QUASI-ANALYTICAL FORMULATION
FOR CALCULATION OF INFILTRATION AND RUNOFF

prepared for

USDA Forest Service
Rocky Mountain Forest and Range Experiment Station
Flagstaff, Arizona

prepared by

Civil Engineering Department
Engineering Research Center
Colorado State University
Fort Collins, Colorado

D. B. Simons
R. M. Li
K. G. Eggert
D. Zachmann

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AUTHORIZATION

This research was sponsored by the USDA Forest Service, Rocky Mountain Forest and Range Experiment Station and supported with Colorado State University matching funds. The investigations were conducted in accordance with the Research Agreement No. 16-763-CA between the Rocky Mountain Forest and Range Experiment Station and Colorado State University. D. Ross Carder was the authorized project leader for the Rocky Mountain Forest and Range Experiment Station and Daryl B. Simons and Ruh-Ming Li were the principal investigators for Colorado State University. The period of agreement was from September, 1977, to December, 1980.

In accordance with the study plan, a procedure was developed for estimating long term water balance and infiltrations. The method uses analytical methods to solve for infiltration and runoff from rainfall. Evapotranspiration processes and vertical redistribution of water is also modeled.

LIST OF SYMBOLS

Symbol	Description	Unit
a	Coordinate change parameter	
a_n	Fourier coefficient	
b	Coordinate change parameter	
C	Water capacity	cm^{-1}
C	Fourier coefficient	
D	Diffusivity	(hr)
E	Potential ET rate	cm/day
G	Green's function	
g	Green's function	
h	Green's function	
K	Hydraulic conductivity	cm/hr*
K_s	Saturated conductivity	cm/hr*
K_N	Natural conductivity	cm/hr*
q	Darcy flux	cm/hr*
r	Rainfall rate	cm/hr
S	Sink strength	$(day)^{-1}$
T	Dimensionless time	
T_p	Dimensionless ponding time	
t	Time	hr*
t_p	Ponding time	hr
u	Transformed dependent variable	
v	Transformed dependent variable	
W_i	Laguerre-Gauss weight	

w_n	Eigenfunction	
Y_i	Laguerre-Gauss abscissa	
Z	Dimensionless vertical coordinate	
Z_o	Dimensionless soil depth	
z	Vertical coordinate	cm
z_o	Soil depth	cm
α	Soil parameter	cm^{-1}
β	Coordinate change parameter	
γ	Soil parameter	hr/cm*
δ	Dirac delta function	
θ	Volumetric water content	
θ_R	Reference water content	
θ_N	Natural saturation	
θ_i	Initial water content	
λ_n	Eigenvalue	
μ_n	Eigenvalue	
τ	Dimensionless time	
ϕ	Porosity	
ψ	Capillary pressure	cm

* During infiltration the time unit is hours, but for the redistribution period time is measured in days.

I. INTRODUCTION

1.1 General

Infiltration is the key process controlling rainfall runoff, sediment yield, and soil water recharge. The infiltration process governs the proportions of the applied water going to overland flow, soil moisture, subsurface flow. Infiltration rates during a rainfall are highly dependent on the antecedent moisture profile in the soil. Thus, to accurately predict infiltration, it is necessary to include in the model not only the rainfall event, but also the redistribution of soil moisture between events.

Infiltration can be classified as either flux controlled or profile controlled. During flux controlled infiltration, the flow of rate into the soil is equal to the rate of application and no runoff results. Flux controlled infiltration occurs during the early stages of a rainfall event or when the rainfall rate is low. When the surface moisture content reaches natural saturation, the rate of flow of water into the soil is controlled by the moisture profile near the soil surface. In this case, the rate of inflow to the soil is less than the rate of application and runoff occurs.

The redistribution of soil water can also be roughly divided into two processes. Immediately following a rainfall, the soil moisture content near the surface is uniformly near natural saturation. Under this condition, water is lost to evapotranspiration (ET) at a rate determined by climatic and cover conditions. As long as the soil can provide water at a rate at least equal to the ET extraction rate, the redistribution process is controlled by the ET flux demand. Within several hours to a few days, depending on the soil type and ET demand, soil moisture content drops to a point where ET extraction can no longer proceed at its initial rate, but is instead controlled by the soil moisture profile.

Redistribution models which assume a constant ET extraction rate and neglect the profile controlled phase provide a too dry moisture profile at the beginning of the next rainfall event and thus tend to underestimate subsequent runoff. Infiltration models which assume instantaneous ponding and do not account for flux controlled infiltration tend to overestimate runoff. Neglecting flux controlled infiltration and profile controlled ET generally produces errors of opposite sign in runoff prediction. However, it is too much to expect these errors to cancel. Therefore, it is desirable to account for both flux and profile controlled infiltration and redistribution.

Infiltration and redistribution models can be classified as physically based or empirically based. Physically based models are derived from Darcy's Law and the principle of mass conservation. Empirically based models rely on observed relationships. Physically based models are usually superior since they are generally easier to calibrate and provide better accuracy.

1.2 Objectives

The overall objective of this report is to provide efficient methods for predicting catchment runoff. Specific objectives are:

1. During a rainfall, determine time to ponding and subsequent runoff.
2. Between rainfalls, describe the redistribution of soil moisture accounting for ET, capillary and gravity effects.

The models which accomplish the above objectives are intended to be used as one component of a more complex watershed model, Simons et al (1977). Thus, it is necessary to develop computer cost-efficient runoff and redistribution models. This cost efficiency is obtained by employing analytical and quasianalytical solution techniques. Some of the solution methods presented here are new while others have been adapted from the recent hydrology literature.

1.3 Soil Characteristics

Infiltration and redistribution patterns are highly soil-type dependent. To mathematically model soil moisutre flow, it is necessary to know the soil's hydraulic properties.

Let ψ (cm) denote capillary pressure head, θ volumetric water cotent, and K (cm/hr) hydraulic conductivity. Under unsaturated flow conditions, θ and K depend on capillary pressure, ψ. Qualitative graphs of θ versus ψ and K versus ψ are shown in Figures 1a and 1b. There ϕ represents the soil's porosity and K_s (cm/hr) saturated hydraulic conductivity. For saturated flow, $\psi \geq 0$, both θ and K are constant with respect to ψ.

The water capacity C (cm^{-1}) is defined by

$$C = \frac{d\theta}{d\psi}$$

which, as Figure 1c shows, is also ψ-dependent. For $\psi \geq 0$, C is identically zero.

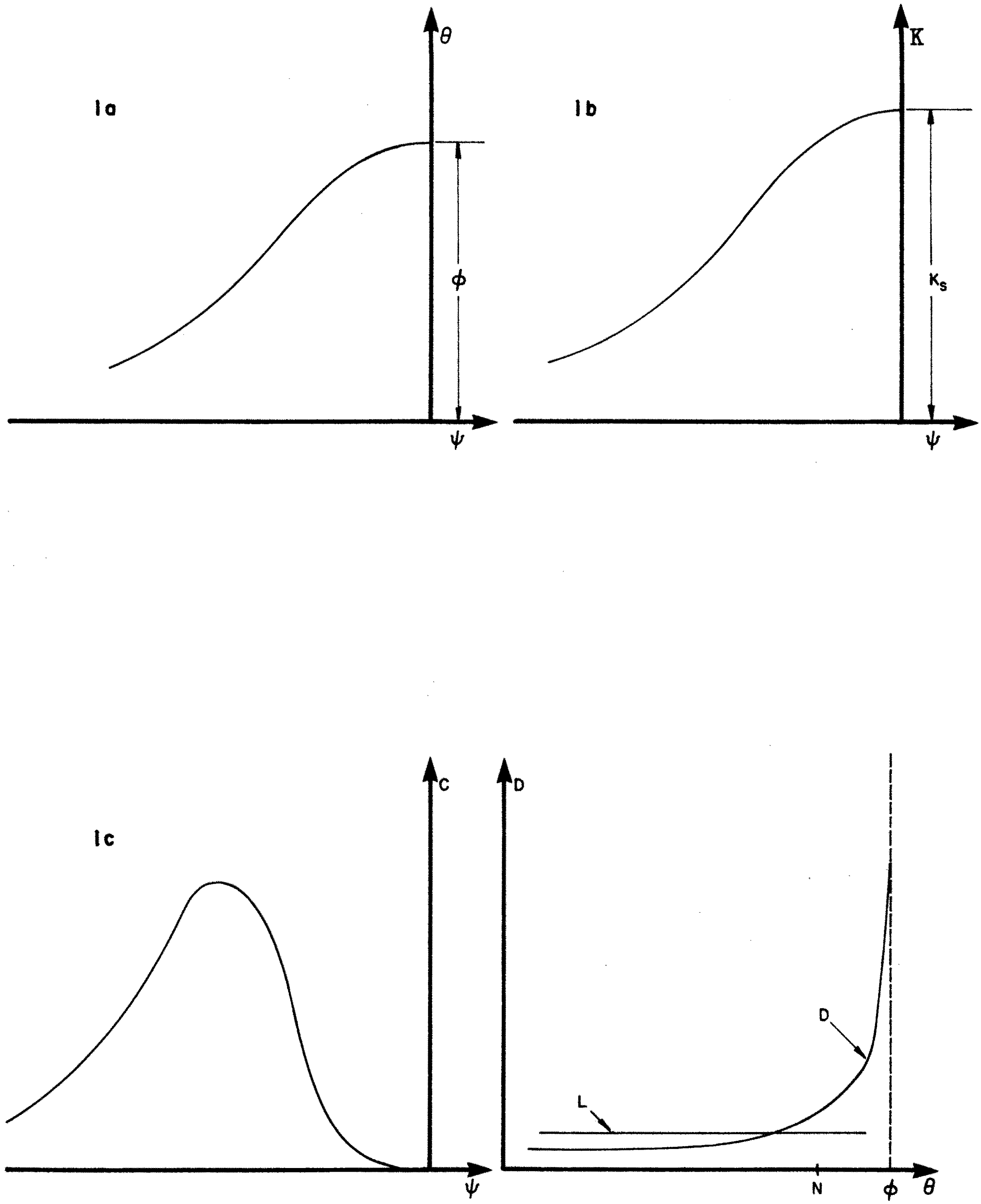

Figure 1. Soil hydraulic functions.

For $\psi < 0$, the soil diffusivity D (cm^2/hr) is defined to be

$$D = \frac{K}{C}$$

Using the θ-ψ relationship, Figure 1a, D can be viewed as a function of θ. A typical D-θ relationship is illustrated by the curve labeled R in Figure 1d. Since, as $\theta \to \phi$, $K \to K_s$, and $C \to 0$, it follows that $D \to \infty$ as $\theta \to \phi$.

1.4 Governing Equations

Let z (cm) denote distance measured downward from the soil surface. Darcy's law states that the flux of water in the z-direction, per unit cross-sectional area is given by

$$-K \left(\frac{\partial \psi}{\partial z} - 1\right)$$

Darcy's law together with the principle of mass conservation yields the balance equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left[K \left(\frac{\partial \psi}{\partial z} - 1\right)\right]$$

Using the identity

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial \psi} \frac{\partial \psi}{\partial t} = C \frac{\partial \psi}{\partial t}$$

θ can be eliminated from the above balance equation to obtain the one-dimensional pressure head flow equation

$$C \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z} \left[K \left(\frac{\partial \psi}{\partial z} - 1\right)\right] \qquad (1.1)$$

Similarly, ψ can be eliminated from the balance equation to obtain the diffusivity form of the flow equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left(D \frac{\partial \theta}{\partial z}\right) - \frac{\partial K}{\partial z} \qquad (1.2)$$

1.5 Simplifying Assumptions

The analysis of Equations 1.1 and 1.2 can be simplified considerably if the real soil being modeled is approximated by an idealized linear soil. In this report, a linear soil is defined to be one with (i) constant diffusivity and (ii) a linear $K - \theta$ relationship. When modeling a real soil by a linear soil, it is not necessary that conditions (i) and (ii) be globally satisfied. It is sufficient that they be approximately satisfied over the relevant range of moisture contents.

The horizontal line labeled L in Figure 1d represents a constant approximation to the real diffusivity curve, D. Clearly, L is not a good approximation to D near $\theta = \phi$. However, to the left of the moisture level labeled θ_N, the approximation of D by L is fairly good.

In terms of θ, K, and ψ a linear soil is one for which

$$\frac{d \ln K}{d\psi} = \alpha \qquad \text{and} \qquad \theta = \gamma K + \theta_R \tag{1.3}$$

where α, γ, and θ_R are parameters dependent on soil type and the relevent range of moisture contents. From the second part of Equation 1.3, condition (ii) is seen to hold. To show that the diffusivity is constant, recall

$$D = \frac{K}{\frac{d\theta}{d\psi}}$$

Equation 1.3 implies

$$\frac{d\theta}{d\psi} = \gamma \frac{dK}{d\psi} = \alpha\gamma K$$

so that if Equation 1.3 hold, then D is constant, $D = 1/(\alpha\gamma)$.

