

DISSERTATION

PSEUDOSTABLE HODGE CLASSES

Submitted by

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ABSTRACT

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We study the relationship between Hodge classes on moduli spaces of pseudostable and stable curves given by the contraction morphism \mathcal{T} . While Mumford's relation does not hold in the pseudostable case, we show that one can express the (pullback via \mathcal{T} of the) Chern classes of $\mathbb{E} \oplus \mathbb{E}^\vee$ solely in terms of strata and ψ classes. We organize the combinatorial structure of the pullback of products of two pseudostable λ classes and obtain an explicit comparison of arbitrary pseudostable and stable quadratic Hodge integrals, as well as certain families of cubic and higher degree pseudostable Hodge integrals.

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Chapter 1

Introduction

The moduli space of genus g curves \mathcal{M}_g is a fundamental object in modern algebraic geometry. Riemann's original insight that one should study the continuous parameters (which he called *moduli* in [26]) governing the deformations of Riemann surfaces has given rise to rich developments in geometry, as well as to several connections to other fields of mathematics and science, including combinatorics, representation theory, integrable systems and string theory (see [15], [2], [27], and [29] for examples of each, respectively).

A milestone in the study of intersection theory of moduli spaces of curves is Deligne and Mumford's introduction of the notion of stable curves [8], which gives rise to a smooth and modular compactification $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$.

The notion of tautological classes [1,24,25], those being Chow (or cohomology) classes defined via the intrinsic geometry of the objects parameterized, identified a deep, and yet accessible slice of the intersection ring of $\overline{\mathcal{M}}_{g,n}$. A set of additive generators for the tautological ring, indexed by decorated graphs is introduced in [17], and their product structure is spelled out via a combinatorial algorithm in [32,33]. The Chern classes λ_j of the Hodge bundle \mathbb{E} , and ψ_i of the cotangent line bundles \mathbb{L}_i are tautological, since the above bundles are constructed from the relative dualizing sheaf ω_π via natural operations of push-forward and pull-back. *Hodge integrals*, referring to any intersection number of lambda and psi classes, have been studied in [10–12]; the degree of a Hodge integral typically refers to the degree of the monomial in lambda classes only, thus linear Hodge integrals consist of intersection numbers with at most one lambda class but an arbitrary number of psi classes. Through the technique of virtual localization [16], Gromov-Witten invariants of targets with a torus action are computed in terms of Hodge integrals; in particular, quadratic Hodge integrals control the Gromov-Witten theory of any toric surfaces. This has provided a fruitful link connecting tautological intersection theory on the moduli spaces of curves with mirror symmetry [6,20].

Mumford's relation [25], which refers to the fact that the positive Chern classes of $\mathbb{E} \oplus \mathbb{E}^\vee$ vanish, has served as an essential tool in mirror symmetry, as it allows to reduce the computations of certain families of Gromov-Witten invariants of threefolds, which in principle are cubic Hodge integrals, in terms of linear Hodge integrals, which are much better understood. This perspective gives a proof of the Aspinwall-Morrison formula [3, 9] for the computation of Gromov-Witten invariants of the resolved conifold.

In [28], Schubert studies an alternative compactification of the moduli space of curves, where the stability condition is tweaked to allow curves with cuspidal singularities, and to disallow curves with elliptic tails. Such curves are called *pseudostable*. The resulting moduli spaces are smooth and proper and have a boundary structure which is not quite normal crossing, but close enough to play well with tropicalization, see [5]. It was later shown in [18] that moduli spaces of pseudostable curves arise in the log minimal model program of $\overline{\mathcal{M}}_{g,n}$, as the birational model following the first wall-crossing.

The study of the tautological intersection theory of moduli spaces of pseudostable curves is initiated in [4]; the strategy is to consider the contraction morphism $\mathcal{T} : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}^{\text{ps}}$ and apply the projection formula to relate tautological cycles on moduli spaces of pseudostable curves to analogous ones on moduli spaces of stable curves. It is shown that psi classes are stable under pullback via \mathcal{T} , and a combinatorial formula expressing the pullback of lambda classes is given, see Theorem 2.2.3:

$$\mathcal{T}^*(\lambda_j) = \sum_{i=0}^j \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{j-i})). \quad (1.0.0.1)$$

It is a simple consequence that stable and pseudostable linear Hodge integrals coincide (Proposition 3.2 of [4]).

Explicit computations in low genus show that Mumford's relation does not hold in the pseudostable case, and pseudostable quadratic Hodge integrals do not agree with their stable counterparts (Proposition 3.5 of [4]). While the comparison formulas for individual lambda classes from [4] provide in principle a solution for computing any pseudostable Hodge integral in term of

stable ones, the combinatorial complexity of a brute force approach makes it not viable, except for very small values of g and n , even with the use of specialized computer software.

The main technical thrust of this work has been to harness the combinatorial complexity of the problem by identifying non-trivial recursive structure. In order to compute the total Chern class of $\mathcal{T}^*(\mathbb{E} \oplus \mathbb{E}^\vee)$ in genus g , we first substitute the relations (1.0.0.1) for each lambda class, then expand the product formally in the decorated graph algebra via the algorithm of [32]. The resulting terms can be then partitioned into three disjoint subsets, each of which may be formally bijected with the terms for the entire computation in genus $g - 1$. After applying the inductive hypothesis, it then becomes manageable to sum the three resulting terms and obtain the pseudostable version of Mumford's relation.

The remarkably simple structure relating quadratic stable and pseudostable Hodge integrals does not carry over to cubic or higher degree Hodge integrals. Nonetheless, the availability of natural combinatorial expressions for some quadratic Hodge parts in terms of boundary and descendants opens up the possibility of approaching the study of specific families of cubic Hodge integrals, by using such relations to reduce the degree.

In particular, we think it should be possible to define notions of *pseudostable Gromov-Witten* invariants, and relate them to their stable counterparts via localization. While a careful study of moduli spaces of pseudostable maps and their virtual classes has still not happened, it is reasonable to expect and almost "guess" what the ingredients contributing to the localized virtual class should be. With this foundational work in place, the pseudostable Gromov-Witten invariants of the resolved conifold should be within reach. It would be interesting to see whether pseudostable Gromov-Witten invariants play a role in the picture of global mirror symmetry, similarly to how other variations of stability conditions do.

The dissertation is organized as follows. In Chapter 2 we give the background necessary to make this work reasonably self-contained: we introduce moduli spaces of stable and pseudostable curves, the combinatorics of the decorated graph algebra for their tautological intersection theory, and commonly-used relations for Hodge integrals. In Chapter 3, we state and prove the

pseudostable version of Mumford's relation, and show as an application some natural relations between λ and ψ classes that arise from it. In Chapter 4, we extract the relationship among stable and pseudostable quadratic Hodge integrals, and as an application compute certain families of pseudostable quadratic Hodge integrals. In Chapter 5, we initiate a study of cubic and higher degree pseudostable Hodge integrals and prove relations for a few families of cubics.

Chapter 2

Background

2.1 The Moduli Spaces of Stable and Pseudostable Curves

Stable and pseudostable curves can have certain types of singularities, and the two of interest are *nodes* and *cusps*. A node is a point locally analytically isomorphic to $x^2 = y^2$, while cusps are locally isomorphic to $x^2 = y^3$. *Special points* of a connected component of a curve are any singularities or marked points of that component.

The moduli spaces of stable and pseudostable curves, denoted $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}^{\text{ps}}$, respectively, both parameterize families of curves of arithmetic genus g with n marked points, and both spaces are of dimension $3g - 3 + n$. Recall that for smooth curves, arithmetic genus is the same as geometric genus, whereas for nodal curves, the arithmetic genus is intuitively the geometric genus after smoothing out the nodes. For cuspidal curves, each cusp adds 1 to the arithmetic genus.

What distinguishes the two types of moduli spaces are their notions of stability, which we now define and discuss.

Definition 2.1.1. A curve of genus g with n marked points is *stable* if it satisfies the following conditions:

1. All marked points are distinct and contained in the smooth locus of the curve.
2. All singularities of the curve are nodes.
3. Each connected component of the curve with genus g and s special points has (g, s) not equal to $(0, 0)$, $(0, 1)$, $(0, 2)$, or $(1, 0)$.

The third condition is equivalent to g and s satisfying $2g - 2 + s > 0$.

Definition 2.1.2. A curve of genus g with n marked points is *pseudostable* if it satisfies the following conditions:

1. All marked points are distinct and contained in the smooth locus of the curve.
2. All singularities of the curve are nodes or cusps.
3. Each connected component of the curve with genus g and s special points has (g, s) not equal to $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, or $(2, 0)$.

The third condition is equivalent to g and s satisfying $g + s > 2$.

Comparing the definitions shows that the two types of curves are largely the same, but with two key differences. One is that pseudostable curves can have cusps, while stable curves cannot. The other difference is that stable curves can have connected components with indices (g, s) equal to $(1, 1)$ or $(2, 0)$, while pseudostable curves cannot. Components with $(g, s) = (1, 1)$ are referred to as *elliptic tails*.

There is a natural map from $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g,n}^{\text{ps}}$, where if a stable curve in $\overline{\mathcal{M}}_{g,n}$ has an elliptic tail, that elliptic tail is contracted to a cusp, thus giving a curve in $\overline{\mathcal{M}}_{g,n}^{\text{ps}}$. We define this map more formally in the following definition.

Definition 2.1.3 (As in [4]). Let C be a stable curve of genus g with marked points q_1, \dots, q_n , so that $[(C, q_1, \dots, q_n)] \in \overline{\mathcal{M}}_{g,n}$. Write

$$C = C_0 \cup E_1 \cup \dots \cup E_r,$$

where E_1, \dots, E_r are the elliptic tails of C . Let \hat{C} be the unique pseudostable curve with marked points $\hat{q}_1, \dots, \hat{q}_n$ that admits a morphism $T : C \rightarrow \hat{C}$ such that

1. T is an isomorphism when restricted to $C \setminus (E_1 \cup \dots \cup E_r)$,
2. $T(E_i)$ is a cusp on \hat{C} for all $i = 1, \dots, r$, and
3. $T(q_i) = \hat{q}_i$.

With this notation, we have the following map on moduli spaces:

$$\begin{aligned} \mathcal{T} : \overline{\mathcal{M}}_{g,n} &\rightarrow \overline{\mathcal{M}}_{g,n}^{\text{ps}} \\ [(C, q_1, \dots, q_n)] &\mapsto [(\hat{C}, \hat{q}_1, \dots, \hat{q}_n)] \end{aligned}$$

The map in Definition 2.1.3 induces a pull-back on the Chow rings of the moduli spaces:

$$\mathcal{T}^* : A^*(\overline{\mathcal{M}}_{g,n}^{\text{ps}}) \rightarrow A^*(\overline{\mathcal{M}}_{g,n}).$$

2.2 Tautological Classes

We introduce two types of tautological classes which are the main characters in this work: the ψ classes are the first Chern class of *cotangent line bundles*, and λ classes are Chern classes of the *Hodge bundle*. We adopt a common shortcut in moduli theory and informally introduce these bundles by describing a natural identification of their fibers with a vector space depending on the geometry of the objects parameterized.

The fiber of the i -th cotangent line bundle \mathbb{L}_i over a point $[(C, q_1, \dots, q_n)] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent line to C at q_i , and ψ_i is defined as the first Chern class of \mathbb{L}_i :

$$\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n}) \text{ for } i = 1, \dots, n.$$

The fiber of the Hodge bundle \mathbb{E} over a point $[(C, q_1, \dots, q_n)]$ is the g -dimensional vector space of global sections of the dualizing sheaf of C , and λ_j is the j -th Chern class of \mathbb{E} , that is,

$$\lambda_j = c_j(\mathbb{E}) \in A^j(\overline{\mathcal{M}}_{g,n}) \text{ for } j = 0, \dots, g.$$

As mentioned in the introduction, one of the most important formulas for λ classes is *Mumford's relation*.

Theorem 2.2.1 (Mumford's relation, [25]). *For stable indices (g, n) ,*

$$(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g) = 1 \in A^*(\overline{\mathcal{M}}_{g,n}).$$

This formula provides a set of relations, which can be observed by grouping together terms of equal codimension in the resulting product:

$$1 + (2\lambda_2 - \lambda_1^2) + \cdots + (-1)^g (2\lambda_g \lambda_{g-2} - \lambda_{g-1}^2) + (-1)^g \lambda_g^2 = 1.$$

Among other relations, this implies the following:

1. $2\lambda_2 = \lambda_1^2$
2. $2\lambda_g \lambda_{g-2} = \lambda_{g-1}^2$ for $g \geq 2$
3. $\lambda_g^2 = 0$ for $g \geq 1$

The final relation, $\lambda_g^2 = 0$ for $g \geq 1$, is particular useful, and is applied profusely throughout the dissertation. For $g = 0$, note that $\lambda_0^2 = 1$ since $\lambda_0 = 1$.

For convenience later on, we introduce the following notation for linear combinations of λ classes:

$$\Lambda_g(k) := \sum_{i=0}^g k^i \lambda_i. \tag{2.2.0.1}$$

With this notation, all the relations of Mumford's formula are compactly encoded by the following:

$$\Lambda_g(1)\Lambda_g(-1) = 1. \tag{2.2.0.2}$$

For $\overline{\mathcal{M}}_{g,n}^{\text{ps}}$, both the cotangent line bundle and Hodge bundle can be defined in the same way, leading to the same definitions of both ψ and λ classes:

$$\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n}^{\text{ps}}) \text{ for } i = 1, \dots, n,$$

$$\lambda_j = c_j(\mathbb{E}) \in A^j(\overline{\mathcal{M}}_{g,n}^{\text{ps}}) \text{ for } j = 0, \dots, g.$$

One way of studying pseudostable ψ and λ classes is by looking at their pullbacks $\mathcal{T}^*(\psi_i)$ and $\mathcal{T}^*(\lambda_j)$, where \mathcal{T} is the map described in Definition 2.1.3. The authors in [4] computed explicit formulas for each of these pullbacks, and this result (which we show in Theorem 2.2.3) utilizes the following notation:

Definition 2.2.2. For $k \geq 1$, let

$$\mathcal{G}^k : \overline{\mathcal{M}}_{g-k, n+k} \times \overline{\mathcal{M}}_{1,1}^{\times k} \rightarrow \overline{\mathcal{M}}_{g,n} \quad (2.2.0.3)$$

be the gluing map that sends (C_0, E_1, \dots, E_k) to the curve which attaches each of E_1, \dots, E_k to C_0 at the marked points $n+1, \dots, n+k$. If $k=0$, then \mathcal{G}^0 is the identity map from $\overline{\mathcal{M}}_{g,n}$ to itself.

Let p_0 be projection onto the factor $\overline{\mathcal{M}}_{g-k, n+k}$ of the domain in (2.2.0.3), and call this factor the *root vertex* of the domain. We denote by p_i the projection onto the i -th factor of $\overline{\mathcal{M}}_{1,1}^{\times k}$ in the domain of (2.2.0.3).

The image of the gluing map in (2.2.0.3) is a stratum parameterizing curves with k elliptic tails. Note that the tautological λ and ψ classes may be pulled back from the factors of the product and pushed forward via \mathcal{G}^k to obtain tautological classes supported on this stratum. This implies the following for products of ψ classes and strata parameterizing curves with k elliptic tails:

$$\psi_i \cdot \mathcal{G}_*^i(1) = \mathcal{G}_*^i(p_0^*(\psi_i)).$$

For products with λ classes, we note that the Hodge bundle splits along the factors of boundary strata (see [11], p. 10). That is, if \mathbb{E}_g is the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$, then

$$\mathbb{E}_g|_{\mathcal{G}_*^i(1)} = \mathbb{E}_{g-i} \oplus \mathbb{E}_1 \oplus \dots \oplus \mathbb{E}_1.$$

This implies

$$\lambda_j \cdot \mathcal{G}_*^i(1) = \sum_{k=0}^{g-i} \binom{i}{j-k} \mathcal{G}_*^i \left(p_0^*(\lambda_k) \prod_{m=1}^{j-k} p_m^*(\lambda_1) \right),$$

where $\binom{i}{j-k}$ accounts for the all the ways that we can distribute $j-k$ factors of λ_1 over the i factors of $\mathcal{G}_*^i(1)$ that correspond to elliptic tails.

We define

$$\psi_{\bullet_i} := p_i^*(\psi_1) \text{ and } \psi_{\star_i} := p_0^*(\psi_{n+i}), \quad (2.2.0.4)$$

so that, for example,

$$\mathcal{G}_*^1(-\psi_{\bullet_1} - \psi_{\star_1}) = \mathcal{G}_*^1(-p_1^*(\psi_1) - p_0^*(\psi_{n+1})).$$

Theorem 2.2.3 (Cavalieri, Gallegos, Ross, Van Over, Wise [4]). *For all psuedostable indices (g, n) , we have*

$$\mathcal{T}^*(\psi_i) = \psi_i \text{ for all } i = 1, \dots, n$$

and

$$\mathcal{T}^*(\lambda_j) = \sum_{i=0}^j \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{j-i})).$$

For notational convenience, we define

$$\hat{\lambda}_j := \mathcal{T}^*(\lambda_j). \quad (2.2.0.5)$$

2.3 Dual Graphs of Boundary Strata

In this section, we introduce notation and procedures for performing tautological intersections via an algebra of *decorated dual graphs*; we follow the treatment in [32].

