC curve. Then, the average marginal profit at W^*,

(6.18) $$\pi_a = MVP_a - MFC_a$$

would be less than the marginal profit of the average weight,

(6.19) $$\pi(W^*) = \mathrm{MVP}(W^*) - \mathrm{MFC}(W^*).$$

This would lead to the conclusion that the average marginal profit would reach zero prior the weight W at which the marginal profit of the average weight is zero. That is, the positive aggregation premium associated with the MVP curve is higher in magnitude than the negative aggregation premium associated with the MFC curve. As in the case of constant marginal factor costs explored earlier, profits for this pen of two animals with increasing marginal factor costs can be expected to be maximized at an average market weight somewhere to the left of W. However, the counter balancing effect of the negative aggregation premium resulting from an upward sloping concave marginal factor cost curve will make that shift less pronounced than the shift from W^* to W^{**} indicated in figure 6.2.

Empirical Application

A panel data set consisting of twelve weight observations individually identified for 350 hogs every 1-3 weeks from 14 days of age to 171 days of age was obtained from Purdue University. The swine in the data set are all gilts taking part in a Purdue University study on antibiotic treatments. Two different genotypes are represented in the data and the pigs are divided into 32 pens of approximately 10-12 pigs per pen. At any point in time, each pen is receiving the same ration fed ad libitum. Exactly half of the animals are given an antibiotic treatment. However, the selection of the treatment animals is done by random draw at the beginning of the trial and again at the beginning of the finishing phase. Therefore, the animals fall into one of four categories concerning antibiotic treatments: (1) treatment in both the nursery and finishing phase, (2) treatment in the nursery and no treatment in finishing, (3) no treatment in the nursery and treatment in finishing, or (4) no treatment in either the nursery or finishing.

The data set is first analyzed to determine one growth path for the average weight of the entire set of 350 hogs. The data set was plagued by a common problem in animal growth modeling. The fastest growing pigs were marketed prior to the twelfth weight observation resulting in a significant amount of missing data. Including all twelve observations to estimate the model parameters would downwardly bias the peak of the sigmoidal growth curve (Craig and Schinckel). Therefore, the group average from observation twelve was not used in the growth curve estimation.

Using the mean values for the entire group at each of the first eleven observations, a Gompertz growth curve (6.1) is fitted to the data. This resulted in the model

(6.1)' $$W_t = 370\left(\frac{12.55}{370}\right)^{e^{-0.0148t}} \qquad (R^2 = 0.9999)$$

as a representation of the growth path of the pen average. The fitted curve from equation (6.1)' is graphed along with the actual data of mean weights in figure 6.4.

The growth curve parameters $A = 370$ and $k = 0.0148$ resulting from the estimation of (6.1)', combine to yield the marginal value product and marginal factor cost equations

(6.4)' $$\mathrm{MVP}(W_t) = -0.006512 \cdot W_t \cdot \ln\left(\frac{W_t}{370}\right)$$

(6.11)' $$\mathrm{MFC}(W_t) = 0.012 \cdot W_t^{0.75}$$

where the output price is assumed constant at P_w = \$0.44 per pound and the feed cost is assumed constant at P_f = \$0.06 per pound. These are graphed in figure 6.5. Solving numerically for the point of intersection in figure 6.5, the result is $W = 230.58$. This represents the profit maximizing market weight for a single average animal. It corresponds to $t = 132.96$ or approximately 147 days of age.

The data set contains an observation of the actual weights at 146 days of age. A Chi-square analysis provides strong evidence to not reject the null hypothesis that these weights are normally distributed (figure 6.6). Analysis of weight data at 132 days of age and 153 days of age also provided evidence of normally distributed weights (p-values of 0.903 and 0.174, respectively). Therefore, the following optimization will be done under the assumption that the animal weights are normally distributed with a mean weight of W_t determined by model equation (6.1)' and a standard deviation of 21.4.

Figure 6.4 Fitted growth path for the average swine weight in the sample data.

