DISSERTATION

VARIATIONS ON METHODS OF LORENTZ AND LORENTZ FOR DIMENSIONS TWO AND THREE

Submitted by

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In partial fulfillment of the requirements

for the degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2003

UMI Number: 3107076

UMI Microform 3107076

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300 North Zeeb Road
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May 12, 2003

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ABSTRACT

VARIATIONS ON METHODS OF LORENTZ AND LORENTZ FOR DIMENSIONS TWO AND THREE

This dissertation will address the problem of polynomial interpolation: finding a polynomial $P(\underline{x})$, which goes through points p_i with multiplicity m_i at each point. Although polynomials are the building block for many numerical methods, such as finite elements and splines, and theorems about approximation of functions or numerical schemes almost always reduce to local interpolation by polynomials, the theory is underdeveloped.

The general problem of computing the dimension of a space of polynomials satisfying certain multiplicity conditions at a set of general points can be formulated in any dimension. This problem, in its most general form, is still unsolved. The only statement known in higher dimension involves the multiplicity two case, which was solved in 1988 by J. Alexander and A. Hirschowitz. Their approach is from an algebraic geometry point of view.

In this dissertation I will discuss this problem and present an alternate approach to the theorem, which I believe to be much more accessible than that given by Alexander and Hirschowitz. Throughout the paper I will use a slight variation of the methods developed by R.A. Lorentz and G.G. Lorentz, with which they have shown the dimension two case.

In the last chapter I apply the methods developed thus far to toric surfaces. I start with a complete analysis of linear systems in $\mathbb{P}^1 \times \mathbb{P}^1$, followed by a discussion of convex polygons, which correspond to certain toric compactifications of $\mathbb{A}^2$. In each case I describe how the method is applied and what the exceptional cases are.

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ACKNOWLEDGEMENTS

Many thanks to Dr. Rick Miranda for his advice, constant encouragement and enthusiasm.

Thanks to my committee members, Dr. Jeanne Duflot, Dr. Holger Kley, and Dr. Charles Anderson, for their contributions and comments.

DEDICATION

To my parents, Ana and Alexandru Radu, and to my husband, Michael Dent.

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Chapter 0

INTRODUCTION

Why are people interested in polynomial interpolation: finding a polynomial $P(\underline{x})$, which goes through points p_i with multiplicity m_i at each point? First, polynomials are the building blocks of many numerical methods. As an example, finite elements and splines, both univariate and multivariate, are piecewise polynomials. Second, theorems about approximation of functions or about numerical schemes almost always reduce to local interpolation by polynomials. Despite their fundamental importance for numerical methods, the theory of polynomial interpolation is underdeveloped, [10]. Basic questions are still left to be answered.

Interpolation goes as far back as ancient Mesopotamia, where linear interpolation seems to have been a commonplace procedure. It was used within the tables to find approximations for values not listed in the table, [2]. In general terms, interpolation is used to find approximate values of a function $f(x)$ for an x between different x-values $x_0, x_1, ..., x_n$ at which the values of $f(x)$ are given. The standard idea in interpolation is to find a polynomial $p_n(x)$ of degree n or less, that assumes the given values. So, $p_n(x_i) = f(x_i)$ for all $i = 0, ..., n$. The polynomial p_n is called the *interpolation polynomial* and the x_i's are called the *nodes*, or the *knots.*

The interpolation polynomial exists and it is unique. There are methods giving formulas for p_n, such as

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{\prod_{i \neq k}(x - x_i)}{\prod_{i \neq k}(x_k - x_i)} f_k \tag{0.1}$$

for all $i = 0, ..., n$, and so it exists. To show uniqueness, suppose there is another polynomial q_n that also satisfies $q_n(x_i) = f(x_i)$, for all $i = 0, ..., n$. Then $p_n - q_n = 0$ at $x_0, x_1, ..., x_n$. However, a polynomial $p_n - q_n$ of degree n (or less) with $n + 1$ roots must be identically zero. Thus $p_n \equiv q_n$, which shows uniqueness.

There are various methods of interpolation such as Lagrange, Newton, Spline and Hermite. In this paper we will only be concerned with Lagrange and mostly Hermite interpolation.

The first chapter introduces the definitions of Lagrange and Bivariate Hermite interpolations. In the case of Hermite interpolation we describe the linear system created. We then look at how one can get information about the interpolating scheme when looking at the determinant of the corresponding matrix. The dimension problem and the connection to algebraic geometry is made in the following section, and a short section on what is known concludes the chapter.

In the second chapter we describe the Hermite interpolation again, but this time from an approximation theory approach. Here we introduce the idea of shifts and talk about their relationship with derivatives of the determinant. The methods used to obtain the results of G.G. Lorentz and R.A. Lorentz, [9], are presented here. Also, the theory needed in the later sections is developed.

The third chapter tackles the dimension two and three parts of the Alexander-Hirschowitz theorem. Here we introduce minimal shifts and coalescence, which are used to prove the two and three dimensional cases.

In the fourth chapter, we extend these ideas of minimal shifts and coalescence to toric surfaces. First we look at surfaces in $\mathbb{P}^1 \times \mathbb{P}^1$ and we do a complete analysis of the linear systems of curves. We then look at general toric surfaces and their corresponding linear systems of curves.

We conclude the paper with a short description of future work. Although we have a complete understanding of the dimension N case, we still need to look more at the end-game in order to finish the proof of the Alexander-Hirschowitz theorem. Also, there is further work to be done for linear systems of curves of toric surfaces in the most general case.

Chapter 1

ALGEBRAIC GEOMETRY APPROACH

1.1 Lagrange interpolation

Given (x_0, f_0), ..., (x_n, f_n) with arbitrarily spaced x_i's, Lagrange had the idea of multiplying each f_i, which is the desired value at x_i, by a polynomial that is one at x_i and zero at all other nodes. Then he took the sum of these $n+1$ polynomials to get the unique interpolation polynomial of degree n or less. Hence, we get the general Lagrange interpolation polynomial in one variable, [8],

$$f(x) \approx p_n(x) = \sum_{k=0}^{n} \frac{\prod_{i \neq k}(x - x_i)}{\prod_{i \neq k}(x_k - x_i)} f_k \tag{1.1}$$

for all $i = 0, ..., n$.

1.2 Uniform Bivariate Hermite Interpolation

Let V_d be the space of bivariate polynomials,

$$P(x, y) = \sum_{i+k \leq d} a_{ik} x^i y^k \tag{1.2}$$

of total degree d or less, where the numbers a_{ik} are the coefficients of each term. Let us fix the following:

1. For each q, $1 \leq q \leq n$, give a finite set A_q of ordered pairs of nonnegative integers (α, β), which are the sets of orders of derivatives.
2. $Z = \{z_q := (x_q, y_q)\} \in \mathbb{C}^2$, $1 \leq q \leq n$, as the interpolation knot points.
3. $c_{\alpha,\beta,q} \in \mathbb{C}$

Denote by $\mathfrak{A} := \{A_q\}_{q=1}^n$. Then, we want to find a polynomial $P \in V_d$ satisfying

$$\left.\frac{\partial^{\alpha+\beta} P}{\partial x^\alpha \partial y^\beta}\right|_{z_q} = c_{\alpha,\beta,q}, \tag{1.3}$$

for all q, $1 \leq q \leq n$ and for all $(\alpha, \beta) \in A_q$.

Thus we have a system of $\sum\limits_{i=1}^{n} |A_q| := |\mathfrak{A}|$ equations with $\dim(V_d)$ unknowns, a_{ik}. We will assume from here on that the two are equal, i.e., $|\mathfrak{A}|$ $= \dim(V_d)$. The monomials in x and y of degree less than or equal to d form an ordered basis for V_d, and so,

$$dim\ V_d = \tfrac{(d+1)(d+2)}{2} = \sum_{i=1}^{n} |A_q|.$$

Let us denote the matrix corresponding to the linear system by $M(Z, \mathfrak{A})$ and its determinant denote by $det\ M(Z, \mathfrak{A}) =: D(Z, \mathfrak{A})$. The columns of $M(Z, \mathfrak{A})$ are indexed by the monomials $x^i y^k$, $i + k \leq d$, in V_d. The rows of $M(Z, \mathfrak{A})$ are doubly indexed by q and then by $(\alpha, \beta) \in A_q$. Thus the $(q, \alpha, \beta) - (i, k)$ entry of $M(Z, \mathfrak{A})$ is given by $\left.\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}(x^i y^k)\right|_{z_q}$. Note that if the coordinates of the z_q's are undetermined, then $D(Z, \mathfrak{A})$ is a polynomial in 2n variables $x_1, ..., x_n, y_1, ..., y_n$. For consistency we will order the monomials and the derivatives in degree lexicographic order.

Example 1.2.1. *Let us consider the following case:* $d = 2$, $n = 1$, $Z = \{(x_1, y_1)\}$, *and* $A_1 = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$. *Then*

$$M(Z, \mathfrak{A}) = \begin{pmatrix} x_1^i y_1^k \\ \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_1} \\ \frac{\partial^2}{\partial x_1^2} \\ \frac{\partial}{\partial x_1 y_1} \\ \frac{\partial^2}{\partial y_1^2} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 0 & 1 & 0 & 2x_1 & y_1 & 0 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

where $i + k \leq d$, $i, k = 0, 1, 2$.

If each set A_q consists of the single lattice point $(0,0)$, then we have Lagrange interpolation at the z_q. If all sets of lattice points are triangles, i.e., the total order of derivatives is no more than a fixed given constant, then we are in the Hermite interpolation case.

Definition 1.2.1. The interpolation problem is called

1. *regular* if the determinant $D(Z, \mathfrak{A})$ is a non-zero constant, i.e., we can solve (1.3) for all $c_{\alpha,\beta,q}$ and for any knot set Z;

2. *almost regular* if the determinant $D(Z, \mathfrak{A})$ is a non-constant polynomial in the x_i's and y_j's, i.e., we can solve (1.3) for a general knot set Z;

3. *singular* if the determinant $D(Z, \mathfrak{A})$ is identically zero, i.e., if for all Z, (1.3) is not solvable for some data $c_{\alpha,\beta,q}$.

For example, note that for the problem considered in Example 1 above, $D(Z, \mathfrak{A}) = 4$ regardless of the values of x_1 and y_1 . Thus, in this case, we have a regular interpolation problem.

Example 1.2.2. *Consider the case of a conic:* $d = 2$, $n = 2$, $Z = \{(x_1, y_1),\ (x_2, y_2)\}$, *and* $A_1 = A_2 = \{(0,0), (1,0), (0,1)\}$.

Then

$$M(Z, \mathfrak{A}) = \begin{pmatrix} x_1^i y_1^k \\ \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_1} \\ x_2^i y_2^k \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 0 & 1 & 0 & 2x_1 & y_1 & 0 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 0 & 1 & 0 & 2x_2 & y_2 & 0 \\ 0 & 0 & 1 & 0 & x_2 & 2y_2 \end{pmatrix}$$

where $i + k \leq d$, *and* $i, k = 0, 1, 2$. *Note that* $D(Z, \mathfrak{A}) \equiv 0$, *for all values of* (x_1, y_1) *and* (x_2, y_2), *which implies that the interpolation problem is singular.*

Example 1.2.3. *Let us now consider the following case:* $d = 2$, $n = 4$, $Z = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$, $A_1 = \{(0,0), (1,0), (0,1)\}$ *and* $A_2 = A_3 = A_4 = \{(0,0)\}$.

Then

$$M(Z, \mathfrak{A}) = \begin{pmatrix} x_1^i y_1^k \\ \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_1} \\ x_2^i y_2^k \\ x_3^i y_3^k \\ x_4^i y_4^k \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 \\ 0 & 1 & 0 & 2x_1 & y_1 & 0 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4 y_4 & y_4^2 \end{pmatrix}$$

where $i + k \leq d$, $i, k = 0, 1, 2$. *In this case* $D(Z, \mathfrak{A})$ *is a non-zero polynomial in the variables* (x_j, y_j), $j = 1, 2, 3, 4$, *which can easily be seen with a program such as Maple. Therefore we have that the interpolation problem is almost regular.*

1.3 Dimension Problem

We want to see what the dimension of the vector space of all interpolating polynomials, whose degree are less than or equal to d, should be. Let us start with some basic facts about projective space.

Definition 1.3.1. A hypersurface X of degree d is a subvariety of $\mathbb{P}^n$ described as the locus of a single homogeneous polynomial F whose total degree is d.

Hence a linear system of hypersurfaces corresponds to a vector space of homogeneous polynomial equations. Now, dehomogenizing the polynomials we arrive at the affine version of the problem. So, with this in mind, we will only describe the dimension for the affine case.

We fix a degree d, and recall that V_d was the space of bivariate polynomials $P(x, y)$ of degree d or less. Now, let us fix points $p_i = (x_i, y_i)$ and integers $m_i \geq 1$, for $i = 1, ..., n$. We ask that the polynomial $P(x, y)$ and all its derivatives up to order $m_i - 1$ vanish at p_i. We often say, in this case, that the polynomial has multiplicity m_i at p_i, i.e., the order of vanishing of $P(x, y)$ at p_i is m_i. The number of conditions on the polynomial which must vanish to order m_i at p_i is exactly the number of terms in the Taylor series expansion of $P(x, y)$ at p_i, up through order $m_i - 1$, and all these coefficients must vanish, [14]. Therefore, the number of conditions is equal to $m_i(m_i + 1)/2$. Letting d be the degree of the polynomial, n be the number of points and m_i be the multiplicity at each point, we obtain the following relation:

$$\frac{(d+1)(d+2)}{2} = \sum_{i=1}^{n} \frac{m_i(m_i+1)}{2} \tag{1.4}$$

and if all multiplicities are equal to m we get

$$\frac{(d+1)(d+2)}{2} = n\binom{m+1}{2}. \tag{1.5}$$

Denote by $\mathcal{L}_d(-\sum_{i=1}^{n} m_i p_i)$ the set of polynomials $P(x,y)$ in V_d with multiplicity at p_i less than or equal to m_i, for all $i = 1, ..., n$.

Lemma 1.3.1. *$\mathcal{L}_d(-\sum_{i=1}^{n} m_i p_i)$ is a linear subspace of V_d.*

Thus the question becomes:

What is the dimension of $\mathcal{L}_d(-\sum_{i=1}^{n} m_i p_i)$?

Let $\phi : V_d \dashrightarrow \mathbb{C}^N$ be the linear map given by $\phi(P(x,y))$ which is equal to a vector with components $\left.\frac{\partial^{\alpha+\beta}P}{\partial x^\alpha \partial y^\beta}\right|_{z_q}$. The matrix of ϕ is given by $M(Z,\mathfrak{A})$, and the kernel of this map is exactly $\mathcal{L}_d(-\sum_{i=1}^{n} m_i p_i)$.

Theorem 1.3.1 (Rank-Nullity Theorem).

$$dim(ker\ \phi) + dim(im\ \phi) = dim(V_d).$$

Now, the dimension of the image of ϕ is equal to the rank of the matrix. Therefore, we have that

$$\begin{aligned} dim(\mathcal{L}_d(-\sum_{i=1}^{n} m_i p_i)) &= dim(V_d) - rank(M(Z,\mathfrak{A})) \\ &= \frac{(d+1)(d+2)}{2} - rank(M(Z,\mathfrak{A})). \end{aligned}$$

Note that the kernel, $\mathcal{L}_d(-\sum_{i=1}^{n} m_i p_i)$, has minimal dimension when the rank of the matrix is maximal. As it turns out, if the points p_i are chosen in special positions then the interpolation conditions may be dependent, thus reducing the rank of the matrix $M(Z,\mathfrak{A})$. So, we want to take the points in *general* position, which guarantees that the the rank is maximal, i.e., the interpolation conditions are as independent as possible, [11].

Example 1.3.1. *Consider the linear case in* $\mathbb{R}^3$*:* $d = 1$*,* $n = 3$*,* $Z = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$ *and* $A_1 = A_2 = A_3 = \{(0,0)\}$*. Then*

$$M(Z, \mathfrak{A}) = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

and in general $rank(M(Z, \mathfrak{A})) = 3$*. However, if two of the points are identical, or if all these points are collinear, then the rank of the matrix drops. So, we want to avoid cases like this.*

1.4 What is known

The problem of computing the dimension of a space of polynomials, satisfying certain multiplicity conditions at a set of general points, can be formulated in any dimension. Denote the space of polynomials of degree at most d in r variables having multiplicity at least m_i at chosen points p_i, $i = 1, ..., n$, by $\mathcal{L}_d^{(r)}(-\sum_{i=1}^n m_i p_i)$. In this general form, the problem of computing the dimension of $\mathcal{L}_d^{(r)}(-\sum_{i=1}^n m_i p_i)$ is unsolved.

The problem concerning the dimensions of the bivariate linear systems of curves with general p_i's is also still open. Work in recent years has focused on the special case of equal multiplicities. This is the system of curves of degree d and multiplicity m at p_i, $\mathcal{L}_d(-\sum_{i=1}^n m p_i)$. The case $m = 2$ has been treated by B. Segre, [13], E. Arbarello, and M. Cornalba, [1], and by A. Hirschowitz, [7]. The case $m = 3$ has been also done by Hirschowitz, [7]. The $m = 4$ case has recently been treated by L. Evain, [6] and by T. Mignon, [12], who have also looked at the general case with all $m_i \leq 4$. R.A. Lorentz and G.G. Lorentz, [9] have shown that for Hermite conditions (1.3) (when all A_q are lower triangles of side length $m - 1$) and the polynomial P is of total degree d, (1.3) is almost regular for all $m = 2, 3, 4$ and all d, except for two cases $m = 2$, $d = 2$ and $m = 2$, $d = 4$. R. Miranda and C. Ciliberto, [3] and [4], have developed a degeneration method with which they were able to verify the dimension problem for constant multiplicities of at most 12. Recently R. Miranda, C. Ciliberto, F. Cioffi and F. Orecchia, [5], have been able to prove the dimension problem for constant multiplicities equal to or less than 20.

Chapter 2

APPROXIMATION THEORY APPROACH

The following is a summary of the article written by G.G. Lorentz and R.A. Lorentz, *Bivariate Hermite Interpolation and Applications to Algebraic Geometry*, [9]. The theory developed here will be needed for the proofs in the next chapter. This chapter provides the background needed to understand the work done in the later sections.

2.1 Introduction and Basic Notation

Suppose we are given a polynomial of the form

$$P(x,y) = \sum_{(i,k)\in S} a_{i,k} x^i y^k$$

We want to solve the bivariate interpolation problem

$$\left.\frac{\partial^{\alpha+\beta} P}{\partial x^\alpha \partial y^\beta}\right|_{z_q} = c_{\alpha,\beta,q}, \tag{2.1}$$

for all q and all $(\alpha,\beta) \in A_q$, where

(1) S is a lower set of lattice points, i.e., $0 \le i' \le i$, $0 \le k' \le k$ and $(i,k) \in S$ implies $(i',k') \in S$.