Each boundary stratum in $\overline{\mathcal{M}}_{g,n}$ is indexed by the dual graph of the general curve parameterized by points of the stratum; vertices V represent connected components, edges E represent nodes, and half-edges H are marked points of the normalization of the curve. To each vertex, we associate a number $g \in \mathbb{Z}_{\geq 0}$ for the genus of that connected component. There is an involution $i : H \rightarrow H$ which exchanges half-edges that are part of the same edge and has as fixed points the marked point

set N . We denote by $\Sigma^*(\overline{\mathcal{M}}_{g,n})$ the set of dual graphs of stable curves of genus g with n marked points.

Definition 2.3.1 (Yang [32]). Given two dual graphs $G, A \in \Sigma^*(\overline{\mathcal{M}}_{g,n})$, a G -structure on A is an identification of a specialization of G with A . Precisely, it is a triple

$$(\alpha : V_A \rightarrow V_G, \beta : H_G \hookrightarrow H_A, \gamma : H_A \setminus \text{im}(\beta) \rightarrow V_G)$$

which satisfies:

1. the map β commutes with the involution ($\beta \circ i_G = i_A \circ \beta$) and induces an isomorphism from the fixed points of G to the fixed points of A ($N_G \xrightarrow{\sim} N_A$);
2. any half-edge $h \in \text{im}(\beta)$ is incident to v if and only if $\beta^{-1}(h)$ is incident to $\alpha(v)$;
3. if $h \in H_A \setminus \text{im}(\beta)$ is incident to v , then $\gamma(h) = \alpha(v)$;
4. if $v \in V_G$, then the preimage $(\alpha^{-1}(v), \gamma^{-1}(v)/i_A)$ is a connected graph of genus $g(v)$.

We find it useful to think of β as identifying directed edges of G with their images, and we will informally talk about identified edges to mean the datum $\beta|_e : e \rightarrow \beta(e)$.

Example 2.3.2. Consider $\mathcal{G}_*^1(1)$ and $\mathcal{G}_*^2(1)$ in $\overline{\mathcal{M}}_{3,0}$, which have the dual graphs G and A depicted in Figure 2.1. There are two G -structures (α, β, γ) on A , one for each of the ways β identifies the edge of G with one of the edges of A . Once β is chosen, the maps α and γ are determined based on the conditions from Definition 2.3.1.

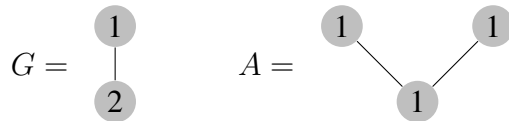


Figure 2.1: The dual graphs for the strata $\mathcal{G}_*^1(1)$ and $\mathcal{G}_*^2(1)$ in $\overline{\mathcal{M}}_{3,0}$.

If a graph A has both a G - and H -structure, then A is said to be a (G, H) -graph with a (G, H) -structure. A (G, H) -graph is *generic* if all its edges are identified with either an edge of G , and edge of H , or both. The set of all generic (G, H) -structures is denoted by $\Gamma(G, H)$. If two (G, H) -structures differ by an automorphism of A , then they are considered *isomorphic*. Lastly, the *common* (G, H) -edges of A are those that are identified with both an edge of G and an edge of H .

Example 2.3.3. Let G and A again be the graphs shown in Figure 2.1, and let $H = G$. As shown in Example 2.3.2, there are two G -structures and two H -structures, so there are four (G, H) -structures on A . For two of these four, one of the edges of A is identified with (the edge of) G , and the other is identified with H , and these are the two generic (G, H) -structures of A . Up to isomorphism, there is only one generic (G, H) -structure since the two differ by an automorphism of A . For this generic (G, H) -graph, there are no common (G, H) -edges.

For strata supporting λ and ψ classes, such as those seen in Theorem 2.2.3, the classes are represented as decorations of the dual graph G : ψ classes on half-edges of G , and λ classes on vertices of G . We denote by θ_v the decorations for a given vertex v :

$$\theta_v = \prod_{i=1}^{n_v} \psi_i^{e_i} \prod_{j=1}^{g_v} \lambda_j^{f_j}. \quad (2.3.0.1)$$

Graphs with these decorations are called *decorated graphs*. We denote \underline{G} to be the underlying graph of G , that is, the graph G without any decorations.

Remark 2.3.4. There are natural geometric reasons why two ψ classes decorating two half-edges that are an orbit of the involution (i.e., (h_-, h_+) belonging to the same edge e) often appear summed together. While technically this sum should correspond to a sum of two graphs, each with one of the half-edges decorated by the ψ class, it is extremely convenient to abuse notation and say that this sum correspond to a single graph with an edge decorated by $(\psi_{h_-} + \psi_{h_+})$. We adopt this convention, extending it by bilinearity to arbitrary polynomial decorations of e by the classes ψ_{h_-}, ψ_{h_+} .

Definition 2.3.5 (Yang [32]). Let G and H denote two decorated graphs with vertex decorations as in (2.3.0.1), and let $\Gamma(G, H)$ denote the set of all generic (G, H) -structures (α, β, γ) . The product of G and H is given by:

$$G \cdot H := \sum_{A \in \Gamma(\underline{G}, \underline{H})} \frac{1}{|\text{Aut}(A)|} F_A(G, H) \cdot A,$$

where

$$f_A(G, v) = \prod_{i=1}^{n_v} \psi_{\beta(i)}^{e_i} \prod_{j=1}^{g_v} \left(\sum_{w \in \alpha^{-1}(v)} \lambda_{j_w} \right)^{f_j},$$

$$F_A(G, H) = \sum_{v \in V_G} f_A(G, v) \sum_{v \in V_H} f_A(H, v) \prod_{e_i = (\bullet_i, \star_i)} (-\psi_{\bullet_i} - \psi_{\star_i}).$$

The last product above is taken over all common $(\underline{G}, \underline{H})$ -edges of A , and $-\psi_{\bullet_i}$ and $-\psi_{\star_i}$ follow the notation introduced in (2.2.0.4).

Yang shows in [32] that this operation turns the set of decorated graphs into an algebra, and that there's a ring surjection from the algebra of decorated dual graphs to the tautological ring of $\overline{\mathcal{M}}_{g,n}$, namely

$$\Phi : \Sigma^*(\overline{\mathcal{M}}_{g,n}) \rightarrow R^*(\overline{\mathcal{M}}_{g,n}), \quad (2.3.0.2)$$

which sends a decorated dual graph to its associated tautological class.

Example 2.3.6. Consider $\mathcal{G}_*^1(1)$ and $\mathcal{G}_*^2(1)$ in $\overline{\mathcal{M}}_{3,0}$, with dual graphs G and H , respectively, shown in Figure 2.2. Up to isomorphism, there are two generic $(\underline{G}, \underline{H})$ -graphs, which we'll denote A_1 and A_2 , depicted in Figure 2.3.

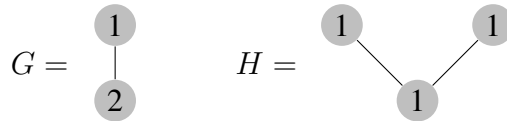


Figure 2.2: The dual graphs for the strata $\mathcal{G}_*^1(1)$ and $\mathcal{G}_*^2(1)$ in $\overline{\mathcal{M}}_{3,0}$.

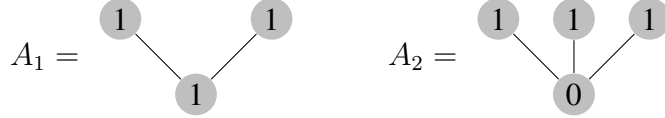


Figure 2.3: Up to isomorphism, the two generic (G, H) -graphs for G and H in Figure 2.2.

Note that $|\text{Aut}(A_1)| = 2!$, but for each common edge of G and H in A_1 , there are $\binom{2}{1} = 2$ ways to choose an edge from H , so the coefficient of these terms is $\frac{2}{2!} = 1$. Similarly, $|\text{Aut}(A_2)| = 3!$, but there are $3!$ ways to order the 3 collective edges of G and H , so the coefficient of A_2 is $\frac{3!}{3!} = 1$.

$$\begin{aligned} G \cdot H &= \frac{2}{2!}(-\psi_{\bullet_1} - \psi_{\star_1})A_1 + \frac{2}{2!}(-\psi_{\bullet_2} - \psi_{\star_2})A_1 + \frac{3!}{3!}A_2 \\ &= (-\psi_{\bullet_1} - \psi_{\star_1})A_1 + (-\psi_{\bullet_2} - \psi_{\star_2})A_1 + A_2 \end{aligned}$$

This result is depicted in Figure 2.4.

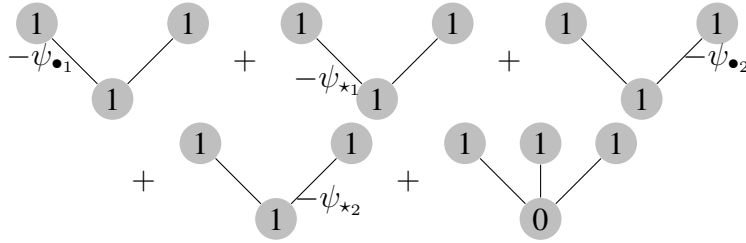


Figure 2.4: The resulting product of G and H .

While this fact is not strictly relevant to this work, we mention for the benefit of the reader that this algorithm has been coded in the freely available program `admcycles` [7], which is a powerful tool for making and checking computations in the tautological ring.

2.4 Hodge Integrals

In this section, we state several commonly-used formulas for *Hodge integrals*. A Hodge integral is the intersection number of λ and ψ classes, denoted

$$\int_{\overline{\mathcal{M}}_{g,n}} F(\lambda_1, \dots, \lambda_g, \psi_1, \dots, \psi_n).$$

Since intersection numbers refer to the 0-dimensional intersection, then the codimension of a term in the integrand must equal the dimension of $\overline{\mathcal{M}}_{g,n}$ to get a nonzero answer. Thus we can generally regard F as a homogeneous polynomial of degree $3g - 3 + n$.

We begin with properties of Hodge integrals of purely ψ classes.

Theorem 2.4.1 (String equation, Witten [31]). *For stable indices (g, n) ,*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \dots \psi_n^{d_n} = \sum_{i=1}^n \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_i^{d_i-1} \dots \psi_n^{d_n},$$

where terms with negative exponents are ignored.

So if there are no factors of ψ_i for some i , we can use the string equation to reduce the number of marked points in the space by 1 and get a sum of Hodge integrals in that space, which will ideally be easier to compute. Note the string equation can also be used with Hodge integrals where the ψ classes are multiplied by λ_g (see section 2.1 of [13]).

Example 2.4.2. As shown below, we can use the string equation to compute the Hodge integral of $\psi_1\psi_2$ over $\overline{\mathcal{M}}_{0,5}$ as the sum of two Hodge integrals over $\overline{\mathcal{M}}_{0,4}$, then use the string equation again to get Hodge integrals over $\overline{\mathcal{M}}_{0,3}$. Since $\overline{\mathcal{M}}_{0,3}$ is a point, that implies $\int_{\overline{\mathcal{M}}_{0,3}} 1 = 1$.

$$\int_{\overline{\mathcal{M}}_{0,5}} \psi_1\psi_2 = \left(\int_{\overline{\mathcal{M}}_{0,4}} \psi_2 \right) + \left(\int_{\overline{\mathcal{M}}_{0,4}} \psi_1 \right) = \left(\int_{\overline{\mathcal{M}}_{0,3}} 1 \right) + \left(\int_{\overline{\mathcal{M}}_{0,3}} 1 \right) = 1 + 1 = 2.$$

Note that the string equation can be used on any Hodge integral when $g = 0$, since there will always be a factor of ψ_i missing for some i due to the dimension of the space being $n - 3$, resulting in the following formula.

Proposition 2.4.3 (see Lemma 1.5.1 of [21] for a proof). *Suppose d_1, \dots, d_n are nonnegative integers such that $d_1 + \dots + d_n = n - 3$. Then*

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{d_1} \dots \psi_n^{d_n} = \binom{n-3}{d_1, \dots, d_n}.$$

Example 2.4.4.

$$\int_{\overline{\mathcal{M}}_{0,10}} \psi_1^4 \psi_2^2 \psi_3 = \binom{7}{4, 2, 1} = \frac{7!}{4!2!1!} = 105.$$

In higher genus, the string equation alone is often not enough, and one formula commonly used alongside the string equation is the *dilaton equation*.

Theorem 2.4.5 (Dilaton equation, Witten [31]). *For stable indices (g, n) ,*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \dots \psi_n^{d_n} \psi_{n+1} = (2g - 2 + n) \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}.$$

So if there is a single factor of ψ_i for some i , we can use the dilaton equation to remove ψ_i and reduce the number of marked points in the space by 1 while introducing a factor of $(2g - 2 + n)$. In the case of $g = 1$, since the dimension of $\overline{\mathcal{M}}_{1,n}$ is n , there will always be some factor of ψ_i or ψ_i^0 , so all such Hodge integrals can be computed with a combination of the string and dilaton equations. It should also be noted that like with the string equation, the dilaton equation also works for Hodge integrals where the ψ classes are multiplied by λ_g (see section 2.1 of [13]).

Before demonstrating an example using the dilaton equation, we first need a base case to compute Hodge integrals with $(g, n) = (1, 1)$.

Proposition 2.4.6 (see Proposition 2.21 of [34] for a proof). *We have*

$$\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \frac{1}{24}.$$

Example 2.4.7. We'll use the above fact, along with the dilaton and string equations, to compute

$$\int_{\overline{\mathcal{M}}_{1,3}} \psi_1^2 \psi_2.$$

Since we have a single factor of ψ_2 , we can use the dilaton equation to get

$$(2 \cdot 1 - 2 + 2) \int_{\overline{\mathcal{M}}_{1,2}} \psi_1^2 = 2 \int_{\overline{\mathcal{M}}_{1,2}} \psi_1^2.$$

From here, we can use the string equation since we have no factors of ψ_2 , which gives us

$$2 \int_{\overline{\mathcal{M}}_{1,1}} \psi_1.$$

From Proposition 2.4.6, we know this Hodge integral is equal to $\frac{1}{24}$, implying that

$$\int_{\overline{\mathcal{M}}_{1,3}} \psi_1^2 \psi_2 = \frac{1}{12}.$$

For $g > 1$, we won't always be able to use the string or dilaton equations. For these cases, we can use the Witten-Kontsevich Theorem (conjectured by Witten in [31], proven by Kontsevich in [22]), which provides a recursive formula for computing Hodge integrals of ψ classes for all stable indices (g, n) . One particular application of the Witten-Kontsevich Theorem is the following formula for Hodge integrals with $n = 1$.

Proposition 2.4.8 (see Example 3.3.5 of [21] for a derivation). *For $g \geq 1$,*

$$\int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{3g-2} = \frac{1}{24^g g!}$$

Moving on to λ classes, relations over $\overline{\mathcal{M}}_{1,1}$ are of particular importance, since elliptic tails have genus 1 with 1 special point, necessitating the following proposition.

Proposition 2.4.9 (see Example 2.26 of [34]). *Over $\overline{\mathcal{M}}_{1,1}$, the Hodge bundle and cotangent line bundle are isomorphic, so $\lambda_1 = \psi_1$ in $A^*(\overline{\mathcal{M}}_{1,1})$. Further, this implies that*

$$\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}.$$

Next, the λ_g -conjecture provides a way to compute any Hodge integral with a factor of λ_g .

Theorem 2.4.10 (λ_g -conjecture, conjectured in [14], proven in [13]). *For $g \geq 1$ and positive integers d_1, \dots, d_n such that $d_1 + \dots + d_n = 2g + n - 3$, we have*

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi_1^{d_1} \dots \psi_n^{d_n} = \binom{2g + n - 3}{d_1, \dots, d_n} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}. \quad (2.4.0.1)$$

The integrals on the right of (2.4.0.1) also have a closed-form formula:

Theorem 2.4.11 (Faber-Pandharipande [11]). *For $g \geq 1$,*

$$\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2} = \frac{2^{2g-1} - 1}{2^{2g-1}} \cdot \frac{|B_{2g}|}{(2g)!},$$

where B_{2g} is the $2g$ -th Bernoulli number.

Using Theorem 2.2.3, the authors in [4] showed that we can evaluate pseudostable Hodge integrals in terms of stable Hodge integrals.