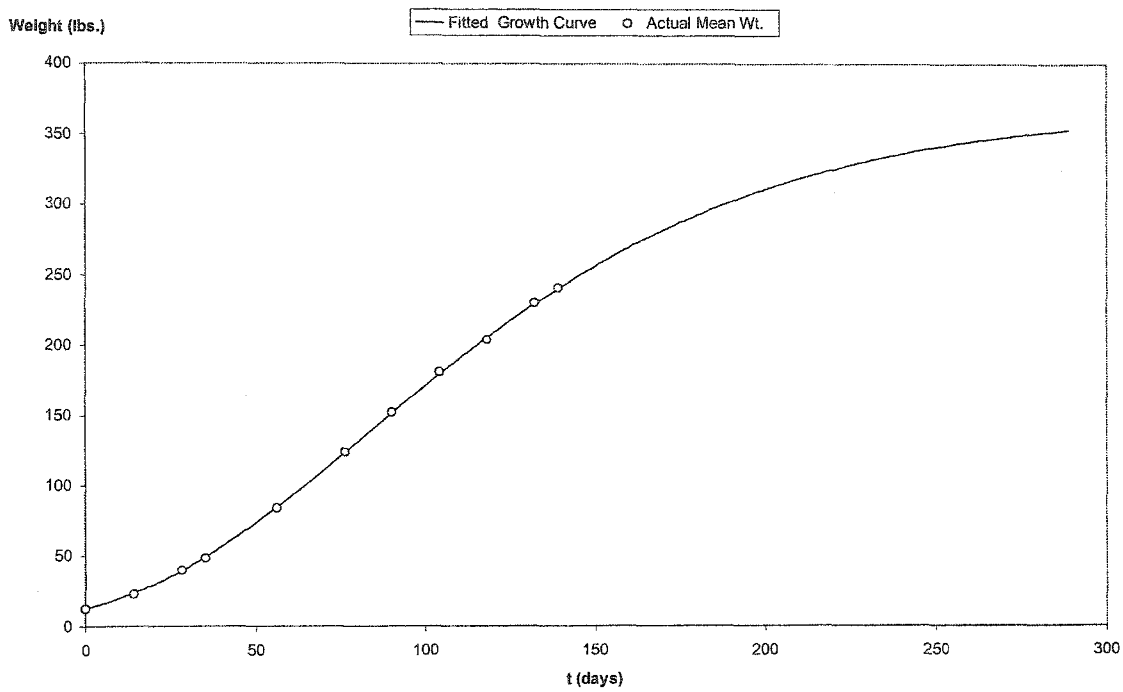


Figure 6.5 Graphical representation of the optimization conditions.

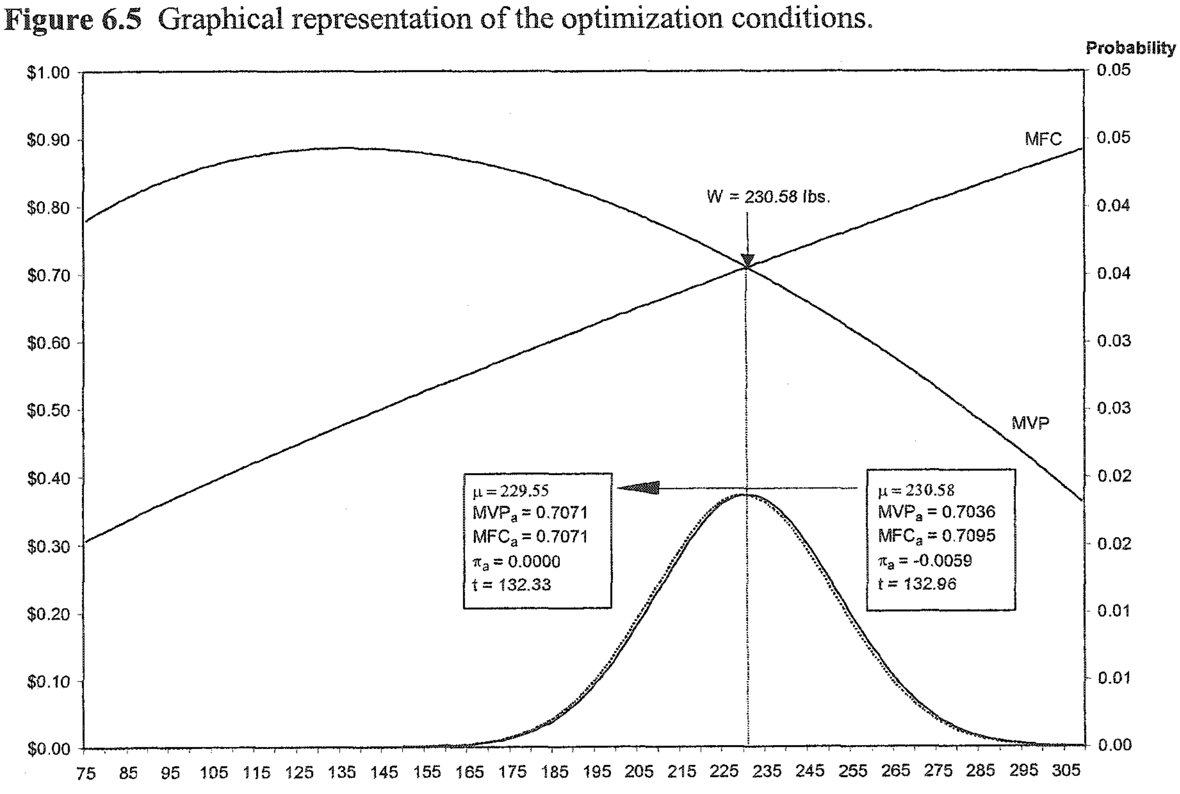
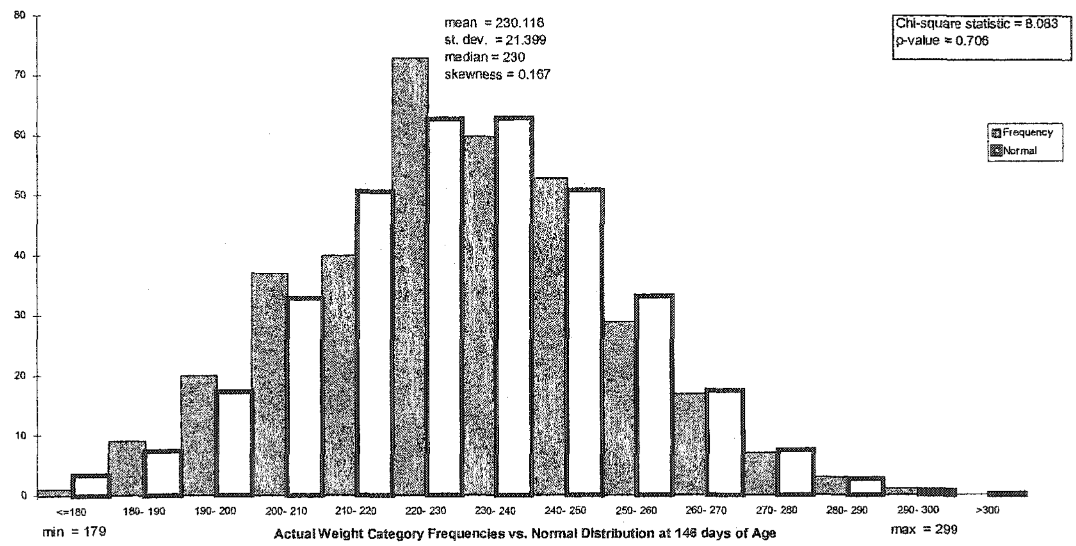