(2) A_q are the sets of orders of derivatives, i.e, for each q, $1 \le q \le n$, we are given a finite set A_q of ordered pairs of nonnegative integers (α,β).

(3) $z_q := (x_q, y_q) \in \mathbb{C}^2$, $1 \leq q \leq n$.

(4) $c_{\alpha,\beta,q} \in \mathbb{C}$.

(5) The number of coefficients, $|S|$, is equal to the number of equations, $\sum |A_q|$, i.e, $|S| = \sum_{i=1}^{n} |A_q| := |\mathfrak{A}|$, where $(\alpha, \beta) \in A_q$, $1 \leq q \leq n$.

Note that (2.1) is a system of $|S|$ equations with $|S| = |\mathfrak{A}|$ unknowns, $a_{i,k}$. The matrix of the system is denoted by $M := M_S(Z, \mathfrak{A})$, where M consists of rows $\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta}(x^i y^k)|_{z_q}$, for all α, $\beta \in A_q$ with $x = x_q$ and $y = y_q$, $1 \leq q \leq n$, which are in degree lexicographical order. Its determinant is given by $D := D_S(Z, \mathfrak{A}) = det(M)$, where $Z = \{z_q\}_{q=1}^{n}$. Note that this determinant is a polynomial in $2n$ variables $x_1, \ldots, x_n, y_1, \ldots, y_n$.

The properties of the interpolation scheme $(S, \mathfrak{A})$ are as follows:

1. The scheme is *regular* if (2.1) is solvable for all $c_{\alpha,\beta,q}$ and for all knots sets Z, i.e., D is a nonzero constant.

2. The scheme is *almost regular* if (2.1) is solvable for at least one $Z \in \mathbb{C}^2$, i.e., D is a nonconstant polynomial.

3. The scheme is *singular* if, for all Z, (2.1) is not solvable for some data $c_{\alpha,\beta,q}$, i.e., $D \equiv 0$.

Note that the interpolation scheme will be defined by means of the knot sets A_q. So we need to decide whether a given interpolation scheme is solvable for the interpolation knots in general position. This means that if the position of the points is altered slightly, the rank of the matrix M does not change.

Theorem 2.1.1. *An almost regular scheme satisfies the Pólya condition*

$$|\mathfrak{A} \cap B| \geqq |B| \tag{2.2}$$

for each lower set $B \subset S$.

Theorem 2.1.2. *A scheme satisfying (2.2) is regular if and only if the knot sets A_q are disjoint.*

2.2 Shifts and Algorithm

This section will formulate a general algorithm which, in some cases, can establish that a given scheme is almost regular. We will start with some definitions and formulas.

Definition 2.2.1. An upper subset $B \subset S$ has the property that if $(i', k') \in B$ and $i' \leq i$, $k' \leq k$, then $(i, k) \in S$ implies that $(i, k) \in B$.

Lemma 2.2.1. *Let A_n be an upper subset of S, $\mathfrak{A} = \{A_q\}_{q=1}^{n}$, and let $\mathfrak{A}' = \{A_q\}_{q=1}^{n-1}$. If $S' = S \backslash A_n$ then $D_S(\mathfrak{A}) = \mu D_{S'}(\mathfrak{A}')$, where $\mu = \pm 1$.*

Corollary 2.2.1. *If the scheme $\{S', \mathfrak{A}'\}$ is almost regular, i.e., $D_{S'}(\mathfrak{A}') \not\equiv 0$, then so is the scheme $\{S, \mathfrak{A}\}$.*

Proof. The two determinants only differ by μ. Q.E.D

We define the shifts on subsets of S, such as one of the knot sets A_q. A *right shift* Λ of A_1 moves a point $(i, k) \in A_1$ to the position $(i+1, k) \notin A_1$. An *upward shift* moves a point $(i, k) \in A_1$ to the position $(i, k+1) \notin A_1$. Recall that S is the set of lattice points and the A_q's are the sets of orders of derivatives which we require to vanish. We want to pave S with the A_q's by performing a series of right and upward shifts, as defined above. Thus, if $(i, k) \in A_q$, for some q, then $(i+1, k)$ corresponds to taking an extra derivative with respect to x of the row in which that point is. Similarly, $(i, k+1)$ corresponds to taking an extra derivative with respect to y of that row. Note that Λ is a function with domain A_1 and range $\Lambda A_1 := A_1^* = (A_1 \backslash (i, k)) \cup (i+1, k)$. Thus $\mathfrak{A}$ is transformed into $\Lambda\mathfrak{A} = \{A_1^*, A_2, ..., A_n\}$. We will always assume that $A_1^* \subset S$. A *multiple shift* Λ^* of order (α, β) is the composition of α right shifts and β upward shifts. The order in which the shifts are performed does not matter, but for consistency, right shifts will be done first.

Definition 2.2.2. Λ^* is a Pólya shift if $\mathfrak{A}^* := \Lambda^*\mathfrak{A}$ satisfies the Pólya condition. (see 2.2)

Let us introduce the following notation: $\frac{\partial}{\partial x} M_S^j(\mathfrak{A}) :=$ partial derivatives with respect to x of the jth row in the matrix $M_S(\mathfrak{A})$.

Lemma 2.2.2. *Let $D = det\ M_S(\mathfrak{A})$. Then*

$$\begin{aligned}\frac{\partial}{\partial x_1} D &= \sum_{\substack{rows\ j \\ in\ A_1\ part}} \left(\begin{array}{c} determinant\ of\ the\ matrix\ formed\ by \\ \frac{\partial}{\partial x_1}\ to\ the\ jth\ row\ of\ M_S(\mathfrak{A})\ in\ A_1\ part \end{array} \right) \\ &= \sum_{\substack{rows\ j \\ in\ A_1\ part}} det\left(\frac{\partial}{\partial x_1} M_S^j\right)\end{aligned}$$

Proof. This is obtained by using the product rule of derivatives. Q.E.D

Example 2.2.1. *Consider the following matrix:*

$$M = \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}$$

where det $M = D$. Then,

$$\begin{aligned}\frac{\partial}{\partial x} D &= \begin{vmatrix} \frac{\partial}{\partial x}s_{11} & \frac{\partial}{\partial x}s_{12} & \frac{\partial}{\partial x}s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} + \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ \frac{\partial}{\partial x}s_{21} & \frac{\partial}{\partial x}s_{22} & \frac{\partial}{\partial x}s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix} + \begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ \frac{\partial}{\partial x}s_{31} & \frac{\partial}{\partial x}s_{32} & \frac{\partial}{\partial x}s_{33} \end{vmatrix} \\ &= det\left(\frac{\partial}{\partial x} M^1\right) + det\left(\frac{\partial}{\partial x} M^2\right) + det\left(\frac{\partial}{\partial x} M^3\right) \\ &= \sum_{j=1}^{3} det\left(\frac{\partial}{\partial x} M^j\right).\end{aligned}$$

Therefore,

$$\frac{\partial}{\partial x} D = \sum_{rows\ j} det\left(\frac{\partial}{\partial x} M^j\right).$$

Corollary 2.2.2.

$$\begin{aligned}\frac{\partial}{\partial y_1} D &= \sum_{\substack{\textit{rows } j \\ \textit{in } A_1 \textit{ part}}} \begin{pmatrix} \textit{determinant of the matrix formed by} \\ \frac{\partial}{\partial y_1} \textit{ to the jth row of } M_S(\mathfrak{A}) \textit{ in } A_1 \textit{ part} \end{pmatrix} \\ &= \sum_{\substack{\textit{rows } j \\ \textit{in } A_1 \textit{ part}}} det\left(\frac{\partial}{\partial y_1} M_S^j\right)\end{aligned}$$

Proof. Similar to the proof of lemma 2.2.2. Q.E.D

Observation: Recall that $M_S(\mathfrak{A})$ is our corresponding matrix, and det $M_S(\mathfrak{A}) = D_S(\mathfrak{A})$. Let $\frac{\partial}{\partial x}\ M_S^j(\mathfrak{A})$ be the matrix formed by taking $\frac{\partial}{\partial x}$ of the jth row (for all columns k). Then

$$\frac{\partial}{\partial x_1}\ M_S^j(\mathfrak{A}) = \underbrace{\mu}_{\substack{\textit{from possible} \\ \textit{reordering of} \\ \textit{the rows}}} \quad M_S(\underbrace{\Lambda\mathfrak{A}}_{\substack{\textit{from the} \\ \textit{right shift}}})$$

where $\Lambda\mathfrak{A} = \{A_1^*, A_2, ..., A_n\}$, and $\mu = \pm 1$. See example 2.2.2 below for an instance where $\mu = -1$.

Reorganizing the sum based on the end result, we obtain

$$\begin{aligned}\frac{\partial}{\partial x_1}\ D_S(\mathfrak{A}) &= \sum_{\substack{\textit{rows } j \\ \textit{in } A_1 \textit{ part}}} det \begin{pmatrix} x_1 \textit{ derivative of row } j \\ \textit{of } M(S,\mathfrak{A}) \textit{ in } A_1 \textit{ part} \end{pmatrix} \\ &= \sum_{(i,k)\in A_1} \mu(i,k)\ det\ M_S(\underbrace{\ \Lambda\mathfrak{A}\ }_{\substack{\textit{right shift} \\ (i,k)\to(i+1,k) \\ \textit{of each row} \\ j \textit{ in } A_1}}).\end{aligned}$$

Therefore, we have that

$$\frac{\partial^\alpha}{\partial x_1^\alpha}\ D_S(\mathfrak{A}) = \underbrace{\sum_{\substack{\textit{rows } j \textit{ in} \\ A_1 \textit{ part}}} \cdots \sum}_{\alpha \textit{ sums}} \mu(i,k)\ det\ M_S(\Lambda^*\mathfrak{A}).$$

Similarly,

$$\begin{aligned}\frac{\partial}{\partial y_1} D_S(\mathfrak{A}) &= \sum_{\substack{rows\ j\\ in\ A_1\ part}} det\begin{pmatrix} y_1\ derivative\ of\ row\ j \\ of\ M_S(\mathfrak{A})\ in\ A_1\ part\end{pmatrix} \\ &= \sum_{(i,k)\in A_1} \mu(i,k)\ det\ M_S(\underbrace{\Lambda\mathfrak{A}}_{\substack{upward\ shift\\ (i,k)\to(i,k+1)\\ of\ each\ row\\ j\ in\ A_1}})\end{aligned}$$

and hence

$$\frac{\partial^\beta}{\partial y_1^\beta} D_S(\mathfrak{A}) = \underbrace{\sum \cdots \sum_{\substack{rows\ j\ in\\ A_1\ part}}}_{\beta\ sums} \mu(i,k)\ det\ M_S(\Lambda^*\mathfrak{A}).$$

Putting all this together we have that

$$\frac{\partial^{\alpha+\beta}}{\partial x_1^\alpha \partial y_1^\beta} D_S(\mathfrak{A}) = \underbrace{\sum \cdots \sum_{\substack{rows\ j\ in\\ A_1\ part}}}_{\alpha\ sums} \underbrace{\sum \cdots \sum_{\substack{rows\ j\ in\\ A_1\ part}}}_{\beta\ sums} \mu(i,k)\ det\ M_S(\Lambda^*\mathfrak{A}).$$

If there is only one possible end result, i.e., one $\Lambda^*\mathfrak{A}$, and if we let $\sigma = \sum \mu(i,k)$, then we can bring the determinant of $M_S(\Lambda^*\mathfrak{A})$ out of the sum. We say that the shifts are *non-cancelling* if $\sigma \neq 0$. Hence, if the shifts Λ^* are non-cancelling, then the scheme $(S, \mathfrak{A})$ is almost regular if the simpler scheme $(S, \mathfrak{A}^*)$ has this property.

Example 2.2.2. *Suppose that* $A_1 = \{(0,0),(1,0),(0,1)\} = A_2$. *Note that*

- $A_1 \to A_1^* := \{(1,0),(1,0),(0,1)\}$ *corresponds to* $\frac{\partial}{\partial x_1}$ *of row 1;*
- $A_1 \to A_1^* := \{(0,0),(2,0),(0,1)\}$ *corresponds to* $\frac{\partial}{\partial x_1}$ *of row 2;*
- $A_1 \to A_1^* := \{(0,0),(1,0),(1,1)\}$ *corresponds to* $\frac{\partial}{\partial x_1}$ *of row 3;*

and thus,

$$\frac{\partial}{\partial x_1}D(A_1,A_2) = \underbrace{D(\{(1,0),(1,0),(0,1)\},A_2)}_{\substack{\textit{collision, i.e.,}\\ \textit{two rows are}\\ \textit{identical so } D=0}} + \underbrace{D(\{(0,0),(2,0),(0,1)\},A_2)}_{\substack{\textit{need to switch}\\ \textit{rows 2 and 3,}\\ \textit{so } \mu(1,0)=-1}}$$

$$+ \underbrace{D(\{(0,0),(1,0),(1,1)\},A_2)}_{\substack{\textit{no switch needed,}\\ \textit{so } \mu(0,1)=1}}$$

$$= 0 + (-1)D(\{(0,0),(2,0),(0,1)\},A_2)$$

$$+ D(\{(0,0),(1,0),(1,1)\},A_2)$$

Similarly,

$$\frac{\partial^2}{\partial x_1^2}D(A_1,A_2)$$

$$= (-1)\frac{\partial}{\partial x_1}D(\{(0,0),(2,0),(0,1)\},A_2) + \frac{\partial}{\partial x_1}D(\{(0,0),(1,0),(1,1)\},A_2)$$

$$= (-1)[\underbrace{D(\{(1,0),(2,0),(0,1)\},A_2)}_{\mu(0,0)=1} + \underbrace{D(\{(0,0),(3,0),(0,1)\},A_2)}_{\mu(2,0)=1}$$

$$+ (-1)\underbrace{D(\{(0,0),(2,0),(1,1)\},A_2)]}_{\mu(0,1)=-1} + [\underbrace{D(\{(1,0),(1,0),(1,1)\},A_2)}_{\textit{collision } D=0}$$

$$+ \underbrace{D(\{(0,0),(2,0),(1,1)\},A_2)}_{\mu(1,0)=1} + \underbrace{D(\{(0,0),(1,0),(1,2)\},A_2)]}_{\mu(1,1)=1}$$

Note that there are two determinants of the form

$$D(\{(0,0),(2,0),(1,1)\},A_2)$$

and combining the coefficients we have that $\sigma = 3 \neq 0$*, in the end result, and all other determinants are different. Hence the shifts are non-cancelling.*

Definition 2.2.3. Define a maximal Pólya UR shift to be a shift that is a Pólya shift of A_i of the highest order that turns A_i into an upper set and that leaves $A_i \subset S$.

Thus, to establish if a scheme $(S, \mathfrak{A})$ is almost regular, we have the following **algorithm**:

1. Take the maximal Pólya UR shift Λ^* of A_1 in S onto $\Lambda^* A_1$, and put

$$S_1 := S \backslash \Lambda^* A_1, \quad \mathfrak{A}_1 := \{A_q\}_{q=2}^n$$

2. Repeat for A_2, A_3, ...

3. If all these shifts are non-cancelling and if this produces a paving of S by disjoint sets $\Lambda^* A_q$, then the scheme is almost regular.

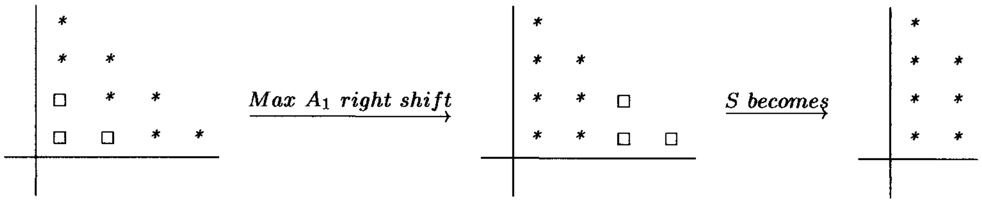
This follows from Theorem 2.1.1 (Pólya condition) and lemma 2.2.2.

Example 2.2.3. *Let* $S = \{x^i y^k : 0 \leq i, k \leq 3\}$, $A_1 = A_2 = A_3 = \{(0,0),(1,0),(0,1)\}$ *and* $A_4 = \{(0,0)\}$. *Suppose* α *is the maximum number of right shifts and* β *is the maximum number of upward shifts so that* $A_i \subset S$. *Recall that if* $A_i \not\subset S$, *then we get a row of zeros in the matrix and hence* $D = 0$.

Now, for $\alpha = 6$ *we have that*

$$\frac{\partial^\alpha}{\partial x_1^\alpha} D_S(\mathfrak{A}) = \sigma D_{S'}(\mathfrak{A}')$$

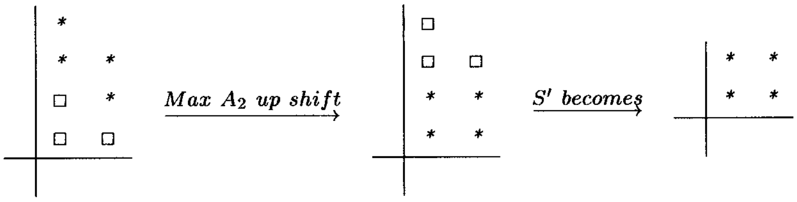
where $A_1^* = \{(2,0),(3,0),(2,1)\}$, *which is an upper subset and we can remove it,* $S' = S \backslash A_1^*$, *and* $\mathfrak{A}' = \mathfrak{A} \backslash A_1^*$.



Similarly, for $\beta = 6$ *we get*

$$\frac{\partial^\beta}{\partial y_2^\beta} D_{S'}(\mathfrak{A}') = \sigma' D_{S''}(\mathfrak{A}'')$$

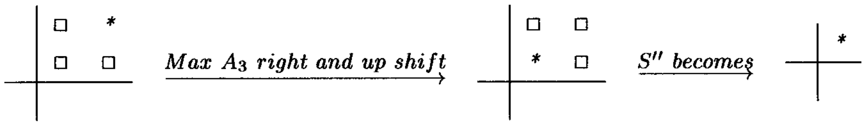
where $A_2^* = \{(0,2),(1,2),(0,3)\}$, *which again is an upper subset and we can remove it,* $S'' = S' \backslash A_2^*$, *and* $\mathfrak{A}'' = \mathfrak{A}' \backslash A_2^*$.



Taking one more x derivative,

$$\frac{\partial}{\partial x_3} D_{S''}(\mathfrak{A}'') = \sigma'' D_{S^{(3)}}(\mathfrak{A}^{(3)})$$

where $A_3^* = \{(1,0),(0,1),(1,1)\}$, which is removed since it is an upper subset, $S^{(3)} = S''\backslash A_3^*$, and $\mathfrak{A}^{(3)} = \mathfrak{A}''\backslash A_3^*$. Note that at this point $S^{(3)} = \{(0,0)\}$.