When Equation 1.3 hold, the nonlinear partial differential equations, Equations 1.1 and 1.2, each reduce to a linear partial differential equation in θ, namely

$$\alpha\gamma \frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2} - \alpha \frac{\partial\theta}{\partial z} \tag{1.4}$$

Equation 1.4 is much easier to analyze than either of Equations 1.1 or 1.2 since analytical solutions of Equation 1.4 can be obtained using Fourier transforms. These analytical solutions are much more computer cost efficient than the finite difference methods required to solve Equations 1.1 and 1.2.

In recent years, Equations 1.3 have been used to approximate real soils by a number of researchers including Lomen and Warrick (1974), Philip (1968, 1971), Reats (1970, 1971, 1972, 1977), Thomas (1972), Warrick (1974), Warrick and Lomen (1977), Zachmann and Thomas (1973), Zachmann (1978). Ben Asher et al (1978) have discussed the validity of the relationships (1.3). They compared the results obtained from a nonlinear model based on actual hydraulic data relating K, θ, and ψ to those obtained from a linear model based on Equations 1.3. For several such comparisons they showed that, in the midrange of moisture content, the results of the linear and nonlinear models were in good agreement. In many field applications, the moisture content stays in this midrange from 40 to 90 percent of total saturation.

The flow patterns to be discussed in this report involve repeated wetting and drying. Therefore, some hysteresis effects are expected. Since the linear model based on Equation 1.4 is not capable of describing hysteresis, it must be assumed that in applications of these methods hysteresis effects are negligible.

During infiltration, when runoff begins, generally the surface water channelizes quickly. Thus, it is assumed that subsequent to ponding time, the pressure effect of ponded water is negligible. It is also necessary to assume the soil is spatially homogenous. This last assumption is not so severe as it first appears since spatial averaging of soil properties tends to smooth out local inhomogeneities.

Under field conditions, the moisture content seldom exceeds what is commonly called natural saturation. Wells and Skaggs (1976) have reported that natural saturation, θ_N, is typically in the range of 80 to 90 percent of total saturation. In order to avoid underestimating runoff, it is assumed in this report that $\theta_N = 0.8\ \phi$ and that runoff can begin when the surface moisture content reaches the value θ_N.

1.6 Calibration

To model a real soil with an idealized linear soil, it is necessary to estimate the parameters α, γ, and θ_R. Mualem (1976) has compiled an extensive catalog of soil hydraulic properties in the form of K-ψ-θ data for some 80 soils. The introduction of the catalog states, "Field oriented scientists may find the catalog helpful when an estimate for the hydraulic properties of some particular soil are required without expensive testing. They can just adapt the data of a similar soil from the collection."

If the soil being modeled is similar to one of the soils in the catalog, the linear model can be calibrated by simple linear regression. To estimate α in the first of Equations 1.3, the K-ψ data yield $(\ln K) - \psi$ data pairs. If these $(\ln K) - \psi$ data points exhibit a roughly linear relationship, then the method of least squares can be used to obtain α. Having found α for a number of soils in the catalog, it appears that α is on the order of 0.1 cm^{-1} for a soil for which K decreases rapidly with decreasing ψ, say a sand. For more clay-like soils, α is generally on the order of 0.01 cm^{-1}.

In many applications, the field conditions are such that the soil moisture content remains in the range of 40 to 90 percent of total saturation. To estimate γ and θ_R consider the cataloged K-θ data pairs over this range. If they exhibit a linear trend, then linear regression can be used to estimate γ and θ_R. Examples of this calibration from catalog data will be given in the following chapters.

An alternative calibration procedure is based on a recently developed parameter identification method, Zachmann et al (1981). A brief outline of the method follows.

To estimate α, γ, and θ_R for a given soil, begin with a vertical column of soil which has an initially constant moisture content, $\theta = \theta_i < \theta_N$ where θ_N is natural saturation. If water is applied to the soil surface at a sufficiently high rate to immediately produce runoff, with an escape provided to prevent ponding, then for $t > 0$, $\theta(0,t) = \theta_N$. Knowing the application rate and observing the runoff rate allows $q(t)$, the flux across the upper surface of the column as a function of time to be calculated. If the lower boundary is far enough removed relative to the duration of the experiment, then the linear flow equations are

$$\alpha\gamma \frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2} - \alpha\frac{\partial\theta}{\partial z}, \qquad z > 0, \quad t > 0 \tag{1.5}$$

$$-\frac{\partial\theta}{\partial z} + \alpha(\theta - \theta_R) = q(t), \qquad z = 0, \quad t > 0 \tag{1.6}$$

$$\theta(z,0) = \theta_i, \qquad z > 0 \tag{1.7}$$

To obtain the boundary conditions, Equation 1.6, Darcy's law, $-D(\partial\theta/\partial z) + K = q$, and Equations 1.3 imply $D = (\alpha\gamma)^{-1}$ and $K = (\theta - \theta_R)/\gamma$ were used.

The solution of Equations 1.5 through 1.7 is clearly dependent on the choice of α, γ, and θ_R. To emphasize this dependence, write the solution of Equations 1.5 through 1.7 as $\theta = \theta(z,t;\ \alpha,\gamma,\theta_R)$. The values of α, γ, and θ_R which characterize the soil are those for which

$$\theta(0,t;\ \alpha,\gamma,\theta_R) = \theta_N, \qquad t > 0 \tag{1.8}$$

The parameter identification procedure is carried out by using an optimization routine to successfully adjust the parameters in Equations 1.5 through 1.7 with the goal of satisfying Equation 1.8. Knowing α from catalog data, approximate values of α, γ, θ_R are a great help in performing the optimization.

II. INFILTRATION AND RUNOFF

2.1 Flux Controlled Infiltration, $z > 0$

Flux controlled infiltration into a linear soil when lower boundary effects can be neglected is governed by

$$\alpha\gamma \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - \alpha \frac{\partial \theta}{\partial z}, \qquad z > 0, \quad t > 0 \tag{2.1}$$

$$\theta(z,0) = \theta_i(z), \qquad z > 0 \tag{2.2}$$

$$- D \frac{\partial \theta}{\partial z} + K = r(t), \qquad z = 0, \quad t > 0 \tag{2.3}$$

where α and γ are soil-type dependent. The initial moisture profile $\theta_i(z)$ is assumed to satisfy

$$\lim_{z \to \infty} \theta_i(z) = \theta_R$$

where θ_R is a constant satisfying $0 \leq \theta_R < \phi$. In Equation 2.3 $r(t)$ (cm/hr) represents the rate of application of water to the soil surface.

Using Equations 1.3 and introducing the dimensionless variable T and Z and a new dependent variable

$$T = \frac{\alpha t}{4\gamma}, \qquad Z = \frac{\alpha z}{2}, \qquad v = D\ell^{T-Z} (\theta \cdot \theta_R) \tag{2.4}$$

allows Equations 2.1 through 2.3 to be transformed to

$$\frac{\partial v}{\partial T} = \frac{\partial^2 v}{\partial Z^2}, \qquad Z > 0, \quad T > 0 \tag{2.5}$$

$$v(Z,0) = v_i(Z), \qquad Z > 0 \tag{2.6}$$

$$\frac{\partial v}{\partial Z} - v = 2\ell^{T} r(T)/\alpha, \qquad Z = 0, \quad T > 0 \tag{2.7}$$

where $v_i(Z) = D\,[\theta_i\,(2Z/\alpha) - \theta_R]\,\ell^{-Z}$ with the diffusivity $D = 1/(\alpha\gamma)$. Equation 2.5 is the one-dimensional linear heat equation and it, together with the initial and boundary conditions (Equations 2.6 and 2.7), can be solved using the Fourier transform. The solution of Equations 2.5 and 2.6 is straightforward, but lengthy. See Warrick (1975) for details.

Solving Equations 2.5 and 2.6 using Equations 1.3 and 2.4 to recover θ yields

$$\theta(Z,T) = \theta_R + M(Z,T,\theta_i) + \gamma N(Z,T,r) \tag{2.8}$$

where

$$M[Z,T,\theta_i(Z)] = \exp(Z-T) \int_0^\infty \{(4\pi T)^{-1/2} [\exp(-\frac{(Z-Z')^2}{4T}) + \exp(-\frac{(Z+Z')^2}{4T})] - \exp(T+Z+Z') \operatorname{erfc}(\frac{(Z+Z')}{2\sqrt{T}} + \sqrt{T})\} (\theta_i(Z) - \theta_R)\, \ell^{-Z'} dZ' \tag{2.9}$$

and

$$N[Z,T,r(T)] = 2\exp(Z-T) \int_0^T \{\exp(\frac{-Z^2}{4(T-\tau)}) \cdot (\pi(T-\tau))^{-1/2} - \exp(T-\tau+Z) \operatorname{erfc}(\frac{Z}{2\sqrt{T-\tau}} + \sqrt{T-\tau})\} r(t)\, \ell^{\tau}\, d\tau \tag{2.10}$$

In Equations 2.9 and 2.10, erfc(x) represents the complementary error function 1 - erf(z) where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \ell^{-y^2} dy$$

For the important special case of constant flux $r(T) \equiv r$ and constant initial water content $\theta_i(Z) \equiv \theta_i$ significant simplifications of Equations 2.9 and 2.10 can be obtained. See Lomen and Warrick (1978).

$$M(Z,T,\theta_i) = \frac{\theta_i - \theta_R}{2}[\operatorname{erfc}(\frac{-Z}{2\sqrt{T}} + \sqrt{T}) + \ell^{2Z} \operatorname{erfc}(\frac{Z}{2\sqrt{T}} + \sqrt{T})] \tag{2.11}$$

$$N(Z,T,r) = r\ [(1/2)\mathrm{erfc}(\frac{Z}{2\sqrt{T}} - \sqrt{T}) + 2\ \sqrt{T/\pi}\ \exp(-(\frac{Z}{2\sqrt{T}} - \sqrt{T})^2)$$

$$-(Z + 2T + 1/2)\ \exp(2Z)\mathrm{erfc}(\frac{Z}{2\sqrt{T}} + \sqrt{T})] \qquad (2.12)$$

Equations 2.8, 2.11, and 2.12 can be used to model flux controlled infiltration due to a step function rainfall hyetograph.

Flux controlled infiltration can be maintained only as long as the surface moisture is below natural saturation.

To determine ponding time T_p when runoff begins and the infiltration becomes profile controlled, it is sufficient to monitor the water content at the soil surface, $Z = 0$. For the case of constant r and constant θ_i, Equations 2.8, 2.11, and 2.12 show that T_p satisfies

$$\theta_N - \theta_R = (\theta_i - \theta_R)\ \mathrm{erfc}(\sqrt{T}) + \gamma r\ [(1/2)\mathrm{erfc}(-\sqrt{T})$$

$$+ 2\sqrt{T/\pi}\ \ell^{-T} - (2T + 1/2)\ \mathrm{erfc}(\sqrt{T})] \qquad (2.13)$$

In practice, the time to ponding is determined by monitoring Equation 2.13 as T increases from zero. Once the ponding time, T_p, has been found from Equation 2.13, then Equations 2.8, 2.11, and 2.12 can be used to determine the moisture profile at the beginning of the profile controlled infiltration.

2.2 Example

Table 1 contains empirical soil data for Yolo Light Clay, Mualem (1976). To two decimal place accuracy, the saturated conductivity, K_s is 0.04 cm/hr and the porosity ϕ is 0.50.

To approximate Yolo Light Clay by a linear soil set

$$\theta/\phi = A[100K/K_s] + B$$

and use linear regression on the data points in Table 1 to obtain

$$\theta = 21.46K + 0.30 \qquad \text{(Yolo Light Clay)}$$

so that in this case $\gamma = 21.46$ hr/cm and $\theta_R = 0.03$. To determine the parameter α in the relationship

Table 1. Soil Data for Yolo Light Clay.

$-\psi$ (cm)	161.00	129.00	100.00	82.00	73.00	64.00	56.00	49.00
$100K/K_s$	1.34	1.90	3.00	4.00	5.50	7.00	8.94	10.4
θ/ϕ	0.622	0.648	0.675	0.701	0.727	0.754	0.780	0.804

$$\frac{d(\ln K)}{d\psi} = \alpha$$

linear regression can be used on the data pairs ψ, $\ln K$ obtained from the first two lines of Table 1 to obtain $\alpha = 0.21\ \text{cm}^{-1}$. For purposes of illustration, take θ_N to be 80 percent of total saturation, namely $\theta_N = 0.04$, and assume that θ_i is equal to θ_R for all Z.

