DISSERTATION

THEORY AND ALGORITHMS FOR w-STABLE IDEALS

Submitted by

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ABSTRACT

THEORY AND ALGORITHMS FOR w-STABLE IDEALS

Strongly stable ideals are a class of monomial ideals which correspond to generic initial ideals in characteristic zero. Such ideals can be described completely by their Borel generators, a subset of the minimal monomial generators of the ideal. In [1], Francisco, Mermin, and Schweig develop formulas for the Hilbert series and Betti numbers of strongly stable ideals in terms of their Borel generators.

In this thesis, a *specialization* of strongly stable ideals is presented which further restricts the subset of relevant generators. A choice of weight vector $w \in \mathbb{N}^n_{>0}$ restricts the set of strongly stable ideals to a subset designated as w-stable ideals. This restriction allows one to further compress the Borel generators to a subset termed the *weighted Borel generators* of the ideal. As in the non-weighted case, formulas for the Hilbert series and Betti numbers of strongly stable ideals can be expressed in terms of their weighted Borel generators. In computational support of this class of ideals, the new Macaulay2 package wStableIdeals.m2 has been developed and segments of its code support computations within the thesis. In a strengthening of combinatorial connections, strongly stable partitions are defined and shown to be in bijection with totally symmetric partitions.

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Chapter 1

Introduction

The main goal of this thesis is to extend results on efficient computations with strongly stable ideals to analogous computations on a restriction of strongly stable ideals, which we call w-stable ideals.

Consider an invertible matrix $\alpha = (a_{ij}) \in GL_n(\mathbb{K})$ for some field $\mathbb{K}$ and the action on the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ by

$$(x_1, \ldots, x_n) \mapsto \Big(\sum_{i=1}^{n} a_{i1} x_i, \ldots, \sum_{i=1}^{n} a_{in} x_i \Big).$$

Given an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, a choice of monomial order $<$ (which respects the variable order $x_1 > x_2 > \cdots > x_n$) determines the initial ideal $in_<(I)$. If $\alpha \in GL_n(\mathbb{K})$ is *generic*, then $in_<(\alpha \cdot I)$ is called the *generic initial ideal*, denoted $gin_<(I)$.

There is a filtration of subgroups

$$T_n(\mathbb{K}) \subset B_n(\mathbb{K}) \subset GL_n(\mathbb{K})$$

where $T_n(\mathbb{K})$ is the group of invertible diagonal matrices and $B_n(\mathbb{K})$ is the group of invertible upper-triangular matrices (called the *Borel subgroup*). By considering the ideals which are fixed by each of these subgroups, we get a filtration going the other direction. All monomial ideals are fixed by $T_n(\mathbb{K})$. Only powers of the maximal ideal $(x_1, \ldots, x_n)$ are fixed by $GL_n(\mathbb{K})$. The intermediate class fixed by $B_n(\mathbb{K})$ are the Borel-fixed ideals and correspond exactly to generic initial ideals. In 1974, Galligo showed that generic initial ideals are Borel-fixed in characteristic zero [2]. Bayer and Stillman extended this result to arbitrary characteristic in 1987 [3]. In 2004, Conca showed that if an ideal is Borel-fixed, then it is its own generic inital ideal [4]. In summary, for any characteristic, the generic initial ideals are exactly the Borel-fixed ideals.

In characteristic zero, the Borel-fixed ideals (and therefore the generic initial ideals) correspond exactly to a class of monomial ideals called strongly stable ideals. Strongly stable ideals (also called Borel ideals) are a class of monomial ideals which are closed under a simple combinatorial operation known as a *Borel move* [5]. In positive characteristic, Borel-fixed ideals are strongly stable if the characteristic is larger than any exponent appearing in any monomial generator of the ideal. This correspondence allows one to study invariants of generic initial ideals by taking advantage of the combinatorial properties of strongly stable ideals.

The combinatorial definition of strongly stable ideals allows one to describe the ideal entirely by a subset of its monomial generators, known as the Borel generators. In [1], Francisco, Mermin, and Schweig show that several computations for strongly stable ideals can be decomposed into computations on the Borel generators of the ideal. In particular, they give formulas for the Hilbert series and Betti numbers of principal strongly stable ideals (those with a single Borel generator).

The main contribution of this thesis is to extend these results to the non-standard graded setting. A choice of weight vector $w = (w_1, \dots, w_n) \in \mathbb{N}^n$ restricts the set of strongly stable ideals to a subset, which we call w-stable ideals (Definition 3.2.1). Since w-stable ideals are also strongly stable, they can be described entirely by their Borel generators. However, because being w-stable is a more strict condition than being strongly stable, w-stable ideals can be described by a subset of the Borel generators along with the choice of weight vector w. We refer to elements of this subset as *weighted* Borel generators (Definition 3.3.2).

Following the approach in [1], we show that principal w-stable ideals can be expressed neatly using a Stanley decomposition (Corollary 2). This decomposition allows us to obtain a nice formula for the Hilbert series of a principal w-stable ideal (see Theorem 3)

$$HS(S/I) = \sum_{s=0}^{d-1} \frac{c_s t^s}{\prod_{j=k_s+1}^{n} (1 - t^{w_j})}.$$

The coefficients c_s appearing in the formula above are tedious to compute as defined, so we introduce a weighted Catalan diagram associated with each principal w-stable ideal (Definition 3.7.1).

The weighted Catalan diagram associated with a principal w-stable ideal records the relevant information of the Stanley decomposition and therefore provides a combinatorial method to compute the coefficients from the Hilbert series formula above (Corollary 9). The combined result is a method for computing the Hilbert series of a principal w-stable ideal directly from its Catalan diagram.

Weighted Catalan diagrams also record information about the largest index of a variable appearing in each of the monomial generators of the principal w-stable ideal. Since this is exactly the information needed to compute the Betti numbers via the Eliahou-Kervaire formula, we can also use weighted Catalan diagrams to compute Betti numbers of principal w-stable ideals. The Poincaré series of an ideal is a compact way of expressing the Betti numbers of that ideal. We give a formula for the Poincaré series of a principal w-stable ideal in terms of its weighted Catalan diagram $C_{w,m}$ (see Theorem 12)

$$P_I^S(t,u) = \sum_{a=d}^{d+max(w)-1} \sum_{b=1}^{n} C_{w,m}(a,b) \cdot ut^a \prod_{k=1}^{b-1}(1+ut^{w_k}).$$

While the Hilbert series and Poincaré series are given in the weighted polynomial ring (where degrees of variables are given by the weight vector $w \in \mathbb{N}^n$), the *total* Betti numbers are independent of w. This allows one to compute the total Betti numbers more efficiently when the set of weighted Borel generators is smaller than the set of classic Borel generators. Examples of this situation are presented in Section 3.10. The section concludes with a conjecture that $in_w(I)$ is w-stable when I is generic non-homogeneous (see Conjecture 3.10.3).

This dissertation is organized as follows. In Chapter 2, we further examine the background information for strongly stable ideals and (classic) Borel generators. Chapter 3 closely follows [6] in presenting w-stable ideals and studying computations in terms of their weighted Borel generators. We end the chapter with the Macaulay2 code for the wStableIdeals.m2 package, which is used throughout the chapter to support computations. The final chapter is dedicated to showing a bijection between strongly stable partitions (defined within) and totally symmetric partitions.

Chapter 2

Background

2.1 Monomial Ideals

Every monomial $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \in \mathbb{K}[x_1,\ldots,x_n]$ can be described by its *exponent vector* $(a_1,a_2,\ldots,a_n) \in \mathbb{N}^n$. This allows visualization of monomials as lattice points in the positive orthant $\mathbb{N}^n$ as in the figure below.

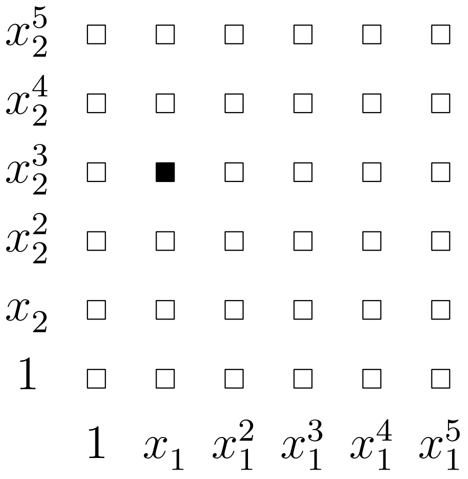


Figure 2.1: The monomial lattice for $\mathbb{K}[x_1,x_2]$ with the monomial $x_1x_2^3$ specified by ■

An ideal $I \subset \mathbb{K}[x_1,\ldots,x_n]$ is said to be a *monomial ideal* if it is generated by monomials. We can use the monomial lattice to visualize the monomials generated by a given monomial.

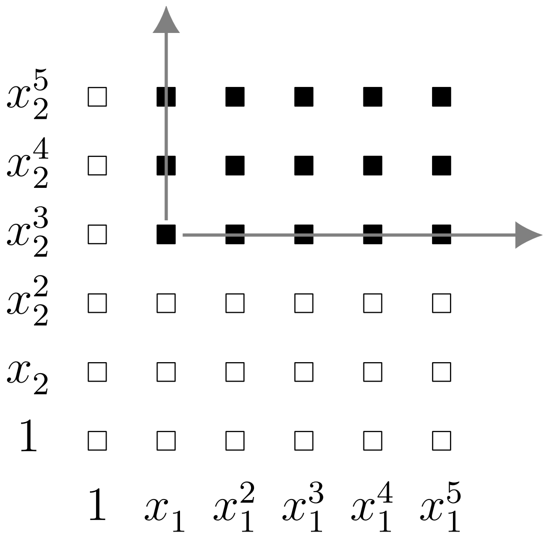


Figure 2.2: A visualization of the monomial ideal $I = (x_1x_2^3) \subset \mathbb{K}[x_1, x_2]$

We can visualize monomial ideals with multiple generators as well.

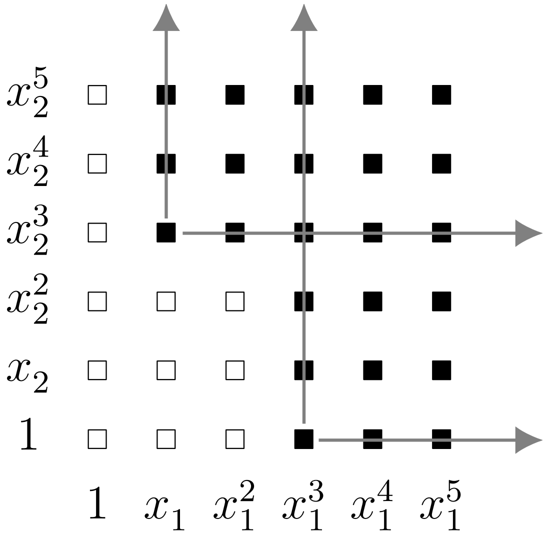


Figure 2.3: A visualization of the monomial ideal $I = (x_1^3, x_1x_2^3) \subset \mathbb{K}[x_1, x_2]$

This relationship between monomial ideals and the monomial lattice allows us to use combinatorial tools to study algebraic properties.

2.2 Initial Ideals

Given a monomial order $<$ and a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$, the *initial term* of f, denoted $in_<(f)$, is the largest monomial (with respect to $<$) of any monomial with nonzero coefficient appearing in f. For an arbitrary ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$, one can consider the *initial ideal* of I as the monomial ideal $in_<(I) = \{in_<(f) : f \in I\}$. The relationship between an ideal I and its initial ideal $in_<(I)$ was first explored in 1927 by Macaulay [7] as a means to describe the possible Hilbert function of ideals.

When $I = (f_1, \ldots, f_r)$ and $in_<(I) = (in_<(f_1), \ldots, in_<(f_r))$, the set of generators is called a $\{f_1, \ldots, f_r\}$ a *Gröbner basis for I*. In 1965 [8], Bruno Buchberger described *Gröbner bases* and an algorithm to compute them. Gröbner bases generalize the notion of row reduction for solving linear systems of equations to a method for solving systems of polynomials of arbitrary degree. As such, they are a foundational tool in the field of computational commutative algebra.

2.3 Generic Initial Ideals

The utility of initial ideals in the theory of ideal computation has motivated much research on initial ideals. The initial ideals obtained from generic ideals are called *generic initial ideals*, which we describe below.

Consider an invertible matrix $g \in GL_n(\mathbb{K})$ for some field $\mathbb{K}$ and the action on the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ by

$$(x_1, \ldots, x_n) \mapsto \Big(\sum_{i=1}^{n} a_{i1}x_i, \ldots, \sum_{i=1}^{n} a_{in}x_i\Big).$$

Given an ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ and a term order (respecting the variable order $x_1 > x_2 > \cdots > x_n$), we can obtain the initial ideal $in_<(I)$. If $g \in GL_n(\mathbb{K})$ is *generic*, then $in_<(gI)$ is called the *generic initial ideal*.

2.4 Borel-Fixed Ideals

There is a filtration of subgroups

$$T_n(\mathbb{K}) \subset B_n(\mathbb{K}) \subset GL_n(\mathbb{K})$$

where $T_n(\mathbb{K})$ is the set of diagonal matrices (often called the torus) and $B_n(\mathbb{K})$ is the set of upper-triangular matrices called the *Borel subgroup*. By considering the ideals which are fixed by each of these subgroups, we get a filtration going the other direction. All monomial ideals are fixed by $T_n(\mathbb{K})$. Only powers of the maximal ideal $(x_1, \ldots, x_n)^d$ are fixed by $GL_n(\mathbb{K})$. The intermediate class fixed by $B_n(\mathbb{K})$ are the Borel-fixed ideals and correspond exactly to generic initial ideals. In 1974, Galligo showed that generic initial ideals are Borel-fixed in characteristic zero [2]. In 1987, Bayer and Stillman extended this result to arbitrary characteristic [3]. In 2004, Conca showed that if an ideal is Borel-fixed, then it is its own generic inital ideal [4]. In summary, for any characteristic, the generic initial ideals are exactly the Borel-fixed ideals.

2.5 Strongly Stable Ideals

Strongly stable ideals are a class of monomial ideals defined as being closed by a simple combinatorial operation known as a *Borel move*, and are exactly the Borel-fixed ideals in characteristic zero [5]. For positive characteristic, Borel-fixed ideals are strongly stable if the characteristic is larger than any exponent appearing in any monomial generator of the ideal. This correspondence allows us to take advantage of the combinatorial properties of strongly stable ideals to study invariants of generic initial ideals. The tight connection between Borel-fixed ideals and strongly stable ideals makes for somewhat confusing language. Some authors (for example [1]) use the term *Borel ideal* for what we call strongly stable ideals here.

We will work over the polynomial ring $R = K[y_1, \ldots, y_n]$ with the variable order $y_1 > y_2 > \cdots > y_n$.

Definition 2.5.1. Let $m \in R$ be a monomial. A *Borel move* on m is an operation that sends m to a monomial

$$m \cdot \frac{y_{i_1}}{y_{j_1}} \cdots \frac{y_{i_s}}{y_{j_s}},$$

where $i_t < j_t$ for all t and $y_{j_1} \cdots y_{j_s}$ divides m.

Given a monomial ideal $J \subset R$, we will use $G(J)$ to denote the minimal set of monomial generators. A monomial ideal $J \subset R$ is called *strongly stable* if it is fixed by Borel moves. Explicitly, for all $m \in J$, if $u \in R$ is a monomial obtained by a Borel move on m, then $u \in J$.

We will now recall the definition of the Borel closure of a set of monomials in R following notation in [1].

Definition 2.5.2. Let $T = \{m_1, \ldots, m_s\} \subset R$ be a set of monomials. Define $Borel(T)$ to be the smallest strongly stable ideal containing T.

Definition 2.5.3. Factor $m_1 = y_{i_1} y_{i_2} \cdots y_{i_r}$ and $m_2 = y_{j_1} y_{j_2} \cdots y_{j_s}$ with $r \geq s$. If $i_k \leq j_k$ for all $k \leq s$, then we say that m_1 precedes m_2 in the *Borel order* and write $m_1 \succ m_2$.

2.6 Borel Generators

Francisco, Mermin, and Schweig describe Stanley decompositions, Hilbert series and Betti number formulas of strongly stable ideals directly from their *Borel generators* in [1]. The Borel generators are a subset of the monomial generators of a strongly stable ideal. Since this is often a proper subset, computations with Borel generators are often more efficient than computing from the monomial generators.

We can revisit the monomial lattice to strengthen our intuition for Borel generators.

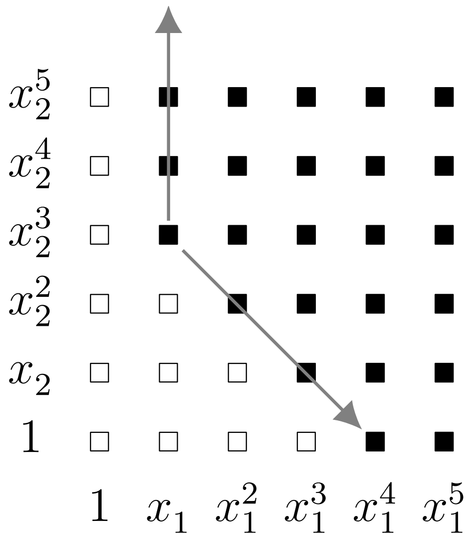


Figure 2.4: A visualization of the monomial ideal $I = Borel(x_1x_2^3) = (x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3) \subset \mathbb{K}[x_1, x_2]$

In the same way that many computations on general monomial ideals can be decomposed into computations on the monomial generators, computations on strongly stable ideals can be decomposed into computations on the Borel generators (see Propositions 3.3.7, 3.3.9, and 3.3.10). In the next chapter, we will further explore Borel generators and see that the incorporation of a weight vector can further restrict this generating set.

Chapter 3

Weighted Borel Generators

3.1 Introduction

Given the motivations for studying strongly stable ideals, it is not surprising that several generalizations of strongly stable ideals have emerged. In his 1994 thesis, Pardue gives a combinatorial definition of p-Borel ideals and shows that they correspond exactly to Borel-fixed ideals in characteristic $p > 0$ [9]. In 2013, Francisco, Mermin, and Schweig define Q-Borel ideals for a poset Q on the variables $x_1, \ldots, x_n$ [10]. Recently (2021), DiPasquale and Nezhad define L-Borel ideals as a generalization of strongly stable ideals for which Borel moves need only be satisfied for a subset of the variables [11]. In 2002, Gasharov, Hibi, and Peeva define (strongly) a-stable ideals for an integer sequence $a = (a_1, \ldots, a_n)$ as the image of (strongly) stable ideals in a polynomial ring with powers of the variables restricted by a [12].