Theorem 2.4.12 (Cavalieri, Gallegos, Ross, Van Over, Wise [4]). *For any polynomial $F \in \mathbb{C}[x_1, \dots, x_g, y_1, \dots, y_n]$, we have*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} F(\lambda_1, \dots, \lambda_g, \psi_1, \dots, \psi_n) = \int_{\overline{\mathcal{M}}_{g,n}} F(\hat{\lambda}_1, \dots, \hat{\lambda}_g, \psi_1, \dots, \psi_n),$$

where

$$\hat{\lambda}_j = \sum_{i=0}^j \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{j-i})).$$

One implication of Theorem 2.4.12 is that all pseudostable Hodge integrals of purely ψ classes are equal to their stable counterparts. The same authors showed that the same is true for any *linear* pseudostable Hodge integrals, which are those with a linear factor of λ_j .

Theorem 2.4.13 (Cavalieri, Gallegos, Ross, Van Over, Wise [4]). *For any $j = 0, \dots, g$ and polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$, we have*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_j F(\psi_1, \dots, \psi_n) = \int_{\overline{\mathcal{M}}_{g,n}} \lambda_j F(\psi_1, \dots, \psi_n).$$

In Chapter 4, we fully classify quadratic pseudostable Hodge integrals, and while they are not strictly equal to their stable counterparts, we show that they can be expressed as sums of stable Hodge integrals. In Chapter 5, we investigate cubic and higher degree pseudostable Hodge integrals.

Chapter 3

Mumford's Relation for Pseudostable λ Classes

In this section, we prove the pseudostable version of Mumford's relation (Theorem 3.0.1) and provide some applications to the computation of pseudostable Hodge integrals.

Theorem 3.0.1. *For all pseudostable indices (g, n) , we have*

$$(1 + \hat{\lambda}_1 + \cdots + \hat{\lambda}_g)(1 - \hat{\lambda}_1 + \cdots + (-1)^g \hat{\lambda}_g) = \sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j}) \right).$$

The proof of Theorem 3.0.1 is rather technical, so before we start, we study toy examples that will help in understanding the general strategy. The proof of Theorem 3.0.1 will be by induction on g , and the following example serves as the base case with $g = 1$.

Example 3.0.2. For genus $g = 1$, and any fixed $n \geq 1$, consider $(1 + \hat{\lambda}_1)(1 - \hat{\lambda}_1) \in A^*(\overline{\mathcal{M}}_{1,n})$, where $\hat{\lambda}_1 = \lambda_1 + \mathcal{G}_*^1(1)$ by Theorem 2.2.3. Making this substitution, we obtain

$$(1 + \lambda_1 + \mathcal{G}_*^1(1))(1 - \lambda_1 - \mathcal{G}_*^1(1)),$$

which using the $\Lambda_g(k)$ notation from (2.2.0.1) is equal to

$$(\Lambda_1(1) + \mathcal{G}_*^1(1))(\Lambda_1(-1) - \mathcal{G}_*^1(1)).$$

This product produces 4 terms:

1. $\Lambda_1(1)\Lambda_1(-1) = 1$, being Mumford's relation for stable curves.
2. $\Lambda_1(1) \cdot -\mathcal{G}_*^1(1) = -\mathcal{G}_*^1(p_1^*(\Lambda_1(1)))$
3. $\mathcal{G}_*^1(1) \cdot \Lambda_1(-1) = \mathcal{G}_*^1(p_1^*(\Lambda_1(-1)))$
4. $\mathcal{G}_*^1(1) \cdot -\mathcal{G}_*^1(1) = -\mathcal{G}_*^1(-\psi_{\bullet_1} - \psi_{\star_1})$

Recall that when $(g, n) = (1, 1)$, we have $\lambda_1 = \psi_{\bullet_1}$ (Proposition 2.4.9), so these four terms sum together and give

$$\begin{aligned}
& 1 - \mathcal{G}_*^1(p_1^*(\Lambda_1(1))) + \mathcal{G}_*^1(p_1^*(\Lambda_1(-1))) - \mathcal{G}_*^1(-\psi_{\bullet_1} - \psi_{\star_1}) \\
&= 1 - \mathcal{G}_*^1(p_1^*(1 + \lambda_1)) + \mathcal{G}_*^1(p_1^*(1 - \lambda_1)) - \mathcal{G}_*^1(-\psi_{\bullet_1} - \psi_{\star_1}) \\
&= 1 + \mathcal{G}_*^1((-1 - \psi_{\bullet_1}) + (1 - \psi_{\bullet_1}) + (\psi_{\bullet_1} + \psi_{\star_1})) \\
&= 1 + \mathcal{G}_*^1(\psi_{\star_1} - \psi_{\bullet_1}),
\end{aligned}$$

which shows that Theorem 3.0.1 holds in the case of $g = 1$.

In the following example, we show a similar computation to the one in Example 3.0.2, but in arbitrary genus g : we wish to illustrate that if we restrict our attention only to strata parameterizing curves with at most one elliptic tail, then the computation is formally identical independently of the genus. This statement will be true for any fixed number of elliptic tails.

Example 3.0.3. For genus $g \geq 2$, consider

$$(\Lambda_g(1) + \mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(1)))) \cdot (\Lambda_g(-1) - \mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(-1)))) \in A^*(\overline{\mathcal{M}}_{g,n}).$$

Distributing the terms gives 4 products:

1.

$$\Lambda_g(1)\Lambda_g(-1) = 1$$

2.

$$\begin{aligned}
\Lambda_g(1) \cdot -\mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(-1))) &= -\mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(-1)\Lambda_{g-1}(1))p_1^*(\Lambda_1(1))) \\
&= -\mathcal{G}_*^1(p_1^*(\Lambda_1(1)))
\end{aligned}$$

(Note: The final equality uses Mumford's relation in genus $g - 1$.)

3.

$$\begin{aligned}\mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(1))) \cdot \Lambda_g(-1) &= \mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(-1)\Lambda_{g-1}(1))p_1^*(\Lambda_1(-1))) \\ &= \mathcal{G}_*^1(p_1^*(\Lambda_1(-1)))\end{aligned}$$

4.

$$\begin{aligned}&\mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(1))) \cdot -\mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(-1))) \\ &= -\mathcal{G}_*^1(p_0^*(\Lambda_{g-1}(-1)\Lambda_{g-1}(1))(-\psi_{\bullet_1} - \psi_{\star_1})) - \mathcal{G}_*^2(p_0^*(\Lambda_g(-1)\Lambda_g(1))) \\ &= -\mathcal{G}_*^1(-\psi_{\bullet_1} - \psi_{\star_1}) - \mathcal{G}_*^2(1)\end{aligned}$$

Notice the only difference between the results above and those from Example 3.0.2 is one additional term in the 4th product, that being $-\mathcal{G}_*^2(1)$. However, all terms supported on strata parameterizing curves with fewer than 2 elliptic tails are in bijection with the 4 terms from Example 3.0.2, and their sum returns the same answer:

$$1 + \mathcal{G}_*^1(\psi_{\star_1} - \psi_{\bullet_1}).$$

We will use this same idea in the proof of Theorem 3.0.1 by showing that the computation in genus g is formally identical to that in genus $g - 1$ when considering terms supported on strata parameterizing curves with fewer than g elliptic tails.

In the following example, we study the terms of the genus 2 case of Theorem 3.0.1 that are supported on strata parameterizing curves with exactly 2 elliptic tails.

Example 3.0.4. Consider the following product in $A^*(\overline{\mathcal{M}}_{2,n})$:

$$(\Lambda_2(1) + \mathcal{G}_*^1(p_0^*(\Lambda_1(1))) + \frac{1}{2!}\mathcal{G}_*^2(1)) \cdot (\Lambda_2(-1) - \mathcal{G}_*^1(p_0^*(\Lambda_1(-1))) + \frac{1}{2!}\mathcal{G}_*^2(1)).$$

The only products which produce terms supported on strata parameterizing curves with 2 elliptic tails are the following:

1.

$$\begin{aligned} & \Lambda_2(1) \cdot \frac{1}{2!} \mathcal{G}_*^2(1) \\ &= \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(1))p_2^*(\Lambda_1(1))) \end{aligned}$$

2.

$$\begin{aligned} & \mathcal{G}_*^1(p_0^*(\Lambda_1(1))) \cdot (-\mathcal{G}_*^1(p_0^*(\Lambda_1(-1)))) \\ &= -\mathcal{G}_*^1(-\psi_{\bullet_1} - \psi_{\star_1}) - \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(1))p_2^*(\Lambda_1(-1))) - \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(-1))p_2^*(\Lambda_1(1))) \end{aligned}$$

3.

$$\begin{aligned} & \mathcal{G}_*^1(p_0^*(\Lambda_1(1))) \cdot \frac{1}{2!} \mathcal{G}_*^2(1) \\ &= \frac{1}{2!} \mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})p_2^*(\Lambda_1(1))) + \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(1))(-\psi_{\bullet_2} - \psi_{\star_2})) \end{aligned}$$

4.

$$\begin{aligned} & \frac{1}{2!} \mathcal{G}_*^2(1) \cdot \Lambda_2(-1) \\ &= \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(-1))p_2^*(\Lambda_1(-1))) \end{aligned}$$

5.

$$\begin{aligned} & \frac{1}{2!} \mathcal{G}_*^2(1) \cdot (-\mathcal{G}_*^1(p_0^*(\Lambda_1(1)))) \\ &= -\frac{1}{2!} \mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})p_2^*(\Lambda_1(-1))) - \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(-1))(-\psi_{\bullet_2} - \psi_{\star_2})) \end{aligned}$$

6.

$$\begin{aligned} & \frac{1}{2!} \mathcal{G}_*^2(1) \cdot \frac{1}{2!} \mathcal{G}_*^2(1) \\ &= \frac{1}{2!} \mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})(-\psi_{\bullet_2} - \psi_{\star_2})) \end{aligned}$$

From the 6 products above, there are 9 terms that are supported on strata parameterizing curves with 2 elliptic tails. These can be partitioned into groups of three based on a common factor on the first elliptic tail.

Group 1, $\Lambda_1(1)$ on the first elliptic tail:

$$\begin{aligned} & \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(1))p_2^*(\Lambda_1(1))) \\ & - \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(1))p_2^*(\Lambda_1(-1))) \\ & \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(1))(-\psi_{\bullet_2} - \psi_{\star_2})) \end{aligned}$$

Group 2, $\Lambda_1(-1)$ on the first elliptic tail:

$$\begin{aligned} & - \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(-1))p_2^*(\Lambda_1(1))) \\ & \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(-1))p_2^*(\Lambda_1(-1))) \\ & - \frac{1}{2!} \mathcal{G}_*^2(p_1^*(\Lambda_1(-1))(-\psi_{\bullet_2} - \psi_{\star_2})) \end{aligned}$$

Group 3, $-\psi_{\bullet_1} - \psi_{\star_1}$ on the first elliptic tail:

$$\begin{aligned} & \frac{1}{2!} \mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})p_2^*(\Lambda_1(1))) \\ & - \frac{1}{2!} \mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})p_2^*(\Lambda_1(-1))) \\ & \frac{1}{2!} \mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})(-\psi_{\bullet_2} - \psi_{\star_2})) \end{aligned}$$

Notice in each group of three, the terms on the second elliptic tail are in bijection with the three terms added together at the end of Example 3.0.2. Thus, the terms in each group sum together in

the same way (up to a negative) on the second elliptic tail:

$$-p_2^*(\Lambda_1(1)) + p_2^*(\Lambda_1(-1)) - (-\psi_{\bullet_2} - \psi_{\star_2}) = \psi_{\star_2} - \psi_{\bullet_2}.$$

Each of the groups are now as follows:

Group 1:

$$-\frac{1}{2!}\mathcal{G}_*^2(p_1^*(\Lambda_1(1))(\psi_{\star_2} - \psi_{\bullet_2}))$$

Group 2:

$$\frac{1}{2!}\mathcal{G}_*^2(p_1^*(\Lambda_1(-1))(\psi_{\star_2} - \psi_{\bullet_2}))$$

Group 3:

$$-\frac{1}{2!}\mathcal{G}_*^2((-\psi_{\bullet_1} - \psi_{\star_1})(\psi_{\star_2} - \psi_{\bullet_2}))$$

Now that the computation has been reduced from 9 terms to 3, notice that these 3 terms all have the same decorations on the second elliptic tail, namely $(\psi_{\star_2} - \psi_{\bullet_2})$, and the decorations on the first elliptic tail are in bijection with the same three terms from the end of Example 3.0.2 as before. Thus, these three terms add together to on the first elliptic tail in the same way:

$$-p_1^*(\Lambda_1(1)) + p_1^*(\Lambda_1(-1)) - (-\psi_{\bullet_1} - \psi_{\star_1}) = \psi_{\star_1} - \psi_{\bullet_1}.$$

This leads in the following sum of the groups above:

$$\frac{1}{2!}\mathcal{G}_*^2((\psi_{\star_1} - \psi_{\bullet_1})(\psi_{\star_2} - \psi_{\bullet_2}))$$

This is exactly the term supported on 2 elliptic tails claimed in the statement of Theorem 3.0.1. We'll use the same idea in the proof by showing that for the genus g computation, the terms supported on g elliptic tails can be partitioned into three groups, each of which is in bijection with the terms supported on on $g - 1$ elliptic tails in the genus $g - 1$ computation.

3.1 Proof

We are now ready to prove Theorem 3.0.1: Mumford's relation for pseudostable λ classes.

Proof. We must compute

$$(1 + \hat{\lambda}_1 + \cdots + \hat{\lambda}_g)(1 - \hat{\lambda}_1 + \cdots + (-1)^g \hat{\lambda}_g), \quad (3.1.0.1)$$

where by Theorem 2.2.3, $\hat{\lambda}_j = \sum_{i=0}^j \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{j-i}))$. Substituting these expressions, exchanging the order of summation, and using the $\Lambda_g(k)$ notation, we can rewrite (3.1.0.1) as

$$\left(\sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\Lambda_{g-i}(1))) \right) \left(\sum_{j=0}^g \frac{(-1)^j}{j!} \mathcal{G}_*^j(p_0^*(\Lambda_{g-j}(-1))) \right). \quad (3.1.0.2)$$

We'll proceed by induction on g . For the base case of $g = 1$, see Example 3.0.2. Assume that the theorem is true for genus $g - 1$, meaning

$$\left(\sum_{i=0}^{g-1} \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\Lambda_{(g-1)-i}(1))) \right) \left(\sum_{j=0}^{g-1} \frac{(-1)^j}{j!} \mathcal{G}_*^j(p_0^*(\Lambda_{(g-1)-j}(-1))) \right) \quad (3.1.0.3)$$

is equal to

$$\sum_{i=0}^{g-1} \frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j}) \right) \quad (3.1.0.4)$$

when working in genus $g - 1$.

In the genus g case, the truncated product with the first $g - 1$ terms is instead written as

$$\left(\sum_{i=0}^{g-1} \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\Lambda_{g-i}(1))) \right) \left(\sum_{j=0}^{g-1} \frac{(-1)^j}{j!} \mathcal{G}_*^j(p_0^*(\Lambda_{g-j}(-1))) \right),$$

with the only difference from (3.1.0.3) being $\Lambda_{g-i}(1)$ and $\Lambda_{g-j}(-1)$ instead of $\Lambda_{(g-1)-i}(1)$ and $\Lambda_{(g-1)-j}(-1)$, respectively. However, the genus of the factor corresponding to the root vertex also increases by 1, so the resulting computation will be formally identical to the genus $g - 1$ case for all terms supported on the strata parameterizing curves with fewer than g elliptic tails. One can

define a bijection between the decorated graphs by assigning to a decorated graph of genus g a corresponding genus $g - 1$ decorated graph where the genus of the root vertex has been lowered by one; this was demonstrated in Example 3.0.3 by showing that the computation in the genus $g \geq 2$ case is the same as the genus 1 case for terms supported on strata parameterizing curves with fewer than 2 elliptic tails. Using the inductive hypothesis, the terms supported on strata parameterizing curves with fewer than g elliptic tails simplify to

$$\sum_{i=0}^{g-1} \frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j}) \right).$$

What remains to show is that all terms supported on strata parameterizing curves with exactly g elliptic tails sum to

$$\frac{1}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \right).$$

Consider all terms in the product (3.1.0.2) which are supported on a stratum parameterizing curves with exactly g elliptic tails. Each term has a dual decorated graph A with g genus 1 vertices, which has a coefficient based on the (G, H) -structure obtained from intersecting two strata G and H with an G -structure and H -structure, respectively. Each coefficient will include $\frac{1}{|\text{Aut}(A)|} = \frac{1}{g!}$ as in Definition 2.3.5.