Finally,

$$\frac{\partial}{\partial y_4} D_{S^{(3)}}(\mathfrak{A}^{(3)}) = \sigma^{(3)} D_{\{1\}}(A_4) = \sigma^{(3)}$$

Therefore, if all our shifts are non-cancelling we get that the determinant is a non-zero constant, and so the interpolation problem is regular.

2.3 Triangles and Hermite Problems

A triangle is defined by $T_m = \{(i,k) \,|\, i \geq 0, k \geq 0, i+k \leq m-1\}$ and has order $|T_m| = \frac{(m)(m+1)}{2}$. An interpolation scheme $(S; A_1, \ldots, A_n)$ is *Hermitian* if S and all A_q are triangles. If in particular $A_1 = A_2 = \cdots = A_n =: T_m$ and $S =: T_s$, we write $(S, \mathfrak{A}) =: (T_s, T_m)$. Then since $|S| = \sum_{q=1}^{n} |A_q|$ we get that

$$s(s+1) = n(m)(m+1). \tag{2.3}$$

where $s = d+1$, d is the degree. With these ideas in mind we arrive at the main theorem of the paper [9].

Theorem 2.3.1. *For* $m = 2, 3, 4$ *and all* d *satisfying (2.3), the scheme* (T_s, T_m) *is almost regular, with the only exceptions of* $m = 2$, $d = 2$, *and of* $m = 2$, $d = 4$.

Example 2.3.1. *Let us consider the following case: degree* = 7, *multiplicity* = 3 *and suppose we have 6 points. Thus,* $S = \{x^i y^k : 0 \leq i, k \leq 7\}$, $A_i = \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2)\}$, $1 \leq i \leq 6$. *Therefore,*

Adding A_1*:*

7								
6								
5								
4								
3								
2	1							
1	1	1						
0	1	1	1					
*	0	1	2	3	4	5	6	7

7								
6								
5								
4								
3								
2						1		
1						1	1	
0						1	1	1
*	0	1	2	3	4	5	6	7

Table 2.1: Max A_1 right shift.

Adding A_2:

7								
5								
4								
3								
2	2					1		
1	2	2				1	1	
0	2	2	2			1	1	1
*	0	1	2	3	4	5	6	7

7								
6								
5								
4								
3								
2					2	1		
1				2	2	1	1	
0			2	2	2	1	1	1
*	0	1	2	3	4	5	6	7

7								
6								
5			2					
4				2				
3				2	2			
2					2	1		
1					2	1	1	
0						1	1	1
*	0	1	2	3	4	5	6	7

Table 2.2: Max A_2 right shift, then max A_2 up shift.

Adding A_3:

7								
6								
5			2					
4				2				
3				2	2			
2	3				2	1		
1	3	3			2	1	1	
0	3	3	3			1	1	1
*	0	1	2	3	4	5	6	7

7								
6								
5			2					
4				2				
3				2	2			
2				3	2	1		
1			3	3	2	1	1	
0			3	3	3	1	1	1
*	0	1	2	3	4	5	6	7

7								
6								
5			2					
4			3	2				
3			3	2	2			
2				3	2	1		
1				3	2	1	1	
0				3	3	1	1	1
*	0	1	2	3	4	5	6	7

Table 2.3: Max A_3 right shift, then max A_3 up shift.

Adding A_4:

7								
6								
5			2					
4			3	2				
3			3	2	2			
2	4			3	2	1		
1	4	4		3	2	1	1	
0	4	4	4	3	3	1	1	1
*	0	1	2	3	4	5	6	7

7								
6								
5			2					
4			3	2				
3			3	2	2			
2			4	3	2	1		
1		4	4	3	2	1	1	
0	4	4	4	3	3	1	1	1
*	0	1	2	3	4	5	6	7

7	4							
6		4						
5		4	2					
4			3	2				
3			3	2	2			
2			4	3	2	1		
1			4	3	2	1	1	
0			4	3	3	1	1	1
*	0	1	2	3	4	5	6	7

Table 2.4: Max A_4 right shift, then max A_4 up shift

Note that in this case there is no room for the next block, A_5, and so we need to make a different move with A_4. Suppose we do max up shifts first and then max right shifts.

Therefore, we get the following

7								
6								
5			2					
4			3	2				
3			3	2	2			
2	4			3	2	1		
1	4	4		3	2	1	1	
0	4	4	4	3	3	1	1	1
*	0	1	2	3	4	5	6	7

7	4							
6	4	4						
5	4	4	2					
4			3	2				
3			3	2	2			
2			4	3	2	1		
1				3	2	1	1	
0				3	3	1	1	1
*	0	1	2	3	4	5	6	7

Table 2.5: Max A_4 up shift, then max A_4 right shift.

Adding A_5: (Note that we can not do the right shifts first, since there will not be enough room left for the last block, A_6)

7	4							
6	4	4						
5	4	4	2					
4			3	2				
3			3	2	2			
2	5		4	3	2	1		
1	5	5		3	2	1	1	
0	5	5	5	3	3	1	1	1
*	0	1	2	3	4	5	6	7

7	4							
6	4	4						
5	4	4	2					
4	5	5	3	2				
3	5	5	3	2	2			
2	5		4	3	2	1		
1			5	3	2	1	1	
0				3	3	1	1	1
*	0	1	2	3	4	5	6	7

7	4							
6	4	4						
5	4	4	2					
4	5	5	3	2				
3	5	5	3	2	2			
2		5	4	3	2	1		
1			5	3	2	1	1	
0				3	3	1	1	1
*	0	1	2	3	4	5	6	7

Table 2.6: Max A_5 up shifts, then max A_5 right shifts

Adding A_6:

7	4							
6	4	4						
5	4	4	2					
4	5	5	3	2				
3	5	5	3	2	2			
2	6	5	4	3	2	1		
1	6	6	5	3	2	1	1	
0	6	6	6	3	3	1	1	1
*	0	1	2	3	4	5	6	7

And so we were able to "pave" S with our triangles A_i, $1 \leq i \leq 6$. *Since all* A_i^**'s are upper subsets they can be removed one by one. Thus, applying the algorithm, we have that the scheme is almost regular.*

Chapter 3

THE ALEXANDER-HIRSCHOWITZ THEOREM

3.1 Basic Definitions and the Theorem

Define the vector space of polynomials of degree at most d in r variables by $\mathcal{L}_d^{(r)}$. Note that the dimension of this space is equal to $\binom{d+r}{r}$. Denote the space of polynomials of degree at most d in r variables having multiplicity m_i at the n chosen general points p_i, $i = 1, \ldots, n$, by $\mathcal{L}_d^{(r)}\Big(-\sum_{i=1}^{n} m_i p_i\Big)$. Note that the expected dimension of this system is given by

$$\mathfrak{e} = \binom{d+r}{r} - \sum_{i=1}^{n} \binom{m_i + r - 1}{r}, \text{ or } 0, \text{ if this is negative.}$$

Main Question: What is the dimension of $\mathcal{L}_d^{(r)}\Big(-\sum_{i=1}^{n} m_i p_i\Big)$ when the points are in general position?

We will say that a system is **special** if it does not have the expected dimension. Also, recall that for a hypersurface to have multiplicity at least two at a point is equivalent to saying that it is **singular** at the point.

Theorem 3.1.1 (Alexander-Hirschowitz). *Fix $r \geq 2$, $d \geq 2$ and consider the linear system*

$$\mathcal{L} = \mathcal{L}_d^{(r)}\Big(-\sum_{i=1}^{n} 2p_i\Big)$$

consisting of hypersurfaces of degree at most d in r variables that are singular at n general points p_i.

(a) *For $d = 2$ the linear system $\mathcal{L}$ is special if and only if $2 \leq n \leq r$.*

(b) *For $d \geq 3$ the linear system $\mathcal{L}$ is special if and only if the triple (r, d, n) is one of the following: $(2,4,5)$, $(3,4,9)$, $(4,4,14)$, $(4,3,7)$.*

3.2 Coalescence in Dimension 2

We want to introduce the idea of coalescence. Consider a general polynomial P in two or more variables. By coalescence we mean replacing two, or more, of the variables with a single one. Let us consider the following example:

Example 3.2.1. *Let $P(x,y,z) = x^3y - xz + 7y - 5$. Coalescing x and y we get the two variable polynomial $P'(x,z) = x^4 - xz + 7x - 5$.*

Denote by $\mathbf{coal}_{12}(\boldsymbol{P})$ the coalescence of the variables (x_1, y_1) and (x_2, y_2) in the polynomial P. We want to use this coalescence when paving our triangle S, because by coalescing A_1 and A_2 we reduce the number of variables in the matrix, and therefore in the determinant $D_S(\mathfrak{A})$, where recall that $\mathfrak{A} = \{A_1, \ldots, A_n\}$.

Lemma 3.2.1.

1.

$$coal_{12}\Big(D_S(\mathfrak{A})\Big) = \begin{cases} D_S\big(A_1 \cup A_2, A_3, \ldots A_n\big), & \text{if } A_1 \cap A_2 = \emptyset; \\ 0, & \text{if } A_1 \cap A_2 \neq \emptyset. \end{cases}$$

2.

$$coal_{12}(P_1 + P_2) = coal_{12}(P_1) + coal_{12}(P_2).$$

Now recall that

$$\frac{\partial^{\alpha+\beta}}{\partial x_1^\alpha \partial y_1^\beta} D_S(\mathfrak{A}) = \sum_{\substack{(\underline{\alpha},\beta) shifts \\ A_1^* of A_1}} D_S(A_1^*, A_2, \ldots, A_n).$$

Applying coalescence to both sides we get that

$$\begin{aligned}\mathrm{coal}_{12}\left(\frac{\partial^{\alpha+\beta}}{\partial x_1^\alpha \partial y_1^\beta} D_S(\mathfrak{A})\right) &= \sum_{\substack{(\underline{\alpha},\beta) shifts \\ A_1^* of A_1}} \mathrm{coal}_{12}\Big(D_S(A_1^*, A_2, \ldots, A_n)\Big) \\ &= \sum_{\substack{(\underline{\alpha},\beta) shifts \\ A_1^* of A_1 \\ such\ that \\ A_1^* \cap A_2 = \emptyset}} D_S(A_1^* \cup A_2, A_3 \ldots, A_n).\end{aligned}$$

Definition 3.2.1. (α, β) is a **minimal shift** for the pair (A_1, A_2) if there is only one (α, β) shift A_1^* of A_1 for which $A_1^* \cap A_2 = \emptyset$.

Corollary 3.2.1. *Suppose* (α, β) *is a minimal shift for* (A_1, A_2). *Then*

$$coal_{12}\left(\frac{\partial^{\alpha+\beta}}{\partial x_1^\alpha \partial y_1^\beta} D_S(\mathfrak{A})\right) = D_S(A_1^* \cup A_2, A_3 \ldots, A_n).$$

where A_1^* *is the result of applying the minimal shift* (α, β) *to* A_1 *such that* $A_1^* \cap A_2 = \emptyset$.

Proof. In general, when we apply coalescence to derivatives of $D_S(\mathfrak{A})$ we end up with a sum of $D_S(A_1^* \cup A_2, A_3 \ldots, A_n)$ over all possible (α, β) shifts A_1^* of A_1, such that $A_1^* \cap A_2 = \emptyset$. However, if the shift is minimal, we only have one shift which implies that there is only one term in the sum. Q.E.D

Now we want to consider "minimal coalescence", i.e., when the shifted A_2 is as close as possible to A_1. We obtain this by performing minimal right and upward shifts on A_2. The goal is to completely pave the triangle S, and we do this one row at a time. Consider the case where S is a triangle of size l and A_i's are triangles of size 2. Then paving S with A_1 and A_2 minimally and coalescing we get

$$\begin{array}{|llll}
1 & 2 & & \\
1 & 1 & 2 & 2 \\
\hline
\end{array}
\quad \xrightarrow{\textit{coalescing}} \quad
\begin{array}{|llll}
1 & 1 & & \\
1 & 1 & 1 & 1 \\
\hline
\end{array}$$

therefore removing the variables (x_2, y_2) from the equation. Repeating for A_3 we get

$$\begin{array}{|llllll}
1 & 1 & 3 & & & \\
1 & 1 & 1 & 1 & 3 & 3 \\
\hline
\end{array}
\quad \xrightarrow{\textit{coalescing}} \quad
\begin{array}{|llllll}
1 & 1 & 1 & & & \\
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

Note that if the length of the first row of S is even, proceeding as described above, we can completely eliminate the first row of S. Suppose that after coalescing A_3 and A_1 above the first row of S is exhausted. Then we still proceed with minimal shifts of A_4

$$\begin{array}{|llllll}
4 & & & & & \\
1 & 1 & 1 & 4 & 4 & \\
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}
\quad \xrightarrow{\textit{coalescing}} \quad
\begin{array}{|llllll}
1 & & & & & \\
1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 1 & 1 & 1 & 1 \\
\hline
\end{array}$$

Now suppose that the length of the first row of S is odd and after performing $l/2$ shifts followed by coalescence, there is one space remaining in the first row. We want to perform minimal non-cancelling shifts which will completely pave the first row. Note that if we start with A_i in initial position, bottom left-hand corner, we can shift to the following non-cancelling result:

$$\begin{array}{c|ccc} & & & \\ & \mathrm{i} & & \\ & \mathrm{i} & \mathrm{i} & \\ \hline & & & \end{array} \quad \xrightarrow{\textit{coalescing}} \quad \begin{array}{c|cccc} & & & & \\ & & & \mathrm{i} & \mathrm{i} \\ & & & & \mathrm{i} \\ \hline & & & & \end{array}$$

Therefore, we can use this move to fill up the remaining spot in the first row.

Example 3.2.2. *Let S have size 6, and we add A_1. Then, shifting A_2 minimally and coalescing we get*

$$\begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & & & & & & \\ 2 & & & & & & \\ 1 & 1 & 2 & & & & \\ 0 & 1 & 1 & 2 & 2 & & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \quad \xrightarrow{\textit{coalescing}} \quad \begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & & & & & & \\ 2 & & & & & & \\ 1 & 1 & 1 & & & & \\ 0 & 1 & 1 & 1 & 1 & & \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

Repeating for A_3

$$\begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & & & & & & \\ 2 & & & & & & \\ 1 & 1 & 1 & 3 & & & \\ 0 & 1 & 1 & 1 & 1 & 3 & 3 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \xrightarrow{\text{coalescing}} \begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & & & & & & \\ 2 & & & & & & \\ 1 & 1 & 1 & 1 & & & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

Repeating for A_4

$$\begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & & & & & & \\ 2 & 4 & & & & & \\ 1 & 1 & 1 & 1 & 4 & 4 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \xrightarrow{\text{coalescing}} \begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & & & & & & \\ 2 & 1 & & & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

and A_5

$$\begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & 5 & & & & & \\ 2 & 1 & 5 & 5 & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \xrightarrow{\text{coalescing}} \begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & 1 & & & & & \\ 2 & 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

Continuing with A_6

$$\begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & 1 & 6 & 6 & & & \\ 2 & 1 & 1 & 1 & 6 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \quad \xrightarrow{\textit{coalescing}} \quad \begin{array}{c|cccccc} 5 & & & & & & \\ 4 & & & & & & \\ 3 & 1 & 1 & 1 & & & \\ 2 & 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

and finally shifting A_7

$$\begin{array}{c|cccccc} 5 & 7 & & & & & \\ 4 & 7 & 7 & & & & \\ 3 & 1 & 1 & 1 & & & \\ 2 & 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array} \quad \xrightarrow{\textit{coalescing}} \quad \begin{array}{c|cccccc} 5 & 1 & & & & & \\ 4 & 1 & 1 & & & & \\ 3 & 1 & 1 & 1 & & & \\ 2 & 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & 1 & \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 \end{array}$$

we have achieved a complete paving of S and the resulting determinant depends only on (x_1, y_1).

We want to know when such a construction is successful. Under what circumstances will it fail?

3.3 Dimension 2 Pyramids

Definition 3.3.1. Fix integers d, k, s and t such that the following inequalities are satisfied:

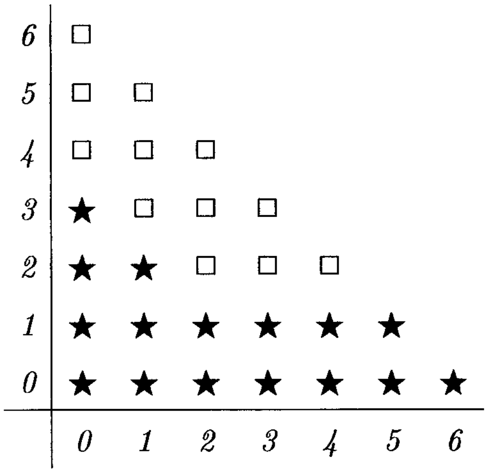
1. $d \geq 0$,
2. $0 \leq k \leq d+1$,
3. $0 \leq s \leq d+1-k$,
4. $0 \leq t \leq \frac{1}{2}s$.

The **pyramid** $\mathcal{P}_d(k,s,t)$ of dimension 2 is the set of indices $\{(i,j) \mid j \geq 0, j \leq k-1, i \geq 0, i+j \leq d\} \cup \{(i,k) \mid 0 \leq i \leq s-1\} \cup \{(i,k+1) \mid 0 \leq i \leq t-1\}$.

Therefore the pyramid $\mathcal{P}_d(k,s,t)$ is contained in the triangle of size $d+1$, where k represents the number of exhausted rows, s is the number of spaces already used in the $k+1$st row and t is the number of filled spaces in the $k+2$nd row. Note that since $t \leq \frac{1}{2}s$ the only pyramids of concern are those in which the t row has less than or equal to half of the number of filled spaces in the s row. Also note that we have the following equalities:

1. $\mathcal{P}_d(k, d+1-k, t) = \mathcal{P}_d(k+1, t, 0)$;
2. $\mathcal{P}_d(d+1, 0, 0) = T_d$, where T_d is a triangle of size $d+1$;
3. $\mathcal{P}_d(0, 2, 1) = T_1$.

Example 3.3.1. *The pyramid* $\mathcal{P}_6(2,2,1)$ *has the following diagram:*



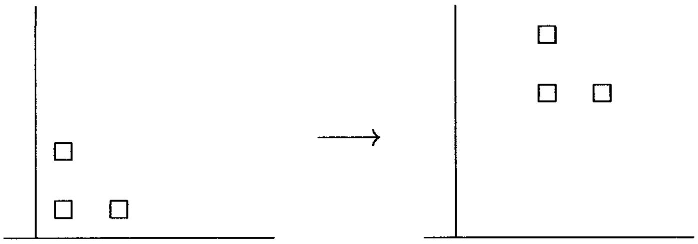
It is contained in a triangle of size 6 and it has the first two rows completely filled, two spaces filled in the $k+1$st row (s row), and one space filled in the $k+2$nd row (t row).

