THESIS

CELLULAR SHEAVES AS A FOUNDATION FOR POLYMER EMBEDDINGS

Submitted by

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ABSTRACT

CELLULAR SHEAVES AS A FOUNDATION FOR POLYMER EMBEDDINGS

Polymers can be represented by a graph, also known as a 1-dimensional cell complex. Since polymers exist in three-dimensional space, we wish to mathematically embed graphs into $\mathbb{R}^3$ (more generally $\mathbb{R}^d$). Cohomology keeps track of the structure of our graph as we embed it. Building upon the framework developed by Cantarella et al. [1], we make a connection to the natural way of finding cohomology through the cohomology of the constant sheaf. Motivated by this perspective, we modify the constant sheaf to fit different embedding restrictions.

ACKNOWLEDGEMENTS

I would like to recognize the help of my advisor, Clayton Shonkwiler. Also to James Wilson, who reminded us of a simple linear algebra fact that helped us notationally.

Further, I would like to dedicate this to my grandparents who could not be here to see me finish.

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Chapter 1

Introduction

The study of embedding polymers has gained attention in the fields of topology and geometry. As science looks to manufacturing resources, a rich mathematical framework is needed. A mathematical object called a graph consisting of vertices and edges is the beginning of such a framework. The vertices of the graph represent monomers, and the naturally occurring bonds between these monomers are represented by edges. We are interested in embeddings of graphs in $\mathbb{R}^d$ to model the existence of polymers in our space where the different embeddings are random variables from a probability distribution on the edge displacements. Some edge displacements are not randomly sampled since edges that form a cycle in the graph must have their edge displacements sum to zero. Recently, polymers with predetermined topologies (forms) have been synthesized [9, 10], in juxtaposition to linear or randomly branched forms studied in the past [11]. A theory of these topological was first developed by James, Guth, and Flory [6], [7], [3] called Phantom Network Theory. In this language, Phantom Network Theory assumes that edge displacements are Gaussian, but more generally it is desirable to have a theory which can handle arbitrary edge distributions [1].

When we choose vertex placements first, the rest of the embedding which consists of defining edge displacements comes without thought. Algebraically, we can think of the vertices as 0-chains and their placements as mapping the 0-chains to $\mathbb{R}^d$. More specifically, the mapping is an element of the cochain group $C^0 = \mathrm{Hom}(C_0(G), \mathbb{R}^d)$. On the other hand, we can start with choosing the edge displacements. We find this to be more advantageous because in polymer models, we want to draw our embedding from a probability distribution on the space of polymer configurations, and the distributions we are interested in are naturally defined in terms of the edge vectors.

For example in "freely-jointed networks" (a generalization of freely-jointed chains), each edge vector should be a unit vector. We can think of the collection of edge vectors as a point in a product of unit spheres, $\prod_{i=1}^{|e|} S^{d-1}$. The allowable edge configuration is then some subset and the probability measure is (re-normalized) Hausdorff measure of the appropriate dimension on the subset. The edges have no other restrictions other than that they form the graph G. As another example, we might assume that each edge is like a harmonic spring. With no graph constraints and an appropriate spring constant, we would choose each edge vector from the d-dimensional standard Gaussian. Since the spring is naturally defined on an edge, it would be hard to adjust this to be defined on the vertices.

In general, the probability distributions to be considered are based on energies. Most of the energies that are used in tractable models only depend on distances between monomers; that is, lengths of edge vectors! Since it is much more convenient to express the probability distributions of interest in terms of edge vectors rather than vertex positions, we choose to do our modeling around the edges of the graph.

Low-energy configurations are optimal and so more likely to happen naturally as opposed to high-energy configurations. The standard model is that the probability P of a state is proportional to $e^{-E/kT}$, where E is the energy of the state, k is the Boltzmann constant, and T is temperature. This is called a Boltzmann distribution of canonical ensemble. For our purposes, T is constant, so this just depends on the energy of the configuration. In the previous example, the energy is the spring energy (notice that the potential energy of a spring of rest length 0 stretched to a length x is $\frac{1}{2}kx^2$; if you assume the spring constant $k = 1$, then $e^{-E} = e^{-\frac{x^2}{2}}$, which is, up to scaling to integrate to 1, exactly the standard Gaussian), but other energies can be used as well.