Figure 6.6 Weight distribution at 146 days of age and Chi-square analysis.



Profit maximization occurs when the average marginal profit, $\pi_a = MVP_a - MFC_a$, is equal to zero. In other words, the optimization problem is to determine the mean weight W_t such that

$$(6.18)' \qquad \pi_a = \int_{W=W_o}^{W=A} \pi(W)\cdot \mathrm{N}(W|W_t,\sigma)dW = 0$$

where $\mathrm{N}(W|W_t,\sigma)$ is a normal probability distribution of W with a mean of W_t and a standard deviation of $\sigma = 21.4$. Dillon and Anderson (1990) point out that only if the probability distribution is of simple form, such as discrete or triangular, is an algebraic expression such as (6.18)$'$ conveniently appraised. We were able to appraise it using the symbolic computational package called *Maple* and numerically find the W_t that made equation (6.18)$'$ hold. However, it was easier to analyze and conduct a sensitivity analysis on the results by converting the normal distribution in (6.18)$'$ into a discrete distribution in an *Excel* spreadsheet. The results were identical, to three decimal places, to those obtained in *Maple*. The details of this conversion are contained in the Appendix and the reader should assume that all results reported here were arrived at using the *Premium Solver for Excel*.

Results

The results indicate that the optimum mean weight is indeed less than the 230.58 lbs. at which the marginal value product curve intersects the marginal factor cost curve. Figure 6.5 shows the implied shift to the left from a mean weight of 230.58 to a mean weight of 229.55 lbs. that is necessary to optimize profits for this group of 350 swine sold as one unit. The point of intersection is calculated as MVP(230.58) = MFC(230.58) = 0.7101, $MVP_a = 0.7036$, and $MFC_a = 0.7095$ dollars per day. The relationship,

$$MVP_a < MFC_a < \mathrm{MVP}(230.58) = \mathrm{MFC}(230.58) \tag{6.20}$$

indicates the concavity of both marginal curves with the marginal value product curve slightly more concave than the marginal factor cost curve. In fact, the closeness of MFC_a to the value of MFC(230.58) indicates that the marginal factor cost curve is nearly linear. Most importantly, however, relationship (6.20) indicates the suboptimality of feeding to a mean weight of 230.58 lbs. The fact that $MVP_a < MFC_a$ at a mean weight of 230.58 lbs. indicates that the group has been fed past the point of profit maximization. How far past is determined by solving equation (6.18)′ for the mean weight W_t.

Solving equation (6.18)′ determines an optimum mean weight of 229.55 lbs. This produces the calculated values $MVP_a = MFC_a = 0.7071$, $\mathrm{MFC}(229.55) = 0.7077$, and

$\mathrm{MVP}(229.55) = 0.7136$. Again, this indicates the relative concavity of the two curves. However, it also displays the difference between the calculated marginal profit, $\pi_a = 0$, and the perceived marginal profit, $\pi(229.55) = 0.7136 - 0.7077 = 0.0059$, indicated by the average weight animal.

The final task is to determine what difference this approximately one pound difference in mean weight makes in the market timing decision. Plugging a mean weight of $W_t = 229.55$ into equation $(6.1)'$ and solving for t yields the market timing of $t = 132.33$ days. This represents an approximately seven-tenths of day difference in the market timing obtained at a mean weight of 230.58 lbs. In other words, at $t = 132.33$ days, the marginal profit to be gained by feeding the pen of animals one more day is zero. Obviously, one would not expect the seven-tenths of a day difference between optimal market timing for the group and optimal market timing for the average animal to significantly impact profits. However, one can envision where relaxation of some of the restrictions of this model as it pertains to swine could lead to situations where this gap is more significant.

Sensitivity Analysis

Based on these results, swine would not justify the cost of accounting for the entire distribution of animals. However, this has a lot to do with the particular circumstances of the swine industry. For example, if the top end of the pen distribution hit weights that received price penalties, it would be very important to determine the tradeoff of discounted hogs to gains in marginal value product by holding one more day. This analysis did not account for weight price penalties because they vary substantially.

However, to illustrate other points, let us relax some of assumptions that were particular to the hog industry in the sensitivity analysis that follows. In particular let us look at the impacts of changing the dispersion level, the location and concavity of the MVP and MFC at equilibrium, and of accounting for every animal individually instead of addressing dispersion through statistical statements about the group.