For a constant rainfall rate, Equation 2.13 can be used to find the dimensionless ponding time, T_p. Then the ponding time in hours can be recovered from

$$t_p = 4\gamma\ T_p/\alpha$$

The calculations required to produce Table 2 were carried out on an Apple II 48K micro-computer. Five-point Gaussian quadrature was used to evaluate the error function in Equation 2.13. Even on this small system, the calculation of ponding time required only a few seconds of computer time. The computer program used to find ponding time is listed in Appendix I.

Once the ponding time has been found, the moisture profile at time T_p must be calculated so it can be used as the initial condition for the ensuing profile controlled infiltration. In the case of constant rainfall rate, $\theta(Z,T_p)$ can be found from Equations 2.8, 2.11, and 2.12. The moisture profile at time 1.9 hours, ponding for the rainfall rate 0.1 cm/hr, is shown in Figure 2. The computer program used to construct this moisture profile in Figure 2 is listed in Appendix II. The calculation and plotting of the profile required only ten seconds on the Apple II.

2.3 Profile Controlled Infiltration, $z > 0$

When the surface moisture content reaches natural saturation, the infiltration rate can no longer be taken equal to the application rate. Profile controlled infiltration into a linear soil when the lower boundary can be neglected is governed by

$$\alpha\gamma \frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2} - \alpha\frac{\partial\theta}{\partial z}, \qquad z > 0, \quad t > t_p \tag{2.14}$$

$$\theta(z,t_p) = \theta_i(z), \qquad z > 0 \tag{2.15}$$

$$\theta(0,t) = \theta_N, \quad t > t_p \tag{2.16}$$

Table 2. Ponding Times for Yolo Light Clay.

r (cm/hr)	0.05	0.10	0.20	0.40	0.60	1.00
t_p (hr)	7.70	1.90	0.47	0.13	0.05	0.02

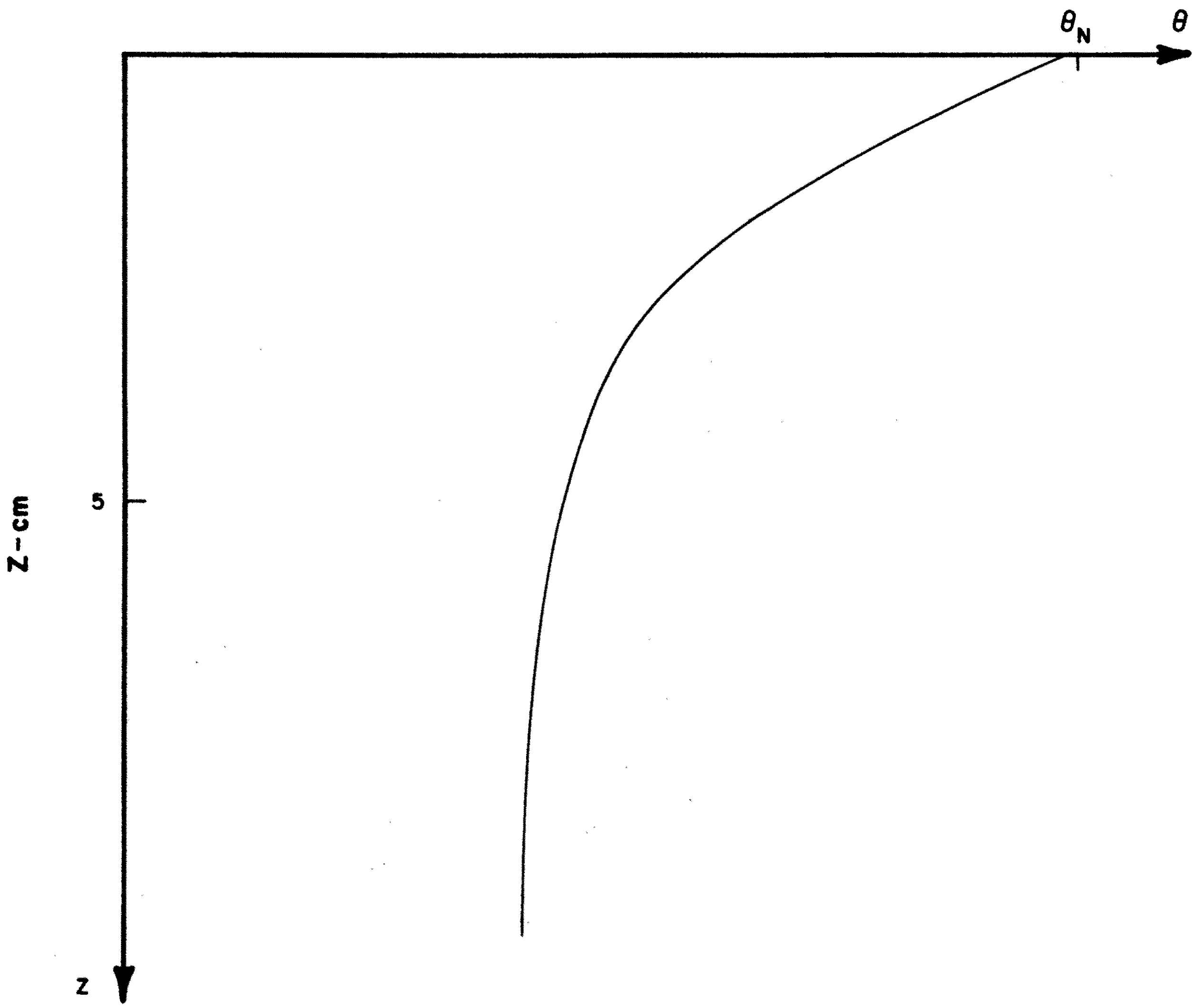

Figure 2. Moisture profile at ponding time for Yolo light clay with r = 0.1 cm/hr.

The initial condition $\theta_i(z)$ is obtained by using Equation 2.13 to find the ponding time T_p and then Equation 2.8 to determine the moisture profile at time of ponding.

To solve Equation 2.14 through 2.16, it is convenient to introduce the dimensionless variables Z and T and the new dependent variable U

$$Z = \frac{\alpha z}{2}, \qquad T = \frac{\alpha(t - t_p)}{4\gamma}, \qquad U = D(\theta - \theta_N)\, \ell^{\,T-Z} \tag{2.17}$$

In terms of T, Z, and U, Equations 2.14 through 2.16 are

$$U_T = U_{ZZ}$$

$$U(Z,0) = D(\theta_i - \theta_N)\, \ell^{-Z}$$

$$U(0,T) = 0$$

A Fourier transform can be used to find $U(Z,T)$, Tychonov and Samarski (1964). From Equations 2.17 and $D = (\alpha\gamma)^{-1}$, θ is seen to be

$$\theta(Z,T) = \theta_N + \ell^{Z-T} \int_0^\infty G(Z,Z',T)(\theta_i(Z') - \theta_N)\, \ell^{-Z'} dZ', \tag{2.18}$$

where,

$$G(Z,Z',T) = (4\pi T)^{-1/2}[\exp(-\frac{(Z - Z')^2}{4T}) - \exp(-\frac{(Z + Z')^2}{4T})] \tag{2.19}$$

Equation 2.19 provides the moisture profile during profile controlled infiltration. In watershed models it is not the moisture profile, but the runoff rate which is of primary concern. The runoff rate can be found from Equations 2.18 and 2.19 by starting with the general Darcy flux

$$- D \frac{\partial\theta}{\partial z} + K$$

When $\theta = \theta_N$, the corresponding value of hydraulic conductivity is K_N which is generally slightly less than K_s. In profile controlled infiltration, the flow rate into the soil is, in terms of dimensionless depth Z,

$$- \frac{1}{2\gamma} \frac{\partial\theta}{\partial Z} + K_N, \qquad Z = 0 \tag{2.20}$$

The partial derivative of θ with respect to Z at $Z = 0$ can be found by differentiating Equations 2.18 and setting $Z = 0$. Let $s(T)$ denote the runoff rate in terms of dimensionless time. Since $G(0,Z',T) = 0$, it follows from Equation 2.18 that

$$s(T) = r(T) - K_N + \frac{\ell^{-T}}{2\gamma} \int_0^\infty G_Z(0,Z',T)(\theta_i(Z') - \theta_N)\ell^{-Z'}\, dZ' \qquad (2.21)$$

where

$$G_Z(0,Z',T) = -(4\pi T)^{-1/2}\, Z' \exp\left(-\frac{Z'^2}{4T}\right)/T$$

Equation 2.21 provides an efficient means for calculating runoff rate during constant rate rainfall. It is straightforward to adapt Equation 2.21 to model runoff under a piecewise constant application rate.

2.4 Example

To illustrate the profile controlled infiltration solution, consider the Yolo Light Clay of Section 2.2 with a constant rainfall rate of 0.1 cm/hr. From Table 2, time to ponding is 1.9 hours so that the initial moisture distribution θ_i is the profile found in Figure 2. Recall that $\gamma = 21.46$ hr/cm, $\alpha = 0.02\ \text{cm}^{-1}$, $\theta_R = 0.3$, $\theta_N = 0.4$, and $K_s = 0.04$ cm/hr. Using two points near the surface on the profile in Figure 2, $d\theta/dz$ at $Z = 0$ is estimated to be -3. From Equation 2.20 the flux across the surface is estimated to be

$$\frac{3}{2\gamma} + K_N = 0.07 + K_N$$

At ponding time, the surface flux equals the rainfall rate. This will be the case if K_N is chosen to be 0.03 cm/hr. Using K_s in place of K_N in Equation 2.21 will generally lead to un underestimate of the runoff.

The calculation of the runoff rate $s(T)$ requires that the improper integral in Equation 2.21 be evaluated. One accurate and efficient method is as follows. First, from Figure 2 determine the depth to which rainfall has advanced during the flux controlled stage of infiltration. In this example, that depth is about $z = 5$ cm or $Z = 0.05$.

The integral in Equation 2.21 is broken into two integrals, one from $Z' = 0$ to 0.05 and the other from $Z' = 0.05$ to infinity. The integral from 0 to 0.05 can be accurately approximated by Simpson's rule. Let $h = 0.05/2$ and use $\theta_i(0) = \theta_N$ to obtain

$$\int_0^{0.05} G_Z(0,Z',T)(\theta_i(Z') - \theta_N)\ell^{-Z'}\, dZ' \doteq (h/3)[G_Z(0,h,T)(\theta_i(h) - \theta_N) \ell^{-h} + G_Z(0,2h,T)(\theta_i(2h) - \theta_N) \ell^{-2h}]$$

For $Z' > 0.05$, $\theta_i(Z') - \theta_N$ is a good approximation $\theta_R - \theta_N$, a constant. Thus, the integral remaining is $\theta_R - \theta_N$ times

$$\int_{0.05}^{\infty} [(4\pi T)^{-1/2} Z' \exp(-\frac{Z'^2}{4T})/T]\ \ell^{-Z'}\ dZ'$$

When T is small, the term in square brackets in the above integral has a large Z' derivative. To avoid having to deal with this large derivative, an integration by parts is used to obtain

$$\int_{0.05}^{\infty} [(4\pi T)^{-1/2} Z' \exp(-\frac{Z'^2}{4T})/T]\ \ell^{-Z'} dZ' = (\pi T)^{-1/2} \exp(\frac{0.025}{4T} - 0.025) + (\pi T)^{-1/2} \int_{0.05}^{\infty} \exp(-\frac{Z'^2}{4T}) \ell^{-Z'} dZ'$$

The last integral in the above equation can be evaluated in terms of the error function. However, less algebra is involved and a satisfactory estimate is obtained if Laguerre-Gauss quadrature is used in which case

$$\int_{0.05}^{\infty} \exp(-\frac{Z'^2}{4T}) \ell^{-Z'} dZ' \doteq \sum_{i=1}^{5} W_i \exp(-\frac{(Y_i + 0.05)^2}{4T})$$

where the Y_i and W_i are shown in Table 3.

Using Equation 2.21 and the above integration techniques yields the runoff values in Table 4. The computer program used to generate Table 4 is listed in Appendix II.

Table 3. Abscissas and Weights for Laguerre-Gauss Quadrature.

i	1	2	3	4	5
Y_i	0.2635	1.4134	3.5964	7.0858	12.6408
W_i	0.5217	0.3987	0.0759	0.0036	0.00002

Table 4. Runoff from Yolo Light Clay with $r = 0.1$ cm/hr.

T	0.007	0.010	0.025	0.050	0.100	0.500
$s(T)$ cm/hr	0.017	0.026	0.046	0.056	0.062	0.069

At the end of the rainfall event, Equations 2.18 and 2.19 can be used to determine the moisture profile at the beginning of the redistribution period. Integration techniques similar to those outlined above are used to evaluate the improper integral in Equation 2.18.

2.5 Flux Controlled Infiltration, $0 < z < z_o$

In some situations it is not reasonable to view the soil as being semi-infinite in z. One common situation is the case of a layer of hardpan not far below the soil surface. In this case, the lower boundary condition has a significant effect on the developing moisture profile and if the semi-infinite theory is applied, an underestimate of runoff will usually result.

