DISSERTATION

THE DIMENSION OF PLANAR LINEAR SYSTEMS

Submitted by

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In partial fulfillment of the requirements

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY JAMES SEIBERT ENTITLED THE DIMENSION OF PLANAR LINEAR SYSTEMS BE ACCEPTED AS FULFILLING IN PART THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

THE DIMENSION OF PLANAR LINEAR SYSTEMS

The space of algebraic curves satisfying multiplicity conditions at specified points is a linear system of plane curves, and is naturally a projective space. In some cases the conditions imposed are dependent, even when the points are chosen in general position, giving the system a larger dimension than is expected. Such linear systems are called special. The Harbourne-Hirschowitz Conjecture hypothesizes that a linear system will be special only if every curve of the system has a multiple of some fixed (-1) curve as a component.

A linear system with a fixed multiplicity, m, assigned to all but one of the points is called a quasi-homogeneous linear system. The (-1) curves which may be contained in a quasi-homogeneous linear system are described herein. All linear systems with $m = 4$ which do contain a multiple (-1) curve are listed. A recent technique described by Rick Miranda and Ciro Ciliberto is then used to prove these are the only quasi-homogeneous linear systems with $m = 4$ which are special.

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Chapter 1

Introduction

Mathematicians have long been interested in the interpolation problem: given points in the plane, can a polynomial curve of a given degree or less be found that passes through those points? More generally, we may ask that the curve vanish m-fold at a point. This means that we require all of the partial derivatives through the $(m-1)$st derivatives to be zero as well. In this case the curve is said to pass through the point with multiplicity m. The question addressed in this paper is "how many" curves are there that satisfy some given conditions of this type. For convenience we prefer to work in projective space, $\mathbb{P}^2$. This means that instead of dealing with general polynomials in two variables of a given degree or less, we homogenize the problem and work with homogeneous polynomials in three variables of some fixed degree.

Consider the collection of all homogeneous polynomials of degree d, together with the zero polynomial. This forms a vector space which will be called $\mathcal{L}_d$, and we can easily compute the dimension. If we take as a basis the monomials of degree d, we see the vector space dimension of $\mathcal{L}_d$ is $\frac{(d+1)(d+2)}{2}$. Because scalar multiples of a polynomial define the same curve, it is more natural to consider this as a projective space (of one lower dimension). Unless otherwise stated, all dimensions will be projective dimensions. Thus we will write

$$\dim(\mathcal{L}_d) = \frac{d(d+3)}{2}.$$

If instead we ask for all homogeneous polynomials of degree d which have multiplicity m at some fixed point p, we get a subset of $\mathcal{L}_d$ denoted $\mathcal{L}_d(-mp)$. This subset is in fact a projective subspace, since differentiation is a linear operation. This is called the linear system of plane curves of degree d which pass through p with multiplicity m. By asking for multiplicity m at a point, we have imposed linear conditions on the coefficients of the polynomial. In the homogeneous case it is enough to require that all of the $(m-1)$st partial derivatives are zero. Thus a point with multiplicity m imposes $\frac{m(m+1)}{2}$ linear conditions.

Consider the projective plane $\mathbb{P}^2$ and $n+1$ points $p_0, p_1, \ldots, p_n$. Consider the linear system of plane curves of degree d which pass through the points p_i with multiplicity m_i for $0 \leq i \leq n$. We denote this system by $\mathcal{L} = \mathcal{L}_d(-\sum_{i=0}^n m_i p_i)$, and define its *virtual dimension*

$$v = v(\mathcal{L}) = \frac{d(d+3)}{2} - \sum_{i=0}^{n} \frac{m_i(m_i+1)}{2}.$$

We have merely subtracted the number of conditions imposed from the dimension of $\mathcal{L}_d$. Since the actual dimension of a linear system cannot be less than -1 (projectively, an empty system has dimension -1), we define the *expected dimension* to be

$$e = e(\mathcal{L}) = \max\{-1, v(\mathcal{L})\}.$$

Of course there may be ways to select the points in special positions that allow the actual dimension of the vector space to be larger than expected. As a simple example, consider the linear system consisting of curves of degree one passing through three points, $\mathcal{L}_1(-p_1 - p_2 - p_3)$. If we choose the three points so that they lie on a line then this system will contain that line, while the system is expected to be empty ($e(\mathcal{L}) = -1$) and for most choices of the three points the system is empty. In general there is a Zariski open set in the parameter space of $(n+1)$-tuples of points where the dimension of $\mathcal{L}_d(-\sum_{i=0}^{n} m_i p_i)$ achieves a minimum. When the points are in this set we will say that they are in *general position*. The dimension of the system with points in general position we call the *dimension* of $\mathcal{L}$ and we write $\ell = \ell(\mathcal{L})$. The actual dimension ℓ must be at least as big as the expected dimension e, with equality when the system is empty or when the conditions imposed are independent. It is possible, however, for the conditions imposed by the points to be dependent even when the points are in general position.

The generic system $\mathcal{L}$ is called *non-special* if it has the expected dimension, that is if $\ell(\mathcal{L}) = e(\mathcal{L})$. If $\ell(\mathcal{L}) > e(\mathcal{L})$ then the conditions are dependent, and the system is said to be *special.* We would like to identify the special linear systems.

As a first example of a special linear system take conics double at two points. $\mathcal{L}_2(-2p_1 - 2p_2)$. This system is expected to be empty, but the actual dimension is zero. The system consists of the line through the two points doubled (defined by squaring the equation for the line through the two points).

The conjecture regarding special linear systems relies on a knowledge of intersection numbers and on the genus formula for curves. In the projective plane, Bezout's Theorem tells us that if two curves do not share a common component then the total number of intersections, counted with multiplicities, is the product of the degrees of the two curves. Take curves $C_1 \in \mathcal{L}_d(-\sum m_i p_i)$ and $C_2 \in \mathcal{L}_e(-\sum n_i p_i)$. If they do not share a component, Bezout's Theorem gives de intersections. At each point p_i, the curves are guaranteed to meet at least $m_i n_i$ times. We can count the "extra intersections" $de - \sum m_i n_i$. If this number is negative, then the conclusion of Bezout's Theorem fails to hold, so the hypothesis must not be satisfied and the two curves must share a component.

Suppose that D_1 is an irreducible curve in $\mathcal{L}_d(-\sum m_i p_i)$ and that D_2 is any other curve in $\mathcal{L}_d(-\sum m_i p_i)$. The extra intersection number for these curves is $d^2 - \sum m_i^2$. If this number is negative one, then D_1 is contained in every curve in $\mathcal{L}_d(-\sum m_i p_i)$. But D_1 has degree d, so $\mathcal{L}_d(-\sum m_i p_i)$ consists of this curve alone.

A smooth complex curve is a Riemann surface, and topologically may be described by its genus. A curve C in $\mathcal{L}_d(-\sum m_i p_i)$ has singularities at points p_i with

$m_i \geq 2$. If this curve is smooth and irreducible after resolving these singularities, then it will have a genus given by the formula of Plücker

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum \frac{m_i(m_i-1)}{2}.$$

An irreducible curve with self-intersection -1 and zero genus is called a (-1) curve. A linear system which contains a (-1) curve, contains no other curves and may be identified with the (-1) curve. Following the convention of earlier work, the word *class* is used to refer to the numerical data of a linear system, namely the degree, d, and the multiplicity numbers, m_i. An irreducible (-1) class refers to the numerical data that satisfies the intersection and genus conditions, and corresponds to a linear system which contains an irreducible curve.

In all known examples, the failure of a system $\mathcal{L}$ to have the expected dimension is due to multiple (-1) curves contained in every curve of $\mathcal{L}$. In other words, there is an irreducible (-1) class $\mathcal{E}$ (consisting of a unique (-1) curve) such that $\mathcal{L} \cdot \mathcal{E} \leq -2$. It is known that a system which contains a multiple (-1) curve is special. Such systems are referred to as (-1) special. It is a conjecture, due to Harbourne [Ha] and Hirschowitz [Hi], that all special systems are (-1) special.

In this paper we discuss systems in which all the m_i are equal to some number m for $i \geq 1$. Such a system is called *quasi-homogeneous* and is denoted $\mathcal{L}(d, m_0, n, m)$. In Chapter 2 we classify all irreducible quasi-homogeneous (-1) curves. In Chapter 3 we complete the analysis of irreducible (-1) curves which may be contained in a quasi-homogeneous linear system. In Chapter 4 we consider quasi-homogeneous linear systems with $m = 4$: we list all systems which contain multiple (-1) curves and proceed to prove that these are the only quasi-homogeneous linear systems with $m = 4$ which are special.

Chapter 2

Quasi-Homogeneous (-1) Classes

The following classification was proved by Ciliberto and Miranda [CM1].

Proposition 2.0.1 *All quasi-homogeneous (-1) classes are those classes of the form* $\mathcal{L}(d, m_0, n, m)$ *with* (d, m_0, n, m) *on the following list.*

(a) *(2,0,5,1) and (1,1,1,1)*

(b) $(e, e-1, 2e, 1)$ *with* $e \geq 1$

(c) *For any* $m \geq 2$, $x \geq 1$, $y \geq 1$ *with*

i) $xy = (m-1)(2m+1)$

ii) $x + m \geq y$

iii) $x - y \equiv m \bmod 2$, *and*

iv) $m | (x + 2y - 1)$

the four-tuple

$$\left(\frac{x+y+3m}{2}, \frac{x-y+m}{2}, \frac{x+2y-1}{m}+4, m\right)$$

Proof: Suppose that $\mathcal{L}(d, m_0, n, m)$ is a quasi-homogeneous (-1) class. Then $\mathcal{L}^2 = -1$ means that

$$d^2 - m_0^2 - nm^2 = -1 \tag{2.1}$$

and a condition equivalent to the genus condition can be written as

$$3d - m_0 - nm = 1 \tag{2.2}$$

Solve 2.2 for m_0 and substitute into 2.1. We get

$$8d^2 - 6dnm + n^2m^2 - 6d + 2nm + nm^2 = 0. \tag{2.3}$$

Using the change of coordinates

$$u = 4d - nm, \qquad v = 2d - nm$$

2.3 becomes

$$uv + (m-1)u - (2m+1)v = 0. \tag{2.4}$$

Change coordinates again by setting

$$x = u - 2m - 1, \qquad y = 1 - m - v$$

This takes 2.4 to

$$xy = (m-1)(2m+1).$$

The case $m = 1$ leads to the classes in parts a) and b). Reversing these changes of coordinates for $m \geq 2$, we get classes as in part c). The conditions in part c) ensure that d, m_0, and n are integral and in ranges that make sense. ∎

The classes in part (a) are obviously irreducible. The class $\mathcal{L}(1, 1, 1, 1)$ is a line through two points, and $\mathcal{L}(2, 0, 5, 1)$ is a conic through five points in general position. The next two lemmas address other simple cases.

Lemma 2.0.1 *Quasi-homogeneous (-1)-classes with*

$$(d, m_0, n, m) = (e, e-1, 2e, 1)$$

are irreducible.

Proof: Performing the standard quadratic transformation centered at m_0 and any two points of multiplicity one leads to the class with

$$(d, m_0, n, m) = (e-1, e-2, 2(e-1), 1),$$

a class of the same form. The class $\mathcal{L}(1, 0, 2, 1)$, the $e = 1$ case, is irreducible, and every other class of this form reduces to the $e = 1$ case after $e - 1$ quadratic transformations. ∎

Lemma 2.0.2 *Quasi-homogeneous (-1)-classes from coming from a factorization $xy = (m-1)(2m+1)$ with $y = 1$ are irreducible.*

Proof: It is easily checked that these classes have the form

$$\mathcal{L}(m^2+m, m^2-1, 2m+3, m).$$

If we apply quadratic transformations $m+1$ times, centered at the points p_0, p_{2i-1}, p_{2i}, for $i = 1, \ldots, m+1$, then this class transforms to

$$\mathcal{L}(m+1, m, 2m+2, 1)$$

which is irreducible by the previous Lemma. ∎

Proposition 2.0.2 *There is at least one irreducible quasi-homogeneous (-1) class for every $n \geq 4$.*

Proof: In the case where n is even we are guaranteed an irreducible class that has the form $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ by Lemma 2.0.1. If n is odd, we are guaranteed an irreducible class of the form $\mathcal{L}(\frac{(n-3)(n-1)}{4}, \frac{(n-5)(n-1)}{4}, n, \frac{n-3}{2})$ by Lemma 2.0.2. ∎

It is enough for us to consider irreducible (-1) classes. A linear system contains a (-1) class if and only if it contains an irreducible (-1) class. It would be desirable to have a classification of irreducible quasi homogeneous (-1) classes. To this end, we begin by investigating some small values of n. From the above list we see that for $n = 1$ there is only the class $\mathcal{L}(1, 1, 1, 1)$, a line through two points. For $n = 2$ and for $n = 4$ there are only the classes $\mathcal{L}(1, 0, 2, 1)$ and $\mathcal{L}(2, 1, 4, 1)$ coming from part (b) of Proposition 2.0.1. $\mathcal{L}(1,0,2,1)$ is again a line through two points. and $\mathcal{L}(2,1,4,1)$ is a conic through five points. These classes are irreducible. There are no classes with $n = 3$. Part (c) of Proposition 2.0.1 only applies when $n \geq 5$.

2.1 Irreducible Classes With $n \leq 7$

When $n = 5$. we have the irreducible class $\mathcal{L}(2,0,5,1)$. a conic through five points. from part (a) of Proposition 2.0.1, and we have the potential for classes coming from a factorization of $(m-1)(2m+1)$. If we set $n = 5$ in the formula

$$n = \frac{x + 2y - 1}{m} + 4$$

and solve for x. we get

$$x = m - 2y + 1.$$

Substituting this into the equation $xy = (m-1)(2m+1)$ we see that classes with $n = 5$ correspond to integer solutions to the following equation. with $m \geq 2$. and $y \geq 1$.

$$2m^2 - my + 2y^2 - y - m - 1 = 0$$

Organize this as a function of y, and apply the quadratic formula to see that

$$y = \frac{(m+1) \pm \sqrt{-15m^2 + 10m + 9}}{4}$$

The discriminant is negative for $m \geq 2$, therefore there are no real solutions for y. let alone integer ones.

For $n = 6$ the analysis is similar. Part (b) of Proposition 2.0.1 gives the irreducible class $\mathcal{L}(3, 2, 6, 1)$. Making a similar substitution to the one in the $n = 5$ case, we are led to looking for integer solutions to

$$2m^2 - 2my + 2y^2 - y - m - 1 = 0.$$

Again the discriminant, when organized as a polynomial in y, is negative for $m \geq 2$. There are no quasi homogeneous (-1)-classes coming from part (c) of Proposition 2.0.1 when $n = 6$.

If $n = 7$ we proceed as before. We look for integer solutions to

$$2m^2 - 3my + 2y^2 - y - m - 1 = 0.$$

This time we get the solution

$$y = \frac{3m + 1 \pm \sqrt{-7m^2 + 14m + 9}}{4}.$$

The discriminant is negative for $m \geq 3$, but positive for $m = 2$. When $m = 2$, $y = \frac{7 \pm 3}{4}$. $y = \frac{7+3}{4}$ is not integral, but $y = \frac{7-3}{4}$ leads to the class $\mathcal{L}(6, 3, 7, 2)$ from the factorization $x = 5, y = 1$. This is the only class with $n = 7$ and it is irreducible by Lemma 2.0.2.

2.2 Irreducible Classes With $n = 8$

The case where $n = 8$ starts the same, but we see a major difference. We have the irreducible class $\mathcal{L}(4, 3, 8, 1)$ coming from (b) since n is even, but the quadratic equation to which we seek integer solutions is no longer an ellipse: it is the hyperbola

$$2m^2 - 4my + 2y^2 - y - m - 1 = 0.$$

We have the possibility for an infinite number of integer solutions, and this possibility is realized. Making the substitutions $m = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$ transforms the equation into

$$2v^2 - u - 1 = 0.$$

The inverse of this substitution sets $v = m - y$ and $u = m + y$. We also have the relation $x = 4m - 2y - 1 = u + 3v + 1$. It is clear that integer solutions in m and y lead to integer solutions u and v. It is also true that u and v transform to integers m and y if and only if u and v are integers such that $u \equiv v \bmod(2)$. Since $2v^2 - u - 1 = 0$, this last condition is satisfied if and only if v is odd.

We must find conditions so that $m \geq 2$ and $y \geq 1$ and $x \geq 1$. To do this let us write these variables in terms of v alone, using $u = 2v^2 - 1$.

$$x = 2v^2 + 3v$$

$$y = v^2 - \frac{v+1}{2}$$

$$m = v^2 + \frac{v-1}{2}$$

In order for $m \geq 2$, $|v|$ must be 3 or greater. This is enough to ensure that y and x are both 1 or more as well.

We must now check that the other conditions from part (c) are satisfied. In particular, $x + m \geq y$, $x - y \equiv m \bmod(2)$, and $m | x + 2y - 1$. We see that

$x + m \geq y$ if and only if $v^2 + 2v \geq 0$. This is true, provided $v \neq -1$. Also, $x - y - m = 3v + 1$, which is even since v is odd. Therefore $x - y \equiv m \bmod(2)$. Finally, $x + 2y - 1 = 4v^2 + 2v - 2 = 4m$, so $m|x + 2y - 1$. Therefore, we do get a quasi homogeneous (-1)-class for all odd v with $|v| \geq 3$.