Much of the work mentioned above generalizes strongly stable ideals by requiring only closure under some subset of Borel moves. In contrast, Dalzotto and Sbarra define *weighted strongly stable ideals* by generalizing Borel moves [13]. They show that weighted strongly stable ideals correspond exactly to the upper triangular automorphisms of a weighted polynomial algebra $\mathbb{K}[x_1, \ldots, x_n]$. This generalization is natural in the sense that it generalizes the correspondence between Borel-fixed ideals and strongly stable ideals to a correspondence between (upper triangular) graded automorphisms and weighted strongly stable ideals. However, as noted in their paper and in [14], this definition is unwieldy because *every* monomial ideal is weighted strongly stable for a suitable choice of weight vector. This makes their weighted strongly stable ideals difficult to work with. For example, determining an analogous resolution to the Eliahou-Kervaire resolution [15] for weighted strongly stable ideals (for arbitrary weight vector) would be equivalent to determining a minimal resolution for *any* monomial ideal. Dalzotto and Sbarra's definition of weighted strongly stable ideals uses *nondecreasing* weight vectors ($w_i \leq w_{i+1}$) while we use *nonincreas-*

ing weight vectors ($w_i \geq w_{i+1}$). On its face, this choice seems arbitrary. However, the classic definition of strongly stable ideals requires a choice of variable order. The relationship between generic initial ideals and strongly stable ideals is dependent on this variable order respecting the monomial order. Dalzotto and Sbarra use the weighted reverse-lexicographic monomial order with weights $w_i \leq w_{i+1}$ but construct Borel moves respecting $x_1 > x_2 > \cdots > x_n$. Dalzotto and Sbarra's choice of orientation of weight vector results in a *weakening* of the Borel move, resulting in a *generalization* of strongly stable ideals. In the present paper, we *strengthen* the Borel move to obtain a *specialization* of strongly stable ideals.

In this chapter, we specialize the notion of a strongly stable ideal to that of a w-stable ideal where $w \in \mathbb{N}^n_{>0}$ is a monotone decreasing tuple of positive integers. The main results are formulas for the Hilbert series (Theorem 3) and Poincaré series (Corollary 13) of principal w-stable ideals. These algebraic results are supported by Theorems 8 and 10, which provide combinatorial methods for computing both formulas. We end by showing how to determine whether a given strongly stable ideal is principally w-stable for some w and how to compute such a weight vector if it exists.

The definition of w-stable ideals presented here restricts the classically the set of strongly stable ideals in a way that preserves many combinatorial attributes of the classically defined strongly stable ideals. In particular, there is an analagous notion of *Borel generators* [1] which we call *weighted Borel generators*. Francisco, Mermin, and Schweig show that many computations on strongly stable ideals can be done directly in terms of Borel generators. In this paper, we show that similar computations can be done on w-stable ideals using weighted Borel generators.

In Section 3.4, we present a Stanley decomposition for w-stable ideals. This decomposition allows us to obtain a formula for the Hilbert series of S/I when I is a principal w-stable ideal and S is considered as a *weighted* polynomial ring in Section 3.5. In Section 3.6, we further explore the combinatorics of principal w-stable ideals and their connection to n-ary trees. In Section 3.7, we see how we can use Catalan diagrams to bookkeep relevant information for principal w-stable ideals. In Section 3.8, we give a formula for the Poincaré series of w-stable ideals in the weighted polynomial ring. Section 3.9 answers the question of whether a given strongly stable ideal is principally w-stable for a suitable weight vector w and gives an algorithm to produce such a weight vector. We end by presenting a class of examples which appear to be well-suited for computations as w-stable ideals and a conjecture that this is not coincidental. Throughout the paper, we use the Macaulay2 package wStableIdeals.m2 to help with computations.

3.2 w-Stable Ideals

Let $\mathbb{N}^n_{>0}$ denote the set of tuples of positive integers. We will call any monotone decreasing tuple of positive integers $w = (w_1, \dots, w_n) \in \mathbb{N}^n_{>0}$ a *weight vector*. Now let $S = \mathbb{K}[x_1, \dots, x_n]$, fix a weight vector $(w_1, \dots, w_n)$, and construct the ring homomorphism

$$\begin{aligned} \psi : S &\to R \\ x_i &\mapsto y_i^{w_i}. \end{aligned}$$

Now, we define w-stable ideals in the polynomial ring S. Note that R has variable order $y_1 > y_2 > \cdots > y_n$. In the last subsection, we defined the Borel closure of a set of monomials in R. Similarly, we can compute a *weighted Borel closure* for a set of monomials $A \subseteq S$ by

$$\overline{A} := \psi^{-1}(Borel(\psi(A))).$$

Monomial ideals in S which are fixed by this closure operation are called w-stable ideals. When $A = \{m\}$ is a singleton, we will use $\overline{m}$ to denote the *principal* w-stable ideal generated by m.

Definition 3.2.1. A monomial ideal $I \subseteq S$ is w-*stable* if $I = \overline{I}$.

Example 3.2.2. Let $w = (2, 1)$ be the weight vector for $S = \mathbb{K}[x_1, x_2]$ and let $I = (x_1, x_2^2) \subset S$. Then $\psi(x_1) = y_1^2$ and $\psi(x_2) = y_2$, so $\psi(I) = (y_1^2, y_2^2) \subset R$. Then

$$\begin{aligned} \psi^{-1}(Borel(\psi(I))) &= \psi^{-1}((y_1^2, y_1y_2, y_2^2)) \\ &= I, \end{aligned}$$

so I is $(2,1)-stable$. Note that the monomial y_1y_2 has empty preimage in S.

We can use the Macaulay2 package wStableIdeals to check our work.

```
i1 : loadPackage "wStableIdeals"

o1 = wStableIdeals

o1 : Package

i2 : K = QQ

o2 = QQ

o2 : Ring

i3 : S = K[x_1,x_2]

o3 = S

o3 : PolynomialRing

i4 : I = ideal(x_1,x_2^2)

                2
o4 = ideal (x , x )
             1   2

o4 : Ideal of S

i5 : borelClosure(I,Weights=>{2,1})

                2
```

```
o5 = ideal (x , x )
             1   2

o5 : Ideal of S

i6 : iswStable(I,w)

o6 = true
```

We used the `borelClosure` method to compute $\overline{I} = \psi^{-1}(Borel(\psi(I)))$. Since the output is equal to I, I is $(2, 1)$-stable. The `iswStable` method computes $\overline{I}$ and checks whether $\overline{I} = I$.

By default, weights in `wStableIdeals.m2` are set to the vector of all ones. When $w = \mathbb{1}$ is the weight vector of all ones (S has standard grading), we recover the classic definition of strongly stable ideals. For any fixed weight vector w, the set of w-stable ideals is a subset of the set of strongly stable ideals as the next proposition shows.

Proposition 3.2.3. *Fix a weight vector w. If $I \subset S$ is w-stable, then I is strongly stable.*

Proof. Let $m \in I$ with $x_j | m$ and $i < j$. We need to show that $m\frac{x_i}{x_j} \in I$. Notice that

$$\begin{aligned} \psi(m\frac{x_i}{x_j}) &= \psi(m)\frac{y_i^{w_i}}{y_j^{w_j}} \\ &= \psi(m)\frac{y_i^{w_j}}{y_j^{w_j}} y_i^{w_i - w_j} \end{aligned}$$

Since $x_j | m \in I$, we have $y_j^{w_j} | \psi(m) \in \psi(I)$, so by (classical) Borel closure,

$$\psi(m) \cdot \frac{y_i^{w_j}}{y_j^{w_j}} \in Borel(\psi(I)).$$

Since $i < j$, we have $w_i \geq w_j$, so $y_i^{w_i - w_j}$ is a monomial. It follows that $\psi(m\frac{x_i}{x_j}) \in Borel(\psi(I))$, so $m\frac{x_i}{x_j} \in I$, and we have shown that I is strongly stable. $\square$

The previous proposition shows that (for a fixed weight vector), the set of w-stable ideals is a subset of the set of strongly stable ideals. From this perspective, w-stable ideals are a refinement of strongly stable ideals. Contrast this with Dalzotto and Sbarra's definition of *weighted strongly stable ideals* in [13] where, for any given monomial ideal, one can find a weight vector which makes the ideal weighted strongly stable. We can see the difference explicitly in the following example.

Example 3.2.4. Let $I = (x_1^2, x_2^2) \subset S = \mathbb{K}[x_1, x_2]$. Using Dalzotto and Sbarra's definition (with order of variables reversed to make their notation match ours), I is weighted strongly stable with respect to $w = (4, 3)$. However, since I is not strongly stable, there exists no weight vector for which I is w-stable.

Definition 3.2.5. Let $u, v \in S$ be monomials. We say that u *precedes* v *in the* w*-Borel order* and write $u \succ_w v$ if $\psi(u) \succ \psi(v)$.

Proposition 3.2.6. *Let* $m, u \in S$ *be monomials. Then* $m \prec_w u$ *if and only if* $u \in \overline{m}$.

Proof. By definition, $m \prec_w u$ if and only if $\psi(m) \prec \psi(u)$. From Lemma 2.11 of [1], we see that $\psi(m) \prec \psi(u)$ if and only if $\psi(u) \in Borel(\psi(m))$. □

Some authors (see [12], [16]) call $\overline{m}$ the *shadow of* m or *big shadow of* m. Rather than giving a combinatorial description of weighted Borel moves corresponding to classical Borel moves, we define a w-Borel move on m as any move $m \mapsto u$ where $m \prec_w u$. The figure below gives a visualization in two variables.

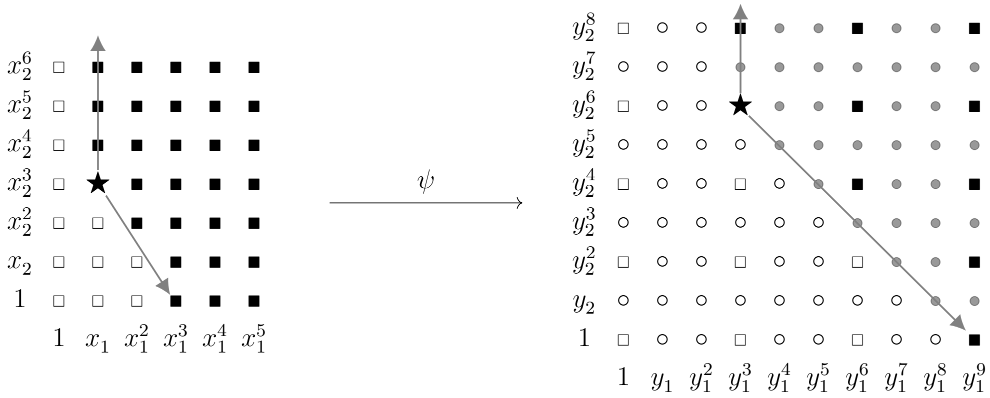


Figure 3.1: A visualization of $\psi : S \to R$ for $w = (3, 2)$

In Figure 3.1, the monomial lattices for $S = \mathbb{K}[x_1, x_2]$ and $R = \mathbb{K}[y_1, y_2]$ are shown on the left and right respectively. We have $m = x_1x_2^3 \in S$ (and its image $\psi(m) \in R$) represented by ★. Monomials in $\overline{m} \subset S$ are represented by ■, while monomials in $S/\overline{m}$ are displayed as □. Monomials in $R \setminus \psi(S)$ are displayed with ○ (and with ● if the monomial is in $Borel(\psi(I))$). Monomials in $\psi(S) \subset R$ are displayed as □ (and with ■ if the monomial is in $Borel(\psi(I))$).

Classically, Borel moves have been defined as those moves which move (in two variables) down and to the right. Rather than define w-Borel moves analagously, we will consider a w-Borel move as a move $m \mapsto u$ for any $u \in \overline{m}$.

Definition 3.2.7. A *w-Borel move* on a monomial $m \in S$ is an operation which sends m to a monomial u where $m \prec_w u$.

While this definition lacks the explicit combinatorial description of classic Borel moves, we can state the analog of the definition of strongly stable ideals as a proposition below.

Proposition 3.2.8. *A monomial ideal I is w-stable if and only if it is fixed by w-Borel moves.*

Proof. Assume $I \subset S$ is w-stable with $m \in I$ a monomial and $u \in S$ such that $u \succ_w m$. We need to show that $u \in I$. We have $\psi(u) \succ \psi(m)$, so $\psi(u) \in Borel(\psi(I))$. Thus, $u \in I$.

Now, assume that $I \subset S$ is closed under w-Borel moves. We need to show that

$$I = \psi^{-1}(Borel(\psi(I))).$$

The forward inclusion is obvious. Let $v \in \psi^{-1}(Borel(\psi(I)))$. Then $\psi(v) \in Borel(\psi(I))$. There must be some $m \in \psi(I)$ such that $\psi(v) \succ \psi(m)$. By definition, $v \succ_w m$. Since I is closed under w-Borel moves, $v \in I$. □

3.3 Weighted Borel Generators

Proposition 3.3.1 (Proposition 2.12 of [1]). *Every strongly stable ideal $J \subset R$ has a unique minimal set of Borel generators, $Bgens(J)$.*

Strongly stable ideals can be described completely by their Borel generators. We have a natural generalization for w-stable ideals.

Definition 3.3.2. Let $I \subset S$ be a w-stable ideal. Then we define the *weighted Borel generators* of I with respect to w as the set

$$Bgens_w(I) = \psi^{-1}(Bgens(Borel(\psi(I))))$$

Since every w-stable ideal $I \subset S$ corresponds to the strongly stable ideal $Borel(\psi(I)) \subset R$, Proposition 3.3.1 gives us the following.

Proposition 3.3.3. *For a fixed weight vector w, every w-stable ideal I has a unique minimal set of weighted Borel generators $Bgens_w(I)$.*

Proposition 3.3.4. *If $m, u \in S$ are monomials such that $m \prec u$, then $m \prec_w u$.*

Proof. We begin by writing both monomials in factored form (following notation in Definition 2.5.3) $m = x_{i_1} \cdots x_{i_s}$, $u = x_{j_1} \cdots x_{j_r}$ and recall that $i_k \leq j_k$ for $1 \leq k \leq s \leq r$. Now, consider

$\psi(m), \psi(u)$ in factored form.

$$\begin{aligned}\psi(m) &= (\overbrace{y_{i_1}y_{i_1}\cdots y_{i_1}}^{w_{i_1}})\,(\overbrace{y_{i_2}y_{i_2}\cdots y_{i_2}}^{w_{i_2}})\cdots(\overbrace{y_{i_s}y_{i_s}\cdots y_{i_s}}^{w_{i_s}})\\ \psi(u) &= (\underbrace{y_{j_1}y_{j_1}\cdots y_{j_1}}_{w_{j_1}})(\underbrace{y_{j_2}y_{j_2}\cdots y_{j_2}}_{w_{j_2}})\cdots(\underbrace{y_{j_s}y_{j_s}\cdots y_{j_s}}_{w_{j_s}})\cdots(\underbrace{y_{j_r}y_{j_r}\cdots y_{j_r}}_{w_{j_r}})\end{aligned}$$

Since $i_k \leq j_k$, we know that $w_{i_k} \geq w_{j_k}$. Thus, $\psi(m) \prec \psi(u)$, so $m \prec_w u$. ☐

Proposition 3.3.5. *If I is w-stable for some weight vector w, then $Bgens_w(I) \subseteq Bgens(I)$.*

Proof. Suppose that $Bgens_w(I)$ contains some $b \in Bgens_w(I) \setminus Bgens(I)$. Then there exists some $m \in Bgens(I)$ such that $m \prec b$. Thus $\psi(m) \prec \psi(b)$ so $b \notin Bgens_w(I)$. ☐

Proposition 5.3 of [17] gives an algorithm for computing Borel generators which can easily be extended to compute weighted Borel generators. That algorithm is implemented in wStableIdeals as the borelGens method.

Example 3.3.6. Let $I = (x_1^2, x_1x_2^2, x_2^4) \subset S = \mathbb{K}[x_1, x_2]$ with weight vector $w = (2, 1)$.

```
i7 : borelGens(ideal(x_1^2,x_1*x_2^2,x_2^4),Weights=>{2,1})


       4
o7 = {x }
       2


o7 : List
```

Many computations on strongly stable ideals can be reduced to computations on *principal* strongly stable ideals as seen in the following three propositions from [1].

Proposition 3.3.7 (Proposition 2.15 of [1]). *Let* $J_1 = Borel(T_1)$ *and* $J_2 = Borel(T_2)$ *for sets of monomials* $T_1, T_2 \subset R$*. Then*

$$J_1 + J_2 = Borel(T_1 \cup T_2).$$

Definition 3.3.8. Let $u = y_{i_1} \cdots y_{i_r}$ and $v = y_{j_1} \cdots y_{j_s}$ be two monomials in factored form in R with $s \leq r$. Put $l_t = min(i_t, j_t)$ with $l_t = s_t$ when $s < t \leq r$. Then $\mu = y_{t_1} \cdots y_{t_r}$ is the *meet of* u *and* v *in the Borel order*, denoted $meet_{Borel}(u, v) = \mu$.

Proposition 3.3.9 (Proposition 2.16 of [1]). *Let* $u = y_{i_1} \cdots y_{i_s}$ *and* $v = y_{j_1} \cdots y_{j_s}$ *with* $r \geq s$*. Set* $k_q = min(i_q, j_q)$ *with* $k_q = i_q$ *if* $s < q \leq r$*. Then set* $t = y_{k_1} \cdots y_{k_r}$ *be the meet of* u *and* v *in the Borel order. Then* $Borel(u) \cap Borel(v) = Borel(t)$*.*

Proposition 3.3.10 (Proposition 2.17 of [1]). *For monomials* $u, v \in R$*,* $Borel(u) \cdot Borel(v) = Borel(uv)$*.*

We can adapt each of the above propositions to the w-stable case, showing that computations on w-stable ideals can be reduced to computations on principal w-stable ideals.