Flux controlled infiltration into a linear soil with an impermeable boundary at $z = z_o$ is governed by

$$\alpha\gamma \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - \alpha \frac{\partial \theta}{\partial z}, \qquad 0 < z < z_o, \quad t > 0 \tag{2.22}$$

$$\theta(z,0) = \theta_i(z), \qquad 0 < z < z_o \tag{2.23}$$

$$- D \frac{\partial \theta}{\partial z} + K = r(t), \qquad z = 0, \quad t > 0 \tag{2.24}$$

$$- D \frac{\partial \theta}{\partial z} + K = 0, \qquad z = z_o, \quad t > 0 \tag{2.25}$$

Again, it is convenient to introduce dimensionless variables T and Z

$$T = \frac{t}{\alpha\gamma z_o^2}, \qquad Z = \frac{z}{z_o}$$

and a new dependent variable

$$v(Z,T) = \exp[b(Z - bT)] \cdot D(\theta - \theta_R), \qquad b = \alpha z_o/2$$

Using Equation 1.3 and the above change of variables allows Equation 2.22 through 2.25 to be transformed to

$$\frac{\partial v}{\partial T} = \frac{\partial^2 v}{\partial Z^2}, \qquad 0 < Z < 1, \quad T > 0 \tag{2.26}$$

$$v(Z,0) = D(\theta_i - \theta_R)\, \ell^{-bZ}, \qquad 0 < Z < 1 \tag{2.27}$$

$$bv - \frac{\partial z}{\partial Z} = z_o\ r(T)\ \ell^{b^2 T}, \qquad Z = 0, \quad T > 0 \tag{2.28}$$

$$bv - \frac{\partial v}{\partial Z} = 0, \qquad Z = 1, \quad T > 0 \tag{2.29}$$

Equations 2.6 through 2.29 can be solved by obtaining a Green's function. Let $g = g(Z,Z',T,T')$ be a Green's function to be determined. Multiplying Equation 2.26 by g integrating from $T = 0$ to $T_o > T'$ and from $Z = 0$ to 1 and using integration by parts to move all derivatives from v onto g yields

$$\int_0^{T_o} \int_0^1 \left(\frac{\partial v}{\partial T} - \frac{\partial^2 v}{\partial Z^2} \right) g\ dZ\ dT = \int_0^{T_o} \int_0^1 - \left(\frac{\partial g}{\partial T} + \frac{\partial^2 g}{\partial Z^2}\right) v\ dZ\ dT$$

$$+ \int_0^1 vg \Big|_0^{T_o} dZ + \int_0^{T_o} \left(v \frac{\partial g}{\partial Z} - g \frac{\partial v}{\partial Z}\right) dT$$

Next, Equations 2.28 and 2.29 are used to eliminate $\partial v/\partial Z$ from the above equation to obtain

$$\int_0^{T_o} \int_0^1 \left(\frac{\partial g}{\partial T} + \frac{\partial^2 g}{\partial Z^2}\right) v\ dZ\ dT = \int_0^1 vg \Big|_0^{T_o} dZ + \int_0^{T_o} \left(\frac{\partial g}{\partial Z} - bg\right) v \Big|_{Z=1} dT$$

$$- \int_0^{T_o} \left[\left(\frac{\partial g}{\partial Z} - bg\right) v - g z_o\ r\ \ell^{b^2 T}\right]_{Z=0} dT$$

From the last equation it follows that if g is required to satisfy

$$\frac{\partial g}{\partial T} + \frac{\partial^2 g}{\partial Z^2} = \delta(Z - Z')\ \delta(T - T'), \qquad 0 < Z, \quad Z' < 1,$$

$$0 < T, \quad T' < T_o \tag{2.30}$$

$$bg - \frac{\partial g}{\partial Z} = 0, \qquad Z = 0 \text{ and } Z = 1 \tag{2.31}$$

$$g = 0, \quad T > T' \tag{2.32}$$

then

$$v(Z,T) = \int_0^1 g(Z,Z',T,0)\ D(\theta_i(Z') - \theta_R)\ \ell^{-bZ'} dZ'$$

$$+ z_o \int_0^T r(T')\ \ell^{b^2T'}\ g(Z,0,T,T')\ dT'$$

In Equation 2.30, δ represents the Dirac delta function which is the distributional derivative of a unit impulse function, Wylie (1975).

The boundary conditions (2.31) motivate looking for g in the form

$$g = \sum_{n=0}^{\infty} a_n(T,T',Z')\ w_n(Z)$$

where the $w_n(Z)$ satisfy $w_n''(Z) = \lambda_n\ w_n(Z)$. A straightforward calculation shows $w_o(Z) = \exp(bZ)$, $\lambda_o = b^2$, and

$$w_n(Z) = n\pi \cos n\pi Z + b \sin n\pi Z, \qquad \lambda_n = -(n\pi)^2, \quad n = 1, 2 \ldots$$

From Equation 2.30 and the orthogonality of the w_n it follows that

$$\left(\frac{\partial a_n}{\partial T} + \lambda_n\ a_n\right) \|w_n\|^2 = w_n(Z')\ \delta(T - T')$$

where $\|w_o\|^2 = 0.5(\ell^{2b} - 1)/b$ and $\|w_n\|^2 = 0.5[(n\pi)^2 + b^2]$, $n = 1, 2 \ldots$

Solving the last equation for a_n completes the determination of g,

$$g(Z,Z',T,T') = \sum_{n=0}^{\infty} \|w_n\|^{-2}\ \ell^{\lambda_n(T-T')}\ w_n(Z)\ w_N(Z')$$

from which it follows that, during flux controlled infiltration with $0 < Z < 1$,

$$\theta(Z,T) = \theta_R + \exp[b(bT - Z)] \int_0^1 g(Z,Z',T,0)(\theta_i(Z) - \theta_R)\ell^{-bZ'} dZ'$$

$$+ \alpha\gamma z_o \exp[b(bT - Z)] \int_0^T g(Z,0,T,T')r(T')\ell^{b^2T'} dT' \qquad (2.33)$$

Equation 2.33 is to be used to model infiltration until ponding time T_p when $\theta(0,T) = \theta_N$.

2.6 Profile Controlled Infiltration, $0 < z < z_o$

Profile controlled infiltration into a linear soil with an impermeable boundary at $z = z_o$ is governed by

$$\alpha\gamma \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - \alpha \frac{\partial \theta}{\partial z}, \qquad 0 < z < z_o, \quad t > t_p \tag{2.34}$$

$$\theta(z,0) = \theta_i(z), \qquad 0 < z < z_o \tag{2.35}$$

$$\theta(0,t) = \theta_N, \qquad t > 0 \tag{2.36}$$

$$- D \frac{\partial \theta}{\partial z} + K = 0, \qquad z = z_o, \quad t > t_p \tag{2.37}$$

The initial moisture distribution $\theta_i(z)$ in Equation 2.35 is the profile at the end of the flux controlled infiltration and is obtained by setting $T = T_p$ in Equation 2.33. Using essentially the same change of variables as in Section 2.5, namely

$$T = \frac{t - t_p}{\alpha\gamma z_o^2}, \qquad Z = \frac{z}{z_o}, \qquad v = \exp[b(Z - bT)]\ D(\theta - \theta_R)$$

where $b = \alpha z_o/2$, allows Equations 2.34 through 2.37 to be transformed to

$$\frac{\partial v}{\partial T} = \frac{\partial^2 v}{\partial Z^2}, \qquad 0 < Z < 1, \quad T > 0 \tag{2.38}$$

$$v(Z,0) = D(\theta_i - \theta_R)\ \ell^{-bZ}, \qquad 0 < Z < 1 \tag{2.39}$$

$$v(0,T) = D(\theta_N - \theta_R)\ \ell^{b^2 T}, \qquad T > 0 \tag{2.40}$$

$$bv - \frac{\partial v}{\partial Z} = 0, \qquad Z = 1, \quad T > 0 \tag{2.41}$$

Let $h = h(Z,Z',T,T')$ be a Green's function to be determined. The calculations between Equations 2.29 and 2.30 together with Equations 2.40 and 2.41 show

$$\int_0^{T_o} \int_0^1 \left(\frac{\partial h}{\partial T} + \frac{\partial^2 h}{\partial Z^2}\right) v \, dZ \, dT = \int_0^1 vh \Big|_0^{T_o} dZ + \int_0^{T_o} \left(\frac{\partial h}{\partial Z} - bh\right) v \Big|_{Z=1} dT$$

$$- \int_0^{T_o} [(D(\theta_N - \theta_R) \, \ell^{-b^2 T} \frac{\partial h}{\partial Z} - h \frac{\partial v}{\partial Z}]\Big|_{Z=0} dT$$

From the above equation it follows that if h satisfies

$$\frac{\partial h}{\partial T} + \frac{\partial^2 h}{\partial Z^2} = \delta(Z - Z') \, \delta(T - T'), \qquad 0 < Z, \quad Z' < 1$$

$$0 < T, \quad T' < T_o \qquad (2.42)$$

$$h = 0, \quad Z = 0 \qquad (2.43)$$

$$bh - \frac{\partial h}{\partial Z}, \qquad Z = 1 \qquad (2.44)$$

$$h = 0, \quad T > T' \qquad (2.45)$$

then the solution of Equations 2.38 through 2.41 is

$$v(Z,T) = \int_0^1 h(Z,Z',T,0) \, D(\theta_i(Z') - \theta_R) \, \ell^{-bZ'} dZ'$$

$$- \int_0^T h_{Z'}(Z,0,T',T) \, D(\theta_N - \theta_R) \, \ell^{b^2 T'} \, dT'$$

Equations 2.43 and 2.44 motivate an expansion for h in the form

$$h = \sum_{n=1}^{\infty} C_n(T,T',Z') \sin \mu_n Z$$

where μ_n is the n-th non-negative root of the transcendental equation $\mu = \tan(\mu/b)$. From the orthogonality of the functions $\sin \mu_n Z$ and Equation 2.42 it follows that

$$\left(\frac{dC_n}{dT} + \mu_n \, C_n\right) \left|\left| \sin \mu_n Z \right|\right|^2 = \sin \lambda_n Z' \cdot \delta(T - T')$$

where

$$||\sin \mu_n Z||^2 = \int_0^1 \sin^2 \mu_n Z$$

Solving for the C_n shows that

$$h(Z,Z',T,T') = \sum_{n=1}^{\infty} ||\sin \mu_n Z||^{-2} \ell^{\mu_n(T-T')} \sin \mu_n Z \sin \mu_n Z'$$

from which it follows that, during profile controlled infiltration with $0 < Z < 1$

$$\theta(Z,T) = \theta_R + \exp[b(Z - bT)] \int_0^1 h(Z,Z',T,0)(\theta_i(Z) - \theta_R)\, \ell^{-bZ'} dZ'$$

$$- \int_0^T h_{Z'}(Z,0,T',T)\, D(\theta_N - \theta_R)\, \ell^{-b^2T'} dT' \qquad (2.46)$$

Equation 2.46 is used to describe infiltration as long as the surface moisture content is held at θ_N. If at any time during the rainfall event the surface moisture content falls below θ_N, then flux controlled infiltration resumes and Equation 2.33 models the flow.

As in Section 2.3, let $s(T)$ be the runoff rate and K_N be the value of the conductivity when $\theta = \theta_N$. It follows from Equations 2.20, 2.46 and the fact that $h_Z(0,0,T',T) = 0$ that

$$s(T) = r(T) - K_N - \frac{\ell^{-b^2T}}{2\gamma} \int_0^1 h_Z(0,Z',T,0)(\theta_i(Z) - \theta_R)\, \ell^{-bZ'} dZ'$$

where $h_Z(0,Z',T,0)$ denotes the partial derivative of h at $Z = 0$, $T' = 0$.

2.7 Example

To illustrate the analytical solutions developed in the last two sections, three soils are considered. The soils are the Yolo Light Clay of Section 2.2, Gravelly Sand G.E. 9, [Reisenauer (1963)(see Table 5)], and Gilat Loam, [Mualem (1976)(see Table 6)].

For Gravelly Sand the saturated conductivity is 1 cm/hr and the porosity $\phi = 0.326$. For the Gilat Loam $K_s = 0.72$ cm/hr and the porosity is 0.44.

Table 5. Soil Data for Gravelly Sand G.E. 9.

$-\psi$ (cm)	90.0	80.0	70.0	55.0	40.0	30.0
$100K/K_s$	3.0	8.0	15.5	34.0	52.5	66.0
θ/ϕ	0.582	0.589	0.705	0.831	0.890	0.908

Table 6. Soil Data for Gilat Loam.

