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ATMOSPHERIC DIFFUSION FROM A POINT SOURCE

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FOREWORD

This report is No. 4 of a series written for the Diffusion Project presently being conducted by the Colorado Agricultural and Mechanical College for the Office of Naval Research. The experimental phase of this project is being carried out in a wind-tunnel at the Fluid Mechanics Laboratory of the College. The project is under the general supervision of Dr. M. L. Albertson, Head of Fluid Mechanics Research of the Civil Engineering Department.

To Dr. M. L. Albertson, and to Dr. D. F. Peterson, Head of the Civil Engineering Department and Chief of the Civil Engineering Section of the Experiment Station, as well as to Professor T. H. Evans, Dean of the Engineering School and Chairman of the Engineering Division of the Experiment Station, the writer wants to express his appreciation for their kind interest in the present work.

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Abstract

The differential equation of diffusion when the wind velocity and the vertical and lateral diffusivities are power functions of height is

$$y^m \frac{\partial c}{\partial x} = D_1 \frac{\partial}{\partial y}\left(y^n \frac{\partial c}{\partial y}\right) + D_2 y^k \frac{\partial^2 c}{\partial z^2}$$

where x, y, and z are measured respectively in the down wind, vertical and cross-wind directions and D_1 and D_2 are physical constants to be defined in the text. Exact solution of this equation for the case of a point source is presented in this paper. In the systematic search for this solution, dimensional analysis has been utilized to the optimum advantage.

1. Introduction

Two-dimensional diffusion, when the wind-velocity and the vertical diffusivities are power functions of height, has been extensively treated by O. G. Sutton (5,1934), W. G. L. Sutton (7,1934) Frost (4, 1946), Calder (1, 1949), and Yih (8, 1951). Three-dimensional diffusion where lateral diffusivity must be considered has been treated by Davis (2, 1947; 3, 1950), and by O. G. Sutton (6, 1947) in the case of a point source, on the assumption that the variation of the wind-velocity with height may be neglected. Thus in comparison with the two-dimensional phenomenon, the three-dimensional one has apparently received only insufficient attention.

This paper is concerned with the atmospheric diffusion from a point source when the wind-velocity and the vertical and lateral diffusivities are power functions of height, the exponents of which (m, n, k in the following) being at first left completely free.

A mathematical solution is found possible for the special case

$$m = k$$

2. The Differential System

With the origin at the point source, and the directions of x, y, and z defined as in the abstract, if the variation of wind-velocity u with y is expressed by

$$\frac{u}{u_1} = \left(\frac{y}{y_1}\right)^m \tag{1}$$

where u_1 is the wind-velocity at y_1 and if the vertical and lateral diffusivities are respectively

$$A_v = A_1\left(\frac{y}{y_1}\right)^n \tag{2}$$

$$A_l = A_2\left(\frac{y}{y_1}\right)^k \tag{3}$$

where again A_1 and A_2 correspond to the height y_1, the equation of diffusion

$$u\frac{\partial c}{\partial x} = \frac{\partial}{\partial y}\left(A_v\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(A_l\frac{\partial c}{\partial z}\right) \tag{4}$$

can be written as

$$y^m\frac{\partial c}{\partial x} = D_1\frac{\partial}{\partial y}\left(y^n\frac{\partial c}{\partial y}\right) + D_2\, y^k\,\frac{\partial^2 c}{\partial z^2} \tag{5}$$

where c is the concentration of the quantity under diffusion and

$$D_1 = \frac{A_1 y_1^{m-n}}{u_1} \qquad D_2 = \frac{A_2 y_1^{m-k}}{u_1} \tag{6}$$

The differential equation (5) is to be solved with the following boundary conditions:

(a) $\frac{\partial c}{\partial y} = 0$ at $y = 0$

(b) $\frac{\partial c}{\partial z} = 0$ at $z = 0$

(c) $c \to c_0$ as $y \to \infty$

(d) $c \to c_0$ as $|z| \to \infty$

(e) $c \to c_0$ as $x \to 0$ for $y > 0$

(f) $c \to c_0$ as $x \to 0$ for $|z| > 0$

and the integral continuity equation

$$(g)\quad \int_{-\infty}^{\infty}\int_{0}^{\infty} u(c-c_0)\,dy\,dz = Q = \text{constant}$$

where c_0 is the ambient concentration, and A is the strength of the point source. (a) stipulates that the ground is impervious to the quantity under diffusion, and (b) follow from symmetry and can be replaced by the more general condition that c should be an even function with respect to z.