Proposition 3.3.11. *Let* $I_1 = \overline{A_1}$ *and* $I_2 = \overline{A_2}$ *for sets of monomials* $A_1, A_2 \subset S$*. Then* $I_1 + I_2 = \overline{A_1 \cup A_2}$*.*

Proof. We begin by noticing that $I_1 = \psi^{-1}(Borel(\psi(A_1)))$ and $I_2 = \psi^{-1}(Borel(\psi(A_2)))$. Now,

$$\begin{aligned}
I_1 + I_2 &= \psi^{-1}(Borel(\psi(A_1))) + \psi^{-1}(Borel(\psi(A_2))) \\
&= \psi^{-1}(Borel(\psi(A_1)) + Borel(\psi(A_2))) \\
&= \psi^{-1}(Borel(\psi(A_1) \cup \psi(A_2))) && \text{(Proposition 3.3.7)} \\
&= \psi^{-1}(Borel(\psi(A_1 + A_2))) \\
&= \overline{A_1 + A_2}.
\end{aligned}$$

$\square$

We can consider the *meet* of u and v in the w-Borel order which we will denote $meet_w(u,v) := \psi^{-1}(meet_{Borel}(\psi(u), \psi(v))$.

Proposition 3.3.12. *Let $u, v \in S$ be monomials with $q = meet_w(u,v)$. Then $\overline{u} \cap \overline{v} = \overline{q}$.*

Proof. Notice that

$$\begin{aligned}
\overline{u} \cap \overline{v} &= \psi^{-1}(Borel(\psi(u))) \cap \psi^{-1}(Borel(\psi(v))) \\
&= \psi^{-1}(Borel(\psi(u)) \cap Borel(\psi(v))) \\
&= \psi^{-1}(Borel(\psi(q))) && \text{(Proposition 3.3.9)} \\
&= \overline{q}.
\end{aligned}$$

$\square$

Example 3.3.13. If $u = x_2^4$ and $v = x_1x_2$ in $S = \mathbb{K}[x_1, x_2]$ with $w = (2,1)$, then $\psi(u) = y_2^4$ and $\psi(v) = y_1^2y_2$. We have $meet(\psi(u), \psi(v)) = y_1^2y_2^2$, so $meet(u,v) = x_1x_2^2$ in the w-Borel order.

Proposition 3.3.14. *For monomials $u, v \in S$, $\overline{u} \cdot \overline{v} = \overline{uv}$.*

Proof. We have $\overline{u} = \psi^{-1}(Borel(\psi(u)))$ and similarly for v. Now,

$$\begin{aligned}
\overline{u} \cdot \overline{v} &= \psi^{-1}(Borel(\psi(u))) \cdot \psi^{-1}(Borel(\psi(v))) \\
&= \psi^{-1}(Borel(\psi(u)) \cdot Borel(\psi(v))) \\
&= \psi^{-1}(Borel((\psi(u) \cdot \psi(v)))) && \text{(Proposition 3.3.10)} \\
&= \psi^{-1}(Borel(\psi(uv))) \\
&= \overline{uv}.
\end{aligned}$$

$\square$

Given that we can reduce computations for arbitrary w-stable ideals to computations on principal w-stable ideals, we will be especially interested in principal w-stable ideals for the remainder of the paper.

3.4 Stanley Decomposition and Hilbert Series

Following Francisco, Mermin, and Schweig in [1], we use truncations of strongly stable ideals to construct a Stanley decomposition. We begin this section by recalling the definition and notation of truncations and a useful lemma from [1].

Definition 3.4.1 (Definition 3.10 of [1])**.** Let $m \in R$ be a monomial with factorization $y_{i_1} \cdots y_{i_r}$ and let d be a positive integer. If $d \leq deg(m)$, define the *d-truncation of m*, denoted $trunc_d(m)$, to be $y_{i_1} \cdots y_{i_d}$. If $d > deg(m)$, then set $trunc_d(m)$ to be m itself. For a monomial ideal $J \subset R$, define the *d-truncation of J* to be the ideal $trunc_d(J) = (trunc_d(m) : m \in J)$.

Lemma 3.4.2 (Lemma 3.12 of [1])**.** *Let $J \subset R$ be a strongly stable ideal. Then*

$$trunc_d(J) = Borel(trunc_d(m) : m \in Bgens(J))).$$

In the classic setting, *truncations* are used to obtain a Stanley decomposition of R/J where J is a strongly stable ideal. We obtain a similar result for w-stable ideals. First, we need a quick proposition.

Proposition 3.4.3. *If I is w-stable, then $S/I = \psi^{-1}(R/Borel(\psi(I)))$.*

Proof. If $u \in \psi^{-1}(R/Borel(\psi(I)))$, then $\psi(u) \notin Borel(\psi(I))$. Clearly, $\psi(u) \notin \psi(I)$, so $u \notin I$.

On the other hand, if $v \in S/I$, then $\psi(v) \notin \psi(I)$. Suppose, however, that $\psi(v) \in Borel(\psi(I))$. Then there exists some $m \in I$ such that $\psi(v) \succ \psi(m)$. Then $v \succ_w m$, but this contradicts I being closed by w-Borel moves. $\square$

Theorem 1. *Let $I \subset S$ be a w-stable ideal generated in degrees less than or equal to d. Let G_s denote the degree s generators of $trunc_s(Borel(\psi(I)))$ which are not in $Borel(\psi(I))$. Then S/I*

has the Stanley decomposition

$$S/I = \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : \psi(u)y_j \notin trunc_{s+1}(Borel(\psi(I)))] \right).$$

Proof. We can start by getting a Stanley decomposition for $R/Borel(\psi(I))$

$$R/Borel(\psi(I)) = \bigoplus_{s=0}^{d-1} \left(\bigoplus_{v \in G_s} u \cdot \mathbb{K}[y_j : vy_j \notin trunc_{s+1}(Borel(\psi(I)))] \right).$$

Since $S/I = \psi^{-1}(R/Borel(\psi(I)))$,

$$\begin{aligned} S/I &= \psi^{-1}(R/Borel(\psi(I))) \\ &= \psi^{-1}\left(\bigoplus_{s=0}^{d-1} \left(\bigoplus_{v \in G_s} v \cdot \mathbb{K}[y_j : vy_j \notin trunc_{s+1}(Borel(\psi(I)))] \right) \right) \\ &= \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \psi^{-1}\left(\mathbb{K}[y_j : my_j \notin trunc_{s+1}(Borel(\psi(I)))] \right) \right) \\ &= \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : my_j \notin trunc_{s+1}(Borel(\psi(I)))] \right). \end{aligned}$$

The first line above is Proposition 3.4.3. The second line uses the formula for Stanley decomposition of $R/Borel(\psi(I))$ found in Theorem 4.1 of [1]. The third line follows from ψ being an algebra homomorphism. To see that the fourth line follows, let $Z_y = \{y_{j_1}, y_{j_2}, \ldots, y_{j_r}\}$ be any subset of the variables $y_1, \ldots, y_n$. Then $\psi^{-1}(\mathbb{K}[Z_y]) = \mathbb{K}[x_{j_1}, x_{j_2}, \ldots, x_{j_r}]$. ☐

Notation 3.4.4. Let u be a monomial with factorization $u = y_{i_1}y_{i_2}\ldots y_{i_k}$ and $i_1 \le i_2 \le \cdots \le i_k$. We will use the notation $max(u) = i_k$ to denote the largest index of any variable dividing u.

Lemma 3.4.5. *Let $m \in R$ be a monomial and assume that $u \in G(Borel(trunc_s(v)))$. Then $uy_j \in Borel(trunc_{s+1}(v))$ if and only if $j \le max(trunc_{s+1}(v))$.*

Proof. First, notice that $uy_j \in Borel(trunc_{s+1}(v))$ if and only if $trunc_{s+1}(v) \prec uy_j$. And this is only possible if $j \le max(trunc_{s+1}(v))$. ☐

Corollary 2. *When $I = \overline{m} \subset S$ is a principal w-stable ideal, we get the following Stanley decomposition*

$$S/\overline{m} = \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : j > max(trunc_{s+1}(\psi(m)))] \right)$$

Proof. Since $I = \overline{m}$ is a principal w-stable ideal, $Borel(\psi(I)) = Borel(\psi(m))$ is a principal strongly stable ideal. By Lemma 3.7.3, it follows that

$$trunc_{s+1}(Borel(\psi(I))) = Borel(trunc_{s+1}(\psi(m))).$$

Thus, the Stanley decomposition of S/I is

$$\begin{aligned} S/I &= \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : \psi(u)y_j \notin trunc_{s+1}(Borel(\psi(I)))] \right) \\ &= \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : \psi(u)y_j \notin Borel(trunc_{s+1}(\psi(m)))] \right) \\ &= \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : j > max(trunc_{s+1}(\psi(m)))] \right) \end{aligned}$$

□

3.5 Hilbert Series

If $I \subset S$ is a principal w-stable ideal and we consider S as being the *weighted* polynomial ring ($deg(x_i) = w_i$), then we get the following formula for the Hilbert series of S/I.

Theorem 3. *Let $I = \overline{m}$ be a principal w-stable ideal generated by a degree d monomial $m \in S$ and let $k_s = max(trunc_{s+1}(\psi(m)))$. Then*

$$HS(S/I) = \sum_{s=0}^{d-1} \frac{c_s t^s}{\prod_{j=k_s+1}^{n} (1 - t^{w_j})}$$

where c_s is the cardinality of $\psi^{-1}(G_s) = \psi^{-1}(G(trunc_s(Borel(\psi(I))))_s \setminus Borel(\psi(m)))$.

Proof. By Corollary 2,

$$S/I = \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : j > max(trunc_{s+1}(\psi(m)))] \right),$$

so we can write the Hilbert series of S/I as

$$\begin{aligned} HS(S/I) &= HS\left(\bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : j > max(trunc_{s+1}(\psi(m)))] \right) \right) \\ &= \sum_{s=0}^{d-1} \left(\sum_{u \in \psi^{-1}(G_s)} HS\Big(u \cdot \mathbb{K}[x_j : j > k_s)] \Big) \right) \Bigg) \end{aligned}$$

Each summand is of the form $u \cdot \mathbb{K}[x_j : j \geq k_s]$ for $u \in \psi^{-1}(G_s)$ and we can write its Hilbert series contribution as

$$\begin{aligned} HS(u \cdot \mathbb{K}[x_j : j \geq k_s]) &= HS(u) \cdot HS(\mathbb{K}[x_j : j \geq k_s]) \\ &= t^{deg(u)} \cdot \frac{1}{\prod_{j=k_s+1}^{n} (1 - t^{w_j})} \\ &= \frac{t^s}{\prod_{j=k_s+1}^{n} (1 - t^{w_j})}. \end{aligned}$$

Summing over all $u \in \psi^{-1}(G_s)$ multiplies the formula above by c_s, and summing over all degrees $0 \leq s \leq d-1$ gives the desired formula. □

The coefficients c_s in the formula above are tedious to compute since we need to compute $G(Borel(trunc_s(m)))$ for $0 \leq s \leq d-1$. In section 3.7, we use *Catalan diagrams* to more efficiently compute these coefficients. But first, we will take a closer look at the combinatorics of principal w-stable ideals and their connection to n-ary trees.

3.6 Principal w-Stable Ideals

Definition 3.6.1. Let $m \in S$ be a monomial and fix a weight vector w. Define $\mathcal{T}^{\infty}_{w,m}$ to be the directed tree with base vertex 1 and a directed edge (v, vx_j) if

$$max(v) \leq j \leq max(trunc_{deg_w(v)+1}(\psi(m))).$$

By convention, we set $max(1) = 1$. We will also use $\mathcal{T}^{d}_{w,m}$ to denote the finite tree obtained by imposing the additional branching condition that $deg_w(v) < d$. When we suppress the exponent, we will take that to mean $\mathcal{T}_{w,m} = \mathcal{T}^{deg_w(m)}_{w,m}$.

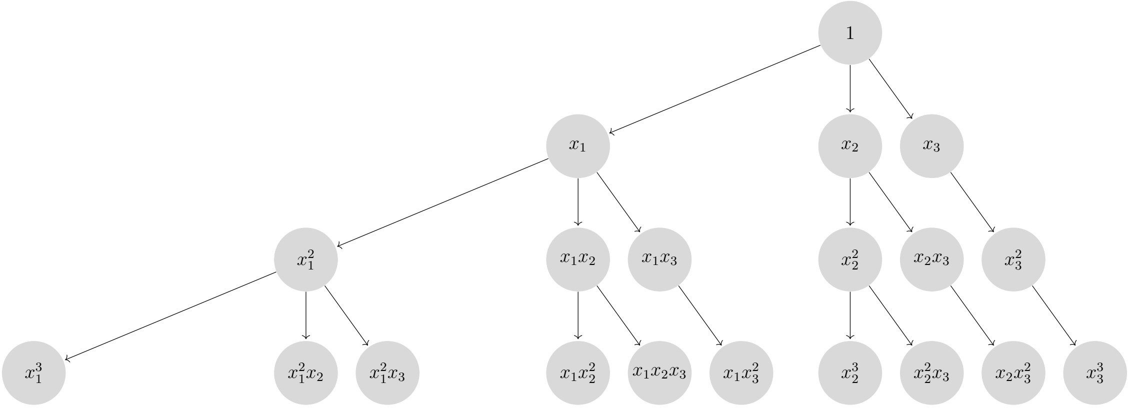


Figure 3.2: $\mathcal{T}_{(1,1,1)}(x_3^3)$

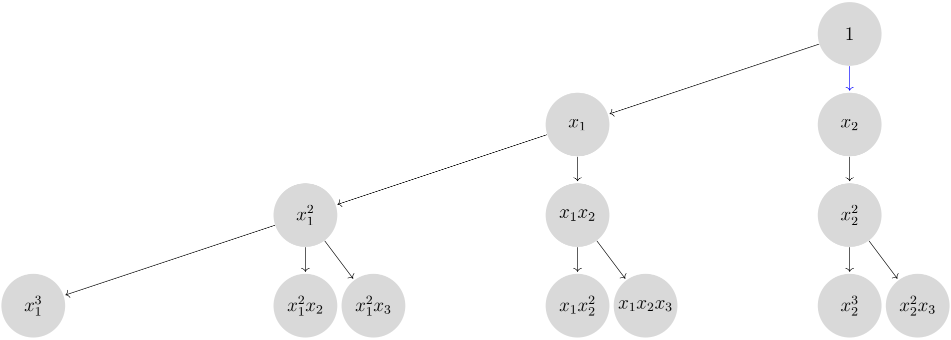


Figure 3.3: $\mathcal{T}_{(1,1,1)}(x_2^2x_3)$

Note that x_1x_3 is not a vertex of $\mathcal{T}_{(1,1,1)}(x_2^2x_3)$, because $max(trunc_{1+1}(x_2^2x_3)) = 2$. Next, consider an example with a nontrivial weight vector.

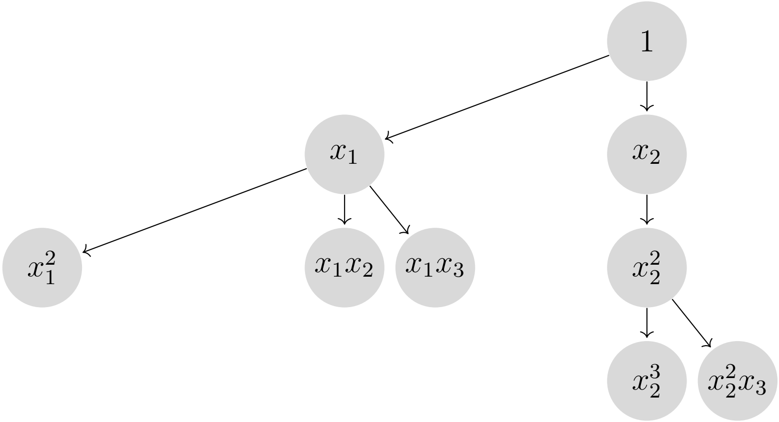


Figure 3.4: $\mathcal{T}_{(4,2,1)}(x_2^2x_3)$

We can get $\mathcal{T}_w(m)$ in Macaulay2 by using the `treeFromMonomial` method.

```
i8 : K[x_1,x_2,x_3]

o8 = QQ[x ..x ]
         1   3

o8 : PolynomialRing

i9 : treeFromMonomial(x_2^2*x_3,Weights=>{4,2,1})

o9 = Digraph{1 => {x , x }          }
                    1   2
                     2
             x  => {x , x x , x x }
              1      1   1 2   1 3
                     2
```

```
             x  => {x }
              2      2
             x x  => {}
              1 2
             x x  => {}
              1 3
              2
             x  => {}
              1
              2      3   2
             x  => {x , x x }
              2      2   2 3
              2
             x x  => {}
              2 3
              3
             x  => {}
              2
o9 : Digraph
```

The following theorem gives a dictionary between $\mathcal{T}_w(m)$ and $\overline{m}$.

Theorem 4. *Let V_d denote the set of vertices of $\mathcal{T}^{\infty}_{w,m}$ whose degree is greater than or equal to d. Then $V_d = \psi^{-1}(Borel(trunc_d(\psi(m))))$.*

Proof. The base case ($d = 0$) is trivial.

($\subseteq$) Let $\tilde{v} \in V_{d+1}$ and write $\tilde{v} = vx_{j_1}x_{j_2}\cdots x_{j_s} \in V_{d+1}$ so that $max(v) \leq j_1 \leq j_2 \leq \cdots \leq j_s$, $deg_w(v) \leq d$, and $deg_w(vj_1) > d$. Then $v \in V_{d-w_{j_1}}$, so $\psi(v) \in Borel(trunc_{d-w_{j_1}}(\psi(m)))$. Now the branch structure of $\mathcal{T}_{w,m}^{\infty}$ gives

$$j_1 \leq max(trunc_{d-w_{j_1}+1}(\psi(m))),$$

so $j_1 \leq max(trunc_{d+1}(\psi(m)))$. Thus, $\psi(vx_{j_1}) \in Borel(trunc_{d+1}(\psi(m)))$. It follows that $\psi(\tilde{v}) \in Borel(trunc_{d+1}(\psi(m)))$.