Let A_2 be a triangle of size 2 and let A_1 be a $\mathcal{P}_d(k,s,t)$ pyramid, for some given d,k,s, and t values. The following lemma gives the minimal shift for the pair (A_1,A_2).

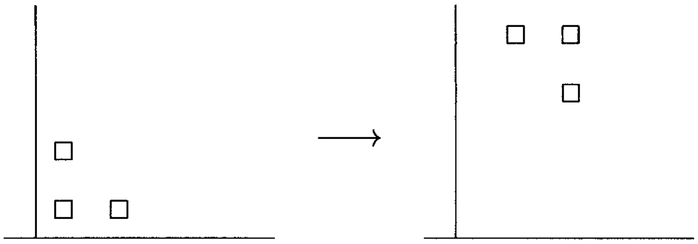
Lemma 3.3.1.

1. *$s \leq d-k-1$ then $(2s+t, 3k)$ is a minimal shift of A_2 and $A_1 \cup A_2^*$ is a $\mathcal{P}_d(k, s+2, t+1)$ pyramid.*
2. *$s = d-k$ then $(s+2t, 3k+1)$ is a minimal shift of A_2 and $A_1 \cup A_2^*$ is a $\mathcal{P}_d(k, s+1, t+2) = \mathcal{P}_d(k+1, t+2, 0)$ pyramid.*

Proof. The results are obtained by keeping track of how A_2 shifts and how this shift influences the number of exhausted rows k and the number of filled spaces in rows $k+1$ and $k+2$. There are two possible ways of shifting A_2 such that the shift is non-cancelling. First we can shift up, then right as needed, maintaining the shape of A_2. Second we can shift up, then right as needed inverting the shape of A_2. Recall that the order in which we shift does not matter, as long as the end result is unique. Thus, we have the following two resulting figures of A_2^*:



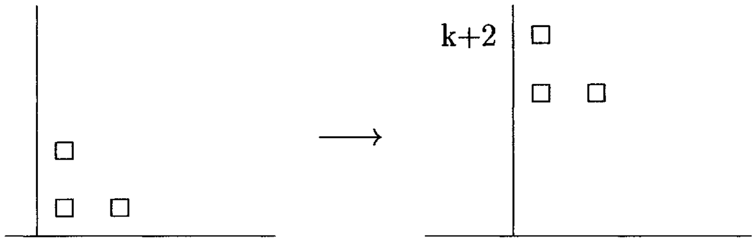
A_2 is shifted by β ups followed by α rights, such that the shape of A_2 is maintained.



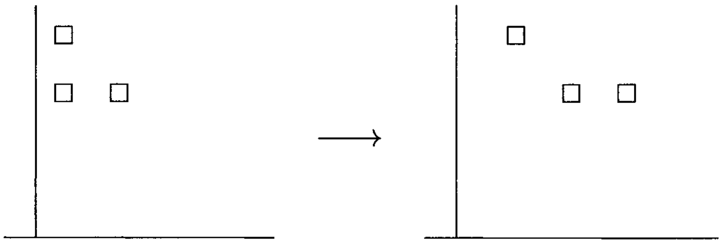
A_2 is shifted by β ups followed by α rights, such that the shape of A_2 is inverted.

Note that we want to use the second way of shifting whenever there is only one space left empty in the k+1st row, since this is the row which is not completely paved.

Suppose there are two spaces left in row $k+1$, i.e., $s \leq d-k-1$. Then we can shift A_2 such that the value of s increases by 2 and that of t increases by 1. This shows that $A_1 \cup A_2^*$ is a $\mathcal{P}_d(k, s+2, t+1)$ pyramid. To see what the minimal shift should be we need to count the spaces we move with each element in A_2. First we shift up to the $k+2$nd row maintaining the shape of A_2, as shown below

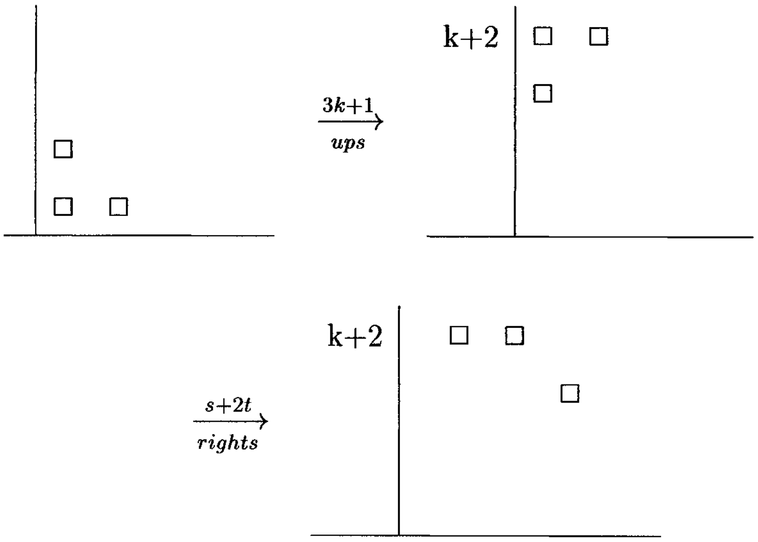


Thus we perform $3k$ upward shifts. Now we move to the edge of A_1, i.e., we move right, just enough to avoid intersection of A_2^* with A_1. Hence we must perform $2s+t$ right shifts.



Therefore the minimal shift $(\alpha, \beta) = (2s+t, 3k)$.

The proof of (2) is similar. Note that this is the case when there is only one space left in the $k+1$st row, and so we need to use the shift which inverts A_2. Thus



in which case the value of s increases by 1, which fills the row, and the value of t increases by 2. Hence $A_1 \cup A_2^*$ is a $\mathcal{P}_d(k, s+1, t+2)$ pyramid. Since the $k+1$st row is now exhausted, we have that $\mathcal{P}_d(k, s+1, t+2) = \mathcal{P}_d(k+1, t+2, 0)$. Q.E.D

Now, we want to see how the pyramid $\mathcal{P}_d(k, s, t)$, for some d, k, s, and t, changes when we do enough shifts of A_2, as above, to exhaust an entire row.

Lemma 3.3.2.

1. *Suppose the number of free places in the $k+1$st row is even, i.e., $d+1-k-s = 2\gamma$. Then after γ minimal shifts, as above, $\mathcal{P}_d(k, s, t)$ becomes the $\mathcal{P}_d(k+1, \gamma+t, 0)$ pyramid.*
2. *Suppose the number of free places in the $k+1$st row is odd, i.e., $d+1-k-s = 2\gamma-1$. Then after γ minimal shifts, as above, $\mathcal{P}_d(k, s, t)$ becomes the $\mathcal{P}_d(k+1, \gamma+t+1, 0)$ pyramid.*

Proof. The proof is a counting argument where we apply the previous lemma γ times. Q.E.D

Corollary 3.3.1.

1. *Starting with $T_1 = \mathcal{P}_d(0, 2, 1)$ if:*
   (a) *d is even, i.e., $d+1-k-s = 2\delta-1$, then after δ minimal shifts we obtain the pyramid $\mathcal{P}_d(1, \delta+2, 0)$;*
   (b) *d is odd, i.e., $d+1-k-s = 2\delta$, then after δ minimal shifts we obtain the pyramid $\mathcal{P}_d(1, \delta+1, 0)$.*
2. *Starting with the $\mathcal{P}_d(k, s, 0)$ pyramid if:*
   (a) *$d+1-k-s$ is even, i.e., $d+1-k-s = 2\delta$, then after δ minimal shifts we obtain the pyramid $\mathcal{P}_d(k+1, \delta, 0)$;*

(b) $d+1-k-s$ is odd, i.e., $d+1-k-s = 2\delta - 1$, then after δ minimal shifts we obtain the pyramid $\mathcal{P}_d(k+1, \delta+1, 0)$.

Thus to get the process of paving started we use part (1) of previous corollary. It can be easily seen that the number of minimal shifts and the resulting pyramid will depend on the parity of d. Once the first row is completed, we repeat part (2) of the corollary as needed. Can we do this and completely pave the triangle T_d, for any given value of d? We will show that for all $d \geq 6$ the process is successful, and for $d \leq 5$ we will show what the exceptions are.

Define a ***DC*** of a polynomial $P(x, y)$ to be the result of a series of derivatives and coalescences.

Lemma 3.3.3 (Reduction to $d - k = 5$). *If A_i's are either T_1's or T_0's, for $i = 1, \ldots, n$, then for $d \geq 6$ there exists a series of DC of $D_{T_d}(A_1, \ldots, A_n)$ which is equal to $D_{T_d}(\mathcal{P}_d(k, s, 0), A'_j s)$, with $j \leq n$, $d - k = 5$ and $0 \leq s \leq 5$.*

Proof. We have already shown in Corollary 3.3.1 on page 39, part (1), that we can always successfully start the paving of T_d by exhausting the first row. Then using part (2) as needed we reduce to the case $d - k = 5$ and $0 \leq s \leq 5$.

Remarks:

1. Note that in the case when $d - k = 5$ and $s = 6$ we get our two exceptions: degree 4 and multiplicity 2, and degree 2 and multiplicity 2. This can be seen by direct calculation:

d - k	5	4	3	2	1	
s	6	0	4	0	3	fails

The construction fails because $s \geq d+1-k$, i.e., the inequality on s is no longer satisfied.

2. To reduce to the case when $d-k=5$ we do induction on $d-k$. Recall that after each step through part (2) of Corollary 3.3.1 $d-k$ decreases by 1. Suppose $d-k=6$. It can be easily seen that for $s=1,2,4$ it reduces to the case $d-k=5$ and $s=3$; for $s=3,5$ it reduces to the case $d-k=5$ and $s=2$; for $s=5$ it reduces to $d-k=5$ and $s=1$; finally, for $s=7$ it reduces to $d-k=5$ and $s=0$. This shows that at the point of arriving at the $d-k=5$ level, we always have $0 \leq s \leq 5$.

Q.E.D

Since the A_i's are either T_1 or T_0, we will denote the determinant $D_{T_d}(A_1,\ldots,A_n)$ by $D_{T_d}(T_1^a T_0^b)$, where $3a+b=\binom{d+2}{2}$ and $b \leq 2$.

Lemma 3.3.4 ($d-k=5$ **case**). *If A_i's are either T_1's or T_0's, for $i=1,\ldots,n$, then there exists a series of DC of $D_{T_d}(T_1^a T_0^b)$, with $b \leq 2$, which is equal to*

1. $D_{T_d}(\mathcal{P}_d(d-5,0,0),T_1^7T_0^0) = D_{T_d}(\mathcal{P}_d(d+1,0,0),T_1^0T_0^0)$; *or*
2. $D_{T_d}(\mathcal{P}_d(d-5,1,0),T_1^6T_0^2) = D_{T_d}(\mathcal{P}_d(d-1,1,0),T_1^0T_0^2)$; *or*
3. $D_{T_d}(\mathcal{P}_d(d-5,2,0),T_1^6T_0^1) = D_{T_d}(\mathcal{P}_d(d,0,0),T_1^0T_0^1)$; *or*
4. $D_{T_d}(\mathcal{P}_d(d-5,3,0),T_1^6T_0^0) = D_{T_d}(\mathcal{P}_d(d+1,0,0),T_1^0T_0^0)$; *or*
5. $D_{T_d}(\mathcal{P}_d(d-5,4,0),T_1^5T_0^2) = D_{T_d}(\mathcal{P}_d(d-1,1,0),T_1^0T_0^2)$; *or*
6. $D_{T_d}(\mathcal{P}_d(d-5,5,0),T_1^5T_0^1) = D_{T_d}(\mathcal{P}_d(d,0,0),T_1^0T_0^1)$,

Proof. We will show this by direct calculation. We apply Corollary 3.3.1 on page 39 to the remaining pyramid $\mathcal{P}_d(k, s, 0)$ where $d - k = 5$. Thus we obtain the following tables:

d - k	5	4	3	2	1	0
s	0	3	1	3	0	1

d - k	5	4	3	2	1	0
s	1	4	2	1	1	done

In this last case, due to the parity of d, in order to complete the construction we must add two simple points to the scheme, i.e., two T_0's. Now for $s = 2$ we have the following table:

d - k	5	4	3	2	1	0
s	2	2	3	2	2	0

Here we only need to add one simple point to complete the paving. For $s = 3, 4, 5$ the table reduces to one of the three case above:

d - k	5	4	3	2	1	0
s	3	2	done			

d - k	5	4	3	2	1	0
s	4	1	done			

d - k	5	4	3	2	1	0
s	5	2	done			

Finally, to see how many T_2's we need for each case we just keep track of the number of minimal shifts performed. Q.E.D

Remark: A similar argument can be applied to the cases when $d-k < 5$. One can easily see by direct calculations, as done above, that the only two cases which fail are $d-k=2$ and $s=0$, and $d-k=4$ and $s=0$. Note that this last one reduces to the first.

Corollary 3.3.2. *$D_{T_d}(A_1^a A_0^b)$ reduces to $D_{T_d}(\mathcal{P}_d(k,s,t), A_1^{a'}, A_0^{b'})$ which is nonzero for all pyramids of dimension two with the following exceptions:*

1. $\mathcal{P}_d(d-4,0,0)$.
2. $\mathcal{P}_d(d-4,2,1)$.
3. $\mathcal{P}_d(d-4,4,2)$.
4. $\mathcal{P}_d(d-2,0,0)$.
5. $\mathcal{P}_d(d-2,2,1)$.

Note that by adding a T_1 to (1) we get (2). By adding a T_1 to (2) we get (3), and so on.

Corollary 3.3.3 (Main theorem). *For all $d \geq 5$ and $d = 1, 3$, $D_{T_d}(T_1^a T_2^b) \neq 0$, where $3a + b = \binom{d+2}{2}$, $b \leq 2$, and the scheme is non-singular.*

3.4 Coalescence in Higher Dimension

What does coalescence mean when we are working with higher dimensions? Consider a general polynomial P in three or more variables. As in the dimension two case, by coalescence we mean replacing three, or more, of the variables with a single one.

Denote by $\mathbf{coal}_{12}\,(\boldsymbol{P})$ the coalescence of the variables (x_1, y_1, z_1) with the variables (x_2, y_2, z_2) in the polynomial P. Again, we want to use the coalescence when paving our pyramid $\mathcal{P}$. By coalescing two smaller pyramids, A_1 and A_2, we reduce the number of variables in the matrix, and therefore in the determinant $D_{\mathcal{P}}(\mathfrak{A})$.

Lemma 3.4.1.

1.

$$coal_{12}\Big(D_{\mathcal{P}}(\mathfrak{A})\Big) = \begin{cases} D_{\mathcal{P}}\big(A_1 \cup A_2, A_3, \dots A_n\big) & \text{if } A_1 \cap A_2 = \emptyset; \\ 0 & \text{if } A_1 \cap A_2 \neq \emptyset. \end{cases}$$

2.

$$coal_{12}\big(P_1 + P_2\big) = coal_{12}\big(P_1\big) + coal_{12}\big(P_2\big).$$

In dimension three, the triangles are going to be replaced by tetrahedrons. So, in this case we need to introduce an outward shift, to account for the extra variable z. We will denote the number of these shifts by γ and it will correspond to taking a derivative with respect to the variable z. Thus the shifts now have the form (α, β, γ), where α and β are as before, right and left shifts, respectively. Therefore, as in the two dimensional case we have that

$$\frac{\partial^{\alpha+\beta+\gamma}}{\partial x_1^\alpha \partial y_1^\beta \partial z_1^\gamma} D_{\mathcal{P}}(\mathfrak{A}) = \sum_{\substack{(\underline{\alpha},\beta,\underline{\gamma}) shifts \\ A_1^* of A_1}} D_{\mathcal{P}}(A_1^*, A_2, \dots, A_n).$$

Applying coalescence to both sides we get that

$$\begin{aligned}
\mathrm{coal}_{12}\left(\frac{\partial^{\alpha+\beta+\gamma}}{\partial x_1^\alpha \partial y_1^\beta \partial z_1^\gamma} D_{\mathcal{P}}(\mathfrak{A})\right) &= \sum_{\substack{(\alpha,\beta,\gamma) shifts \\ A_1^* of A_1}} \mathrm{coal}_{12}\Big(D_{\mathcal{P}}(A_1^*, A_2, \ldots, A_n)\Big) \\
&= \sum_{\substack{(\alpha,\beta,\gamma) shifts\ A_1^* \\ of A_1\ such\ that \\ A_1^* \cap A_2 = \emptyset}} D_{\mathcal{P}}(A_1^* \cup A_2, A_3 \ldots, A_n) \\
&= \sum_{\substack{(\alpha,\beta,\gamma) shifts\ A_1^* \\ of A_1\ such\ that \\ A_1^* \cap A_2 = \emptyset}} \sigma(A_1^*) D_{\mathcal{P}}(A_1^* \cup A_2, A_3 \ldots, A_n) \\
&= \sigma(A_1^*) \sum_{\substack{(\alpha,\beta,\gamma) shifts\ A_1^* \\ of A_1\ such\ that \\ A_1^* \cap A_2 = \emptyset}} D_{\mathcal{P}}(A_1^* \cup A_2, A_3 \ldots, A_n).
\end{aligned}$$

Recall that $M_{\mathcal{P}}(\mathfrak{A})$ was the matrix corresponding to our polynomial system. Now each shift A_1^* of A_1 will correspond to possible reorderings of the rows of $M_{\mathcal{P}}(\mathfrak{A})$. Hence for each such A_1^* we will get that $M_{\mathcal{P}}(\mathfrak{A}) = \mu(A_1^*) M_{\mathcal{P}}(A_1^*, A_2, \ldots, A_n)$, where $\mu(A_1^*) = \pm 1$. Since there might be more than one such A_1^* result when shifting A_1, we get that $\sigma(A_1^*) = \sum \mu(A_1^*)$. We say that the shift is non-cancelling if $\sigma(A_1^*) \neq 0$. This was not an issue for the dimension two case due to the multiplicity always being equal to 2. However, as we have seen in the previous chapter, if the multiplicity is greater than 2, the possibility of cancelling shifts becomes an issue even in dimension 2.

Definition 3.4.1. (α, β, γ) is a **minimal shift** for the pair (A_1, A_2) if there is only one (α, β, γ) shift A_1^* of A_1 for which $A_1^* \cap A_2 = \emptyset$ and it is non-cancelling.