In order to list these classes, let us set $v = -2z - 1$. Then for $z \geq 1$ and $z \leq -2$ we get the classes

$$\mathcal{L}(12z^2 + 8z, 4z^2 - 1, 8, 4z^2 + 3z).$$

When $z = -1$, we get the class $\mathcal{L}(4, 3, 8, 1)$, which is the irreducible quasi homogeneous (-1) class mentioned above. All of these classes transform to each other. The z class can be brought to the $z + 1$ class by the following sequence of twelve standard Cremona transformations. The underlined multiplicities correspond to the centers of the next transformation.

$$\begin{array}{rl}
12z^2 + 8z : & 4z^2 - 1, \underline{\underline{\underline{(4z^2 + 3z)}}}^8 \\
12z^2 + 7z : & 4z^2 - 1, (4z^2 + 2z)^3, \underline{\underline{\underline{(4z^2 + 3z)}}}^5 \\
12z^2 + 5z : & \underline{4z^2 - 1}, (4z^2 + 2z)^3, \underline{\underline{(4z^2 + z)}}^3, (4z^2 + 3z)^2 \\
12z^2 + 8z + 1 : & (4z^2 + 4z + 1)^2, (4z^2 + 3z)^3, \underline{\underline{(4z^2 + 2z)}}^3, \underline{4z^2 + z} \\
12z^2 + 11z + 2 : & (4z^2 + 5z + 1)^2, (4z^2 + 4z + 1)^3, \underline{\underline{(4z^2 + 3z)}}^3, \\
 & \underline{4z^2 + 2z} \\
12z^2 + 14z + 4 : & (4z^2 + 6z + 2)^2, 4z^2 + 5z + 2, (4z^2 + 5z + 1)^2, \\
 & \underline{\underline{(4z^2 + 4z + 1)}}^3, \underline{4z^2 + 3z} \\
12z^2 + 17z + 6 : & (4z^2 + 7z + 3)^2, (4z^2 + 6z + 2)^3, \underline{4z^2 + 5z + 2} \\
 & \underline{(4z^2 + 5z + 1)}^2, \underline{4z^2 + 4z + 1} \\
12z^2 + 20z + 8 : & 4z^2 + 8z + 4, 4z^2 + 8z + 3, (4z^2 + 7z + 3)^3, \\
 & \underline{\underline{(4z^2 + 6z + 2)}}^3, \underline{4z^2 + 5z + 1} \\
12z^2 + 23z + 11 : & (4z^2 + 9z + 5)^2, (4z^2 + 8z + 4)^2, 4z^2 + 8z + 3, \\
 & \underline{\underline{(4z^2 + 7z + 3)}}^3, \underline{4z^2 + 6z + 2} \\
12z^2 + 26z + 14 : & (4z^2 + 10z + 6)^2, (4z^2 + 9z + 5)^3, \underline{\underline{(4z^2 + 8z + 4)}}^2
\end{array}$$

$$
\begin{array}{ll}
 & 4z^2 + 8z + 3, \underline{4z^2 + 7z + 3} \\
12z^2 + 29z + 17: & (4z^2 + 11zz + 7)^2, (4z^2 + 10z + 6)^3, \underline{\underline{(4z^2 + 9z + 5)^3}} \\
 & 4z^2 + 8z + 3 \\
12z^2 + 31z + 19: & (4z^2 + 11z + 7)^5, \underline{\underline{(4z^2 + 10z + 6)^3}}, 4z^2 + 8z + 3 \\
12z^2 + 32z + 20: & (4z^2 + 11z + 7)^8, 4z^2 + 8z + 3
\end{array}
$$

The last class can be written as

$$\mathcal{L}(12(z+1)^2 + 8(z+1), 4(z+1)^2 - 1, 8, 4(z+1)^2 + 3(z+1)).$$

Since each Cremona transformation is its own inverse, reversing these transformations takes the z class to the $z-1$ class. Therefore all of these classes are birationally equivalent to the $z = -1$ class $\mathcal{L}(4, 3, 8, 1)$, and are irreducible. All quasi homogeneous (-1) classes with $n = 8$ are irreducible. This, combined with the previous section, proves the following theorem.

Theorem 2.2.1 *There is exactly one quasi-homogeneous (-1)-class for $n = 1, 2, 4, 5, 6,$ and 7, and it is irreducible in each case. All quasi-homogeneous (-1)-classes with $n = 8$ are of the form $\mathcal{L}(12z^2 + 8z, 4z^2 - 1, 8, 4z^2 + 3z)$, for $z \leq -1$ or $z \geq 1$, and are irreducible.*

2.3 Irreducible Classes With $n = 9$

We employ the same technique as above. The quadratic we look to solve is

$$2m^2 - 5my + 2y^2 - m - y - 1 = 0.$$

This factors as

$$(2y - m + 1)(y - 2m - 1) = 0.$$

Suppose $y = 2m + 1$. Then $x = m - 1$, and $x + m = 2m - 1 < 2m + 1 = y$. There are no solutions for $y = 2m + 1$. So we suppose $m = 2y + 1$. Then $y = \frac{m-1}{2}$ and $x = 4m + 2$. We have $x + m = 5m + 2$ which is greater than y since $m \geq 2$. The requirement that $x - y \equiv m \text{ mod}(2)$ forces y to be odd, since x is even and m is odd. If we write $y = 2z - 1$ for $z \geq 1$, and write everything in terms of z, the results are the classes $\mathcal{L}(15z - 3, 9z - 1, 9, 4z - 1)$. The class $\mathcal{L}(12, 8, 9, 3)$, when $z = 1$, is the irreducible class guaranteed by Proposition 2.0.2. This is the only irreducible class with $n = 9$. All other classes Cremona reduce to something which splits, as follows.

$15z - 3:$	$\underline{9z - 1}, \underline{\underline{(4z - 1)^9}}$
$13z - 3:$	$\underline{7z - 1}, (2z - 1)^2, \underline{\underline{(4z - 1)^7}}$
$11z - 3:$	$\underline{5z - 1}, (2z - 1)^4, \underline{\underline{(4z - 1)^5}}$
$9z - 3:$	$3z - 1, (2z - 1)^6, \underline{\underline{\underline{(4z - 1)^3}}}$
$6z - 3:$	$\underline{3z - 1}, \underline{\underline{(2z - 1)^6}}, (z - 1)^3$
$5z - 3:$	$\underline{\underline{\underline{(2z - 1)^5}}}, (z - 1)^5$
$4z - 3:$	$\underline{(z - 1)^8}, \underline{\underline{(2z - 1)^2}}$
$3z - 3:$	$(z - 1)^9, -1$

Reversing this reduction shows the class $\mathcal{L}(12, 8, 9, 3)$ is the class which splits. This is verified easily by looking at the index of intersection.

$$12(15z - 3) - 8(9z - 1) - (9)(3)(4z - 1) = -1$$

We have proved the following.

Proposition 2.3.1 $\mathcal{L}(12, 8, 9, 3)$ *is the lone irreducible quasi homogeneous (-1)-classes with* $n = 9$. *All others are reducible and contain this class as a component.*

2.4 Irreducible Classes with $n = 10$

With $n = 10$ we proceed as before. We have the irreducible class $\mathcal{L}(5, 4, 10, 1)$ from Proposition 2.0.2. This time we get the quadratic

$$2m^2 - 6my + 2y^2 - m - y - 1 = 0. \tag{2.5}$$

and solutions y of the form

$$y = \frac{6m + 1 \pm \sqrt{20m^2 + 20m + 9}}{4}.$$

It is possible to classify all integer solutions to this using the theory of Pell's equations, but this method does not generalize easily to larger values of n. Instead, we can try to prove something more general that does not depend on y being an integer. The following lemma allows us to do this.

Lemma 2.4.1 *Solutions to equation 2.5 of the form*

$$y = \frac{6m + 1 + \sqrt{20m^2 + 20m + 9}}{4}$$

do not lead to quasi-homogeneous (-1) classes. In particular, they fail to satisfy $x + m \geq y$ (necessary for m_0 to be non-negative).

Proof: Suppose that y has the above form. Then $x = (10 - 4)m + 1 - 2y$ has the form

$$x = \frac{6m + 1 - \sqrt{20m^2 + 20m + 9}}{2}.$$

We can directly check the inequality $x + m \geq y$. This simplifies to

$$10m + 1 \geq 3\sqrt{20m^2 + 20m + 9}.$$

Both sides of this inequality are positive for $m \geq 2$, so we may square both sides and simplify again. This inequality holds if and only if

$$0 \geq 80m^2 + 160m + 80.$$

This is clearly false for all $m \geq 2$. ∎

This Lemma is enough to prove the following result.

Proposition 2.4.1 $\mathcal{L}(5, 4, 10, 1)$ *is the only irreducible quasi-homogeneous (-1)-class with* $n = 10$. *All others are reducible and contain this class as a component.*

Proof: We must show that the index of intersection of any (-1) class of the form $\mathcal{L}(d, m_0, 10, m)$ and the class $\mathcal{L}(5, 4, 10, 1)$ is negative. That is,

$$5d - 4m_0 - 10m < 0.$$

First, note that $\mathcal{L}(5, 4, 10, 1)$ is the only class with $m = 1$ and $n = 10$. Thus, we may assume that $m \geq 2$ and the class $\mathcal{L}(d, m_0, 10, m)$ comes from a factorization $xy = (m-1)(2m+1)$. The previous lemma allows us to assume that

$$y = \frac{6m + 1 - \sqrt{20m^2 + 20m + 9}}{4}.$$

and

$$x = \frac{6m + 1 + \sqrt{20m^2 + 20m + 9}}{2}.$$

Now we may write d and m_0 in terms of m, as well.

$$d = \frac{x + y + 3m}{2} = \frac{30m + 3 + \sqrt{20m^2 + 20m + 9}}{8}$$

$$m_0 = \frac{x - y + m}{2} = \frac{10m + 1 + 3\sqrt{20m^2 + 20m + 9}}{8}$$

Make these substitutions, and simplify $5d - 4m_0 - 10m$. We get

$$5d - 4m_0 - 10m = \frac{30m + 11 - 7\sqrt{20m^2 + 20m + 9}}{8}.$$

We wish to show that this quantity is negative. This will be negative if

$$30m + 11 < 7\sqrt{20m^2 + 20m + 9}.$$

Both sides of this inequality are positive for $m \geq 2$, so we may square both sides and simplify. This inequality holds if

$$(30m + 11)^2 - 49(20m^2 + 20m + 9) = -80m^2 - 320m - 320 < 0.$$

This inequality is true for $m \geq 2$. If these numbers are integers, this implies that every quasi homogeneous (-1) class with $n = 10$ contains the class $\mathcal{L}(5, 4, 10, 1)$. ∎

2.5 Irreducible Classes with $n \geq 11$

For $n = 9$ and $n = 10$ we saw that the irreducible class given by Proposition 2.0.2 was the unique irreducible class for that value of n. We are now in a position to attempt to prove this in general.

For a given n, solve $n = \frac{x+2y-1}{m} + 4$ for x. Substitute $x = (n-4)m - 2y + 1$ into the equation $xy = (m-1)(2m+1)$, and solve for y.

$$y = \frac{(n-4)m + 1 \pm \sqrt{n(n-8)m^2 + 2nm + 9}}{4}$$

Lemma 2.5.1 *Factorizations $xy = (m-1)(2m+1)$ with $m \geq 2$ and*

$$y = \frac{(n-4)m + 1 + \sqrt{n(n-8)m^2 + 2nm + 9}}{4}$$

do not correspond to quasi-homogeneous (-1)-classes for $n \geq 11$. Specifically, a solution of this form will not satisfy the condition $x + m \geq y$.

Proof: If y has the above form, then

$$x = \frac{(n-4)m + 1 - \sqrt{n(n-8)m^2 + 2nm + 9}}{2}.$$

The inequality $x + m \geq y$ simplifies to

$$nm + 1 \geq 3\sqrt{n(n-8)m^2 + 2nm + 9}.$$

Both sides of this inequality are positive for $n \geq 11$ and $m \geq 2$. Square both sides and simplify again to see that this inequality holds if and only if

$$0 \geq 8n(n-9)m^2 + 16nm + 80.$$

This is clearly false for $n \geq 11$ and $m \geq 2$. ∎

This lemma allows us to assume that for any class coming from a factorization $xy = (m-1)(2m+1)$ with $m \geq 2$.

$$y = \frac{(n-4)m + 1 - \sqrt{n(n-8)m^2 + 2nm + 9}}{4}.$$

and therefore

$$x = \frac{(n-4)m + 1 + \sqrt{n(n-8)m^2 + 2nm + 9}}{2}.$$

This lets us write d and m_0 in terms of m and n.

$$d = \frac{x + y + 3m}{2} = \frac{3mn + 3 + \sqrt{n(n-8)m^2 + 2nm + 9}}{8} \tag{2.6}$$

$$m_0 = \frac{x - y + m}{2} = \frac{mn + 1 + 3\sqrt{n(n-8)m^2 + 2nm + 9}}{8} \tag{2.7}$$

Proposition 2.5.1 *For a given even $n \geq 12$, $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ is the unique irreducible quasi homogeneous (-1) class. Every other quasi-homogeneous (-1) class $\mathcal{L}(d, m_0, n, m)$ with even $n \geq 12$ contains the class $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ as a component.*

Proof: First note that for any even $n \geq 12$, $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ is the only class with $m = 1$. This class is irreducible by Proposition 2.0.2. Thus, we conclude that any other class $\mathcal{L}(d, m_0, n, m)$ comes from a factorization $xy = (m-1)(2m+1)$ with $m \geq 2$. Use the substitutions 2.6 and 2.7 to check that the index of intersection of any other class $\mathcal{L}(d, m_0, n, m)$ with $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$,

$$\frac{n}{2}d - \frac{n-2}{2}m_0 - nm,$$

is negative. The index of intersection becomes

$$\frac{n(n-7)m + (n+1) - (n-3)\sqrt{n(n-8)m^2 + 2nm + 9}}{8}.$$

This will be negative if and only if

$$n(n-7)m+(n+1) < (n-3)\sqrt{n(n-8)m^2+2nm+9}.$$

For $n \geq 12$ and $m \geq 2$, both sides of this inequality are positive, so we may square both sides. This inequality is true if and only if

$$0 < 8n(n-9)m^2+32nm+8(n-2)(n-5).$$

This is clearly true for $n \geq 12$ and $m \geq 2$. If these numbers are integers, this implies that every quasi homogeneous (-1) class with even $n \geq 12$ contains the class $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ as a component. This makes $\mathcal{L}(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ the unique irreducible class for a given even $n \geq 12$. ∎

Proposition 2.5.2 *For a given odd $n \geq 11$, the unique irreducible quasi homogeneous (-1) class is $\mathcal{L}(\frac{(n-3)(n-1)}{4}, \frac{(n-5)(n-1)}{4}, n, \frac{(n-3)}{2})$. Any other quasi-homogeneous (-1) class $\mathcal{L}(d, m_0, n, m)$ with odd $n \geq 11$ contains this class as a component.*

Proof: Note that for odd $n \geq 11$ there are no classes with $m = 1$. Therefore, all quasi homogeneous (-1) classes with odd $n \geq 11$ come from a factorization $xy = (m-1)(2m+1)$ with $m \geq 2$. Use the substitutions 2.6 and 2.7 to check that the index of intersection of any other class $\mathcal{L}(d, m_0, n, m)$ with a class of the form $\mathcal{L}(\frac{(n-3)(n-1)}{4}, \frac{(n-5)(n-1)}{4}, n, \frac{(n-3)}{2})$,

$$\frac{(n-3)(n-1)}{4}d - \frac{(n-5)(n-1)}{4}m_0 - n\frac{(n-3)}{2}m,$$

is negative. The index of intersection becomes

$$\frac{n(n^2-11n+26)m+n(n-3)-(n-6)(n-1)\sqrt{n(n-8)m^2+2nm+9}}{16}.$$

This will be negative if and only if

$$n(n^2-11n+26)m+n(n-3)<(n-6)(n-1)\sqrt{n(n-8)m^2+2nm+9}.$$

Both sides of this inequality are positive for $n \geq 11$ and $m \geq 2$, so we may square both sides and simplify. This inequality holds if and only if

$$0<32n(n-9)m^2+4n(n^2-3n+18)m+4(n^2-9n+9)(2n^2-12n+9).$$

A quick check using the quadratic formula shows that all terms on the right are positive for $n \geq 11$ and $m \geq 2$. If these numbers are all integers, this implies that every quasi homogeneous (-1) class with odd $n \geq 11$ contains the class $\mathcal{L}(\frac{(n-3)(n-1)}{4}, \frac{(n-5)(n-1)}{4}, n, \frac{(n-3)}{2})$ as a component. This means that the class $\mathcal{L}(\frac{(n-3)(n-1)}{4}, \frac{(n-5)(n-1)}{4}, n, \frac{(n-3)}{2})$ is the unique irreducible class for a given odd $n \geq 11$. ∎

2.6 Conclusion

We have completely classified irreducible quasi homogeneous (-1) classes. To complete the proof of the following theorem, one must only check that the irreducible classes we found for small values of n are represented in this list.

Theorem 2.6.1 *All irreducible quasi homogeneous (-1) classes are those classes $\mathcal{L}(d, m_0, n, m)$ with (d, m_0, n, m) on the following list*

(a) (1,1,1,1)

(b) $(\frac{n}{2}, \frac{n-2}{2}, n, 1)$ *for n even,* $n \geq 2$.

(c) $(\frac{(n-3)(n-1)}{4}, \frac{(n-5)(n-1)}{4}, n, \frac{n-3}{2})$ *for n odd,* $n \geq 5$.

(d) For $n = 8$, *there is an infinite family of irreducible classes. For all integers* $z \neq 0$ *we have the class*

$$(12z^2 - 8z, 4z^2 - 1, 8, 4z^2 - 3z).$$

Chapter 3

Compound (-1) Configurations

Let A be a (-1)-curve of degree δ and multiplicity μ_i at point p_i, for i from 0 to n. Suppose that A is contained as a component of a quasi-homogeneous linear system, $\mathcal{L} = \mathcal{L}(d, m_0, n, m)$. For any permutation $\sigma \in \Sigma_n$ let A_σ be the curve of degree δ, multiplicity μ_0 at p_0, and multiplicity $\mu_{\sigma(i)}$ at p_i for $i \geq 1$. Then if $\mathcal{L}$ contains A as a component, $\mathcal{L}$ must also contain A_σ for all $\sigma \in \Sigma_n$. In this situation the A_σ curves must be disjoint, for if two (-1)-curves meet on a rational surface their union moves in a linear system. The Picard group of the blowup surface has rank $n + 2$, and the largest negative definite subspace has dimension $n + 1$. This means that there can be at most $n + 1$ of these disjoint (-1)-curves.

