Groundwater

#2

# PREDICTION OF BEGINNING AND DURATION OF ICE COVER

by

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## SYNOPSIS

Ice cover is related to the capacity of streams to assimilate wastes because it cuts off air contact, and winter conditions may, in certain circumstances, produce worse oxygen deficits than summer conditions, in spite of the slower rates of deoxygenation and the higher oxygen saturation values of cold waters. The annual variation of stream water temperature can be well represented by a sine curve for most streams[2]. However, streams in cold regions may be frozen over for as much as six months per year. In order to represent the annual variation of stream water temperature for these streams, a modification of the sine curve is necessary. The validity of the modified curve for streams that are frozen over for a portion of a year is indicated in that

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[2] Ward, J. C., "Annual Variation of Stream Water Temperature," Journal of the Sanitary Engineering Division, ASCE, Volume 89, No. SA6, Proc. Paper 3710, December, 1963, pages 1-16. Closure, Volume 91, No. SA1, Proc. Paper 4213, February 1965, pages 69-74. Digest, Transactions, ASCE, Volume 130, 1965, pages 258-260.

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the duration and beginning date of ice cover is reasonably well predicted, and the stream water temperatures for the rest of the year are also predicted with a fair amount of accuracy.

The effects of thermal pollution on the sine curve are reviewed, and the possible effect on ice cover is indicated.

The possible application of the sine curve to lakes and reservoirs is illustrated.

Notation: The symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in the Appendix.

## INTRODUCTION

It has been shown[2] that the following equation closely fits the annual variation of temperature at a given point on a stream:

$$T_c = a\left[\sin(bx + c)\right] + \overline{T}_c \tag{1}$$

where $T_c$ = calculated temperature of the stream water, degrees Fahrenheit (°F)

$a$ = amplitude, °F

$b$ = 0.987 degrees per day (or 0.0172 radians per day)

$x$ = number of days since October 1 ($x = 1$ for October 1), days

$c$ = phase coefficient, degrees

$\overline{T}_c$ = arithmetic mean or average value of $T_c$ (if all values of $T_c$ are distributed at uniform intervals of time throughout the year), °F.

Equation 1 may also be stated as follows:

$$T_c = p \sin bx + q \cos bx + \overline{T}_c \tag{2}$$

where

$$c = \tan^{-1}\left(\frac{q}{p}\right) \tag{3}$$

and

$$a = \sqrt{p^2 + q^2} \tag{4}$$

where p and q = parameters, $^0$F.

The standard error of estimate for a since curve is:

$$S_T = \sqrt{\frac{\Sigma(T - T_c)^2}{n - 5}} \tag{5}$$

where $S_T$ = standard error of estimate, $^0$F

$T$ = observed water temperature, $^0$F

$n$ = number of pairs of observed values of $T$ and $x$, dimensionless.

The index of correlation is

$$\rho = \sqrt{1 - \left(\frac{n - 1}{n - 5}\right)\left[\frac{\Sigma(T - T_c)^2}{\Sigma(T - \overline{T}^2)}\right]} \tag{6}$$

where $\rho$ = index of correlation, dimensionless

$\overline{T}$ = arithmetic mean or average value of $T$, $^0$F

The least squares normal equations for equation 2 are:

$$p\Sigma \sin^2 bx + q\Sigma \sin bx \cos bx + \overline{T}_c \Sigma \sin bx = \Sigma T \sin bx \tag{7}$$

$$p\Sigma \sin bx \cos bx + q\Sigma \cos^2 bx + \overline{T}_c \Sigma \cos bx = \Sigma T \cos bx \tag{8}$$

$$p\Sigma \sin bx + q\Sigma \cos bx + \overline{T}_c n = \Sigma T \tag{9}$$

which can be written as follows:

$$pa_{11} + qa_{12} + \overline{T}_c\, a_{13} = a_{14} \tag{10}$$

$$pa_{21} + qa_{22} + \overline{T}_c\, a_{23} = a_{24} \tag{11}$$

$$pa_{31} + qa_{32} + \overline{T}_c\, a_{33} = a_{34} \tag{12}$$

where

$$a_{11} = \Sigma \sin^2 bx \tag{13}$$

$$a_{12} = \Sigma \sin bx \cos bx = a_{21} \tag{14}$$

$$a_{13} = \Sigma \sin bx = a_{31} \tag{15}$$

$$a_{14} = \Sigma T \sin bx \tag{16}$$

$$a_{22} = \Sigma \cos^2 bx \tag{17}$$

$$a_{23} = \Sigma \cos bx = a_{32} \tag{18}$$

$$a_{24} = \Sigma T \cos bx \tag{19}$$

$$a_{33} = n \tag{20}$$

and

$$a_{34} = \Sigma T \tag{21}$$

If $\gamma$ is defined as follows,

$$\gamma \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) +$$
$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22}) \tag{22}$$

then

$$p = \frac{\begin{vmatrix} a_{14} & a_{12} & a_{13} \\ a_{24} & a_{22} & a_{23} \\ a_{34} & a_{32} & a_{33} \end{vmatrix}}{\gamma} \tag{23}$$

$$q = \frac{\begin{vmatrix} a_{11} & a_{14} & a_{13} \\ a_{21} & a_{24} & a_{23} \\ a_{31} & a_{34} & a_{33} \end{vmatrix}}{\gamma} \tag{24}$$

and

$$\overline{T}_c = \frac{\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \end{vmatrix}}{\gamma} \tag{25}$$

## ILLUSTRATIVE EXAMPLE

The optimum method of solution of the above equations is by means of a computer using all values of $T > 32^0F$. However, a computer is not always available. In this case, an approximate method for solution of the above equations may be used. This method is illustrated in the following example.

The Geological Survey records temperature to the nearest $^0F$ and the thermometers used for determining water temperature are accurate to $\pm 0.5^0F$. Therefore, the error in temperature measurements will be a maximum of $\pm 1^0F$. When only one temperature observation is made each day, water temperatures are taken at about the same time each day, if practicable, so that the data will be relatively unaffected by diurnal variations in temperature. At stations where thermographs are located, the records consist of maximum and minimum temperatures for each day. If a stream is frozen over, a temperature of $32^{0}F$ is assumed although the temperature of the ice may be much lower.

Because pure water can exist as both a solid and liquid at $32^{0}F$, it is not clear from temperature records when a stream actually has an ice cover. In addition, temperatures of less than $32^{0}F$ are almost never reported (very rarely $31^{0}F$), yet one would expect the temperature of the ice cover to be $< 32^{0}F$.

Another interesting phenomena observed on some streams in cold climates is several ice covers. In other words, when one penetrates the top ice cover, there is a space between it and the next one underneath, etc., and there exists several layers of ice.

In order to illustrate the application of the above equations, the maximum daily temperatures for the 1962 water year for Henry's Fork near Rexburg, Idaho, given in Table 1, will be used[3]. Because 365 values of $x$ and $T$ involve far too many calculations, unless a computer

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[3] Quality of Surface Waters of the United States, 1962, Parts 9-14, Colorado River Basin to Pacific Slope Basins in Oregon and Lower Columbia River Basin, Geological Survey Water-Supply Paper 1945, United States Government Printing Office, Washington, 1964, page 578.