($\supseteq$) Let $\tilde{u} \in \psi^{-1}(Borel(trunc_{d+1}(\psi(m))))$ and write $\tilde{u} = ux_{j_1}x_{j_2}\cdots x_{j_s}$ so that $deg_w(u) \leq d$ and $deg_w(ux_{j_1}) > d$. Then $\psi(ux_{j_1}) = trunc_{deg_w(u)+w_{j_1}}(\nu)$ for some $\nu \in Borel(\psi(m))$. Thus,

$$\begin{aligned} j_1 &= max(trunc_{deg_w(u)+w_{j_1}}(\nu)) \\ &= max(trunc_{deg_w(u)+1}(\nu)) \\ &\leq max(trunc_{deg_w(u)+1}(\psi(m)), \end{aligned}$$

so $ux_{j_1} \in V_{deg_w(u)+w_{j_1}}$. Now, we can repeat the procedure above for $j_2, \ldots, j_s$ to get $\tilde{u} \in V_{deg_w(\tilde{u})} \subseteq V_{d+1}$. □

Corollary 5. *The set $sink(\mathcal{T}_w(m))$ is equal to $G(\overline{m})$.*

Proof. By the previous theorem, we know that $\overline{m} = V_{deg_w(m)}$. Since our tree is $\mathcal{T}_w(m) = \mathcal{T}_w^{deg_w(m)}(m)$, branches end when a vertex reaches $deg_w(m)$. □

We will now translate our earlier results on Stanley decompositions and Hilbert series for principal w-stable ideals in terms of $\mathcal{T}_w(m)$.

Theorem 6. *Let V, E denote the sets of vertices and (directed) edges of $\mathcal{T}_w(m)$. The Stanley decomposition of $S/\overline{m}$ is given by*

$$S/\overline{m} = \bigoplus_{u \in V \setminus sink(\mathcal{T}_w(m))} u \cdot \mathbb{K}[x_j : max(u) \leq j \text{ and } (u, ux_j) \notin E]$$

Proof. By Corollary 2, we have

$$S/\overline{m} = \bigoplus_{s=0}^{d-1} \left(\bigoplus_{u \in \psi^{-1}(G_s)} u \cdot \mathbb{K}[x_j : j > max(trunc_{s+1}(\psi(m))] \right)$$

Notice that

$$\begin{aligned} \psi^{-1}(G_s) &= \psi^{-1}(G(trunc_s(Borel(\psi(m)))) \setminus Borel(\psi(m))) \\ &= \psi^{-1}(G(Borel(trunc_s(\psi(m))))) \setminus \psi^{-1}(Borel(\psi(m))) \\ &= \psi^{-1}(G(Borel(trunc_s(\psi(m))))) \setminus \overline{m} \end{aligned}$$

Now, if we sum over all $s = 0, \ldots, d-1$, we obtain all vertices except those in $\overline{m}$ which is exactly $sink(\mathcal{T}_w(m))$, so

$$S/\overline{m} = \bigoplus_{u \in V \setminus sink(\mathcal{T}_w(m))} u \cdot \mathbb{K}[x_j : max(u) \leq j \text{ and } (u, ux_j) \notin E]$$

□

We can also interpret the Hilbert series of $S/\overline{m}$ in terms of $\mathcal{T}_{w,m}$.

Theorem 7. *Let V, E denote the sets of vertices and edges of $\mathcal{T}_{w,m}$. The Hilbert series of $S/\overline{m}$ is given by*

$$\sum_{s=0}^{d-1} \frac{c_s t^s}{\prod_{k_s+1}^{n}(1 - t^{w_j})}$$

where c_s is the number of degree s vertices in $\mathcal{T}_{w,m}$ and k_s is the largest index such that $(v, vx_{k_s}) \in E$ for any degree s vertex $v \in V$.

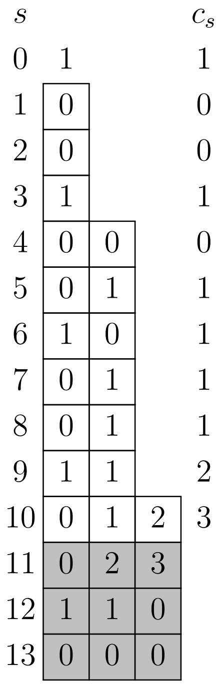
3.7 Catalan Diagrams

In this section, we generalize the notion of a Catalan diagram (see Definition 5.2 of [1]) to a weighted Catalan diagram with respect to a given weight vector. The combinatorial structure of a principal w-stable ideal $\overline{m}$ allows us to efficiently track of the degree and maximal index of monomials in truncations of $\overline{m}$ using the weighted Catalan diagram $C_{w,m}$.

Definition 3.7.1. For a fixed weight vector w and monomial $m \in S$, we can form $C_{w,m}$, the *weighted Catalan diagram* of m with respect to w, as follows. Construct the classic Catalan diagram with shape $\psi(m)$ with an additional $max(w) - 1$ rows of length n at the bottom. Then fill in the entries with the following recursion.

$$C_{w,m}(0,1) = 1$$

$$C_{w,m}(a,b) = \begin{cases} \sum_{k=1}^{b} C_{w,m}(a - w_b, k) & 0 \leq a - w_b < d \text{ and } max(trunc_{a-w_b+1}(\psi(m))) \geq b \\ 0 & \text{otherwise} \end{cases}$$



s				c_s
0	1			1
1	0			0
2	0			0
3	1			1
4	0	0		0
5	0	1		1
6	1	0		1
7	0	1		1
8	0	1		1
9	1	1		2
10	0	1	2	3
11	0	2	3	
12	1	1	0	
13	0	0	0	

Figure 3.5: The Catalan diagram $C_{(3,2,1)}(x_1 x_2^3 x_3^2)$

The rows of degree $s \geq deg(m)$ are shaded gray. The reason for this distinction will become clear in the next section. We can use the `catalanDiagram` method to compute the Catalan diagram in Macaulay2.

```
i10 : catalanDiagram(x_1*x_2^3*x_3^2,Weights=>{3,2,1})

o10 =  | 1 0 0 |
       | 0 0 0 |
       | 0 0 0 |
       | 1 0 0 |
       | 0 0 0 |
       | 0 1 0 |
       | 1 0 0 |
       | 0 1 0 |
       | 0 1 0 |
       | 1 1 0 |
       | 0 1 2 |
       | 0 2 3 |
       | 1 1 0 |
       | 0 0 0 |

               14        3
o10 : Matrix ZZ   <-- ZZ
```

Definition 3.7.2. Let $I \subset S$ be a w-stable ideal with minimal generators $G(I) = \{m_1, \ldots, m_s\}$. For each $1 \leq i \leq n$ and each degree d, let $q_i^d(I)$ be the cardinality of the set $\{m \in G(I) : max(m) = i \text{ and } deg(m) = d\}$.

Theorem 8. *Fix a weight vector w and a monomial $m \in S$. Then for $1 \leq a \leq d$,*

$$C_{w,m}(a,b) = q_b^a(\psi^{-1}(trunc_a(Borel(\psi(m)))).$$

Proof. We will prove this by induction on $1 \leq a \leq d + max(w) - 1$. First, notice that

$$C_{w,m}(1,b) = \begin{cases} 1 & w_b = 1 \\ 0 & w_b > 1 \end{cases}$$

On the other hand, we have $trunc_1(Borel(\psi(m))) = (y_1, y_2, \dots, y_{min(m)})$, so

$$\psi^{-1}(trunc_1(Borel(\psi(m)))) = (x_1, x_2, \dots, x_{min(m)}).$$

Since $\psi^{-1}(y_i) = x_i$, it follows that

$$q_b^1(\psi^{-1}(trunc_1(Borel(\psi(m))))) = \begin{cases} 1 & w_b = 1 \\ 0 & w_b > 1 \end{cases}$$

for $1 \leq b \leq min(m)$. The base case is finished.

Now, assume that $C_{w,m}(l,k) = q_k^l(\psi^{-1}(trunc_l(Borel(\psi(m)))))$ for $1 \leq l < a \leq d$ and for all k. Any monomial in $G(\psi^{-1}(trunc_a(Borel(\psi(m)))))$ must be of the form $ux_b = \psi^{-1}(\psi(u)y_b^{w_b})$ for some $u \in \psi^{-1}(trunc_{a-w_b}(Borel(\psi(m))))$ with $deg(u) = a - w_b$ and $max(u) \leq b$. Therefore,

$$\begin{aligned} q_b^a(\psi^{-1}(trunc_a(Borel(\psi(m))))) &= \sum_{k=1}^{b} q_k^{a-w_b}(\psi^{-1}(trunc_{a-w_b}(Borel(\psi(m))))) \\ &= \sum_{k=1}^{b} C_{w,m}(a-w_b,k) \\ &= C_{w,m}(a,b). \end{aligned}$$

□

Corollary 9. *The coefficients c_s for $0 \leq s \leq d-1$ in Theorem 5 are given by the sum of the entries in the s^{th} row of $C_{w,m}$.*

This gives us a combinatorial method for computing the Hilbert series of principal w-stable ideals. The following theorem will help us do something similar for Betti numbers in the next section.

Lemma 3.7.3. *Fix a weight vector w and a monomial $m \in S$. Then, for $d \leq deg(m)$, we have*

$$\psi^{-1}(G(trunc_d(Borel(\psi(m))))_d) = G(\psi^{-1}(trunc_d(Borel(\psi(m)))))_d$$

Proof. ($\subseteq$) Let $u \in \psi^{-1}(G(trunc_d(Borel(\psi(m))))_d)$ so that $\psi(u)$ is a degree d monomial generator of $trunc_d(Borel(\psi(m)))$. Then $u \in \psi^{-1}(trunc_d(Borel(\psi(m))))$ also has degree d and must be a generator, because every degree d monomial of $\psi^{-1}(trunc_d(Borel(\psi(m))))$ is a generator. ($\supseteq$) Let $v \in G(\psi^{-1}(trunc_d(Borel(\psi(m)))))_d$. Then $\psi(v) \in trunc_d(Borel(\psi(m)))$ with degree d and is therefore a generator. It follows that $v \in \psi^{-1}(G(trunc_d(Borel(\psi(m))))_d)$. □

Theorem 10. *Fix a weight vector w and a monomial $m \in S$ with $d = deg_w(m)$. For $d \leq a \leq d + max(w) - 1$,*

$$C_{w,m}(a,b) = q^a_b(\overline{m}).$$

Proof. The base case is $a = d$, where it suffices to show that

$$G(\overline{m})_d = G(\psi^{-1}(trunc_d(Borel(\psi(m)))))_d.$$

We have

$$\begin{aligned}
u \in G(\overline{m})_d = G(\psi^{-1}(Borel(\psi(m)))) &\iff \psi(u) \in G(Borel(\psi(m)))_d \\
&\iff \psi(u) \in G(trunc_d(Borel(\psi(m))))_d \\
&\iff u \in \psi^{-1}(G(trunc_d(Borel(\psi(m))))_d) \\
&\iff u \in G(\psi^{-1}(trunc_d(Borel(\psi(m)))))_d,
\end{aligned}$$

where the last line follows from Lemma 3.7.3. Since $G(\overline{m})_d = G(\psi^{-1}(trunc_d(Borel(\psi(m)))))_d$, we have $q_b^d(\overline{m}) = q_b^d(\psi^{-1}(trunc_d(Borel(\psi(m)))))$. From the previous theorem, we know that $C_{w,m}(d, b) = q_b^d(\psi^{-1}(trunc_d(Borel(\psi(m)))))$, so we can conclude that $C_{w,m}(d, b) = q_b^d(\overline{m})$. For the inductive step, we should assume that $C_{w,m}(l, b) = q_b^l(\overline{m})$ for $d \leq l < a < d+max(w)-1$. And we want to prove that $C_{w,m}(a, b) = q_b^a(\overline{m})$. We will consider two cases:

- $a - w_b < d$
- $a - w_b \geq d$

In the case $a - w_b < d$, any degree a generator is of the form ux_b where

$$u \in \psi^{-1}(G(trunc_{a-w_b}(Borel(\psi(m))))_{a-w_b}).$$

Thus,

$$\begin{aligned}
q_b^a(\overline{m}) &= \sum_{k=1}^{b} q_k^{a-w_b}(\psi^{-1}(trunc_{a-w_b}(Borel(\psi(m))))) \\
&= \sum_{k=1}^{b} C_{w,m}(a - w_b, k) \\
&= C_{w,m}(a, b).
\end{aligned}$$

In the case $a - w_b \geq d$, we see that $C_{w,m}(a, b) = 0$ by definition. Additionally, any monomial generator of $\psi^{-1}(trunc_a(Borel(\psi(m))))$ with maximum index b must be of the form vx_b for some degree $a - w_b$ generator $v \in G(\psi^{-1}(trunc_{a-w_b}(Borel(\psi(m)))))$. But since $a - w_b \geq d$, we have $v \in \overline{m}$, so $vx_b \notin G(\overline{m})$. $\square$

3.8 Betti Numbers

In this section, we (again) consider S to be the weighted polynomial ring with $deg(x_i) = w_i$. Since w-stable ideals are strongly stable, the Eliahou-Kervaire resolution gives the minimal free resolution. In particular, we can compute the total Betti numbers as follows.

Theorem 11 (Eliahou-Kervaire)**.** *Let I be a (strongly) stable ideal minimally generated by $G(I) = \{m_1, \dots, m_r\}$. Then*

$$b_i(I) = \sum_{j=1}^{r} \binom{max(m_j) - 1}{i - 1} \tag{3.1}$$

We can see that the total Betti numbers $b_i(I)$ depend only on the maximum index of each generator. For a principal strongly stable ideal (or any strongly stable ideal generated entirely in degree d), we can rewrite this formula in terms of $q_i^d(I)$.

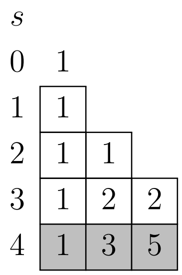
Proposition 3.8.1 (Proposition 6.4 of [1])**.** *Suppose that I is a strongly stable ideal generated entirely in degree d. Then*

$$b_i(I) = \sum_{j=1}^{n} \binom{j-1}{i-1} q_j^d(I) \tag{3.2}$$

Example 3.8.2. Let $w = (1, 1, 1)$, $m = x_1x_2x_3^2$, and

$$I = \overline{m} = (x_1x_2x_3^2, x_1^2x_3^2, x_1x_2^2x_3, x_1^2x_2x_3, x_1^3x_3, x_1x_2^3, x_1^2x_2^2, x_1^3x_2, x_1^4)$$

The Catalan diagram is shown in Figure **??** below.



s			
0	1		
1	1		
2	1	1	
3	1	2	2
4	1	3	5

Figure 3.6: $C_{(1,1,1),x_1x_2x_3^2}$

The last row of $C_{(1,1,1),x_1x_2x_3^2}$ tells us that we have $q_1^4(I) = 1, q_2^4(I) = 3, q_3^4(I) = 5$. We can apply this information to Equation (3.2) to obtain the Betti numbers.

$$\begin{aligned} b_1(I) &= 1\binom{0}{0} + 3\binom{1}{0} + 5\binom{2}{0} = 9 \\ b_2(I) &= 1\binom{0}{1} + 3\binom{1}{1} + 5\binom{2}{1} = 13 \\ b_3(I) &= 1\binom{0}{2} + 3\binom{1}{2} + 5\binom{2}{2} = 5 \end{aligned}$$

We can verify this with the Betti table obtained by Macaulay2.

```
i11 : I = borelClosure(ideal(x_1*x_2*x_3^2))

                 2   2 2     2     2       3       3   2 2   3     4
o11 = ideal (x x x , x x , x x x , x x x , x x , x x , x x , x x , x )
             1 2 3   1 3   1 2 3   1 2 3   1 3   1 2   1 2   1 2   1

o11 : Ideal of QQ[x ..x ]
                   1   3

i12 : betti res I

             0 1  2 3
```

```
o12 = total: 1 9 13 5

          0: 1 .  . .

          1: . .  . .

          2: . .  . .

          3: . 9 13 5


o12 : BettiTally
```

The resolution in the previous example is linear (all nonzero Betti numbers are in the same row of Betti table) and $b_i(I) = b_{i,i+d}(I)$. This happens because principal strongly stable ideals are generated entirely in degree d. Since a principal w-stable ideal $I = \overline{m}$ can have generators of multiple degrees greater than or equal to $d = deg(m)$, we can't use Proposition 3.8.1 for arbitrary w-stable ideals. We combine the Koszul complexes corresponding to the $q_j^d(I)$ to obtain a new formula (Theorem 12) for the Poincaré series of w-stable ideals. For a principal w-stable ideal, we can write the Poincaré series (and therefore the graded Betti numbers) in terms of $C_{w,m}$ (Corollary 13).

Theorem 12. *Let $I \subset S$ be a w-stable ideal. Then the graded Poincaré series for S/I is*

$$P_I^S(u,t) = \sum_{m \in G(I)} ut^{deg(m)} \prod_{k=1}^{max(m)-1} (1 + ut^{w_k}) \tag{3.3}$$

Proof. First, we will denote the set of subsets of w truncated at q with length p by $A_{p,q} \subset \mathcal{P}(\{w_1, \dots, w_q\})$.