3. The Solution

To facilitate the systematic search for a similarity solution (Ähnlichkeitslösung), a dimensional analysis will be performed first, which, in conjunction with considerations of the powers of x, A_1, A_2, u_1, and y_1, will afford the most adequate transformation to be made in order that the solution will be the simplest. The pertinent variables are

$$c,\ c_0,\ Q,\ A_1,\ A_2,\ u_1,\ y_1,\ x,\ y,\ z$$

A dimensional analysis yields the relationship

$$\frac{c-c_0}{c_0} = F\left(\frac{Q}{A_1 c_0 x},\ \frac{u_1 x}{A_1},\ \frac{u_1 x}{A_2},\ \frac{y_1}{x},\ \frac{y}{x},\ \frac{z}{x}\right) \tag{7}$$

To obtain a similarity - solution, one makes the following substitution:

$$\varphi = \frac{c-c_0}{c_0} = \frac{Q}{A_1 c_0 x}\left(\frac{u_1 x}{A_1}\right)^{\alpha}\left(\frac{u_1 x}{A_2}\right)^{\beta}\left(\frac{y_1}{x}\right)^{\gamma} f(\eta, \zeta) \tag{8}$$

where

$$\eta = \left(\frac{u_1 x}{A_1}\right)^{p}\left(\frac{y_1}{x}\right)^{q}\frac{y}{x} \tag{9}$$

$$\zeta = \left(\frac{u_1 x}{A_2}\right)^{r}\left(\frac{y_1}{x}\right)^{s}\frac{z}{x} \tag{10}$$

and where the exponents $\alpha, \beta, \gamma, p, q, r$ and s are to be determined. Before proceeding further with the solution, it may be noticed here that the power of $Q/A_1 c_0 x$ is 1 in circumspection of (g), and that a pair of fixed values for η and ζ defines a space curve which is the intersection of two parabolic cylinders.

$$y = K_1 x^{1+q-p} \tag{11}$$

$$z = K_2 x^{1+s-r} \tag{12}$$

The set of all the curves defined by (11) and (12) for various values of K_1 and K_2 will be dense in the three-dimensional space under consideration. On any two such curves, the values of φ will always bear the same ratio for any value of x. This is the reason why the solution having the form of (8) is called a similarity-solution.

One now proceeds to determine the exponents in (8), (9), and (10). Substituting (8) in (5) and demanding equal powers in u_1, y_1, and x and equal joint powers in A_1 and A_2 one has

$$p = \frac{1}{m-n+2} \qquad q = \frac{n-m}{m-n+2} \tag{13}$$

$$r = \frac{k-n+2}{2(m-n+2)} \qquad s = \frac{k-m}{m-n+2} \tag{14}$$

so that

$$2p - q - 1 = 0 \tag{15}$$

$$2r - s - 1 = 0 \tag{16}$$

The exponents α, β, and γ are left undetermined by this procedure, and will be determined by (9) which gives, after (8) has been substituted into (g):

$$\alpha + \beta + 1 - p(m+1) - r = 0$$

$$-m - q(m+1) - s + \gamma = 0$$

$$-1 - \alpha + p(m+1) = 0$$

$$-\beta + r = 0$$

$$(m+1)(-p+q+1) + (-r+s+1) + \alpha + \beta - \gamma - 1 = 0$$

and

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \eta^{m} f(\eta,\zeta)\, d\eta\, d\zeta = 1 \tag{17}$$

The five equations given by (9) involving the unknown exponents α, β, γ are not independent, and are satisfied by

$$\beta = r \tag{18}$$

$$\alpha = p(m+1)-1 \tag{19}$$

$$\gamma = m+s+q(m+1) \tag{20}$$

so that

$$\alpha+\beta+\gamma-1 = -\left[p(m+1)+r\right] \tag{21}$$

With the exponents given in (13), (14), (18), (19), and (20) in terms of m, n, and k, substitution of (8) in (5) results in the following equation:

$$(\alpha+\beta-\gamma-1)f + (p-q-1)\eta f_{\eta} + (r-s-1)\zeta f_{\zeta}$$
$$= \eta^{n-m} f_{\eta\eta} + n\eta^{n-m-1} f_{\eta} + \sigma^{1-2r}\eta^{k-m} f_{\zeta\zeta} \tag{22}$$

where

$$\sigma = A_2/A_1$$

and where subscripts denote partial differentiations. In virtue of (15), (16), and (21), (22) can be written

$$-\left[p(m+1)+r\right]f - p\eta f_{\eta} - r\zeta f_{\zeta} = \eta^{n-m} f_{\eta\eta} + n\eta^{n-m-1} f_{\eta} + \sigma^{-s}\eta^{k-m} f_{\zeta\zeta} \tag{23}$$

This is the differential equation that has to be solved in general. For the case $m = k$, make the transformation

$$\xi = \sigma^{s/2}\zeta \tag{24}$$

(23) can now be written.

$$-\left[p(m+1)+r\right]f - p\eta f_{\eta} - r\xi f_{\xi} = \eta^{n-m} f_{\eta\eta} + n\eta^{n-m-1} f_{\eta} + f_{\xi\xi} \tag{25}$$

(25) being in a form suitable for separation of variables, one assumes

$$f = Y(\eta) Z(\xi) \tag{26}$$

Substitution into (25) gives

$$-[p(m+1)+r] - \frac{p\eta Y'}{Y} - \frac{\eta^{n-m} Y'' + n\eta^{n-m-1} Y'}{Y}$$
$$= \frac{Z''}{Z} + \frac{r\xi Z'}{Z} = \lambda$$

which can be written into the two separate equations

$$[p(m+1)+r] Y + p\eta Y' + (\eta^{n-m} Y'' + n\eta^{n-m-1} Y') = 0 \tag{27}$$

$$Z'' + r\xi Z' - \lambda Z = 0 \tag{28}$$

where the primes denote ordinary differentation: with respect to η for Y and with respect to ξ for Z. The boundary conditions for (27) are

(h) $Y'(0) = 0$

(i) $Y(\infty) = 0$

and those for (28) are

(j) $Z'(0) = 0$

(k) $Z(\infty) = 0$

One first considers the system (28), (j) and (k). A first integration gives

$$Z' + r\xi Z - (\lambda + r)\int_0^{\xi} Z\, d\xi = 0 \tag{28}$$

the lower limit being chosen equal to zero since $Z'(0) = 0$.