Partition the multiplicities μ_i, for i from 1 to n, into the largest subsets on which the multiplicities are constant. If these subsets have size

$$k_1 \leq k_2 \leq \ldots \leq k_s,$$

then the number of distinct permutations (and the number of distinct A_σ's) is

$$\frac{n!}{k_1!k_2!\dots k_s!}.$$

This must be less than or equal to $n+1$. There are only two possibilities. Either $s = 1$ and A is itself quasi-homogeneous, or $s = 2$, $k_1 = 1$ and $k_2 = n-1$. In the second case A is a curve of degree δ, multiplicity μ_0 at p_0, multiplicity μ_1 at p_1, and some constant multiplicity μ at p_i for i from 2 to n. Copying the notation for quasi-homogeneous linear systems, let us say $A = \mathcal{L}(d, m_0, m_1, n, m)$. This will occasionally be referred to as a quasi-quasi-homogeneous (-1) class. (Note that this renames the degree and multiplicity of A, but changes the meaning of the variable n; n is now two less than the total number of points instead of one less than the total number of points, as in the quasi homogeneous case.)

If A is contained in a quasi homogeneous linear system $\mathcal{L}$, then $\mathcal{L}$ will also contain the sum of the curves A_σ. This sum is itself quasi homogeneous, with the form

$$\mathcal{L}((n+1)d, (n+1)m_0, n+1, m_1 + nm).$$

We will refer to such a sum as a (quasi homogeneous) (-1) *configuration*. The configuration is said to be *compound* if it consists of more than one (-1) curve. We will classify the irreducible curves A, and list the corresponding compound (-1) configurations.

We have several conditions on the degree and multiplicities of A. Since A is a

(-1)-class we have

$$d^2 - m_0^2 - m_1^2 - nm^2 = -1. \tag{3.1}$$

A condition equivalent to the genus condition (specifically, that $\mathcal{L} \cdot K = -1$ on the blow-up surface) is

$$3d - m_0 - m_1 - nm = 1. \tag{3.2}$$

Recall that the classes A_σ, arrived at by permuting the points p_1 through p_n, must be disjoint. This may be expressed in terms of their index of intersection as

$$d^2 - m_0^2 - 2m_1m - (n-1)m^2 = 0. \tag{3.3}$$

Subtracting equation 3.1 from 3.3 gives

$$(m_1 - m)^2 = 1,$$

which implies that $m_1 = m \pm 1$.

Solve equation 3.2 to get $m_0 = 3d - m_1 - nm - 1$. This transforms equation 3.1 into

$$8d^2 - 6dnm_1 + n^2m^2 - 6d + 2nm + nm^2 + 2m - 6dm_1 + 2m_1^2 + 2nm_1m = 0.$$

We have two cases to consider, $m_1 = m \pm 1$. We will examine these cases separately.

3.1 The Case $m_1 = m - 1$ for Small n

If $m_1 = m - 1$, we need $m \geq 1$ or else m_1 will be negative. When $m_1 = m - 1$, equation 3.1 becomes

$$8d^2 - 6dm - 6dnm + 2m^2 + 3nm^2 + n^2m^2 - 2m = 0. \tag{3.4}$$

This may be rewritten as

$$(2d - (n+1)m)(4d - (n+1)m) + (n+1)m^2 - 2m = 0.$$

suggesting the change of coordinates

$$u = 2d - (n+1)m,$$

$$v = 4d - (n+1)m.$$

Note that $v - 2u = (n+1)m$. Equation 3.4 is transformed into

$$uv + vm - 2um - 2m = 0.$$

Finally, the change of coordinates

$$x = -u - m,$$

$$y = v - 2m$$

takes this equation to

$$xy = 2m(m-1). \tag{3.5}$$

Reversing these changes of coordinates yields

$$d = \frac{x + y + 3m}{2},$$

$$m_0 = \frac{y - x + m}{2},$$

$$m_1 = m - 1,$$

and

$$n = \frac{2x+y}{m} + 3.$$

This gives the following classification.

Proposition 3.1.1 *The (-1) classes $\mathcal{L}(d, m_0, m_1, n, m)$ with $m_1 = m - 1$ are those with (d, m_0, m_1, n, m) of the form*

$$\left(\frac{x+y+3m}{2}, \frac{y-x+m}{2}, m-1, \frac{2x+y}{m} + 3, m\right).$$

for integers x and y satisfying the following conditions

a) $xy = 2m(m-1)$ *for* $m \geq 1$

b) $y - x \equiv m \mod 2$

c) $m | 2x + y$

d) $y + m \geq x$

e) $x + y + 3m \geq 2$

f) $\frac{2x+y}{m} \geq -2$

Proof: By the preceding paragraph, such a class $\mathcal{L}(d, m_0, m_1, n, m)$ will satisfy equations 3.1, 3.2, and 3.3. If these numbers are integral and in ranges that make sense, they will correspond to (-1) classes. We see that d and m_0 will be integral if and only if x and y are integers such that $y - x \equiv m \mod 2$, and that n will be integral if and only if $m|2x + y$. The remaining conditions ensure that d, n and m will be greater than or equal to 1, while m_0 and m_1 will be zero or greater. ∎

Let us consider first the case $m = 1$. Equation 3.5 is $xy = 0$ in this case. so either $x = 0$ or $y = 0$. Suppose that $x = 0$. Then

$$d = \frac{y+3}{2},$$

$$m_0 = \frac{y+1}{2},$$

$$m_1 = 0,$$

and

$$n = y + 3.$$

This gives a (-1) curve of the desired form for every odd $y \geq -1$. These may be written in a more familiar form as $\mathcal{L}(e, e-1, 0, 2e, 1)$, for all $e \geq 1$. This class and it's permutations sum to the compound (-1) configuration $\mathcal{L}(e(2e+1), (e-1)(2e+1), 2e+1, 2e)$.

Suppose now that $y = 0$. Then

$$d = \frac{x+3}{2},$$

$$m_0 = \frac{1-x}{2},$$

$$m_1 = 0,$$

and

$$n = 2x + 3.$$

We need both d and m_0 integral, $d > 0$ and $m_0 \geq 0$. This happens only for $x = \pm 1$. The case $x = 1$ leads to the class $\mathcal{L}(2, 0, 0, 5, 1)$, and $x = -1$ the class

$\mathcal{L}(1,1,0,1,1)$. These are the only classes with $m = 1$. They lead to the compound (-1) configurations $\mathcal{L}(12,0,6,5)$ and $\mathcal{L}(2,2,2,1)$ respectively.

Now consider $m \geq 2$. Solve the equation $n = \frac{2x+y}{m} + 3$ for y. Substitute into equation 3.5 and solve for x to get

$$x = \frac{(n-3)m \pm \sqrt{(n-7)(n+1)m^2 + 16m}}{4}. \quad (3.6)$$

For a given n, x must at least satisfy equation 3.6, and be integral. A simple check shows that there are no real solutions for x with $m \geq 2$ and $n \leq 5$. When $n = 6$, we have

$$x = \frac{3m \pm \sqrt{-7m^2 + 16m}}{4}.$$

This has only $m = 2$, $x = 2$ as a solution, leading to the class $\mathcal{L}(5,1,1,6,2)$. This system is irreducible, as it Cremona reduces to a line through two points. It produces the compound (-1) configuration $\mathcal{L}(35,7,7,13)$. For $n \geq 7$ there are potentially an infinite number of solutions.

3.2 The Case $m_1 = m + 1$ for Small n

In this case, we may consider all values $m \geq 0$. When $m_1 = m + 1$ the equation 3.1 becomes

$$8d^2 - 12d + 4 + 6m + 4nm - 6dm - 6dnm + 2m^2 + 3nm^2 + n^2m^2 = 0. \quad (3.7)$$

If $m = 0$, this equation factors as $4(d-1)(2d-1) = 0$, which forces d to be 1. This brings us to the class $A = \mathcal{L}(1, m_0, 1, n, 0)$. Now, $A \cdot A = -1$ implies that m_0 is also equal to 1. Thus we have the class $\mathcal{L}(1, 1, 1, n, 0)$ for any $n \geq 0$, corresponding to the (-1) configurations $\mathcal{L}(e, e, e, 1)$ for all $e \geq 1$. The configuration is compound if $e \geq 2$.

For $m \geq 1$, equation 3.7 may be rewritten as

$$(2d - (n+1)m)(4d - (n+1)m) + (m-2)(4d - (n+1)m)$$

$$-(2m+2)(2d - (n+1)m) + 2m + 4 = 0,$$

again suggesting the change of coordinates

$$u = 2d - (n+1)m,$$

$$v = 4d - (n+1)m.$$

This transforms equation 3.7 into

$$uv + (m-2)v - 2(m+1)u + 2(m+2) = 0. \tag{3.8}$$

Change coordinates again, using

$$x = -u - (m-2),$$

$$y = v - 2(m+1).$$

Equation 3.8 becomes

$$xy = 2m^2. \tag{3.9}$$

Reversing these changes yields

$$d = \frac{x + y + 3m}{2},$$

$$m_0 = \frac{y - x + m}{2},$$

$$m_1 = m + 1,$$

and

$$n = \frac{2x + y - 2}{m} + 3.$$

We have the following classification.

Proposition 3.2.1 *The (-1) classes $\mathcal{L}(d, m_0, m_1, n, m)$ with $m_1 = m + 1$ are those with (d, m_0, m_1, n, m) either of the form*

$$(1, 1, 1, n, 0), \quad n \geq 0, \text{ or}$$

$$\left(\frac{x + y + 3m}{2}, \frac{y - x + m}{2}, m + 1, \frac{2x + y - 2}{m} + 3, m\right)$$

for integers x and y satisfying the following conditions.

a) $xy = 2m^2$ *for* $m \geq 1$

b) $y - x \equiv m \bmod 2$

c) $m | 2x + y - 2$

d) $y + m \geq x$

e) $x + y + 3m \geq 2$

f) $\frac{2x+y-2}{m} \geq -2$

Proof: By the preceding paragraph, such a class $\mathcal{L}(d, m_0, m_1, n, m)$ will satisfy equations 3.1, 3.2, and 3.3. If these numbers are integral and in ranges that make sense, they will correspond to (-1) classes. Condition b) guarantees that d and

m_0 will be integral. Condition c) ensures that n will be integral. The remaining conditions force d, n, and m to be greater than or equal to 1, and m_0 and m_1 to be zero or greater. ∎

We begin the analysis in much the same way as before. Solve the equation $n = \frac{2x+y-2}{m} + 3$ for y, substitute into $xy = 2m^2$, and solve for x to see that

$$x = \frac{2 + (n-3)m \pm \sqrt{(n-7)(n+1)m^2 + 4(n-3)m + 4}}{4}. \tag{3.10}$$

For a given n, x must at least satisfy this equation and be integral. For n from 1 through 4 there are no real solutions for x with $m \geq 1$. When $n = 5$, the only solution is $m = 1$, $x = 1$, which gives the irreducible system $\mathcal{L}(3,1,2,5,1)$, and the compound (-1) configuration $\mathcal{L}(18,6,6,7)$. When $n = 6$, there are two possibilities. Either $m = 1$, $x = 2$, and the system is $\mathcal{L}(3,0,2,6,1)$, or $m = 2$, $x = 2$, with the system $\mathcal{L}(6,2,3,6,2)$. These are both irreducible and lead to the compound (-1) configurations $\mathcal{L}(21,0,7,8)$ and $\mathcal{L}(42,14,7,15)$ respectively. For $n \geq 7$ there are potentially an infinite number of solutions.

3.3 The Case $n = 7$

As in the quasi-homogeneous case, these linear systems with nine points (i.e. when n is seven) are unusual. We will see infinitely many irreducible (-1) classes. We must still consider separately $m_1 = m \pm 1$.

3.3.1 $m_1 = m - 1$

For $n = 7$, the (-1) classes with $m_1 = m - 1$ come from solutions to $x = m \pm \sqrt{m}$ (from equation 3.6). Clearly, m must be a perfect square. Set $m = z^2$ for $z \geq 1$, so that $x = z^2 \pm z$.

For $x = z^2 - z$, $y = 2z(z+1)$, and we get the class with

$$d = \frac{6z^2 + z}{2},$$

$$m_0 = \frac{2z^2 + 3z}{2},$$

$$m_1 = z^2 - 1,$$

and $m = z^2$. Of course, $n = 7$. From the equations for d and m_0 we see that z must be even. Make the substitution $z = 2a$ for $a \geq 1$, and these classes take the form

$$d = 12a^2 + a,$$

$$m_0 = 4a^2 + 3a,$$

$$m_1 = 4a^2 - 1,$$

$m = 4a^2$ and $n = 7$.

For $x = z^2 + z$, $y = 2z(z-1)$, and we get the class with

$$d = \frac{6z^2 - z}{2},$$

$$m_0 = \frac{2z^2 - 3z}{2},$$

$$m_1 = z^2 - 1,$$

$m = z^2$, and $n = 7$. Again z must be even, so substitute $z = 2a$ for $a \geq 1$ to rewrite this class as

$$d = 12a^2 - a.$$

$$m_0 = 4a^2 - 3a,$$

$$m_1 = 4a^2 - 1,$$

$m = 4a^2$ and $n = 7$.

These two sets of classes may be combined, giving the following proposition.

Proposition 3.3.1 *All (-1) classes with $n = 7$ and $m_1 = m - 1$ are of the form*

$$\mathcal{L}(12a^2 + a, 4a^2 + 3a, 4a^2 - 1, 7, 4a^2)$$

for some $a \neq 0$.

The irreducibility of these classes is discussed in section 3.3.3.

3.3.2 $m_1 = m + 1$

For $n = 7$ and $m \geq 1$, (-1) classes with $m_1 = m + 1$ come from solutions to

$$x = \frac{2m + 1 \pm \sqrt{4m + 1}}{2}$$

(from equation 3.10). We see that $4m + 1$ must be a perfect square. Set $m = \frac{a^2-1}{4}$. Then $x = \frac{a^2+1\pm 2a}{4}$. Clearly a must be odd. Let $a = 2z + 1$, and this simplifies to give $x = z^2$ or $x = (z + 1)^2$, and $m = z(z + 1)$.

Consider the case $x = (z+1)^2$. This gives $y = 2z^2$. The calculation $d = \frac{x+y+3m}{2}$ gives $d = \frac{6z^2+5z+1}{2}$. This requires that z be odd as well. Let $z = 2p+1$. and we are finally led to the class with

$$d = 12p^2 + 17p + 6.$$

$$m_0 = 4p^2 + 3p.$$

$$m_1 = 4p^2 + 6p + 3.$$

$$m = 4p^2 + 6p + 2.$$

and $n = 7$. for all $p \geq 0$.

The case $x = z^2$ works out similarly. We are led to the conclusion that z is even and make the substitution $z = 2p$. This gives the class with

$$d = 12p^2 + 7p + 1.$$

$$m_0 = 4p^2 + 5p + 1.$$

$$m_1 = 4p^2 + 2p + 1.$$

$$m = 4p^2 + 2p.$$

and $n = 7$. for all $p \geq 1$.

These two sets of classes can also be combined. Replace p by $-p-1$ in the second set of classes to get the following. Notice that the case $p = -1$ is the class $\mathcal{L}(1,1,1,7,0)$, a line through two points, so this does give a (-1) class as well (the (-1) class with $n = 7$ and $m = 0$).

Proposition 3.3.2 *All (-1) classes with $n = 7$ and $m_1 = m + 1$ are of the form*

$$\mathcal{L}(12p^2 + 17p + 6, 4p^2 + 3p, 4p^2 + 6p + 3, 7, 4p^2 + 6p + 2)$$

for some $p \in \mathbb{Z}$.

3.3.3 Irreducibility

Proposition 3.3.3 *All classes*

$$\mathcal{L}(12a^2 + a, 4a^2 + 3a, 4a^2 - 1, 7, 4a^2)$$

for $a \neq 0$, and

$$\mathcal{L}(12p^2 + 17p + 6, 4p^2 + 3p, 4p^2 + 6p + 3, 7, 4p^2 + 6p + 2)$$

for all $p \neq -1$ are irreducible.

Proof: To see this, it is enough to show that one class is irreducible, and that all others reduce to each other.

The class corresponding to $a = 1$, $\mathcal{L}(13, 7, 3, 7, 4)$, reduces to a line through two points by the following Cremona transformations (centered at the first three points in each case).