Fix a monomial $m \in G(I)$. Let $d = deg(m)$ and $q = max(m) - 1$. We will now describe the Koszul complex arising from m. The Eliahou-Kervaire resolution tells us that $\beta_{i,j}(I)$ counts the number of $S(-j)$ appearing in the i^{th} piece of the resolution. Since m contributes the resolution of

$(x_1, \dots, x_q)$ shifted by d, we have the Koszul complex below

$$S(-d) \leftarrow \bigoplus_{a \in A_{1,q}} S(-sum(a) - d) \leftarrow \bigoplus_{a \in A_{2,q}} S(-sum(a) - d) \leftarrow \cdots \leftarrow \bigoplus_{a \in A_{q,q}} S(-sum(a) - d)$$

From this, we can see that the contribution to $\beta_{i,j}$ from m is exactly the number of partitions of $j - d$ using distinct entries from $\{w_1, w_2, \dots, w_q\}$. The generating function for the number of partitions using distinct entries from $\{w_1, w_2, \dots, w_q\}$ is given by

$$\prod_{k=1}^{q} (1 + ut^{w_k}),$$

where the u exponent corresponds to the number of summands and the t exponent corresponds to the size of the partition $j - d$. In order to make the t exponent correspond to j, we multiply the product by t^d. Similarly, since the i^{th} piece of the resolution corresponds to partitions with $i - 1$ summands, we multiply the product by u.

$$ut^d \prod_{k=1}^{q} (1 + ut^{w_k}).$$

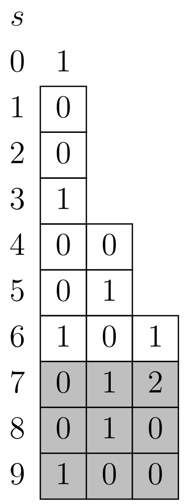
The expression above is the generating function for the graded Betti numbers of $(x_1, \dots, x_q)$. Now, we can sum over $m \in G(I)$ as in the Eliahou-Kervaire formula and we're done. □

Corollary 13. *If $I = \overline{m}$ is a principal w-stable ideal with $d = deg(m)$, then*

$$P_I^S(t, u) = \sum_{a=d}^{d+max(w)-1} \sum_{b=1}^{n} C_{w,m}(a, b) \cdot ut^a \prod_{k=1}^{b-1} (1 + ut^{w_k}) \tag{3.4}$$

While the Poincaré series for a principal w-stable ideal is not as aesthetically pleasing as (3.3), it allows us to compute $P_I^S(t, u)$ directly from the Catalan diagram.

Example 3.8.3. Let $w = (3, 2, 1)$, $m = x_1 x_2 x_3^2$, and $I = \overline{m} = (x_1 x_2 x_3^2, x_1^2 x_3, x_1 x_2^2, x_1^2 x_2, x_1^3)$. Notice that $deg(m) = 7$ and $deg(x_1^2 x_2) = 8$ and $deg(x_1^3) = 9$. Now, we build the Catalan diagram.



s			
0	1		
1	0		
2	0		
3	1		
4	0	0	
5	0	1	
6	1	0	1
7	0	1	2
8	0	1	0
9	1	0	0

Figure 3.7: $C_{(3,2,1),x_1x_2x_3^2}$

To obtain the Catalan diagram above, we add $max(w_i) - 1 = 3 - 1 = 2$ rows and fill them in as usual except we ignore any entries in rows $\geq d = deg(m) = 7$. Then we can see both $deg(m_j)$ and $max(m_j)$ for each of the five generators directly from the extended Catalan diagram.

Now, we can use Corollary 13 to compute the graded Poincaré series.

$$\begin{aligned} P_I^S(t,u) &= \sum_{a=7}^{9}\sum_{b=1}^{3} C_{w,m}(a,b)\cdot ut^a \prod_{k=1}^{b-1}(1+ut^{w_k}) \\ &= ut^7(1+ut^3) + 2ut^7(1+ut^3)(1+ut^2) + ut^8(1+ut^3) + ut^9 \\ &= 3ut^7 + ut^8 + ut^9 + 2u^2t^9 + 3u^2t^{10} + u^2t^{11} + 2u^3t^{12} \end{aligned}$$

The `poincareSeries` method from `wStableIdeals` allows us to compute the Poincaré series of I, which we can compare with the Betti table above.

```
i13 : poincareSeries(x_1*x_2*x_3^2,Weights=>{3,2,1})

         12 3     11 2      10 2      9 2     9      8       7
o13 = 2t   u  + t   u  + 3t   u  + 2t u  + t u + t u + 3t u
```

```
o13 : ZZ[t..u]
```

We can check that this agrees with the Betti table.

```
i14 : w = {3,2,1}

o14 = {3, 2, 1}

o14 : List

i15 : S = K[x_1,x_2,x_3,Degrees=>w]

o15 = S

o15 : PolynomialRing

i16 : I = borelClosure(ideal(x_1*x_2*x_3^2),Weights=>w)

                  2    2        2    2      3
o16 = ideal (x x x , x x , x x , x x , x )
              1 2 3   1 3   1 2   1 2   1

o16 : Ideal of S

i17 : betti res I

              0 1 2 3
o17 = total:  1 5 6 2
           0: 1 . . .
```

```
           1: . . . .

           2: . . . .

           3: . . . .

           4: . . . .

           5: . . . .

           6: . 3 . .

           7: . 1 2 .

           8: . 1 3 .

           9: . . 1  2


o17 : BettiTally
```

If we fix a that different choices of weight vector produce different Betti tables, but the total Betti numbers $b_i(I)$ are unaffected.

```
i18 : w = {1,1,1}


o18 = {1, 1, 1}


o18 : List


i19 : S = K[x_1,x_2,x_3,Degrees=>w]


o19 = S


o19 : PolynomialRing


i20 : I = borelClosure(ideal(x_1*x_2*x_3^2),Weights=>{3,2,1})


                 2  2      2  2     3
```

```
o20 = ideal (x x x , x x , x x , x x , x )
              1 2 3   1 3   1 2   1 2   1

o20 : Ideal of S

i21 : betti res I

             0 1 2 3
o21 = total: 1 5 6 2
          0: 1 . . .
          1: . . . .
          2: . 4 4 1
          3: . 1 2 1

o21 : BettiTally
```

Since the total Betti numbers do not depend on the weight vector, a good choice of weight vector w (where I is w-stable and the cardinality of $Bgens_w(I)$ is small relative to the cardinality of $Bgens(I)$) reduces the number of computations required to obtain the (total) Betti numbers.

3.9 Principal Cones

Question 3.9.1. Fix a strongly stable ideal I. For which weight vectors (if any) is I principally w-stable?

This section is dedicated to exploring the question above. Theorem 14 provides an answer to this question in the form of an algorithm. This algorithm produces the space of weight vectors for which an Artinian strongly stable ideal is principally w-stable.

Using the language of n-ary trees, we can ask if a given tree $\mathcal{T}$ is equal to $\mathcal{T}_{w,m}$ for some m and w. It's clear what m should be. It must be the smallest lex vertex in $sink(\mathcal{T})$. Determining w is a more difficult task and we devote the rest of the section to it.

Definition 3.9.2. Let I be a strongly stable ideal. Define $\mathcal{T}_I$ as the tree with base vertex 1 and a directed edge (v, vx_j) if $vx_j = trunc_{deg_{\Bbbk}(vx_j)}(g)$ for some $g \in G(I)$ with $g \neq v$.

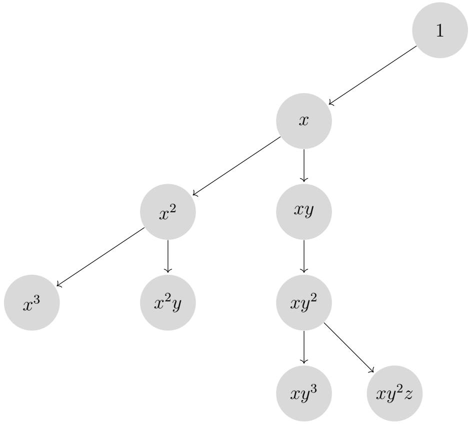


Figure 3.8: $\mathcal{T}_I$ for $I = (x^3, x^2y, xy^3, xy^2z)$

We can use $\mathcal{T}_I$ to obtain the space of weight vectors for which I is principally w-stable. First, we introduce some notation.

Lemma 3.9.3. *Let* $m = \mathbf{x}^{(a_1,\ldots,a_n)}$ *and* $v = \mathbf{x}^{(b_1,\ldots,b_n)}$ *be monomials. For any* $1 \leq k \leq n$, $k = max(trunc_{deg_w(v)+1}(\psi(m)))$ *if and only if both of the following hold.*

$$\sum_{i=1}^{k-1} w_i a_i \leq \sum_{i=1}^{n} w_i b_i$$
$$\sum_{i=1}^{k} w_i a_i > \sum_{i=1}^{n} w_i b_i$$

Proof. First, we get a condition to ensure that k is small enough.

$$\begin{aligned} k \leq max(trunc_{deg_w(v)+1}(\psi(m))) &\iff \sum_{i=1}^{k-1} w_i a_i < \Big(\sum_{i=1}^{n} w_i b_i\Big) + 1 \\ &\iff \sum_{i=1}^{k-1} w_i a_i \leq \sum_{i=1}^{n} w_i b_i. \end{aligned}$$

We get a second condition when we force k to be large enough.

$$\begin{aligned} k \geq max(trunc_{deg_w(v)+1}(\psi(m))) &\iff \sum_{i=1}^{k} w_i a_i \geq \Big(\sum_{i=1}^{n} w_i b_i\Big) + 1 \\ &\iff \sum_{i=1}^{k} w_i a_i > \sum_{i=1}^{n} w_i b_i. \end{aligned}$$

$\square$

When k is the largest index such that (v, vx_k) is an edge of $\mathcal{T}_I$, denote the space of weight vectors satisfying the two inequalities in Lemma 3.9.3 by $\tau_{m,v}$. Intuitively, $\tau_{m,v}$ is the space of weight vectors for which the branching structure at v is compatible with $\mathcal{T}_{w,m}$.

Let $u = \mathbf{x}^\alpha, v = \mathbf{x}^\beta \in S$ be monomials. Then let $\sigma_{u,v}$ denote the half-space defined by

$$\sum_{i=1}^{n} w_i(b_i - a_i) \geq 0.$$

Similarly, let $\rho_{u,v}$ denote the half-space

$$\sum_{i=1}^{n} w_i(b_i - a_i) < 0.$$

Intuitively, $\sigma_{m,v}$ is the half-space of weight vectors for which $deg_w(m) \leq deg_w(v)$ and $\rho_{m,v}$ is the space for which $deg_w(m) > deg_w(v)$.

Theorem 14. *Let I be a strongly stable ideal with $m \in G(I)$ the smallest lex generator of I. Let V be the vertex set of $\mathcal{T}_I$. For each $u \in V \setminus sink(\mathcal{T}_\mathcal{I})$, let k_u denote the largest variable branching from u. Denote the set of vertices which share an edge with a vertex in $sink(\mathcal{T}_I)$ by $subsink(\mathcal{T}_I)$.*

Then I is principally w-stable if and only if

$$w \in \mathcal{P}_I = \left(\bigcap_{v \in sink(\mathcal{T}_I)} \sigma_{m,v} \right) \cap \left(\bigcap_{u \in subsink(\mathcal{T}_I)} \rho_{m,v} \right) \cap \left(\bigcap_{u \in V \setminus sink(\mathcal{T}_I)} \tau_{m,u,k_u} \right).$$

Proof. The first intersection makes sure that degrees of sinks are larger than or equal to m. The second intersection ensures that the degrees of subsinks are less than the degree of m, and the third intersection guarantees that the branching structure of $\mathcal{T}_I$ and $\mathcal{T}_{w,m}$ agree at each branching vertex. □

We will refer to $\mathcal{P}_I$ as the *principal region* of I and its closure as the *principal cone*. Any lattice point in the interior of the principal cone gives a weight vector w for which $I = \overline{m}$ where m is the smallest lex monomial in $G(I)$.

We can use the `principalCone` method to compute the principal cone of a strongly stable ideal.

```
i20 : S = K[x,y,z]

o20 = S

o20 : PolynomialRing

i21 : I = ideal(x^3,x^2*y,x*y^3,x*y^2*z)

              3   2       3     2
o21 = ideal (x , x y, x*y , x*y z)

o21 : Ideal of S

i22 : P = principalCone I
```

```
o22 = Cone{...1...}

o22 : Cone

i22 : rays P

o22 = | 1 2 2 |
      | 1 1 1 |
      | 0 0 1 |

               3       3
o22 : Matrix ZZ  <-- ZZ

i23 : principalWeightVector I

o23 = | 5 |
      | 3 |
      | 1 |

               3       1
o23 : Matrix ZZ  <-- ZZ
```

The `wStableIdeals` package uses the method `principalCone` to compute the principal cone of I. Then `principalWeightVector` returns a vector in the interior of P_I.

3.10 Applications of w-Stable Ideals

In [18], Onn and Sturmfels describe *corner cuts* as subsets of nonnegative integer vectors which are linearly separable from their complement. Theorems 2.0 and 5.0 of [18] show that vanishing ideals of generic configurations of k points in affine n-space have corner cut initial ideals. This is one class of examples which seem to be well-suited for computations as w-stable ideals, as illustrated below.

Table 3.1: Weight vectors and generating sets for initial ideals in Figure **??**

	w	I	$Bgens(I)$	$Bgens_w(I)$
a	$(8,8,1)$	(x,y,z^8)	$\{y,z^8\}$	$\{z^8\}$
b	$(7,6,1)$	(x,y^2,yz,z^7)	$\{x,yz,z^7\}$	$\{z^7\}$
c	$(5,5,1)$	(x^2,xy,y^2,xz,yz,z^6)	$\{yz,z^6\}$	$\{z^6\}$
d	$(6,4,1)$	(x,y^2,yz^2,z^6)	$\{x,y^2,yz^2,z^6\}$	$\{z^6\}$
e	$(4,3,1)$	(x^2,xy,y^2,xz,yz^2,z^5)	$\{xz,y^2,yz^2,z^5\}$	$\{z^5\}$
f	$(10,5,2)$	(x,y^2,yz^3,z^5)	$\{x,y^2,yz^3,z^5\}$	$\{z^5\}$
g	$(8,3,2)$	(x,y^3,y^2z,yz^3,z^4)	$\{x,y^2z,z^4\}$	$\{z^4\}$
h	$(5,3,2)$	$(x^2,xy,xz,y^3,y^2z,yz^2,z^4)$	$\{xz,yz^2,z^4\}$	$\{yz^2,z^4\}$
i	$(2,2,1)$	$(x^2,xy,y^2,xz^2,yz^2,z^4)$	$\{y^2,yz^2,z^4\}$	$\{z^4\}$

Figure **??** shows the (Schlegel diagram of the) Gröbner fan of $J = (F_1, F_2, F_3)$ where the F_i are generic, non-homogeneous polynomials of degree 2. The 9 maximal cones with interior points contained in the strict interior of the region where $x > y > z$ are labelled $a, \ldots, i$. Note that b does not appear in the figure but is used to denote the maximal cone adjacent to cones a, c, and d. One can check that each initial ideal is w-stable with respect to the corresponding weight vector in the interior of its maximal cone. This motivates the following.

Table 3.1 shows a weight vector w in the interior, the ideal $I = in_<(J)$, $Bgens(I)$, and $Bgens_w(I)$ for each maximal cone. In all cases except h, the initial ideal I can be expressed as a principal w-stable ideal by using the corresponding weight vector in the first column. Clearly, an Artinian principal w-stable ideal is a corner cut. However, the next example shows that no weight vector exists for which the corner cut $(x^2, xy, xz, y^3, y^2z, yz^2, z^4)$ is principally w-stable.

Example 3.10.1. The ideal $I = (x^2, xy, xz, y^3, y^2z, yz^2, z^4) \subset \mathbb{K}[x, y, z]$ is a corner cut, because it is the initial ideal of a generic, non-homogeneous ideal (corresponding to h in Table 3.1). Since z^4 cannot generate yz^2 without also generating y^2, there exists no weight vector w such that $I = \overline{z^4}$.

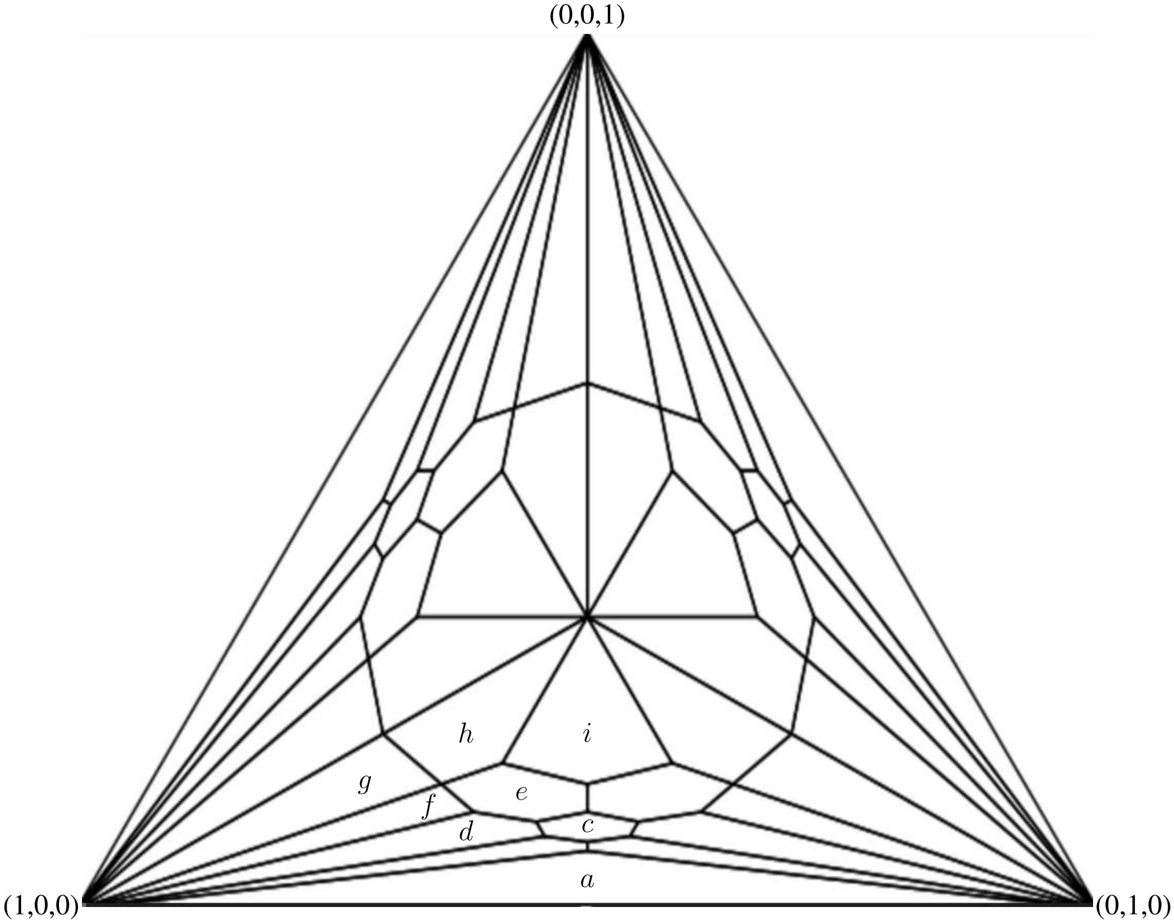


Figure 3.9: Gröbner fan of generic non-homogeneous ideal of type $(2, 2, 2)$

Question 3.10.2. Assume $I \subset \mathbb{K}[x_1, \ldots, x_n]$ is a corner cut. Is there an upper bound on the minimal number of weighted Borel generators (for a suitable choice of weight vector) for I?