TABLE 1. -MAXIMUM DAILY TEMPERATURES

| x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F | x, days | T, °F |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Oct. | | Nov. | | Dec. | | Jan. | | Feb. | | Mar. | | Apr. | | May | | June | | July | | Aug. | | Sept. | |
| 001 | 49 | 032 | 40 | 062 | 35 | 093 | 32 | 124 | 32 | 152 | 32 | 183 | 46 | 213 | 48 | 244 | 54 | 274 | 66 | 305 | 66 | 336 | 60 |
| 002 | 50 | 033 | 39 | 063 | 35 | 094 | 32 | 125 | 32 | 153 | 32 | 184 | 46 | 214 | 48 | 245 | 55 | 275 | 66 | 306 | 67 | 337 | 61 |
| 003 | 50 | 034 | 37 | 064 | 35 | 095 | 32 | 126 | 32 | 154 | 32 | 185 | 46 | 215 | 50 | 246 | 55 | 276 | 64 | 307 | 67 | 338 | 61 |
| 004 | 50 | 035 | 37 | 065 | 35 | 096 | 32 | 127 | 32 | 155 | 32 | 186 | 47 | 216 | 52 | 247 | 49 | 277 | 62 | 308 | 67 | 339 | 61 |
| 005 | 52 | 036 | 37 | 066 | 34 | 097 | 32 | 128 | 32 | 156 | 32 | 187 | 47 | 217 | 52 | 248 | 46 | 278 | 64 | 309 | 67 | 340 | 62 |
| 006 | 52 | 037 | 36 | 067 | 34 | 098 | 32 | 129 | 32 | 157 | 32 | 188 | 45 | 218 | 52 | 249 | 46 | 279 | 67 | 310 | 65 | 341 | 62 |
| 007 | 48 | 038 | 37 | 068 | 32 | 099 | 32 | 130 | 32 | 158 | 32 | 189 | 46 | 219 | 53 | 250 | 51 | 280 | 68 | 311 | 67 | 342 | 62 |
| 008 | 46 | 039 | 37 | 069 | 32 | 100 | 32 | 131 | 32 | 159 | 32 | 190 | 46 | 220 | 53 | 251 | 56 | 281 | 68 | 312 | 67 | 343 | 60 |
| 009 | 46 | 040 | 37 | 070 | 32 | 101 | 32 | 132 | 32 | 160 | 32 | 191 | 44 | 221 | 53 | 252 | 57 | 282 | 69 | 313 | 67 | 344 | 58 |
| 010 | 45 | 041 | 37 | 071 | 32 | 102 | 32 | 133 | 32 | 161 | 33 | 192 | 45 | 222 | 52 | 253 | 59 | 283 | 70 | 314 | 67 | 345 | 56 |
| 011 | 45 | 042 | 38 | 072 | 32 | 103 | 32 | 134 | 32 | 162 | 34 | 193 | 46 | 223 | 51 | 254 | 59 | 284 | 71 | 315 | 67 | 346 | 56 |
| 012 | 46 | 043 | 38 | 073 | 32 | 104 | 32 | 135 | 32 | 163 | 35 | 194 | 48 | 224 | 51 | 255 | 59 | 285 | 70 | 316 | 68 | 347 | 56 |
| 013 | 50 | 044 | 37 | 074 | 32 | 105 | 32 | 136 | 32 | 164 | 34 | 195 | 50 | 225 | 49 | 256 | 59 | 286 | 67 | 317 | 68 | 348 | 56 |
| 014 | 52 | 045 | 37 | 075 | 32 | 106 | 32 | 137 | 32 | 165 | 34 | 196 | 52 | 226 | 48 | 257 | 60 | 287 | 63 | 318 | 68 | 349 | 59 |
| 015 | 52 | 046 | 37 | 076 | 32 | 107 | 32 | 138 | 32 | 166 | 35 | 197 | 52 | 227 | 45 | 258 | 60 | 288 | 63 | 319 | 69 | 350 | 60 |
| 016 | 52 | 047 | 36 | 077 | 32 | 108 | 32 | 139 | 32 | 167 | 36 | 198 | 52 | 228 | 48 | 259 | 56 | 289 | 64 | 320 | 69 | 351 | 60 |
| 017 | 52 | 048 | 34 | 078 | 32 | 109 | 32 | 140 | 32 | 168 | 37 | 199 | 52 | 229 | 51 | 260 | 58 | 290 | 65 | 321 | 67 | 352 | 60 |
| 018 | 52 | 049 | 34 | 079 | 32 | 110 | 32 | 141 | 32 | 169 | 38 | 200 | 52 | 230 | 51 | 261 | 60 | 291 | 66 | 322 | 67 | 353 | 60 |
| 019 | 49 | 050 | 35 | 080 | 32 | 111 | 32 | 142 | 32 | 170 | 38 | 201 | 53 | 231 | 53 | 262 | 62 | 292 | 66 | 323 | 67 | 354 | 61 |
| 020 | 50 | 051 | 34 | 081 | 32 | 112 | 32 | 143 | 32 | 171 | 38 | 202 | 53 | 232 | 53 | 263 | 64 | 293 | 68 | 324 | 67 | 355 | 61 |
| 021 | 50 | 052 | 35 | 082 | 32 | 113 | 32 | 144 | 32 | 172 | 38 | 203 | 49 | 233 | 50 | 264 | 64 | 294 | 70 | 325 | 67 | 356 | 59 |
| 022 | 45 | 053 | 34 | 083 | 32 | 114 | 32 | 145 | 32 | 173 | 38 | 204 | 49 | 234 | 50 | 265 | 63 | 295 | 71 | 326 | 66 | 357 | 59 |
| 023 | 41 | 054 | 36 | 084 | 32 | 115 | 32 | 146 | 32 | 174 | 41 | 205 | 51 | 235 | 52 | 266 | 63 | 296 | 71 | 327 | 65 | 358 | 60 |
| 024 | 41 | 055 | 36 | 085 | 32 | 116 | 32 | 147 | 32 | 175 | 41 | 206 | 51 | 236 | 52 | 267 | 66 | 297 | 70 | 328 | 64 | 359 | 60 |
| 025 | 41 | 056 | 37 | 086 | 32 | 117 | 32 | 148 | 32 | 176 | 41 | 207 | 51 | 237 | 51 | 268 | 66 | 298 | 69 | 329 | 64 | 360 | 60 |
| 026 | 41 | 057 | 37 | 087 | 32 | 118 | 32 | 149 | 32 | 177 | 42 | 208 | 50 | 238 | 53 | 269 | 66 | 299 | 69 | 330 | 64 | 361 | 60 |
| 027 | 41 | 058 | 37 | 088 | 32 | 119 | 32 | 150 | 32 | 178 | 42 | 209 | 50 | 239 | 53 | 270 | 66 | 300 | 67 | 331 | 64 | 362 | 59 |
| 028 | 41 | 059 | 37 | 089 | 32 | 120 | 32 | 151 | 32 | 179 | 42 | 210 | 46 | 240 | 49 | 271 | 65 | 301 | 66 | 332 | 62 | 363 | 58 |
| 029 | 40 | 060 | 37 | 090 | 32 | 121 | 32 | | | 180 | 41 | 211 | 44 | 241 | 49 | 272 | 64 | 302 | 66 | 333 | 59 | 364 | 57 |
| 030 | 40 | 061 | 36 | 091 | 32 | 122 | 32 | | | 181 | 43 | 212 | 45 | 242 | 50 | 273 | 65 | 303 | 66 | 334 | 59 | 365 | 57 |
| 031 | 40 | | | 092 | 32 | 123 | 32 | | | 182 | 45 | | | 243 | 54 | | | 304 | 66 | 335 | 59 | | |
| Average values of x and T | | | | | | | | | | | | | | | | | | | | | | | |
| 16 | 47 | 46 | 37 | 77 | | 108 | | 137 | | 167 | | 198 | 48 | 228 | 51 | 258 | 59 | 289 | 67 | 320 | 66 | 350 | 59 |
| Hypothetical average values of x | | | | | | | | | | | | | | | | | | | | | | | |
| 15 | | 46 | | 76 | | 106 | | 137 | | 167 | | 198 | | 228 | | 258 | | 289 | | 319 | | 350 | |

is used, it is desirable to obtain fewer values of x and T that will be representative of several days each.

One approximation is to use the average monthly temperatures even though the temperature variation during a month is not arithmetically linear. However, for this approximation, no monthly averages that include any daily temperatures of $32^0$F can be used. Another source of error is due to the fact that each monthly average does not contain the same number of days. However, the middle day of each month is reasonably close to the value of x obtained if the year were divided into 12 equal parts as is shown in Table 1. For plotting purposes, the middle day of each month (average value of x ) should be used for observed data, and the hypothetical average values of x should be used for calculated temperatures.

Table 2 illustrates the method of application. It should be noted that in columns 3 through 9 it was assumed that the year was divided into 12 equal parts which is another approximation. The reason for this approximation is to simplify the calculations when $n = 12$, because when $n = 12$ , the sums of columns 3, 4 and 5 are all zero, and the sums of columns 6 and 7 are both 6, so that equations 23, 24 and 25 reduce to