Let the graph G correspond to a term in the left sum of (3.1.0.2) and the graph H correspond to a term in the right sum of (3.1.0.2), and consider the last edge and genus one vertex of the graph A . If the last edge of A is only from the G -structure, then its coefficient will decorate the vertex with $\Lambda_1(-1)$. If the last edge of A is only from the H -structure, then its coefficient will have $-\Lambda_1(1)$ on the genus one vertex. If the last edge of A is from both the G - and H -structure, then the edge is decorated by $-(\psi_{\bullet_1} - \psi_{\star_1})$.

Consider the set of all possible graphs such that the coefficient that is pulled back from the last factor (i.e., decorating the last genus one vertex) is $\Lambda_1(-1)$. If we factor out this coefficient, what remains will be those coefficients that were pulled back from the other factors. As in Example 3.0.4, removing the last edge from every decorated graph gives a natural bijection between these

remaining coefficients and the coefficients of the factors of the term supported on $g - 1$ elliptic tails; we have shown inductively in (3.1.0.4) those add up to:

$$F_{g-1} := \prod_{j=1}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}),$$

decorating the first $g - 1$ edges of the dual graph. The same argument holds for the collection of all terms where the last vertex is decorated by $-\Lambda_1(1)$ or by $-(\psi_{\bullet_1} - \psi_{\star_1})$. By induction, summing all these terms, the first $g - 1$ edges will be decorated by the same coefficient of F_{g-1} , so we can sum over the last factor to get

$$-p_g^*(\Lambda_1(1)) + p_g^*(\Lambda_1(-1)) - (-\psi_{\bullet_g} - \psi_{\star_g}) = \psi_{\star_g} - \psi_{\bullet_g}.$$

Along with F_{g-1} and $\frac{1}{g!}$, this returns the desired coefficient for strata parameterizing curves with exactly g elliptic tails:

$$\frac{1}{g!} F_{g-1} (\psi_{\star_g} - \psi_{\bullet_g}) = \frac{1}{g!} \prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}),$$

which concludes the proof. □

3.2 Applications

Much like the classic version of Mumford's relation, Theorem 3.0.1 provides a set of relations for λ classes in $A^*(\overline{\mathcal{M}}_{g,n}^{\text{ps}})$ for all pseudostable indices (g, n) .

Corollary 3.2.1. *For all pseudostable indices (g, n) and $i = 0, \dots, g$, we have*

$$\sum_{j=\max\{0, 2i-g\}}^{\min\{2i, g\}} (-1)^j \hat{\lambda}_{2i-j} \hat{\lambda}_j = \frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j}) \right).$$

Proof. With Theorem 3.0.1, we know that the codimension $2i$ terms of

$$(1 + \hat{\lambda}_1 + \dots + \hat{\lambda}_g)(1 - \hat{\lambda}_1 + \dots + (-1)^g \hat{\lambda}_g) \tag{3.2.0.1}$$

are equal to the codimension $2i$ term of

$$\sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j}) \right). \quad (3.2.0.2)$$

The codimension $2i$ terms of (3.2.0.1) are of the form

$$(-1)^j \hat{\lambda}_{2i-j} \hat{\lambda}_j,$$

where $\max\{0, 2i-g\} \leq j \leq \min\{2i, g\}$. The reason for $2i-g$ in the lower bound is that $\hat{\lambda}_{2i-j} = 0$ when $2i-j > g$. Similarly, we include g in the upper bound of j because $\hat{\lambda}_j = 0$ for $j > g$.

Finally, the codimension $2i$ term of (3.2.0.2) is

$$\frac{1}{i!} \mathcal{G}_*^i \left(\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j}) \right)$$

since the codimension of each of $\mathcal{G}_*^i(1)$ and $\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j})$ is i . □

The following example highlights a few specific relations from Corollary 3.2.1.

Example 3.2.2. Codimension $2g$ ($i = g$ in Corollary 3.2.1):

$$\hat{\lambda}_g^2 = \frac{(-1)^g}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \right).$$

Codimension $2g-2$ ($i = g-1$ in Corollary 3.2.1):

$$2\hat{\lambda}_g \hat{\lambda}_{g-2} - \hat{\lambda}_{g-1}^2 = \frac{(-1)^g}{(g-1)!} \mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}) \right).$$

Codimension 2 ($i = 1$ in Corollary 3.2.1):

$$2\hat{\lambda}_2 - \hat{\lambda}_1^2 = \frac{1}{2} \mathcal{G}_*^1 (\psi_{\star_1} - \psi_{\bullet_1}).$$

We can also apply Theorem 3.0.1 to Hodge integrals:

Proposition 3.2.3. *For all pseudostable indices (g, n) ,*

$$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \frac{(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{\prod_{j=1}^n (1 - \psi_j)} = \sum_{i=0}^g \frac{(-1)^i}{24^i i!} \int_{\mathcal{M}_{g-i, n+i}} \frac{1}{\prod_{j=1}^n (1 - \psi_j)},$$

where the denominator should be intended as a shorthand for the corresponding expansion as a product of geometric series.

Proof. We start by applying Theorem 3.0.1:

$$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \frac{(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{\prod_{j=1}^n (1 - \psi_j)} = \int_{\mathcal{M}_{g,n}} \sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i \left(\frac{\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j})}{\prod_{j=1}^n (1 - p_0^*(\psi_j))} \right).$$

Notice that in the product $\prod_{j=1}^i (\psi_{\star_j} - \psi_{\bullet_j})$, the integral vanishes unless there is a class $-\psi_{\bullet_j}$ on each flag adjacent to the genus one tail-vertex. This reduces the Hodge integral to

$$\begin{aligned} & \int_{\mathcal{M}_{g,n}} \sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i \left(\frac{\prod_{j=1}^i (-\psi_{\bullet_j})}{\prod_{j=1}^n (1 - p_0^*(\psi_j))} \right) \\ &= \sum_{i=0}^g \frac{1}{i!} \left(\int_{\mathcal{M}_{1,1}} -\psi_1 \right)^i \left(\int_{\mathcal{M}_{g-i, n+i}} \frac{1}{\prod_{j=1}^n (1 - \psi_j)} \right). \end{aligned} \quad (3.2.0.3)$$

The first integral of (3.2.0.3) is known to be $-\frac{1}{24}$ (Proposition 2.4.6), simplifying (3.2.0.3) to

$$\sum_{i=0}^g \frac{(-1)^i}{24^i i!} \int_{\mathcal{M}_{g-i, n+i}} \frac{1}{\prod_{j=1}^n (1 - \psi_j)},$$

completing the proof. □

If we restrict Proposition 3.2.3 to just a single ψ class, that is, replace $\frac{1}{\prod_{j=1}^n (1 - \psi_j)}$ by $\frac{1}{1 - \psi_1}$, the resulting Hodge integrals can be easily computed with the string equation (Theorem 2.4.1), leading to the following corollary.

Corollary 3.2.4. *For all pseudostable indices (g, n) , we have*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{1 - \psi_1} = 0.$$

Proof. Using Proposition 3.2.3 implies

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{1 - \psi_1} = \sum_{i=0}^g \frac{(-1)^i}{24^i i!} \int_{\overline{\mathcal{M}}_{g-i,n+i}} \frac{1}{1 - \psi_1}.$$

Since the dimension of $\overline{\mathcal{M}}_{g-i,n+i}$ is $3g - 2i + n - 3$, the only non-vanishing term in the geometric series expansion of $\frac{1}{1 - \psi_1}$ is $\psi_1^{3g-2i+n-3}$, simplifying our work to

$$\sum_{i=0}^g \frac{(-1)^i}{24^i i!} \int_{\overline{\mathcal{M}}_{g-i,n+i}} \psi_1^{3g-2i+n-3}. \quad (3.2.0.4)$$

Using the string equation (Theorem 2.4.1), we can reduce the number of marked points in the integral to get the following:

$$\left(\sum_{i=0}^{g-1} \frac{(-1)^i}{24^i i!} \int_{\overline{\mathcal{M}}_{g-i,1}} \psi_1^{3g-3i-2} \right) + \frac{(-1)^g}{24^g g!} \int_{\overline{\mathcal{M}}_{0,3}} 1. \quad (3.2.0.5)$$

We can evaluate the integrals in the summation of (3.2.0.5) using an application of the Witten-Kontsevich Theorem for one-pointed psi class intersection numbers (Proposition 2.4.8):

$$\int_{\overline{\mathcal{M}}_{g-i,1}} \psi_1^{3(g-i)-2} = \frac{1}{24^{g-i} (g-i)!}. \quad (3.2.0.6)$$

Since $\overline{\mathcal{M}}_{0,3}$ is a point, then the latter integral of (3.2.0.5) is equal to 1. Together with (3.2.0.6), this simplifies (3.2.0.5) to

$$\begin{aligned}
\left(\sum_{i=0}^{g-1} \frac{(-1)^i}{24^i i!} \cdot \frac{1}{24^{g-i} (g-i)!} \right) + \frac{(-1)^g}{24^g g!} &= \left(\sum_{i=0}^{g-1} \frac{(-1)^i}{24^g i! (g-i)!} \right) + \frac{(-1)^g}{24^g g!} \\
&= \left(\sum_{i=0}^g \frac{(-1)^i}{24^g i! (g-i)!} \right) \\
&= \frac{1}{24^g} \sum_{i=0}^g \frac{(-1)^i}{i! (g-i)!} \\
&= 0.
\end{aligned} \tag{3.2.0.7}$$

□

The summation in (3.2.0.7) highlights another result for specific terms of Corollary 3.2.4:

Corollary 3.2.5. *For all pseudostable indices (g, n) and $k = 0, \dots, g$,*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \left(\sum_{j=\max\{0, 2k-g\}}^{\min\{2k, g\}} (-1)^j \lambda_{2k-j} \lambda_j \right) \psi_1^{3g-2k+n-3} = \frac{(-1)^k}{24^g k! (g-k)!}.$$

Proof. We observe first of all that the Hodge part in the parenthesis is the degree $2k$ part of $(1 + \lambda_1 + \dots + \lambda_g)(1 - \lambda_1 + \dots + (-1)^g \lambda_g)$. Proceeding as in the proof of Corollary 3.2.4 above, we have

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \left(\sum_{j=\max\{0, 2k-g\}}^{\min\{2k, g\}} (-1)^j \lambda_{2k-j} \lambda_j \right) \psi_1^{3g-2k+n-3} = \sum_{i=0}^g \frac{(-1)^i}{24^i i!} \int_{\overline{\mathcal{M}}_{g-i, n+i}} \psi_1^{3g-2k+n-3}. \tag{3.2.0.8}$$

By dimension reasons the only non-vanishing summand corresponds to $i = k$. We recognize this term as the k -th summand of (3.2.0.7), which proves the result. □

The following example highlights specific cases of Corollary 3.2.5.

Example 3.2.6. For $i = g$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^2 \psi_1^{g-3+n} = \frac{1}{24^g g!}.$$

For $i = g - 1$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} (2\lambda_g \lambda_{g-2} - \lambda_{g-2}^2) \psi_1^{g-1+n} = \frac{-1}{24^g (g-1)!}.$$

For $i = 1$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} (2\lambda_2 - \lambda_1^2) \psi_1^{3g-5+n} = \frac{-1}{24^g (g-1)!}.$$

Remark 3.2.7. We observe a curious duality among the following stable and pseudostable Hodge integrals:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^2 \psi_1^{g-3+n} = \int_{\overline{\mathcal{M}}_{g,n}} \frac{(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{1 - \psi_1}, \quad (3.2.0.9)$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g^2 \psi_1^{g-3+n} = \int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{1 - \psi_1}. \quad (3.2.0.10)$$

For $g = 0$ and any value of $n \geq 3$, both lines are giving the equation $1 = 1$. In higher genus, (3.2.0.9) evaluates to $\frac{1}{24^g g!}$: we saw the pseudostable integral in Example 3.2.6, whereas the stable evaluation is obtained from (3.2.0.6) after applying Mumford's relation to the numerator. The second line (3.2.0.10) evaluates to 0: in the stable case this follows from Mumford's relation ($\lambda_g^2 = 0$), whereas the pseudostable integral is Corollary 3.2.4.

3.3 Relation for $\hat{\lambda}_g \hat{\lambda}_{g-1}$

Notice that all the relations provided by Theorem 3.0.1 are for terms of even codimension. In this section, we prove a relation for $\hat{\lambda}_g \hat{\lambda}_{g-1}$, which has codimension $2g - 1$.

Theorem 3.3.1. *For all pseudostable indices (g, n) ,*

$$\hat{\lambda}_g \hat{\lambda}_{g-1} = \left(\sum_{k=0}^{g-1} \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k} \lambda_{(g-1)-k}) \prod_{j=1}^k (\psi_{\bullet_j} - \psi_{\star_j}) \right) \right) + \frac{1}{(g-1)!} \mathcal{G}_*^g \left(\prod_{j=1}^{g-1} (\psi_{\bullet_j} - \psi_{\star_j}) \right).$$

Proof. We start by applying Theorem 2.2.3:

$$\hat{\lambda}_g \hat{\lambda}_{g-1} = \left(\sum_{r=0}^g \frac{1}{r!} \mathcal{G}_*^r(p_0^*(\lambda_{g-r})) \right) \left(\sum_{s=0}^{g-1} \frac{1}{s!} \mathcal{G}_*^s(p_0^*(\lambda_{(g-1)-s})) \right).$$

The result of the above product contains many terms supported on strata parameterizing curves with k elliptic tails, where $\max\{r, s\} \leq k \leq \min\{r + s, g\}$. Such terms occur in the expansion of the term indexed by (r, s) if the integer tuple (r, s) satisfies the following conditions:

$$0 \leq r \leq k, \quad 0 \leq s \leq k, \quad r + s \geq k. \quad (3.3.0.1)$$

For each pair (r, s) , we describe the coefficient of the k -tails stratum in the following product:

$$\frac{1}{r!} \mathcal{G}_*^r(p_0^*(\lambda_{g-r})) \cdot \frac{1}{s!} \mathcal{G}_*^s(p_0^*(\lambda_{(g-1)-s})). \quad (3.3.0.2)$$

Using notation from [32], let G and H be the two decorated graphs representing $\mathcal{G}_*^r(p_0^*(\lambda_{g-r}))$ and $\mathcal{G}_*^s(p_0^*(\lambda_{(g-1)-s}))$, respectively, and let A be the unique isomorphism class of decorated graphs in the resulting product supported on a stratum parameterizing curves with k elliptic tails. Our goal is to identify its decorations and numerical coefficient. Of the k edges of A , there are $k - s$ edges from only the G -structure, $k - r$ edges from only the H -structure, and $N := r + s - k$ edges from both the G and H -structures. We call these three types of edges G -edges, H -edges and (G, H) -edges, respectively. All the edges, together with their adjacent vertices have associated decorations. We break the computation in four parts.

DECORATIONS OF (G, H) -EDGES. The decoration for each of these edges is $(-\psi_\bullet - \psi_\star)$, and with there being $N := r + s - k$ such edges, the contribution is

$$\mathcal{G}_*^k \left(\prod_{j=1}^N (-\psi_{\bullet_j} - \psi_{\star_j}) \right). \quad (3.3.0.3)$$

DECORATIONS OF H -EDGES. Since G supports a λ_{g-r} class on its root vertex, and A has a root vertex of genus $g - k$, then λ_{g-r} will split along the H -edges of A in the following way:

$$\mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=N+1}^{N+(k-r)} p_j^*(\lambda_1) \right). \quad (3.3.0.4)$$

DECORATIONS OF G -EDGES. Since H supports a $\lambda_{(g-1)-s}$ class on its root vertex, and A has a root vertex of genus $g - k$, then λ_{g-s} will split along the G -edges of A in one of the following two ways:

$$\mathcal{G}_*^k \left(p_0^*(\lambda_{(g-1)-k}) \prod_{j=N+(k-r)+1}^k p_j^*(\lambda_1) \right) \quad (3.3.0.5)$$

or

$$\mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=N+(k-r)+1}^{k-1} p_j^*(\lambda_1) \right). \quad (3.3.0.6)$$

However, (3.3.0.6) will vanish for all $k \leq g - 1$ when combined with the decorations from (3.3.0.4) since $\lambda_{g-k}^2 = 0$ in $\overline{\mathcal{M}}_{g-k, n+k}$ due to Mumford's relation:

$$\mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) p_0^*(\lambda_{g-k}) \prod_{j=N+1}^{k-1} p_j^*(\lambda_1) \right) = 0.$$

For $k = g$, we have $\lambda_{g-k}^2 = 1$, so (3.3.0.6) will not vanish. However, (3.3.0.5) will vanish for this case since $\lambda_{-1} = 0$. For now, we focus only on $k \leq g - 1$, and $k = g$ will follow similarly.