As $\xi \to \infty$, one has $Z \to 0$, $Z' \to 0$, and

$$\int_0^{\infty} Z\, d\xi \neq 0 \tag{29}$$

since otherwise the integral in (17) would vanish. On the other hand, $\int_0^{\infty} Z d\xi$ should be finite on account of (17). Consequently, if Z behaves as ξ^{-b} for large ξ, b must be larger than 1 so that $\xi Z \to 0$ as $\xi \to \infty$.

If Z vanish exponentially as $\xi \to \infty$, then also $\xi Z \to 0$ as $\xi \to \infty$. These seem to be the only cases in which Z can vanish at infinity, and for each of these cases the first two terms of (28) vanish while the third one does not unless $\lambda = -r$ on account of (29). The satisfaction of (28) and (k) therefore requires

$$\lambda = -r \tag{30}$$

Thus there is only one single eigenvalue for the parameter λ. With (30), integration of (28) gives

$$Z = K \exp(-r\xi^2/2) \tag{31}$$

where K is an arbitrary constant to be determined by (17). Substituting (30) into (27) and multiplying throughout by η^m, one has

$$\frac{m+1}{m-n+2}\eta^m Y + \frac{1}{m-n+2}\eta^{m+1}Y' + (\eta^n Y'' + n\eta^{n-1}Y') = 0$$

a first integration of which gives

$$\frac{1}{m-n+2}\eta^{m+1}Y + \eta^n Y' = 0$$

the constant of integration being zero since $Y'(0) = 0$ and $Y(0)$ is finite. A second integration gives

$$Y = \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2}\right) \tag{32}$$

the constant factor being absorbed in K of (31).

The constant K can be determined by (17) which can be written as

$$K\sigma^{-5/2}\int_{-\infty}^{\infty}\int_0^{\infty} \eta^m \exp\left(-\frac{\eta^{m-n+2}}{(m-n+2)^2} - \frac{r\xi^2}{2}\right)d\eta\, d\xi = 1 \tag{33}$$

Evaluation of (33) gives

$$\sigma^{-\frac{5}{2}}\left(\frac{2\pi}{r}\right)^{\frac{1}{2}}\int_0^{\infty}\eta^m \exp\left(-\frac{\eta}{(m-n+2)^2}\right)d\eta = 1$$

or

$$(m-n+2)^{1-2a} K\sigma^{-\frac{5}{2}}\left(\frac{2\pi}{r}\right)^{\frac{1}{2}}\Gamma(a) = 1$$

which gives

$$K = \frac{\sigma^{5/2}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}(m-n+2)^{2a-1}}{\Gamma(a)} \tag{34}$$

where

$$a = \frac{m+1}{m-n+2} \tag{35}$$

and

$$\Gamma(a) = \int_0^\infty w^{a-1} e^{-w}\, dw$$

is the gamma function.

Equations (31), (32), (34), and (35) give the function f by means of (26), which in conjunction with (24) and (8), yields the solution. The exponents α, β, γ, p, q, r and s being given in terms of m, n, and k by (13), (14), (18), (19), and (20), and σ having been defined to be A_2/A_1. As has been stated, the exponents n and m = k (and in fact also the parameter σ) are left free to be determined by measurements.

4. Acknowledgment

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5. References

1. Calder, K. L.: Quart. Journ. Mech. and Applied Math. 2, p. 153, 1949

2. Davis, R. D.: Proc. Roy. Soc. A. 190, p. 232, 1947

3. Davis, R. D.: Three-Dimensional Turbulence and Evaporation in the Lower Atmosphere, I and II, Quart. Journ. Mech. and Applied Math., Vol III, Pt.I, pp. 51-73, 1950.

4. Frost, R. : Turbulence and Diffusion in the Lower Atmosphere. Proc. of Royal Soc. A. 186, pp. 20-35, 1946.

5. Sutton, O.G.: Wind Structure and Evaporation in a Turbulent Atmosphere, Proc. Royal Soc. Vol. A 146, pp. 701-722, 1934

6. Sutton, O.G.: Quart. Journ. Roy. Met. Soc. 73. p. 257, 1947

7. Sutton, W.G.L.: On the Equation of Diffusion in a Turbulent Medium. Proc. Royal Soc. A. 182, pp. 48-75, 1943

8. Yih, C. S. : On a Differential Equation of Atmospheric Diffusion, 1951, to be published in the Transactions of the American Geophysical Union.