The laws of physics do not care about rigid motions and so the probability distributions should be invariant under translation of vertices. If we had a translation-invariant measure on the vertex positions, the distribution of the center of mass should be proportional to the Lebesgue measure on $\mathbb{R}^d$, every possible center of mass as likely as all others. This is an issue since the Lebesgue measure is neither a probability measure nor finite. Edge displacement vectors, however, are translation-invariant, motivating us further to study graph embeddings using edge vectors rather than vertex positions.

In this paper, we examine a framework introduced by Cantarella, Deguchi, Shonkwiler, and Uehara [1, 2] in section 2.2 and later offer a new framework for polymer embeddings that allows vertices to be restricted to different subspaces of $\mathbb{R}^d$ while ultimately remaining edge-centric.

Chapter 2

A Framework for Graph Embeddings

As just described, it is more convenient to talk about sampling graph embeddings at the level of edge displacements rather than vertex positions. Going from vertex placements to an embedding is easier; edge vectors to an embedding is harder to do. So as we do things with our focus on edges, we need to handle the constraints on edge displacements imposed by the topology of the graph. This turns out to be very natural if we interpret graph embeddings as cochains on the graph which are valued in $\mathbb{R}^d$. In this chapter, we'll first describe these cochains at a formal level, then connect back to graph embeddings. See Hatcher [5] for a standard mathematical treatment of chains, cochains, homology, and cohomology, or Sunada [8, Chapter 4] for a discussion specific to graphs.

2.1 Chains and Boundaries

Before talking about cochains, we should start with chains. Let our connected, directed graph be G with the vertex set V and the edge set E.

Definition 1. *A* ***0-chain*** *is a linear combination of vertices. Said algebraically, a* 0*-chain is*

$$x = \sum_{v \in V} k_v v.$$

Definition 2. *The set* $\mathbf{C_0}(G, \mathbb{R})$ *is the vector space generated by* 0*-chains. Since the vertices form a basis, this satisfies the following conditions:*

1. $\sum_{v \in V}^{\textit{finite}} k_v v = 0$ *if and only if* $k_v = 0 \; \forall v \in V$

2. $\sum_{v \in V}^{\textit{finite}} k_v v \pm \sum_{v \in V}^{\textit{finite}} y_v v = \sum_{v \in V}^{\textit{finite}} (k_v \pm y_v) v.$

Definition 3. *A* ***1-chain*** *is a linear combination of edges. Said algebraically, a* 1*-chain is*

$$w = \sum_{e \in E} k_e e.$$

Note that because we defined our edge set to be oriented edges, these representations are unique. To represent the inverse of an edge, we write $\overline{e}$.

Definition 4. *The group* $\mathbf{C_1}(G, \mathbb{R})$ *is the vector space generated by* 1*-chains.*

In cell complexes (and graph theory), we define the boundary of an edge as the difference of its end vertex and its start vertex.

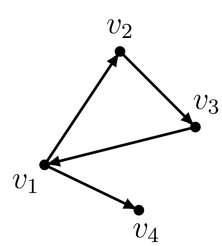
$$\partial : C_1 \to C_0 \qquad \qquad \partial[e] := z(e) - a(e)$$

where $z(e)$ is terminal vertex of the edge and $a(e)$ is origin vertex of the edge (reminiscent of "from a to z").

We can extend this linearly to the groups C_0 and C_1 where

$$\partial\Big(\sum_{e \in E} k_e e\Big) = \sum_{e \in E} k_e \partial(e).$$

Example 1. *With the graph below, we name edges by their boundary;* e_{12} *is the edge between* v_1 *and* v_2*,* e_{14} *is the edge between* v_1, v_4*.*



Then for $\sigma = 3e_{12} - 2e_{23} + e_{31} + 5e_{14} \in C_1$,

$$\begin{aligned}\partial(\sigma) &= 3(v_2 - v_1) - 2(v_3 - v_2) + v_1 - v_3 + 5(v_4 - v_1)\\ &= -7v_1 + 5v_2 + -3v_3 + 5v_4 \in C_0.\end{aligned}$$

Not every linear combination of vertices in C_0 is the boundary of an element in C_1. This gives our map interesting structure and insight into our graph. For each graph, we can write the boundary map as a matrix with respect to the basis of C_0 and C_1. For this example, each row is a vertex and each column an edge. If v_i is $z(e_j)$, $(i, j) = 1$. If v_i is $a(e_j)$ then $(i, j) = -1$. If v_i and e_j are not adjacent, the entry is zero.