NUMERICAL COEFFICIENT. There are many different but isomorphic (G, H) -structures on the graph \underline{A} . One possible way to organize the count for $k \leq g - 1$ is the following. We first count the possible ways to obtain the (G, H) -edges: one must choose N of the k edges of A , then N of the r and s edges of G and H , respectively; finally there are $N!$ ways to map the N chosen edges of both G and H to the chosen edges of A . This gives us the following count:

$$\binom{k}{N} \binom{r}{N} \binom{s}{N} N! N!,$$

which simplifies to

$$\frac{k!r!s!}{N!(k-N)!(r-N)!(s-N)!}$$

Having dealt with the (G, H) -edges, now we have to choose $r - N$ of the $k - N$ remaining edges of A to create subsets of size $r - N$ and $s - N$; then there are $(r - N)!$ and $(s - N)!$ bijections of the chosen subsets of A into the chosen subsets of G and H , respectively. This gives us the following count:

$$\binom{k-N}{r-N} (r-N)!(s-N)!,$$

which simplifies to $(k - N)!$. Multiplying the two, we obtain the total number of (G, H) -structures on \underline{A} . The numerical coefficient for \underline{A} in the multiplication (4.1.0.3) is given by

$$\frac{1}{r!} \cdot \frac{1}{s!} \cdot \frac{1}{k!} \cdot \frac{k!r!s!(k-N)!}{N!(k-N)!(r-N)!(s-N)!} = \frac{1}{N!(r-N)!(s-N)!}, \quad (3.3.0.7)$$

where the factor of $\frac{1}{k!}$ is from $|\text{Aut}(A)| = k!$, following the algorithm for multiplication of graphs G and H shown in [32].

TOTAL CONTRIBUTION. Combining all types of decorations, and the numerical coefficient, the (isomorphism class of the) decorated graph A for $k \leq g - 1$ is given by

$$\frac{1}{N!(r-N)!(s-N)!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}\lambda_{(g-1)-k}) \prod_{j=1}^N (-\psi_{\bullet_j} - \psi_{\star_j}) \prod_{j=N+1}^{N+(k-r)} p_j^*(\lambda_1) \prod_{j=N+(k-r)+1}^k p_j^*(\lambda_1) \right)$$

Since $\lambda_1 = \psi_1$ on $\overline{\mathcal{M}}_{1,1}$, the above term is equal to

$$\frac{1}{N!(r-N)!(s-N)!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}\lambda_{(g-1)-k}) \prod_{j=1}^N (-\psi_{\bullet_j} - \psi_{\star_j}) \prod_{j=N+1}^{N+(k-r)} \psi_{\bullet_j} \prod_{j=N+(k-r)+1}^k \psi_{\bullet_j} \right). \quad (3.3.0.8)$$

We will get the term in (3.3.0.8) for all pairs (r, s) satisfying the conditions in (3.3.0.1), which is equivalent to (r, s) satisfying $0 \leq r \leq k$ and $k - r \leq s \leq k$. Summing over all such terms,

notice that

$$\sum_{r=0}^k \sum_{s=k-r}^k \left(\binom{k}{N} \binom{k-N}{r-N} \prod_{j=1}^N (-\psi_{\bullet_j} - \psi_{\star_j}) \prod_{j=N+1}^{N+(k-r)} \psi_{\bullet_j} \prod_{j=N+(k-r)+1}^k \psi_{\bullet_j} \right) = \prod_{j=1}^k (\psi_{\bullet_j} - \psi_{\star_j}). \quad (3.3.0.9)$$

This is because we can reindex the ψ classes to get isomorphic strata (e.g. the stratum supporting $(-\psi_{\bullet_1} - \psi_{\star_1})\psi_{\bullet_2}\psi_{\bullet_3}$ is isomorphic to the stratum supporting $(-\psi_{\bullet_2} - \psi_{\star_2})\psi_{\bullet_3}\psi_{\bullet_1}$). Since we chose N of k edges to be decorated with $(-\psi_{\bullet_j} - \psi_{\star_j})$, and $r - N$ of the remaining $k - N$ edges to be decorated with ψ_{\bullet_j} , the number of ways to reindex the ψ classes is $\binom{k}{N} \binom{k-N}{r-N}$.

Using (3.3.0.9), we can sum (3.3.0.8) for all (r, s) satisfying the conditions in (3.3.0.1) to get

$$\frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}\lambda_{(g-1)-k}) \prod_{j=1}^k (\psi_{\bullet_j} - \psi_{\star_j}) \right).$$

Thus terms of $\hat{\lambda}_g \hat{\lambda}_{g-1}$ up to $k = g - 1$ are

$$\sum_{k=0}^{g-1} \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}\lambda_{(g-1)-k}) \prod_{j=1}^k (\psi_{\bullet_j} - \psi_{\star_j}) \right).$$

The term for $k = g$ follows the same process, but the graph \underline{A} is that from (3.3.0.6), and the coefficient is $\frac{1}{(g-1)!}$ instead of $\frac{1}{g!}$ due to only $g - 1$ of the edges being decorated with ψ classes:

$$\frac{1}{(g-1)!} \mathcal{G}_*^g \left(\prod_{j=1}^{g-1} (\psi_{\bullet_j} - \psi_{\star_j}) \right).$$

This is the final term in the statement of the theorem, proving the result. □

Chapter 4

Quadratic Pseudostable Hodge Integrals

In this section, we give a formula for computing all quadratic pseudostable Hodge integrals in terms of their stable counterparts, Theorem 4.1.1. We then show in Theorem 4.1.2 that the formulas arising from Theorem 4.1.1 are nicely encoded as a relation among generating functions for quadratic Hodge integrals. We begin with a Lemma which will be used to simplify some of the algebra in the proof of Theorem 4.1.1.

Lemma 4.0.1. *For all non-negative integers k ,*

$$\sum_{(r,s) \in T} \frac{(-1)^{r+s-k}}{(r+s-k)!(k-s)!(k-r)!} = \frac{1}{k!},$$

where $T \subset \mathbb{Z}^2$ is the set of integer tuples (r, s) with $0 \leq r \leq k, 0 \leq s \leq k$, and $r + s \geq k$.

Proof. Note that T can equivalently be defined by all (r, s) where $0 \leq r \leq k$ and $k - r \leq s \leq k$.

So the sum is

$$\sum_{r=0}^k \left(\sum_{s=k-r}^k \frac{(-1)^{r+s-k}}{(r+s-k)!(k-s)!(k-r)!} \right) = \sum_{r=0}^k \frac{1}{(k-r)!} \left(\sum_{s=k-r}^k \frac{(-1)^{r+s-k}}{(r+s-k)!(k-s)!} \right).$$

For convenience, define $N := r + s - k$, which leads to

$$\sum_{r=0}^k \frac{1}{(k-r)!} \left(\sum_{N=0}^r \frac{(-1)^N}{N!(r-N)!} \right).$$

The inner sum is equal to 0, except when $r = 0$, in which case it is equal to 1. This allows us to reduce the work to just that case:

$$\sum_{r=0}^0 \frac{1}{(k-r)!} = \frac{1}{k!},$$

proving the result. □

4.1 Theorem: Relation to Quadratic Stable Hodge Integrals

Theorem 4.1.1. *For all pseudostable indices (g, n) , $i, j = 0, \dots, g$, and for any polynomial $F \in \mathbb{C}[a_1, \dots, a_n]$,*

$$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \lambda_i \lambda_j F(\psi_1, \dots, \psi_n) = \sum_{k=0}^{\min\{i,j\}} \frac{1}{24^k k!} \int_{\mathcal{M}_{g-k, n+k}} \lambda_{i-k} \lambda_{j-k} F(\psi_1, \dots, \psi_n).$$

Proof. From Theorem 2.2.3, we know

$$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \lambda_i \lambda_j F(\psi_1, \dots, \psi_n) = \int_{\mathcal{M}_{g,n}} \hat{\lambda}_i \hat{\lambda}_j F(\psi_1, \dots, \psi_n),$$

where

$$\hat{\lambda}_i = \sum_{r=0}^i \frac{1}{r!} \mathcal{G}_*^r(p_0^*(\lambda_{i-r})) \text{ and } \hat{\lambda}_j = \sum_{s=0}^j \frac{1}{s!} \mathcal{G}_*^s(p_0^*(\lambda_{j-s})).$$

Consider the product

$$\hat{\lambda}_i \hat{\lambda}_j = \left(\sum_{r=0}^i \frac{1}{r!} \mathcal{G}_*^r(p_0^*(\lambda_{i-r})) \right) \left(\sum_{s=0}^j \frac{1}{s!} \mathcal{G}_*^s(p_0^*(\lambda_{j-s})) \right). \quad (4.1.0.1)$$

The result of (4.1.0.1) contains many terms supported on strata parameterizing curves with k elliptic tails, where $\max\{r, s\} \leq k \leq \min\{r + s, g\}$. Such terms occur in the expansion of the term indexed by (r, s) if the integer tuple (r, s) satisfies the following conditions:

$$0 \leq r \leq k, \quad 0 \leq s \leq k, \quad r + s \geq k, \quad (4.1.0.2)$$

i.e., it belongs to the indexing set T from Lemma 4.0.1. For each pair (r, s) in T , we describe the coefficient of the k -tails stratum in the following product:

$$\frac{1}{r!} \mathcal{G}_*^r(p_0^*(\lambda_{i-r})) \cdot \frac{1}{s!} \mathcal{G}_*^s(p_0^*(\lambda_{j-s})). \quad (4.1.0.3)$$

Using notation from [32], let G and H be the two decorated graphs representing $\mathcal{G}_*^r(p_0^*(\lambda_{i-r}))$ and $\mathcal{G}_*^s(p_0^*(\lambda_{i-s}))$, respectively, and let A be the unique isomorphism class of decorated graphs in the resulting product supported on a stratum parameterizing curves with k elliptic tails. Our goal is to identify its decorations and numerical coefficient. Of the k edges of A , there are $k-s$ edges from only the G -structure, $k-r$ edges from only the H -structure, and $N := r+s-k$ edges from both the G and H -structures. We call these three types of edges G -edges, H -edges and (G, H) -edges, respectively. All the edges, together with their adjacent vertices have associated decorations. We break the computation in three parts.

DECORATIONS OF (G, H) -EDGES. The decoration for each of these edges is $(-\psi_{\bullet} - \psi_{\star})$, and with there being $N := r+s-k$ such edges, the contribution is

$$\mathcal{G}_*^k \left(\prod_{t=1}^N (-\psi_{\bullet_t} - \psi_{\star_t}) \right). \quad (4.1.0.4)$$

We observe that when integrating over $\overline{\mathcal{M}}_{g,n}$, any non-vanishing term must have a class $-\psi_{\bullet_t}$ on each flag adjacent to the genus one tail-vertex, thus the only non-vanishing contribution to the Hodge integral from (4.1.0.4) is

$$(-1)^N \mathcal{G}_*^k \left(\prod_{t=1}^N \psi_{\bullet_t} \right). \quad (4.1.0.5)$$

DECORATIONS OF G -EDGES OR H -EDGES. By how the Hodge bundle splits along boundary strata, the decorations of the tail vertices of these edges are either 1 or λ_1 . After integrating over $\overline{\mathcal{M}}_{g,n}$, all terms that have edges with a coefficient of 1 associated to any of the outer vertices vanish by dimension reasons, so we can disregard them here. It follows that for the contributing terms, on the root vertex there is the following coefficient:

$$\lambda_{(i-r)-(k-r)} \lambda_{(j-s)-(k-s)} = \lambda_{i-k} \lambda_{j-k}.$$

Thus the contribution from these edges is

$$\mathcal{G}_*^k \left(p_0^*(\lambda_{i-k}\lambda_{j-k}) \prod_{t=N+1}^k p_t^*(\lambda_1) \right). \quad (4.1.0.6)$$

NUMERICAL COEFFICIENT. There are many different but isomorphic (G, H) -structures on the graph \underline{A} . One possible way to organize the count is the following. We first count the possible ways to obtain the (G, H) -edges: one must choose N of the k edges of A , then N of the r and s edges of G and H , respectively; finally there are $N!$ ways to map the N chosen edges of both G and H to the chosen edges of A . This gives us the following count:

$$\binom{k}{N} \binom{r}{N} \binom{s}{N} N! N! (-1)^N,$$

which simplifies to

$$\frac{k! r! s! (-1)^N}{N! (k - N)! (r - N)! (s - N)!}.$$

Having dealt with the (G, H) -edges, now we have to choose $r - N$ of the $k - N$ remaining edges of A to create subsets of size $r - N$ and $s - N$; then there are $(r - N)!$ and $(s - N)!$ bijections of the chosen subsets of A into the chosen subsets of G and H , respectively. This gives us the following count:

$$\binom{k - N}{r - N} (r - N)! (s - N)!,$$

which simplifies to $(k - N)!$. Multiplying the two, we obtain the total number of (G, H) -structures on \underline{A} . The numerical coefficient for \underline{A} in the multiplication (4.1.0.3) is given by

$$\frac{1}{r!} \cdot \frac{1}{s!} \cdot \frac{1}{k!} \cdot \frac{k! r! s! (-1)^N (k - N)!}{N! (k - N)! (r - N)! (s - N)!} = \frac{(-1)^N}{N! (r - N)! (s - N)!}, \quad (4.1.0.7)$$

where the factor of $\frac{1}{k!}$ is from $|\text{Aut}(A)| = k!$, following the algorithm for multiplication of graphs G and H shown in [32].

TOTAL CONTRIBUTION. Combining all types of decorations, and the numerical coefficient, the (isomorphism class of the) decorated graph A is given by

$$\frac{(-1)^N}{N!(r-N)!(s-N)!} \mathcal{G}_*^k \left(p_0^*(\lambda_{i-k}\lambda_{j-k}) \prod_{t=1}^N \psi_{\bullet_t} \prod_{t=N+1}^k p_t^*(\lambda_1) \right). \quad (4.1.0.8)$$

Multiplying (4.1.0.8) by $F(\psi_1, \dots, \psi_n)$ and integrating over $\overline{\mathcal{M}}_{g,n}$ yields

$$\begin{aligned} & \frac{(-1)^N}{N!(r-N)!(s-N)!} \cdot \int_{\overline{\mathcal{M}}_{g,n}} \mathcal{G}_*^k \left(p_0^*(\lambda_{i-k}\lambda_{j-k}) \prod_{t=1}^N \psi_{\bullet_t} \prod_{t=N+1}^k (p_t^*(\lambda_1)) \right) F(\psi_1, \dots, \psi_n) \\ &= \frac{(-1)^N}{N!(r-N)!(s-N)!} \cdot \left(\int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{i-k}\lambda_{j-k} F(\psi_1, \dots, \psi_n) \right) \left(\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 \right)^N \left(\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 \right)^{k-N} \\ &= \frac{(-1)^{r+s-k}}{24^k (r+s-k)!(k-s)!(k-r)!} \int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{i-k}\lambda_{j-k} F(\psi_1, \dots, \psi_n). \end{aligned} \quad (4.1.0.9)$$

The contribution to the Hodge integral by the k -tail stratum, is obtained by adding (4.1.0.9) for the range of pairs (r, s) from (4.1.0.2):

$$\sum_{(r,s) \in T} \frac{(-1)^{r+s-k}}{24^k (r+s-k)!(k-s)!(k-r)!} \int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{i-k}\lambda_{j-k} F(\psi_1, \dots, \psi_n),$$

where T is the set of integer tuples (r, s) with $0 \leq r \leq k, 0 \leq s \leq k$, and $r+s \geq k$. Using Lemma 4.0.1, this sum simplifies to

$$\frac{1}{24^k k!} \int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{i-k}\lambda_{j-k} F(\psi_1, \dots, \psi_n).$$

Thus we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \hat{\lambda}_i \hat{\lambda}_j F(\psi_1, \dots, \psi_n) = \sum_{k=0}^{\min\{i,j\}} \frac{1}{24^k k!} \int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{i-k}\lambda_{j-k} F(\psi_1, \dots, \psi_n),$$

where the upper bound on the sum of $\min\{i, j\}$ is due to $\lambda_{i-k} = 0$ and $\lambda_{j-k} = 0$ when $i - k < 0$ and $j - k < 0$, respectively. \square

We now present a comparison between all pseudostable and stable quadratic Hodge integrals using generating functions. While we used λ_i and λ_j in Theorem 4.1.1, we instead use λ_{g-i} and λ_{g-j} in the generating functions for convenience.

Theorem 4.1.2. *Let x, y, z, t_m , for m any positive integer, be formal variables. Define*

$$\mathcal{F}^{ps}(x, y, z, \{t_m\}) := \sum_{i,j,g,n=0}^{\infty} x^i y^j z^g \int_{\mathcal{M}_{g,n}^{ps}} \frac{\lambda_{g-i} \lambda_{g-j}}{\prod_{m=1}^n (1 - t_m \psi_m)},$$

where the denominator should be intended as a shorthand for the corresponding expansion as a product of geometric series. Define $\mathcal{F}(x, y, z, \{t_m\})$ similarly for stable Hodge integrals. In both cases we adopt the convention that indices (g, n) for which we have not defined a corresponding moduli space correspond to zero summands. Then:

$$\mathcal{F}^{ps}(x, y, z, \{t_m\}) = e^{z/24} \mathcal{F}(x, y, z, \{t_m\}).$$

Proof. Let $\underline{\mu} = (\mu_1, \dots, \mu_n)$ be a vector of non-negative integers and denote by $t^{\underline{\mu}}$ the monomial $\prod_{m=1}^n t_m^{\mu_m}$. Given $i, j, g, \underline{\mu}$, the coefficient of the monomial $x^i y^j z^g t^{\underline{\mu}}$ in $\mathcal{F}^{ps}(x, y, z, \{t_m\})$ is equal to

$$\int_{\mathcal{M}_{g,n}^{ps}} \lambda_{g-i} \lambda_{g-j} \prod_{m=1}^n \psi_m^{\mu_m}, \quad (4.1.0.10)$$

and is non-zero only if $|\underline{\mu}| = g + i + j + n - 3$. Note that the same condition holds in the stable case.