Recall that a monomial order $<$ on $S = \mathbb{K}[x_1, \ldots, x_n]$ is a *total* order on the set of monomials which respects multiplication ($u \leq v \implies ut \leq vt$ for any monomials t, u, v). For a given monomial order $<$ and a fixed ideal $I \subset S$, one can obtain the monomial ideal $in_<(I)$. A weight vector $w = (w_1, \ldots, w_n)$ gives a *partial* order on the set of monomials. Note that the initial ideal of I with respect to w, denoted $in_w(I)$ might not be a monomial ideal. For background on initial ideals and their relationship to the Gröbner fan, see [19].

Conjecture 3.10.3. Fix a generic non-homogeneous ideal I and a monomial order $<$. If $w \in \mathbb{N}^n_{>0}$ is a weight vector and $in_w(I) = in_<(I)$, then $in_<(I)$ is w-stable.

3.11 Conclusion

Restrictions on the structure of monomial ideals can reduce the number of monomial generators needed to describe a fixed monomial ideal I. Francisco, Mermin, and Schweig show how to use the structure of strongly stable ideals to compress the generating set from the classic monomial generators $G(I)$ to a subset $Bgens(I)$. In this paper, we showed how a choice of weight vector w restricts the set of strongly stable ideals to a subset of monomial ideals called w-stable ideals. This restriction allows us to describe w-stable ideals completely in terms of their weighted Borel generators $Bgens_w(I)$. In summary, for a w-stable ideal I, we have

$$Bgens_w(I) \subseteq Bgens(I) \subseteq G(I)$$

In Section 3.3, we showed that computations on w-stable ideals can be decomposed into computations on principal w-stable ideals whose combinatorial structure was examined in Section 3.6. Formulas for the Stanley decomposition, Hilbert series, and Betti numbers of principal w-Stable ideals were developed in Sections 3.4, 3.5, and 3.8, respectively. In Section 3.9, we showed how to compute the space of weight vectors for which a given strongly stable ideal is principally w-stable. In Section 3.10, we explored generic, non-homogeneous initial ideals and presented evidence that using a weight vector w corresponding to the given monomial order produces a w-stable ideal (Conjecture 3.10.3).

The following section is the code for the Macaulay2 package wStableIdeals.m2 which was used throughout the chapter.

3.12 wStableIdeals.m2

```
newPackage(
    "wStableIdeals",
```

```
    Version => "1.0",
    Date => "July 2024",
    Headline => "Computations for w-Stable Ideals",
    Authors => {{   Name => "Seth Ireland",
                    Email => "seth.ireland@colostate.edu",
                    HomePage => "sethireland.com"   }},
    AuxiliaryFiles => false,
    DebuggingMode => false,
    PackageExports => {"Graphs","Polyhedra","SRdeformations"},
    Keywords => {"Combinatorial Commutative Algebra"}
    )

export {
    "borelClosure",
    "borelGens",
    "iswStable",
    "treeFromMonomial",
    "catalanDiagram",
    "poincareSeries",
    "principalCone",
    "principalWeightVector"
    }


--------------------------------------------------------
--------------------------------------------------------
-- CONSTRUCTING W-STABLE IDEALS AND BASIC COMPUTATIONS
--------------------------------------------------------
--------------------------------------------------------

psiMap = method();
psiMap := (S,R,degs) -> (
    L := {};
```

```
    n := numgens S;
    ys := gens R;
    for i from 0 to n-1 do (
        L = append(L,ys_i^(degs_i));
        );
    psi := map(R,S,L)
    );

borelClosure = method(Options => {Weights => null});
borelClosure Ideal := Ideal => opts -> I -> (
    S := ring I;
    n := numgens S;
    w := if opts.Weights === null
        then (for i from 1 to n list 1) else opts.Weights;
    startIdeal := monomialIdeal I;
    K := coefficientRing S;
    R := K[vars (1..n)];
    psi := psiMap(S,R,w);
    psI := monomialIdeal psi(startIdeal);
    psIbar := borel psI;
    Ibar := preimage_psi(psIbar)
    );

iswStable = method();
iswStable (Ideal,List) := Boolean => (I,w) -> (
    Ibar := borelClosure(I,Weights=>w);
    I==Ibar);

hatShift = method();
hatShift RingElement := RingElement => mon -> (
    expVec := (exponents mon)_0;
    n := #expVec;
    hatExpVec := {};
```

```
    count := 0;
    for i from 0 to n-1 do (
        count = count + expVec_i;
        hatExpVec = append(hatExpVec,count);
        );
    monHat := vectorToMonomial(vector hatExpVec,ring mon)
    );

borelGens = method(Options => {Weights => null});
borelGens Ideal := List => opts -> J -> (
    S := ring J;
    K := coefficientRing S;
    n := numgens S;
    R := K[vars(1..n)];
    w := if opts.Weights===null
        then apply(n,i->1) else opts.Weights;
    psi := psiMap(S,R,w);
    I := psi(J);
    G := (entries gens I)_0;
    Ghat := {};
    for g in G do (
        Ghat = append(Ghat,hatShift(g));
        );
    Ihat := ideal(Ghat);
    bgensHat := set (entries mingens Ihat)_0;
    bgens := {};
    for u in G do (
        uHat := hatShift(u);
        if bgensHat#?uHat then bgens = append(bgens,u);
        );
    Bgens := for b in bgens list
        ((gens (preimage_psi(ideal(b))))_0)_0;
    if not iswStable(J,w) then (
```

```
        print(toString(J) | " is not " | toString(w) | "-stable.");
        Bgens = null;
        );
    Bgens);


------------------------------------------------------------
------------------------------------------------------------
-- CATALAN DIAGRAMS
------------------------------------------------------------
------------------------------------------------------------

catalanDiagram = method(Options => {Weights=>null});
catalanDiagram RingElement := Matrix => opts -> m -> (
    S := ring m;
    K := coefficientRing S;
    n := numgens S;
    R := K[vars(1..n)];
    w := if opts.Weights===null
        then apply(n,i->1) else opts.Weights;
    psi := psiMap(S,R,w);
    M := psi(m);
    fM := factoredIndices(M);
    weirdfM := join(fM,apply(w_0-1,i->n-1));
    d := #fM;
    C := mutableMatrix(ZZ,#weirdfM+1,n);
    C_(0,0) = 1;
    for i from 1 to numrows(C)-1 do (
        for j from 0 to weirdfM_(i-1) do (
            if (0<=i-w_j and i-w_j<d) then (
                if weirdfM_(i-w_j) >= j then (
                    rowAbove := for k from 0 to j
                        list C_(i-w_j,k);
```

```
                    C_(i,j) = sum(rowAbove);
                    );
                );
            );
        );
    matrix C);


poincareSeries = method(Options => {Weights=>null});
poincareSeries RingElement := RingElement => opts -> m -> (
    n := numgens ring m;
    w := if opts.Weights===null
        then apply(n,i->1) else opts.Weights;
    C := catalanDiagram(m,Weights=>w);
    runningSum := {};
    rng := ZZ[vars(19..20)];
    t := (gens rng)_0;
    u := (gens rng)_1;
    n := #w;
    d := numRows C - w_0;
    for a from d to d+w_0-1 do (
        for b from 1 to n do (
            thing := C_(a,b-1)*u*t^a*product(
                for k from 1 to b-1 list (1+u*t^(w_(k-1)))
                );
            runningSum = append(runningSum,thing);
            );
        );
    sum(runningSum));


--------------------------------------------------------
--------------------------------------------------------
```

```
-- CONSTRUCTING TREES
--------------------------------------------------------
--------------------------------------------------------

maxIndex = method();
maxIndex RingElement := ZZ => (m) -> (
    expVec := (exponents m)_0;
    maxNonzero := position(expVec,i->i!=0,Reverse=>true);
    maxNonzero);

factoredIndices = method();
factoredIndices RingElement := List => m -> (
    expVec := (exponents m)_0;
    factorList := {};
    n := #expVec;
    for i from 0 to n-1 do (
        for j from 0 to expVec_i-1 do (
            factorList = append(factorList,i);
        );
    );
    factorList);

treeFromMonomial = method(Options => {Weights=>null});
treeFromMonomial RingElement := Digraph => opts -> m -> (
    S := ring m;
    K := coefficientRing S;
    n := numgens S;
    w := if opts.Weights===null
        then apply(n,i->1) else (opts.Weights);
    R := K[vars(1..n)];
    psi := psiMap(S,R,w);
    psim := psi(m);
    fm := factoredIndices(psim);
```

```
    d := #fm;
    gs := gens S;
    trunk := for i from 0 to fm_0 list ({sub(1,S),gs_i});
    tree := digraph(trunk);
    tf := true;
    while tf do (
        leafs := for v in (sinks tree) list
            (if #factoredIndices(psi(v))<d then v else continue);
        if #leafs == 0 then tf = false;
        newLeafs := {};
        for leaf in leafs do (
            fleaf := factoredIndices(psi(leaf));
            maxBranch := fm_(#fleaf);
            minBranch := maxIndex(leaf);
            for i from minBranch to maxBranch do (
                newVert := leaf*gs_i;
                tree = addVertex(tree,newVert);
                tree = addEdge(tree,set{newVert,leaf});
                newLeafs = append(newLeafs,newVert);
                );
            );
        leafs = newLeafs;
        );
    tree);

factoredGens = method();
factoredGens Ideal := Matrix => I -> (
    G := sortLex((entries gens I)_0);
    -- start by making a matrix of factored indices of g\in G
    fG := {};
    fG2 := {};
    longest := 0;
    for g in G do (
```

```
        fg := factoredIndices(g);
        gLength := #fg;
        if gLength > longest then longest = gLength;
        fG = append(fG,fg);
        );
    for fg in fG do (
        extra := apply(longest-#fg,i->-1);
        fg2 := join(fg,extra);
        fG2 = append(fG2,fg2);
        );
    fG2);

treeFromIdeal = method();
treeFromIdeal Ideal := Digraph => I -> (
    S := ring I;
    n := numgens S;
    gs := gens S;
    G := factoredGens(I);
    k := #(G_0);
    branches := {};
    for m in G do (
        branches = append(branches,{sub(1,S),gs_(m_0)});
        for i from 0 to k-2 do (
            if m_(i+1) >= 0 then (
                someList := for j from 0 to i list gs_(m_j);
                prev := product(someList);
                next := prev*gs_(m_(i+1));
                branches = append(branches,{prev,next});
                );
            );
        );
    tree := digraph(toList set branches);
    tree);
```

```
--------------------------------------------------------
--------------------------------------------------------
-- CONVEX GEOMETRY
--------------------------------------------------------
--------------------------------------------------------

sortLex = method();
sortLex List := List => (A) -> (
    S := ring A_0;
    K := coefficientRing S;
    gs := gens S;
    S2 := K[gs,MonomialOrder=>Lex];
    A2 := for a in A list ( sub(a,S2) );
    A3 := sort A2;
    A4 := for a in A3 list ( sub(a,S) );
    A4);

fundRegion = n -> (
    Rays := {};
    for i from 0 to n-1 do (
        iRay := {};
        for j from 0 to i do (
            iRay = append(iRay,1);
            );
        for j from i+1 to n-1 do (
            iRay = append(iRay,0);
            );
        Rays = append(Rays,iRay);
        );
    coneFromVData(transpose(matrix Rays)));
```

```
subsink = method();
subsink Digraph := List => T -> (
    sink := set sinks T;
    verts := set vertices T;
    checkVerts := verts - sink;
    checkVerts = toList checkVerts;
    subsink := {};
    for u in checkVerts do (
        kids := toList children(T,u);
        for kid in kids do (
            if isSubset({kid},sink) then (
                subsink = append(subsink,u);
                );
            );
        );
    toList set subsink);

-- gets space where degree of v is larger or equal to degree of u
sigmaUV = method();
sigmaUV (RingElement,RingElement) := List => (u,v) -> (
    a := (exponents u)_0;
    b := (exponents v)_0;
    ineq := for i from 0 to #a-1 list ( b_i - a_i );
    ineq);

-- gets space where branching structure of T_{w,m} matches at v
tauMV = method();
tauMV (RingElement,RingElement,ZZ) := List => (m,v,k) -> (
    a := (exponents m)_0;
    b := (exponents v)_0;
    ineq1 := for i from 0 to k-2 list ( b_i - a_i );
    ineq2 := for i from k-1 to #a-1 list ( b_i );
```

```
    ineq := join(ineq1,ineq2);
    ineq);

principalCone = method();
principalCone Ideal := Cone => I -> (
    m := (sortLex((entries gens I)_0))_0;
    S := ring I;
    n := numgens S;
    gs := gens S;
    tree := treeFromIdeal(I);
    verts := vertices(tree);
    sink := sinks(tree);
    branchPoints := toList( set verts - set sink );
    ineqs := {};
    -- make sure every branch is correct
    for v in branchPoints do (
        k := 1;
        for j from 0 to n-1 do (
            if isSubset({v*gs_j},verts) then k = j+1;
            );
        ineqs = append(ineqs,tauMV(m,v,k));
        ineqs = append(ineqs,-1*tauMV(m,v,k+1));
        );
    -- make sure sinks have degree greater than or equal to m
    for v in sink do (
        ineqs = append(ineqs,sigmaUV(m,v));
        );
    subb := subsink tree;
    -- make sure subsinks have dgree less than m
    for u in subb do (
        ineqs = append(ineqs,sigmaUV(u,m));
        );
    f := fundRegion(n);
```

```
    hdata := matrix ineqs;
    returnCone := intersection(f,coneFromHData(hdata));
    if dim returnCone < n then (
        returnCone = coneFromVData transpose matrix {apply(n,i->0)}
        );
    returnCone);

principalWeightVector = method();
principalWeightVector Ideal := List => I -> (
    c := principalCone I;
    n := numgens ring I;
    r := rays c;
    A := (transpose matrix {apply(n,i->0)}) | r;
    p := convexHull A;
    i := 2;
    tf := true;
    returnPt := null;
    if dim c < n then (
        tf = false;
        print(toString I | " is not principally w-stable.")
        );
    while tf do (
        B := A*i;
        i = i+1;
        p = convexHull B;
        tf = (interiorLatticePoints p == {});
        if tf == false then returnPt = (interiorLatticePoints p)_0;
        );
    returnPt);


--------------------------------------------------------
--------------------------------------------------------
```

```
-- DOCUMENTATION
--------------------------------------------------------
--------------------------------------------------------

beginDocumentation()

doc ///
  Key
    wStableIdeals
  Headline
    Basic computations with w-stable ideals
  Description
    Text
      {\bf Overview:}

      w-stable ideals are a specialization of strongly stable ideals.

      {\bf References:}

      [Ire24] S. Ireland: Weighted Borel Generators, {\it},
      @BR{}@ Available at @HREF{"https://arxiv.org/abs/2408.04120"}@.

      [FMS11] C.A. Francisco, J. Mermin, J. Schweig: Borel generators,
      {\it Journal of Algebra}, 332(1), 522-542, 2011.
      @BR{}@ Available at @HREF{"https://arxiv.org/abs/1006.1436"}@.



      {\bf Key user functions:}

        {\it Weighted Borel Generators:}

          @TO borelClosure@ -- Compute the Borel closure of
```

```
                            -- a given monomial ideal

          @TO iswStable@ -- Test whether a monomial ideal is w-stable
                         -- with respect to a given weight vector

          @TO borelGens@ -- Compute the Borel generators
                         -- of a strongly stable ideal

        {\it Principal w-Stable Ideals:}

          @TO treeFromMonomial@ -- Compute the tree associated with
                                -- a principal w-stable ideal

          @TO catalanDiagram@ -- Compute the Catalan diagram associated
                              -- with a principal w-stable ideal

          @TO poincareSeries@ -- Compute the Poincare series
                              -- of a principal w-stable ideal

          @TO principalCone@ -- Compute the principal cone
                             -- of a strongly stable ideal

          @TO principalWeightVector@ -- Compute a weight vector w
                                     -- for which a given strongly
                                     -- stable ideal is
                                     -- principally w-stable
///


doc ///
  Key
    borelClosure
  Headline
```

```
    Compute the Borel closure of a monomial ideal
  Usage
    borelClosure I
  Inputs
    I : Ideal
  Outputs
    : Ideal
  Description
   Text
      Returns the Borel closure of the input ideal

   Example
     ZZ/101[x,y,z];
     borelClosure(ideal(x*y*z))
///

doc ///
  Key
    Weights
    [borelClosure,Weights]
  Headline
    Option to set the weight vector for the Borel closure
  Description
   Text
    This option can be used to specify the weight vector
    when taking a Borel closure

    The default is the vector of ones
    (corresponding to the classic Borel closure)

   Example
     ZZ/101[x,y,z];
     borelClosure(ideal(x*y*z),Weights=>{3,2,1})
```

```
///


doc ///
  Key
    iswStable
  Headline
    Test whether an ideal is w-stable for a given weight vector
  Usage
    iswStable(I,w)
  Inputs
    I : Ideal
    w : List
  Outputs
    : Boolean
  Description
   Text
      Returns Boolean value of whether I is w-stable

   Example
     ZZ/101[x,y];
     iswStable(ideal(y^4,x*y^2,x^2),{2,1})

   Example
     ZZ/101[x,y];
     iswStable(ideal(y^4,x*y^2,x^2),{3,1})
///


doc ///
  Key
    borelGens
  Headline
```

```
    Compute the Borel generators of a strongly stable ideal
  Usage
    borelGens I
  Inputs
    I : Ideal
  Outputs
    : List
  Description
   Text
      Returns the Borel generators of a strongly stable ideal

   Example
     ZZ/101[x,y];
     borelGens ideal(y^4,x*y^2,x^2)
///

doc ///
  Key
    Weights
    [borelGens,Weights]
  Headline
    Option to set the weight vector for computing
    weighted Borel generators
  Description
   Text
    This option can be used to specify the weight vector
    when computing Borel generators

    The default is the vector of ones
    (corresponding to the classic Borel generators)

    Given a weight vector for which the ideal is not
    w-stable, returns null
```

```
   Example
     ZZ/101[x,y];
     borelGens(ideal(y^4,x*y^2,x^2),Weights=>{2,1})

   Example
     ZZ/101[x,y];
     borelGens(ideal(y^4,x*y^2,x^2),Weights=>{3,1})
///


doc ///
  Key
    treeFromMonomial
  Headline
    Compute the n-ary tree associated with a given monomial
  Usage
    treeFromMonomial m
  Inputs
    m : RingElement
  Outputs
    : Digraph
  Description
   Text
      Computes the n-ary tree corresponding to m

   Example
     ZZ/101[x,y];
     treeFromMonomial x*y^3

   Example
     ZZ/101[x,y,z];
     treeFromMonomial y^2*z
```

```
///

doc ///
  Key
    Weights
    [treeFromMonomial,Weights]
  Headline
    Option to set the weight vector for treeFromMonomial
  Description
   Text
    This option can be used to specify the weight vector
    when computing n-ary trees associated with monomials

    The default is the vector of ones
    (corresponding to the classic Borel generators)

   Example
     ZZ/101[x,y];
     treeFromMonomial(x*y^3,Weights=>{3,1})

   Example
     ZZ/101[x,y,z];
     treeFromMonomial(y^2*z,Weights=>{4,2,1})
///



doc ///
  Key
    catalanDiagram
  Headline
    Compute the Catalan diagram associated with a given monomial
  Usage
    catalanDiagram m
```

```
  Inputs
    m : RingElement
  Outputs
    : Matrix
  Description
   Text
      Computes the Catalan diagram corresponding to a given monomial

   Example
     ZZ/101[x,y];
     catalanDiagram x*y^3

   Example
     ZZ/101[x,y,z];
     catalanDiagram y^2*z
///

doc ///
  Key
    Weights
    [catalanDiagram,Weights]
  Headline
    Option to set the weight vector for treeFromMonomial
  Description
   Text
    This option can be used to specify the weight vector
    when computing Catalan diagrams

    The default is the vector of ones
    (corresponding to the classic Catalan diagrams)

   Example
     ZZ/101[x,y];
```

```
      catalanDiagram(x*y^3,Weights=>{3,1})

    Example
      ZZ/101[x,y,z];
      catalanDiagram(y^2*z,Weights=>{4,2,1})
///



doc ///
  Key
    poincareSeries
  Headline
    Compute the Poincare series associated with a given monomial
  Usage
    catalanDiagram m
  Inputs
    m : RingElement
  Outputs
    : RingElement
  Description
   Text
      Computes the Poincare series of the principal
      strongly stable ideal generated by m

   Example
     ZZ/101[x,y];
     poincareSeries x*y^3

   Example
     ZZ/101[x,y,z];
     poincareSeries y^2*z
///
```

```
doc ///
  Key
    Weights
    [poincareSeries,Weights]
  Headline
    Option to set the weight vector for poincareSeries
  Description
   Text
    This option can be used to specify the weight vector
    when computing Poincare series

    The default is the vector of ones
    (corresponding to the standard grading)

   Example
     ZZ/101[x,y];
     poincareSeries(x*y^3,Weights=>{3,1})

   Example
     ZZ/101[x,y,z];
     poincareSeries(y^2*z,Weights=>{4,2,1})
///



doc ///
  Key
    principalCone
  Headline
    Compute the principal cone of a given strongly stable ideal
  Usage
    principalCone I
  Inputs
    I : Ideal
```

```
  Outputs
    : Cone
  Description
   Text
      Computes the principal cone of I

   Example
     ZZ/101[x,y];
     principalCone(ideal(y^4,x*y^2,x^2))

   Example
     ZZ/101[x,y,z];
     principalCone(ideal(x^3,x^2*y,x*y^3,x*y^2*z))
///



doc ///
  Key
    principalWeightVector
  Headline
    Compute a principal weight vector
  Usage
    principalCone I
  Inputs
    I : Ideal
  Outputs
    : Cone
  Description
   Text
      Computes a weight vector for which a given strongly stable ideal
      is principally w-stable (if one exists)

      This method gets a weight vector from the
```

```
      interior of the principal cone

   Example
     ZZ/101[x,y];
     principalWeightVector ideal(y^4,x*y^2,x^2)

   Example
     ZZ/101[x,y,z];
     principalWeightVector ideal(x^3,x^2*y,x*y^3,x*y^2*z)
///



---------------------------------------------------------
---------------------------------------------------------
-- TESTS AND END
---------------------------------------------------------
---------------------------------------------------------



TEST ///
        S = ZZ/101[x,y,z];
        G = {z^5};
        w = {4,3,1};
        I = borelClosure(ideal(G),Weights=>w);
        assert ( borelGens(I,Weights=>w) == G )
///

end

restart
installPackage "wStableIdeals"
check "wStableIdeals"
```