$-\psi$ (cm)	77.50	72.50	66.00	60.00	54.50	49.60	44.70	39.00
$100K/K_s$	1.29	2.58	4.79	8.33	13.30	20.80	32.50	47.90
θ/ϕ	0.590	0.636	0.682	0.727	0.773	0.818	0.864	0.909

Recall from Section 2.2 that when a linear soil was calibrated to the soil data in Table 1, the relationships

$$\theta = \gamma K + 0.30, \qquad \gamma = 21.46 \text{ hr/cm} \qquad \text{(Yolo Light Clay)}$$

$$\frac{d(\ln K)}{d\psi} = \alpha = 0.02 \text{ cm}^{-1} \qquad \text{(Yolo Light Clay)}$$

were obtained. Using the calibration method outlined in Section 2.2 to fit a linear soil to the Gravelly Sand and the Gilat Loam yields

$$\theta = \gamma K + 0.191, \qquad \gamma = 0.18 \text{ hr/cm} \qquad \text{(Gravelly Sand G.E. 9)}$$

$$\frac{d(\ln K)}{d\psi} = \alpha = 0.05 \text{ cm}^{-1} \qquad \text{(Gravelly Sand G.E. 9)}$$

$$\theta = \gamma K + 0.284, \qquad \gamma = 0.39 \text{ hr/cm} \qquad \text{(Gilat Loam)}$$

$$\frac{d(\ln K)}{d\psi} = \alpha = 0.09 \text{ cm}^{-1} \qquad \text{(Gilat Loam)}$$

As an illustration of the analytical solutions of Sections 2.5 and 2.6, consider three 50 cm columns of the above soils. For purposes of comparison, assume that initially the water in each column is in static equilibrium with $\psi = -100$ cm at the soil surface.

Figure 3 shows how the three soils respond to the rainfall hyetograph indicated by the piecewise constant solid curve. As expected, the Yolo Light Clay has the smallest ponding time. In fact, for the clay, runoff begins almost immediately. The runoff rate is easily obtained from Figure 3 by subtracting the infiltration rate from the application rate. The Gravelly Sand has the largest ponding time. If the rainfall rate during the last half hour had been only slightly less, then the infiltration for the Gravelly Sand during that period would have switched from profile to flux controlled. Table 7 shows that during this rainfall, the cumulative inflow into the sand is much greater than into the clay. In all respects, the response of the loam falls between the sand and the clay.

The computer program used to construct Figure 3 and Table 7 is listed in Appendix IV. The calculations required to construct Figure 3 required nine seconds on a CDC Cyber 171.

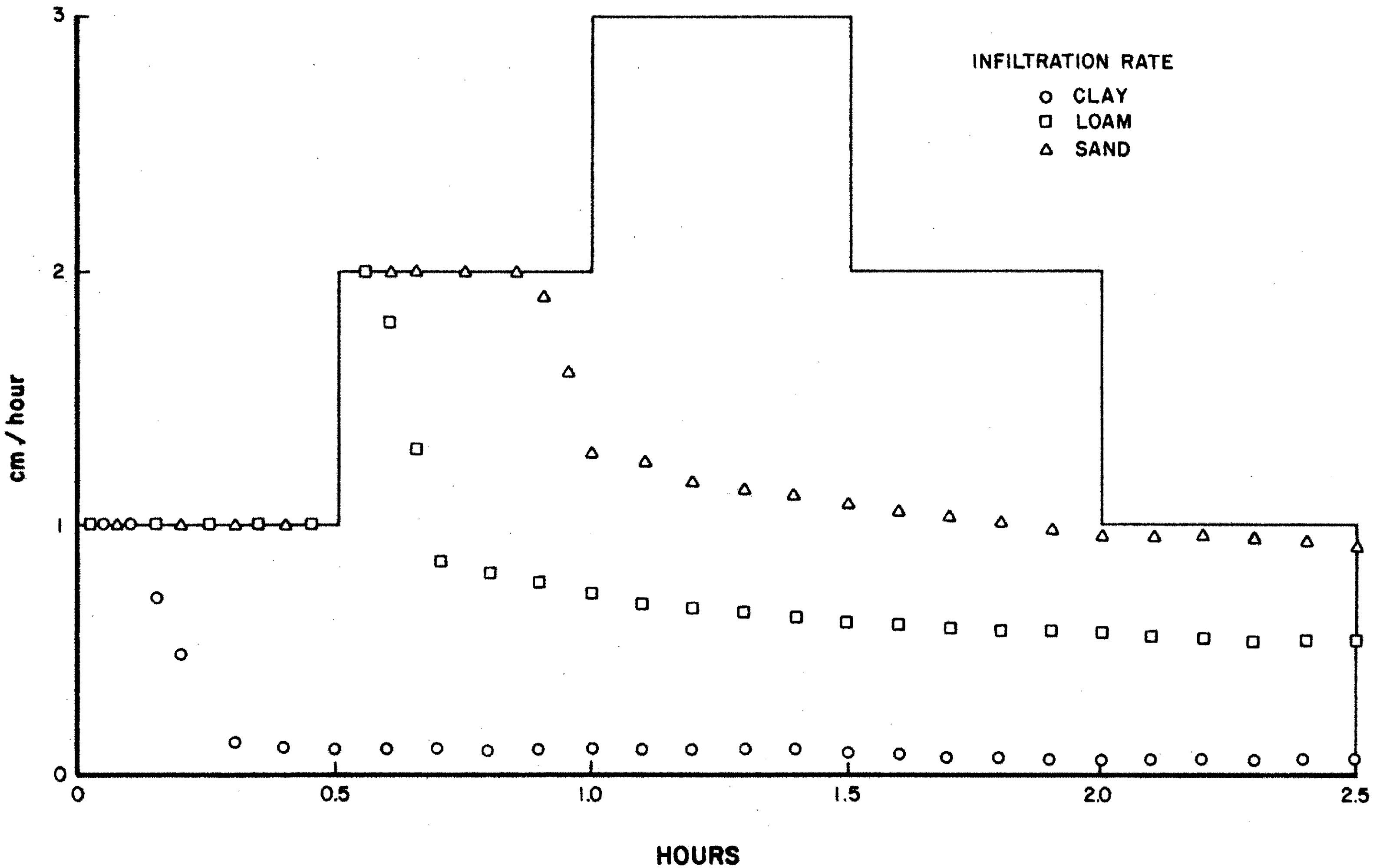

Figure 3. Infiltration rate for clay, loam and sand.

Table 7. Cumulative Infiltration During Rainfall.

Time	0	0.50	0.10	1.50	2.00	2.50
Clay (cm H_2O)	0	0.17	0.23	0.28	0.32	0.36
Sand (cm H_2O)	0	0.50	1.42	2.12	2.71	3.15
Loam (cm H_2O)	0	0.50	1.21	1.64	2.02	2.39

III. REDISTRIBUTION

3.1 Evaporation

Kolasew (1941) divided soil water evaporation into three stages. The first is the constant rate or the energy-limiting stage. When the soil can no longer supply water to the surface fast enough to use all the available energy, the profile-limiting stage begins. The third phase occurs when the soil surface is very dry and the evaporation rate is low and decreasing almost linearly with time. In Figure 4, adapted from Gardner (1981), these stages are indicated by A, B, and C, respectively.

Let θ_R be the effective residual, reference water content in Equation 1.3. To approximate the evaporation rate curve A-B-C of Figure 4, the outward flux at the surface is set equal to $E(\theta - \theta_R)$,

$$D \frac{\partial \theta}{\partial z} - K = E(\theta - \theta_R), \qquad z = 0 \tag{3.1}$$

The constant E (cm/day) is chosen so that $E(\theta_N - \theta_R)$ agrees with the energy-limiting evaporation rate. The flux boundary condition Equation 3.1 yields an evaporation rate curve qualitatively as indicated by the dashed curve in Figure 4.

3.2 Extraction by Roots

Experimental work of Herkelrath et al (1977) indicates that in many cases the rate at which roots extract water from the soil decreases with decreasing moisture content. Moltz and Remson (1970, 1971) have used a moisture dependent distributed sink to simulate uptake by roots.

If the extraction rate varies linearly with water content then a sink of the form $S(\theta - \theta_R)$ can be used to account for removal of water by roots where $S(\theta_N - \theta_R)$ is the maximum extraction rate. Let z_1 and z_2 be two soil depths, $z_2 > z_1$. Darcy's law and conservation of mass imply

$$\frac{d}{dt} \int_{z_1}^{z_2} \theta dz = \int_{z_1}^{z_2} \frac{d}{dz} [D \frac{\partial \theta}{\partial z} - K] \, dz + \int_{z_1}^{z_2} S(\theta_N - \theta_R) \, dz \tag{3.2}$$

so

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} [D \frac{\partial \theta}{\partial z} - K] + S(\theta_N - \theta_R)$$

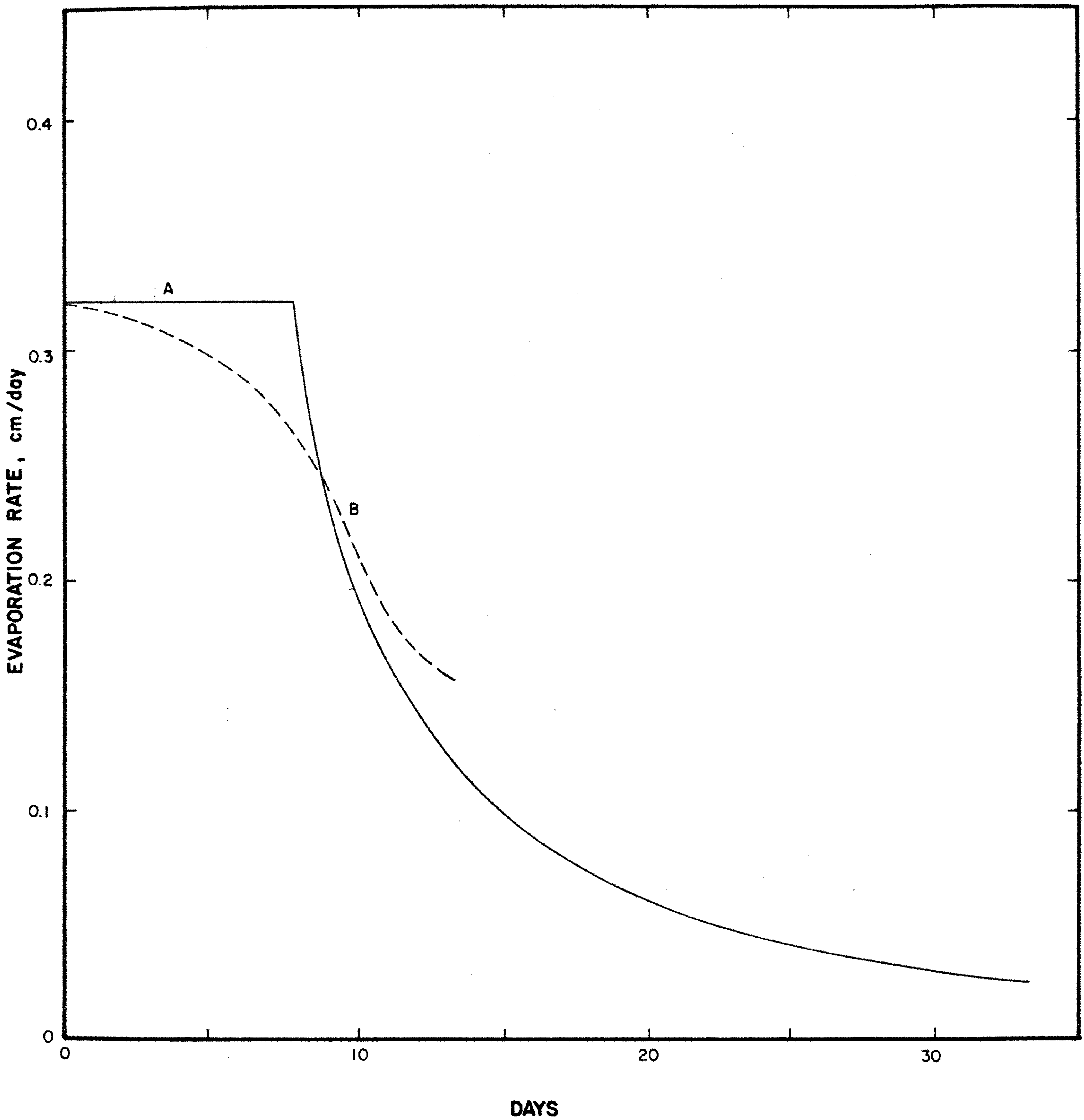

Figure 4. Typical evaporation rate curve.

For the case of a linear soil with $D = (\alpha\gamma)^{-1}$ the above equation is

$$\alpha\gamma \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - \alpha \frac{\partial \theta}{\partial z} + \alpha\gamma\ S(\theta - \theta_R)$$

Since the sink is distributed it follows that the units on S are $(\text{day})^{-1}$. The constant S is negative and such that $S(\theta_N - \theta_R)$ is the maximum extraction rate. S is dependent on the range of moisture content, soil type, and root system being modeled.