We now compute the coefficient of the same monomial in the expression $e^{z/24} \mathcal{F}(x, y, z, \{t_m\})$. We first expand the product to obtain:

$$\begin{aligned} e^{z/24} \mathcal{F}(x, y, z, \{t_m\}) &= \left(\sum_{g=0}^{\infty} \frac{1}{24^g g!} z^g \right) \left(\sum_{i,j,g,n=0}^{\infty} x^i y^j z^g \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g-i} \lambda_{g-j}}{\prod_{m=1}^n (1 - t_m \psi_m)} \right) \\ &= \sum_{i,j,g,n=0}^{\infty} x^i y^j z^g \left(\sum_{k=0}^g \frac{1}{24^k k!} \int_{\mathcal{M}_{g-k,n}} \frac{\lambda_{(g-k)-i} \lambda_{(g-k)-j}}{\prod_{m=1}^n (1 - t_m \psi_m)} \right). \end{aligned}$$

The coefficient of the monomial $x^i y^j z^g t^\mu$ is

$$\sum_{k=0}^{\min\{g-i, g-j\}} \frac{1}{24^k k!} \int_{\mathcal{M}_{g-k, n+k}} \lambda_{(g-i)-k} \lambda_{(g-j)-k} \prod_{m=1}^n \psi_m^{\mu_m} \quad (4.1.0.11)$$

The reason for $n + k$ instead of n is because that is the only number of marked points which returns a nonzero integral for dimension reasons. Further, notice that while the upper bound of k in the inner sum is g , any value of k greater than $\min\{g - i, g - j\}$ will make $\lambda_{(g-k)-i}$ or $\lambda_{(g-k)-j}$ equal to zero, so we can instead use $\min\{g - i, g - j\}$ for the upper bound.

We now observe that the equality of (4.1.0.10) and (4.1.0.11) is precisely the statement of Theorem 4.1.1, thus concluding the proof. □

4.2 Applications

We conclude the chapter with two applications: first we observe the how simple and explicit the equality of generating function from Theorem 4.1.2 is when restricting attention to Hodge integrals with a λ_g^2 term and a single ψ class. Next we give a closed formula the evaluation of arbitrary $\lambda_g \lambda_{g-1}$ pseudostable Hodge integrals.

Example 4.2.1. We demonstrate Theorem 4.1.2 for $x = 0, y = 0$, and $\{t_m\} = (t, 0, 0, 0, \dots)$. That is, we explicitly observe

$$\sum_{g,n=0}^{\infty} z^g \int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^2}{1 - t\psi_1} = e^{z/24} \left(\sum_{g,n=0}^{\infty} z^g \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g^2}{1 - t\psi_1} \right). \quad (4.2.0.1)$$

Starting on the left side of (4.2.0.1), the only term of the integrand which will give a nonzero result is $\lambda_g^2 (t\psi_1)^{g+n-3}$, so

$$\sum_{g,n=0}^{\infty} z^g \int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^2}{1 - t\psi_1} = \sum_{g,n=0}^{\infty} z^g t^{g+n-3} \int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^2 \psi_1^{g+n-3}. \quad (4.2.0.2)$$

Looking at the right side of (4.2.0.1), note that $\lambda_g^2 = 0$ except when $g = 0$, in which case $\lambda_0^2 = 1$. The only term in the expansion of $\frac{1}{1-t\psi_1}$ which produces a nonzero result is $(t\psi_1)^{g+n-3}$, implying

$$e^{z/24} \left(\sum_{g,n=0}^{\infty} z^g \int_{\overline{\mathcal{M}}_{g,n}} \frac{\lambda_g^2}{1-t\psi_1} \right) = e^{z/24} \left(\sum_{n=3}^{\infty} t^{n-3} \int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{n-3} \right) = e^{z/24} \sum_{n=3}^{\infty} t^{n-3}. \quad (4.2.0.3)$$

The last equality of (4.2.0.3) used the string equation (Theorem 2.4.1) to evaluate each of the integrals to 1. By Theorem 4.1.2,

$$\sum_{g,n=0}^{\infty} \left(\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^2 \psi_1^{g+n-3} \right) z^g t^{g+n-3} = \frac{e^{z/24}}{1-t}. \quad (4.2.0.4)$$

The coefficient of $z^g t^{g+n-3}$ in (4.2.0.4) is $\frac{1}{24^g} \cdot \frac{1}{g!}$, which aligns with what we found in Example 3.2.6, where we computed these integrals explicitly.

Proposition 4.2.2. *For all pseudostable indices (g, n) ,*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \sum_{k=0}^{g-1} \frac{1}{24^k k!} \cdot \frac{(2(g-k)-3+(n+k))! |B_{2(g-k)}|}{2^{2(g-k)-1} (2(g-k))! \prod_{j=1}^{n+k} (2d_j - 1)!!},$$

where $B_{2(g-k)}$ represents the $2(g-k)$ -th Bernoulli number.

Proof. Applying Theorem 4.1.1 to the quadratic pseudostable Hodge integral gives

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \sum_{k=0}^{g-1} \frac{1}{24^k k!} \int_{\overline{\mathcal{M}}_{g-k,n+k}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_{g-k} \lambda_{(g-k)-1}.$$

Each of the integrals in the sum can be evaluated with the Faber intersection number conjecture (conjectured in [10], proven in [23]):

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \frac{(2g-3+n)! |B_{2g}|}{2^{2g-1} (2g)! \prod_{j=1}^n (2d_j - 1)!!},$$

where B_{2g} is the $2g$ -th Bernoulli number. Replace g with $g - k$, and this yields the desired result.

□

Chapter 5

Higher Degree Pseudostable Hodge Integrals

In this chapter, we investigate certain cubic and higher degree pseudostable Hodge integrals.

5.1 Cubics With Mumford's Relation

The following pseudostable Hodge integral is a cubic which utilizes Mumford's relation for pseudostable λ classes (Theorem 3.0.1).

Theorem 5.1.1. *For all pseudostable indices (g, n) with $n \geq 1$, the pseudostable Hodge integral*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g(1 + \lambda_1 + \cdots + \lambda_g)(1 - \lambda_1 + \cdots + (-1)^g \lambda_g)}{1 - \psi_1}$$

is equal to

$$\sum_{k=0}^g \frac{1}{24^k} \binom{2g + n - 3 - k}{k} \frac{|2^{2(g-k)-1} - 1|}{2^{2(g-k)-1}} \frac{|B_{2(g-k)}|}{(2(g-k))!},$$

where the denominator in the integrand should be intended as a shorthand for the corresponding expansion as a product of geometric series, and $B_{2(g-k)}$ represents the $2(g-k)$ -th Bernoulli number.

Proof. We start by applying Theorem 2.2.3 and Theorem 3.0.1 to the numerator of the integrand:

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\left(\sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{g-i}))\right) \left(\sum_{k=0}^g \frac{1}{k!} \mathcal{G}_*^k \left(\prod_{j=1}^k (\psi_{\star_j} - \psi_{\bullet_j})\right)\right)}{1 - \psi_1} \quad (5.1.0.1)$$

We'll focus on the product of specific terms of each sum in the numerator of (5.1.0.1):

$$\frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{g-i})) \cdot \frac{1}{k!} \mathcal{G}_*^k \left(\prod_{j=1}^k (\psi_{\star_j} - \psi_{\bullet_j})\right). \quad (5.1.0.2)$$

Let G and H be the two decorated graphs representing $\mathcal{G}_*^i(p_0^*(\lambda_{g-i}))$ and $\mathcal{G}_*^k\left(\prod_{j=1}^k(\psi_{\star_j} - \psi_{\bullet_j})\right)$, respectively, and let A be the unique isomorphism class of decorated graphs in the resulting product supported on a stratum parameterizing curves with m elliptic tails. Our goal is to identify its decorations and numerical coefficient. Of the m edges of A , there are $m - k$ edges from only the G -structure, $m - i$ edges from only the H -structure, and $N := i + k - m$ edges from both the G and H -structures. We call these three types of edges G -edges, H -edges and (G, H) -edges, respectively. All the edges, together with their adjacent vertices have associated decorations. We break the computation in four parts.

DECORATIONS OF (G, H) -EDGES. The decoration for each of these edges is $(-\psi_{\bullet} - \psi_{\star})$, and with there being $N := i + k - m$ such edges, the contribution is

$$\mathcal{G}_*^m \left(\prod_{j=1}^N (-\psi_{\bullet_j} - \psi_{\star_j}) \right). \quad (5.1.0.3)$$

From (5.1.0.2), there will also be a $(\psi_{\star} - \psi_{\bullet})$ associated to each of these edges, and together with (5.1.0.3), this gives

$$\mathcal{G}_*^m \left(\prod_{j=1}^N ((-\psi_{\bullet_j} - \psi_{\star_j})(\psi_{\star_j} - \psi_{\bullet_j})) \right) = \mathcal{G}_*^m \left(\prod_{j=1}^N (\psi_{\bullet_j}^2 - \psi_{\star_j}^2) \right). \quad (5.1.0.4)$$

However, on $\overline{\mathcal{M}}_{1,1}$, we know that $\psi_{\bullet_j} = \lambda_1$ and $\lambda_1^2 = 0$, so $\psi_{\bullet_j}^2 = 0$. Further, all terms without a class ψ_{\bullet_j} on each flag adjacent to the genus one tail-vertex will vanish, so the $-\psi_{\star_j}^2$ terms are eliminated as well. The conclusion from this case is that in order to get a non-vanishing strata, the graph A must have no common (G, H) -edges, so $N = 0$.

DECORATIONS OF G -EDGES. Since $N := i + k - m = 0$, the number of edges of A identified only with G is $m - k = (i + k) - k = i$, which is the number of edges of G . However, none of these edges will have a ψ_{\bullet} or λ_1 on the flags adjacent to the genus one tail-vertices, so these terms will vanish as well. Like the last case, we can eliminate all terms other than $i = 0$.

For $i = 0$, the product in (5.1.0.2) is simply

$$\lambda_g \cdot \frac{1}{k!} \mathcal{G}_*^k \left(\prod_{j=1}^k (\psi_{\star_j} - \psi_{\bullet_j}) \right). \quad (5.1.0.5)$$

Since there is no longer a dual graph G , we can ignore the previous computations regarding A , and simply evaluate the product in (5.1.0.5).

Since the root vertex of the boundary strata in (5.1.0.5) has genus $g - k$, then λ_g splits in the following way:

$$\frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=1}^k p_j^*(\lambda_1) (\psi_{\star_j} - \psi_{\bullet_j}) \right). \quad (5.1.0.6)$$

Recalling that on $\overline{\mathcal{M}}_{1,1}$, we have $\psi_{\bullet_j} = \lambda_1$ and $\lambda_1^2 = 0$, we can reduce (5.1.0.6) in the following way:

$$\begin{aligned} \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=1}^k p_j^*(\lambda_1) (\psi_{\star_j} - \psi_{\bullet_j}) \right) &= \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=1}^k \psi_{\bullet_j} (\psi_{\star_j} - \psi_{\bullet_j}) \right) \\ &= \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=1}^k (\psi_{\bullet_j} \psi_{\star_j} - \psi_{\bullet_j}^2) \right) \\ &= \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=1}^k (\psi_{\bullet_j} \psi_{\star_j}) \right). \end{aligned}$$

This computation simplifies the integral in (5.1.0.1) to

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\sum_{k=0}^g \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k}) \prod_{j=1}^k (\psi_{\bullet_j} \psi_{\star_j}) \right)}{1 - \psi_1}. \quad (5.1.0.7)$$

Note that the dimension of the root vertex of the boundary strata is $3g + n - 3 - 2k$, and the total codimension of the classes associated to that component is g , so the only nonvanishing term of $\frac{1}{1-\psi_1}$

is $\psi_1^{2g+n-3-2k}$, reducing (5.1.0.7) to

$$\begin{aligned}
& \int_{\overline{\mathcal{M}}_{g,n}} \sum_{k=0}^g \frac{1}{k!} \mathcal{G}_*^k \left(p_0^*(\lambda_{g-k} \psi_1^{2g+n-3-2k}) \prod_{j=1}^k (\psi_{\bullet_j} \psi_{\star_j}) \right) \\
&= \sum_{k=0}^g \frac{1}{k!} \left(\int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{g-k} \psi_1^{2g+n-3-2k} \psi_{n+1} \cdots \psi_{n+k} \right) \left(\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 \right)^k \\
&= \sum_{k=0}^g \frac{1}{24^k k!} \left(\int_{\overline{\mathcal{M}}_{g-k,n+k}} \lambda_{g-k} \psi_1^{2g+n-3-2k} \psi_{n+1} \cdots \psi_{n+k} \right).
\end{aligned}$$

The integrals above can be simplified with the dilaton equation (Theorem 2.4.5):

$$\begin{aligned}
&= \left(\sum_{k=0}^g \frac{1}{24^k k!} \frac{(2(g-k) - 2 + (n+k-1))!}{(2(g-k) - 2 + n - 1)!} \int_{\overline{\mathcal{M}}_{g-k,n}} \lambda_{g-k} \psi_1^{2g+n-3-2k} \right) \\
&= \left(\sum_{k=0}^g \frac{1}{24^k k!} \frac{(2g+n-3-k)!}{(2g+n-3-2k)!} \int_{\overline{\mathcal{M}}_{g-k,n}} \lambda_{g-k} \psi_1^{2g+n-3-2k} \right) \\
&= \left(\sum_{k=0}^g \frac{1}{24^k} \binom{2g+n-3-k}{k} \int_{\overline{\mathcal{M}}_{g-k,n}} \lambda_{g-k} \psi_1^{2g+n-3-2k} \right).
\end{aligned}$$

From here, we can use the string equation (Theorem 2.4.1) to reduce the number of marked points:

$$= \left(\sum_{k=0}^{g-1} \frac{1}{24^k} \binom{2g+n-3-k}{k} \int_{\overline{\mathcal{M}}_{g-k,1}} \lambda_{g-k} \psi_1^{2(g-k)-2} \right) + \left(\frac{1}{24^g} \binom{g+n-3}{g} \int_{\overline{\mathcal{M}}_{0,3}} 1 \right).$$

The former integrals are computed with Theorem 2.4.11, and the latter integral is 1 since $\overline{\mathcal{M}}_{0,3}$ is a point:

$$\begin{aligned}
&= \left(\sum_{k=0}^{g-1} \frac{1}{24^k} \binom{2g+n-3-k}{k} \frac{2^{2(g-k)-1} - 1}{2^{2(g-k)-1}} \frac{|B_{2(g-k)}|}{(2(g-k))!} \right) + \frac{1}{24^g} \binom{g+n-3}{g} \\
&= \sum_{k=0}^g \frac{1}{24^k} \binom{2g+n-3-k}{k} \frac{|2^{2(g-k)-1} - 1|}{2^{2(g-k)-1}} \frac{|B_{2(g-k)}|}{(2(g-k))!},
\end{aligned}$$

where $B_{2(g-k)}$ represents the $2(g-k)$ -th Bernoulli number. □

For ease of reference, we write out in the table below the results of Theorem 5.1.1 for specific pairs (g, n) , all of which have been verified with `admcycles` [7].

$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \frac{\lambda_g \Lambda_g(1) \Lambda_g(-1)}{1 - \psi_1}$	$n = 1$	$n = 2$	$n = 3$	$n \geq 3$
$g = 0$			1	1
$g = 1$		$\frac{1}{24}$	$\frac{1}{12}$	$\frac{n-1}{24}$
$g = 2$	$\frac{17}{5760}$	$\frac{3}{640}$	$\frac{47}{5760}$	$\frac{5n^2 - 5n + 17}{5760}$
$g = 3$	$\frac{31}{120960}$	$\frac{437}{967680}$	$\frac{383}{483840}$	$\frac{35n^3 + 322n + 387}{2903040}$

Much like with Theorem 3.0.1, we can identify specific codimension terms from Theorem 5.1.1 to determine additional formulas for cubic pseudostable Hodge integrals.