Dispersion

The baseline example for hogs turned out to show that market timing based on the average size is probably a sufficient decision rule. However, how would the market timing change for a pen that is more heterogeneous, such as is commonly seen with cattle or with smaller operations? One way to represent more heterogeneity is by expanding the variance in our model. In the previous example, it was assumed that the standard deviation was constant at 21.4. Table 6.1 summarizes the results if the standard deviation were assumed to be held constant at 15, 20, 25, 30, or 35 lbs.

Note that even with a standard deviation of 35 lbs., the difference in the optimal market timing is less than two days. The timing differential does appear to be growing as

Table 6.1 Sensitivity to Dispersion

	Standard Deviation				
	15	20	25	30	35
W*	230.58	230.58	230.58	230.58	230.58
MVP(W*)	0.7101	0.7101	0.7101	0.7101	0.7101
MFC_a	0.7098	0.7096	0.7093	0.7089	0.7085
MVP_a	0.7069	0.7044	0.7012	0.6973	0.6926
W**	230.08	229.68	229.16	228.52	227.76
Δt	-0.31	-0.55	-0.87	-1.27	-1.73

the dispersion is widened but the growth is relatively slow. There are limits to how large one can make the standard deviation and still make logical sense in the context presented here but other scenarios may not be so limited.

Location

The model results were also limited by the fact that marginal curves intersect at approximated $0.62A$. This is a relatively linear portion of the MVP curve. How would the results change if the intersection occurred in a more concave region of the MVP curve? To test the sensitivity of the results to the location of the intersection, the price of feed is varied from \$0.06 per pound to \$0.10 per pound in table 6.2. One can see that even with the intersection occurring in the very concave region of 155 pounds, the timing differential Δt is only about eight-tenths of a day.

The results do appear to be more sensitive to changes in dispersion than to changes in location. However, the timing differential is not very large in either case individually. One would expect that if these two changes occurred simultaneously, their effects would compound and, in fact, they do. However, even with P_f = \$0.10 and the

Table 6.2 Sensitivity to Location

	P_f				
	0.06	0.07	0.08	0.09	0.10
W*	230.58	210.40	191.07	172.58	154.93
MVP(W*)	0.7101	0.7734	0.8223	0.8571	0.8783
MFC_a	0.7095	0.7727	0.8213	0.8558	0.8767
MVP_a	0.7036	0.7663	0.8144	0.8484	0.8686
W**	229.55	209.25	189.79	171.15	153.31
Δt	-0.63	-0.65	-0.68	-0.73	-0.81

standard deviation equal to 35 pounds, Δt is only –2.46 days. A more compelling question might be: What is the value associated with shifting from W^* to W^{**}? This issue is examined in the next section.

Comparison Across Levels of Information

The theoretical discussion was started by defining three levels of information: basic, intermediate, and full. Basic information leads to the solution depicted as W^* in figure 6.2. The model presented above incorporates intermediate information by using knowledge of the distribution to arrive at the solution W^{**}. Now, define the solution arrived at by incorporating full knowledge as W^{***}. The objective of this section is to assess the marginal effect these increasing levels of knowledge have on net profit.

To do this, growth function (6.1) was estimated for each of the 350 individual animals. Then values for W^*, W^{**}, and W^{***} were determined for each of the 32 pens. A net profit per animal for each pen i was calculated at each of the three solution weights and the results defined as π_i^*, π_i^{**}, and π_i^{***}, respectively. Calculating the average results across all pens produced a net profit per animal per pen. These values are summarized in Table 6.3 as π^*, π^{**}, and π^{***}, respectively.

Table 6.3 Comparison of average net profit per head incorporating different levels of information in the profit maximization model. (Assuming P_w = \$0.44 and P_f = \$0.06.)

π - max	π^*	π^{**}	π^{***}
\$ 51.22	\$ 50.40	\$ 50.48	\$ 50.57

Table 6.3 compares these results to the theoretical profit per head maximum, π-max, which was calculated assuming each animal could be sold as an individual at its profit maximizing weight. Note that incorporating only a basic level of information in determining the sell date for whole pens, the net profit per head is $50.40. This represents an $0.82 drop from the maximum profit obtained by selling individual animals. Incorporating an intermediate level of information and using our model to determine the whole pen sell dates, raised net profit per head to $50.48. This represents a marginal gain of only $0.08 per head. However, even using full information to determine sell dates, the best one can hope for when selling whole pens is a net profit per head of $50.57. This represents a marginal gain of just $0.17 per head over using only basic information to determine sell dates.

An analysis of variance on the data leads to the conclusion that π^*, π^{**}, and π^{***} are not significantly different from one another. Therefore, while our model incorporating intermediate information does not produce results significantly different from a model using only basic information, it does not appear that significant results are there to be found. Even using full information, the results are not significantly different and the intermediate result appears to capture approximately half of what there is to gain.