3.3 Redistribution, $z > 0$

If the evaporation rate and the root extraction rate are as described in Sections 3.1 and 3.2 and lower boundary effects can be neglected, then for a linear soil the moisture content is governed by

$$\alpha\gamma \frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial z^2} - \alpha \frac{\partial \theta}{\partial z} + \alpha\gamma\ S(\theta - \theta_R), \qquad z > 0, \quad t > 0 \tag{3.3}$$

$$D \frac{\partial \theta}{\partial z} - K = E(\theta - \theta_R), \qquad z = 0, \quad t > 0 \tag{3.4}$$

$$\theta(z,0) = \theta_i(z), \qquad z > 0 \tag{3.5}$$

where time is measured from the end of the last rainfall. In accordance with Equations 1.3, the diffusivity is $D = 1/(\alpha\gamma)$. The parameters α, γ, and θ_R are, as before, soil type and calibration range dependent. The saturated potential evaporation rate determines E while the root extraction rate fixes the constant S.

By introducing new variables, Equation 3.3 can be transformed into a standard initial-boundary value problem for the heat equation. Let

$$Z = \frac{\alpha\beta z}{4}, \qquad \tau = \frac{\alpha\beta^2 t}{16\gamma}, \qquad v = D(\theta - \theta_R)\ \ell^{(a\tau + bZ)} \tag{3.6}$$

where

$$\beta = (1 + 2\gamma E), \qquad b = 1/\beta, \qquad a = \frac{4}{\beta^2}\left(\frac{4\gamma 5}{\alpha} - 1\right) \tag{3.7}$$

In terms of the new variables Z, T, v, defined by Equations 3.6 and 3.7, Equations 3.3 through 3.5 can be written in the form

$$\frac{dv}{d\tau} = \frac{d^2v}{dZ^2}, \qquad Z > 0, \quad \tau > 0 \tag{3.8}$$

$$\frac{\partial v}{\partial Z} - 2v = 0, \qquad Z = 0, \quad \tau > 0 \tag{3.9}$$

$$v = D(\theta_i - \theta_R)\, \ell^{bZ}, \qquad Z > 0, \quad \tau > 0 \tag{3.10}$$

The solution to Equations 3.8 through 3.10 is known, Warrick (1975), to be

$$v(Z,T) = D\ell^{Z-T} \int_0^\infty \{[(4\pi T)^{-1/2} (\exp(-\frac{(Z-Z')^2}{4\tau}) + \exp(-\frac{(Z-Z')^2}{4\tau}))$$

$$- \exp(T + Z + Z')\operatorname{erfc}(\frac{Z+Z'}{2\sqrt{\tau}} + \sqrt{\tau}\ )]\ (\theta_i - \theta_R)\ell^{bZ'}\}\ \ell^{-Z'} dZ'$$

From Equation 3.6 it follows that

$$\theta(Z,\tau) = \exp(-a\tau - bZ)\ D^{-1}\ v(Z,\tau) + \theta_R$$

The last two equations provide the solution θ in terms of Z and τ. The scaled variables Z and τ needed to transform Equations 3.3 through 3.5 to Equations 3.8 through 3.10 are not as convenient for practical use as the original variables z and t. For most practical values of t, α, β, and γ, τ is quite small and it is easier to work with t. Using Equations 3.6 and 3.7 to recover the variables z and t yields

$$\theta(z,t) = \exp[\ \frac{\alpha t}{4}\ (1 - \frac{4\gamma S}{\alpha}) - \frac{\alpha z}{4}\ ]\ u(z,t) + \theta_R \tag{3.11}$$

where $u(z,t)$ is given by

$$u(z,t) = \exp[\frac{\alpha\beta z}{4} - \frac{\alpha\beta^2 t}{16\gamma}] \frac{\alpha\beta}{4} \int_0^\infty \{ [(\pi \frac{\alpha\beta^2 t}{4\gamma})^{-1/2} (\exp(-\alpha \frac{(z - z')^2}{4\gamma t})$$

$$+ \exp(- \frac{\alpha(z + z')^2}{4\gamma t})) - \exp(\frac{\alpha\beta^2 t}{16\gamma} + \frac{\alpha\beta}{4}(z + z'))$$

$$\mathrm{erfc}(\sqrt{\frac{\alpha}{\gamma t}} \frac{(z + z')}{2} + \frac{\beta\sqrt{\alpha\gamma t}}{4})](\theta_i(z') - \theta_R) \ell^{\frac{\alpha z'}{4}} \}$$

$$\ell^{\frac{-\alpha\beta z'}{4}} dz' \qquad (3.12)$$

At first the integral in Equation 3.12 appears formidable. However, it can be easily approximated using Laguerre-Gauss quadrature. Let $F(\hat{z},t,z')$ denote the expression in {} in Equation 3.12 where $\hat{z}$ represents the grouping $\alpha\beta z/4$. If y_i and w_i are as in Table 3, then

$$u(z,t) \doteq \exp[\hat{z} - \frac{\alpha\beta^2 t}{16\gamma}] \sum_{i=1}^{5} w_i F(\hat{z},t,y_i) \qquad (3.13)$$

Using the above approximation to u requires only 21 exponential and five error function evaluations per θ evaluation. This allows the moisture profile to be economically updated during redistribution.

3.4 Redistribution, $0 < z < z_o$

If the evaporation rate and the root extraction rate are as described in Sections 3.1 and 3.2, then redistribution of water in a linear soil with an impermeable boundary at $z = z_o$ is governed by

$$\alpha\gamma \frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial z^2} - \alpha \frac{\partial\theta}{\partial z} - \alpha\gamma S(\theta - \theta_R), \quad 0 < z < z_o, \quad t > 0 \qquad (3.14)$$

$$D \frac{\partial\theta}{\partial z} - K = E(\theta - \theta_R), \quad z = 0, \quad t > 0 \qquad (3.15)$$

$$D \frac{\partial\theta}{\partial z} - K = 0, \quad z = z_o, \quad t > 0 \qquad (3.16)$$

$$\theta(z,0) = \theta_i(z), \quad 0 < z < z_o \qquad (3.17)$$

where time is measured from the end of the last rainfall.

Introducing the new variables of Equations 3.6 and 3.7 transforms Equations 3.14 through 3.16 to

$$\frac{\partial v}{\partial T} = \frac{\partial^2 v}{\partial Z^2}, \qquad 0 < Z < Z_o, \quad T > 0 \tag{3.18}$$

$$\frac{\partial v}{\partial Z} - 2v = 0, \qquad z = 0, \quad T > 0 \tag{3.19}$$

$$\beta \frac{\partial v}{\partial Z} - 5v = 0, \qquad Z = Z_o, \quad T > 0 \tag{3.20}$$

$$v = D(\theta_i - \theta_R)\, \ell^{bZ}, \qquad 0 < Z < Z_o, \quad T > 0 \tag{3.21}$$

where $Z_o = \alpha\beta z_o/4$.

Equations 3.18 through 3.21 can be solved by a simple eigenfunction expansion. Let the functions $w_n(Z)$ be chosen such that

$$w''_n(Z) = \lambda_n\, w_n(Z), \qquad w'_n(0) - 2w_n(0) = 0,$$

$$\beta w'_n(Z_o) - 5w_n(Z_o) = 0 \tag{3.22}$$

Equation 3.22 constitutes a regular Sturm-Liouville eigenvalue problem and therefore it is guaranteed, Wylie (1975), that there are an infinite number of nontrivial solutions $w_n(Z)$, $n = 0, 1, 2 \ldots,$ satisfying

$$\int_0^{Z_o} w_n(z)\, w_m(Z)\, dZ = 0, \qquad m \neq n \tag{3.23}$$

To determine the eigenpair $w_o(Z)$, λ_o, set $\lambda_o = \mu_o^2$ so from the first of Equations 3.22 it follows that

$$w_o(Z) = A\, \ell^{\mu Z} + 3\ell^{-\mu Z}$$

Now the above and the last two of Equations 3.22 imply that

$$-\frac{(2 - \mu_o)(5 + \beta\mu_o)}{(2 + \mu_o)(5 - \beta\mu_o)}\, \ell^{-2\mu_o Z_o} = 1 \tag{3.24}$$

which has a unique positive solution, near $\mu = 5/\beta$, except in the special case that $2 - \mu = 5 - \beta\mu$ in which no zeroth eigenpair exists. In the exceptional case that $2\beta = 10Z_o + 5$ the next eigenpair is $w = 2Z + 1$, $\lambda = 0$.

Otherwise, the remaining eigenpairs can be found by setting $\lambda_n = -\mu_n^2$ to obtain from the first of Equations 3.22

$$w_n = A_n \cos \mu_n Z + B_n \sin \mu_n Z$$

which together with the last two of Equations 3.22 implies

$$\beta\mu_n^2 \cos 2\mu_n Z_o = 5\mu_n \cos \mu_n Z_o + 10 \sin \mu_n Z_o \tag{3.25}$$

By solving Equations 3.24 and 3.25, the eigenvalues and eigenfunctions can be determined. In practice only the first dozen or so eigenpairs are required. Equations 3.24 and 3.25 can be solved quickly and economically by elementary numerical methods such as Newton's method. Having found the eigenpair w_n, λ_n, the solution of Equations 3.18 through 3.21 is almost immediate. Set

$$v = \sum_{n=0}^{\infty} a_n(T)\, w_n(Z)$$

Equations 3.18 and 3.23 imply $a'_n = \lambda_n a_n$ so

$$v = \sum_{n=0}^{\infty} C_n\, \ell^{\lambda_n T}\, w_n(Z)$$

Finally the sequence of constants C_n is fixed by requiring

$$D(\theta_i - \theta)\, \ell^{bZ} = \sum_{n=0}^{\infty} C_n\, w_n(Z)$$

from which using Equation 3.23, it follows that

$$C_n = ||w_n||^{-2} \int_0^{Z_o} D(\theta_i - \theta)\ell^{bZ}\, w_n(Z)\, dZ$$

where

$$||w_n||^2 = \int_0^{Z_o} w_n^2(Z)\, dZ$$

With the determination of $v(Z,T)$ complete, the moisture content function $\theta(Z,T)$ can be easily recovered from Equation 3.6.

3.5 Example

To illustrate the solution developed in the last section, consider the 50 cm column of Yolo Light Clay at the end of the rainfall event described by the solid, piecewise constant curve in Figure 3. The parameter E is taken to be 2.5 cm/day or zero. Since $\theta_N - \theta_R = 0.1$, it follows from Section 3.1 that a positive value of E in Table 8 represents a potential evaporation rate of 0.25 cm/day. The constant S is chosen to be either zero or -0.7 day^{-1} so that in Table 8 a negative value of S corresponds to a distributed sink which each day extracts 20 percent of the moisture in excess of $\theta = 0.3$.

The body of Table 8 contains the moisture content at the 10 cm depth. The day following the rainfall is indicated across the top row. Root extraction and ET patterns are specified in the first column. The program used to construct Table 8 is listed in Appendix IV.

Table 8. Moisture Content at 10 cm in Yolo Light Clay During Redistribution.

t (days)	0	1	2	3	4	5	6	7
$E = 0, \quad S = 0$	0.302	0.332	0.329	0.328	0.326	0.325	0.324	0.323
$E = 0, \quad S < 0$	0.302	0.323	0.317	0.314	0.311	0.309	0.306	0.306
$E > 0, \quad S = 0$	0.302	0.329	0.323	0.317	0.311	0.306	0.302	0.300
$E > 0, \quad S < 0$	0.302	0.320	0.311	0.305	0.301	0.300	0.300	0.300

IV. SUMMARY

Using an idealized linear soil to model real soils makes it possible to develop analytical solutions of the equations governing soil mositure flow. In this report, solutions have been obtained for both flux and profile controlled infiltration into soils either semi-infinite or finite in depth. Equations to predict the beginning of runoff during a rainfall and the subsequent runoff rate were developed.

The redistribution of soil water between rainfall events was also considered. To model both the energy-limited and profile-controlled ET processes, the removal rate at the surface was taken to be moisture dependent. Extraction by roots was accounted for by a distributed moisture dependent sink in the flow equation. Analytical solutions of the redistribution equations were developed for soils semi-infinite or finite in depth.

One of the foremost considerations in this report was computational efficiency. The solutions presented here can be quickly evaluated. Many of the calculations require only the error function. Instead of using tabular values, the error function was evaluated by five-point Gauss-Legendre quadrature. In those cases where an improper integral must be evaluated, the calculation was based on a five-point Laguerre-Gauss quadrature. The solutions for soils of finite dpeth are eigenfunction expansions. The finite Fourier transforms involved then were evaluated by Filon integration.

The solutions developed in this report are much more computationally efficient than the usual finite difference approach to soil moisture modeling. The algorithms presented here are not based on marching through time. For example, during redistribution a moisture profile can be calculated at any time knowing only the profile at the end of the rainfall event and the ET and soil parameters. In contrast, finite difference methods require that profiles be calculated at many intermediate times.