Corollary 5.1.2. *For all pseudostable indices (g, n) , $n \geq 1$, and $k = 0, \dots, g$, the pseudostable Hodge integral*

$$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \left(\sum_{j=\max\{0,2k-g\}}^{\min\{2k,g\}} (-1)^j \lambda_{2k-j} \lambda_j \right) \lambda_g \psi_1^{2g+n-3-2k}$$

is equal to

$$\frac{1}{24^k} \binom{2g+n-3-k}{k} \frac{|2^{2(g-k)-1} - 1|}{2^{2(g-k)-1}} \frac{|B_{2(g-k)}|}{(2(g-k))!},$$

where $B_{2(g-k)}$ represents the $2(g-k)$ -th Bernoulli number.

Proof. We observe first of all that the Hodge part in the parenthesis is the degree $2k$ part of $(1 + \lambda_1 + \dots + \lambda_g)(1 - \lambda_1 + \dots + (-1)^g \lambda_g)$. Proceeding as in the proof of Theorem 5.1.1 above, we know

$$\int_{\mathcal{M}_{g,n}^{\text{ps}}} \left(\sum_{j=\max\{0,2k-g\}}^{\min\{2k,g\}} (-1)^j \lambda_{2k-j} \lambda_j \right) \lambda_g \psi_1^{2g+n-3-2k}$$

is equal to

$$\sum_{i=0}^g \frac{1}{24^i i!} \left(\int_{\overline{\mathcal{M}}_{g-i, n+i}} \lambda_{g-k} \psi_1^{2g+n-3-2k} \psi_{n+1} \cdots \psi_{n+k} \right).$$

By dimension reasons the only non-vanishing summand corresponds to $i = k$. We recognize this term as the k -th summand in the statement of Theorem 5.1.1, which proves the result. \square

The following example highlights specific cases of Corollary 5.1.2.

Example 5.1.3. For $k = g$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^3 \psi_1^{n-3} = \frac{(-1)^g}{24^g} \binom{g+n-3}{g}.$$

For $k = g - 1$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} (2\lambda_g \lambda_{g-2} - \lambda_{g-1}^2) \lambda_g \psi_1^{n-1} = \frac{(-1)^g}{24^g} \binom{g+n-2}{g-1}.$$

For $k = 1$ and $g \geq 1$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} (2\lambda_2 - \lambda_1^2) \lambda_g \psi_1^{2g-5+n} = \frac{1}{24} \binom{2g+n-4}{1} \frac{|2^{2g-3} - 1|}{2^{2g-3}} \frac{|B_{2g-2}|}{(2g-2)!}.$$

Remark 5.1.4. We can make further assumptions about the number of marked points to see the following:

$$\int_{\overline{\mathcal{M}}_{g,3}^{\text{ps}}} \lambda_g^3 = \frac{(-1)^g}{24^g}.$$

$$\int_{\overline{\mathcal{M}}_{g,1}^{\text{ps}}} (2\lambda_g \lambda_{g-2} - \lambda_{g-1}^2) \lambda_g = \frac{(-1)^g}{24^g}.$$

We end the section with one more cubic pseudostable Hodge integral that is able to utilize an application of Mumford's relation for pseudostable λ classes (Theorem 3.0.1).

Corollary 5.1.5. *For all pseudostable indices (g, n) ,*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} (2\lambda_g \lambda_{g-1} \lambda_{g-2} - \lambda_{g-1}^3) \psi_1^n = \frac{(-1)^g}{24^g} \left(\binom{n+g-2}{g-1} - \binom{n+g-2}{g-2} \right).$$

Proof. Notice that

$$\begin{aligned} \int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} (2\lambda_g \lambda_{g-1} \lambda_{g-2} - \lambda_{g-1}^3) \psi_1^n &= \int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_{g-1} (2\lambda_g \lambda_{g-2} - \lambda_{g-1}^2) \psi_1^n \\ &= \int_{\overline{\mathcal{M}}_{g,n}} \hat{\lambda}_{g-1} (2\hat{\lambda}_g \hat{\lambda}_{g-2} - \hat{\lambda}_{g-1}^2) \psi_1^n. \end{aligned}$$

Substituting the relation for $2\hat{\lambda}_g \hat{\lambda}_{g-2} - \hat{\lambda}_{g-1}^2$ from Example 3.2.2 shows that the integral above is equal to

$$\frac{(-1)^g}{(g-1)!} \int_{\overline{\mathcal{M}}_{g,n}} \hat{\lambda}_{g-1} \left(\mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}) \right) \right) \psi_1^n. \quad (5.1.0.8)$$

For precisely the same reasons as in the proof of Theorem 5.1.1, the only non-vanishing term of $\hat{\lambda}_{g-1}$ in the integrand of (5.1.0.8) is λ_{g-1} . Thus (5.1.0.8) is equal to

$$\frac{(-1)^g}{(g-1)!} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_{g-1} \left(\mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}) \right) \right) \psi_1^n. \quad (5.1.0.9)$$

For simplicity, we'll focus on the product

$$\lambda_{g-1} \left(\mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}) \right) \right). \quad (5.1.0.10)$$

The term λ_{g-1} splits along the factors of the stratum in (5.1.0.10) in one of two different ways: either λ_1 for each of the $g-1$ factors of $\overline{\mathcal{M}}_{1,1}$, or λ_1 on the root vertex and λ_1 for $g-2$ of the $g-1$ factors of $\overline{\mathcal{M}}_{1,1}$. For the latter, there are $\binom{g-1}{g-2} = g-1$ ways to do this, so (5.1.0.10) is equal to

$$\mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}) p_j^*(\lambda_1) \right) + (g-1) \mathcal{G}_*^{g-1} \left(p_0^*(\lambda_1) (\psi_{\star_1} - \psi_{\bullet_1}) \prod_{j=2}^{g-1} (\psi_{\star_j} - \psi_{\bullet_j}) p_j^*(\lambda_1) \right). \quad (5.1.0.11)$$

Recalling that $p_j^*(\lambda_1) = \psi_{\bullet_j}$ and $\psi_{\bullet_j}^2 = 0$ for $j \geq 1$, and that any non-vanishing strata classes must have a class ψ_{\bullet_j} on each flag adjacent to the genus one tail-vertices, the non-vanishing terms of

(5.1.0.11) are

$$\begin{aligned}
& \mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} \psi_{\star_j} \psi_{\bullet_j} \right) + (g-1) \mathcal{G}_*^{g-1} \left(p_0^*(\lambda_1) (-\psi_{\bullet_1}) \prod_{j=2}^{g-1} \psi_{\star_j} \psi_{\bullet_j} \right) \\
&= \mathcal{G}_*^{g-1} \left(\prod_{j=1}^{g-1} \psi_{\star_j} \psi_{\bullet_j} \right) - (g-1) \mathcal{G}_*^{g-1} \left(p_0^*(\lambda_1) \prod_{j=2}^{g-1} \psi_{\star_j} \prod_{j=1}^{g-1} \psi_{\bullet_j} \right). \tag{5.1.0.12}
\end{aligned}$$

Since the two terms of (5.1.0.12) have identical decorations (ψ_{\bullet_j}) for the $g-1$ factors of $\overline{\mathcal{M}}_{1,1}$, we can sum over the only other factor, that being the root vertex, to get

$$\begin{aligned}
& \mathcal{G}_*^{g-1} \left(\left(\prod_{j=1}^{g-1} (\psi_{\star_j}) - (g-1) p_0^*(\lambda_1) \prod_{j=2}^{g-1} \psi_{\star_j} \right) \prod_{j=1}^{g-1} \psi_{\bullet_j} \right) \\
&= \mathcal{G}_*^{g-1} \left((\psi_{\star_1} - (g-1) p_0^*(\lambda_1)) \prod_{j=2}^{g-1} \psi_{\star_j} \prod_{j=1}^{g-1} \psi_{\bullet_j} \right). \tag{5.1.0.13}
\end{aligned}$$

Substituting (5.1.0.13) into (5.1.0.9) gives

$$\begin{aligned}
& \frac{(-1)^g}{(g-1)!} \int_{\overline{\mathcal{M}}_{g,n}} \mathcal{G}_*^{g-1} \left((\psi_{\star_1} - (g-1) p_0^*(\lambda_1)) \prod_{j=2}^{g-1} \psi_{\star_j} \prod_{j=1}^{g-1} \psi_{\bullet_j} \right) \psi_1^n \\
&= \frac{(-1)^g}{(g-1)!} \int_{\overline{\mathcal{M}}_{g,n}} \mathcal{G}_*^{g-1} \left(p_0^*(\psi_1^n) (\psi_{\star_1} - (g-1) p_0^*(\lambda_1)) \prod_{j=2}^{g-1} \psi_{\star_j} \prod_{j=1}^{g-1} \psi_{\bullet_j} \right) \\
&= \frac{(-1)^g}{(g-1)!} \left(\int_{\overline{\mathcal{M}}_{1,n+g-1}} \psi_1^n (\psi_{n+1} - (g-1) \lambda_1) \prod_{j=2}^{g-1} \psi_{n+j} \right) \left(\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 \right)^{g-1} \\
&= \frac{(-1)^g}{24^{g-1} (g-1)!} \left(\int_{\overline{\mathcal{M}}_{1,n+g-1}} \psi_1^n (\psi_{n+1} - (g-1) \lambda_1) \prod_{j=2}^{g-1} \psi_{n+j} \right). \tag{5.1.0.14}
\end{aligned}$$

Using the dilaton equation (Theorem 2.4.5), the Hodge integral in (5.1.0.14) is equal to

$$\begin{aligned}
& \frac{(-1)^g (n+g-2)!}{24^{g-1} (g-1)! n!} \left(\int_{\overline{\mathcal{M}}_{1,n+1}} \psi_1^n (\psi_{n+1} - (g-1) \lambda_1) \right) \\
&= \frac{(-1)^g (n+g-2)!}{24^{g-1} (g-1)! n!} \left(\int_{\overline{\mathcal{M}}_{1,n+1}} \psi_1^n \psi_{n+1} - \int_{\overline{\mathcal{M}}_{1,n+1}} (g-1) \psi_1^n \lambda_1 \right). \tag{5.1.0.15}
\end{aligned}$$

For the former Hodge integral of (5.1.0.15), we can use the dilaton equation once again to get

$$\frac{(-1)^g(n+g-2)!}{24^{g-1}(g-1)!n!} \left(n \int_{\overline{\mathcal{M}}_{1,n}} \psi_1^n - \int_{\overline{\mathcal{M}}_{1,n+1}} (g-1)\psi_1^n \lambda_1 \right).$$

Both Hodge integrals above can be computed with the string equation (Theorem 2.4.1) to get

$$\begin{aligned} & \frac{(-1)^g(n+g-2)!}{24^{g-1}(g-1)!n!} \left(\frac{n}{24} - \frac{g-1}{24} \right) \\ &= \frac{(-1)^g(n+g-2)!}{24^g(g-1)!n!} (n - (g-1)) \\ &= \frac{(-1)^g}{24^g} \left(\binom{n+g-2}{g-1} - \binom{n+g-2}{g-2} \right). \end{aligned}$$

□

Example 5.1.6. Using Corollary 5.1.5 with $n = 0$, we see that

$$\int_{\overline{\mathcal{M}}_{g,0}^{\text{ps}}} (2\lambda_g \lambda_{g-1} \lambda_{g-2} - \lambda_{g-1}^3) = \frac{(-1)^{g-1}}{24^g}.$$

5.2 Relations and Hodge Integrals for $\hat{\lambda}_g^k$

Theorem 5.2.1. *For all pseudostable indices (g, n) and $k \geq 2$,*

$$\hat{\lambda}_g^k = \frac{(-1)^{g(k-1)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-2} \right).$$

Proof. We'll proceed by induction on k . For the base case of $k = 2$, see Example 3.2.2. For the inductive step, assume that the theorem is true for $k - 1$. That is, assume that

$$\hat{\lambda}_g^{k-1} = \frac{(-1)^{g(k-2)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-3} \right).$$

Notice

$$\hat{\lambda}_g^k = \hat{\lambda}_g(\hat{\lambda}_g^{k-1}),$$

and we can substitute Theorem 2.2.3 for $\hat{\lambda}_g$ and the inductive step for $\hat{\lambda}_g^{k-1}$:

$$\left(\sum_{i=0}^g \frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{g-i})) \right) \left(\frac{(-1)^{g(k-2)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-3} \right) \right).$$

Consider the product using the i -th term in the sum above:

$$\left(\frac{1}{i!} \mathcal{G}_*^i(p_0^*(\lambda_{g-i})) \right) \left(\frac{(-1)^{g(k-2)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-3} \right) \right). \quad (5.2.0.1)$$

Using notation from [32], let G and H be the two decorated graphs representing the first and second boundary strata in (5.2.0.1), respectively, and let A be the unique isomorphism class of decorated graphs in the resulting product, which will be supported on a stratum parameterizing curves with g elliptic tails. Our goal is to identify its decorations and numerical coefficient. Of the g edges of A , there are $g - i$ edges from only the H -structure, and i edges from both the G and H -structures. We call these three types of edges G -edges, H -edges and (G, H) -edges, respectively. All the edges, together with their adjacent vertices, have associated decorations. We break the computation in three parts.

DECORATIONS OF (G, H) -EDGES. The decoration for each of these edges is $(-\psi_{\bullet} - \psi_{\star})$, and with there being i such edges, the contribution is

$$\mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\bullet_j} - \psi_{\star_j}) \right). \quad (5.2.0.2)$$

DECORATIONS OF H -EDGES. Since G supports a λ_{g-i} class on its root vertex, and A has a root vertex of genus 0, then λ_{g-i} will split along the $g - i$ edges of A which are identified only with H . This will result in a coefficient of λ_1 associated to the tail vertices of these edges of A . Thus the contribution from these edges is

$$\mathcal{G}_*^g \left(\prod_{j=i+1}^g p_j^*(\lambda_1) \right).$$

NUMERICAL COEFFICIENT. There are many different but isomorphic (G, H) -structures on the graph \underline{A} . One possible way to organize the count is the following. We first count the possible ways to obtain the (G, H) -edges: one must choose i of the g edges of A , then i of the i and g edges of G and H , respectively; finally there are $i!$ ways to map the i chosen edges of both G and H to the chosen edges of A . This gives us the following count:

$$\binom{g}{i} \binom{i}{i} \binom{g}{i} i! i!,$$

which simplifies to

$$\frac{(g!)^2}{((g-i)!)^2}.$$

Having dealt with the (G, H) -edges, we now note that there are $(g-i)!$ bijections of the remaining $g-i$ edges of A into the remaining $g-i$ edges of H . Multiplying the two, we obtain the total number of (G, H) -structures on \underline{A} . The numerical coefficient for \underline{A} in the multiplication (5.2.0.1) is given by

$$\frac{1}{i!} \cdot \frac{(-1)^{g(k-2)}}{g!} \cdot \frac{1}{g!} \cdot \frac{(g!)^2 (g-i)!}{((g-i)!)^2} = \frac{(-1)^{g(k-2)}}{i!(g-i)!}, \quad (5.2.0.3)$$

where the extra factor of $\frac{1}{g!}$ is from $|\text{Aut}(A)| = g!$, following the algorithm for multiplication of graphs G and H shown in [32].

TOTAL CONTRIBUTION. Combining all types of decorations, and the numerical coefficient, the (isomorphism class of the) decorated graph A is given by

$$\frac{(-1)^{g(k-2)}}{i!(g-i)!} \mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\bullet_j} - \psi_{\star_j}) \prod_{j=i+1}^g p_j^*(\lambda_1) \prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-3} \right). \quad (5.2.0.4)$$

To simplify the tautological classes of (5.2.0.4), recall that on $\overline{\mathcal{M}}_{1,1}$, we have $\lambda_1 = \psi_{\bullet_1}$ (see [30] or [34]), and also that $\lambda_1^2 = 0$ due to Mumford's relation. These two facts allow us to reduce

(5.2.0.4) in the following way:

$$\begin{aligned}
& \frac{(-1)^{g(k-2)}}{(g-i)!i!} \mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\bullet_j} - \psi_{\star_j}) \prod_{j=i+1}^g p_j^*(\lambda_1) \prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-3} \right) \\
&= \frac{(-1)^{g(k-2)}}{(g-i)!i!} \mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\bullet_j} - \psi_{\star_j}) (\psi_{\star_j} - \psi_{\bullet_j}) \prod_{j=i+1}^g p_j^*(\lambda_1) (\psi_{\star_j} - \psi_{\bullet_j}) \prod_{j=1}^g \psi_{\star_j}^{k-3} \right) \\
&= \frac{(-1)^{g(k-2)}}{(g-i)!i!} \mathcal{G}_*^g \left(\prod_{j=1}^i (\psi_{\bullet_j}^2 - \psi_{\star_j}^2) \prod_{j=i+1}^g (\psi_{\bullet_j} \psi_{\star_j} - \psi_{\bullet_j}^2) \prod_{j=1}^g \psi_{\star_j}^{k-3} \right) \\
&= \frac{(-1)^{g(k-2)}}{(g-i)!i!} \mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\star_j}^2) \prod_{j=i+1}^g (\psi_{\bullet_j} \psi_{\star_j}) \prod_{j=1}^g \psi_{\star_j}^{k-3} \right).
\end{aligned}$$

So far, we have shown

$$\hat{\lambda}_g^k = \sum_{i=0}^g \frac{(-1)^{g(k-2)}}{(g-i)!i!} \mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\star_j}^2) \prod_{j=i+1}^g (\psi_{\bullet_j} \psi_{\star_j}) \prod_{j=1}^g \psi_{\star_j}^{k-3} \right). \quad (5.2.0.5)$$

We can take this further by noticing that

$$\sum_{i=0}^g \left(\binom{g}{i} \prod_{j=1}^i (-\psi_{\star_j}^2) \prod_{j=i+1}^g (\psi_{\bullet_j} \psi_{\star_j}) \right) = \prod_{j=1}^g (-\psi_{\star_j}^2 + \psi_{\bullet_j} \psi_{\star_j}).$$

This is because in the product $\prod_{j=1}^g (\psi_{\bullet_j} \psi_{\star_j} - \psi_{\star_j}^2)$, we can renumber the indices of the ψ classes to get isomorphic decorated strata (e.g. the class supporting $-\psi_{\star_1}^2 \psi_{\bullet_2} \psi_{\star_2}$ is isomorphic to the class supporting $-\psi_{\star_2}^2 \psi_{\bullet_1} \psi_{\star_1}$). Since $-\psi_{\star_j}^2 + \psi_{\bullet_j} \psi_{\star_j}$ is a binomial, the number of ways to do this is $\binom{g}{i}$, where i is number of factors of $-\psi_{\star_j}^2$.