$$p = \frac{\Sigma T \sin bx}{6} \qquad (\text{if } n = 12) \qquad (26)$$

$$q = \frac{\Sigma T \cos bx}{6} \qquad (\text{if } n = 12) \qquad (27)$$

TABLE 2. EXAMPLE CALCULATIONS FOR MONTHLY AVERAGES

| Month (1) | T, $^0F$ (2) | sin bx (3) | cos bx (4) | sin bx times cos bx (5) | $\sin^2$ bx (6) | $\cos^2$ bx (7) | T sin bx, $^0F$ (8) | T cos bx, $^0F$ (9) | $T - \overline{T}$, $^0F$ (10) | $(T - \overline{T})^2$, $^0F^2$ (11) | p sin bx, $^0F$ (12) | q cos bx, $^0F$ (13) | $T_c$, $^0F$ (14) | $T - T_c$, $^0F$ (15) | $(T - T_c)^2$, $^0F^2$ (16) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Oct. | 47 | 0.259 | 0.966 | 0.250 | 0.067 | 0.933 | 12.2 | 45.4 | -7 | 49 | -4.6 | 6.84 | 48.5 | -1 | 1 |
| Nov. | 37 | 0.707 | 0.707 | 0.500 | 0.500 | 0.500 | 26.2 | 26.2 | -17 | 289 | -12.6 | 5.00 | 38.7 | 2 | 4 |
| Dec. | | 0.966 | 0.259 | 0.250 | 0.933 | 0.067 | | | | | -17.2 | 1.83 | 30.9 | | |
| Jan. | | 0.966 | -0.259 | -0.250 | 0.933 | 0.067 | | | | | -17.2 | -1.83 | 27.3 | | |
| Feb. | | 0.707 | -0.707 | -0.500 | 0.500 | 0.500 | | | | | -12.6 | -5.00 | 28.7 | | |
| Mar. | | 0.259 | -0.966 | -0.250 | 0.067 | 0.933 | | | | | -4.6 | -6.84 | 34.8 | | |
| Apr. | 48 | -0.259 | -0.966 | 0.250 | 0.067 | 0.933 | -12.4 | -46.4 | -6 | 36 | 4.6 | -6.84 | 44.1 | 4 | 16 |
| May | 51 | -0.707 | -0.707 | 0.500 | 0.500 | 0.500 | -36.0 | -36.0 | -3 | 9 | 12.6 | -5.00 | 53.9 | -3 | 9 |
| June | 59 | -0.966 | -0.259 | 0.250 | 0.933 | 0.067 | -57.0 | -15.3 | 5 | 25 | 17.2 | -1.83 | 61.7 | -3 | 9 |
| July | 67 | -0.966 | 0.259 | -0.250 | 0.933 | 0.067 | -64.7 | 17.4 | 13 | 169 | 17.2 | 1.83 | 65.3 | 2 | 4 |
| Aug. | 66 | -0.707 | 0.707 | -0.500 | 0.500 | 0.500 | -46.6 | 46.6 | 12 | 144 | 12.6 | 5.00 | 63.9 | 2 | 4 |
| Sept. | 59 | -0.259 | 0.966 | -0.250 | 0.067 | 0.933 | -15.3 | 57.0 | 5 | 25 | 4.6 | 6.84 | 57.8 | 1 | 1 |
| $\Sigma$ or $a_{ij}$ | 434 | -2.898 | 1.673 | 0.750 | 3.567 | 4.433 | -193.6 | 94.9 | 2 | 746 | 0.0 | 0.00 | 555.6 | 0 | 48 |
| $a_{ij}$ | $a_{34}$ | $a_{13}=a_{31}$ | $a_{23}=a_{32}$ | $a_{12}=a_{21}$ | $a_{11}$ | $a_{22}$ | $a_{14}$ | $a_{24}$ | | | | | | | |

and

$$\overline{T}_c = \frac{\Sigma T}{n} \qquad (\text{if } n = 12) \qquad (28)$$

respectively.

The sums of columns 3 through 7 in Table 2 do not include the values listed for December through March. $\overline{T}$ is calculated from the following equation:

$$\overline{T} = \frac{\Sigma T}{n} \qquad (29)$$

The equation obtained for the calculations shown in Table 2 is:

$$T_{max} = -19.2 \sin(0.987x - 21.7) + 46.3 \qquad (30)$$

where $T_{max}$ = calculated maximum daily temperature of the stream water, $^0F$.

Values of $S_T$ and $\rho$ obtained were 4.00 $^0F$ and 0.922, respectively. The equation obtained by analyzing the minimum daily temperatures in the same manner is

$$T_{min} = -18.0 \sin(0.987x - 22.3) + 45.1 \qquad (31)$$

where $T_{min}$ = calculated minimum daily temperature of the stream water, $^0F$.

The values of $S_T$ and $\rho$ obtained were -3.78$^0F$ and 0.922, respectively.

Equations 30 and 31 are plotted in Fig. 1 along with the observed average monthly maximum and minimum stream water temperatures. From a practical standpoint, it would appear that the average of the maximum and minimum daily temperatures might suffice. Approximately 68.2% of the daily maximum and minimum stream water temperatures

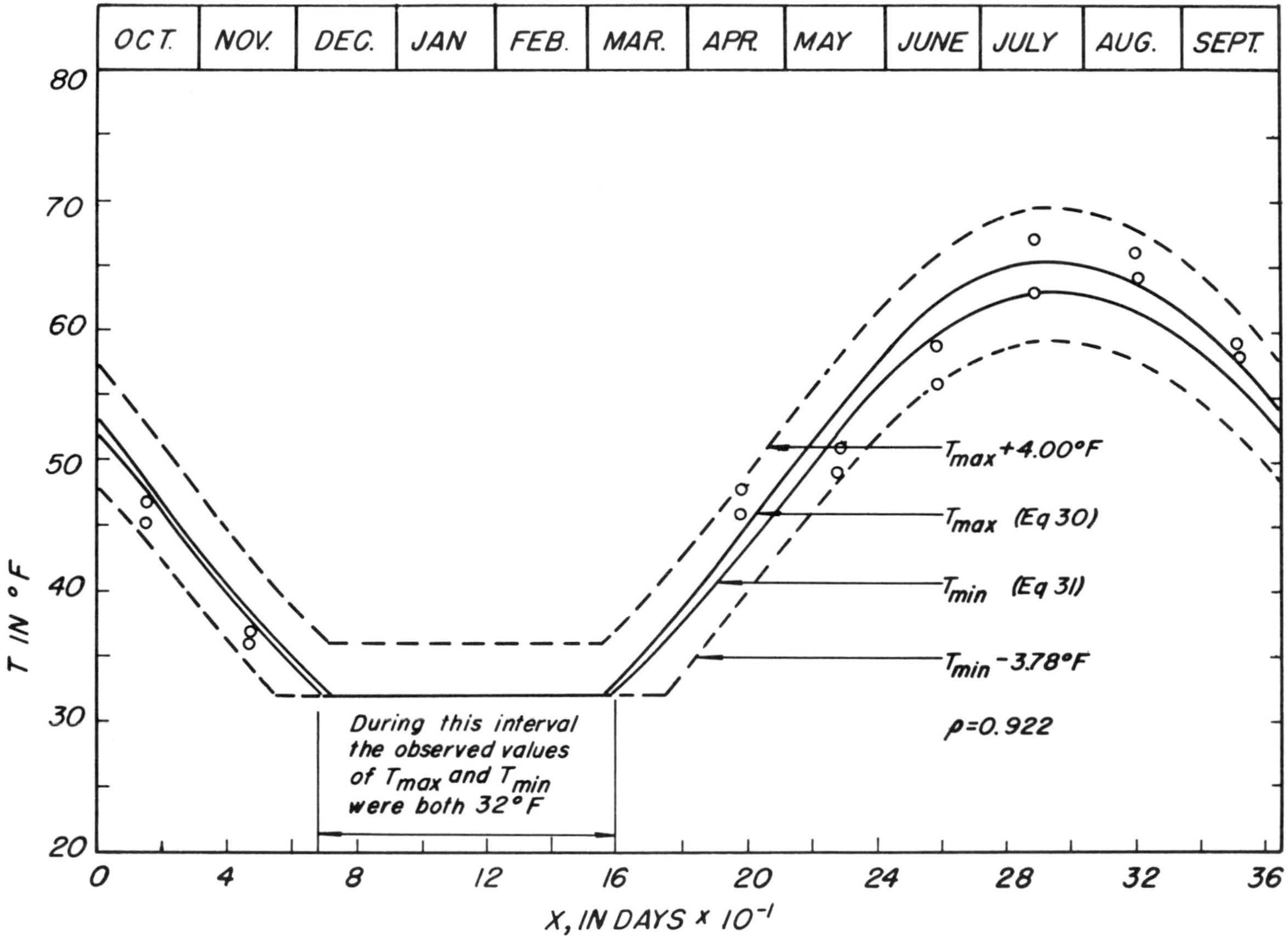


Figure 1.- 1962 Water year temperatures for Henry's Fork near Rexburg, Idaho

should fall between the two dashed curves, $T_{max} + 4.00^0F$ and $T_{min} - 3.78^0F$ (the $T_{min} - 3.78\ ^0F$ dashed curve is $32^0F$ from $x = 70$ to $x = 158$).

There are 8 points on the sine curve that materially aid in plotting. The maximum and minimum values of $T_c$ given by equation 1 ($\overline{T}_c \pm a$) occur when

$$bx + c = \sin^{-1} (1 \text{ and } -1) = 90^0 \text{ and } 270^0 \tag{32}$$

The two points of inflection occur when $T_c = \overline{T}_c$, or

$$bx + c = \sin^{-1} 0 = 0^0 \text{ and } 180^0 \tag{33}$$

The values of $x$ corresponding to $T_c = 32^0F$ occur when

$$bx + c = \sin^{-1} \left( \frac{32 - \overline{T}_c}{a} \right) = \alpha \text{ and } 180^0 - \alpha \tag{34}$$

where $\alpha$ = acute angle, degrees.