Corollary 3.4.1. *Suppose* (α, β, γ) *is a minimal shift for* (A_1, A_2). *Then*

$$coal_{12}\left(\frac{\partial^{\alpha+\beta+\gamma}}{\partial x_1^\alpha \partial y_1^\beta \partial z_1^\gamma} D_{\mathcal{P}}(\mathfrak{A})\right) = D_{\mathcal{P}}(A_1^* \cup A_2, A_3 \ldots, A_n),$$

up to constant, where A_1^* *is the result of applying the minimal shift* (α, β, γ) *to* A_1 *such that* $A_1^* \cap A_2 = \emptyset$ *and non-cancelling.*

Corollary 3.4.2. *If* $D_{\mathcal{P}}(A_1^* \cup A_2, A_3 \ldots, A_n)$ *is nonzero, then so is* $D_{\mathcal{P}}(\mathfrak{A})$.

Thus we want to consider "minimal coalescence", i.e., when the shifted A_2 is as close as possible to A_1. We obtain this by performing minimal right, upward and outward shifts on A_2. The goal is to reduce the slices of the pyramid $\mathcal{P}$ one at a time.

3.5 Dimension 3 Pyramid

Definition 3.5.1. Fix integers $d, f, k, s, t, k', s', t'$ such that the following inequalities are satisfied:

1. $d \geq 0$,
2. $0 \leq f \leq d+1$,
3. k, s, t are such that we have the pyramid $\mathcal{P}_{d-f}(k, s, t)$ of dimension two.
4. k', s', t' are such that we have the pyramid $\mathcal{P}_{d-f-1}(k', s', t')$ of dimension two.
5. $\left|\mathcal{P}_{d-f-1}(k', s', t')\right| = \frac{1}{3}\left|\mathcal{P}_{d-f}(k, s, t)\right|$

The **pyramid** $\mathcal{P}_d(f, k, s, t, k', s', t')$ of dimension 3 is the set of nonnegative indices $\{(i, j, h) \mid i + j \leq d - f, j \leq k - 1, h \leq d - f\} \cup \{(i, k, h) \mid 0 \leq i \leq s - 1, h \leq d - f\} \cup \{(i, k+1, h) \mid 0 \leq i \leq t - 1, h \leq d - f\} \cup \{(i, k', h) \mid 0 \leq i \leq s' - 1, h \leq d - f - 1\} \cup \{(i, k'+1, h) \mid 0 \leq i \leq t' - 1, h \leq d - f - 1\}$.

The pyramid $\mathcal{P}_d(f, k, s, t, k', s', t')$ is contained in the tetrahedron of size $d+1$, which we will denote by H_d. Thus the pyramid $\mathcal{P}_d(f, k, s, t, k', s', t')$

of dimension three is contained in $d+1$ parallel slices, each of which is a triangle of sizes $d+1, d, \ldots, 1$, respectively. The pyramid has the first f slices completed, a pyramid of dimension two, $\mathcal{P}_{(1)} := \mathcal{P}_{d-f}(k, s, t)$, on the $f+1$st slice, and a pyramid of dimension two, $\mathcal{P}_{(2)} := \mathcal{P}_{d-f-1}(k', s', t')$, on the $f+2$nd slice. Note that the calculations reduce to looking at two pyramids of dimension two which are in a special relationship with each other, namely, if the two pyramids were on the same slice then $\mathcal{P}_{(2)} \subset \mathcal{P}_{(1)}$, and furthermore, $|\mathcal{P}_{(2)}| \leq \frac{1}{3}|\mathcal{P}_{(1)}|$. Also, we obtain the following equalities:

1. $\mathcal{P}_d(0,0,2,1,0,1,0)) = H_1$, the tetrahedron of size 2.

2. $\mathcal{P}_d(d+1,0,0,0,0,0,0) = H_d$, the tetrahedron of size d+1.

Let A_2 be a tetrahedron of size 2 and let A_1 be a pyramid of dimension 3 given by $\mathcal{P}_d(f, k, s, t, k', s', t')$ and contained in H_d, for some $d, f, k, s, t, k', s', t'$ values. The following lemma describes the minimal shift for the pair (A_1, A_2). Recall that the minimal shift has the form (α, β, γ), where α, β and γ are the right, upward and outward shifts, respectively. Again, the order in which we perform the shifts does not matter, and so we are free to choose it.

Lemma 3.5.1. *In the dimension three case we get the following minimal shifts:*

1. *If $k \leq d-f-1$ and $s \leq d-f-k-1$, then $(2s+t+s', 3k+k', 4f)$ is a minimal shift of A_2 and $A_1 \cup A_2^*$ is a $\mathcal{P}_d(f, k, s+2, t+1, k', s'+1, t')$ pyramid.*

2. *If $k \leq d-f-1$ and $s = d-f-k$, then $(s+2t+s', 3k+k'+1, 4f)$ is a minimal shift of A_2 and $A_1 \cup A_2^*$ is a $\mathcal{P}_d(f, k, s+1, t+2, k', s'+1, t') = \mathcal{P}_d(f, k+1, t+2, 0, k'+1, t', 0)$ pyramid.*

3. *If* $k = d - f - 1$, $s = d - f - k$, *and* $s' \le d - f - k - 2$, *then* $(s + t + 2s', 2k + 2k', 4f + 1)$ *is a minimal shift of* A_2 *and* $A_1 \cup A_2^*$ *is a* $\mathcal{P}_d(f, k, s + 1, t + 1, k', s' + 2, t') = \mathcal{P}_d(f + 1, k', s' + 2, t', 0, 0, 0)$ *pyramid.*

4. *If* $k = d - f - 1$, $s = d - f - k$, *and* $s' = d - f - k' - 1$, *then* $(s+t+s'+t', 2k+2k'+1, 4f+1)$ *is a minimal shift of* A_2 *and* $A_1 \cup A_2^*$ *is a* $\mathcal{P}_d(f, k, s+1, t+1, k', s'+1, t'+1) = \mathcal{P}_d(f+1, k'+1, t'+1, 0, 0, 0, 0)$ *pyramid.*

5. *If* $k = d - f$ *and* $s' \le d - f - k' - 2$ *then* $(2s' + t', k + 3k', 4f + 2)$ *is a minimal shift of* A_2 *and* $A_1 \cup A_2^*$ *is a* $\mathcal{P}_d(f, k, s+1, 0, k', s'+2, t'+1) = \mathcal{P}_d(f + 1, k' + 1, t' + 1, 0, 0, 0, 0)$ *pyramid.*

6. *If* $k = d - f$ *and* $s' = d - f - k' - 1$ *then* $(s' + 2t', k + 3k' + 1, 4f + 2)$ *is a minimal shift of* A_2 *and* $A_1 \cup A_2^*$ *is a* $\mathcal{P}_d(f, k, s+1, 0, k', s'+1, t'+2) = \mathcal{P}_d(f + 1, k' + 1, t' + 2, 0, 0, 0, 0)$ *pyramid.*

Proof. Note that we are looking at two pyramids of dimension 2, namely $\mathcal{P}_{(1)} = \mathcal{P}_{d-f}(k, s, t)$ and $\mathcal{P}_{(2)} = \mathcal{P}_{d-f-1}(k', s', t')$. First we need to shift outward and reach these two pyramids, i.e., move away from the completed slices. Hence we need to shift the tetrahedron of size two, A_2. Recall that this tetrahedron is made up of a triangle of size two, which we will denote by B_1, and a triangle of size one, which we will denote by B_0. So, we will follow a similar argument to that of dimension two, but we keep track of two pyramids, $\mathcal{P}_{(1)}$ and $\mathcal{P}_{(2)}$, and two triangles, B_1 and B_0. There are three separate cases which need to be considered: $4f$ outward shifts, $4f+1$ outward shifts, and finally $4f+2$ outward shifts.

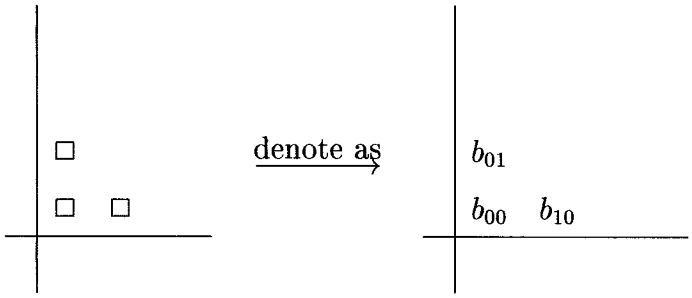
Case I: In this case we have enough space left in pyramid $\mathcal{P}_{(1)}$ to minimally shift B_1. To move away from the completed slices we need to perform exactly $\gamma = 4f$ outward shifts. Note that in $\mathcal{P}_{(2)}$ we only have to worry about B_0, i.e., a single point. So, when performing up and right shifts on B_0 it will only contribute by s' to α and by k' to β. Now looking at $\mathcal{P}_{(1)}$ we have the two scenarios discussed in dimension two:

(a) $s \leq d - f - k - 1$, i.e., at least two spaces are left in this row. So, we get a $2s + t$ contribution to α and $3k$ contribution to β.

(b) $s = d - f - k$, i.e., exactly one space left in this row. Here we get $s + 2t$ contributions to α and $3k + 1$ contribution to β.

Hence this proves statements (1) and (2).

Case II: Here we only have two spaces left in $\mathcal{P}_{(1)}$, i.e., $k = d - f - 1$ and $s = d - f - k$, and so we need to perform one extra outward shift. Thus to move away from the completed slices γ has to be $4f + 1$. First we need to decide which box will move outward. If we perform the extra shift on B_0 then we have too many boxes left in $\mathcal{P}_{(1)}$, which means that we must move a box of B_1 outward. Consider the notation for B_1 illustrated below:



If we perform the extra outward shift on the b_{00} box, this will cause a collision with B_0. So, the only choices left which might work are either moving b_{01} or moving b_{10}. In either case we have two cases to consider. These are similar to those of dimension two, only this time we are looking at pyramid $\mathcal{P}_{(2)}$.

(a) $s' \leq d - f - k' - 2$, i.e., at least two spaces are left in this row. With a careful analysis of all possible shifts, one can see that the only non-cancelling shift exists when we move out the b_{10} box. So we have $s + t + 2s'$ overall contributions to α and $2k + 2k'$ contributions to β.

(b) $s' = d - f - k' - 1$, i.e., exactly one space left in this row. In this case we need to move out the b_{01} box, this being the only move with does not result in a collision or a cancelling shift. So we have $s+t+s'+t'$ contributions to α and $2k+2k'+1$ contributions to β,

which proves statements (3) and (4) of the lemma.

Case III: Here we only have once space left in $\mathcal{P}_{(1)}$, so we must perform two extra outs. Again, the question is which boxes should be moved out? Clearly, if b_{00} is moved we will have a collision with the box in B_0. Hence the only choice we have left is to move b_{01} and b_{10}, which will make γ be equal to $4f + 2$. Note that now we have a triangle of size two in $\mathcal{P}_{(2)}$ and a triangle of size one in $\mathcal{P}_{(1)}$. Using the same notation as before, we now have to shift B_1 in $\mathcal{P}_{(2)}$ and B_0 in $\mathcal{P}_{(1)}$. This is very similar to the first case, where B_0 contributes s to α and k to β. Hence if

(a) $s' \leq d - f - k' - 2$, i.e., at least two spaces are left in this row, so α needs to be equal to $2s' + t'$ and β equal to $k + 3k'$.

(b) $s' = d - f - k' - 1$, i.e., exactly one space left in this row. Thus α has to be $s' + 2t'$ and β has to be $k + 3k' + 1$.

This proves the last two statements.

Q.E.D

Remark: With one exception, we can fill up the pyramid $\mathcal{P}_1$ as in the two dimensional case. While $\mathcal{P}_{(1)}$ is being filled with B_1's, $\mathcal{P}_{(2)}$ is being filled with B_0's, i.e., one box at a time. The exception happens when we have $\mathcal{P}_{(1)} = \mathcal{P}_{d-f}(d-f-2,2,1)$, i.e., there are three boxes left in $\mathcal{P}_{(1)}$, each on different rows and different columns. This is similar to the exception discussed in two dimensions case. Therefore, there is no minimal shift which could move the entire A_i in the position needed to fill these last there spaces. So, we break it up into two separate shifts using one of the last four statements of the lemma stated above. Thus we either leave one box in $\mathcal{P}_{(1)}$ and move two extra in $\mathcal{P}_{(2)}$, followed by the appropriate shift described in statements (3) and (4), or vice versa. Thus the exception encountered in dimension two is no longer a problem.

Since we look at each slice separately, and work with two pyramids of dimension two, we know by previous dimension case that the paving can always be started. Thus we use the two dimensional case to fill the slices of our pyramid one at a time. Can this be done to completely pave the tetrahedron H_d, for any value of d? As before, the process is successful for all $d-f \geq 6$, and for $d-f < 6$ we will treat each case separately, and show the exceptions.

Lemma 3.5.2. *If $d-f \geq 6$ then there exists a minimal shift for any A_i. Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are being increased, by iterating, there exists a series of DC of $D_{H_d}(A_1,\cdots,A_n)$ which is equal to*

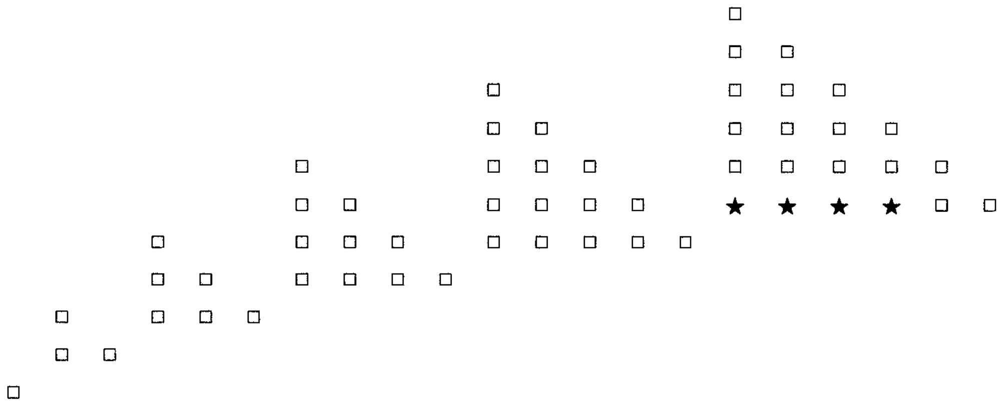
$$D_{H_d}\big(\mathcal{P}_d(f,k,s,0,k',s',0),A_j's\big),$$

with $j \leq n$, $d-f=5$.

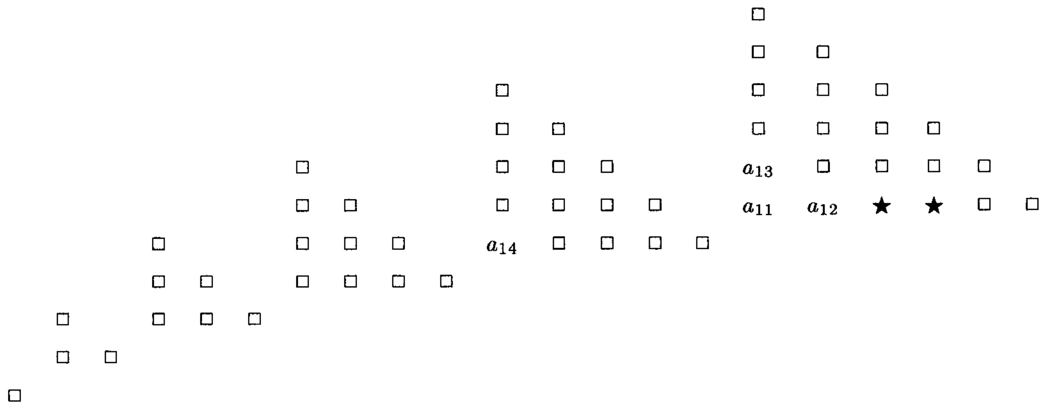
Proof. Again the proof follows from that of the two dimensional case. Since the slice $d-f$ gets modified by at most a T_1, and the slice $d-f-1$ gets modified by at least a T_0 and at most a T_1, we know that the paving up to this point will be successful. Q.E.D

Lemma 3.5.3 (**$d-f \leq 5$ case**). *Let $\mathcal{P} = \mathcal{P}_d(f,k,s,t,k',s',t')$ be a pyramid such that $d-f \leq 5$, $0 \leq k \leq 6$, $0 \leq s \leq 6-k$, $0 \leq t \leq 5-k$, $0 \leq k' \leq 5$, $0 \leq s' \leq 5-k'$, and $0 \leq t' \leq 4-k'$. If A_i's are either H_1's or T_0's, for $i = 1,\ldots,n$, then there exists a series of DC of $D_{\mathcal{P}}(A_1,A_2,\ldots,A_n)$ which completely pave $\mathcal{P}$.*

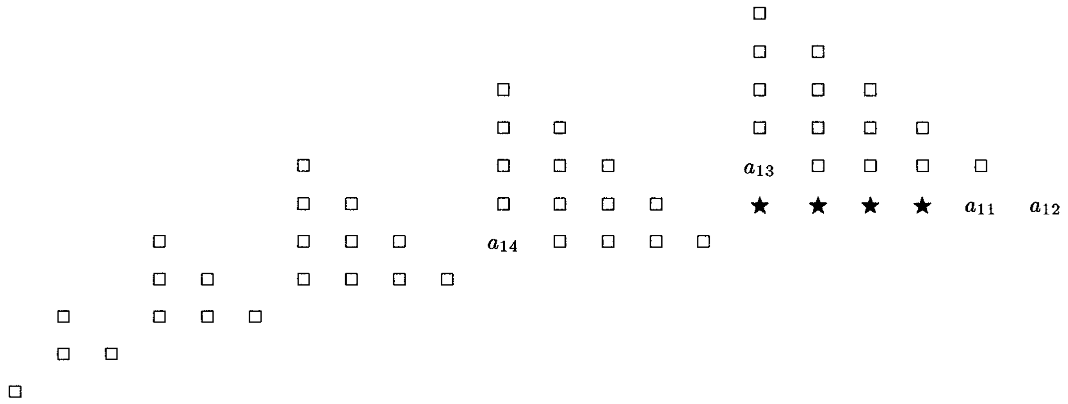
Proof. As in the dimension two case this is shown by direct calculation. Consider the following case as an example of this is accomplished. Suppose $d-f=5$, $k=0$, $s=4$, and $t=0$. Hence we have the following pyramid of dimension 3 to start with, and note that the back face is partially filled, as a result of previous shifts, where the completed faces have been removed.