Summary and Conclusions

The concept of variability in livestock pens is being addressed through ultrasounding and video taping animals (e.g. Brethour, 1989). However, no literature could be found that expresses the problem of variability in a flexible, formal manner that would allow someone to determine which method of addressing variability is superior. This research provides useful insight into the microeconomics surrounding the optimal market timing for whole pens of livestock. In the presence of a nonlinear marginal value product curve, the marginal value associated with the average output level is not representative of the average marginal value product for the pen. The degree of this separation is dependent upon the degree of concavity in the marginal value product curve and the degree of dispersion associated with the distribution of output levels existing in the pen. This separation may be partially offset by an analogous concavity in the marginal factor cost curve associated with the decreasing marginal increase in the cost of feeding a growing animal. The net effect can be expected to be such that the optimal market timing for the pen taken as a whole arrives prior to the optimal market timing for the average sized animal in the pen.

This empirical application to swine verified our theoretical construct incorporating knowledge of the current output distribution with knowledge of the average growth path to determine market dates for whole pens. However, the result was insignificantly different in terms of optimal market timing and profit per head when compared to a model using only the basic knowledge of the average growth path. However, even using the full information of individual growth paths, the gains were not significant and our model incorporating an intermediate level of information appears to capture approximately half the gains that are possible.

The insignificance of the differential in market timing for our baseline case study data is not totally unexpected. The swine industry has homogenized the genetics to the point that few distinguishable breeds exist in the feeding sector. Therefore, one would expect the average pig to be very representative of the group. Furthermore, the timing of the optimum market weight within the growth cycle of a pig is such that the concavity of the marginal curves is minimal. And finally, the direction of the error in the MVP was offset by the direction of the error in MFC.

The theoretical construct of our model appears to be sound. It just so happens that for swine, and particularly our data set, marketing groups of animals based on the average curves appears to be an economically sound technique. Future research applying the principals of our model to more diverse production populations may yield more significant insights into market timing decisions.

Chapter 7

Summary and Concluding Remarks

If there are two things that one can be certain of in this world they are that, first, the world is filled with uncertainty and, second, nonlinear relationships are everywhere. The concept of diminishing marginal returns has been around for a long time dating back to at least Daniel Bernoulli's (1738) risk-averse utility function. The concept is so commonplace that economists sometimes downplay its impact by treating it in a routine fashion in a world filled with certainty. However, those diminishing marginal returns result in nonlinear relationships that interact with a real world that is filled with uncertainty. This interaction provides an abundance of opportunities to apply Jensen's inequality and the concepts developed in this dissertation.

It was not the intent of this dissertation to revolutionize the production of livestock. In the examples of swine and cattle shown here, the impact of Jensen's inequality was relatively modest. Nevertheless, in a large operation, even a modest impact can multiply into a significant economic benefit. In chapter six, I showed through a full economic analysis that a one-half day difference in market timing may be enough to account for the effect of Jensen's inequality in the production of a group of swine. In chapter five, I showed that a three-day difference might be more appropriate for cattle. These differences are not enough to revolutionize an industry. However, the direction of these differences is significant and important because of its consistency. For both cattle and swine, the results indicate that earlier market timing at lighter weights may be appropriate to account for the Jensen effect in group optimization.

By accounting for the entire distribution as output from production, this dissertation addressed the issue of variable productivity from a different perspective than it has been done in the past. This approach seems appropriate in agriculture and a number of empirical settings where production is done in groups. Optimizing the time on feed for a group of livestock based upon the productivity of the expected output ignores the productivity information contained in the distribution itself. I showed that there is an economic penalty for ignoring this information.

My objective was not to derive specific economic significance resulting from employing these concepts in the livestock industry but rather to develop the microeconomic principles necessary to discuss it. At the center of the theory developed in this dissertation are Jensen's inequality and the nonlinear marginal value functions. In chapter four, I introduced the terminology of the Jensen effect to describe the difference between the average marginal value for a group and the marginal value for the average individual of the group. I related the Jensen effect to an aggregation premium. It is the aggregation premium that explains the difference between the average individual of a group and the aggregate equivalent individual that represents the average marginal productivity of the group. In the livestock application examples, it is the aggregation premium that explains the difference between the lighter average weight at the earlier market timing determined by optimal conditions and the heavier average weight associated with the point of intersection of the marginal curves.

Knowledge of the aggregation premium is important to any decision-maker seeking a group optimization result. In the presence of a decreasing, concave marginal value curve with respect to a random variable, the decision-maker knows that the aggregation premium will be positive and increasing with respect to increasing uncertainty in the random variable. Therefore, if the uncertainty increases, the optimizing decision-maker should decrease output to compensate for the larger aggregation premium. Unfortunately, in the livestock examples developed in this dissertation, the aggregation premium was relatively small. The best explanation for that is to look at the concavity of the marginal curves. For both cattle and swine, the marginal curves are relatively linear around the time they are harvested. This near linearity results in a very small Jensen effect and, subsequently, a small aggregation premium. Subsequently, the aggregate equivalent individual for the group differed very little from the average individual in the group.

Although the impacts in the liveweight examples presented here turned out to be small, there may be other applications within the livestock industry of more significance. However, it is likely that the most significant applications of these concepts will lie outside of the production livestock arena. In chapter two, I outlined some of the areas where Jensen's inequality has already been cited as having a significant impact. These areas include risk-sensitive behavior such as that exhibited by the foraging animals in the experiments of Caraco, Martindale, and Whittam (1980) and the ecological impacts of variable conditions. Policies and actions taken based on expected values may miss their mark in terms of impact if Jensen's inequality is not accounted for in the process. It is my hope that the concepts developed in this dissertation will help in that evaluation process.