13:	7, 4, 4, 4, 4, 4, 4, 4, 3
11:	5, 4, 4, 4, 4, 4, 3, 2, 2
9:	4, 4, 4, 3, 3, 2, 2, 2, 2
6:	3, 3, 2, 2, 2, 2, 1, 1, 1
4:	2, 2, 2, 1, 1, 1, 1, 1
2:	1, 1, 1, 1, 1
1:	1, 1

The following transformations, centered at the underlined points, take the class corresponding to a to the class corresponding to $a + 1$.

$$
\begin{array}{rl}
12a^2 + a: & 4a^2 + 3a, 4a^2 - 1, \underline{\underline{(4a^2)}}^7 \\
12a^2 + 2a: & 4a^2 + 3a, 4a^2 - 1, (4a^2 + a)^3, \underline{\underline{(4a^2)}}^4 \\
12a^2 + 4a: & 4a^2 + 3a, \underline{4a^2 - 1}, \underline{(4a^2 + a)}^3, (4a^2 + 2a)^3, \underline{4a^2} \\
12a^2 + 7a + 1: & (4a^2 + 3a)^2, 4a^2 + 4a + 1, \underline{\underline{(4a^2 + a)}}^2, \\
 & \underline{(4a^2 + 2a)}^3, 4a^2 + 3a + 1 \\
12a^2 + 10a + 2: & (4a^2 + 3a)^2, (4a^2 + 4a + 1)^3, 4a^2 + 5a + 1, \\
 & \underline{\underline{(4a^2 + 2a)}}^2, \underline{4a^2 + 3a + 1} \\
12a^2 + 13a + 3: & \underline{\underline{(4a^2 + 3a)}}^2, \underline{(4a^2 + 4a + 1)}^3, (4a^2 + 5a + 1)^3, \\
 & 4a^2 + 6a + 2 \\
12a^2 + 16a + 5: & (4a^2 + 6a + 2)^3, 4a^2 + 7a + 3, \underline{\underline{(4a^2 + 4a + 1)}}^2, \\
 & \underline{(4a^2 + 5a + 1)}^3 \\
12a^2 + 19a + 7: & \underline{(4a^2 + 6a + 2)}^3, (4a^2 + 7a + 3)^3, 4a^2 + 8a + 3, \\
 & \underline{\underline{(4a^2 + 5a + 1)}}^2 \\
12a^2 + 22a + 10: & \underline{\underline{(4a^2 + 6a + 2)}}^2, \underline{(4a^2 + 7a + 3)}^3, 4a^2 + 8a + 3, \\
 & (4a^2 + 8a + 4)^2, 4a^2 + 9a + 5 \\
12a^2 + 25a + 13: & (4a^2 + 9a + 5)^3, 4a^2 + 10a + 6, \underline{\underline{(4a^2 + 7a + 3)}}^2, \\
 & 4a^2 + 8a + 3, \underline{(4a^2 + 8a + 4)}^2 \\
12a^2 + 28a + 16: & (4a^2 + 9a + 5)^3, \underline{\underline{(4a^2 + 10a + 6)}}^3, 4a^2 + 8a + 3, \\
 & 4a^2 + 11a + 7, 4a^2 + 8a + 4 \\
12a^2 + 26a + 14: & \underline{\underline{(4a^2 + 9a + 5)}}^3, 4a^2 + 8a + 3, 4a^2 + 11a + 7, \\
 & (4a^2 + 8a + 4)^4 \\
12a^2 + 25a + 13: & 4a^2 + 11a + 7, 4a^2 + 8a + 3, (4a^2 + 8a + 4)^7
\end{array}
$$

This last class may be rewritten as

$$\mathcal{L}(12(a+1)^2 + (a+1), 4(a+1)^2 + 3(a+1), 4(a+1)^2 - 1, 7, 4(a+1)^2),$$

the class corresponding to $a + 1$. Since Cremona transformations are their own inverses, reversing this reduction will take the class corresponding to a to the class corresponding to $a - 1$, as well. This proves that all classes of this form transform to each other, and are therefore all irreducible since the $a = 1$ case is irreducible.

The following transformations, centered at the underline points, take the class corresponding to p to the class corresponding to $p + 1$.

$12p^2 + 17p + 6$:	$4p^2 + 3p, \underline{4p^2 + 6p + 3}, \underline{\underline{(4p^2 + 6p + 2)^7}}$
$12p^2 + 16p + 5$:	$4p^2 + 3p, 4p^2 + 5p + 2, (4p^2 + 5p + 1)^2,$ $\underline{\underline{\underline{(4p^2 + 6p + 2)^5}}}$
$12p^2 + 14p + 4$:	$\underline{4p^2 + 3p}, 4p^2 + 5p + 2, (4p^2 + 5p + 1)^2,$ $\underline{\underline{(4p^2 + 4p + 1)^3}}, (4p^2 + 6p + 2)^2$
$12p^2 + 17p + 6$:	$(4p^2 + 6p + 2)^3, \underline{4p^2 + 5p + 2}, \underline{(4p^2 + 5p + 1)^2},$ $(4p^2 + 7p + 3)^2, \underline{4p^2 + 4p + 1}$
$12p^2 + 20p + 8$:	$\underline{\underline{(4p^2 + 6p + 2)^3}}, 4p^2 + 8p + 4, \underline{4p^2 + 5p + 1},$ $4p^2 + 8p + 3, (4p^2 + 7p + 3)^3$
$12p^2 + 23p + 11$:	$\underline{4p^2 + 6p + 2}, (4p^2 + 8p + 4)^2, 4p^2 + 8p + 3,$ $\underline{\underline{(4p^2 + 7p + 3)^3}}, (4p^2 + 9p + 5)^2$
$12p^2 + 26p + 14$:	$(4p^2 + 9p + 5)^3, \underline{\underline{(4p^2 + 8p + 4)^2}}, 4p^2 + 8p + 3,$ $\underline{4p^2 + 7p + 3}, (4p^2 + 10p + 6)^2$
$12p^2 + 29p + 17$:	$\underline{\underline{(4p^2 + 9p + 5)^3}}, (4p^2 + 11p + 7)^2, \underline{4p^2 + 8p + 3},$ $(4p^2 + 10p + 6)^3$
$12p^2 + 32p + 21$:	$\underline{4p^2 + 9p + 5}, (4p^2 + 11p + 7)^3, \underline{\underline{(4p^2 + 10p + 6)^3}},$ $(4p^2 + 12p + 9)^2$
$12p^2 + 35p + 25$:	$(4p^2 + 12p + 9)^3, \underline{\underline{(4p^2 + 11p + 7)^3}}, \underline{4p^2 + 10p + 6},$ $(4p^2 + 13p + 10)^2$
$12p^2 + 38p + 30$:	$\underline{\underline{\underline{(4p^2 + 12p + 9)^3}}}, (4p^2 + 14p + 12)^2, 4p^2 + 11p + 7$ $4p^2 + 13p + 11, (4p^2 + 13p + 10)^2$
$12p^2 + 40p + 33$:	$4p^2 + 11p + 7, \underline{4p^2 + 13p + 11}, \underline{\underline{(4p^2 + 13p + 10)^2}}$ $(4p^2 + 14p + 12)^5$
$12p^2 + 41p + 35$:	$4p^2 + 11p + 7, 4p^2 + 14p + 13, (4p^2 + 14p + 12)^7$

This last line may be written as

$$\mathcal{L}(12(p + 1)^2 + 17(p + 1) + 6, 4(p + 1)^2 + 3(p + 1), 4(p + 1)^2 + 6(p + 1) + 3,$$

$$7, 4(p+1)^2 + 6(p+1) + 2),$$

the class corresponding to $p+1$. This shows how all of these classes transform to each other.

The class corresponding to $p = 0$ is $\mathcal{L}(6,0,3,7,2)$. This class is shown to be irreducible by the following transformations (all centered at the first three points).

6:	3, 2, 2, 2, 2, 2, 2, 2
5:	2, 2, 2, 2, 2, 2, 1, 1
4:	2, 2, 2, 1, 1, 1, 1, 1
2:	1, 1, 1, 1, 1
1:	1, 1

Since the case $p = 0$ is irreducible, all of these classes are irreducible. ∎

These collections of irreducible (-1) classes sum to the compound (-1) configurations

$$\mathcal{L}(8(12a^2 + a), 8(4a^2 + 3a), 8, 8 \cdot 4a^2 - 1),$$

for $a \neq 0$, and the (-1) configurations

$$\mathcal{L}(8(12p^2 + 17p + 6), 8(4p^2 + 3p, 8, 8(4p^2 + 6p + 2) + 1)$$

for all p (compound if $p \neq -1$).

3.4 The Case $n = 8$

The classes with $n = 8$ are generally reducible and contain the irreducible quasi homogeneous class with the same total number of points, the class $\mathcal{L}(12, 8, 9, 3)$.

The exceptions occur only when $m = 1$ and $m_1 = m - 1$, and in the case with $m = 0$ and $m_1 = m + 1$. The method used to prove this conjecture can be extended to prove a similar conjecture for all classes with $n \geq 8$.

3.4.1 $m_1 = m - 1$

The classes with $m_1 = m - 1$ and $m = 1$ were discussed separately in Section 3.1. All other classes with $m_1 = m - 1$ come from factorizations $xy = 2m(m - 1)$ with $m \geq 2$. These classes will have the form

$$d = \frac{x + y + 3m}{2}$$

$$m_0 = \frac{y - x + m}{2}$$

$$m_1 = m - 1$$

$$n = \frac{2x + y}{m} + 3 = 8$$

As before, solve this last equation for y and solve $xy = 2m(m - 1)$ for x to get

$$x = \frac{5m \pm \sqrt{9m^2 + 16m}}{4}.$$

Lemma 3.4.1 *Factorizations with* $x = \frac{5m+\sqrt{9m^2+16m}}{4}$ *do not lead to (-1) classes. In particular they fail to satisfy condition d) of Proposition 3.1.1, that* $y + m \geq x$ *(necessary for* $m_0 \geq 0$*).*

Proof: If x has the above form, the corresponding y has the form

$$y = \frac{5m - \sqrt{9m^2 + 16m}}{2}.$$

In order for this to lead to a (-1) class we must have $x \leq y + m$. So we check

$$\frac{5m + \sqrt{9m^2 + 16m}}{4} \leq \frac{5m - \sqrt{9m^2 + 16m}}{2} + m.$$

This simplifies to

$$3\sqrt{9m^2 + 16m} \leq 9m.$$

For positive m this inequality is clearly false, and this completes the proof. ∎

Proceeding as in the above proof we check that classes with $x = \frac{5m - \sqrt{9m^2 - 16m}}{4}$ do satisfy the condition $x \leq y + m$. We will assume that x has this form, and that $y = \frac{5m + \sqrt{9m^2 + 16m}}{2}$. This allows us to write

$$d = \frac{27m + \sqrt{9m^2 + 16m}}{8}, \tag{3.11}$$

$$m_0 = \frac{9m + 3\sqrt{9m^2 + 16m}}{8}. \tag{3.12}$$

A (-1) class $\mathcal{L}(d, m_0, m_1, 8, m)$ will contain the curve $\mathcal{L}(12, 8, 9, 3)$ if the index of intersection is negative, that is, if $12d - 8m_0 - 3m_1 - 24m < 0$.

Proposition 3.4.1 *All (-1) classes with $m_1 = m - 1$, $m \geq 2$, and $n = 8$ contain the curve $\mathcal{L}(12, 8, 9, 3)$.*

Proof: Write $12d - 8m_0 - 3m_1 - 24m$ in terms of m, using the equations 3.11 and 3.12. After simplifying, we get

$$12d - 8m_0 - 3m_1 - 24m = \frac{9m + 6 - 3\sqrt{9m^2 + 16m}}{2}.$$

This expression is clearly negative for all $m \geq 2$. ∎

3.4.2 $m_1 = m + 1$

Classes with $m_1 = m + 1$ and $m \geq 1$ come from factorizations $xy = 2m^2$. Such a factorization leads to the class with

$$d = \frac{x + y + 3m}{2}$$

$$m_0 = \frac{y - x + m}{2}$$

$$m_1 = m + 1$$

$$n = \frac{2x + y - 2}{m} + 3 = 8$$

This last equation. again forces x to have the form $x = \frac{5m+2\pm\sqrt{9m^2+20m+4}}{4}$.

Lemma 3.4.2 *Factorizations with* $x = \frac{5m+2+\sqrt{9m^2+20m+4}}{4}$ *do not lead to (-1) classes. In particular, they fail to satisfy* $x \leq y + m$.

Proof: If x has the above form then $y = \frac{5m+2-\sqrt{9m^2+20m+4}}{2}$. Check that

$$\frac{5m + 2 + \sqrt{9m^2 + 20m + 4}}{4} \leq \frac{5m + 2 - \sqrt{9m^2 + 20m + 4}}{2} + m.$$

This simplifies to

$$3\sqrt{9m^2 + 20m + 4} \leq 9m + 2,$$

and this is false for positive m. ∎

We may assume, therefore, that $x = \frac{5m+2-\sqrt{9m^2+20m+4}}{4}$. For this x, we have that

$y = \frac{5m+2+\sqrt{9m^2+20m+4}}{2}$. This allows us to write

$$d = \frac{27m + 6 + \sqrt{9m^2 + 20m + 4}}{8}. \tag{3.13}$$

$$m_0 = \frac{9m + 2 + 3\sqrt{9m^2 + 20m + 4}}{8}. \tag{3.14}$$

Proposition 3.4.2 *All (-1) classes with* $m_1 = m + 1$, $m \geq 1$, *and* $n = 8$ *contain the curve* $\mathcal{L}(12, 8, 9, 3)$.

Proof: We check that for $\mathcal{L}(d, m_0, m_1, 8, m)$, the index of intersection is negative. Substituting equations 3.13 and 3.14, we see that the index of intersection is

$$12d - 8m_0 - 3m_1 - 24m = \frac{9m + 8 - 3\sqrt{9m^2 + 20m + 4}}{2}.$$

This is negative for all $m \geq 1$. ∎

3.5 The Case n Even and Larger than Eight

We attempt to use the method of the preceding section to prove that in general the classes $\mathcal{L}(d, m_0, m_1, n, m)$ with n even and larger than eight contain the irreducible quasi homogeneous (-1) class with the same total number of points. When n is even, $n + 1$ is odd, and the irreducible quasi homogeneous (-1) class is

$$\mathcal{L}(\frac{n(n-2)}{4}, \frac{n(n-4)}{4}, n + 1, \frac{n-2}{2}).$$

The method consists of writing d, m_0, and m_1 in terms of m and n, then showing that the index of intersection

$$\frac{n(n-2)}{4}d - \frac{n(n-4)}{4}m_0 - \frac{n-2}{2}m_1 - n\frac{n-2}{2}m$$

is negative.

3.5.1 $m_1 = m - 1$

We analyzed separately the (-1) classes with $m_1 = m - 1$ and $m = 1$. The (-1) classes with $m_1 = m - 1$ and $m \geq 2$ come from factorizations $xy = 2m(m-1)$. For a given n, x must satisfy

$$x = \frac{(n-3)m \pm \sqrt{(n-7)(n+1)m^2 + 16m}}{4}.$$

Lemma 3.5.1 *Factorizations with* $x = \frac{(n-3)m+\sqrt{(n-7)(n+1)m^2+16m}}{4}$ *do not lead to (-1) classes. In particular they do not satisfy* $m_0 \geq 0$.

Proof: If x has the above form, then $y = \frac{(n-3)m-\sqrt{(n-7)(n+1)m^2+16m}}{2}$. In order to lead to a (-1) class (with $m_0 \geq 0$), we must have that $x \leq y + m$. Substitute the expressions for x and y and simplify. We see that $x \leq y + m$ if and only if

$$3\sqrt{(n-7)(n+1)m^2 + 16m} \leq (n+1)m.$$

Both side of this inequality are positive, so we may square both sides and simplify again to see that this inequality hold if and only if

$$8(n-8)(n+1)m^2 + 16m \leq 0.$$

For $m \geq 2$ and $n \geq 8$, this inequality is false. Therefore, x with the above form does not lead to a (-1) class. ∎

Proposition 3.5.1 *All (-1) classes $\mathcal{L}(d, m_0, m_1, n, m)$, with n even, $n \geq 8$, $m_1 = m - 1$, and $m \geq 2$ are reducible and contain the irreducible quasi homogeneous class $\mathcal{L}(\frac{n(n-2)}{4}, \frac{n(n-4)}{4}, n+1, \frac{n-2}{2})$.*

Proof: We may assume that $x = \frac{(n-3)m - \sqrt{(n-7)(n+1)m^2+16m}}{4}$. For this x, we have that $y = \frac{(n-3)m+\sqrt{(n-7)(n+1)m^2+16m}}{2}$. Use these expressions to write

$$d = \frac{3(n+1)m + \sqrt{(n-7)(n+1)m^2+16m}}{8}. \tag{3.15}$$

$$m_0 = \frac{(n+1)m + 3\sqrt{(n-7)(n+1)m^2+16m}}{8}. \tag{3.16}$$

Now check that the index of intersection is negative.

$$\frac{n(n-2)}{4}d - \frac{n(n-4)}{4}m_0 - \frac{n-2}{2}m_1 - n\frac{n-2}{2}m$$

simplifies to

$$\frac{(n+1)(n^2-9n+16)m + 8(n-2) + n(5-n)\sqrt{(n-7)(n+1)m^2+16m}}{16}.$$

This will be negative if

$$(n+1)(n^2-9n+16)m + 8(n-2) < n(n-5)\sqrt{(n-7)(n+1)m^2+16m}.$$

Both sides of this inequality are positive for $n \geq 8$ and $m \geq 2$. Square both sides and simplify. The index of intersection will be negative if and only if

$$-32(n+1)(n-8)m^2 - 32(n^2-n+16)m + 64(n-2)^2 < 0.$$

For a given value of $n \geq 8$, this expression is maximized when $m = 2$. If this expression is negative when $m = 2$, it will be negative for all $m \geq 2$. When $m = 2$ this inequality reduces to

$$-64(2n^2 - 11n - 4) < 0.$$

This is true for all $n > \frac{11+\sqrt{153}}{4}$, so certainly true for all $n \geq 8$. The index of intersection of $\mathcal{L}(d, m_0, m_1, n, m)$, with n even and $n \geq 8$ and $m_1 = m - 1$, with $\mathcal{L}(\frac{n(n-2)}{4}, \frac{n(n-4)}{4}, n+1, \frac{n-2}{2})$ is negative. ∎

3.5.2 $m_1 = m + 1$

The (-1) classes with $m_1 = m + 1$ and $m = 0$ were taken care of earlier. The (-1) classes with $m_1 = m + 1$ and $m \geq 1$ come from factorizations $xy = 2m^2$. For a given n, x must satisfy

$$x = \frac{(n-3)m + 2 \pm \sqrt{(n-7)(n+1)m^2 + 4(n-3)m + 4}}{4}.$$

Lemma 3.5.2 *Factorizations with* $x = \frac{(n-3)m+2+\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}}{4}$ *do not lead to (-1) classes. In particular they do not satisfy* $m_0 \geq 0$.