Finally, the value of $T_c$ given by equation 1 for $x = 0$ and 365 days is

$$T_c = a \sin c + \overline{T}_c \tag{35}$$

The calculated and observed values of these 8 points given by equations 32 through 35 are shown in Table 3.

The $T_{max}$ and $T_{min}$ curves are roughly $0.6^0$ or 0.6 days out of phase in that the maximum value of $T_{max}$ occurs roughly 14 hours before the maximum value of $T_{min}$. However, if one assumes that the value of $c$ is $-22^0$, then the diurnal variation in temperature is

$$T_{max} - T_{min} = -1.2 \sin (0.987x - 22) + 1.2 \tag{36}$$

which indicates a probable maximum daily variation in temperature of $2.4^0F$ at $x = 296$ and a minimum variation of $0^0F$ at $x = 113$. The

TABLE 3. - EIGHT USEFUL PLOTTING POINTS

| Point | Calculated | | | | Observed | | | |
|---|---|---|---|---|---|---|---|---|
| | max | | min | | max | | min | |
| | x, days | $T_c$, °F | x, days | $T_c$, °F | x, days | T, °F | x, days | T, °F |
| Maximum | 296 | 65.5 | 297 | 63.1 | 296[4] | 71 | 316[5] | 67 |
| Minimum | 113 | 27.1[6] | 114 | 27.1[6] | 113 | 32 | 114 | 32 |
| Points of inflection | 22 | 46.3 | 23 | 45.1 | 22 | 45 | 23 | 40 |
| | 203 | 46.3 | 205 | 45.1 | 203 | 49 | 205 | 48 |
| Stream is frozen over from | 71 | 32 | 70 | 32 | 68 | 32 | 67 | 32 |
| to | 156 | 32 | 158 | 32 | 160 | 32 | 162 | 32 |
| Beginning and | 0 | 53.4 | 0 | 51.9 | 1 | 49 | 1 | 46 |
| End of sine curve | 365 | 53.4 | 365 | 51.9 | 365 | 57 | 365 | 54 |

actual differences between the average monthly maximum and minimum temperatures are given in Table 4 along with the calculated values taken from Fig. 1. It should be noted that the average monthly stream water temperature has no meaning when temperatures of 32°F and other temperatures occur in the same month.

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[4] A temperature of 71°F was also observed at x = 284 and x = 295.

[5] A temperature of 67°F was also observed at x = 317, 318 and 320.

[6] These values have no real physical meaning when they are less than 32°F.

TABLE 4. - $T_{max}$ MINUS $T_{min}$

| | $T_{max} - T_{min}$ | |
|---|---|---|
| Month | Observed | Calculated |
| October | 2 | 1.3 |
| November | 1 | 0.7 |
| January | 0 | 0 |
| February | 0 | 0 |
| April | 2 | 1.1 |
| May | 2 | 1.7 |
| June | 3 | 2.2 |
| July | 4 | 2.4 |
| August | 2 | 2.3 |
| September | 1 | 1.9 |

## YEARLY VARIATIONS

In general, yearly variations in the values of $a$, $c$, $\overline{T}_c$, $S_T$ and $\rho$ will depend on the size of the stream, among other things, but are usually not very great unless the stream has been affected by other than natural phenomena. Still, it is extremely desirable to use all values of $T > 32^0F$ for all years of record if a computer is available. An exception to this would be the case when it is suspected that the stream's temperature has been affected by some man-made activity such as thermal pollution.

If a computer is not available, it still would be preferable to work with values of $T$ that represent all years of record, such as the average value of the average monthly temperatures. However, it is important to be sure that no temperatures of $32^0F$ are included in any averages.

As on extreme example of yearly variations, the maximum daily temperatures for the 1959 water year for Henry's Fork near Rexburg, Idaho[7], (this is the same stream station that was previously discussed in connection with the 1962 water year), were analyzed. During the 1959 water year, temperatures of $32^0F$ were recorded on only three successive days, x = 104 through 106 (temperatures of $33^0F$ were observed for every day for $91 \leqq x < 104$ and $106 < x \leqq 138$). However, the index of correlation between equation 30 ($T_{max}$ for the 1962 water year) and the maximum temperature data for the 1959 water year was 0.917 with a value of $S_T$ of $\pm 5.14^0F$. While $\rho$ is about the same, $S_T$ is roughly 20% greater. The equation obtained for the maximum temperature data of the 1959 water year is:

$$T_{max} = -18.2 \left[\sin(0.987x - 18.0)\right] + 49.7 \qquad (37)$$

Values of $S_T$ and $\rho$ obtained were $\pm 1.96^0F$ and 0.988, respectively. Equations 30 and 37 are both plotted in Fig. 2 along with the observed average monthly maximum stream water temperatures for the 1959 water year.

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[7] Quality of Surface Waters of the United States, 1959, Parts 9-14, Colorado River Basin to Pacific Slope Basins in Oregon and Lower Columbia River Basin, Geological Survey Water-Supply Paper 1645, United States Government Printing Office, Washington, 1966, page 428.

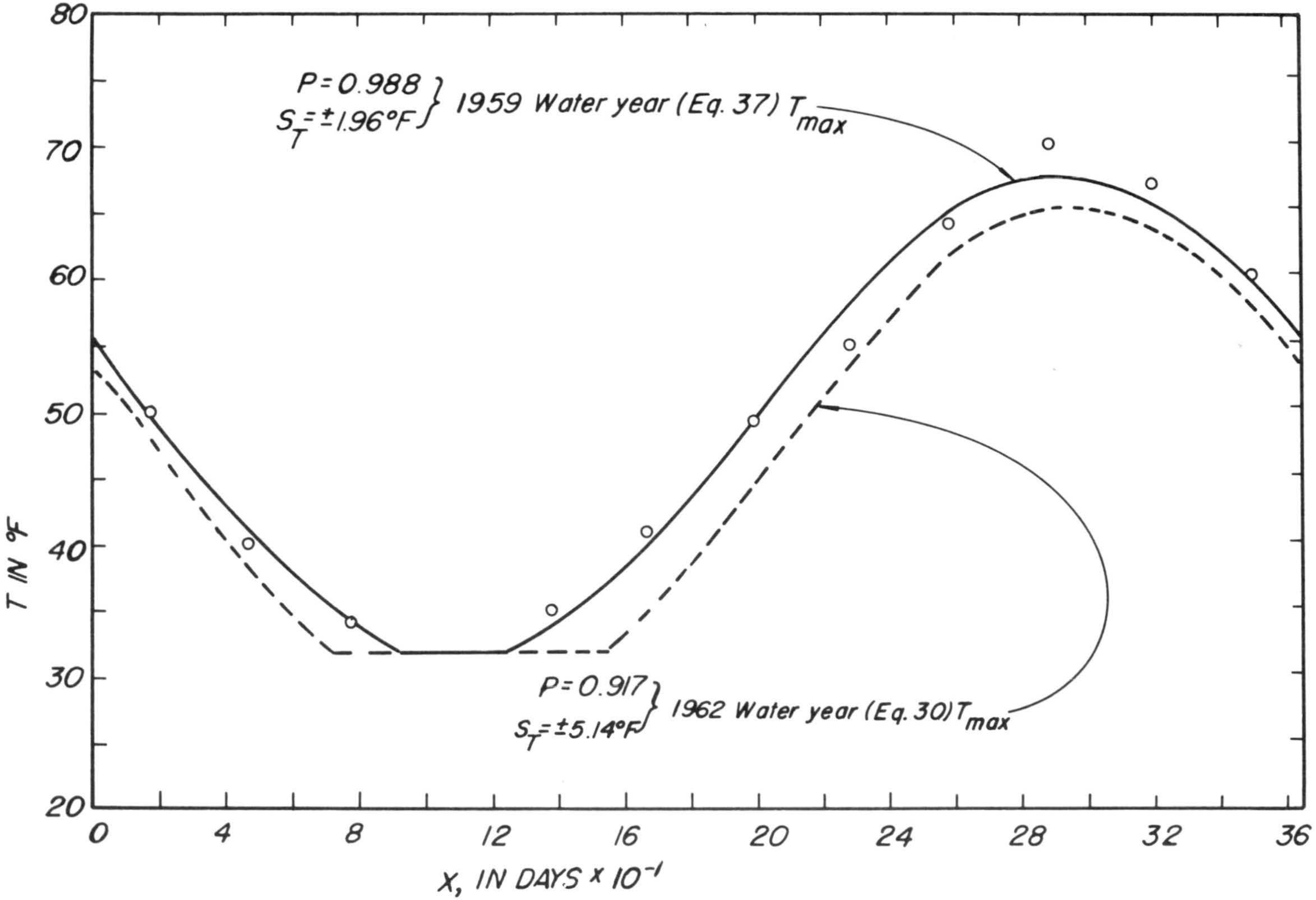


Figure 2.– 1959 Water year maximum temperatures for Henry's Fork near Rexburg, Idaho

Equations 30 and 37 are only $3.7^0$ or roughly 1% out of phase. Assuming that the value cf c is $-19.9^0$, equation 37 minus equation 30 gives

$$\Delta T_{max} = 1.0 \left[\sin(0.987x - 19.9)\right] + 3.4 \qquad (38)$$

where $\Delta T_{max}$ = difference between maximum daily temperatures of the stream water for two different water years, $^0$F.