$$\partial = \begin{pmatrix} -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Redoing the calculation from earlier is then much easier,

$$\begin{pmatrix} -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ -3 \\ 5 \end{pmatrix}.$$

With this, we know that for $w \in C_1$ such that $\partial(w) = x = K_1v_1 + K_2v_2 + K_3v_3 + K_4v_4$, the coefficients K_i must be such that

$$K_1 = -k_1 + k_3 - k_4$$
$$K_2 = k_1 - k_2$$
$$K_3 = k_2 - k_3$$
$$K_4 = k_4.$$

We can find x so that this is not the case: $x = 0v_1 - 1v_2 - 4v_3 + 3v_4$.

We may extend ∂ to a chain complex,

$$0 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial_0} 0$$

and with this chain complex comes the homology of our graph. As mentioned before, we find it more advantageous to work with cochains so instead of going into detail on the chain complex, we leave it here.

If we would like to switch our focus to the edges, we may take the natural transpose map $\partial^T : C_0 \to C_1$.

Example 2. *Continuing example 1,*

$$\partial^T = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

and so when our input is the vector $\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^T$*, we expect our output to be the edges adjacent to* v_1 *with the correct orientation. Observe,*

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Reading the resulting matrix, $v_1 = a(e_{12}) = z(e_{31}) = a(e_{14})$ *which is true.*

2.2 Cochains and Embeddings

Given a chain complex like the one described in the previous section, we get a corresponding cochain complex by applying the Hom functor. Let us first define some of the groups involved.

Definition 5. *The **cochain group*** $C^i(G, \mathbb{R}^d)$ *is the space of linear maps from the chain group* $C_i(G, \mathbb{R})$ *to* $\mathbb{R}^d$*, also denoted* $\mathrm{Hom}(C_i(G, \mathbb{R}), \mathbb{R}^d)$.

Definition 6. *By definition 5, there is a **first cochain group*** $\mathbf{C^1} = \mathrm{Hom}(C_1(G, \mathbb{R}), \mathbb{R}^d)$*. We denote linear maps in this group as a* $d \times |E|$ *matrix* W*.*

Definition 7. *By definition 5, there is a **zero-th cochain group*** $\mathbf{C^0} = \mathrm{Hom}(C_0(G, \mathbb{R}), \mathbb{R}^d)$*. We denote linear maps in this group as a* $d \times |V|$ *matrix* X*.*

Note we can write elements of C^0, C^1 as real matrices with respect to the basis we choose for C_0 and C_1.

Interpreting the linear maps, C^0 is any collection of assignments of vectors in $\mathbb{R}^d$ to vertices. As mentioned before, any such assignment determines an "embedding" of the graph, explaining the connection between graph embeddings and cochains with coefficients in $\mathbb{R}^d$. Now, elements in C^1 assign vectors to edges, but we are more strict with these. The assignments we like are the matrices $W \in C^1$ whose column vectors are the displacement vectors that come from some actual embedding of the graph; calling back to Example 1 where we see not all w are in the image. For example, if we are looking to embed a 3-cycle, if we were given

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

there is no way to arrange the basis vectors to make a cycle. We can see this geometrically since the total displacement around a cycle in a proper embedding should be 0, and these three vectors do not add to 0. We will call W which stem from an embedding of vertices X, *compatible* with X.

In the matrix $W \in C^1$, the (i, j) entry W_{ij} gives the ith coordinate of the vector assigned to the jth edge. Since maps $X \in C^0$ send vertices to vectors, which we will usually think of as vertex positions, X_{ij} gives the ith coordinate of the vector assigned to the jth vertex. With this interpretation, W and X are compatible if $W_{ij} = X_{i,z(j)} - X_{i,a(j)}$ for all i, j.