For example, suppose a policymaker wishes to reduce pollution by cutting down on emissions from certain industrial plants. Suppose the proposed instrument for effecting this reduction is a new technology that all plants will be required to adopt. Furthermore, suppose that the effectiveness of this new technology is dependent upon the weather which, of course, is full of uncertainty. If the relationship between the weather and the marginal effectiveness of the new technology is nonlinear, then a policy based upon expected weather conditions may lead to erroneous results. If based upon expected weather conditions, calculations of the effectiveness of another marginal unit of regulation implemented through the new technology will differ from its expected marginal value. With knowledge of the Jensen effect and the nonlinear relationship, the policymaker can adjust the regulation up or down to account for an aggregation premium and implement a policy better suited to achieve the stated goals.

There are other applications within production agriculture that could prove to be important. For example, in crop production anhydrous ammonia is often applied as a fertilizer. A manifold is included on the application equipment to regulate the flow of the fertilizer to the applicator knives that incorporate the fertilizer into the soil. Tests conducted over the past few years have shown that some applicator knives will receive up to two to three times the flow of fertilizer as other applicator knives on the same piece of equipment (Fee). The problem is in the manifold's ability to regulate the flow of anhydrous ammonia, which becomes a combination of liquid and gas as it enters the manifold itself.

Improvements are being made in manifold technology. However, the prudent producer should account for these uncertainties when making application decisions. A nonlinear marginal value curve with respect to fertilizer as an input would indicate the presence of the Jensen effect. Therefore, the marginal value of the expected application rate will differ from the expected marginal value from the distribution of application rates that actually exist. Adjusting the mean application rate by an associated aggregation premium will result in the optimality conditions being met that maximize the expected returns to the producer.

Improvements in technology are being made every day. The improvements made in the genetics and production systems used in the livestock industry over the last forty years have been staggering. These improvements will continue to have profound effects on future profits to the industry. Somewhere along the way, Jensen's inequality may play a more significant role in livestock production. However, because of the near linearity of the marginal product curves around the time of harvest, the Jensen effect had a relatively minor impact on the optimality conditions of the examples presented in this dissertation. Nevertheless, the livestock production setting proved to be a convenient platform from which to develop the theory surrounding the Jensen effect and related aggregation concepts. No matter how improved technology becomes, a person only need to look outside at the weather to know that some uncertainty will always exist. It is my hope that the concepts and terminology introduced in this dissertation will aid in the discussion and further research into variable productivity and the effect of Jensen's inequality. Doing so will ensure that the goal of this dissertation has been met.

References

Abel, A. B. "Optimal Investment Under Uncertainty." *The American Economic Review.* Volume 73, Issue 1 (March 1983): pp. 228-233.

Agnihothri, S., J. S. Lee, and J. Kim. "Lot sizing with random yields and tardiness costs." *Computers and Operations Research.* 27 (2000): pp. 437-459.

Basarab, J.A., J.R. Brethour, D.R. ZoBell, and B. Graham. "Sorting feeder cattle with a system that intergrates ultrasound backfat and marbline estimates with a model that maximizes feedlot profitability in value-based marketing." *Can. J. Anim. Sci.* 79(1999): pp. 327-334.

Basarab, J.A., D. Milligan, J.J. McKinnon, and B.E. Thorlakson. "Potential use of video imaging and real-time ultrasound on incoming feeder steers to improve carcass uniformity." *Can. J. Anim. Sci.* 77(1997): pp. 385-392.

Batra, R.N. and A. Ullah. "Competitive Firm and the Theory of Input Demand under Price Uncertainty." *The Journal of Political Economy.* Volume 82 Issue 3 (May-June 1974): pp. 537-548.

Behan, D. F. "Determining the Economic Value of Human Life." Available at http://www.behan.ws/lifevalue.htm.

Bernoulli, D. "Exposition of a New Theory on the Measurement of Risk." *Econometrica.* Volume 30, Issue 1 (January 1954): pp. 23-36. Translation of Bernoulli, D. "Specimen theoriae novae de mensura sorties." *Papers Imp. Acad. Sci. St. Petersburg.* 5 (1738): pp. 175-192.

Bertalanffy, L. von. "Quantitative Laws in Metabolism and Growth." *The Quarterly Review of Biology.* Volume 32, Number 3 (September 1957): pp. 217-231.

Boland, M.A., P.V. Preckel, and A.P. Schinckel. "Optimal Hog Slaughter Weights Under Alternative Pricing Systems." *J. Agr. and Applied Econ.* Volume 25, Number 2 (December, 1993): pp. 148-163.

Breiman, L. "Investment Policies for Expanding Businesses Optimal in a Long-run Sense." *Nav. Res. Logistics Q.* 7 (1960): pp. 647-651.

Breiman, L. "Optimal Gambling Systems for Favorable Games." In *Fourth Berkeley Symposium on Mathematical Statistics and Probabiliity*, pp. 65-78. Berkeley: University of California Press, 1961.

Brethour, J.R. "Using Ultrasound Technology to Increase Cattle Feeding Profit." *Kansas Agricultural Experiment State Report of Progress 570.* 1989.

Brody, S. *Bioenergetics and Growth.* New York: Rheinhold Pub. Corp., 1945.