According to Fig. 2, $0^0F \leqq \Delta T_{max} \leqq 5^0F$, and the difference of $5^0$F occurs when $x = 156$.

## NUMBER OF OBSERVATIONS REQUIRED

Theoretically, only 3 observations are necessary to evaluate a, c and $\overline{T}_c$ in equation 1, whereas at least 6 observations are required if values are to be obtained for $S_T$ and $\rho$ (equations 5 and 6, respectively). From a practical standpoint, these 6 observations should be as well spaced as possible and representative of as much data as feasible.

In order to illustrate the results obtained when only 6 representative observations are used, the daily temperature data for the Colorado River at Hot Sulphur Springs, Colorado[8] was analyzed. During the 1962 water year, daily temperatures of $32^0$F were recorded for all but two days in November, all of December through February, for 21 days in March and for the first day of April. Therefore, only

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[8] Reference 3, page 35.

the average monthly data for October and May through September could be used, giving the minimum of 6 representative observations. The equation obtained is:

$$T_c = -22.0 \left[\sin (0.987x - 29.9)\right] + 42.1 \tag{39}$$

Values of $S_T$ and $\rho$ obtained were $\pm 2.00^0$F and 0.967, respectively. Equation 39, along with the observed average monthly temperatures, are plotted in Fig. 3.

The plotting of average monthly temperatures is somewhat misleading in that the actual daily variations are likely to be greater. For example, only about 2/3 of the daily temperature observations would be expected to fall between the curves $T \pm S_T$. Table 5 gives the significance of various statistical limits. From Fig. 3, it is apparent that the predicted duration of $32^0$F daily temperatures is about 10% too short, the predicted beginning date is about a month late (26 days), and the predicted end of $32^0$F daily temperatures is about 10% late. Perhaps these errors would have been reduced if all 192 values of $T > 32^0$F had been used, rather than 6 representative values.

From Fig. 3, one could infer that about 68.2% of the time, one would expect a beginning date of ice cover of $x = 58 \pm 6$ days and an end of ice cover at $x = 183 \pm 6$ days, giving a duration of about $126 \pm 8$ days, during which no temperatures greater than $34^0$F would be observed.

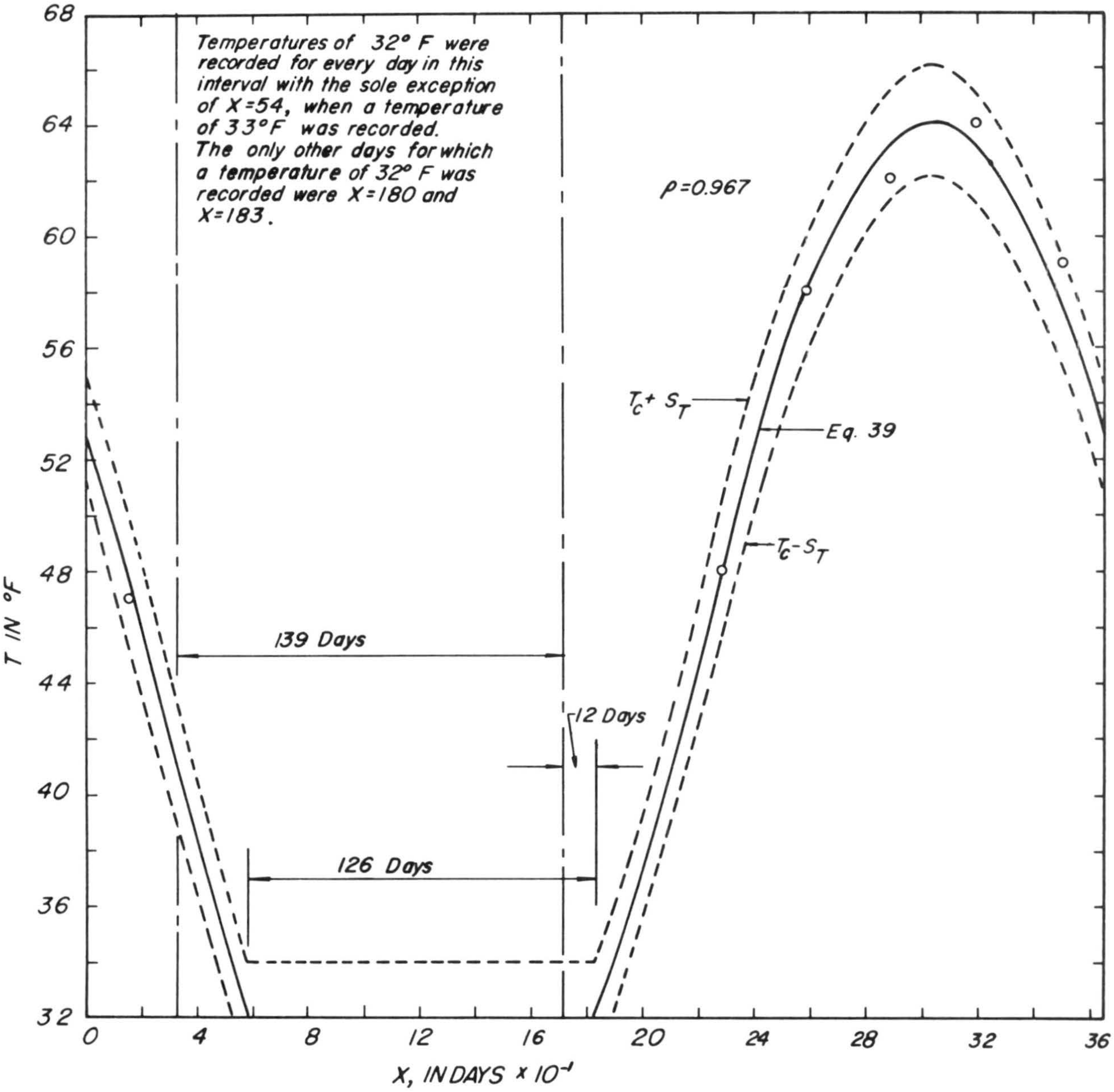


*Figure 3. — 1962 water year temperatures for the Colorado River at Hot Sulphur Springs, Colo.*

TABLE 5. - SIGNIFICANCE OF STATISTICAL LIMITS

| Curves | Percentage of daily temperature observations included |
|---|---|
| $T \pm S_T$ | 68.2 |
| $T \pm 2S_T$ | 95.4 |
| $T \pm 3S_T$ | 99.730 |
| $T \pm 4S_T$ | 99.994 |

## THERMAL POLLUTION

Curtis, Doyle and Whetstone[9] have illustrated the effect of thermal pollution by applying equation 1 to points above and below the points of pollution. The equations they obtained were, respectively,

$$T_c = -20.2 \left[\sin(bx - 20.2)\right] + 57.9 \qquad (40)$$

with $\rho = 0.952$ and $S_T = 4.59^0F$, and

$$T_c = -21.7 \left[\sin(bx - 30.8)\right] + 60.8 \qquad (41)$$

with $\rho = 0.969$ and $S_T = \pm 3.91^0F$. Equations 40 and 41 intersect at roughly $x = 130$ and $x = 240$ and give approximately the same value of $T_c$ in the interval $130 \leqq x \leqq 240$. In this interval, the stream flow was roughly 3 to 4 times the quantity of water used for cooling purposes.

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[9] Curtis, L. W., Doyle, T. J. and Whetstone, G. W., Discussion of "Annual Variation of Stream Water Temperature," by J. C. Ward. Discussion is in Journal of the Sanitary Engineering Division, ASCE, Volume 90, No. SA4, Proc. Paper 4006, August, 1964, pages 92-97.

While the maximum probable temperature was increased 4.4°F, the minimum probable temperature was increased only 1.4°F. Even though the effect of thermal pollution on ice cover is unknown, it appears that probably the effect would be to delay the beginning and shorten the duration. If this is the case, then some thermal pollution might be desirable in some streams.