Proof: If x has the above form, then $y = \frac{(n-3)m+2+\sqrt{(n-7)(n+1)m^2+16m}}{2}$. In order to lead to a (-1) class (with $m_0 \geq 0$), we must have that $x \leq y + m$. Substitute the expressions for x and y and simplify. We see that $x \leq y + m$ if and only if

$$3\sqrt{(n-7)(n+1)m^2 + 4(n-3)m + 4} \leq (n+1)m + 2.$$

Both sides of this inequality are positive. Square both sides and simplify to see that this inequality is true if and only if

$$8(n+1)(n-8)+16(2n-7)m+32 \leq 0.$$

This inequality does not hold for any $n \geq 8$ and $m \geq 1$. ∎

Proposition 3.5.2 *All (-1) classes $\mathcal{L}(d, m_0, m_1, n, m)$, with even $n \geq 8$, $m_1 = m+1$, and $m \geq 1$ are reducible and contain the irreducible quasi homogeneous class $\mathcal{L}(\frac{n(n-2)}{4}, \frac{n(n-4)}{4}, n+1, \frac{n-2}{2})$.*

Proof: We may assume that $x = \frac{(n-3)m+2-\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}}{4}$. For this x, we have that $y = \frac{(n-3)m+2+\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}}{2}$. Use these expressions to write

$$d = \frac{3(n+1)m+6+\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}}{8}. \tag{3.17}$$

$$m_0 = \frac{(n+1)m+2+3\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}}{8}. \tag{3.18}$$

Now check that the index of intersection is negative.

$$\frac{n(n-2)}{4}d - \frac{n(n-4)}{4}m_0 - \frac{n-2}{2}m_1 - n\frac{n-2}{2}m$$

simplifies to

$$\frac{(n+1)(n^2-9n+16)m+2(n^2-5n+8)}{16}$$
$$+\frac{n(5-n)\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}}{16}.$$

This will be negative if

$$(n+1)(n^2-9n+16)m + 2(n^2-5n+8)$$

$$< n(n-5)\sqrt{(n-7)(n+1)m^2+4(n-3)m+4}.$$

Both sides of this inequality are positive, so we may square both sides and simplify. The index of intersection will be negative if and only if

$$-32(n+1)(n-8)m^2 - 32(n^2+3n-16)m + 64(n-1)(n-4) < 0.$$

It is clear that if this inequality holds for all $n \geq 8$ and $m = 1$, then it will be true for all $n \geq 8$ and all $m \geq 1$. When $m = 1$, this inequality simplifies to

$$-64(3n-16) < 0.$$

This is clearly true for all $n \geq 8$. Therefore, the index of intersection of the class $\mathcal{L}(d, m_0, m_1, n, m)$, with even $n \geq 8$ and $m_1 = m + 1$, with the quasi homogeneous (-1) curve of the form $\mathcal{L}(\frac{n(n-2)}{4}, \frac{n(n-4)}{4}, n+1, \frac{n-2}{2})$ is negative.

∎

3.6 The Case n Odd and Larger than Eight

Using the Lemmas 3.5.1 and 3.5.2, we can prove that almost every class with n odd and larger than eight, $\mathcal{L}(d, m_0, m_1; n, m)$, contains the irreducible quasi homogeneous (-1) class with the same total number of points. When n is odd, $n+1$ is even. and the irreducible quasi homogeneous (-1) class is

$$\mathcal{L}(\frac{(n+1)}{2}, \frac{(n-1)}{2}, n+1, 1).$$

3.6.1 $m_1 = m - 1$

Proposition 3.6.1 *All (-1) classes $\mathcal{L}(d, m_0, m_1, n, m)$, with n odd, $n \geq 8$, $m_1 = m - 1$, and $m \geq 2$ are reducible and contain the irreducible quasi homogeneous class $\mathcal{L}(\frac{(n+1)}{2}, \frac{(n-1)}{2}, n+1, 1)$.*

Proof: Use the expressions 3.15 and 3.16 to write d and m_0 in terms of m and n. Write the index of intersection

$$\frac{(n+1)}{2}d - \frac{(n-1)}{2}m_0 - m_1 - nm.$$

This simplifies to

$$\frac{(n-6)(n+1)m + 8 + (2-n)\sqrt{(n-7)(n+1)m^2 + 16m}}{8}.$$

This will be negative if and only if

$$(n-6)(n+1)m + 8 < (n-2)\sqrt{(n-7)(n+1)m^2 + 16m}.$$

Both sides of this inequality are positive for $n \geq 8$ and $m \geq 2$. Square both sides and simplify. This inequality will hold if and only if

$$-8(n+1)(n-8)m^2 - 16(n+10)m + 64 < 0.$$

It is enough to check this is true when $m = 1$. When $m = 1$, this inequality reduces to

$$-8(n-1)(n-4) < 0.$$

This is true for all $n \geq 8$, therefore the index of intersection will be negative for all $n \geq 8$ and $m \geq 2$. ∎

3.6.2 $m_1 = m + 1$

Proposition 3.6.2 *All (-1) classes $\mathcal{L}(d, m_0, m_1, n, m)$, with odd $n \geq 8$, $m_1 = m+1$ and with $m \geq 1$ are reducible and contain the irreducible quasi homogeneous class $\mathcal{L}(\frac{(n+1)}{2}, \frac{(n-1)}{2}, n+1, 1)$.*

Proof: Use the expressions 3.17 and 3.18 to write d and m_0 in terms of m and n. Write the index of intersection

$$\frac{(n+1)}{2}d - \frac{(n-1)}{2}m_0 - m_1 - nm.$$

This simplifies to

$$\frac{(n-6)(n+1)m + 2(n-2) + (2-n)\sqrt{(n-7)(n+1)m^2 + 4(n-3)m + 4}}{8}.$$

This will be negative if and only if

$$(n-6)(n+1)m + 2(n-2) < (n-2)\sqrt{(n-7)(n+1)m^2 + 4(n-3)m + 4}.$$

Both sides of this inequality are positive. Square both sides and simplify to see that this inequality is true if and only if

$$-8(n+1)(n-8)m^2 - 48(n-2)m < 0.$$

This inequality holds for $n > 8$ and $m \geq 1$, therefore the index of intersection is negative. ∎

3.7 Conclusion

Combining the results of this chapter gives the following list of compound (-1) configurations.

Theorem 3.7.1 *All (-1) configurations constructed from irreducible quasi-quasi-homogeneous (-1) classes are on the following list.*

a) The system $\mathcal{L}(12, 0, 6, 5)$

b) The system $\mathcal{L}(18, 6, 6, 7)$

c) The system $\mathcal{L}(21, 0, 7, 8)$

d) The system $\mathcal{L}(35, 7, 7, 13)$

e) The system $\mathcal{L}(42, 14, 7, 15)$

f) Systems $\mathcal{L}(2e^2 + e, 2e^2 - e - 1, 2e + 1, 2e)$, *for* $e \geq 1$

g) Systems $\mathcal{L}(e, e, e, 1)$, *for* $e \geq 1$

h) Systems $\mathcal{L}(8(12a^2 + a), 8(4a^2 + 3a), 8, 8(4a^2) - 1)$

for $a \neq 0$

i) Systems $\mathcal{L}(8(12p^2 + 17p + 6), 8(4p^2 + 3p), 8, 8(4p^2 + 6p + 2) + 1)$

for $p \neq -1$.

These (-1) configurations are all compound except case g) when $e = 1$.

Chapter 4

Special Systems With $m = 4$

A linear system $\mathcal{L}$ is *(-1) special* if there are (-1) curves $A_1, \ldots, A_r$ such that $\mathcal{L} \cdot A_j = -N_j$ with $N_j \geq 1$ for every j and $N_j \geq 2$ for some j, with the residual system $\mathcal{M} = \mathcal{L} - \sum_j N_j A_j$ having non-negative virtual dimension and non-negative intersection with every (-1) curve A_j. The (-1) curves A_j must be pairwise disjoint.

It is known that every (-1) special system is special. The main conjecture is that every special system is (-1) special. We begin by classifying the (-1) special linear systems.

4.1 (-1) Special Systems

Suppose that $\mathcal{L}(d, m_0, n, 4)$ is a (-1) special linear system. Then $\mathcal{L}$ must be one of two possible forms. The first possibility is that

$$\mathcal{L} = \mathcal{M} + N \cdot C,$$

where $v(\mathcal{M}) \geq 0$ and $\mathcal{M} \cdot C = 0$. In this case N is 2, 3, or 4, and the (-1) curve or configuration $C = \mathcal{L}(\delta, \mu_0, n, 1)$, or N is 2 and $C = \mathcal{L}(\delta, \mu_0, n, 2)$. The second possibility is that

$$\mathcal{L} = 2C_1 + 2C_2.$$

In this case C_1 and C_2 must be disjoint (-1) curves or configurations.

The following is a complete list of (-1) curves and configurations with $m \leq 2$.

$\mathcal{L}(2, 0, 5, 1)$		
$\mathcal{L}(e, e-1, 2e, 1)$	$e \geq 1$	
$\mathcal{L}(e, e, e, 1)$	$e \geq 1$	compound for $e > 2$
$\mathcal{L}(6, 3, 7, 2)$		
$\mathcal{L}(3, 0, 3, 2)$		compound

Note that the only disjoint curves on this list are $\mathcal{L}(e, e-1, 2e, 1)$ and $\mathcal{L}(2e, 2e, 2e, 1)$.

We may now classify (-1) special curves with $m = 4$.

Theorem 4.1.1 *The quasi homogeneous (-1) special systems with $m = 4$ are those systems $\mathcal{L}(d, m_0, n, 4)$ on the following list:*

$d - m_0$		
0	$\mathcal{L}(d, d, e, 4)$	$d \geq 4e \geq 4$
1	$\mathcal{L}(d, d-1, e, 4)$	$d \geq \frac{7}{5}e \geq 1$
2	$\mathcal{L}(d, d-2, e, 4)$	$d \geq \frac{9e+1}{3} \geq \frac{10}{3}$
2	$\mathcal{L}(6e, 6e-2, 2e, 4)$	$e \geq 1$
3	$\mathcal{L}(5e, 5e-3, 2e, 4)$	$e \geq 1$
3	$\mathcal{L}(5e+1, 5e-2, 2e, 4)$	$e \geq 1$
4	$\mathcal{L}(4e, 4e-4, 2e, 4)$	$e \geq 1$
4	$\mathcal{L}(4e+1, 4e-3, 2e, 4)$	$e \geq 1$
4	$\mathcal{L}(4e+2, 4e-2, 2e, 4)$	$e \geq 1$
5	$\mathcal{L}(5, 0, 2, 4)$	
5	$\mathcal{L}(6, 1, 2, 4)$	
5	$\mathcal{L}(8, 3, 4, 4)$	
5	$\mathcal{L}(9, 4, 4, 4)$	
5	$\mathcal{L}(12, 7, 6, 4)$	
5	$\mathcal{L}(15, 10, 8, 4)$	
6	$\mathcal{L}(6, 0, 2, 4)$	
6	$\mathcal{L}(6, 0, 3, 4)$	
6	$\mathcal{L}(8, 2, 4, 4)$	
6	$\mathcal{L}(12, 6, 7, 4)$	
7	$\mathcal{L}(9, 2, 5, 4)$	
8	$\mathcal{L}(8, 0, 5, 4)$	
8	$\mathcal{L}(9, 1, 5, 4)$	
9	$\mathcal{L}(9, 0, 5, 4)$	

Proof: Suppose that $\mathcal{L} = \mathcal{M} + NC$ with $N = 2, 3,$ or 4. We begin with $N = 4$.

Suppose that $\mathcal{L}(2, 0, 5, 1)$ splits off 4 times from $\mathcal{L}(d, m_0, 5, 4)$. Then the residual system $\mathcal{M} = \mathcal{L}(d-8, m_0, 5, 0)$. $\mathcal{M}$ is disjoint from $\mathcal{L}(2, 0, 5, 1)$ only if $d = 8$, and so $v(\mathcal{M}) = 0$ implies that $m_0 = 0$. Thus we have the class $4 \cdot \mathcal{L}(2, 0, 5, 1) = \mathcal{L}(8, 0, 5, 4)$ with virtual dimension -6, but dimension 0.

Suppose that $A = \mathcal{L}(e, e-1, 2e, 1)$ splits off 4 times from $\mathcal{L}(d, m_0, 2e, 4)$. Then $\mathcal{M} = \mathcal{L}(d - 4e, m_0 - 4e + 4, 2e, 0)$. We require that

$$\mathcal{M} \cdot A = e(d - m_0 - 8) + m_0 + 4 = 0$$

so that $m_0 > d-8$. Then $v(\mathcal{M}) \geq 0$ implies that $m_0 \leq d-4$. There are four possibilities for m_0.

$m_0 = d-7$: $\mathcal{M} \cdot A = e(d-m_0-8) + m_0 + 4 = 0$ implies that $e = d-3$. Then $v(\mathcal{M}) = -12d + 39 \geq 0$ implies that $d \leq 3$. This, however, makes $m_0 < 0$.

$m_0 = d-6$: $\mathcal{M} \cdot A = e(d-m_0-8) + m_0 + 4 = 0$ implies that $d = 2e+2$. But $d \geq 4e$ and $e \geq 1$, so $e = 1$ and $d = 4$. Again this makes $m_0 < 0$.

$m_0 = d-5$: $\mathcal{M} \cdot A = e(d-m_0-8) + m_0 + 4 = 0$ implies that $d = 3e+1$. But $d \geq 4e$ and $e \geq 1$, so $e = 1$ and $d = 4$. This makes $m_0 = -1$.

$m_0 = d-4$: $\mathcal{M} \cdot A = e(d-m_0-8) + m_0 + 4 = 0$ implies that $d = 4e$. This gives the system $\mathcal{L}(4e, 4e-4, 2e, 4)$, for all $e \geq 1$.

Suppose that $A = \mathcal{L}(e, e, e, 1)$ splits off 4 times from $\mathcal{L}(d, m_0, e, 4)$. Then $\mathcal{M} = \mathcal{L}(d-4e, m_0-4e, 2e, 0)$. We require that

$$\mathcal{M} \cdot A = e(d-4e) + e(m_0-4e) = 0,$$

so that $d = m_0$. Now $v(\mathcal{M}) \geq 0$ whenever $d \geq 4e$. This gives the system $\mathcal{L}(d, d, e, 4)$ whenever $d \geq 4e \geq 4$.

This completes the $N = 4$ analysis.

Suppose that $A = \mathcal{L}(2, 0, 5, 1)$ splits off 3 times from $\mathcal{L}(d, m_0, 5, 4)$. Then $\mathcal{M} = \mathcal{L}(d-6, m_0, 5, 1)$, and $\mathcal{M} \cdot A = 2d - 17$. This can never be zero, so this case cannot occur.

Suppose that $A = \mathcal{L}(e, e-1, 2e, 1)$ splits off 3 times from $\mathcal{L}(d, m_0, 2e, 4)$. Then

$\mathcal{M} = \mathcal{L}(d - 3e, m_0 - 3e + 3, 2e, 1)$. We require that

$$\mathcal{M} \cdot A = e(d - m_0 - 8) + m_0 + 3 = 0.$$

This forces $m_0 > d - 8$. Then $v(\mathcal{M}) \geq 0$ requires $m_0 \leq d - 3$, and there are five possibilities for m_0.

$m_0 = d - 7$: $\mathcal{M} \cdot A = d - e - 4 = 0$ implies that $m_0 = e - 3$, but $m_0 \geq 3e - 3$. This case cannot occur.

$m_0 = d - 6$: $\mathcal{M} \cdot A = d - 2e - 3 = 0$ implies that $m_0 = 2e - 3$, but $m_0 \geq 3e - 3$. This case cannot occur.

$m_0 = d - 5$: $\mathcal{M} \cdot A = d - 3e - 2 = 0$ implies that $d = 3e + 2$ and $m_0 = 3e - 3$. In this case $\mathcal{M} = \mathcal{L}(2, 0, 2e, 1)$ and $v(\mathcal{M}) \geq 0$ forces e to be 1 or 2. These are the systems $\mathcal{L}(5, 0, 2, 4)$ and $\mathcal{L}(8, 3, 4, 4)$.

$m_0 = d - 4$: $\mathcal{M} \cdot A = d - 4e - 1 = 0$, so $d = 4e + 1$ and $m_0 = 4e - 3$. Now $\mathcal{M} = \mathcal{L}(e + 1, e, 2e, 1)$ has virtual dimension 2 for all e. Therefore we have the (-1) special system $\mathcal{L}(4e + 1, 4e - 3, 2e, 4)$ for all $e \geq 1$.

$m_0 = d - 3$: $\mathcal{M} \cdot A = d - 5e = 0$, so $d = 5e$ and $m_0 = 5e - 3$. Now $\mathcal{M} = \mathcal{L}(2e, 2e, 2e, 1)$ has virtual dimension 0 for all e. This gives the system $\mathcal{L}(5e, 5e - 3, 2e, 4)$ for all $e \geq 1$.

Suppose that $A = \mathcal{L}(e, e, e, 1)$ splits off 3 times from $\mathcal{L}(d, m_0, e, 4)$. Then $\mathcal{M} = \mathcal{L}(d - 3e, m_0 - 3e, 2e, 1)$. We require that

$$\mathcal{M} \cdot A = e(d - m_0 - 1) = 0,$$

so that $m_0 = d - 1$. Now $v(\mathcal{M}) = 2d - 7e \geq 0$ if $d \geq \frac{7}{2}e$. This gives the systems $\mathcal{L}(d, d-1, e, 4)$ for every $d \geq \frac{7}{2}e \geq \frac{7}{2}$.

This completes the $N = 3$ analysis.

Suppose that $A = \mathcal{L}(2, 0, 5, 1)$ splits off twice from $\mathcal{L}(d, m_0, 5, 4)$. Then $\mathcal{M} = \mathcal{L}(d-4, m_0, 5, 2)$, and $\mathcal{M} \cdot A = 2d - 18$, so $d = 9$. $v(\mathcal{M}) \geq 0$ implies that $0 \leq M - 0 \leq 2$. This gives the systems $\mathcal{L}(9, 0, 5, 4)$, $\mathcal{L}(9, 1, 5, 4)$, and $\mathcal{L}(9, 2, 5, 4)$.

Suppose that $A = \mathcal{L}(e, e-1, 2e, 1)$ splits off twice from $\mathcal{L}(d, m_0, 2e, 4)$. Then $\mathcal{M} = \mathcal{L}(d-2e, m_0 - 2e + 2, 2e, 2)$ and $\mathcal{M} \cdot A = e(d - m_0 - 8) + m_0 + 2$. This forces $m_0 > d - 8$. We also have $m_0 \leq d - 2$ so that $v(\mathcal{M}) \geq 0$. There are six possibilities for m_0.