No claim is made here that these linear flow equations can model all soils over all moisture levels, nor that the profile and runoff predictions from these flow equations will always be in very close agreement with the actual profiles and runoff. However, it is expected that over the midrange of moisture content for soils which are not highly hysteretic, these models will perform well. Even in those cases where these solutions do not produce quantitatively accurate results, valuable information regarding the relative impact of different management decisions and natural phenomena can be gained.

The solutions developed here are modular and can easily be interfaced with each other or with existing watershed models. The computer time involved is sufficiently small that it is feasible to apply these techniques to a fairly large watershed.

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APPENDIX I

Program Theta Top *

* This program is interactive and is written in Applesoft.

```
1      GOSUB 200
5      AL = .02:
       GM = 21.46:
       TR = .3:
       TN = .4:
       RR = .1
10     INPUT "ENTER T  ";T
15     T = AL * T / 4 / GM:
       X =  SQR (T)
20     GOSUB 90
30     U = (.5 + 2 * T) * (ERF - 1) + 2 * X *  EXP ( - T * T) /  SQR (PI) + .5
       * (1 + ERF)
35     U = TR + GM * RR * U
40     PRINT "THETA AT THE SURFACE IS ";U
50     IF U < TN
          THEN
        PRINT "***PONDING HAS NOT OCCURRED***"
60     IF U > TN
          THEN
        PRINT "***PONDING HAS ALREADY OCCURRED***"
70     PRINT "  "
80     GOTO 10
90     IF X > 3
          THEN
       ERF = 1:
       IF X > 3
          THEN
       30
100    G1 = .90618:
       G2 = .53847:
       G3 = 0:
       G4 =  - G2:
       G5 =  - G1:
       PI = 4 *  ATN (1)
110    W1 = .23693:
       W2 = .47863:
       W3 = .56889:
       W4 = W2:
       W5 = W1
130    DEF  FN E(Y) =  EXP ( - X * X * (Y + 1) * (Y + 1) / 4)
140    ERF = X * (W1 * ( FN E(G1)) + W2 * ( FN E(G2)) + W3 * ( FN E(G3)) + W4 *
       ( FN E(G4)) + W5 * ( FN E(G5))) /  SQR (PI)
150    RETURN
200    PRINT "THE USER MUST SPECIFY ALPHA (AL),GAMMA (GM),THETA SUB R&N (TR
       &TN) AND THE RAINFALL RATE (RR) IN CM/HR.  THIS IS DONE IN STATEMENT 5"
210    PRINT "  "
220    PRINT "WHEN ASKED TO ENTER T, INPUT TIME SINCE BEGINNING OF RAINFALL IN
       HRS."
230    PRINT "  "
240    RETURN
```

APPENDIX II

Program Profile *

* This program is interactive and is written in Applesoft.

II.1

```
1       GOSUB 400
5       AL = .02:
        GM = 21.46:
        TR = .3:
        TN = .4:
        D = 10:
        ZL = D * AL / 2:
        RR = .1
7       INPUT "ENTER T ";T:
        TP =  SQR (AL * T / 4 / GM)
10      FOR I = 1 TO 11:
           Z = .1 * (I - 1) * ZL
15         X = Z / 2 / TP - TP
17         GOSUB 90:
           TH = ERF / 2 + 2 * TP *  EXP ( - X * X) /  SQR (PI)
25         X = TP + Z / 2 / TP
30         GOSUB 90:
           TH = TH - (Z + 2 * TP * TP + .5) *  EXP (2 * Z) * ERF
35         TH = TR + GM * RR * TH:
           IF X > 3
              THEN
           TH = TR
40         IF I = 1
              THEN
            GOSUB 300
45         GOSUB 310
47         PRINT TH
50      NEXT
55      PRINT "MOISTURE PROFILE AT TIME ";T;" FOR RAINFALL RATE OF ";RR;"CM/HR
        AND 0<Z<";D;" CM"
60      PRINT "  ":
        GOTO 7
90      IF X > 3
           THEN
        ERF = 1:
        IF X > 3
           THEN
        150
100     G1 = .90618:
        G2 = .53847:
        G3 = 0:
        G4 =  - G2:
        G5 =  - G1:
        PI = 4 *  ATN (1)
110     W1 = .23693:
        W2 = .47863:
        W3 = .56889:
        W4 = W2:
        W5 = W1
130     DEF  FN E(Y) =  EXP ( - X * X * (Y + 1) * (Y + 1) / 4)
140     ERF = X * (W1 * ( FN E(G1)) + W2 * ( FN E(G2)) + W3 * ( FN E(G3)) + W4 *
        ( FN E(G4)) + W5 * ( FN E(G5))) /  SQR (PI)
150     ERF = 1 - ERF:
        RETURN
300     HGR :
        HPLOT 279,0 TO 0,0 TO 0,159:
        HPLOT 275,5 TO 275,0:
        RETURN
310     YI =  INT (159 * Z / ZL):
        XI =  INT (110 * ((TH - TR) / (TN - TR))):
        XI = XI + 165:
```

APPENDIX III

Program Runoff 2 *

* This program is interactive and is written in Applesoft.

```
1       GOSUB 400
100     DIM Y(5),W(5):
        KN = .03:
        R = .1:
        TR = .3:
        TN = .4:
        GM = 21.46
110     Y(1) = .263560:
        Y(2) = 1.413403:
        Y(3) = 3.596426:
        Y(4) = 7.085810:
        Y(5) = 12.640801
120     W(1) = .521756:
        W(2) = .398667:
        W(3) = .0759424:
        W(4) = .00361176:
        W(5) = .0000234
130     PI = 4 *  ATN (1):
        T1 = .38789:
        T2 = .3788:
        WD = .05:
        H = WD / 2
140     INPUT "ENTER TIME T":T
145     IF T < .005
           THEN
        S = R - KN -  EXP ( - T) * .07:
        IF T < .005
           THEN
        210
150     DEF  FN GZ(Z) =  - Z *  EXP ( - Z * Z / 4 / T) / T /  SQR (PI * T) / 2
160     I1 = ( FN GZ(H) *  EXP ( - H) * (T1 - TN) +  FN GZ(2 * H) *  EXP ( - 2
        H) * (T2 - TN)) * H / 3
170     DEF  FN G(Z) =  EXP ( - Z * Z / 4 / T) /  SQR (PI * T)
180     I2 =  EXP ( - .05) *  FN G(.05)
190     FOR N = 1 TO 5:
           I2 = I2 - W(N) *  FN G(Y(N) + .05):
        NEXT :
        I2 = I2 * (TR - TN)
200     S = R - KN +  EXP ( - T) * (I1 + I2) / .5 / GM
210     PRINT "AT TIME ";T;" WITH A RAINFALL RATE OF ":R;"CM/HR THE RUNOFF RATE
        IN CM/HR PER UNIT SURFACE AREA IS ":S
220     PRINT "  "
230     GOTO 140
400     PRINT "THE USER MUST SUPPLY VALUES OF GAMMA (GM), THETA SUB R&N
        (TR&TN),NATURAL CONDUCTIVITY (KN) AND RAINFALL RATE (R).  THIS IS DONE
        IN STATEMENT 100."
410     PRINT "  "
420     PRINT "PRIOR TO EXECUTING THIS PROGRAM 'PROFILE' MUST BE USED TO
        DETERMINE THE POSITION OF THE WETTING FRONT AT THE TIME OF PONDING AND
        THE TWO VALUES OF MOISTURE CONTENT ON THAT PROFILE NEEDED FOR THE
        SIMPSON-RULE INTEGRATION."
430     PRINT "  "
440     PRINT "WHEN ASKED TO INPUT T, THE APPROPRIATE DIMENSIONLESS VALUE OF T
        IS TO BE ENTERED."
450     PRINT "  "
460     RETURN
```

```
     HPLOT  TO XI,YI:
     RETURN
00   PRINT "THE USER MUST SUPPLY VALUES OF ALPHA (AL), GAMMA (GM). THETA SUB
     R&N (TR&TN), RAINFALL RATE (RR) IN CM/HR AND DEPTH (D) OF PROFILE TO BE
     PLOTTED.  THIS IS DONE IN STATEMENT 5."
10   PRINT "  "
20   PRINT "WHEN ASKED TO INPUT T, ENTER T IN HRS SINCE BEGINNING OF RAINFALL"
30   PRINT "  "
40   PRINT "THIS PROFILE GENERATOR CAN BE USED ONLY PRIOR TO PONDING.  IF A
     TIME GREATER THEN PONDING TIME IS ENTERED AN ERROR MESSAGE WILL RESULT"
50   PRINT "  "
60   RETURN
```