## ANNUAL VARIATION OF WATER TEMPERATURE IN LAKES AND RESERVOIRS

In storage reservcirs, at least one additional quanitity is added to those discussed above, depth. In general, the behavior of reservoirs that are frozen over for part of the year is as follows:[10]

1. In the spring, the temperature may be approximately the same at all depths.

2. In the summer, the temperature decreases with depth, but the temperature gradient may not be constant. Summer stratification may begin in May or June and may persist for 4 to 6 months. From 30 to 50 feet below the surface, the temperature may be approximately constant. Below this there may be a thermocline about 10 to 20 feet thick in which the temperature decreases at a rate of 0.5°F per foot or more.

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[10] Kittrell, F. W., "Thermal Stratification in Reservoirs," Symposium on Streamflow Regulaticn for Quality Control, Public Health Service Publication No. 999-WP-30, June 1965, pages 57-76.

3. In the fall, the temperature may again be approximately constant for all depths.

4. Because 39.2°F is the temperature at which water has its maximum density, if there is an ice cover, the temperature may increase from 32°F just below the ice to 39.2°F at a greater depth, and the temperature gradient may vary with depth.

The above is, of course, highly simplified and numerous exceptions may occur, as Kittrell has pointed out.[10]

Lake Mead,[11] in general, does not behave very much like the above model because it apparently has no ice cover during any part of the year. In general, probably the behavior of reservoirs that are not frozen over for part of the year may have summer stratification as previously described, but the temperature may be approximately constant for all depths only during the winter or early spring. Because temperature measurements were made at irregular intervals of time, the calculations for the 5 foot depth are shown in Table 6.

The results for all depths analyzed are given in Table 7. $\sigma_T$ is the arithmetic standard deviation in °F and is given by:

$$\sigma_T = \sqrt{\frac{\Sigma(T - \overline{T})^2}{n - 1}} \qquad (42)$$

The results for the 5 foot depth indicate a behavior similar to streams, and presumably the beginning and duration of ice cover could

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[11] Reference 3, pages 177-182.

TABLE 6. - EXAMPLE CALCULATIONS FOR LAKE MEAD

| Month (1) | Day (2) | x, days (3) | bx, degrees (4) | T, °F (5) | sin bx (6) | cos bx (7) | sin bx times cos bx (8) | $\sin^2$ bx (9) | $\cos^2$ bx (10) | T sin bx, °F (11) | T cos bx, °F (12) | $T-\overline{T}$, °F (13) | $(T-\overline{T})^2$, $°F^2$ (14) | p sin bx, °F (15) | q cos bx, °F (16) | $T_c$, °F (17) | $T-T_c$, °F (18) | $(T-T_c)^2$, $°F^2$ (19) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Oct. | 3 | 3 | 3 | 74.1 | 0.052 | 0.999 | 0.052 | 0.003 | 0.998 | 3.9 | 74.0 | 7.1 | 50.4 | -0.43 | 9.94 | 75.5 | -1.4 | 1.96 |
| Dec. | 19 | 80 | 79 | 56.0 | 0.982 | 0.191 | 0.188 | 0.964 | 0.036 | 55.0 | 10.7 | -11.0 | 121.0 | -8.08 | 1.90 | 59.8 | -3.8 | 14.45 |
| Jan. | 30 | 122 | 120 | 54.4 | 0.862 | -0.506 | -0.436 | 0.744 | 0.256 | 46.9 | -27.6 | -12.6 | 158.8 | -7.09 | -5.04 | 53.9 | 0.5 | 0.25 |
| Mar. | 1 | 152 | 150 | 54.1 | 0.500 | -0.866 | -0.433 | 0.250 | 0.750 | 27.0 | -46.9 | -12.9 | 166.4 | -4.11 | -8.62 | 53.3 | 0.8 | 0.64 |
| Apr. | 3 | 185 | 183 | 56.6 | -0.045 | -0.999 | 0.045 | 0.002 | 0.998 | -2.5 | -56.5 | -10.4 | 108.2 | 0.37 | -9.94 | 56.5 | 0.1 | 0.01 |
| Apr. | 26 | 208 | 205 | 63.8 | -0.427 | -0.904 | 0.386 | 0.182 | 0.826 | -27.3 | -57.6 | -3.2 | 10.2 | 3.51 | -8.89 | 60.6 | 3.2 | 10.25 |
| June | 8 | 251 | 248 | 69.4 | -0.925 | -0.379 | 0.350 | 0.855 | 0.144 | -64.2 | -26.3 | 2.4 | 5.8 | 7.60 | -3.78 | 69.8 | -0.4 | 0.16 |
| June | 28 | 271 | 267 | 70.4 | -0.999 | -0.044 | 0.044 | 0.998 | 0.002 | -70.3 | -3.1 | 3.4 | 11.6 | 8.22 | -0.44 | 73.8 | -3.4 | 11.57 |
| July | 25 | 298 | 294 | 75.8 | -0.913 | 0.409 | -0.373 | 0.834 | 0.167 | -69.1 | 31.0 | 8.8 | 77.5 | 7.50 | 4.07 | 77.6 | -1.8 | 3.26 |
| Aug. | 30 | 334 | 330 | 81.5 | -0.505 | 0.863 | -0.436 | 0.255 | 0.744 | -41.1 | 70.4 | 14.5 | 210.2 | 4.15 | 8.59 | 78.7 | 2.8 | 7.85 |
| Sept. | 28[12] | 363 | 358 | 80.6 | -0.030 | 1.000 | -0.030 | 0.001 | 1.000 | -2.4 | 80.6 | 13.6 | 185.0 | 0.25 | 9.95 | 76.2 | 4.4 | 19.38 |
| $\Sigma$ or $a_{ij}$ | | | | 736.7 | -1.448 | -0.236 | -0.643 | 5.088 | 5.921 | -144.1 | 48.7 | -0.3 | 1,105.1 | 11.89 | -2.26 | 728.1 | 1.0 | 69.78 |
| $a_{ij}$ | | | | $a_{34}$ | $a_{13}=a_{31}$ | $a_{23}=a_{32}$ | $a_{12}=a_{21}$ | $a_{11}$ | $a_{22}$ | $a_{14}$ | $a_{24}$ | | | | | | | |

[12] The temperature measurements for depths 5 through 175 feet were made September 26, and the measurements for 200 through 475 feet were made on September 30. The maximum depth at which temperature measurements were made during the 1962 water year was 483 feet.

TABLE 7. - SUMMARY OF TEMPERATURE ANALYSES FOR LAKE MEAD

| Depth, feet | a, $^0$F | c, degrees | $\overline{T}_c$, $^0$F | $\overline{T}_c$+a, $^0$F | $\overline{T}_c$-a, $^0$F | $\overline{T}$, $^0$F | $\sigma_T$, $^0$F | $S_T$, $^0$F | $\rho$ |
|---|---|---|---|---|---|---|---|---|---|
| 5 | -12.93 | -50.5 | 66.0 | 53.1 | 78.9 | 67.0 | ±10.51 | ±3.41 | 0.946 |
| 50 | -13.2 | -54.9 | 64.7 | 51.5 | 77.9 | 65.6 | ±10.62 | ±2.60 | 0.970 |
| 100 | - 7.71 | -51.2 | 60.3 | 52.6 | 68.0 | 60.8 | ± 6.91 | ±2.44 | 0.936 |
| 150 | - 5.03 | -69.8 | 57.6 | 52.6 | 62.6 | 57.9 | ± 4.30 | ±2.29 | 0.845 |
| 375 | - 0.348 | -12.6 | 53.8 | 53.5 | 54.1 | 53.8 | ± 1.21 | ±1.52 | |
| Average | | | | 52.7 | | | | | |

be predicted in the same manner, but this should be verified experimentally. At the other extreme, the correlation at the 375 foot depth is so poor that it is more accurate to represent the temperature at this depth as $\overline{T} \pm \sigma_T$ $^0$F. The observed values of 53.8 ± 1.2$^0$F correspond very closely to the observed temperatures in the Colorado River just downstream from Lake Mead[13] which varied from 52$^0$F to 55$^0$F with an average of about 54.2$^0$F for the 1962 water year.

Some of the data in Table 7 (1962 water year) are plotted in Fig. 4 along with observed data for the 1959 water year[14] at the 5 foot depth. The sine curve for the 50 foot depth is not plotted because it is very close to the sine curve for the 5 foot depth.

It should be pointed out that, in general $\overline{T}$ and $\overline{T}_c$ will not be the same, but may be rather close if temperature measurements are made at equal intervals of time distributed uniformly throughout the year and if no temperatures of 32$^0$F or less are recorded. Furthermore, $\overline{T}_c$ is always calculated using equation 25 and never by dividing the sum of the $T_c$ values by the number of $T_c$ values, although again, if the $T_c$ values are distributed uniformly at equal intervals of time throught the year, the results will be about the same.

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[13] Reference 3, page 183.

[14] Reference 7, pages 138-144.