$m_0 = d - 7$: $\mathcal{M} \cdot A = d - e - 5 = 0$, so $d = e + 5$ and $m_0 = e - 2$. But $m_0 \geq 2e - 2$, so this case cannot occur.

$m_0 = d - 6$: $\mathcal{M} \cdot A = d - 2e - 4 = 0$, so $d = 2e + 4$ and $m_0 = 2e - 2$. Now $v(\mathcal{M}) = 14 - 6e \geq 0$ implies that e is 1 or 2. This leads to the systems $\mathcal{L}(6, 0, 2, 4)$ and $\mathcal{L}(8, 2, 4, 4)$.

$m_0 = d - 5$: $\mathcal{M} \cdot A = d - 3e - 3 = 0$, so $d = 3e + 3$ and $m_0 = 3e - 2$. Then $v(\mathcal{M}) = 9 - 2e \geq 0$ implies that e is 1, 2, 3, or 4. These systems are $\mathcal{L}(6, 1, 2, 4)$, $\mathcal{L}(9, 4, 4, 4)$, $\mathcal{L}(12, 7, 6, 4)$, and $\mathcal{L}(15, 10, 8, 4)$.

$m_0 = d - 4$: $\mathcal{M} \cdot A = d - 4e - 2 = 0$, so $d = 4e + 2$ and $m_0 = 4e - 2$. In this case $v(\mathcal{M}) = 5$ for all e, giving the systems $\mathcal{L}(4e + 2, 4e - 2, 2e, 4)$ for all $e \geq 1$.

$m_0 = d - 3$: $\mathcal{M} \cdot A = d - 5e - 1 = 0$, so $d = 5e + 1$ and $m_0 = 5e - 2$. This time

$v(\mathcal{M}) = 2$ for all e, giving the systems $\mathcal{L}(5e+1, 5e-2, 2e, 4)$ for all $e \geq 1$.

$m_0 = d-2$: $\mathcal{M} \cdot A = d - 6e = 0$, so $d = 6e$ and $m_0 = 6e-2$, but $v(\mathcal{M}) = -2$ for all e. This means that $\mathcal{L}(e, e-1, 2e, 1)$ cannot split off by itself in this case. We will see later that this case does occur when $\mathcal{L}(e, e-1, 2e, 1)$ and $\mathcal{L}(2e, 2e, 2e, 1)$ both split.

Suppose that $A = \mathcal{L}(e, e, e, 1)$ splits off twice from $\mathcal{L}(d, m_0, e, 4)$. Then $\mathcal{M} = \mathcal{L}(d-2e, m_0-2e, e, 2)$ and $\mathcal{M} \cdot A = e(d - m_0 - 2)$. This forces $m_0 = d-2$. Now $v(\mathcal{M}) = 3d - 9e - 1 \geq 0$ occurs whenever $d \geq \frac{9e+1}{3}$. This gives a (-1) special system $\mathcal{L}(d, d-2, e, 4)$ whenever $d \geq \frac{9e+1}{3} \geq \frac{10}{3}$.

Suppose that $A = \mathcal{L}(6, 3, 7, 2)$ splits off twice from $\mathcal{L}(d, m_0, 7, 4)$. Then $\mathcal{M} \cdot A = 6d - 3m_0 - 54$ and $m_0 = 2d - 18$. Now $v(\mathcal{M}) \geq 0$ only if $d \leq 12$, but $d \geq 12$ since A splits off twice. Therefore $d = 12$ and the system is $\mathcal{L}(12, 6, 7, 4) = 2A$

Suppose that $A = \mathcal{L}(3, 0, 3, 2)$ splits off twice from $\mathcal{L}(d, m_0, 3, 4)$. Then $\mathcal{M} \cdot A = 3(d-6) = 0$ implies that $d = 6$, and $v(\mathcal{M}) \geq 0$ implies that $m_0 = 0$, giving the system $\mathcal{L}(6, 0, 3, 4) = 2A$.

This completes the analysis of the case where one curve splits off twice. The final possibility is that $\mathcal{L} = \mathcal{M} + 2N_1 + 2N_2$, where N_1 and N_2 are disjoint (-1) curves or configurations. As mentioned previously, $\mathcal{L}(e, e-1, 2e, 1)$ and $\mathcal{L}(2e, 2e, 2e, 1)$ are the only possibilities for N_1 and N_2. In this situation, the residual system is $\mathcal{M} = \mathcal{L}(d-6e, m_0 - 6e + 2, 2e, 0)$. N_1 and N_2 intersect $\mathcal{M}$ as

$$\mathcal{M} \cdot \mathcal{L}(2e, 2e, 2e, 1) = 2e(d - m_0) - 4e = 0.$$

$$\mathcal{M} \cdot \mathcal{L}(e, e-1, 2e, 1) = e(d - m_0 - 8) + m_0 + 2 = 0.$$

From the first equation we get that $m_0 = d - 2$. From the second we then have $d = 6e$. Now $v(\mathcal{M}) \geq 0$ implies that $m_0 = 6e - 2$. This leads to the (-1) special systems $2N_1 + 2N_2 = \mathcal{L}(6e, 6e-2, 2e, 4)$ for every $e \geq 1$.

∎

4.2 Large m_0

The cases with $m_0 \geq d - 5$ are dealt with by the following lemmas from [CM1] (modified to the case $m = 4$).

Lemma 4.2.1 *Let $\mathcal{L} = \mathcal{L}(d, d-4, n, 4)$, with $d \geq 4$. Write $d = 4q + \mu$ with $0 \leq \mu \leq 3$, and $n = 2h + \epsilon$, with $\epsilon \in \{0, 1\}$. Then the system $\mathcal{L}$ is special if and only if $q = h$, $\epsilon = 0$ and $\mu \leq 2$.*

In particular, $\mathcal{L}(d, d-4, n, 4)$ is special if and only if it is one of the following types:

$$\mu = 0 \qquad \mathcal{L}(4q, 4q-4, 2q, 4)$$

$$\mu = 1 \qquad \mathcal{L}(4q+1, 4q-3, 2q, 4)$$

$$\mu = 1 \qquad \mathcal{L}(4q+2, 4q-2, 2q, 4)$$

These agree with the list in theorem 4.1.1.

Lemma 4.2.2 *Let* $\mathcal{L} = \mathcal{L}(d, d-4+k, n, 4)$ *with* $k \geq 1$, *and let*

$$\mathcal{L}' = \mathcal{L}(d-kn, d-kn-4+k, n, 4-k).$$

Then $\dim \mathcal{L} = \dim \mathcal{L}'$ *and* $\mathcal{L}$ *is non-special unless either*

(a) $k \geq 2$ *and* $\mathcal{L}'$ *is nonempty and non-special, or*

(b) $\mathcal{L}'$ *is special.*

Again, we compare these results with the list in Theorem 4.1.1. If $k = 1$, $\mathcal{L} = \mathcal{L}(d, d-3, n, 4)$ and $\mathcal{L}' = \mathcal{L}(d-n, d-n-3, n, 3)$, a system with $m = 3$. $\mathcal{L}$ is special if and only if $\mathcal{L}'$ is special. The special quasi homogeneous systems with $m \leq 3$ are classified in [CM1]. $\mathcal{L}'$ is special if and only if it is of the form $\mathcal{L}(3e, 3e-3, 2e, 3)$, or $\mathcal{L}(3e+1, 3e-2, 2e, 3)$. These lead to systems $\mathcal{L}(5e, 5e-3, 2e, 4)$ and $\mathcal{L}(5e+1, 5e-2, 2e, 4)$. These appear on the list in Theorem 4.1.1, and they are the only classes on the list with $d - m_0 = 3$.

If $k = 2$, $\mathcal{L} = \mathcal{L}(d, d-2, n, 4)$ and $\mathcal{L}' = \mathcal{L}(d-2n, d-2n-2, n, 2)$. $\mathcal{L}'$ is nonempty and non-special if $d \geq \frac{9n+1}{3}$. $\mathcal{L}'$ is special if and only if it has the form $\mathcal{L}(2e, 2e-2, 2e, 2)$. This gives $\mathcal{L} = \mathcal{L}(6e, 6e-2, 2e, 4)$. These agree with the list in Theorem 4.1.1.

If $k = 3$, $\mathcal{L} = \mathcal{L}(d, d-1, n, 4)$ and $\mathcal{L}' = \mathcal{L}(d-3n, d-3n-1, n, 1)$. $\mathcal{L}'$ is non-special since $m = 1$ and nonempty if $d \geq \frac{7n}{2}$. If $k = 4$, $\mathcal{L} = \mathcal{L}(d, d, n, 4)$ and $\mathcal{L}' = \mathcal{L}(d-4n, d-4n, n, 0)$. $\mathcal{L}'$ is non-special since $m = 0$ and nonempty if $d \geq 4n$. These conditions match those on the list in Theorem 4.1.1.

Lemma 4.2.3 *Let $\mathcal{L} = \mathcal{L}(d, d-5, n, 4)$ with $d \geq 5$. Write $d = 3q + \mu$ with $0 \leq \mu \leq 2$, and $n = 2h + \epsilon$ with $\epsilon \in \{0, 1\}$. Then the system $\mathcal{L}$ is non-special unless*

(a) $q = h + 1$, $\mu = \epsilon = 0$ and $h \leq 4$, or

(b) $q = h$, $\epsilon = 0$, and $4q \leq \mu(\mu + 3)$.

Case (a) gives the systems $\mathcal{L}(3h+3, 3h-2, 2h, 4)$ where $1 \leq h \leq 4$. These appear on the list in Theorem 4.1.1. In case (b), $\mu = 0$ and $\mu = 1$ both force $d \leq 5$, so we are left with $\mu = 2$ and $q = 1$ or 2. These are the systems $\mathcal{L}(5, 0, 2, 4)$ and $\mathcal{L}(8, 3, 4, 4)$. These account for all the systems with $d - m_0 = 5$ on the list in Theorem 4.1.1.

The main conjecture holds for quasi homogeneous systems with $m = 4$ and $m_0 \geq d - 5$.

4.3 The Degeneration

To finish the classification of special systems we use the degeneration of the plane described in detail by Ciliberto and Miranda [CM1]. The general plan relies on the fact that if the points are in special position, the dimension of the system can only increase. We attempt to find a special position for the points that allows us to calculate the dimension, but such that the dimension is not greater than the expected dimension. We are able to do so because the degeneration affords us a great deal of flexibility.

Briefly, we consider $V = \mathbb{P}^2 \times \mathbb{A}^1$ and X, the blow-up in this three-fold of a line in $\mathbb{P}^2 \times \{0\}$. We have the projections $p_1 : V \to \mathbb{A}^1$ and $p_2 : V \to \mathbb{P}^2$, the blowup map $f : X \to V$, and the compositions $\pi_1 = p_1 \circ f : X \to \mathbb{A}^1$ and $\pi_2 = p_2 \circ f : X \to \mathbb{P}^2$. Let X_t be the fiber of π_1 over t in $\mathbb{A}^1$. Then $X_t \cong \mathbb{P}^2$ if $t \neq 0$. In X_0, the degeneration produces two surfaces, a plane $\mathbb{P} = \mathbb{P}^2$ and a Hirzebruch surface $\mathbb{F} = \mathbb{F}_1$, joined transversely along a curve R. R is the line L in $\mathbb{P}$ and the exceptional divisor E of $\mathbb{F}$.

The Picard group of X_0 is the fibered product of Pic($\mathbb{P}$) and Pic($\mathbb{F}$). That is, a line bundle $\mathcal{X}$ on X_0 is equivalent to a line bundle $\mathcal{X}_{\mathbb{P}}$ on $\mathbb{P}$ and a line bundle $\mathcal{X}_{\mathbb{F}}$ on $\mathbb{F}$ which agree when restricted to R. This means we must have that $\mathcal{X}_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(d)$ and $\mathcal{X}_{\mathbb{F}} \cong \mathcal{O}_{\mathbb{F}}(cH - dE)$ for some c and d. Denote this bundle on X_0 by $\mathcal{X}(c, c-d)$.

We also have $\mathbb{P}$ and $\mathbb{F}$ as divisors on X, and the corresponding bundles $\mathcal{O}_X(\mathbb{P})$ and $\mathcal{O}_X(\mathbb{F})$. The bundle $\mathcal{O}_X(\mathbb{P})$ is disjoint from the fibers X_t for $t \neq 0$, but restricts to $\mathbb{P}$ as $\mathcal{O}_{\mathbb{P}}(-1)$ and restricts to $\mathbb{F}$ as $\mathcal{L}_{\mathbb{F}}(E)$.

Denote by $\mathcal{O}_X(d)$ the line bundle $\pi_2^*(\mathcal{O}_{\mathbb{P}^2}(d))$. For $t \neq 0$ the restriction of $\mathcal{O}_X(d)$ to X_t is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d)$. The restriction of $\mathcal{O}_X(d)$ to X_0 is the bundle $\mathcal{X}(d, 0)$. The bundle $\mathcal{X}(d, 0)$ restricts to $\mathbb{P}$ as $\mathcal{O}_{\mathbb{P}}(d)$ and to $\mathbb{F}$ as $\mathcal{O}_{\mathbb{F}}(dH - dE)$.

Let $\mathcal{O}_X(d, k)$ be the line bundle $\mathcal{O}_X(d) \otimes \mathcal{O}_X(k\mathbb{P})$. When $\mathcal{O}_X(d, k)$ is restricted to X_t for $t \neq 0$ the result is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(d)$, as before, but the restriction to X_0 has changed. The restriction of $\mathcal{O}_X(d, k)$ to X_0 is isomorphic to $\mathcal{X}(d, k)$, which restricts to $\mathbb{P}$ as $\mathcal{O}_{\mathbb{P}}(d-k)$ and restricts to $\mathbb{F}$ as $\mathcal{O}_{\mathbb{F}}(dH - (d-k)E)$.

In this way, all of the bundles $\mathcal{X}(d, k)$ on X_0 are seen as flat limits of the bundles

$\mathcal{O}_{\mathbb{P}^2}(d)$ on the general fiber X_t of this degeneration. This is part of the flexibility this degeneration provides. The rest of the flexibility lies in the position of the points.

Take integers n and b such that $0 \le b \le n$. consider $n - b + 1$ general points $p_0, p_1, \ldots, p_{n-b}$ in $\mathbb{P}$ and b general points $p_{n-b+1} \ldots, p_n$ in $\mathbb{F}$. These can be realized as the limits of $n + 1$ general points $p_{0,t}, p_{1,t} \ldots, p_{n,t}$ in X_t. Then we have the linear systems $\mathcal{L}_t(d, m_0, n, m) = \mathcal{L}(d, m_0, n, m)$ in $X_t \cong \mathbb{P}_2$ for $t \ne 0$.

On X_0 we have the system formed by divisors in $\mathcal{X}(d, k)$ having a point of multiplicity m_0 at p_0 and multiplicity m at points $p_1, \ldots, p_n$. This system will be called $\mathcal{L}_0 := \mathcal{L}_0(d, k, m_0, n, b, m)$. Any one of these systems may be considered as the flat limit on X_0 of the system $\mathcal{L}_t = \mathcal{L}(d, m_0, n, m)$. We will say that $\mathcal{L}_0$ is obtained from $\mathcal{L}$ by a (k, b) *degeneration.*

The linear system $\mathcal{L}_0$ restricts to $\mathbb{P}$ as the system $\mathcal{L}_{\mathbb{P}} := \mathcal{L}(d - k, m_0, n - b, m)$ and restricts to $\mathbb{F}$ as the system $\mathcal{L}_{\mathbb{F}} := \mathcal{L}(d, d-k, b, m)$. Divisors in the linear system $\mathcal{L}_0$ come in three types. The first type consists of a divisor $C_{\mathbb{F}}$ on $\mathbb{F}$ in the system $|dH - (d - k)E|$ and a divisor $C_{\mathbb{P}}$ on $\mathbb{P}$ in the system $|(d - k)H|$, both of which satisfy the multiple point conditions, and which restrict to the same divisor on the curve R.

The second type is a divisor corresponding to a section of the bundle which is identically zero on $\mathbb{P}$, and gives a divisor in the system $\mathcal{L}_{\mathbb{F}}$ which contains the exceptional curve E as a component. A divisor in $\mathcal{L}_{\mathbb{F}}$ which contains E is an element of the system $E + \mathcal{L}(d, d - k + 1, b, m)$. Since we are interested only in the dimension of this kernel system, we will denote it by $\hat{\mathcal{L}}_{\mathbb{F}}$ and (abusing notation) write $\hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d - k + 1, b, m)$.

The third type is similar. It corresponds to a section of the bundle which is identically zero on $\mathbb{F}$, and gives a divisor in the system $\mathcal{L}_{\mathbb{P}}$ which contains the line L as a component. That is, it comes from an element of the system $L + \mathcal{L}(d - k - 1, m_0, n - b, m)$. We will denote this kernel system by $\hat{\mathcal{L}}_{\mathbb{P}}$ and abuse notation further to write $\hat{\mathcal{L}}_{\mathbb{F}} = \mathcal{L}(d, d - k + 1, b, m)$.

The four main linear systems are collected in the following table. They are all quasi homogeneous, and in our case have $m = 4$. They also have smaller data than the original system, and give us the chance to argue by induction.