Chapter 4

Strongly Stable Partitions

4.1 Introduction

Artinian monomial ideals in $K[x_1, \dots, x_d]$ correspond naturally to d-dimensional partitions by considering the monomials which are not in the ideal. We can think of these as d-dimensional stacks of blocks in a (d-dimensional) corner. Totally symmetric partitions are d-dimensional partitions which are fixed by the symmetric group S_d.

In this chapter, we define strongly stable partitions (Definition 4.2.5), show that they naturally correspond to strongly stable ideals (Proposition 4.4.5), and finally give an explicit bijection with totally symmetric partitions which preserves the side length n of the smallest box containing the partitions (Theorem 15). Note that Klivans was aware of the correspondence between totally symmetric partitions and shifted simplicial complexes in 2003 [20].

Remark 4.1.1. The reader should be aware that the notation in this chapter is not affiliated with previous notation.

4.2 d-dimensional Partitions

Let d be a positive integer and let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

Definition 4.2.1. A *d-dimensional partition* is a finite subset $P \subset \mathbb{N}^d$ such that

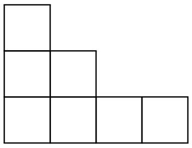
$$(a_1, a_2, \dots, a_j, \dots, a_d) \in P \implies (a_1, a_2, \dots, a_j - 1, \dots, a_d) \in P$$

for all $1 \leq j \leq d$ if $a_j > 0$. Refer to the elements of a partition as *cells*.

Denote the set of d-dimensional partitions with largest coordinate of any cell equal to $n - 1$ by $\mathcal{P}_d(n)$. We think of these as d-dimensional partitions which fit inside a d-dimensional box of side length n and also touch at least one edge of the box. Note that the "boxes" we refer to are

$n \times \cdots \times n$ boxes of dimension d (every side of the box is the same length). Integer partitions correspond exactly with the 2-dimensional partitions.

Example 4.2.2. For example, consider the integer partition 7=3+2+1+1. We can represent the integer partition with the Ferrers diagram below,



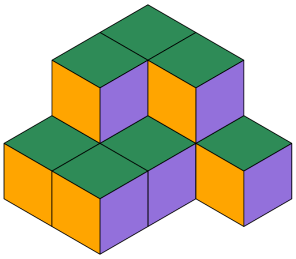
which depicts the 2-dimensional partition

$$P = \{(0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(3,0)\} \in \mathcal{P}_2(4).$$

Example 4.2.3. A 3-dimensional partition is commonly called a *plane partition*. An example of a plane partition of 10 is

$$P = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,1,0),(1,0,1),(1,2,0),(0,2,0),(2,0,0)\} \in \mathcal{P}_3(3)$$

which we can visualize with the diagram below



Plane partitions are also commonly represented in matrix notation. In matrix notation, the partition above could be written

$$\begin{array}{ccc} 2 & 2 & 1 \\ 2 & 1 & \\ 1 & 1 & \end{array}$$

Definition 4.2.4. Let P be a d-dimensional partition. For a cell $\alpha = (a_1, \dots, a_d) \in P$, define the *$j$th arm length* to be the largest integer h_j such that $\alpha = (a_1, a_2, \dots, a_j + h_j, \dots, a_d) \in P$. Denote the vector of arm lengths by $H(\alpha) = (h_1, \dots, h_d)$ and call this the *Hook vector* of α.

Definition 4.2.5. A *strongly stable partition* is a partition for which every cell's Hook vector is weakly increasing.

We will denote the set of strongly stable partitions which fit inside a box of side length n by

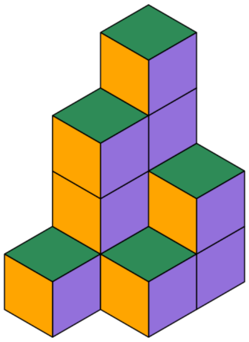
$$\tilde{\mathcal{B}}_d(n) := \{P \mid P \in \mathcal{P}_d(n) \text{ and } P \text{ is strongly stable}\}$$

Example 4.2.6. An example of a 2-dimensional strongly stable partition $P \in \tilde{\mathcal{B}}_2(7)$ is given below with Hook vectors given inside each cell.

0,0			
0,1			
0,2			
1,3	0,0		
2,4	1,1	0,0	
2,5	1,2	0,1	
3,6	2,3	1,2	0,0

Note that 2-dimensional strongly stable partitions are exactly the integer partitions with no repeats. These are commonly called *strict* partitions.

Example 4.2.7. An example of a 3-dimensional strongly stable partition $P \in \tilde{\mathcal{B}}_3(4)$ is given below. One can verify that the Hook vectors of each cell are weakly increasing.

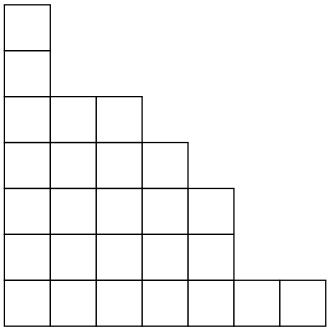


Definition 4.2.8. A *totally symmetric partition* is a partition for which $\alpha \in P \implies \pi(\alpha) \in P$ for all $\pi \in S_d$.

We denote the set of totally symmetric partitions which fit inside a box of side length n by

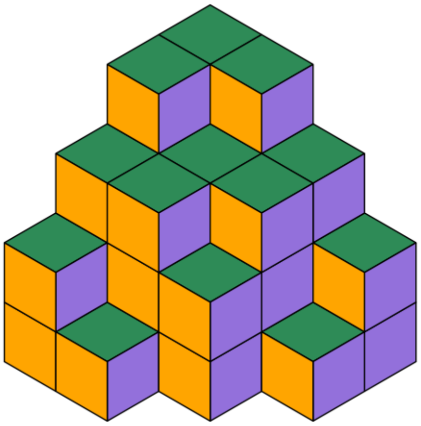
$$\tilde{\mathcal{T}}_d(n) := \{P \mid P \in \mathcal{P}_d(n) \text{ and } P \text{ is totally symmetric}\}$$

Example 4.2.9. An example of a 2-dimensional totally symmetric partition $P \in \tilde{\mathcal{T}}_2(7)$ is given below.



Note that 2-dimensional totally symmetric partitions are commonly called *self-conjugate* partitions.

Example 4.2.10. An example of a 3-dimensional totally symmetric partition (totally symmetric plane partition) $P \in \tilde{\mathcal{T}}_3(4)$ is shown below.



4.3 Symmetric Monomial Ideals

Definition 4.3.1. A monomial ideal I is *symmetric* if it is closed under the action of the symmetric group S_d on the variables.

Proposition 4.3.2. *A monomial ideal I is symmetric iff $G(I)$ is closed under the action of the symmetric group S_d on the variables.*

Proof. ($\Rightarrow$) Let $g \in G(I)$ and $\pi \in S_d$. Since I is symmetric, $\pi(g) \in I$. To see that $\pi(g) \in G(I)$, suppose not. Then there exists j so that $\frac{\pi(g)}{x_j} \in I$. It follows that $\pi^{-1}(\frac{\pi(g)}{x_j}) = \frac{g}{x_{\pi^{-1}(j)}}$ so $g \notin G(I)$. Contradiction.

($\Leftarrow$) Let $u \in I$. Then there exists some $g \in G(I)$ so that $g \mid u$. It follows that $\pi(g) \mid \pi(u)$ for all $\pi \in S_d$, so $\pi(u) \in I$. $\square$

Given a set of monomials $A = \{\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_r}\}$, we can *symmetrize* this set of monomials by letting S_d act on the variables. Denote this set by

$$sym(A) := \{\mathbf{x}^{\pi(\alpha_j)} \mid \mathbf{x}^{\alpha_j} \in A; \pi \in S_d\}$$

When $A = \{\mathbf{x}^{\alpha}\}$ consists of a single monomial, we refer to $sym(A)$ as the *orbit* of $\mathbf{x}^{\alpha}$.

Definition 4.3.3. A monomial $\mathbf{x}^{\alpha}$ is a *pure power* of x_j if $\alpha = (a_1, \ldots, a_j, \ldots, a_d)$ with $a_i = 0$ for $i \neq j$.

Definition 4.3.4. An ideal $I \subset K[x_1, \ldots, x_d]$ is *Artinian* if the Krull dimension of R/I is zero.

Note that a monomial ideal is Artinian if and only if I contains a pure power of x_j for $1 \leq j \leq d$. Denote the set of Artinian ideals with the largest degree of any pure power equal to n by $\mathcal{A}_d(n)$. Denote the set of Artinian strongly stable ideals with pure power $x_d^n \in G(I)$ by

$$\mathcal{B}_d(n) := \{I \in \mathcal{A}_d(n) \mid I \text{ is strongly stable}\}$$

Similarly, use

$$\mathcal{T}_d(n) := \{I \in \mathcal{A}_d(n) \mid I \text{ is symmetric}\}$$

to denote the set of Artinian symmetric monomial ideals with pure power $x_d^n \in G(I)$.

Proposition 4.3.5. *Let* $I \in \mathcal{B}_d(n)$ *with pure power* $x_d^n \in G(I)$*. Then for every pure power* $x_j^{k_j} \in G(I)$*, we have* $k_j \leq n$*.*

Proof. We can use Borel moves to get $x_j^n \in I$ for all j. So every pure power $x_j^{k_j} \in G(I)$ must satisfy $k_j \leq n$. □

Proposition 4.3.6. *Let* $I \in \mathcal{T}_d(n)$ *with pure power* $x_d^n \in G(I)$*. Then for every pure power* $x_j^{k_j} \in G(I)$*, we have* $k_j = n$*.*

Proof. This follows from Proposition 4.3.2. □

4.4 Partitions Correspond to Artinian Monomial Ideals

Proposition 4.4.1. *If* $I \in \mathcal{A}_d(n)$*, then* $P = \{\alpha \mid \mathbf{x}^\alpha \notin I\} \in \mathcal{P}_d(n)$*.*

Proof. First, we need to show that $P = \{\alpha \mid \mathbf{x}^\alpha \notin I\}$ is a partition. Just notice that if $\alpha = (a_1, \ldots, a_d) \in P$, then $\mathbf{x}^\alpha \notin I$. It follows that $\mathbf{x}^{(a_1,\ldots,a_j-1,\ldots,a_d)} \notin I$ so $(a_1, \ldots, a_j - 1, \ldots, a_d) \in P$. Since I is Artinian, P is finite.

For every cell $\gamma = (c_1, \ldots, c_d) \in P$, we have $c_i < n$, because I contains a pure power of every variable with degree less than or equal to n. Since $x_d^n \in G(I)$, $(0, 0, \ldots, 0, n-1) \in P$. □

Proposition 4.4.2. *If* $P \in \mathcal{P}_d(n)$*, then* $\{\mathbf{x}^\alpha \mid \alpha \notin P\} \in \mathcal{A}_d(n)$*.*

Proof. Let $I = \{\mathbf{x}^\alpha \mid \alpha \notin P\}$. It is clear that I is a monomial ideal. Since $(0, 0, \ldots, n, \ldots, 0, 0) \notin P$, $x_j^n \in I$ for $1 \leq j \leq d$.

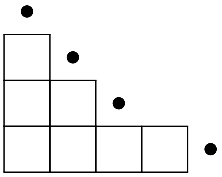
There is some cell $\gamma = (c_1, \ldots, c_j, \ldots, c_d) \in P$ with $c_j = n - 1$ for some $1 \leq j \leq d$. By the definition of d-dimensional partition, $(0, 0, \ldots, n-1, \ldots, 0) \in P$, so $x_j^n \in G(I)$. □

The two propositions above allow us to define a map $\varphi : \mathcal{A}_d(n) \to \mathcal{P}_d(n)$ by

$$\varphi(I) = \{\alpha \in \mathbb{N}^d \mid \mathbf{x}^\alpha \notin I\}$$

with inverse $\varphi^{-1} : \mathcal{P}_d(n) \to \mathcal{A}_d(n)$ given by $\varphi^{-1}(P) = \{\mathbf{x}^\alpha \mid \alpha \notin P\}$.

Example 4.4.3. Let $I = (x^4, x^2y, xy^2, y^3) \in \mathcal{A}_2(4)$. Then $\varphi(I)$ is the partition from Example 2.2 and is shown below with • used to represent generators of I.



Example 4.4.4. Let $I = (x^3, x^2y, y^3, x^2z, xyz, y^2z, z^2) \in \mathcal{A}_3(3)$. Then $\varphi(I)$ is the plane partition from Example 4.2.3.

We will show that we can restrict φ to a bijection between strongly stable ideals and strongly stable partitions and between symmetric monomial ideals and totally symmetric partitions.

Proposition 4.4.5. *A monomial ideal I is strongly stable iff $\varphi(I)$ is a strongly stable partition.*

Proof. ($\Rightarrow$) Let $\beta \in \varphi(I)$ so that $\mathbf{x}^\beta \notin I$ and consider $H(\beta) = (h_1, \ldots, h_d)$. Fix $1 \leq i < j \leq d$. We have $(b_1, \ldots, b_i, \ldots, b_j + h_j + 1, \ldots, b_d) \notin \varphi(I)$, and we can use the fact that I is strongly stable to get

$$\mathbf{x}^{(b_1,\ldots,b_i,\ldots,b_j+h_j+1,\ldots,b_d)} \in I \implies \mathbf{x}^{(b_1,\ldots,b_i+h_j+1,\ldots,b_j,\ldots,b_d)} \in I$$

so $(b_1, \ldots, b_i + h_j + 1, \ldots, b_j, \ldots, b_d) \notin P$. Therefore, $h_i < h_j + 1 \implies h_i \leq h_j$.