APPENDIX IV

Program Runoff

PROGRAM RUNOFF

```
      PROGRAM RUNOFF(INPUT,OUTPUT,TAPE6=OUTPUT)
      REAL K
      COMMON/BLOCK1/A,B,DZ,DD,T,TS,PI,D,RR,IND,JIND,RO,US
      COMMON/BLOCK2/U(21),Q(25),K(25)
      COMMON/BLOCK3/NTD,DTD,TAU,E
      COMMON/BLOCK4/GAM,P,ALP
      DIMENSION R(5)
      DIMENSION H(11)
      PRINT(6,*)"DO YOU WANT A LIST OF SYMBOLS? 1=YES, 2=NO"
      READ*,NN
      IF(NN.GT.1) GO TO 10
      PRINT(6,*)"D:  DEPTH OF SOIL IN CM"
      PRINT(6,*)"ALPHA:  SOIL PARAM.   (ABOUT .01 FOR CLAY,.1 FOR SAND)"
      PRINT(6,*)"P:  PACKING PARAMETER   ( 0.<P<.01 )"
      PRINT(6,*)"GAMMA:  DTHETA/DK   ( ABOUT .1)"
      PRINT(6,*)"SATK:  SATURATED CONDUCTIVITY"
      PRINT(6,*)"T1:  TIME OVER WHICH RAIN RATE ASSUMED CONSTANT"
      PRINT(6,*)"R(I):  RAIN RATE(CM/HR) DURING ITH T1 TIME INTERVAL"
      PRINT(6,*)"NTD:  NO. OF TIMES DURING DRYING CYCLE"
      PRINT(6,*)"DTD:  DELTA T DURING DRYING CYCLE"
      PRINT(6,*)"TAU:  TRANSPIRATION PARAMETER (TAU ABOUT .01)"
      PRINT(6,*)"E:  EVAPORATION PARAMETER   (E ABOUT .1)"
      PRINT(6,*)"W:  EXP(ALPHA*H)   (H: CAPILLARY PRESSURE HEAD)"
      PRINT(6,*)"HS:  H VALUE AT 90% SATURATION"
10    CONTINUE
      PRINT(6,*)"ENTER D, ALPHA, P, GAMMA, SATK"
      READ*,D,ALP,P,GAM,SK
      WRITE 11
11    FORMAT(*ENTER T1 AND R(I)  I=1,5*)
      READ*,T1,R(1),R(2),R(3),R(4),R(5)
      WRITE(6,*) "ENTER INITIAL H AT X=0 AND HS."
      READ*,WCI,HS
      WRITE 13
13    FORMAT(*ENTER NUMBER OF TIMES PER T1 INTERVAL*)
      READ*,NT
      PRINT(6,*)"ENTER NTD,DTD,TAU,E"
      READ*,NTD,DTD,TAU,E
      PI = 3.141592654
      A = D*D*ALP*GAM
      B = .5*D*(ALP+P)
      US = SK*EXP(ALP*HS)/ALP
      DT = T1/FLOAT(NT)
      DZ = .05
      C1 = SK*EXP(ALP*WCI)/ALP
      DO 14 I = 1,21
14    U(I) = C1*EXP(2.*B*FLOAT(I-1)*DZ)
      WRITE(6,110)WCI
110   FORMAT(*AT T=0 WATER CONTENT IN CM IS  *,F10.6,5X,*AND W IS*)
      DO 3 I = 1,11
3     H(I) = U(2*I-1)/US
      WRITE(6,101) (H(I),I=1,11)
      DO 49 II = 1,5
      JIND = 1
```

```
31      X = B
        DO 17 I=1,20
        TOP = X*COSH(X)*COSH(X)-B*SINH(X)*COSH(X)
        BOT = COSH(X)*COSH(X)  - B
17      X = X - TOP/BOT
        S1 = 0.
        S2 = 0.
        DO 18 I=1,10
        ZE = FLOAT(2*I)*DZ
        ZO = FLOAT(2*I-1)*DZ
        S1 = S1 +4.*U(2*I)*SINH(X*ZE)*EXP(-B*ZE)
18      S2 = S2 + 2.*U(2*I-1)*SINH(X*ZO)*EXP(-B*ZO)
        QQ = DZ*(S1+S2+U(21)*SINH(X)*EXP(-B))/3.
        EB = .5*(EXP(X+B)-1.)/(X+B)-.5*(EXP(B-X)-1.)/(B-X)
29      DO 1 J=1,21
        Z = DZ*FLOAT(J-1)
        IF(B-1.) 39,40,41
39      SUM1 = 0.
        SUM2 = 0.
        IF (J.GT.1) GO TO 42
        SUM3 = 0.
        SUM4 = 0.
        GO TO 42
40      C = (TS - T)/A
        SUM1 = 3.*EXP(C)*QQ*Z
        SUM2 = 3.*EXP(C)*EB*Z
        IF(J.GT.1) GO TO 42
        IF (J.GT.1) GO TO 42
        SUM3 = 3.*EXP(C)*QQ
        SUM4 = 3.*EXP(C)
        GO TO 42
41      C = (B*B-X*X)*(TS-T)/A
        WNS = -.5 + SINH(2.*X)/(4.*X)
        SUM1 = EXP(C)*QQ*SINH(X*Z)/WNS
        SUM2 = EXP(C)*EB*SINH(X*Z)/WNS
        IF(J.GT.1) GO TO 42
        SUM3 = (B*B-X*X)*EXP(C)*QQ*EB/WNS
        SUM4 = (B*B-X*X)*EB*EB/WNS
42      DO 3 I=1,25
        L = K(I)
        MI = L*L + B*B
        WIS = .5 - SIN(2.*L)/(4.*L)
        TOP = L*(1.-EXP(B)*COS(L))+B*EXP(B)*SIN(L)
        BRAC = TOP/MI
        C = MI*(TS-T)/A
        IF(C.LT.-200.) GO TO 11
        SUM1 = SUM1+EXP(C)*Q(I)*SIN(L*Z)/WIS
        SUM2 = SUM2+EXP(C)*BRAC*SIN(L*Z)/WIS
        IF(J.GT.1) GO TO 3
        SUM3 = SUM3+MI*Q(I)*EXP(C)*BRAC/WIS
        SUM4 = SUM4+MI*EXP(C)*BRAC*BRAC/WIS
3       CONTINUE
11      V1(J) = EXP(B*Z)*SUM1
        V2(J) = EXP(B*Z)*SUM2
1       U(J) = US*EXP(2.*B*Z) + V1(J) - US*V2(J)
```

```
    3 F(I)=EXP(-B*DZ*FLOAT(I-1))*U(I)
      DO 5 I=1,25
      TH = K(I)*DZ
      ATH = (TH*(TH+.5*SIN(2.*TH))-2.*(SIN(TH))**2)/TH**3
      BTH = (2.*(TH*(1.+(COS(TH))**2)-SIN(2.*TH)))/TH**3
      GTH = 4.*(SIN(TH)-TH*COS(TH))/TH**3
      SE = 0.
      SO = F(2)*SIN(K(I)*DZ)
      DO 7 J=1,9
      SE = SE + F(2*J+1)*SIN(K(I)*DZ*FLOAT(2*J))
    7 SO = SO + F(2*(J+1))*SIN(K(I)*DZ*FLOAT(2*J+1))
      SE = SE + .5*F(21)*SIN(K(I))
    5 Q(I)=DZ*(ATH*(F(1)-F(21)*COS(K(I)))+BTH*SE+GTH*SO)
      IF(IND.EQ.0)GO TO 17
      S1 = 0.
      S2 = 0.
      DO 9 I=1,10
      S1 = S1 + 4.*U(2*I)
9     S2 = S2 + 2.*U(2*I-1)
      Q0 = DZ*(S1 + S2 - U(1) + U(21))/3.
      DO 11 I = 1,25
      TH = K(I)*DZ
      TH2 = TH*TH
      TH3 = TH*TH2
      ATH = 1./TH + .5*SIN(2.*TH)/TH2 - (2.*SIN(TH)*SIN(TH))/TH3
      BTH = (2.+2.*COS(TH)*COS(TH))/TH2 - 2.*SIN(2.*TH)/TH3
      GTH = 4.*SIN(TH)/TH3 - 4.*COS(TH)/TH2
      CE = .5*F(1)
      CO = F(2)*COS(K(I)*DZ)
      DO 13 J= 1,9
      CE = CE+F(2*J+1)*COS(K(I)*DZ*FLOAT(2*J))
13    CO = CO + F(2*(J+1))*COS(K(I)*DZ*FLOAT(2*J+1))
      CE = CE + .5*F(21)*COS(K(I))
      Q1 = DZ*(ATH*F(21)*SIN(K(I)) + BTH*CE + GTH*CO)
      C(I) = Q1
11    Q(I) = Q1 + B*Q(I)/K(I)
17     CONTINUE
      RETURN
      END
      SUBROUTINE ROOT
      REAL K
      COMMON/BLOCK1/A,B,DZ,Q0,T,TS,PI,D,RR,IND,JIND,RO,US
      COMMON/BLOCK2/U(21),Q(25),K(25)
      X=.1
      DO 7 I=1,25
      DO 9 J=1,20
      X=X+PI*FLOAT(I)
      IF(B.LT.1.) X=X-PI
      X=ATAN(X/B)
    9 CONTINUE
      X=X+PI*FLOAT(I)
      IF(B.LT.1.) X=X-PI
      K(I)=X
      X=X-PI*FLOAT(I)
    7 CONTINUE
```

```
      IF(II.EQ.1) RO = 0.
      IF(II.EQ.1) TS = 0.
      IF(II.EQ.1) GO TO 51
      IF(R(II)-R(II-1))37,38,37
37    TS = FLOAT(II-1)*T1
38    CONTINUE
      IF(R(II)-R(II-1).EQ.0.) JIND = 0
51    CONTINUE
      RR = R(II)
      DO 48 J = 1,NT
      T = FLOAT(II-1)*T1 + FLOAT(J)*DT
      IF(US-U(1)) 29,29,31
29    IF(RO)31,31,30
30    CALL SAT
      DO 15 N = 1,11
      ZP=.1*FLOAT(N)
15    H(N) = U(2*N-1)/(US*EXP(ZP*P))
      SUM = U(21) - U(1)
      DO 47 I=1,10
47    SUM = SUM +4.*U(2*I) + 2.*U(2*I-1)
      WAT = D*GAM*ALP*SUM/60.
      WRITE(6,100) T,WAT
100   FORMAT(*AT TIME T=*,F5.2,*  H2O CONTENT IS*,F5.2,*  AND W IS*)
      WRITE(6,101) (H(N),N=1,11)
101   FORMAT(11F7.2)
      WRITE(6,103) RO,RR
103   FORMAT(*RUNOFF IS*,F10.4,*    RAIN RATE IS   *,F10.4)
      IF(RO.LE.0.) TS=T
      IF(RO.LE.0.) JIND = 1
      GO TO 48
31    CALL FLUX
      DO 16 N=1,11
16    H(N) = U(2*N-1)/US
      SUM = U(21) - U(1)
      DO 46 I=1,10
46    SUM = SUM +4.*U(2*I) + 2.*U(2*I-1)
       WAT = D*GAM*ALP*SUM/60.
      WRITE(6,100) T,WAT
      WRITE (6,101) (H(N) , N=1,11)
      WRITE(6,104) RR
104   FORMAT(*THERE IS NO RUNOFF.  THE RAIN RATE IS   *,F10.4)
      IF(US-U(1))39,48,48
39    TS=T
      JIND=1
      RO = .00001
48    CONTINUE
49    CONTINUE
      CALL DRY
      END
      SUBROUTINE QUAD
      REAL K
      COMMON/BLOCK1/A,B,DZ,QO,T,TS,PI,D,RR,IND,JIND,RO,US
      COMMON/BLOCK2/U(21),Q(25),K(25)
      DIMENSION F(21),C(25)
      DO 3 I=1,21
```

```
      RETURN
      END
      SUBROUTINE FLUX
      REAL K,MI,L
      COMMON/BLOCK1/A,B,DZ,QO,T,TS,PI,D,RR,IND,JIND,RO,US
      COMMON/BLOCK2/U(21),Q(25),K(25)
      IF(IND.EQ.0)  JIND = 1
      IND = 1
      IF(JIND -1) 25,24,25
24    DO 12 I=1,25
12    K(I) = FLOAT(I)*PI
      CALL QUAD
      JIND = 0
25    WNS = (EXP(2.*B)-1.)/(2.*B)
      DO 1 J = 1,21
      Z = DZ*FLOAT(J-1)
      SUM1 = QO*EXP(B*Z)/WNS
      SUM2 = EXP(B*Z)*(T-TS)/(A*WNS)
      DO 3 I=1,25
      L = PI*FLOAT(I)
      MI = L*L + B*B
      WIS = MI/(2.*L*L)
      WIZ = COS(L*Z) + B*SIN(L*Z)/L
      C = MI*(T-TS)/A
      IF(C.GT.200.) GO TO 11
      SUM1 = SUM1 + EXP(-C)*Q(I)*WIZ/WIS
      SUM2 = SUM2 - (EXP(-C)-1.)*WIZ/(MI*WIS)
      GO TO 3
11    SUM2 = SUM2 + WIZ/(MI*WIS)
3     CONTINUE
      U(J) = EXP(B*Z)*(SUM1 + D*RR*SUM2)
1     CONTINUE
      RETURN
      END
      SUBROUTINE SAT
      REAL K,MI,L
      COMMON/BLOCK1/A,B,DZ,QO,T,TS,PI,D,RR,IND,JIND,RO,US
      COMMON/BLOCK2/U(21),Q(25),K(25)
      DIMENSION V1(21),V2(21),W(11)
      IF(IND.EQ.1) JIND = 1
      IND = 0
      IF(JIND.EQ.1) CALL ROOT
      IF(JIND.EQ.1) CALL QUAD
      IF (JIND.EQ.0) GO TO 29
      IF(B-1.) 29,30,31
30    S1 = 0.
      S2 = 0.
      DO 9 I=1,10
      ZE = FLOAT(2*I)*DZ
      ZO = FLOAT(2*I-1)*DZ
      S1 = S1 + 4.*U(2*I)*ZE/EXP(ZE)
9     S2 = S2 + 2.*U(2*I-1)*ZO/EXP(ZO)
      QO = DZ*(S1+S2+U(21)*EXP(-B))/3.
      EB = 1.
      GO TO 29
```

```
      FLUX = (US*SUM4 - SUM3)/D
      RO = RR - FLUX
      JIND = 0
      RETURN
      END
      SUBROUTINE DRY
      REAL K,MI,L
      COMMON/BLOCK1/A,B,DZ,QO,T,TS,PI,D,RR,IND,JIND,RO,US
      COMMON/BLOCK2/U(21),Q(25),K(25)
      COMMON/BLOCK3/NTD,DTD,TAU,E
      COMMON/BLOCK4/GAM,P,ALP
      DIMENSION H(11)
      DO 12 I = 1,25
12    K(I) = FLOAT(I)*PI
      IND = 1
      CALL QUAD
      WNS =(EXP(2.*B) - 1.)/(2.*B)
      DO 59 JJ = 1,NTD
      T = FLOAT(JJ)*DTD
      DO 58 J = 1,21
      Z = DZ*FLOAT(J-1)
      MI = TAU/A
      SUM1 = QO*EXP(B*Z)*EXP(-MI*T)/WNS
      SUM2 = (1.-EXP(-MI*T))*EXP(B*Z)/(MI*WNS)
      DO 57 I = 1,25
      L = PI*FLOAT(I)
      MI = L*L +B*B +TAU
      C = MI*T/A
      WIS = MI/(2.*L*L)
      WIZ = COS(L*Z) +B*SIN(L*Z)/L
      IF(C.GT.200.) GO TO 56
      SUM1 = SUM1 + EXP(-C)*Q(I)*WIZ/WIS
      SUM2 = SUM2 + ((1.-EXP(-C)))/(MI*WIS)
      GO TO 57
56    SUM2 = SUM2 + WIZ/(MI*WIS)
57    CONTINUE
      IF(U(1) -.1*US) 8,8,6
7     U(J) = EXP(B*Z)*SUM1
      GO TO 58
8     U(J) = EXP(B*Z)*(SUM1 - B*SUM2)
58    CONTINUE
      DO 11 N = 1,11
      ZP = .1*FLOAT(N)
11    H(N) = U(2*N-1)/(US*EXP(P*ZP))
      SUM = U(21)-U(1)
      DO 47 I=1,10
47    SUM = SUM + 4.*U(2*I) + 2.*U(2*I-1)
      WAT = D*GAM*ALP*SUM/60.
      WRITE(6,100) T,WAT
100   FORMAT(F5.2,* HRS AFTER STORM H2O CONTENT IS*,F5.2,*AND W IS*)
      WRITE(6,102) (H(N),N=1,11)
102   FORMAT(11F7.2)
59    CONTINUE
      RETURN
      END
```