$\hat{\mathcal{L}}_{\mathbb{F}}$:	d	$d - k + 1$	b	4
$\mathcal{L}_{\mathbb{F}}$:	d	$d - k$	b	4
$\mathcal{L}_{\mathbb{P}}$:	$d - k$	m_0	$n - b$	4
$\hat{\mathcal{L}}_{\mathbb{P}}$:	$d - k - 1$	m_0	$n - b$	4

The following notation will be used.

v	the virtual dimension of the general system.
$v_{\mathbb{P}}$	the virtual dimension of the system on $\mathbb{P}$.
$v_{\mathbb{F}}$	the virtual dimension of the system on $\mathbb{F}$.
$\hat{v}_{\mathbb{P}}$	the virtual dimension of the kernel system on $\mathbb{P}$.
$\hat{v}_{\mathbb{F}}$	the virtual dimension of the kernel system on $\mathbb{F}$.
ℓ	the dimension of the general system.
$\ell_{\mathbb{P}}$	the dimension of the system on $\mathbb{P}$.
$\ell_{\mathbb{F}}$	the dimension of the system on $\mathbb{F}$.
$\hat{\ell}_{\mathbb{P}}$	the dimension of the kernal system on $\mathbb{P}$.
$\hat{\ell}_{\mathbb{F}}$	the dimension of the kernal system on $\mathbb{F}$. and
ℓ_0	the dimension of the system $\mathcal{L}_0$.

We have that $\ell \leq \ell_0$ by semi-continuity, and we will attempt to exploit the inequality $v \leq \ell \leq \ell_0$ to show that the linear system $\mathcal{L}$ has the expected dimension.

Note the following relationships between the virtual dimensions of the main linear systems.

Lemma 4.3.1 *The following are identities in the variables d, m_0, n, k, and b.*

a. $v_{\mathbb{P}} + v_{\mathbb{F}} = v + d - k$.

b. $\hat{v}_{\mathbb{P}} + v_{\mathbb{F}} = v - 1$.

c. $v_{\mathbb{P}} + \hat{v}_{\mathbb{F}} = v - 1$.

4.4 Classification of Special Systems with $m = 4$

One of the main results of [CM1] is the following computation of the dimension of $\mathcal{L}_0$.

Theorem 4.4.1 *Let* $r_{?} = \ell_{?} - \hat{\ell}_{?} - 1$ *and* $r_{\bar{?}} = \ell_{\bar{?}} - \hat{\ell}_{\bar{?}} - 1$. *Then*

(a) If $r_{?} + r_{\bar{?}} \leq d - k - 1$, *then*

$$\ell_0 = \hat{\ell}_{?} + \hat{\ell}_{\bar{?}} + 1.$$

(b) If $r_{?} + r_{\bar{?}} \geq d - k - 1$, *then*

$$\ell_0 = \ell_{?} + \ell_{\bar{?}} - d + k.$$

The dimension computed in part (b) of the Theorem is the virtual dimension of the system $\mathcal{L}$ by Lemma 4.3.1, and will be used to show that a non-empty system has the expected dimension. Part (a) of the Theorem is more useful for showing that a system is empty. The next two lemmas make this explicit.

Lemma 4.4.1 *Let* $\mathcal{L}$ *be a quasi homogeneous linear system with negative virtual dimension,* $v \leq -1$. *If integers* k *and* b *can be found such that when a* (k, b) *degeneration is executed*

(a) The systems $\mathcal{L}_{\bar{?}}$ *and* $\mathcal{L}_{?}$ *are all non-special, and*

(b) the kernel systems $\hat{\mathcal{L}}_{\bar{?}}$ *and* $\hat{\mathcal{L}}_{?}$ *are empty,*

then $\mathcal{L}$ *is empty.*

Proof: If either $\mathcal{L}_{\searrow}$ or $\mathcal{L}_{\nearrow}$ is empty then $\mathcal{L}$ is empty as well since the kernel systems are empty. If $\mathcal{L}_{\searrow}$ and $\mathcal{L}_{\nearrow}$ are not empty, then

$$r_{\nearrow} + r_{\searrow} = \ell_{\nearrow} + \ell_{\searrow} = v_{\nearrow} + v_{\searrow} = v + d - k \leq d - k - 1.$$

The first equality follows from (b). The second is true because the systems are non-special and not empty. The third equality is Lemma 4.3.1 a. The final inequality holds by assumption $v \leq -1$. Therefore Theorem 4.4.1 (a) applies and $\ell_0 = \hat{\ell}_{\nearrow} + \hat{\ell}_{\searrow} + 1 = -1$. Now $\mathcal{L}$ must be empty since $-1 \leq \ell \leq \ell_0 = -1$. ∎

Lemma 4.4.2 *Let $\mathcal{L}$ be a quasi homogeneous linear system with virtual dimension $v \geq -1$. If integers $k < d$ and b can be found such that when a (k, b) degeneration is executed*

(a) The systems $\hat{\mathcal{L}}_{\searrow}$, $\mathcal{L}_{\searrow}$, $\mathcal{L}_{\nearrow}$, and $\hat{\mathcal{L}}_{\nearrow}$ are all non-special, and

(b) the systems $\mathcal{L}_{\searrow}$ and $\mathcal{L}_{\nearrow}$ have virtual dimension at least -1,

then $\mathcal{L}$ has the expected dimension.

Proof: The proof relies on the identities from Lemma 4.3.1. We claim that with the given hypotheses, $\hat{\ell}_{\nearrow} + \hat{\ell}_{\searrow} \leq v - 1$. There are three possibilities. If both $\hat{\mathcal{L}}_{\nearrow}$ and $\hat{\mathcal{L}}_{\searrow}$ are empty, then $\hat{\ell}_{\nearrow} + \hat{\ell}_{\searrow} = -2 \leq v - 1$ since $v \geq -1$. If both systems are non-empty and non special, then $\hat{\ell}_{\nearrow} = \hat{v}_{\nearrow}$ and $\hat{\ell}_{\searrow} = \hat{v}_{\searrow}$. Then using the three identities we get

$$\hat{\ell}_{\nearrow} + \hat{\ell}_{\searrow} = \hat{v}_{\nearrow} + \hat{v}_{\searrow} = v - 1 - d + k \leq v - 1.$$

The inequality holds since $k < d$. If one of the systems is empty and the other is not, identities b) and c) give

$$\hat{\ell}_{\supset} + \hat{\ell}_{\subset} = -1 + \hat{v}_{\subset} = v - 1 - 1 - v_{\supset} \leq v - 1.$$

or

$$\hat{\ell}_{\supset} + \hat{\ell}_{\subset} = \hat{v}_{\supset} - 1 = v - 1 - 1 - v_{\subset} \leq v - 1.$$

The inequalities follow from hypothesis (b). Now,

$$\begin{aligned}
r_{\supset} + r_{\subset} &= \ell_{\supset} - \hat{\ell}_{\supset} - 1 + \ell_{\subset} - \hat{\ell}_{\subset} - 1 & \\
&= v_{\supset} + v_{\subset} - \hat{\ell}_{\supset} - \hat{\ell}_{\subset} - 2 & \text{by hypothesis} \\
&= v + d - k - (\hat{\ell}_{\supset} + \hat{\ell}_{\subset}) - 2 & \text{by Lemma 4.3.1 a)} \\
&\geq v + d - k - (v - 1) - 2 & \text{by the claim} \\
&= d - k - 1.
\end{aligned}$$

We apply Theorem 4.4.1 (b) to get $\ell_0 = \ell_{\supset} + \ell_{\subset} - d + k = v_{\supset} + v_{\subset} - d + k = v$, by Lemma 4.3.1 a. Finally, we have $v \leq \ell \leq \ell_0 = v$. Therefore $\ell = v$ and $\mathcal{L}$ is non special. ∎

These are the basic tools in the proof of the Main Conjecture.

Theorem 4.4.2 *A system $\mathcal{L}(d, m_0, n, 4)$ is special if and only if it is a (-1) special system, i.e., it is one of the systems listed in Theorem 4.1.1.*

Proof: We may assume that $m_0 \leq d - 6$ and that $d \geq 6$. The proof is by induction on d. Assume the theorem is true for smaller values of d. Then assume that $\mathcal{L}$ is not (-1) special, and prove that it is non-special.

Begin with the case $v \leq -1$. We perform a (k, b) degeneration with $k = 3$. This gives the following important systems.

$$\begin{array}{lcccc} \hat{\mathcal{L}}_{\mathbb{F}}: & d & d-2 & b & 4 \\ \mathcal{L}_{\mathbb{F}}: & d & d-3 & b & 4 \\ \mathcal{L}_{\mathbb{P}}: & d-3 & m_0 & n-b & 4 \\ \hat{\mathcal{L}}_{\mathbb{P}}: & d-4 & m_0 & n-b & 4 \end{array}$$

We wish to find a b so that both $\hat{\mathcal{L}}_{\mathbb{P}}$ and $\hat{\mathcal{L}}_{\mathbb{F}}$ are empty, and all four systems are non-special. We pick $b > \frac{d}{3}$ so that $\hat{\mathcal{L}}_{\mathbb{F}}$ is not (-1) special, and thus non-special by section 4.2. Then $b > \frac{3d-1}{10}$, which makes $\hat{v}_{\mathbb{F}} \leq -1$. Therefore $\hat{\mathcal{L}}_{\mathbb{F}}$ is empty. If, in addition, $b \leq \frac{4d-2}{10}$, then $\hat{v}_{\mathbb{P}} \leq -1$ since $v - \hat{v}_{\mathbb{P}} = 4d - 2 + 10b$ and $v \leq -1$. Now if $\hat{\mathcal{L}}_{\mathbb{P}}$ is not (-1) special (and non-special by induction), it will be empty. If $m_0 < d - 6$ this can be ensured by choosing b so that $n - b$ is odd. This is also enough to conclude that $\mathcal{L}_{\mathbb{P}}$ is not (-1) special (and non-special by induction). In order for $\mathcal{L}_{\mathbb{F}}$ to be not (-1) special (and thus non-special by section 4.2), we choose $b < \frac{4d-4}{10}$.

An integer b, satisfying the inequalities

$$\frac{3d-1}{10} < \frac{d}{3} < b < \frac{4d-4}{10} < \frac{4d-2}{10},$$

and such that $n - b$ is odd, may be found when d is 29, 32, 34, 35, or $d \geq 37$. This choice of b makes all of the systems involved non-special and the kernel systems empty. We apply Lemma 4.4.1. This proves that $\mathcal{L}$ is empty, provided $d \geq 37$, $m_0 < d - 6$ and the conjecture holds for lower d.

If $m_0 = d - 6$, $\hat{\mathcal{L}}_{\geq}$ might be (-1) special even when $n - b$ is odd, but only if $n - b \leq \frac{3(d-4)-1}{9}$. If $\hat{\mathcal{L}}_{\geq}$ is not (-1) special, Lemma 4.4.1 can be used to conclude that $\mathcal{L}$ is empty. We would like

$$n > \frac{3(d-4)-1}{9} + b.$$

For a given d and m_0, we only need to prove the theorem for the smallest n which makes v negative. For $m_0 = d - 6$, this value is the smallest integer n such that $n > \frac{7d-15}{10}$. It is clear that we need to choose b as small as possible. We will pick $\frac{d}{3} < b \leq \frac{d}{3} + 2$ such that $n - b$ is odd. Then $\frac{3(d-4)-1}{9} + b \leq \frac{6d+5}{9}$. Now

$$n > \frac{7d-15}{10} > \frac{6d+5}{9}$$

whenever $d \geq 62$. This guarantees that $\hat{\mathcal{L}}_{\geq}$ is not (-1) special when $m_0 = d - 6$ and $d \geq 62$. For values of d between 62 and 39, one may directly check that this choice of b makes $\hat{\mathcal{L}}_{\geq}$ not (-1) special for the smallest value of n. The same is true for $d = 38, 37, 35$, and 34. We are assuming the theorem for smaller values of d, so Lemma 4.4.1 applies and these systems are empty. The only outstanding case with $d \geq 37$ is $d = 39$ and $m_0 = 33$.

For $d = 39$ and $m_0 = 33$, the smallest n which makes v negative is $n = 26$. We choose $k = 3$ and $b = 13$. This makes $\hat{\mathcal{L}}_{\leq}$ non-special and empty by section 4.2 and $\hat{\mathcal{L}}_{\geq}$ non-special and empty by induction. $\mathcal{L}_{\leq}$ is not (-1) special since $b < \frac{4d-4}{10}$ and non-special by section 4.2. $\mathcal{L}_{\geq}$ is not (-1) special since $n - b$ is odd for $n = 26$ and non-special by induction. Now all the systems non-special and the kernel systems empty so we use Lemma 4.4.1 to conclude that this system is empty. We have proven the theorem for all $d \geq 37$ provided it is true for smaller values of d. The theorem will be proved case by case for smaller values of d.

If $d = 36$, we perform a $(3, 13)$ degeneration giving the systems

$\hat{\mathcal{L}}_{\mathbb{F}}$:	36	34	13	4
$\mathcal{L}_{\mathbb{F}}$:	36	33	13	4
$\mathcal{L}_{\mathbb{P}}$:	33	m_0	$n - 13$	4
$\hat{\mathcal{L}}_{\mathbb{P}}$:	32	m_0	$n - 13$	4

We are assuming that $m_0 \leq 30$. If all these systems are non-special and the kernel systems are empty then we use Lemma 4.4.1 to conclude that $\mathcal{L}$ is empty. The kernel systems have negative virtual dimension, so are empty if they are non-special. The systems $\hat{\mathcal{L}}_{\mathbb{F}}$ and $\mathcal{L}_{\mathbb{F}}$ are not special by section 4.2. The system $\hat{\mathcal{L}}_{\mathbb{P}}$ is (-1) special only if it has the form $\mathcal{L}(32, 28, 16, 4)$ or $\mathcal{L}(32, 30, x, 4)$ with $x \leq 10$. The first comes from the system $\mathcal{L} = \mathcal{L}(36, 28, 29, 4)$ which has virtual dimension $v = 6$, contrary to hypothesis. The second type comes from a system $\mathcal{L} = \mathcal{L}(36, 30, x + 13, 4)$ with virtual dimension $v = 107 - 10x$. Again, v is positive (since $x \leq 10$), contrary to hypothesis. The system $\mathcal{L}_{\mathbb{P}}$ is (-1) special only if it has the form $\mathcal{L}(33, 29, 16, 4)$. This comes from the system $\mathcal{L} = \mathcal{L}(36, 29, 29, 4)$ with virtual dimension $v = -23$. The system $\mathcal{L}_{\mathbb{P}} = \mathcal{L}(33, 29, 16, 4)$ is special, but Cremona reduces to the class of a line, and so has dimension 2. $\mathcal{L}_{\mathbb{F}}$ is non-special with virtual dimension 11. So in this case, $r_{\mathbb{P}} + r_{\mathbb{F}} = \ell_{\mathbb{P}} + \ell_{\mathbb{F}} = 2 + 11 < d - k - 1 = 32$ and we may appeal directly to Theorem 4.4.1 (a) to conclude that $\mathcal{L}$ is empty.

The next open case is $d = 33$. (The cases $d = 34$ and $d = 35$ were mentioned earlier.) We perform a (3.11) degeneration giving the systems

$\hat{\mathcal{L}}_{\mp}$:	33	31	11	4
$\mathcal{L}_{\mp}$:	33	30	11	4
$\mathcal{L}_{\supset}$:	30	m_0	$n - 11$	4
$\hat{\mathcal{L}}_{\supset}$:	29	m_0	$n - 11$	4

The kernel systems have negative virtual dimension, hence they are empty if they are non-special. $\hat{\mathcal{L}}_{\mp}$ and $\mathcal{L}_{\mp}$ are non-special by section 4.2. If $\mathcal{L}_{\supset}$ and $\hat{\mathcal{L}}_{\supset}$ are non-special we apply Lemma 4.4.1 to conclude that $\mathcal{L}$ is empty. $\hat{\mathcal{L}}_{\supset}$ is only (-1) special if it of the form $\mathcal{L}(29, 25, 14, 4)$ or $\mathcal{L}(29, 27, x, 4)$ where $x \leq 9$. Both of these come from systems with positive virtual dimension. $\mathcal{L}_{\supset}$ is only (-1) special only if it is $\mathcal{L}(30, 27, 12, 4)$ or $\mathcal{L}(30, 26, 14, 4)$. In these cases we appeal directly to Theorem 4.4.1 (a).

In both cases $\mathcal{L}_{\mp}$ is non-special of dimension 19. $\mathcal{L}(30, 27, 12, 4)$ Cremona reduces to the zero dimensional space of constant polynomials. Now $r_{\supset} + r_{\mp} = \ell_{\supset} + \ell_{\mp} = 0 + 19 < d - k - 1 = 29$, so we use Theorem 4.4.1 (a) to conclude that $\mathcal{L}$ is empty. In the other case, we notice that $\mathcal{L}(30, 26, 14, 4)$ Cremona reduces to the 5 dimensional space of quadratics. This means that $r_{\supset} + r_{\mp} = \ell_{\supset} + \ell_{\mp} = 5 + 19 < d - k - 1 = 29$. and Theorem 4.4.1 (a) tells us that $\mathcal{L}$ is empty.

When $d = 32$ we pick b as before ($\frac{d}{3} < b < \frac{4d-4}{10}$ and $n - b$ odd). This allows us to apply Lemma 4.4.1 unless $\mathcal{L} = \mathcal{L}(32, 26, n, 4)$. It is enough to prove $\mathcal{L}(32, 26, 21, 4)$ is empty. We perform a $(3, 11)$ degeneration in this case and apply Lemma 4.4.1.

For $d = 31$, a $(3, 11)$ degeneration produces systems which are not (-1) special and kernel systems which are empty, unless $\mathcal{L} = \mathcal{L}(31, 24, 25, 4)$ or $\mathcal{L}(31, 25, x+11, 4)$ with $x \leq 8$. The later does not have negative virtual dimension. The former degenerates as

$\hat{\mathcal{L}}_{\mathbb{F}}$:	31	29	11	4
$\mathcal{L}_{\mathbb{F}}$:	31	28	11	4
$\mathcal{L}_{\mathbb{P}}$:	28	24	14	4
$\hat{\mathcal{L}}_{\mathbb{P}}$:	27	24	14	4

$\mathcal{L}_{\mathbb{F}}$ is non-special of dimension $\ell_{\mathbb{F}} = 11$. $\mathcal{L}_{\mathbb{P}}$ Cremona reduces to the zero dimensional class of a quadruple line. In this case, $r_{\mathbb{P}} + r_{\mathbb{F}} = \ell_{\mathbb{P}} + \ell_{\mathbb{F}} = 11 + 0 < d - k - 1 = 27$, and Theorem 4.4.1 (a) tells us that $\mathcal{L}$ is empty.