($\Leftarrow$) Let $\mathbf{x}^\alpha \in G(I)$ such that $x_j \mid \mathbf{x}^\alpha$. Then $\frac{\mathbf{x}^\alpha}{x_j} = \mathbf{x}^{(a_1,\ldots,a_j-1,\ldots,a_d)} \notin I$, so $\tilde{\alpha} = (a_1, \ldots, a_j - 1, \ldots, a_d) \in \varphi(I)$. If $H(\tilde{\alpha}) = (h_1, \ldots, h_d)$, then $h_j = 1$ because $\alpha \notin P$. Since $\varphi(I)$ is a strongly stable partition, $h_i \leq h_j = 1$ for $i < j$. In particular, $(a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_d) \notin I$. So $\mathbf{x}^\alpha \frac{x_i}{x_j} \in I$. ☐

Proposition 4.4.6. *A monomial ideal I is symmetric iff $\varphi(I)$ is a totally symmetric partition.*

Proof. ($\Rightarrow$) Let $\beta \in \varphi(I)$ and let $\pi \in S_d$. Since $\mathbf{x}^\beta \notin I$, $\mathbf{x}^{\pi(\beta)} \notin I$, so $\pi(\beta) \in \varphi(I)$.

($\Leftarrow$) Let $\mathbf{x}^\alpha \in I$. Then $\alpha \notin \varphi(I) \implies \pi(\alpha) \notin \varphi(I)$, so $\mathbf{x}^{\pi(\alpha)} \in I$. ☐

In this section, we have shown that $\varphi\mid_{\mathcal{B}_d(n)} \colon \mathcal{B}_d(n) \to \tilde{\mathcal{B}}_d(n)$ and $\varphi\mid_{\mathcal{T}_d(n)} \colon \mathcal{T}_d(n) \to \tilde{\mathcal{T}}_d(n)$ are bijections. In the next section, we show a bijection $\mathcal{B}_d(n) \to \mathcal{T}_d(n)$.

4.5 Bijection Between Monomial Ideals

Denote the set of all monomials in $K[x_1, \ldots, x_d]$ by M_d and the set of monomials with weakly increasing exponent vector by F_d. Then use $\mathcal{F}_d(n)$ to denote the set of minimal subsets of F_d which contain x_d^n and no coordinate of any exponent vector exceeding n. Define a map $\psi : M_d \to F_d$ by

$$\psi(\mathbf{x}^{(a_1,a_2,\ldots,a_d)}) = \mathbf{x}^{(a_1,a_1+a_2,a_1+a_2+a_3,\ldots,a_1+a_2+\cdots+a_d)}$$

and notice that $\psi^{-1} : F_d \to M_d$ is given by

$$\psi^{-1}(\mathbf{x}^{(a_1,\ldots,a_d)}) = \mathbf{x}^{(a_1,a_2-a_1,a_3-a_2,\ldots,a_d-a_{d-1})}$$

Proposition 4.5.1. $\psi(m\frac{x_{q+1}}{x_q}) = \frac{\psi(m)}{x_q}$

Proof.

$$\begin{aligned}\psi(m\frac{x_{q+1}}{x_q}) &= \psi(\mathbf{x}^{(a_1,\ldots,a_q-1,a_{q+1}+1,\ldots,a_d)}) \\ &= \mathbf{x}^{(a_1,a_2-a_1,a_3-a_2,\ldots,a_q-a_{q-1}-1,a_{q+1}-a_q,a_{q+2}-a_{q+1},\ldots,a_d-a_{d-1})} \\ &= \frac{\psi(m)}{x_q}\end{aligned}$$

□

For a subset of monomials $A = \{\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_r}\}$, we will use $\psi(A) := \{\psi(\mathbf{x}^{\alpha_1}), \ldots, \psi(\mathbf{x}^{\alpha_r})\}$. For an ideal I, we will use $\psi(I)$ to refer to the ideal generated by $\psi(G(I))$.

Proposition 4.5.2. *For a strongly stable ideal I, $m \in I \iff \psi(m) \in \psi(I)$.*

Proof. ($\Rightarrow$) If $m \in I$, then there exists $g \in G(I)$ so that $g \mid m$. If $g = \mathbf{x}^{(c_1,\ldots,c_d)}$ and $m = \mathbf{x}^{(a_1,\ldots,a_d)}$, then $c_i \leq a_i$ for $1 \leq i \leq d$. It follows that $c_1 + \cdots + c_i \leq a_1 + \cdots + a_i$ for all i. So $\psi(g) \mid \psi(m)$. ($\Leftarrow$) Assume $\psi(m) \in \psi(I)$ and let $k_i := a_1 + \cdots + a_i - (c_1 + \cdots + c_i)$ so that

$$\mathbf{x}^{(k_1,\ldots,k_d)}\psi(g) = \psi(m)$$

Note that $k_i = a_i - c_i + k_{i-1}$ for $i = 2, \dots, d$. Then we have

$$\begin{aligned}
\tilde{g} &:= g\frac{x_1^{k_1}}{x_2^{k_1}}\frac{x_2^{k_2}}{x_3^{k_2}}\cdots\frac{x_{d-1}^{k_{d-1}}}{x_d^{k_{d-1}}} \\
&= g\frac{x_1^{k_1-k_2}}{x_2^{k_1-k_2}}\frac{x_1^{k_2-k_3}}{x_3^{k_2-k_3}}\cdots\frac{x_1^{k_{d-2}-k_{d-1}}}{x_{d-1}^{k_{d-2}-k_{d-1}}}\frac{x_1^{k_{d-1}}}{x_d^{k_{d-1}}} \\
&= \mathbf{x}^{(c_1+k_1, c_2+k_2-k_1, c_3+k_3-k_2, \dots, c_{d-1}+k_{d-1}-k_{d-2}, c_d-k_{d-1})} \\
&= \mathbf{x}^{(a_1, a_2, \dots, a_{d-1}, c_d-k_{d-1})}
\end{aligned}$$

since $c_i + k_i - k_{i-1} = c_i + (a_i - c_i + k_{i-1}) - k_{i-1} = a_i$. For the last coordinate, we have

$$\begin{aligned}
c_d - k_{d-1} &= c_d - (a_1 + \cdots + a_{d-1} - (c_1 + \cdots + c_{d-1})) \\
&= c_1 + \cdots + c_d - (a_1 + \cdots + a_{d-1}) \\
&\leq a_d
\end{aligned}$$

We have shown that $\tilde{g} \mid m$. Now, we claim that $\tilde{g} \in I$ because it is a Borel move from g. To see this, notice that $k_{i-1} - k_i = c_i - a_i$ for $i = 2, \dots, d$ and $k_{d-1} = a_1 + \cdots + a_{d-1} - (c_1 + \cdots + c_{d-1}) \leq c_d$. So g is divisible by each of the denominators in the second line of $\tilde{g}$. □

Proposition 4.5.3. *If I is a strongly stable ideal, then $Bgens(I) = \psi^{-1}(min(\psi(G(I))))$.*

Proof. ($\subseteq$) Let $m \in Bgens(I) \subset G(I)$. For any $x_q \mid \psi(m)$, $\frac{\psi(m)}{x_q} = \psi(m\frac{x_{q+1}}{x_q}) \notin \psi(I)$ because $m\frac{x_{q+1}}{x_q} \notin I$. Therefore, $\psi(m) \in min(\psi(G(I)))$.

($\supseteq$) Let $m \in \psi^{-1}(min(\psi(G(I))))$. Then $\psi(m)$ is minimal, so $\frac{\psi(m)}{x_q} = \psi(m\frac{x_{q+1}}{x_q}) \notin \psi(I)$. It follows that $m\frac{x_{q+1}}{x_q} \notin I$. □

Note that this gives an algorithm to compute $Bgens(I)$ given a generating set $G(I)$. As a result of Proposition 4.3.5, for any $I \in \mathcal{B}_d(n)$, we have

$$\mathbf{x}^\alpha \in G(I) \implies deg(\alpha) \leq n.$$

It follows that every coordinate of $\psi(\mathbf{x}^\alpha)$ is less than or equal to n. This observation combined with the previous proposition allows us to define a map

$$\begin{aligned} \Lambda : \mathcal{B}_d(n) &\to \mathcal{F}_d(n) \\ I &\mapsto \psi(Bgens(I)). \end{aligned}$$

Proposition 4.5.4. *$\Lambda : \mathcal{B}_d(n) \to \mathcal{F}_d(n)$ is a bijection.*

Proof. We claim that $\Lambda^{-1} : \mathcal{F}_d(n) \to \mathcal{B}_d(n)$ is given by $S \mapsto Borel(\psi^{-1}(S))$. Since $x_d^n \in S$, $\psi^{-1}(x_d^n) = x_d^n \in \psi^{-1}(S)$. There are no other pure powers of x_d in $\psi^{-1}(S)$, so $x_d^n \in G(I)$. It follows that $Borel(\psi^{-1}(S)) \in \mathcal{B}_d(n)$ and

$$\begin{aligned} (\Lambda^{-1} \circ \Lambda)(I) &= Borel(\psi^{-1}(\psi(Bgens(I)))) \\ &= I. \end{aligned}$$

□

Proposition 4.5.5. *If $I \subset K[x_1, \ldots, x_d]$ is a symmetric monomial ideal, then $sym(G(I) \cap F_d) = G(I)$.*

Proof. ($\subseteq$) If $u \in sym(G(I) \cap F_d)$, then $u = \pi(m)$ for some $m \in G(I) \cap F_d$ and some $\pi \in S_d$. Since $G(I)$ is closed under operations of S_d, $u \in G(I)$.
($\supseteq$) If $u \in G(I)$, then there exists some $\pi \in S_d$ so that $\pi(u) \in G(I) \cap F_d$, so $u \in sym(G(I) \cap F_d)$. □

For an ideal $I \in \mathcal{T}_d(n)$, define a map $\Omega : \mathcal{F}_d(n) \to \mathcal{T}_d(n)$ by

$$\Omega(A) = ideal(sym(A))$$

Proposition 4.5.6. *$\Omega : \mathcal{F}_d(n) \to \mathcal{T}_d(n)$ is a bijection*

Proof. We claim that the inverse $\Omega^{-1} : \mathcal{T}_d(n) \to \mathcal{F}_d(n)$ is given by $\Omega^{-1}(I) = G(I) \cap F_d$. Notice that for $I \in \mathcal{T}_d(n)$, we have $x_d^n \in G(I)$, so $x_d^n \in G(I) \cap F_d$. It follows that $\Omega^{-1}(I) \in \mathcal{F}_d(n)$ and

$$\begin{aligned} \Omega(\Omega^{-1}(I)) &= \Omega(G(I) \cap F_d) \\ &= ideal(sym(G(I) \cap F_d)) \\ &= I. \end{aligned}$$

□

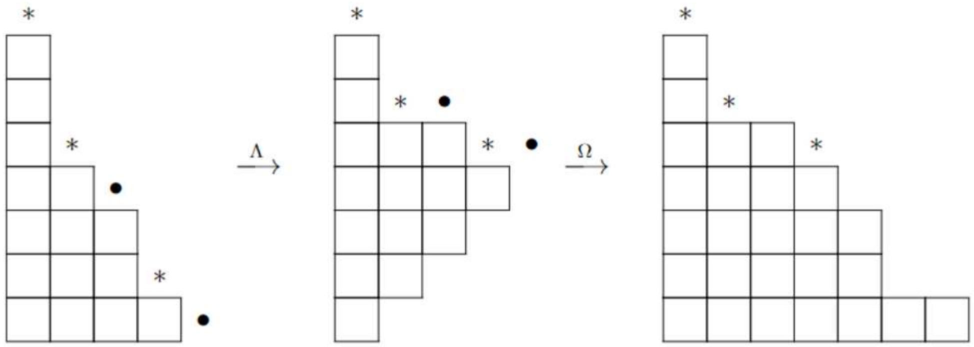
Theorem 15. *There is a bijection between d-dimensional strongly stable partitions and d-dimensional totally symmetric partitions which preserves the side length n of the minimal d-dimensional box containing the partitions.*

Proof. We have the diagram of bijections below. The vertical bijections were given in section 4 and the horizontal map $\Omega \circ \Lambda$ was shown to be a bijection by Propositions 4.5.4 and 4.5.6.

$$\begin{tikzcd} \mathcal{B}_d(n) \arrow[r, "\Omega\circ\Lambda"] \arrow[d, "\varphi"'] & \mathcal{T}_d(n) \arrow[d, "\varphi"] \\ \tilde{\mathcal{B}}_d(n) \arrow[r] & \tilde{\mathcal{T}}_d(n) \end{tikzcd}$$

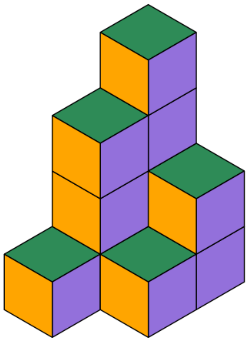
□

Example 4.5.7. Consider the strongly stable ideal $I = (x^4, x^3y, x^2y^3, xy^4, y^7) \in \mathcal{B}_2(7)$. The corresponding partition $\varphi(I) \in \tilde{\mathcal{B}}_2(7)$ from Example 4.2.6 is shown on the left below. Minimal generators of I which are not in $Bgens(I)$ are represented with •. Elements of $Bgens(I)$ are represented by $*$. The middle diagram depicts $\psi(I)$ and illustrates that $\psi(Bgens(I)) = min(\psi(G(I)))$. The diagram on the right is the corresponding totally symmetric partition.



Remark 4.5.8. For any $\mathbf{x}^\alpha \notin I$, we have $\psi(\mathbf{x}^\alpha) \notin \Lambda(I)$. Since $\psi(\mathbf{x}^\alpha)$ is the unique representative of its orbit, the number of orbits of monomials not in $\Omega(\Lambda(I))$ is exactly the total number of monomials in the original ideal I. In partition language, this means that the number of cells in the strongly stable partition is exactly the number of orbits in the totally symmetric partition. In the example above, there are 15 cells in the strongly stable partition on the left which correspond to 15 orbits in the totally symmetric partition on the right.

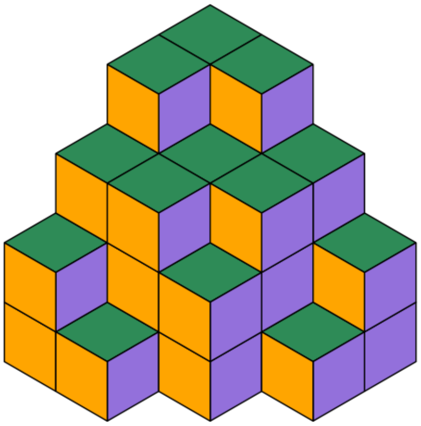
Example 4.5.9. Consider the strongly stable partition $P \in \tilde{\mathcal{B}}_3(4)$ from Example 4.2.7 shown below.



One can check that $I = (x^2, xy^2, y^3, xyz, y^2z, xz^2, yz^3, z^4) \in \mathcal{B}_3(4)$ is the corresponding strongly stable ideal ($\varphi(I) = P$) and that $Bgens(I) = \{x^2, xz^2, y^2z, z^4\}$. To compute the totally symmetric partition corresponding to P, we first compute

$$\begin{aligned}\Omega(\Lambda(I)) &= \Omega(\psi(Bgens(I))) \\ &= \Omega(\{x^2y^2z^2, xyz^3, y^2z^3, z^4\}) \\ &= ideal(sym(\{x^2y^2z^2, xyz^3, y^2z^3, z^4\}))\end{aligned}$$

Finally, notice that $\varphi(ideal(sym(\{x^2y^2z^2, xyz^3, y^2z^3, z^4\}))) \in \tilde{\mathcal{T}}_3(4)$ is the partition shown below.



4.6 Enumerations

In this last section, we will explore the enumeration of strongly stable partitions (equivalently totally symmetric partitions) in boxes. Let

$$B_d(n) := \sum_{k=0}^{n} |\tilde{\mathcal{B}}_d(k)|$$

denote the number of d-dimensional strongly stable partitions which fit in a box of side length n. Similarly, use

$$T_d(n) := \sum_{k=0}^{n} |\tilde{\mathcal{T}}_d(k)|$$

By convention, we count the empty partition. As a result, we have $B_d(0) = T_d(0) = 1$ since the empty partition fits inside a box of side length 0. As a result of Theorem 15,

$$B_d(n) = T_d(n)$$

for positive integers d, n.

In [21], Hawkes shows a bijection between d-dimensional totally symmetric partitions inside a box of side length n and $(n-1)$-dimensional totally symmetric partitions inside a box of side length $d+1$. As a result, $T_d(n) = T_{n-1}(d+1)$ for $n \geq 2$. Combining this with the formula above, we have

$$B_d(n) = B_{n-1}(d+1)$$

for positive integers d, n.

Lastly, we consider some formulae for $B_d(n)$ when d is fixed. When $d = 1$, partitions are simply nonnegative integers, and every 1-dimensional partition is (trivially) strongly stable.

$$B_1(n) = T_1(n) = n + 1$$

The number of 2-dimensional strongly stable partitions which fit inside a box of side length n (strict integer partitions with largest part n) is given by

$$B_2(n) = T_2(n) = 2^n,$$

because for $1 \leq i \leq n$, we have only the choice of whether or not to include i in the integer partition.

In [22], Stembridge proved that the number of totally symmetric plane partitions which fit inside a box of side length n is given by the product formula

$$T_3(n) = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}$$

and so we have a formula for $B_3(n)$.

In fact, in the case $d = 3$, an even stronger result has been shown for totally symmetric plane partitions. The q-TSPP formula

$$\sum_{P \in \tilde{\mathcal{T}}_3(n)} q^{|P/S_3|} = \prod_{1 \le i \le j \le k \le n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}$$

was shown to be the orbit-counting generating function for totally symmetric plane partitions fitting inside an $n \times n \times n$ box in [23]. Remark 5.8 implies that this same formula is a *cell*-counting generating function for strongly stable plane partitions.

There is no known formula for $B_d(n) = T_d(n)$ for $d \geq 4$ [24].

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