When we apply the Hom functor to ∂, we get a new function,

$$\partial^* : C^0 \to C^1 \qquad \text{defined by} \qquad (\partial^* X)(w) = X(\partial(w))$$

with $w \in C_1$. Observing the action on a single edge,

$$\begin{aligned} \partial^*(X)(e_i) &= X(z(e_i)) - X(a(e_i)) \\ &= X(z(e_i) - a(e_i)) \\ &= X(\partial(e_i)). \end{aligned}$$

Thus when we extend linearly, our induced map ∂^* is equivalent to finding the position vectors of the boundary vertices of the edges in w. Thus, W and X are compatible if $W = \partial^* X$. Since $(\partial^* X)(w) = X(\partial(w))$ and $W = \partial^* X$, W and X are also compatible if $W = X\partial$.

We then observe the following cochain complex,

$$0 \xrightarrow{\partial_2^*} C^0 \xrightarrow{\partial^*} C^1 \xrightarrow{\partial_0^*} 0$$

Definition 8. *The ith **cohomology group** with coefficients in $\mathbb{R}^d$ is* $H^i(G, \mathbb{R}^d) = \dfrac{\ker \partial_{i+1}^*}{\operatorname{im} \partial_i^*}$.

Definition 9. *By definition 8,* $H^1(G, \mathbb{R}^d) = \frac{\ker \partial_2^*}{\operatorname{im} \partial^*}$. *Note that* $\ker \partial_2^* = C^1$ *and so* $H^1 = \frac{C^1}{\operatorname{im} \partial^*}$. *We call this the **first cohomology group** and write* $\mathbf{H^1}$ *when clear.*

Definition 10. *By definition 8,* $H^0(G, \mathbb{R}^d) = \frac{\ker \partial^*}{\operatorname{im} \partial_0^*}$. *Note that* $\operatorname{im} \partial_0^* = 0$ *and so* $H^0(G, \mathbb{R}^d) = \ker \partial^*$. *We call this the **zero-th cohomology group** and write* $\mathbf{H^0}$ *when clear.*

Analogous definitions hold with any coefficients from an abelian group, but we are interested in using $\mathbb{R}^d$ as coefficients because these correspond to embeddings of the graph in to $\mathbb{R}^d$. Since the composition of linear maps is linear, ∂^* is a linear map from the $d|V|$-dimensional vector space C^0 to the $d|E|$-dimensional vector space C^1, so it can be represented by a $d|E| \times d|V|$ matrix. Specifically, if $\partial^* X = W$, meaning that W is compatible with X or, in cohomological terms, W represents the class 0 in cohomology, then

$$(\partial^T \otimes I_d) \operatorname{vec}(X) = \operatorname{vec}(W)$$

where the dimension of the matrices are $d|E| \times d|V|$, $d|V| \times 1$, and the resulting $d|E| \times 1$. Hence, $\partial^T \otimes I_d$ is the matrix representation of ∂^*, which we now focus on.

From [2],

Lemma 1 ([2, Lemma 6])**.**

$$\ker(\partial^T \otimes I_d) = \ker \partial^T \otimes \mathbb{R}^d \textit{ and}$$

$$\operatorname{im}(\partial^T \otimes I_d) = \operatorname{im} \partial^T \otimes \mathbb{R}^d.$$

Proposition 1 ([2, Proposition 7])**.**

$$\ker(\partial^T \otimes I_d) = \ker \partial^T \otimes \mathbb{R}^d = \{1_{1\times|V|} \otimes u \in Mat_{d\times|V|} : u \in \mathbb{R}^d\}$$
$$\operatorname{im}(\partial^T \otimes I_d) = (\ker \partial)^{\perp} \otimes \mathbb{R}^d$$

while $\dim(\operatorname{im} \partial^T \otimes I_d) = d(|E| - \textit{number of cycles in G})$ *and* $\dim(\ker \partial^T \otimes I_d) = d$*.*

We expect the dimension of the cohomology groups with coefficients in $\mathbb{R}^d$ to be d times the dimension of our original map. We can see so with these calculations,

$$H^0(G, \mathbb{R}^d) = \ker(\partial^T \otimes I_d) = \ker \partial^T \otimes \mathbb{R}^d = H^0(G, \mathbb{R}) \otimes \mathbb{R}^d$$
$$H^1(G, \mathbb{R}^d) = \frac{C^1}{\operatorname{im}(\partial^T \otimes I_d)} = \frac{C^1}{\operatorname{im} \partial^T} \otimes \mathbb{R}^d = H^1(G, \mathbb{R}) \otimes \mathbb{R}^d.$$

The number of components in our connected graph is 1, and we can see this reflected in the dimension of H^0 being d, and the dimension of H^1 is d times the number of cycles.