If $d = 30$, we perform a $(3, 11)$ degeneration. This produces systems which are not (-1) special and kernel systems which are empty unless $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(26, 22, 12, 4)$, $\mathcal{L}(26, 23, 10, 4)$, or $\mathcal{L}(26, 24, x, 4)$, where $x \leq 8$. These exceptions all come from systems with positive virtual dimension, therefore Lemma 4.4.1 handles all cases with $d = 30$.

When $d = 29$ we pick b as before ($\frac{d}{3} < b < \frac{4d-4}{10}$ and $n - b$ odd). This satisfies the conditions of Lemma 4.4.1 unless $\mathcal{L} = \mathcal{L}(29, 23, n, 4)$. It is enough to show that $\mathcal{L}(29, 23, 19, 4)$ is empty. A $(3, 10)$ degeneration allows us to use Lemma 4.4.1 in this instance as well.

For $d = 28$, a $(3,10)$ degeneration satisfies the conditions of Lemma 4.4.1 in all but five cases, only two of which have negative virtual dimension. In these cases we use Cremona reduction to find the dimension of $\mathcal{L}_?$ and apply Theorem 4.4.1 (a). The $d = 27$ case has one exception when a $(3,9)$ degeneration is used, and it is handled in the same way. A $(3,9)$ degeneration also suffices to prove the theorem in case $d = 26$ and 25.

When $d = 24$, a $(3,9)$ degeneration works unless $\mathcal{L}$ is of the form $\mathcal{L}(24,18,17,4)$, $\mathcal{L}(24,17,19,4)$, or $\mathcal{L}(24,16,19,4)$. In the first two of these exceptions $\mathcal{L}_?$ is special and we proceed as above using Cremona reduction to find the dimension. In the final case, it is $\hat{\mathcal{L}}_?$ which is special and we must find another approach. A $(4,10)$ degeneration allows us to use Lemma 4.4.1 to conclude that $\mathcal{L}$ is empty in this case as well.

If $d = 23$, we use a $(3,8)$ degeneration. This lets us apply Lemma 4.4.1 in all but two cases. In these cases, we use Cremona reduction and Theorem 4.4.1 (a). For $d = 22$, a $(3,8)$ degeneration allows us to apply Lemma 4.4.1 unless $\mathcal{L} = \mathcal{L}(22,16,14,4)$. This system Cremona reduces to the system $\mathcal{L}(8,0,15,2)$. The theorem is true for $m = 2$ ([CM1]), and this system is not (-1) special, so this system is empty. Therefore $\mathcal{L}$ is empty, as well.

When $18 \leq d \leq 21$, we use a $(3,7)$ degeneration. We may appeal to Lemma 4.4.1 in all but a finite number of cases. In all but one of these cases we use Cremona reduction to determine the dimension of $\mathcal{L}_{\mathbb{P}}$ and apply Theorem 4.4.1 (a). In case $\mathcal{L} = \mathcal{L}(19, 12, 15, 4)$, we use a (3, 8) degeneration and Lemma 4.4.1. If $d = 17$, a (3, 6) degeneration succeeds. If $d = 16$ there are eleven possibilities for m_0. In each case, it is enough to prove the theorem for the smallest n which makes v negative and the system not (-1) special. We have the following systems

$$\begin{array}{llll}
\mathcal{L}(16, & 10, & 10, & 4) \\
\mathcal{L}(16, & 9, & 11, & 4) \\
\mathcal{L}(16, & 8, & 12, & 4) \\
\mathcal{L}(16, & 7, & 13, & 4) \\
\mathcal{L}(16, & 6, & 14, & 4) \\
\mathcal{L}(16, & 5, & 14, & 4) \\
\mathcal{L}(16, & 4, & 15, & 4) \\
\mathcal{L}(16, & 3, & 15, & 4) \\
\mathcal{L}(16, & 2, & 15, & 4) \\
\mathcal{L}(16, & 1, & 16, & 4) \\
\mathcal{L}(16, & 0, & 16, & 4)
\end{array}$$

The first system Cremona reduces to a homogeneous system with $m = 2$, which is empty [CM1]. A (3, 6) degeneration is used for all but the third and ninth systems. For these degenerations, $\mathcal{L}_{\mathbb{P}}$ is special but we use Cremona transformations to find its dimension and apply Theorem 4.4.1 (a). The remaining systems yield to a (4, 7) degeneration and Lemma 4.4.1.

For $d = 15$ or 14, a (3, 5) degeneration works. A (3, 5) degeneration also works when $d = 13$ in all but one case. The case $\mathcal{L} = \mathcal{L}(13, 5, 9, 4)$ did not yield to any of the approaches used so far. To see that this system is empty, Maple was used to analyze the 105×105 matrix constructed using the points $p_0 = (0, -3)$, $p_1 = (8, 3)$, $p_2 = (4, -4)$, $p_3 = (-5, -5)$, $p_4 = (-5, -2)$, $p_5 = (3, -1)$, $p_6 = (-5, -9)$,

$p_7 = (8, 5)$, $p_8 = (5, 8)$, and $p_9 = (-1, 4)$. The matrix was found to have full rank, therefore there are no thirteenth degree polynomials passing through these points with multiplicities. This implies that $\mathcal{L}(13, 5, 9, 4)$ is empty for points in general position.

In the $d = 12$ case, we again list the systems corresponding to possible values of m_0 and the critical n for each. We have the following seven systems.

$$\begin{array}{llll} \mathcal{L}(12, & 6, & 8, & 4) \\ \mathcal{L}(12, & 5, & 8, & 4) \\ \mathcal{L}(12, & 4, & 9, & 4) \\ \mathcal{L}(12, & 3, & 9, & 4) \\ \mathcal{L}(12, & 2, & 9, & 4) \\ \mathcal{L}(12, & 1, & 9, & 4) \\ \mathcal{L}(12, & 0, & 10, & 4) \end{array}$$

It is sufficient to show that the second, sixth, and seventh systems are empty, as these imply the rest are empty. The second system Cremona reduces to $\mathcal{L}(8, 1, 8, 3)$, which is non-special and empty by [CM1]. To show the sixth is empty, we employ a (4, 5) degeneration and Lemma 4.4.1. Finally, we use a (4, 5) degeneration with the seventh system. $\mathcal{L}_{\mathbb{P}}$ Cremona reduces to a quadruple line, and so has dimension 0. We apply Theorem 4.4.1 (a).

The $d = 11$ case proceeds similarly. There are six possibilities $0 \leq m_0 \leq 5$, and we only need to check the smallest n in each. We only need to show that $\mathcal{L}(11, 4, 7, 4)$ and $\mathcal{L}(11, 0, 8, 4)$ are empty since these imply the other four are empty. In both of these cases we use a (3, 4) degeneration. $\mathcal{L}_{\mathbb{F}}$ is special, but Cremona reduces to the two dimensional class of a line. In both cases $\mathcal{L}_{\mathbb{P}}$ is not (-1) special with virtual dimension 4. We apply Theorem 4.4.1 (a).

In the $d = 10$ case, for similar reasons, we only need to show that $\mathcal{L}(10, 3, 6, 4)$ and $\mathcal{L}(10, 0, 7, 4)$ are empty. The first Cremona reduces to the empty system of constant polynomials with a double point. The second reduces to the empty system of quadratics with a triple point. When $d = 9$, it suffices to show that $\mathcal{L}(9, 3, 5, 4)$ and $\mathcal{L}(9, 0, 6, 4)$ are empty. The former reduces to the empty system of lines through three general points. The latter reduces to the empty system of non-zero constant polynomials passing through three points. For $d = 8$, we need to show that $\mathcal{L}(8, 1, 5, 4)$ and $\mathcal{L}(8, 0, 6, 4)$ are empty. The first reduces to the empty system of non-zero constant polynomials passing through a point. The second we may conclude is empty by applying a (3, 3) degeneration and Lemma 4.4.1. Finally, we must show that $\mathcal{L}(7, 0, 4, 4)$ and $\mathcal{L}(6, 0, 4, 4)$ are empty. $\mathcal{L}(7, 0, 4, 4)$ reduces to the empty system of quadratics with a quadruple point. $\mathcal{L}(6, 0, 4, 4)$ reduces to the empty system of constant polynomials with a quadruple point.

Now we consider the $v \geq -1$ case. Again, we assume that $m_0 \leq d - 6$ and that $d \geq 6$. We assume that $\mathcal{L}$ is not (-1) special with $v \geq -1$ and prove that it is non special. For each d and m_0 it is enough to prove the theorem for the largest n which makes the virtual dimension $v \geq 0$. If $\mathcal{L}(d, m_0, n, 4)$ is non empty and non special, then $\mathcal{L}(d, m_0, n', 4)$ will be non-empty and non-special for $n' < n$. This is because the conditions imposed on curves of degree d in $\mathcal{L}(d, m_0, n', 4)$ are a subset of the independent conditions for the system $\mathcal{L}(d, m_0, n, 4)$. For $m_0 = d - 6$, $v \geq 0$ when

$n \leq \frac{7d-15}{10}$. For lower m_0, the largest value of n which makes $v \geq 0$ must be at least as big as in the $m_0 = d-6$ case. Therefore, we will assume that $n > \frac{7d-15}{10} - 1$.

We use a $(3.b)$ degeneration where b is chosen to satisfy the hypotheses of Lemma 4.4.2. $\hat{v}_{\leq}$ is not (-1) special when $b > \frac{d}{3}$. $v_{\leq}$ is not (-1) special if $b < \frac{2d-2}{5}$. $v_{\geq}$ is not (-1) special when $n-b$ is odd. $\hat{v}_{\geq}$ is not (-1) special if $n-b$ is odd and $m_0 \neq d-6$. Now $\hat{v}_{\leq}$ and $v_{\leq}$ are non-special and $v_{\geq}$ and $\hat{v}_{\geq}$ are non special by the inductive hypothesis. We need $b \leq \frac{2d-1}{5}$ to make $v_{\leq} \geq -1$. To force $v_{\geq} \geq -1$, the second identity in the proof of Lemma 4.4.2 gives us that it is enough to have that $v \geq -1$ and $\hat{v}_{\leq} \leq -1$. The former is true by hypothesis and the later is true when $b > \frac{3d}{10}$. All of this may be achieved (when $m_0 < d-6$) by choosing

$$\frac{d}{3} < b < \frac{2d-2}{5}$$

so that $n-b$ is odd. As before, we can do this for $d = 29, 32, 34, 35$, or whenever $d \geq 37$. Recall that $n > \frac{7d-25}{10}$, so that this choice makes $b < n$ for all $d \geq 8$.

When $m_0 = d-6$, $\hat{v}_{\geq}$ will be (-1) special if $n - b \leq \frac{3(d-4)-1}{9}$. For $m_0 = d-6$ the largest n making $v \geq 0$ is the integer $\frac{7d-25}{10} < n \leq \frac{7d-15}{10}$. We need $b < n - \frac{3(d-4)-1}{9}$. Putting these last two inequalities together, we see that we should choose $b < \frac{33d-95}{90}$. This guarantees that $\hat{v}_{\geq}$ is non-special even when $m_0 = d-6$.

If we select $\frac{d}{3} < b < \frac{33d-95}{90}$ such that $n-b$ is odd, then we may appeal to Lemma 4.4.2 to conclude that $\mathcal{L}$ is non-special. This may be done for $d \geq 91$. For $37 \leq d \leq 90$, the hypotheses of Lemma 4.4.2 are satisfied (for $m_0 = d-6$ and the largest n which makes $v \geq -1$) if we choose $\frac{d}{3} \leq b \leq \frac{d}{3} + 1$, for all but $d = 46$ and $d = 40$. For the system $\mathcal{L}(46, 40, 30, 4)$, we perform a (3.16) degeneration. $\hat{\mathcal{L}}_{\mathbb{F}}$ is empty, $\mathcal{L}_{\mathbb{F}}$ is non-special of dimension 21. $\mathcal{L}_{\mathbb{P}}$ is non-special (by induction) of dimension 29, and $\hat{\mathcal{L}}_{\mathbb{P}}$ is expected to be empty, but is (-1) special. $\hat{\mathcal{L}}_{\mathbb{P}} = \mathcal{L}(42, 40, 14, 4)$ Cremona reduces to the zero dimensional space of constants. We compute $r_{\mathbb{P}} + r_{\mathbb{F}} = (29 - 0 - 1) + (21 - (-1) - 1) = 49 > d - k - 1 = 46 - 3 - 1$. Therefore, $\ell_0 = 21 + 29 - 46 + 3 = 7$ by Theorem 4.4.1 (b), and this is also the virtual dimension. The $d = 40$ case follows in the same fashion. Use a (3.14) degeneration. $\hat{\mathcal{L}}_{\mathbb{P}}$ is special, but Cremona reduces to the zero dimensional space of constants. The hypothesis of Theorem 4.4.1 (b) are satisfied and $v = \ell_0$ since $v_{\mathbb{P}} = \ell_{\mathbb{P}}$ and $v_{\mathbb{F}} = \ell_{\mathbb{F}}$.

The theorem is proved for $d \geq 37$, provided it is true for smaller values of d. We now prove the theorem for $d < 37$ case by case. In each case we choose b so that $v_{\mathbb{F}}$ and $v_{\mathbb{P}}$ are at least -1, then check that all the systems are not (-1) special. If they are all not (-1) special we may apply Lemma 4.4.2. When $d = 36$ we use a (3.13) degeneration. We analyzed this degeneration already in the $v \leq -1$ case. If the four main systems are all non-special and $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ have dimension at least -1 we apply Theorem 4.4.1 (b) to conclude that $\mathcal{L}$ has the expected dimension. Now, b was chosen so that $\mathcal{L}_{\mathbb{P}}$ and $\mathcal{L}_{\mathbb{F}}$ have virtual dimension at least -1, so it is enough that the systems are all non-special. We saw earlier that this only fails (when $v \geq -1$) if $\mathcal{L} = \mathcal{L}(36, 28, 29, 4)$ or if $\mathcal{L}$ is of the form $\mathcal{L}(36, 30, x + 13, 4)$ where $x \leq 10$. In the second case it is enough to prove the theorem for the largest n making $v \geq -1$, so we may assume $\mathcal{L} = \mathcal{L}(36, 30, 23, 4)$. In both cases $\hat{\mathcal{L}}_{\mathbb{P}}$ is special, but we may use Cremona transformations to find it's dimension. Then directly apply Theorem 4.4.1 (b).

For $d = 35$. we may choose b to be 12 or 13 to make $n - b$ odd (for the largest n making v at least 0) and use Lemma 4.4.2. If $d = 34$. this b also works unless $\mathcal{L} = \mathcal{L}(34.28.22.4)$. In this case we appeal to Theorem 4.4.1 (b) (using Cremona reduction to find the dimension of $\hat{\mathcal{L}}_{\mathbb{P}}$. When $d = 33$ we use a (3.11) degeneration and Lemma 4.4.2 unless $\mathcal{L} = \mathcal{L}(33, 25, 25, 4)$ or is of the form $\mathcal{L}(33.27.x + 11.4)$ where $x \leq 9$. In the first case we appeal to Theorem 4.4.1 (b). In the second case. note that we only care about the largest n (the largest x). This is the system $\mathcal{L} = \mathcal{L}(33.27.20.4)$, but $\mathcal{L}(33.27, 21, 4)$ is non-empty and non special (by Lemma 4.4.2) so $\mathcal{L}$ is as well.

When $d = 32$. we may choose $b = 11$ or 12 so that $n - b$ is odd for the largest n making v at least 0 and use Lemma 4.4.2. For $d = 31$ a (3.11) degeneration allows us to apply Lemma 4.4.2. A (3, 11) degeneration also allows us to apply Lemma 4.4.2 in all but three cases. For these exceptions. we use Cremona reduction to find the dimension of the special systems and use Theorem 4.4.1 (b). If $d = 29$. we let b be either 10 or 11 so that $n - b$ is odd and use Lemma 4.4.2 unless $\mathcal{L} = \mathcal{L}(29.23.18.4)$. In this case we use a (3, 10) degeneration and Theorem 4.4.1 (b).

For every d from 17 to 28 the process is the same. For each $m_0 \leq d - 6$ and the corresponding largest n, there is a b (between $\frac{3d}{10}$ and $\frac{2d-1}{5}$ so that $v_{\mathbb{F}}$ and $v_{\mathbb{P}}$ are at least -1) such that either a $(3, b)$ degeneration satisfies the hypothesis of Lemma

4.4.2, or so that we may use Cremona reduction to find the dimension of the special systems and apply Theorem 4.4.1 (b).

When $d = 16$, we may use a $(3,5)$ degeneration and Theorem 4.4.1 (b) in all but two cases. In these cases ($\mathcal{L} = \mathcal{L}(16,6,13,4)$ or $\mathcal{L}(16,1,15,4)$) a $(4,7)$ degeneration satisfies the requirements of Lemma 4.4.2. For $d = 15$ or 14, a $(3,5)$ degeneration works.

If $d = 13$ there are eight cases to check. For $m_0 = 7$ or 6 the systems (with the largest n) Cremona reduce to a known case and are non-special. Either a $(3,4)$ or a $(3,5)$ degeneration works for the rest of the cases except $m_0 = 2$. To prove the theorem for the system $\mathcal{L}(13,2,10,4)$ a $(6,7)$ degeneration satisfies the hypotheses of Lemma 4.4.2. When $d = 12$ there are six cases to check. The top two ($m_0 = 5$ or 6) Cremona reduce to a known case and are non-special. A $(3,4)$ or a $(4,5)$ degeneration works for the rest.

For $6 \leq d \leq 11$, $0 \leq m_0 \leq d-6$, and the corresponding largest n making v at least 0 and the system non-special, the linear systems all Cremona reduce to known cases and are non-special.

Therefore, all linear systems $\mathcal{L}(d, m_0, n, m)$ which are not (-1) special are non-special. In other words the only special quasi homogeneous linear systems with $m = 4$ are the (-1) special systems listed in Theorem 4.1.1.

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