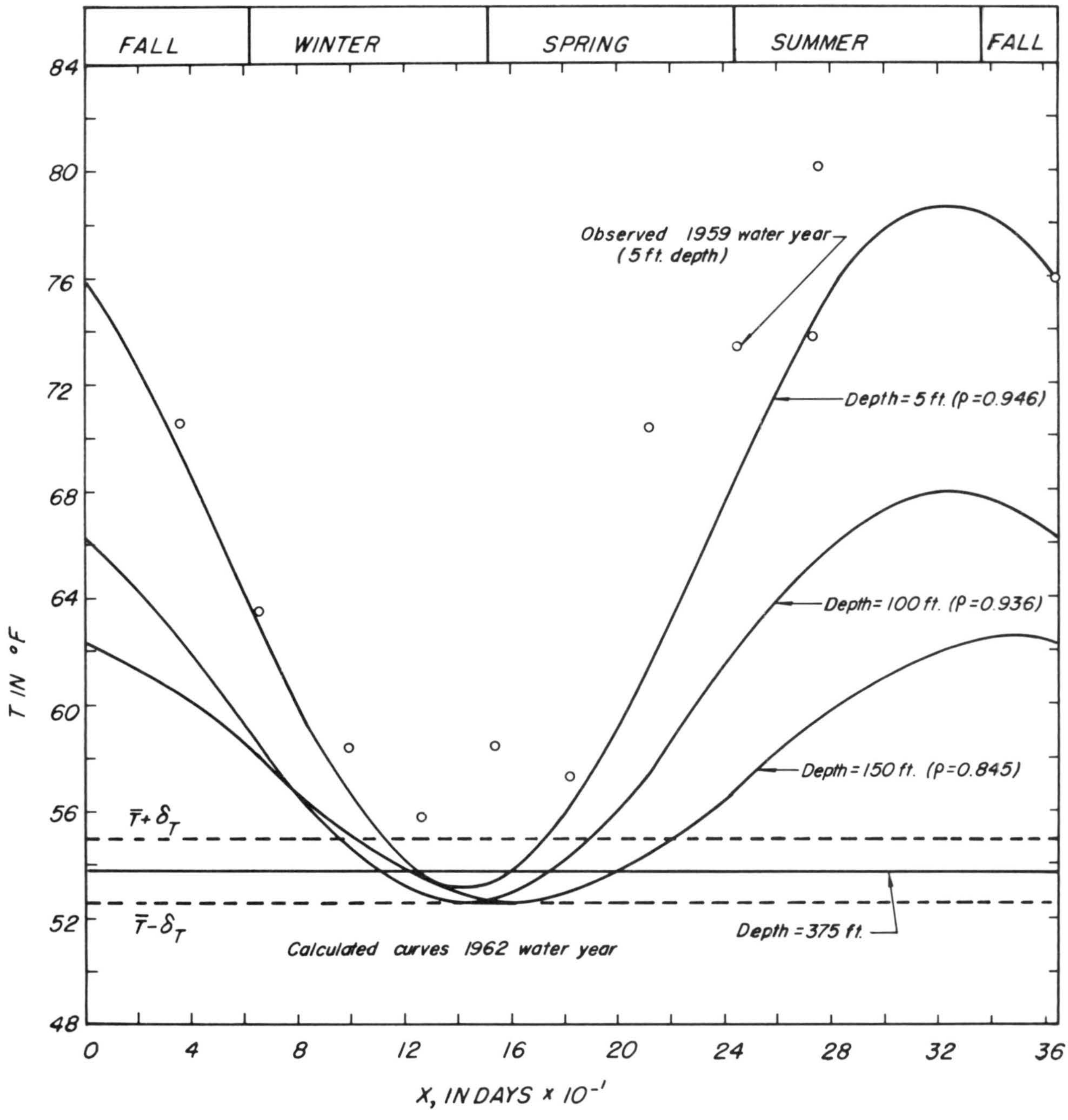


Figure 4.-Lake Mead water temperature

From Table 7, it appears that

$$\overline{T} \approx \overline{T}_c \tag{43}$$

Equation 43 might have been more accurate if the temperature measurements had been made at equal intervals of time distributed uniformly throughout the year. Also, it appears that possibly

$$\overline{T}_c + a \approx \overline{T}_D \tag{44}$$

where $\overline{T}_D$ = maximum average value of $\overline{T}$, $^0F$.

The observed values of $\overline{T}$ are plotted in Fig. 5 and indicate that $\overline{T}_D$ is approximately 53.8$^0$F for Lake Mead. Also, it appears that $\overline{T}$ is approximately constant for all depths in excess of about 275 feet,and for depths greater than 275 feet,

$$\overline{T} \approx \overline{T}_D \qquad \text{(only for depths} \gtreqless 275 \text{ feet)} \tag{45}$$

Although occasional temperature measurements were made at depths greater than 425 feet, there were not enough measurements to determine a reasonably accurate value of $\overline{T}$. In addition, the measurements were not distributed throughout the year. However, none of these measurements varied more than 0.4$^0$F from the value of T observed at the 375 foot depth.

For depths less than about 275 feet, it might be possible to express a, c and $\overline{T}_c$ as a function of depth, but Fig. 5 and Table 7 indicate that this may not be a straight forward proposition. Perhaps values of a and $\overline{T}_c$ could be roughly estimated from Fig. 5, but the value of c would still be undetermined.

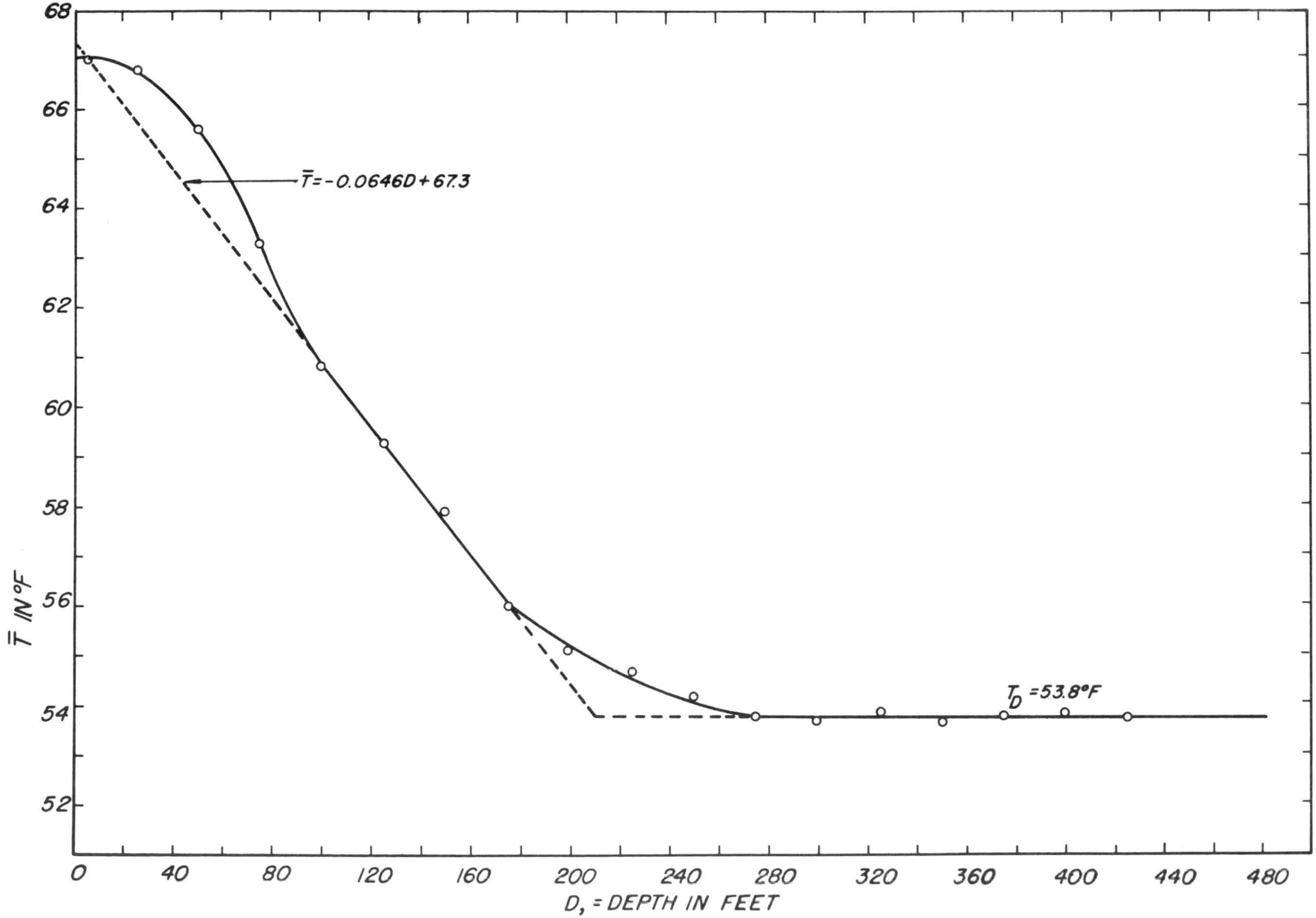


Figure 5.- Lake Mead average water tempemperature for 1962 water year

While $S_T$ appears to decrease with depth, $\sigma_T$ also seems to show somewhat the same general trend with the result that $\rho$ generally decreases with depth.