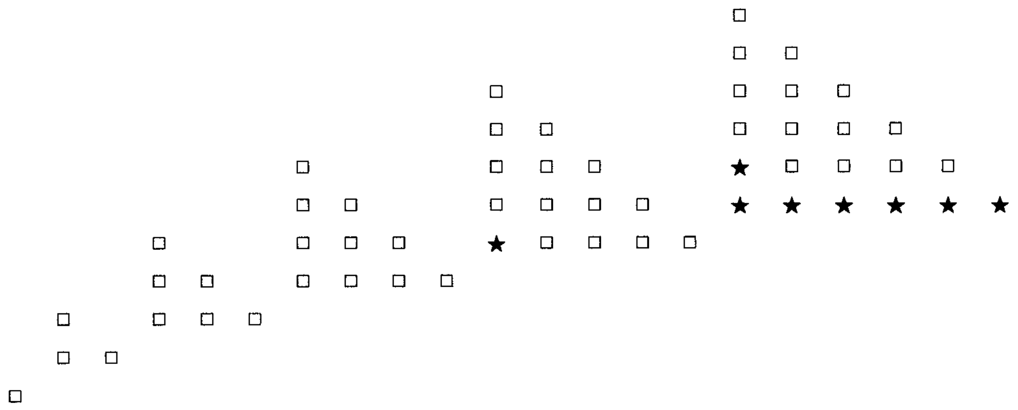
There are 52 empty boxes which implies that we will need 13 tetrahedrons of size 2, denoted by H_1. To begin the process we introduce the first tetrahedron A_1. Recall that as in dimension 2 we always start with the A_i's in the bottom back corner as shown below:



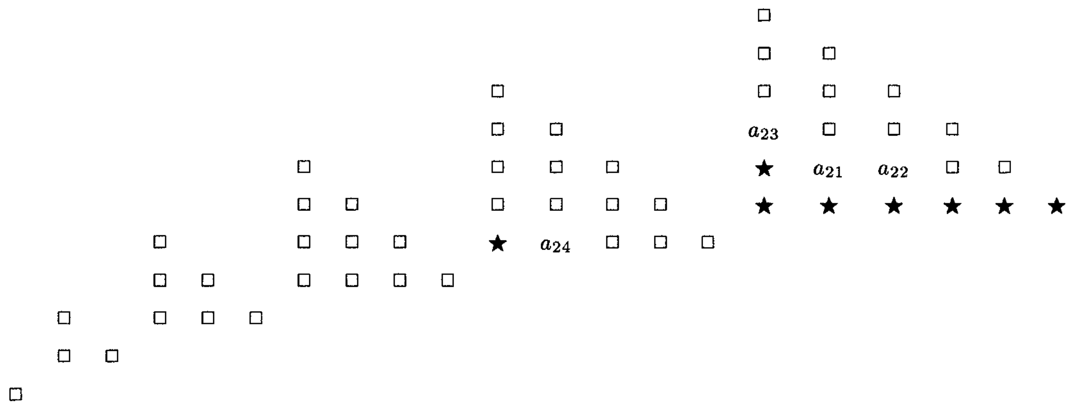
Now we need to perform the required minimal shifts, enough to avoid collision with the filled boxes. Thus we get



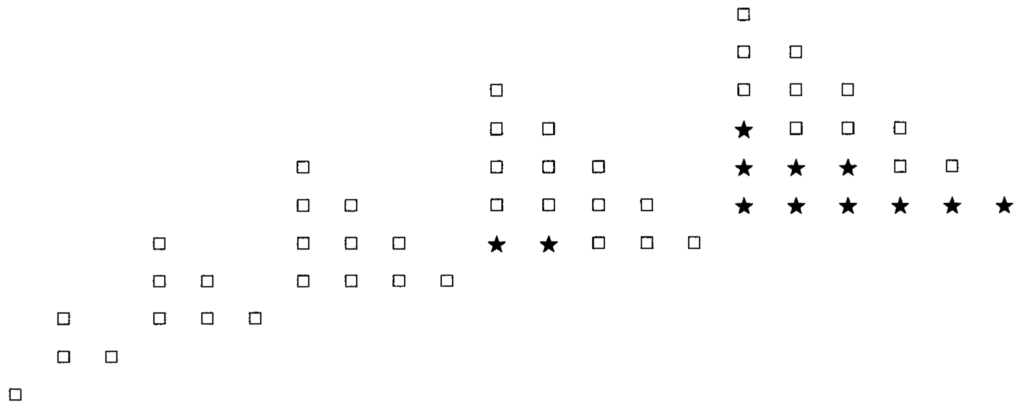
And now we coalesce to get



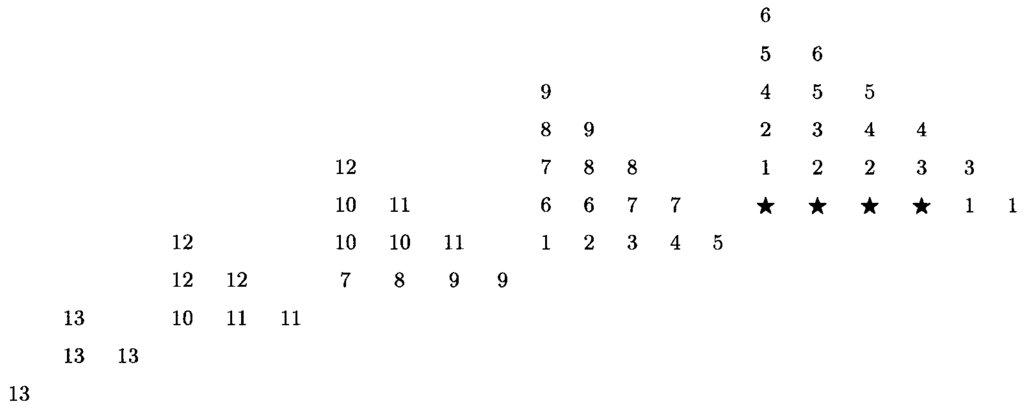
Repeating the process for A_2 we get the following picture after shifting,



and the following after coalescing



Note that we use the necessary minimal shifts that are given by Lemma 3.5.1 on page 47 for each A_i, $0 \le i \le 13$. Repeating the process above for the remaining A_i's, we completely fill in the pyramid. Thus we get the following final picture:



For simplicity, we used i to denote the shifted elements of A_i, where $0 \le i \le 13$. All other cases, 52 in all, are treated in a similar manner to the one described above, thus completing the proof.

Q.E.D

As before, since the A_i's are either H_1 or T_0, we will denote the determinant $D_{H_d}(A_1, \ldots A_n)$ by $D_{H_d}(H_1^a T_0^b)$, where $4a + b = \binom{d+3}{3}$ and $b \le 3$.

Corollary 3.5.1 (Main Theorem). *In dimension three, for all d,* $D_{H_d}(H_1^a T_0^b)$*, where* $4a+b=\binom{d+2}{2}$ *and* $b \leq 3$*, is nonzero and the scheme is non-singular.*

Remark: According to the Alexander-Hirschowitz theorem, in dimension three the exceptions are the following:

1. $d=4$, $r=3$, and $n=9$
2. $d=2$, $r=3$, and $2 \leq n \leq r$.

In the case of the pyramid of degree 4 we have at most

$$\binom{6}{2}+\binom{5}{2}+\binom{4}{2}+\binom{3}{2}+\binom{2}{2}=35$$

empty boxes. Hence we need 8 tetrahedrons of size 2, H_1's, and 3 T_0's to completely fill the pyramid. By adding the three single points we guarantee that the matrix is square. Thus the exceptional case given by the theorem is not a problem. In the case of pyramids of degree 2 we have at most

$$\binom{4}{2}+\binom{3}{2}+\binom{2}{2}=10$$

empty boxes. Hence we need two tetrahedrons of size 2, H_1's, and two T_0's to completely fill the pyramid. As above, by adding the two single points we guarantee that the matrix is square, and the exceptional case given by the theorem is no longer a problem.

It is important to notice that earlier cases never reduce to these exceptions since we will never completely fill a face f without partially filling the $f+1$st face.

Chapter 4

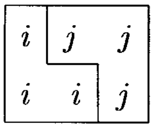
TORIC SURFACES

4.1 Linear Systems of Curves in $\mathbb{P}^1 \times \mathbb{P}^1$

Until now we have discussed problems in dimension two involving only monomials of degree less than or equal to d, i.e., isosceles triangles. We want to develop a few lemmas and then the main theorem about the multiplicity two problem where we want to be able to completely fill a rectangle of size $m \times n$. This corresponds to a linear system of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with prescribed double points. In particular the rectangle of size $m \times n$ represents the monomials forming the basis for the system of curves having bidegree $(m-1, n-1)$.

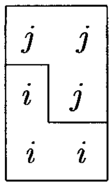
Let us start with some of the basic cases. Note that the first such case is when the rectangle has size $2 \times n$ because if the rectangle had size $1 \times n$ then we could not have multiplicity two. Let $R_{(m,n)}$ denote the rectangle of size $m \times n$. There are going to be two rectangles which are going to be particularly useful when we shift.

Type I: In this case we shift A_i and A_j such that we get the following rectangle:

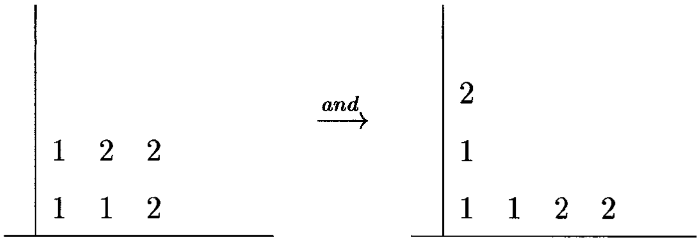


i	j	j
i	i	j

Type II: In this case we shift A_i and A_j such that we get the following rectangle:



Type II rectangles always give unique minimal shifts. However, Type I rectangles give minimal shifts which are not unique. Let α be the number of filled spaces in row i and let β be the number of filled spaces in row $i+1$, the two rows that we are currently working on. If $\alpha = 2\beta$ then we do not get an unique minimal shift. With the same number of upward and right shifts we get two different final positions. For example,



are both final positions obtained by using one upward shift and four right shifts. Therefore, we need to be careful when using Type I rectangles. This problem occurs only when we are working close to the left edge of our rectangle.

In what follows, a and b are going to be positive integers, where a is the number of T_1's and b is the number of T_0's used in our paving of the rectangle $R_{(m,n)}$.

Lemma 4.1.1 **($2 \times n$ Case).** *$D_{R_{(2,n)}}(T_1^a, T_0^b) \neq 0$ for all integer values of $n > 1$, where $b = 0, 1,$ or 2, and the scheme is nonsingular.*

Proof. We will prove this based on how n behaves modulo 3.

1. $n = 0 \pmod 3$

   In this case we can fill $R_{(2,n)}$ using $n/3$ smaller rectangles of Type I.

2. $n = 1 \pmod 3$, i.e., $n \geq 4$

   In this case we break the rectangle into $R_{(2,n-1)}$ and $R_{(2,1)}$. Now we use part (1) to fill the first rectangle and we fill the remaining rectangle with two multiplicity one points, i.e, two T_0's.

3. $n = 2 \pmod 3$, i.e., $n \geq 2$

   In this case we break the rectangle into $R_{(2,n-2)}$ and $R_{(2,2)}$. Now we use part (a) to fill the first rectangle and we fill the second rectangle as follows:

i	T_0
i	i

where $i = \frac{n-2}{3} + 1$.

Q.E.D

Remark: In this case we do not have problems with minimal shifts that are not unique.

Lemma 4.1.2 **($3 \times n$ Case).** *$D_{R_{(3,n)}}(T_1^a) \neq 0$ for all even integer values of $n > 2$ and the scheme is nonsingular.*

Proof. This is shown based on the parity of n. In this case we fill $R_{(3,n)}$ using $n/2$ rectangles of Type II. Q.E.D

Remark: If n is odd the system corresponds to the curves of bidegree $(2, n-1) = (2, 2k)$ where k is a nonzero positive integer. Then there exists a curve of degree $(1, k)$ passing through the points with multiplicity one at each point. Doubling this curve we get our curve of bidegree $(2, 2k)$ which passes through the points with multiplicity two. Therefore, the determinant will be identically zero.

Lemma 4.1.3 ($4 \times n$ **Case**)**.** *$D_{R_{(4,n)}}(T_1^a, T_0^b) \neq 0$ for all integer values of $n > 3$, where $b = 0, 1$, or 2, and the scheme is nonsingular.*

Proof. Since we are looking only at cases where $n > 3$, then 4×4 is the smallest case we have to consider. We use shifts similar to the ones used for the triangles and get the following result:

4	5	5	T_0
3	4	4	5
1	2	3	3
1	1	2	2

Therefore, we are able to completely fill the rectangle using only one T_0 at the end.

The 4×5 rectangle we fill as follows. We split it in $R_{(4,2)}$ and $R_{(4,3)}$. We fill the first using one rectangle of Type II and two T_0's, and we fill the second using two Type I rectangles. Thus we are able to fill it and avoid any non-unique shifts.

Now for the 4×6 rectangle we use our algorithm and we get

6	6	7	7	8	8
4	5	5	6	7	8
1	2	3	4	4	5
1	1	2	2	3	3

and therefore we do not use any T_0's and we avoid non-unique shifts.

To fill the next rectangle, 4 × 7, we split it into $R_{(4,4)}$ and $R_{(4,3)}$. The first we fill as above, and the second is filled using two rectangles of Type I.

Thus if $n = 0 \pmod 3$ we split the rectangle into $R_{(4,6)}$ and $R_{(4,n-6)}$. The first is filled as above and the second is filled using Type I rectangles. If $n = 1 \pmod 3$ we split it into $R_{(4,4)}$, which is filled as above, and $R_{(4,n-4)}$, which is filled using rectangles of Type I, since now $n - 4 = 0 \pmod 3$. Finally, if n=2 (mod 3) we split it into $R_{(4,2)}$ and $R_{(4,n-2)}$. The first is filled using one rectangle of Type II and two T_0's. $R_{(4,n-2)}$ is filled using rectangles of Type I, since $n - 2 = 0 \pmod 3$. Q.E.D

Now we need to look at the case when we have a rectangle of the form $R_{(5,n)}$.

Lemma 4.1.4 (5 × n **Case).** *$D_{R_{(5,n)}}(T_1^a, T_0^b) \neq 0$ for all integer values of $n > 4$, where $b = 0, 1,$ or 2, and the scheme is nonsingular.*

Proof. We will show this based on the parity of n. Let $k \in \mathbb{N}^*$. Suppose $n = 2k$. Then in this case we split the problem into two rectangles, namely $R_{(3,n)}$ and $R_{(2,n)}$. Since n is even we know that we can fill the first by Lemma 4.1.2 part (1) and we can do this without the use of any T_0's. We can fill the second using Lemma 4.1.1 part (1). Thus we don't need to worry about having more than two T_0's overall. Since $R_{(2,n)}$ is the top slice we do not have to worry about shifts that are not unique.

By the above discussion we know we can fill the $R_{(5,6)}$ rectangle without the use of any T_0's. So, we will be done with the proof if we can fill the following rectangles: $R_{(5,5)}$, $R_{(5,7)}$, $R_{(5,8)}$, $R_{(5,9)}$, $R_{(5,10)}$. This follows by induction on n. If we can fill $R_{(5,n)}$ then this implies that we can fill $R_{(5,n+6k)}$, where k is a positive integer without using more than two T_0's. It is easy to see by above that we can fill $R_{(5,8)}$. Thus all we have left to prove is that we can fill $R_{(5,5)}$, $R_{(5,7)}$, and $R_{(5,9)}$.

Let us start with the $R_{(5,5)}$ rectangle. The method for filling this rectangle is similar to the one used above for $R_{(4,4)}$. Thus we get the following diagram:

7	7	8	8	T_0
5	6	6	7	8
4	4	5	5	6
1	2	3	3	4
1	1	2	2	3

For $R_{(5,7)}$ we could not split this into $R_{(5,6)}$ and $R_{(5,1)}$ since we can not fill the second rectangle under any circumstances. Let us split the rectangle into $R_{(5,5)}$ and $R_{(5,2)}$. We know by the above discussion that we can fill both of these. The only thing that we still have to check is that we do not use more than two T_0's. To fill $R_{(5,5)}$ we need to use one T_0. Now to fill $R_{(5,2)}$ we do a further split and look at $R_{(3,2)}$ and $R_{(2,2)}$. The first can be filled using a Type II rectangle, and therefore no T_0's, and the second can be filled using a T_1 and a T_0. Thus we only use two T_0's to fill out rectangle $R_{(5,7)}$.

Finally in order to fill $R_{(5,9)}$ we note that there are a few possible splits. If we try to split it into $R_{(5,6)}$ and $R_{(5,3)}$, we note that the second rectangle is the exceptional case 3 × odd. If we try to split it into $R_{(5,5)}$ and $R_{(5,4)}$ we get into trouble because we used one T_0 to fill $R_{(5,5)}$ and two T_0's to fill $R_{(5,4)}$ which means that we could have use an extra T_1. So, we will try to fill $R_{(5,9)}$ directly, and we get the following:

11	12	12	13	13	14	14	15	15
8	9	10	11	11	12	13	14	15
6	7	7	8	8	9	9	10	10
1	2	3	4	5	5	6	6	7
1	1	2	2	3	3	4	4	5

This completes the proof. Q.E.D

Lemma 4.1.5 **(6 × n Case).** $D_{R_{(6,n)}}(T_1^a, T_0^b) \neq 0$ *for all integer values of* $n > 5$, *where* $b = 0, 1, 2$, *and the scheme is nonsingular.*

Proof. We show this based on the parity of n. Let $k \in \mathbb{N}^*$, and suppose $n = 2k$. Then we can split the rectangle into two $R_{(3,n)}$ slices. By Lemma 4.1.2 we know we can fill this without using any additional T_0's.

Now suppose that $n = 2k + 1$. In this case we split the rectangle into $R_{(6,n-3)}$ and $R_{(6,3)}$. The first can be filled by above, since now $n - 3$ is even, and the second can be filled using three $R_{(2,3)}$ rectangles. Q.E.D

The last basic case that we will need is the $7 \times n$ case. The lemma and proof for this follows.

Lemma 4.1.6 **(7 × n Case).** $D_{R_{(7,n)}}(T_1^a, T_0^b) \neq 0$ *for all integer values of* $n > 6$, *where* $b = 0, 1, 2$, *and the scheme is nonsingular.*

Proof. We will show this based on the parity of n. Let $k \in \mathbb{N}^*$. Suppose $n = 2k$. Then in this case we split the problem into two rectangles, namely $R_{(3,n)}$ and $R_{(4,n)}$. Since n is even we know that we can fill the first by Lemma 4.1.2, and we can do this without the use of any T_0's. We can fill the second using Lemma 4.1.3. Thus we don't need to worry about having more than two T_0's overall.

Remark: By the above discussion we know we can fill the $R_{(7,6)}$ rectangle without the use of any T_0's. So, we will be done with the proof if we can fill the following rectangles: $R_{(7,7)}$, $R_{(7,8)}$, $R_{(7,9)}$. As before this follows by induction on n. If we can fill $R_{(7,n)}$ then this implies that we can fill $R_{(7,n+6k)}$, where k is a positive integer without using more than two T_0's. It is easy to see by above that we can fill $R_{(7,8)}$. Thus all we have left to prove is that we can fill $R_{(7,7)}$, and $R_{(7,9)}$.