Thus we have a way to find edge displacements from vertex positions as well as a way to study the bigger structure. This is not surprising and is the natural way to make an embedding, as mentioned in the introduction.

If we want to start with edge displacements instead and find corresponding vertex positions, we look at $(\partial^T \otimes I_d)^+ : C^1 \to C^0$, the Moore–Penrose pseudoinverse. We must ask ourselves which vertex positions will we receive since a graph translated down 5 units has the same displacement vectors as the original. Cantarella at al. [2] offers this result,

Lemma 2 ([2, Lemma 8])**.**

$$\operatorname{im}(\partial^T \otimes I_d)^+ \cong \{X \in C^0 : \textit{ the center of mass, } \frac{1}{|V|}\sum_i x_i = 0\}$$

$$\ker(\partial^T \otimes I_d)^+ \cong \ker \partial \times \mathbb{R}^d$$

The interesting result to note here is that out of all the translations of a set of vectors, the image of the pseudoinverse gives vertex positions such that the center of mass at the origin. Also said, this shifts our vertex positions, our data, into a centered point cloud. As we have seen and as [2] shows in their study, many results of embeddings can be studied at this stage.

Observing our forward structure, we assign each vertex a position vector from $\mathbb{R}^d$ then compute edge displacement vectors in the natural way, find the kernel and image, and thus the cohomology groups. A sheaf provides a different structure to do the same thing. Abstractly, we assign vector spaces to each cell, maps are established between vertices and edges, then sheaf cohomology groups are found. With the correct choice of maps, we imitate regular cohomology. We will look at sheaves generally first before we get to the choice of maps that bridge the gap between sheaf cohomology and regular cohomology.

Chapter 3

Cellular Sheaves

3.1 Definition and Examples

(Cellular) Sheaves are a relatively new concept in mathematics. Generally, cellular sheaves assign functions to each cell in a cellular complex and assign maps for each incidence relation. We are able to calculate the cohomology of a sheaf: while regular cohomology gives information of the space, sheaf cohomology gives information on how the functions assigned to the space behave. The material in this chapter is adapted from Chapter 9 of Ghrist's book *Elementary Applied Topology* [4].

Definition 11. *A **cellular sheaf** $\mathcal{F}$ over a cellular complex X is generated by an assignment to each cell $\sigma \in X$ of an abelian group $\mathcal{F}(\sigma)$ and to each face $\sigma \trianglelefteq \tau$ of τ a restriction map, $\mathcal{F}(\sigma \trianglelefteq \tau) : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ where $\mathcal{F}(\sigma \trianglelefteq \tau)$ is a homomorphism such that the faces of faces satisfy,*

$$\rho \trianglelefteq \sigma \trianglelefteq \tau \implies \mathcal{F}(\rho \trianglelefteq \tau) = \mathcal{F}(\sigma \trianglelefteq \tau) \circ \mathcal{F}(\rho \trianglelefteq \sigma).$$

Since $\tau \trianglelefteq \tau$, the induced homomorphism is the identity, $\mathcal{F}(\tau \trianglelefteq \tau) = id$. Another thing to note is that since in this paper we are dealing with 1-simplices, we do not have to worry about "faces of faces" and the satisfaction of composition. Since we work with vector spaces and linear maps, the following definition is sufficient for this paper.

Definition 12. *A **cellular sheaf** $\mathcal{F}$ over X is generated by an assignment to each cell $\sigma \in X$ of a vector space $\mathcal{F}(\sigma)$ and to each $v_i \trianglelefteq e_i$ a linear restriction map $\mathcal{F}(v_i \trianglelefteq e_i) : \mathcal{F}(v_i) \to \mathcal{F}(e_i)$.*

We could assign random vector spaces and linear maps and extrapolate information from that, but the real meaning is found within certain sheaves. One such example is the constant sheaf, discussed in Chapter 4.

With each sheaf, we assign an overall matrix in the way we do for the boundary and coboundary maps. To form the matrix, you take the coboundary matrix ∂^T and in each incidence instance, instead of the entry being 1 or -1, we have $\pm\mathcal{F}(\sigma \trianglelefteq \tau)$.