This fact simplifies (5.2.0.5) in the following way:

$$\begin{aligned}
\hat{\lambda}_g^k &= \sum_{i=0}^g \frac{(-1)^{g(k-2)}}{(g-i)!i!} \mathcal{G}_*^g \left(\prod_{j=1}^i (-\psi_{\star_j}^2) \prod_{j=i+1}^g (\psi_{\bullet_j} \psi_{\star_j}) \prod_{j=1}^g \psi_{\star_j}^{k-3} \right) \\
&= \frac{(-1)^{g(k-2)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (-\psi_{\star_j}^2 + \psi_{\bullet_j} \psi_{\star_j}) \prod_{j=1}^g \psi_{\star_j}^{k-3} \right) \\
&= \frac{(-1)^{g(k-2)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (-\psi_{\star_j}^2 + \psi_{\bullet_j} \psi_{\star_j}) \psi_{\star_j}^{k-3} \right) \\
&= \frac{(-1)^{g(k-2)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (-\psi_{\star_j} + \psi_{\bullet_j}) \psi_{\star_j}^{k-2} \right) \\
&= \frac{(-1)^{g(k-1)}}{g!} \mathcal{G}_*^g \left(\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-2} \right),
\end{aligned}$$

completing the proof by induction. □

Corollary 5.2.2. *For all pseudostable indices (g, n) and $k \geq 2$,*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^k}{\prod_{j=1}^n (1 - \psi_j)} = \frac{(-1)^{gk}}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{\prod_{j=n+1}^{n+g} \psi_j^{k-2}}{\prod_{j=1}^n (1 - \psi_j)} \right),$$

where the denominator should be intended as a shorthand for the corresponding expansion as a product of geometric series.

Proof. Applying Theorem 5.2.1 gives

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^k}{\prod_{j=1}^n (1 - \psi_j)} = \int_{\overline{\mathcal{M}}_{g,n}} \frac{(-1)^{g(k-1)}}{g!} \mathcal{G}_*^g \left(\frac{\prod_{j=1}^g (\psi_{\star_j} - \psi_{\bullet_j}) \psi_{\star_j}^{k-2}}{\prod_{j=1}^n (1 - p_0^*(\psi_j))} \right) \quad (5.2.0.6)$$

Any non-vanishing strata classes must have a class $-\psi_{\bullet_j}$ on each flag adjacent to the genus one tail-vertices, simplifying (5.2.0.6) to

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{(-1)^{g(k-1)}}{g!} \mathcal{G}_*^g \left(\frac{\prod_{j=1}^g (-\psi_{\bullet_j} \psi_{\star_j}^{k-2})}{\prod_{j=1}^n (1 - p_0^*(\psi_j))} \right) \quad (5.2.0.7)$$

Distributing the integral across the factors gives

$$\frac{(-1)^{g(k-1)}}{g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{\prod_{j=n+1}^{n+g} \psi_j^{k-2}}{\prod_{j=1}^n (1-\psi_j)} \right) \left(\int_{\overline{\mathcal{M}}_{1,1}} -\psi_1 \right)^g = \frac{(-1)^{gk}}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{\prod_{j=n+1}^{n+g} \psi_j^{k-2}}{\prod_{j=1}^n (1-\psi_j)} \right).$$

□

The integrals of Corollary 5.2.2 for $k = 2$ and $k = 3$ are computable with the string and dilaton equations, respectively.

Example 5.2.3. Using Corollary 5.2.2 with $k = 2$ returns

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^2}{\prod_{j=1}^n (1-\psi_j)} = \frac{1}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{1}{\prod_{j=1}^n (1-\psi_j)} \right).$$

The non-vanishing terms are all the monomials of ψ classes in degree $n + g - 3$, and each of these can be evaluated with an application of the string equation (Proposition 2.4.3):

$$\int_{\overline{\mathcal{M}}_{0,n+g}} \prod_{i=1}^n \psi_i^{d_i} = \binom{n+g-3}{d_1, \dots, d_n}. \quad (5.2.0.8)$$

Using an application of the multinomial theorem [19], the sum of all such multinomial coefficients is

$$\sum_{d_1 + \dots + d_n = n+g-3} \binom{n+g-3}{d_1, \dots, d_n} = n^{n+g-3}.$$

Thus

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^2}{\prod_{j=1}^n (1-\psi_j)} = \frac{n^{n+g-3}}{24^g g!}.$$

Note that for a single ψ class, meaning replace $\frac{1}{\prod_{j=1}^n (1-\psi_j)}$ with ψ_1^{g-3+n} , this result coincides with the result from Example 3.2.6.

Example 5.2.4. Using Corollary 5.2.2 with $k = 3$ returns

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^3}{\prod_{j=1}^n (1-\psi_j)} = \frac{(-1)^g}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{\prod_{j=n+1}^{n+g} \psi_j}{\prod_{j=1}^n (1-\psi_j)} \right).$$

All ψ classes with an exponent of 1 can be removed with the Dilaton equation (Theorem 2.4.5), and this also introduces a coefficient:

$$\begin{aligned} \frac{(-1)^g}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{\prod_{j=n+1}^{n+g} \psi_j}{\prod_{j=1}^n (1-\psi_j)} \right) &= \frac{(-1)^g (g+n-3)!}{24^g g! (n-3)!} \int_{\overline{\mathcal{M}}_{0,n}} \frac{1}{\prod_{j=1}^n (1-\psi_j)} \\ &= \frac{(-1)^g}{24^g} \binom{g+n-3}{g} \int_{\overline{\mathcal{M}}_{0,n}} \frac{1}{\prod_{j=1}^n (1-\psi_j)}. \end{aligned}$$

Just as in Example 5.2.3, we can use the string equation and the multinomial theorem together to fully evaluate the integral above, showing that

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \frac{\lambda_g^3}{\prod_{j=1}^n (1-\psi_j)} = \frac{(-1)^g n^{n-3}}{24^g} \binom{g+n-3}{g}.$$

For a single ψ class, meaning replace $\frac{1}{\prod_{j=1}^n (1-\psi_j)}$ by $\psi_1^{n+g(3-k)-3}$, the integral of Corollary 5.2.2 can be easily computed for any $k \geq 2$ with the string theorem, leading to the following corollary.

Corollary 5.2.5. *For all pseudostable indices (g, n) and $k \geq 2$,*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^k \psi_1^{n+g(3-k)-3} = \frac{(-1)^{gk} (n+g-3)!}{24^g g! (n+g(3-k)-3)! ((k-2)!)^g}.$$

Proof. Following the proof of Corollary 5.2.2, it can be shown that

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^k \psi_1^{n+g(3-k)-3} = \frac{(-1)^{gk}}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \psi_1^{n+g(3-k)-3} \prod_{j=n+1}^{n+g} \psi_j^{k-2} \right). \quad (5.2.0.9)$$

Using an application of the string equation (Proposition 2.4.3), we can see that (5.2.0.9) is equal to

$$\frac{(-1)^{gk}}{24^g g!} \binom{n+g-3}{n+g(3-k)-3, k-2, \dots, k-2} = \frac{(-1)^{gk} (n+g-3)!}{24^g g! (n+g(3-k)-3)! ((k-2)!)^g}.$$

□

Remark 5.2.6. For $k = 2$ and $k = 3$, note that Corollary 5.2.5 coincides with the results from Example 3.2.6 and Example 5.1.3, respectively.

In the following example, we explicitly show Corollary 5.2.5 for $k = 4, 5, 6$.

Example 5.2.7. For $k = 4$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^4 \psi_1^{n-g-3} = \frac{(n+g-3)!}{24^g g! (n-g-3)! (2!)^g}.$$

For $k = 5$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^5 \psi_1^{n-2g-3} = \frac{(-1)^g (n+g-3)!}{24^g g! (n-2g-3)! (3!)^g}.$$

For $k = 6$:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^6 \psi_1^{n-3g-3} = \frac{(n+g-3)!}{24^g g! (n-3g-3)! (4!)^g}.$$

Example 5.2.8. If we make further assumptions for the number of marked points n , we see the following:

$$\int_{\overline{\mathcal{M}}_{g,g+3}^{\text{ps}}} \lambda_g^4 = \frac{(2g)!}{24^g g! (2!)^g}.$$

$$\int_{\overline{\mathcal{M}}_{g,2g+3}^{\text{ps}}} \lambda_g^5 = \frac{(-1)^g (3g)!}{24^g g! (3!)^g}.$$

$$\int_{\overline{\mathcal{M}}_{g,3g+3}^{\text{ps}}} \lambda_g^6 = \frac{(4g)!}{24^g g! (4!)^g}.$$

5.3 Comparisons to Stable Hodge Integrals

The following table compares some of the pseudostable Hodge integrals in this chapter to their stable counterparts. The left column represent the integrand, the middle column is the resulting Hodge integral over $\overline{\mathcal{M}}_{g,n}^{\text{ps}}$, and the right column is the same Hodge integral over $\overline{\mathcal{M}}_{g,n}$. Further, we define

$$C_g := \frac{|2^{2g-1} - 1| |B_{2g}|}{2^{2g-1} (2g)!},$$

where B_{2g} is the $2g$ -th Bernoulli number. Finally, for this table, we make the assumption that $g \geq 1$ so that λ_g is non-trivial.

	$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}}$	$\int_{\overline{\mathcal{M}}_{g,n}}$
$\frac{\lambda_g \Lambda_g(1) \Lambda_g(-1)}{1 - \psi_1}$	$\sum_{k=0}^g \frac{1}{24^k} \binom{2g+n-3-k}{k} C_{g-k}$	C_g
$(2\lambda_g \lambda_{g-2} - \lambda_{g-1}^2) \lambda_g \psi_1^{n-1}$	$\frac{(-1)^g}{24^g} \binom{g+n-2}{g-1}$	0
$(2\lambda_2 - \lambda_1^2) \lambda_g \psi_1^{2g-5+n}$	$\frac{1}{24} \binom{2g+n-4}{1} C_{g-1}$	0
$(2\lambda_g \lambda_{g-1} \lambda_{g-2} - \lambda_{g-1}^3) \psi_1^n$	$\frac{(-1)^g}{24^g} \left(\binom{n+g-2}{g-1} - \binom{n+g-2}{g-2} \right)$	0
$\lambda_g^3 \psi_1^{n-3}$	$\frac{(-1)^g}{24^g} \binom{g+n-3}{g}$	0
$\lambda_g^k \psi_1^{n+g(3-k)-3}$	$\frac{(-1)^{gk} (n+g-3)!}{24^g g! (n+g(3-k)-3)! ((k-2)!)^g}$	0
$\frac{\lambda_g^3}{\prod_{j=1}^n (1 - \psi_j)}$	$\frac{(-1)^g n^{n-3}}{24^g} \binom{g+n-3}{g}$	0
$\frac{\lambda_g^k}{\prod_{j=1}^n (1 - \psi_j)}$	$\frac{(-1)^{gk}}{24^g g!} \left(\int_{\overline{\mathcal{M}}_{0,n+g}} \frac{\prod_{j=n+1}^{g+n} \psi_j^{k-2}}{\prod_{j=1}^n (1 - \psi_j)} \right)$	0

While almost all the stable Hodge integrals above evaluate to 0, this is simply due to utilizing Mumford's relation for stable λ classes, and we do not expect this in general. We end with a conjecture for general cubic pseudostable Hodge integrals.

Conjecture 5.3.1. *For all pseudostable indices (g, n) and $j_1, j_2, j_3 \in \{0, \dots, g\}$, and for any polynomial $F \in \mathbb{C}[a_1, \dots, a_n]$, the pseudostable Hodge integral*

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_{j_1} \lambda_{j_2} \lambda_{j_3} F(\psi_1, \dots, \psi_n)$$

is equal to

$$\sum_{k=0}^g \frac{1}{24^k k!} \left(\sum_{k_1, k_2, k_3} D \cdot \int_{\mathcal{M}_{g-k, n+3k-(k_1+k_2+k_3)}} \lambda_{j_1-k_1} \lambda_{j_2-k_2} \lambda_{j_3-k_3} F(\psi_1, \dots, \psi_n) \right),$$

where

$$D = \binom{k}{(k_1 + k_2 + k_3) - 2k} \cdot \frac{(-1)^{(k_1+k_2+k_3)-2k} (2g + n - 3 - k)!}{(2g + n - 3 + k - (k_1 + k_2 + k_3))!},$$

and the values k_1, k_2, k_3 satisfy the following conditions:

$$\max\{0, j_i + k - g\} \leq k_i \leq \min\{k, j_i\},$$

$$k_1 + k_2 + k_3 \geq 2k.$$

Remark 5.3.2. While we do not offer a proof of Conjecture 5.3.1, the constant D is due to $-\psi_{\star_j}$ appearing on the root vertex a total of $(k_1 + k_2 + k_3) - 2k$ times, then using the dilaton equation (Theorem 2.4.5) to remove them from the Hodge integral.

It can be shown that Conjecture 5.3.1 agrees with all results shown for cubic and lower degree pseudostable Hodge integrals. We demonstrate two examples.

Example 5.3.3. If $j_3 = 0$ in Conjecture 5.3.1, then we have a quadratic pseudostable Hodge integral, so we should retrieve the result from Theorem 4.1.1. Since $j_3 = 0$, then $k_3 = 0$ as well from the conditions in Conjecture 5.3.1. This implies that $k_1 + k_2 \geq 2k$, but this only happens when $k_1 = k$ and $k_2 = k$, which is exactly what is stated by Theorem 4.1.1. Further, in this case the value D is

$$D = \binom{k}{0} \cdot \frac{(-1)^0 (2g + n - 3 - k)!}{(2g + n - 3 + k)!} = 1,$$

which also aligns with Theorem 4.1.1. So Conjecture 5.3.1 agrees with all quadratic and lower degree pseudostable Hodge integrals.

Example 5.3.4. If $j_1, j_2, j_3 = g$, then the conditions of Conjecture 5.3.1 would imply $k_1, k_2, k_3 = k$.

If the conjecture were true, this would imply the following:

$$\int_{\overline{\mathcal{M}}_{g,n}^{\text{ps}}} \lambda_g^3 F(\psi_1, \dots, \psi_n) = \sum_{k=0}^g \frac{C}{24^k k!} \cdot \int_{\overline{\mathcal{M}}_{g-k, n+k}} \lambda_{g-k}^3 F(\psi_1, \dots, \psi_n), \quad (5.3.0.1)$$

where

$$D = \binom{k}{3g-2k} \cdot \frac{(-1)^{3g-2k} (2g+n-3-k)!}{(-g+n-3+k)!}$$

Note that $\lambda_{g-k}^3 = 0$, except when $k = g$, in which case it is equal to 1. So (5.3.0.1) is equal to

$$\frac{(-1)^g (g+n-3)!}{24^g g! (n-3)!} \int_{\overline{\mathcal{M}}_{0, n+g}} F(\psi_1, \dots, \psi_n).$$

For $F(\psi_1, \dots, \psi_n) = \frac{1}{\prod_{j=1}^n (1-\psi_j)}$, this is precisely what was shown in Example 5.2.4, so Conjecture 5.3.1 agrees with that result.

We leave the proof (or disproof) of Conjecture 5.3.1 as an exercise for the reader.

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