Figure 6 is a hypothetical application of equation 1 to reservoirs that are frozen over for part of the year. Even if equation 1 could be applied in this manner, actual calculated values will undoubtedly differ from Fig. 6 in several respects. The temperature of the fall overturning is likely to be greater than that of the spring overturning and neither is likely to occur at the points of inflection. The values of $\overline{T}_c$ and c will probably vary with depth and all of the curves will probably be shifted to the right. It is unlikely that all curves will intersect at only 2 points. If a, c and $\overline{T}_c$ could be expressed as functions of depth, it would be hypothetically possible to predict the thickness of the ice cover as a function of time, but this is extremely doubtful. In fact, it may be that the application of equation 1 to this type of reservoir, or even reservoirs in general, may be totally misleading and could obscure very important phenomena.

## CONCLUSIONS

While equation 1 has been shown to be applicable to most streams that are not frozen over during any part of the year[2], there is still insufficient evidence upon which to base any conclusions about

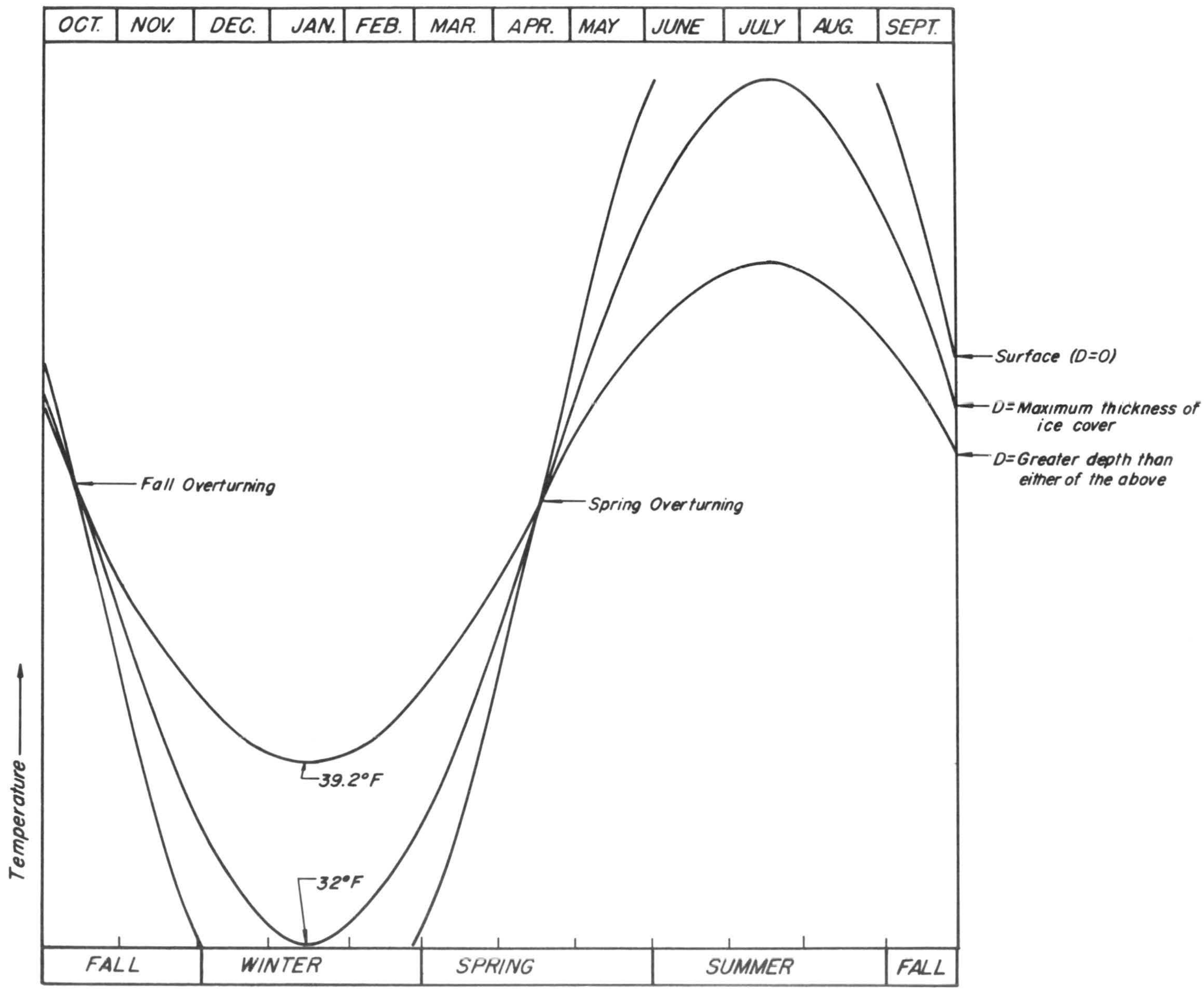


*Figure 6.– Possible idealized behavior of reservoirs that are frozen over for part of the year*

the utility of equation 1 for predicting the beginning and duration of ice cover in streams. This latter possibility has merely been indicated.

In order for equation 1 to be of real value with respect to thermal pollution, a, c and $\overline{T}_c$ will have to be quantitatively related to the physical parameters of streams. However, the detection of thermal pollution may be possible by either or both of two methods:

1. examination of temperature records both before and after the suspected beginning of thermal pollution, and

2. examination of temperature records both above and below the suspected points of pollution.

Application of equation 1 to lakes and reservoirs has been shown to be a possibility, but a great deal of additional study is strongly indicated. It is possible that equation 1 may be applicable only to lakes and reservoirs that are not frozen over during any portion of the year, or applicable only to lakes and reservoirs that are frozen over for a portion of the year, or neither or both.

In all of the above cases, it is very desirable to utilize all temperature data available and avoid using representative values obtained from individual observations. However, this is only feasible when a computer is available. However, analysis without a computer has been shown to be possible in every case, but in addition to other disadvantages, analysis without a computer increases the possibility of errors in calculated values.

One problem still unresolved is the determination of whether a temperature reading of $32^0$F means ice or liquid. It would be of considerable assistance, if, in the future, when temperatures of $32^0$F are recorded, a notation is made to the effect that the reading represented the actual temperature of the water, or was inferred because of ice cover. Finally, it is certainly possible that only portions of the surfaces of streams and reservoirs may be covered with ice of varying thicknesses, and that the fraction covered will be an additional variable.

## ACKNOWLEDGMENTS

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# APPENDIX - NOTATION

The following symbols have been adopted for use in this paper:

$a$ = amplitude of sine curve, in $^0$F;

$a_{ij}$ = element of determinant;

$b$ = $0.987^0$ per day

$c$ = phase coefficient of sine curve, in degrees;

$D$ = depth of a lake or reservoir, in feet;

$^0$F = degrees Fahrenheit;

$i$ = 1, 2 or 3, dimensionless;

$j$ = 1, 2, 3 or 4, dimensionless;

$n$ = number of pairs of observed values of $T$ and $x$, dimensionless;

$p$ = parameter, in $^0$F;

$q$ = parameter, in $^0$F;

$S_T$ = standard error of estimate, in $^0$F;

$T$ = observed temperature of the stream, lake or reservoir water, in $^0$F;

$\overline{T}$ = airthmetic mean or average value of $T$, in $^0$F;

$T_c$ = calculated value of $T$, $^0$F;

$\overline{T}_c$ = arithmetic mean or average value of $T_c$ (if all values of $T_c$ are distributed at uniform intervals of time throughout the year), in $^0$F;

$\overline{T}_D$ = minimum average value of $\overline{T}$ for a lake or reservoir, in $^0$F;

$T_{max}$ = maximum daily temperature of the stream water, in $^0$F;

$T_{min}$ = minimum daily temperature of the stream water, in $^0F$;

x = number of days since October 1 (x = 1 for October 1), in days;

$\alpha$ = acute angle, in degrees;

$\gamma$ = determinant of order 3, dimensionless;

$\Delta T_{max}$ = difference between maximum daily temperatures of the stream water for two different water years, in $^0F$;

$\rho$ = index of correlation ($\rho$ has a maximum possible value of one for perfect correlation), dimensionless;

$\Sigma$ = sum;

$\sigma_T$ = aithmetic standard deviation, in $^0F$.