Let us start with the $R_{(7,7)}$ rectangle. The method for filling this rectangle is similar to the one used above for $R_{(5,5)}$. Thus we get the following:

14	14	15	15	16	16	T_0
11	12	13	13	14	15	16
9	10	11	11	12	12	13
6	7	8	9	9	10	10
5	6	6	7	7	8	8
1	2	3	4	4	5	5
1	1	2	2	3	3	4

Finally in order to fill $R_{(7,9)}$ we note that there are a few possible splits. However, the split which will work the easiest will be to do a horizontal split into $R_{(5,9)}$ and $R_{(2,9)}$. By Lemma 4.1.4 we know we can fill the first without problems and furthermore, without using any additional T_0's. The second rectangle can be filled using Lemma 4.1.1, part (1), and again we don't use any T_0's. This completes the proof of this case. Q.E.D

These are all the basic rectangles we will need in order to prove the general theorem.

Corollary 4.1.1 (Main theorem). $D_{R_{(m,n)}}(T_1^a, T_0^b) \neq 0$ *for all integer values of* $m, n > 1$, *with the exception* $m = 3$ *and* n *odd, where* $b = 0, 1,$ *or* 2, *and the scheme is nonsingular.*

Proof. Without loss of generality suppose that $m \leq n$, and assume that $7 \leq m \leq n$, since all other cases can be shown using the previous five lemmas. We have four cases cases to consider and in each case we will look at the behavior of m or n modulo 3. Suppose

1. m and n are both even
   (a) $n = 0 \bmod 3$

      In this case we can split the rectangle into $R_{(m,6)}$ and $R_{(m,n-6)}$. The second rectangle can be filled using Type I rectangles since $n - 6$ is divisible by 3. The rectangle $R_{(m,6)}$ is filled using the algorithm: two spaces on row i and one space on row $i + 1$, until we reach the last two rows. Once here we will have three empty spaces on row m. To complete the rectangle we use three inverted triangles. Note that here there are no problems with shifts that are not unique.

   (b) $n = 1 \bmod 3$

      Here we split the rectangle into $R_{(m,4)}$ and $R_{(m,n-4)}$. The rectangle $R_{(m,n-4)}$ can be filled using Type I rectangles since $n - 4 = 0 \pmod 3$. To fill the first rectangle we use the algorithm and 4.1.3, and we use at most two T_0's.

   (c) $n = 2 \bmod 3$

      Here we do a similar split and we look at $R_{(m,2)}$ and $R_{(m,n-2)}$. The second rectangle can be filled without using any T_0's by part (a) above, and the first can be filled using Lemma 4.1.1.

2. m is even and n is odd
   (a) $n = 0 \bmod 3$

      This is similar to part (1a) above.

   (b) $n = 1 \bmod 3$

      This is similar to part (1b) above.

   (c) $n = 2 \bmod 3$

   This is similar to part (1c) above.

3. m is odd and n is even

   (a) $m = 0 \bmod 3$

   In this case we can split the rectangle into $n/2$ columns of $R_{(m,2)}$ rectangles which we know we can fill without using any T_0's by Lemma 4.1.1 part (1).

   (b) $m = 1 \bmod 3$

   In this case we split the rectangle into $R_{(4,n)}$ and $R_{(m-4,n)}$. The first can be filled using 4.1.3. The second is filled using part (a) above.

   (c) $m = 2 \bmod 3$

   Here we split the rectangle into $R_{(m-2,n)}$ and $R_{(2,n)}$. The first rectangle can be filled using part (a) and the second using Lemma 4.1.1. Here the $2 \times n$ rectangle is at the top and so the shifts are all unique.

4. m and n are both odd

   (a) $n = 0 \bmod 3$

   In this case we will do a horizontal split and get rectangles $R_{(5,n)}$ and $R_{(m-5,n)}$. The first rectangle can be filled using Lemma 4.1.4, and the second can be filled with rectangles of Type I, since $m-5$ is now even and n is divisible by 3.

   (b) $n = 1 \bmod 3$

   Here we can split the rectangle into $R_{(3,n-7)}$, $R_{(m-3,n-7)}$ and $R_{(m,7)}$. The first can be filled without using any T_0's by Lemma 4.1.2, since $n-7$ is even. The second rectangle is an even $\times$ even where 3 divides $n-7$. Thus we can fill this using (1a), and again we don't use any T_0's. Finally, the last rectangle is filled using Lemma 4.1.6.

(c) $n = 2 \bmod 3$

In this case we split the rectangle into $R_{(3,n-5)}$, $R_{(m-3,n-5)}$ and $R_{(m,5)}$. The first can be filled without using any T_0's by Lemma 4.1.2, since $n-5$ is even. The second rectangle is an even × even where 3 divides $n-5$. Thus we can fill this using (1a), and again we don't use any T_0's. Finally, the last rectangle is filled using Lemma 4.1.4.

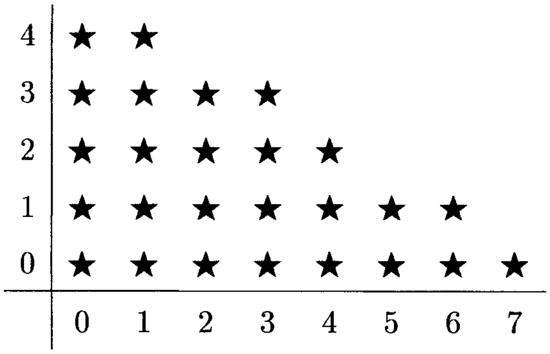
Q.E.D

4.2 Linear Systems of Curves in General Toric Surfaces

We have seen that for problems involving bivariate polynomials the method is successful for both linear systems of bivariate polynomials whose monomials are less that or equal to degree d, and also for linear systems of curves in $\mathbb{P}^1 \times \mathbb{P}^1$. We want to see if the methods developed so far can also be extended to general toric surfaces. We will not describe the details of what a toric surface is, but for an introduction to the theory of toric surfaces one can read [15].

In this situation we are looking at general convex polygons, where the degrees of the monomials are given by the integer lattice points of the polygon. We are going to concentrate on the special case where the polygon is a general right triangle, i.e., we no longer require that it is an isosceles triangle, but we ask that it is convex. We will leave the general convex polygons for future work.

We will denote a general convex triangle by $T_{(d_0,d_1,\ldots,d_{n-1})}$, where the d_i's give the number of integer lattice points on the $i+1$st row of the triangle. Then the lattice point $(i,j) \in T_{(d_0,d_1,\ldots,d_{n-1})}$ if and only if $0 \leq j \leq n-1$ and $0 \leq i \leq d_j - 1$. For example, the triangle $T_{(8,7,5,4,2)}$ has the following lattice of integer points.



It is important to note that since we want $T_{(d_0,d_1,\ldots,d_{n-1})}$ to be convex we must have that:

1. The d_i's form a decreasing sequence of positive integers,

$$d_0 \geq d_1 \geq \cdots \geq d_{n-1} \geq 1.$$

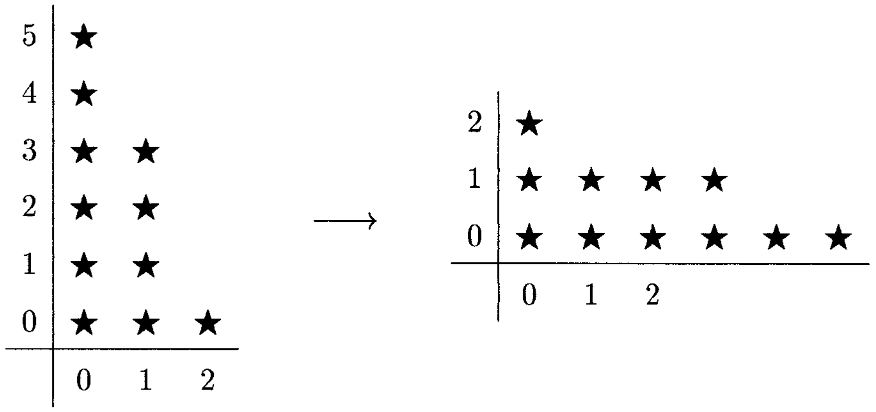
2. Given any three levels $j_1 < j_2 < j_3$,

$$d_{j_2} > \frac{d_{j_3} - d_{j_1}}{j_3 - j_1} j_2 + d_{j_3} - 1 + \frac{j_3(d_{j_3} - d_{j_1})}{j_3 - j_1},$$

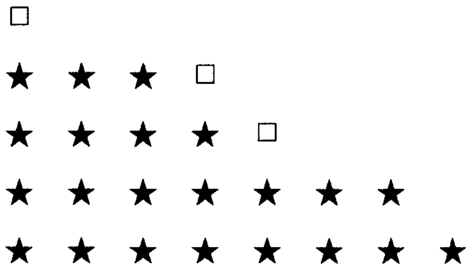
   i.e., the point (d_{j_2}, j_2) is <u>above</u> the line joining the points (d_{j_1}, j_1) and (d_{j_3}, j_4).

Clearly if all d_i's are equal to 1, then we have a $1 \times n$ rectangle which as we have seen in Section 4.1 we can not impose multiplicity two on such curves.

The previous definition of a pyramid also holds in this case. We still have k filled rows and two partially filled rows s and t with the property that $0 \leq t \leq \frac{1}{2}s$. Just as before the order in which we shift does not matter, and so we are free to choose it. Because of this we can always choose as the base of the triangle to be the longest side. For example, if we have $T_{(3,2,2,2,1,1)}$ we can rewrite it as $T_{(6,4,1)}$, and work with the latter triangle. Pictorially this is



The previous discussion of pyramids of dimension 2, isosceles triangles, are subsets of these general triangles and therefore it is safe to assume that the exceptions which occurred there also occur in this case. From the dimension 2 problem we know that there is no minimal shift when we have three empty spaces on separate rows and columns. This is the case when we have the following picture



We already know that in this case there is no unique minimal shift which fills in the three remaining spaces. However, as long as $d_{n-2} \neq 3$, $d_{n-1} \neq 2$, and $d_n \neq 1$, i.e., we do not have an isosceles triangle, we can get around this problem. We use T_1's as prescribed by the algorithm until we are left with exactly one box in row $n-2$, and exactly one box in row $n-1$. Note that at this point row n is completely empty. We then remove the last used triangle and replace it with an inverted one. So we are left with two empty spaces in row $n-2$, and no empty spaces in row $n-1$. Finally we use a T_1 to finish row $n-1$ and the pyramid is then reduced by two rows. The problem case is therefore averted, except in the following situation. We are not able to use the inverted triangle when $2d_{n-1} - 1 = d_{n-2}$. This is because if we have empty spaces on row d_{n-2}, two empty spaces left on row d_{n-1}, and row d_n is empty, then there are two possible end results using the same number or up shifts and right shifts, and therefore the shift is not unique.

With this convention, we notice that we can always start filling in the pyramid and the only rows to which we need to pay closer attention are the last three. The reason for this is that we apply the algorithm us before, we either use T_1's or inverted T_1's, i.e, the two rows that we are working on are being filled two and one, or one and two spaces at a time, respectively.

Lemma 4.2.1 (Reduction to last three rows). *Given a pyramid* $\mathcal{P}$ *with 4 or more rows, there exists a series of DC of* $D(\mathcal{P})$ *which is equal to* $D(\mathcal{P}')$*, where* $\mathcal{P}'$ *is a pyramid with exactly three not completely filled rows.*

Also, we know that there is no minimal shift when we have at least three empty spaces left on the top row, or one empty space left on each of the top three rows, all in the first column. These last two cases do not occur when we are working with isosceles triangles, but both of them can happen for general convex triangles. The case of three spaces left in the same column, and one on each of the top three rows, can be ignored. The reason for this is that in this situation we have a triangle with a large number of d_i's, all of which are small, i.e., a triangle with small base and high altitude. By reversing the base and the altitude we have a triangle with a small number of rows, and a large base, as we have seen above. Thus we have eliminated the possibility of being left with three or more empty spaces, all in the same first column.

As before, if the determinant of the pyramid made up by the last three rows is nonzero, where row $n-2$ may be partially filled, then so is the determinant of the entire pyramid. This is due to the fact that we can remove from the matrix the rows and columns corresponding to the filled rows of the pyramid. If $\mathcal{P}'$ denotes the remaining last three rows and $\mathcal{P}$ denotes the original pyramid, we have that if

$$D(\mathcal{P}', T_1^{a'} T_0^{b'}) \neq 0$$

then

$$D(\mathcal{P}, T_1^a T_0^b) \neq 0,$$

where a, a', b, b' are positive integers with $b, b' \leq 2$.

Proposition 4.2.1 (Reduction to two rows). *Let d_{n-2}, d_{n-1}, and d_n be the lengths of the last three rows, respectively. The number of spaces that have been filled in row $n-2$ is denoted by s, and the number of filled spaces in row $n-1$ is denoted by t. To reduce the pyramid one more row, we have the following cases to consider:*

1. *$d_{n-2} - s$ is odd and $t = \frac{1}{2}(d_{n-2} - s - 1)$*
    - *(a) $d_{n-1} > t + 1$*
    - *(b) $d_{n-1} = t + 1$, $t \geq 1$, and $2 \leq d_n \leq 6$*
    - *(c) $d_{n-1} = t + 1$, $t \geq 1$, and $d_n = 1$ or $d_n \geq 7$*
    - *(d) $d_{n-1} = t + 1$, and $t = 0$*
    - *(e) $d_{n-1} = t$, and $d_n \leq 4$*
    - *(f) $d_{n-1} = t$, and $d_n > 4$*
    - *(g) $d_{n-1} < t$*
2. *$d_{n-2} - s$ is even and $t = \frac{1}{2}(d_{n-2} - s)$*
    - *(a) $d_{n-1} > t$*
    - *(b) $d_{n-1} = t$, and $d_n \leq 2$*
    - *(c) $d_{n-1} = t$, and $d_n > 2$*
    - *(d) $d_{n-1} < t$*

In each of these cases we can reduce the pyramid by at least one row, unless $d_{n-2} = 2d_{n-1} - 1$.

Proof. We will discuss each case separately.

1. $d_{n-2} - s$ is odd and $t = \frac{1}{2}(d_{n-2} - s - 1)$

   (a) $d_{n-1} > t + 1$

   To fill row $n - 2$ we need at least $\frac{1}{2}(d_{n-2} - s - 1) + 2 = t + 2$ spaces in row $n - 1$. Since $d_{n-1} > t + 1$, we can complete the row and therefore reduce the pyramid by one row. We are then left with $d_{n-1} - t$ spaces in row $n - 1$ and d_n spaces in row n.

   (b) $d_{n-1} = t + 1$, $t \geq 1$, and $2 \leq d_n \leq 6$

   Here $d_{n-1} = t + 1 = \frac{1}{2}(d_{n-2} - s - 1) + 1$, and so we can fill row $n - 2$ with the exception of the last space. This also fills row $n - 1$, except for the last space. To finish the pyramid we use at most two inverted T_1's and at most two T_0's.

   (c) $d_{n-1} = t + 1$, $t \geq 1$, and $d_n = 1$ or $d_n \geq 7$

   This is similar to the above. We have one empty space, $(d_{n-2} - 1, n - 2)$, in row $n - 2$, and one empty space, $(d_{n-1} - 1, n - 1)$, in row $n - 1$. Also, row n is completely empty. If $d_{n-2} = d_{n-1}$, and $d_n = 1$, then we can complete the pyramid. If $d_{n-2} \neq d_{n-1}$ and $d_n = 1$ we have the case where there are three empty spaces on each row and each in different columns, and so we can complete the pyramid. If $d_n \geq 7$ we can complete rows $n-2$ and $n-1$, and reduce the pyramid to the last row. However, we are left with at least three empty spaces in row n and so we can not complete the pyramid.

   (d) $d_{n-1} = t + 1$, and $t = 0$

   This means that $d_{n-1} = 1$ which implies that $d_n = 1$. Then we have one box left on each of the last three rows, and we can complete the pyramid using a T_1.

   (e) $d_{n-1} = t$, and $d_n \leq 4$

   In this case row $n-2$ is reduced to on empty space, and row $n-1$ is completely filled. We can use an inverted T_1 to complete the row $n-2$ and two T_0's to complete the pyramid.

   (f) $d_{n-1} = t$, and $d_n > 4$

   This is similar to the above. Once we fill row $n-2$ however, we have at least three empty spaces left in row n, and we know that there is no minimal shift possible which can complete the pyramid.

   (g) $d_{n-1} < t$

   Here row $n-1$ fills up before row $n-2$ is completed. So, the pyramid reduces to two rows, row $n-2$ which has $d_{n-2} - s$ spaces empty and row n which has d_n empty spaces.

2. $d_{n-2} - s$ is even and $t = \frac{1}{2}(d_{n-2} - s)$

   (a) $d_{n-1} > t$

   To complete row $n-2$ we need $\frac{1}{2}(d_{n-2} - s) = t$ empty spaces. Since $d_{n-1} > t$, we can reduce the pyramid by one row. We are left with $d_{n-1} - t$ spaces in row $n-1$ and row n is completely empty.

   (b) $d_{n-1} = t$, and $d_n \leq 2$

   In this case rows $n-2$ and $n-1$ fill up at the same time. So we are only left with the last row and since $d_n \leq 2$, we need at most two T_0's to completely fill the pyramid.

   (c) $d_{n-1} = t$, and $d_n > 2$

   This is similar to the case above. However, in this case we have at least three empty spaces in row n, and we know that there is no minimal shift that can fill the pyramid.

(d) $d_{n-1} < t$

Here row $n-1$ is completed before row $n-2$. Therefore, we are able to reduce the pyramid to two rows. Row $n-2$ has $d_{n-2}-2$ empty spaces left and row n is empty.

Q.E.D

The proposition above shows in fact that we can always reduce the pyramid to the last two rows, except in the case where $d_{n-2} = 3$, $d_{n-1} = 2$, and $d_n = 1$. Once here we have six cases to consider:

1. Rows $n-1$ and n are left, with t filled spaces in row $n-1$ and none in row n.
   (a) $d_{n-1} - t > d_n$
   (b) $d_{n-1} - t = d_n$
   (c) $d_{n-1} - t < d_n$
2. Rows $n-2$ and n are left, with s filled spaces in row $n-2$ and none in row n.
   (a) $d_{n-2} - s > d_n$
   (b) $d_{n-2} - s = d_n$
   (c) $d_{n-2} - s < d_n$

The analysis for (1) and (2) is similar, all we have to do is replace $d_{n-1} - t$ with $d_{n-2} - s$ in the notation. Therefore, we will only explicitly look at (1).