Brown, J. E., H. A. Fitzhugh, and T. C. Cartwright. "A comparison of nonlinear models for describing weight-age relationships in cattle." *Journal of Animal Science.* 42(1976): pp. 810-818.

Caraco, T. "On Foraging Time Allocation in a Stochastic Environment." *Ecology.* Volume 61, Number 1 (February 1980): pp. 119-128.

Caraco, T., S. Martindale, and T. S. Whittam. "An Empirical Demonstration of Risk-Sensitive Foraging Preferences." *Anim. Behav.* 28 (1980): pp. 820-830.

Chavas, J. P., J. Kliebenstein, and T. D. Crenshaw. "Modeling Dynamic Agricultural Production Response: The Case of Swine Production." *Amer. J. Agr. Econ.* 67(1985): pp. 636-646.

Crabtree, J.R. "Feeding Strategy Economics in Bacon Pig Production." *J. Agr. Econ.* 28(1977): pp. 39-54.

Craig, B.A. and A.P. Schinckel. "Nonlinear Mixed Effects Model for Swine Growth." *The Professional Animal Scientist*, 17(2001):256-260.

Dillon, J.L. and J.R. Anderson. *The Analysis of Response in Crop and Livestock Production. 3rd Ed.* Oxford: Pergamon Press, 1990.

Elton, E. J. and M. J. Gruber. "On the Maximization of the Geometric Mean with Lognormal Return Distribution." *Management Science.* Volume 21, Number 4, Application Series (December 1974): pp. 483-488.

Fee, R. "New NH_3 manifolds do better." *Successful Farming.* Volume 101, Number 1 (January 2003): 46-48.

Gompertz, B. "On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Method of Determining the Value of Life Contingencies." *Philosophical Transactions of the Royal Society of London.* 115(1825): pp. 513-583.

Grauer, R. R. "A Comparison of Growth Optimal and Mean Variance Investment Policies." *The Journal of Financial and Quantitative Analysis.* Volume 16, Issue 1 (March 1981): pp. 1-21.

Hakansson, N. H. "Capital Growth and the Mean-Variance Approach to Portfolio Selection." *The Journal of Financial and Quantitative Analysis*. Volume 6, Number 1 (January 1971): pp. 517-557.

Hansen, F. "Operator Inequalities Associated with Jensen's Inequality." In *Survey on Classical Inequalities*, T. M. Rassias (ed.), pp. 67-98, The Netherlands: Kluwer Academic Publishers, 2000.

Hardaker, J. B., R. B. M. Huirne, and J. R. Anderson. *Coping with Risk in Agriculture*. United Kingdom: CAB International, 1997.

Hardy, G. H., J. E. Littlewood, and G. Polya. *Inequalities*. London: Cambridge University Press, 1989.

Hartman, R. "The Effects of Price and Cost Uncertainty on Investment." *Journal of Economic Theory*. 5 (1972): pp. 258-266.

Heady, E. O., S. T. Sonka, and F. Dahm. "Estimation and Application of Gain Isoquants in Decision Rules For Swine Producers." *J. Agr. Econ.* 27(1976): pp. 235-42.

Hölder, O. "Ueber einen Mittelwertsatz." *Göttinger Nachr.* (1889): pp. 38-47.

Horowitz, I. *Decision Making and the Theory of the Firm*. New York: Holt, Rinehart, and Winston, Inc., 1970.

Jean, W. H. "The Geometric Mean and Stochastic Dominance." *The Journal of Finance*. Volume 35, Number 1 (March 1980): pp. 151-158.

Jensen, J. L. "Sur les functions convexes et les inégualités entre les valeurs moyennes." *Acta Math.* 30 (1906): pp. 175-193.

Kleiber, M. *The Fire of Life*. New York, NY: Wiley, 1961.

Latané, H. A. "Criteria for Choice Among Risky Ventures." *The Journal of Political Economy*. Volume 67, Issue 2 (April 1959): pp. 144-155.

Latané, H. A. "Individual Risk Preference in Portfolio Selection." *The Journal of Finance*. Volume 15, Number 1 (March 1960): pp. 45-52.

Le Dividich, J. "A Review – Neonatal and Weaner Pig: Management to Reduce Variation." In *Manipulating Pig Production VII*, P.D. Cranwell (Ed.), pp. 135-155, Werribee, Victoria 3030, Australia: Aust. Pig Sci. Assoc., 1999.

Markowitz, H. "Portfolio Selection." *The Journal of Finance*. Volume 7, Number 1 (March 1952): pp. 77-91.

McCall, J. J. "Probabilistic Microeconomics." *The Bell Journal of Economics and Management Science*. Volume 2, Issue 2 (Autumn, 1971): pp. 403-433.

Mittlehammer, R. C. *Mathematical Statistics for Economics and Business*. New York: Springer-Verlag, 1996.

Mossin, J. *Theory of Financial Markets*. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1973.

Nelder, J. A. "The Fitting of a Generalization of the Logistic Curve." *Biometrics*. Volume 17, Issue 1 (March 1961): pp. 89-110.

Oliver, F. R. "Another Generalisation of the Logistic Growth Function." *Econometrica*. Volume 37, Issue 1 (January 1969): pp. 144-147.