Proposition 4.2.2 (Last two rows). *Let $d_{n-1} - t$ and d_n be the number of unfilled spaces in the last two rows, respectively. To complete the pyramid we have the following cases to consider:*

1. $d_{n-1} - t > d_n$ and $d_{n-1} - t$ is odd

   (a) $\frac{1}{2}(d_{n-1} - t - 1) \leq d_n \leq \frac{1}{2}(d_{n-1} - t - 1) + 4$
   (b) $d_n < \frac{1}{2}(d_{n-1} - t - 1)$ *or* $d_n > \frac{1}{2}(d_{n-1} - t - 1) + 4$

2. $d_{n-1} - t > d_n$ *and* $d_{n-1} - t$ *is even*
   (a) $d_n = \frac{1}{2}(d_{n-1} - t)$
   (b) $d_n = \frac{1}{2}(d_{n-1} - t) - 1$
   (c) $d_n < \frac{1}{2}(d_{n-1} - t) - 1$
   (d) $d_n = \frac{1}{2}(d_{n-1} - t) + 1$ *or* $d_n = \frac{1}{2}(d_{n-1} - t) + 2$
   (e) $d_n = \frac{1}{2}(d_{n-1} - t) + 3$
   (f) $d_n > \frac{1}{2}(d_{n-1} - t) + 3$

3. $d_{n-1} - t = d_n$

4. $d_{n-1} - t < d_n$ *and* d_n *is odd*
   (a) $\frac{1}{2}(d_n - 1) \leq d_{n-1} - t \leq \frac{1}{2}(d_n - 1) + 4$
   (b) $d_{n-1} - t < \frac{1}{2}(d_n - 1)$ *or* $d_{n-1} - t > \frac{1}{2}(d_n - 1) + 4$

5. $d_{n-1} - t < d_n$ *and* d_n *is even*
   (a) $d_{n-1} - t = \frac{1}{2}d_n$
   (b) $d_{n-1} - t = \frac{1}{2}d_n - 1$
   (c) $d_{n-1} - t < \frac{1}{2}d_n - 1$
   (d) $d_{n-1} - t = \frac{1}{2}d_n + 1$ *or* $d_{n-1} - t = \frac{1}{2}d_n + 2$
   (e) $d_{n-1} - t = \frac{1}{2}d_n + 3$
   (f) $d_{n-1} - t > \frac{1}{2}d_n + 3$

We can complete the pyramid in all cases except for (1b), (2c) and (2e), (4b), (5c) and (5e).

Proof. Once again we will look at each individual case.

1. $d_{n-1} - t > d_n$ and $d_{n-1} - t$ is odd

   (a) $\frac{1}{2}(d_{n-1} - t - 1) \leq d_n \leq \frac{1}{2}(d_{n-1} - t - 1) + 4$

   In this case we can completely fill row $n-1$, and we use at most two T_0's to complete the pyramid.

   (b) $d_n < \frac{1}{2}(d_{n-1} - t - 1)$ or $d_n > \frac{1}{2}(d_{n-1} - t - 1) + 4$

   If $d_n < \frac{1}{2}(d_{n-1} - t - 1)$ then row n fills before row $n-1$ is completed, and we are left with at least three empty spaces in row $n-1$. Therefore, the pyramid can not be completely filled. If $d_n > \frac{1}{2}(d_{n-1} - t - 1) + 4$, then row $n-1$ is filled, but we are left with at least three empty spaces in row n, and so the pyramid can not be completed.

2. $d_{n-1} - t > d_n$ and $d_{n-1} - t$ is even

   (a) $d_n = \frac{1}{2}(d_{n-1} - t)$

   As row $n-1$ gets filled, row n also fills up. Pyramid is completed filled when row $n-1$ is exhausted.

   (b) $d_n = \frac{1}{2}(d_{n-1} - t) - 1$

   In this case row n fills up and we are left with two empty spaces in row $n-1$. To complete the pyramid, we use two T_0's.

   (c) $d_n < \frac{1}{2}(d_{n-1} - t) - 1$

   Here row n fills up and there are at least three empty spaces left in row $n-1$. Since there is no minimal shift, the pyramid can not be completed.

   (d) $d_n = \frac{1}{2}(d_{n-1} - t) + 1$ or $d_n = \frac{1}{2}(d_{n-1} - t) + 2$

   There is enough room in row n to allow us to complete row $n-1$. However, there will be one or two empty spaces left in row

n which we can fill using at most two T_0's. Thus the pyramid is completely filled.

   (e) $d_n = \frac{1}{2}(d_{n-1} - t) + 3$

   This is similar to above, except that when row $n-1$ is completed there will be three empty spaces left in row n, and we can not completely fill the pyramid.

   (f) $d_n > \frac{1}{2}(d_{n-1} - t) + 3$

   In this case we can fill rows $n-1$ and n, and we use at most two T_0's. To complete the pyramid we will have to use some T_1's and some inverted T_1's, since the size of row n is 1 or 2 smaller than that of row $n-1$.

3. $d_{n-1} - t = d_n$

   This is a $2 \times d_n$ rectangle, which we know we can completely fill using at most two T_0's, for any value of d_n.

4. $d_{n-1} - t < d_n$ and d_n is odd

   (a) $\frac{1}{2}(d_n - 1) \leq d_{n-1} - t \leq \frac{1}{2}(d_n - 1) + 4$

   (b) $d_{n-1} - t < \frac{1}{2}(d_n - 1)$ or $d_{n-1} - t > \frac{1}{2}(d_n - 1) + 4$

5. $d_{n-1} - t < d_n$ and d_n is even

   (a) $d_{n-1} - t = \frac{1}{2}d_n$

   (b) $d_{n-1} - t = \frac{1}{2}d_n - 1$

   (c) $d_{n-1} - t < \frac{1}{2}d_n$

   (d) $d_{n-1} - t = \frac{1}{2}d_n + 1$ or $d_{n-1} - t = \frac{1}{2}d_n + 2$

   (e) $d_{n-1} - t = \frac{1}{2}d_n + 3$

   (f) $d_{n-1} - t > \frac{1}{2}d_n + 3$

Note that (4) and (5) are similar to (1) and (2), respectively, where the roles of $d_{n-1} - t$ and d_n are reversed.

Q.E.D

Corollary 4.2.1 (Exceptions). *$D(\mathcal{P}, T_1^a T_0^b) \neq 0$, where $b \leq 2$, and the scheme is nonsingular, except in the following cases:*

1. *$d_{n-2} - s$ is odd and $t = \frac{1}{2}(d_{n-2} - s - 1)$*
    - *(a) $d_{n-1} > t + 1$, $d_{n-1} - t > d_n$, $d_{n-1} - t$ is even and $d_n < \frac{1}{2}(d_{n-1} - t) - 1$*
    - *(b) $d_{n-1} > t+1$, $d_{n-1} - t > d_n$, $d_{n-1} - t$ is even and $d_n < \frac{1}{2}(d_{n-1} - t) + 3$*
    - *(c) $d_{n-1} > t+1$, $d_{n-1} - t > d_n$, $d_{n-1} - t$ is odd and $d_n < \frac{1}{2}(d_{n-1} - t - 1)$ or $d_n > \frac{1}{2}(d_{n-1} - t - 1) + 4$*
    - *(d) $d_{n-1} > t + 1$, $d_{n-1} - t < d_n$, d_n is even and $d_{n-1} - t < \frac{1}{2}d_n - 1$*
    - *(e) $d_{n-1} > t + 1$, $d_{n-1} - t < d_n$, d_n is even and $d_{n-1} - t < \frac{1}{2}d_n + 3$*
    - *(f) $d_{n-1} > t + 1$, $d_{n-1} - t < d_n$, d_n is odd and $d_{n-1} - t < \frac{1}{2}(d_n - 1)$ or $d_{n-1} - t > \frac{1}{2}(d_n - 1) + 4$*
    - *(g) $d_{n-1} = t + 1$, $t \geq 1$ and $d_n \geq 7$*
    - *(h) $d_{n-1} = t$ and $d_n > 4$*
    - *(i) $d_{n-1} < t$, $d_{n-2} - s > d_n$, $d_{n-2} - s$ is even and $d_n < \frac{1}{2}(d_{n-2} - s) - 1$*
    - *(j) $d_{n-1} < t$, $d_{n-2} - s > d_n$, $d_{n-2} - s$ is even and $d_n < \frac{1}{2}(d_{n-2} - s) + 3$*
    - *(k) $d_{n-1} < t$, $d_{n-2} - s > d_n$, $d_{n-2} - s$ is odd and $d_n < \frac{1}{2}(d_{n-2} - s - 1)$ or $d_n > \frac{1}{2}(d_{n-2} - s - 1) + 4$*
    - *(l) $d_{n-1} < t$, $d_{n-2} - s < d_n$, $d_{n-2} - s$ is even and $d_{n-2} - s < \frac{1}{2}d_n - 1$*
    - *(m) $d_{n-1} < t$, $d_{n-2} - s < d_n$, $d_{n-2} - s$ is even and $d_{n-2} - s < \frac{1}{2}d_n + 3$*
    - *(n) $d_{n-1} < t$, $d_{n-2} - s < d_n$, $d_{n-2} - s$ is odd and $d_{n-2} - s < \frac{1}{2}(d_n - 1)$ or $d_{n-2} - s < \frac{1}{2}(d_n - 1) + 4$*

2. $d_{n-2} - s$ *is even and* $t = \frac{1}{2}(d_{n-2} - s)$

  (a) $d_{n-1} > t$, $d_{n-1} - t > d_n$, $d_{n-1} - t$ *is even and* $d_n < \frac{1}{2}(d_{n-1} - t) - 1$

  (b) $d_{n-1} > t$, $d_{n-1} - t > d_n$, $d_{n-1} - t$ *is even and* $d_n < \frac{1}{2}(d_{n-1} - t) + 3$

  (c) $d_{n-1} > t$, $d_{n-1} - t > d_n$, $d_{n-1} - t$ *is odd and* $d_n < \frac{1}{2}(d_{n-1} - t - 1)$ *or* $d_n < \frac{1}{2}(d_{n-1} - t - 1) + 4$

  (d) $d_{n-1} > t$, $d_{n-1} - t < d_n$, d_n *is even and* $d_{n-1} - t < \frac{1}{2}d_n - 1$

  (e) $d_{n-1} > t$, $d_{n-1} - t < d_n$, d_n *is even and* $d_{n-1} - t < \frac{1}{2}d_n + 3$

  (f) $d_{n-1} > t$, $d_{n-1} - t < d_n$, d_n *is odd and* $d_{n-1} - t < \frac{1}{2}(d_n - 1)$ *or* $d_{n-1} - t < \frac{1}{2}(d_n - 1) + 4$

  (g) $d_{n-1} = t$ *and* $d_n > 2$

  (h) $d_{n-1} < t$, $d_{n-2} - s > d_n$, $d_{n-2} - s$ *is even and* $d_n < \frac{1}{2}(d_{n-2} - s) - 1$

  (i) $d_{n-1} < t$, $d_{n-2} - s > d_n$, $d_{n-2} - s$ *is even and* $d_n < \frac{1}{2}(d_{n-2} - s) + 3$

  (j) $d_{n-1} < t$, $d_{n-2} - s > d_n$, $d_{n-2} - s$ *is odd and* $d_n < \frac{1}{2}(d_{n-2} - s - 1)$ *or* $d_n < \frac{1}{2}(d_{n-2} - s - 1) + 4$

  (k) $d_{n-1} < t$, $d_{n-2} - s < d_n$, d_n *is even and* $d_{n-2} - s < \frac{1}{2}d_n - 1$

  (l) $d_{n-1} < t$, $d_{n-2} - s < d_n$, d_n *is even and* $d_{n-2} - s < \frac{1}{2}d_n + 3$

  (m) $d_{n-1} < t$, $d_{n-2} - s < d_n$, d_n *is odd and* $d_{n-2} - s < \frac{1}{2}(d_n - 1)$ *or* $d_{n-2} - s < \frac{1}{2}(d_n - 1) + 4$

The linear system of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are subsets of the general toric case. The only exception encountered in the previous section was the case of the $3 \times n$ rectangle, where n was odd. It is easy to see that this is the exception given by (1f) above.

Chapter 5

FUTURE WORK

5.1 Alexander-Hirschowitz Theorem and Dimension N

The definition of the N-dimensional pyramid is straight forward. First we want to understand the space in which such a pyramid lives.

Definition 5.1.1. Define the **N-hedron** of degree d to be the set of indices $\{(i_1, \ldots, i_N) \,|\, i_j \geq 0, \text{for all j, and} \sum i_j \leq d\}$.

Note that the N-hedron is made up of slices each of which is an (N-1)-hedron, and each slice is a subset of the N-hedron with constant i_N. If the degree of the N-hedron is d, then there are a total of $d+1$ slices each being (N-1)-hedrons of degree $d+1, d, \ldots, 1$, respectively. For example, a pyramid of dimension three is contained in the 3-hedron (tetrahedron) which is made up of slices that are 2-hedrons (triangles). Now we are ready to define the pyramid.

Definition 5.1.2. A pyramid of dimension N and degree d, contained in the N-hedron of degree d, is inductively defined to be the subset made up of f completed slices of degrees $d, d-1, \ldots, d-f+1$, respectively, a pyramid of dimension $N-1$ and degree $d-f$ at the $f+1$st slice, and a "smaller" pyramid of dimension $N-1$ and degree $d-f-1$ at the $f+2$nd slice, which is $\frac{1}{N}$th of the size of the previous one.

It is also easy to see that we can reduce the pyramid one face at a time and ignore from the calculations the faces that have been completed.

Lemma 5.1.1. *Let $\mathcal{P}$ be a pyramid of dimension N and denote by $\mathcal{P}'$ the pyramid resulting when the completed faces of $\mathcal{P}$ have been removed. Then, as in dimension two, we have that if*

$$D(\mathcal{P}', T_1^a T_0^b) \neq 0$$

then

$$D(\mathcal{P}, T_1^a T_0^b) \neq 0.$$

Proof. To see this we look at the matrix of $\mathcal{P}$ which has the following general form

$$\mathcal{M} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

Here M_1 is made up of the rows corresponding to the shifted A_i's and M_2 corresponds to the remaining A_i's which have not yet been shifted. Due to coalescence, M_1 has upper triangular form with constants on the "diagonal". Here by diagonal we mean the diagonal of our entire matrix $\mathcal{M}$. Now if we look at M_2 this is the matrix corresponding to $\mathcal{P}'$ where none of the A_i's have been moved yet. To fill in this second pyramid we must take extra derivatives of the rows in the M_2 part of the matrix. Since we shift in $\mathcal{P}'$ only, none of the faces of $\mathcal{P}$ which have already been filled will be affected. This is to say that M_1 will not be affected by taking extra derivatives and so it will remain as before. Therefore, taking these additional derivatives of the rows of M_2 will result in a matrix of the following form:

$$\mathcal{M}^* = \begin{bmatrix} M_1^* & * \\ 0 & M_2^* \end{bmatrix}$$

and both M_1^* and M_2^* are upper triangular sub-matrices. Also note that when we only consider the pyramid $\mathcal{P}'$, the rows on columns used to partially fill $\mathcal{P}$ will be removed from the matrix. This means that we are only left with the M_2^* part of the matrix. Hence if

$$det(M_2^*) = D(\mathcal{P}', T_1^a T_0^b) \neq 0$$

then

$$det(\mathcal{M}) = D(\mathcal{P}, T_1^a T_0^b) \neq 0.$$

Q.E.D

To completely analyze the N-dimensional case we need to look at pyramids of small degree and make sure that there are non-cancelling shifts with which we can completely fill the pyramid. Due to the lack of exceptions for large dimension, it will be sufficient to analyze the end-game for $d \leq 3$. Based on the work done so far in this paper, there is no evidence that this paving can not be done, and that the proof, which will follow by induction of N, can not be completed. All these cases will need to be done "by hand", which will be left for future work.

5.2 General Toric Surfaces

In Section 4.2 on page 68 we discussed the toric surfaces which have as lattice points a triangular convex polygon. We were able to determine in general what the exceptions could be. It is still left to check and see if these exception actually occur, i.e., is it possible that the convexity restrictions will remove some of these, and also under what circumstances polygons will reduce to these.

Also, of interest are the general affine toric surfaces. The varieties corresponding to these surfaces have lattice points of monomials with the shape of a general convex polygon. Here by general we mean that it no longer has the triagular form discussed earlier. This case will also be left for later work.

Bibliography

[1] E. Arbarello and M. Cornalba: *Combinatorial and Algebro-Geometric Cohomology Classes on the Moduli Spaces of Curves*
J. Alg. Geom. 5 (1996), 705-749

[2] C.B. Boyer: *A History of Mathematics*
John Wiley & Sons, Inc., 1991

[3] C. Ciliberto and R. Miranda: *Degenerations of Planar Linear Systems*
J. Reine Angew. Math. 501 (1998), 191-220

[4] C. Ciliberto and R. Miranda: *Linear Systems of Plane Curves with Base Points of Equal Multiplicity*
Trans. Amer. Math. Soc. 352 (2000), 4037-4050

[5] C. Ciliberto, F. Cioffi, R. Miranda, and F. Orecchia: *Bivariate Hermite interpolation and linear systems of plane curves with base fat points*
To appear in: The Asian Symposium on Computer Mathematics [ASCM] 2003

[6] L. Evain: *Une minoration du degré des courbes planes à singularités imposées*
Bull. S.M.F., 126 (1998), 525-543.

[7] A. Hirschowitz: *Une Conjecture pour la Cohomologie des Diviseurs sur les Surfaces Rationnelles Génériques*
J. Reine Angew. Math. 397 (1989), 208-213

[8] R.A. Horn and C.R. Johnson: *Matrix Analysis*
The Press Syndicate of the University of Cambridge, 1990

[9] G.G. Lorentz and R.A. Lorentz: *Bivariate Hermite Interpolation and Applications to Algebraic Geometry*
Numer. Math. 57 (1990) 669 - 680.

[10] R.A. Lorentz: *Multivariate Hermite Interpolation by Algebraic Polynomials: A Survey*
Journal of Computational and Applied Mathematics 122 (2000) 167 - 201

[11] R. Miranda: *Linear Systems of Plane Curves*
Notices AMS 46 (1999) 192 - 202.

[12] T. Mignon: *Systémes linéaires de courbes planes*
Thése, Univ. de Nice (1997).

[13] B. Segre: *Alcune questioni su insiemi finiti di punti in geometria algebrica*
Atti Convegno Intern. di Geom. Alg. di Torino (1961), 15-33.

[14] I. Shafarevich: *Basic Algebraic Geometry 2: Varieties in Projective Space*
Springer - Verlag, 1997. (2nd edition)

[15] W. Fulton: *Introduction to Toric Varieties*
Princeton University Press, 1993.