Parks, J. R. *A Theory of Feeding and Growth of Animals*. New York: Springer-Verlag, 1982.

Parsons, C. Personal communication. October 17, 2001.

Pope, R.D. and R.A. Kramer. "Production Uncertainty and Factor Demands for the Competive Firm." *Southern Economic Journal*. 46(1979):489-501.

Pope, III, C.A. and C.R. Shumway. "Management of Intensive Forage-Beef Production Under Yield Uncertainty." *Southern Journal of Agricultural Economics*. December, 1984: pp. 37-43.

Pringle, L. "Operations Research: The Productivity Engine." *ORMS Today*. Volume 27, Number 3 (June 2000): pp. 28-31.

Real, L. "On uncertainty and the law of diminishing returns in evolution and behavior." In *Limits to action: The allocation of individual behavior*, J. E. R. Staddon (ed.), pp. 37-64. New York: Academic Press, 1980.

Richards, F. J. "A Flexible Growth Function for Empirical Use." *Journal of Experimental Botany*. Volume 10, Number 29 (June 1959): pp. 290-230.

Robison, L.J. and P.J. Barry. *The Competitive Firm's Response to Risk.* New York: MacMillan Publishing Company, 1987.

Rosenblatt, M. J. and H. L. Lee. "Economic Production Cycles with Imperfect Production Processes." *IIE Transactions*. 18 (March 1986): pp. 48-55.

Rothschild, M. and J. E. Stiglitz. "Increasing Risk: I. A Definition." *Journal of Economic Theory*. 2 (1970): pp. 225-243.

Ruel, J. J. and M. P. Ayres. "Jensen's inequality predicts effects of environmental variation." *TREE.* Volume 14, Number 9 (September 1999): pp. 361-366.

Samuelson, P. A. "Lifetime Portfolio Selection By Dynamic Stochastic Programming." *The Review of Economics and Statistics*, Volume 51, Issue 3 (August 1969): pp. 239-246.

Sandmo, A. "On the Theory of the Competitive Firm Under Price Uncertainty." *The American Economic Review.* Volume 61, Issue 1 (March 1971): pp. 65-73.

Savage, L. J. *The Foundations of Statistics*. New York: John Wiley & Sons, Inc., 1954.

Schinckel, A. P. Personal communication. February 28, 2000.

Small, J. P. "The Timing and Scale of Investment Under Uncertainty." CRNEC, University of Auckland, June 1, 1999.

Smallwood, P. D. "An Introduction to Risk Sensitivity: The Use of Jensen's Inequality to Clarify Evolutionary Arguments of Adaptation and Constraint." *Amer. Zool.* 36 (1996): pp. 392-401.

Smith, G.C., J.W. Savell, H.G. Dolezal, T.G. Field, D.R. Gill, D.B. Griffin, D.S. Hale, J.B. Morgan, S.L. Northcutt, and J.D. Tatum. *Improving the Quality, Consistency, Competitiveness and Market-Share of Beef: Executive Summary*. National Beef Quality Audit. Colorado State University, Oklahoma State University and Texas A&M University, December 1995.

Stearns, S. C. "Daniel Bernoulli (1738): evolution and economics under risk." *J. Biosci.* Volume 25, Number 3 (September 2000): pp. 221-228.

Walker, J. L. "Economic Returns to Ultrasound Technology in the Timing and Sorting of Feedlot Cattle: A Study in Value-Based Marketing." *Masters Thesis*. Colorado State University, Fort Collins, CO. 1999.

Walters, A.A. "Marginal Productivity and Probability Distributions of Factor Services." *The Economic Journal.* Volume 70, Issue 278 (June 1960): pp. 325-330.

Wang, Chia-Hsing and B. Roe. "Profit Maximization Behavior On the Fed Cattle Grid: The Value of Pre-harvest and Genetic Sorting with Implications for Aggregate Supply." Paper presented at the *American Agricultural Economics Association Annual Meetings*, Chicago, IL. August 5-8, 2001.

Whittemore, C. *The Science and Practice of Pig Production*. Essex, England: Longman Scientific & Technical, 1993.

Winsor, C. P. "The Gompertz Curve as a Growth Curve." *Proceedings of the National Academy of Sciences of the United States of America*. Volume 18, Issue 1 (Jan. 15, 1932): pp. 1-8.

Yao, D. D. "Optimal Run Quantities for an Assembly System with Random Yields." *IIE Transactions*. Volume 20, Number 4 (December 1988): pp. 399-403.

Appendix

Conversion of the Normal Distribution into Discrete Form

The normal distribution in equation (6.18)′ was converted into discrete form by calculating

$$\mathrm{N}_w(W_t,\sigma)=\int_{W=W-0.5}^{W=W+0.5}\mathrm{N}(W|W_t,\sigma)dW$$

for $W=W_o$ to A. Then the average of the marginal value products and marginal factor costs can be calculated as

$$MVP_a(W_t)=\sum_{W=W_o}^{W=A}\mathrm{MVP}(W)\cdot\mathrm{N}_w(W_t,\sigma)$$

$$MFC_a(W_t)=\sum_{W=W_o}^{W=A}\mathrm{MFC}(W)\cdot\mathrm{N}_w(W_t,\sigma)$$

with the marginal profit

$$\pi_a(W_t)=MVP_a(W_t)-MFC_a(W_t).$$