Example 3. *Observe the following sheaf,*

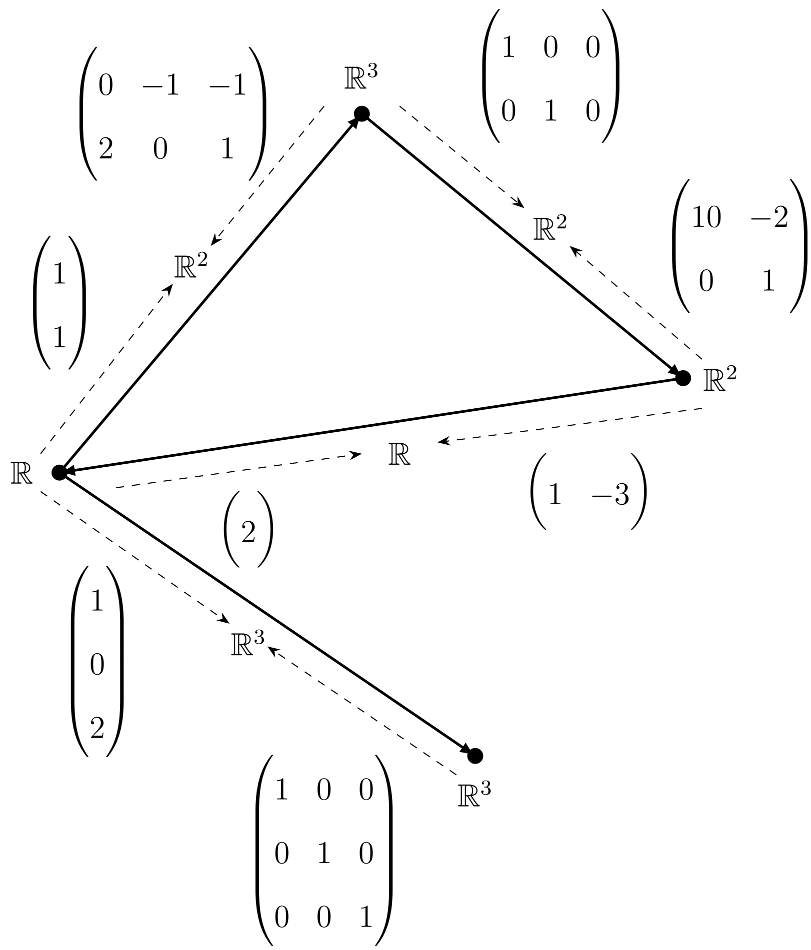


Figure 3.1: Sheaf: An assignment of a vector space to each cell and a linear map in each case of incidence.

Since the coboundary matrix $\partial^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$*, the corresponding sheaf coboundary matrix is (with sectioning added for understanding)*

$$\left(\begin{array}{ccc|c|cc|ccc} 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 10 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & -1 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Next, we define a term related to the vector spaces themselves:

Definition 13. *A **global section** of a sheaf $\mathcal{F}$ is an assignment of values to each cell from the vector space assigned to it that are compatible with the restriction maps.*

Naturally, assigning every edge and vertex a value of 0 gives a global section. If the graph is a tree, finding a nontrivial global section is much easier than on a graph with a cycle. With a tree, there is only one path to get from one vertex to another and so the path of linear maps is easily satisfied. When there is more than one path from one vertex to another, it is harder to find values that satisfy the cycle of restriction maps that occur.

A global section leads to finding an element in the kernel of the coboundary map. Ignoring the values on the edges, we take the elements assigned to the vertices and stack them into a vector, in the order in which they are represented in the coboundary matrix. We notate this as $\operatorname{vec} X$, calling back to the structure of our map from [2]. Since the vertices on the boundary of an edge have the same image in their respective restriction maps, the sheaf coboundary matrix subtracts the image in the sheaf from itself. It is for this reason that it's said that finding the cohomology of a sheaf is a study of its global sections (see the Lemma below). We officially define sheaf cohomology in the next section.

Lemma 3 ([4, Lemma 9.5])**.** *Sheaf cohomology in grading zero classifies global sections:*

$$H^0(X;\mathcal{F}) = \mathcal{F}(X).$$

Let us see how global sections may come together or fail to do so. We will start by finding a global section for a similar graph to figure 3.1. We choose an element at a random vertex and work our way through the maps to find values for the other cells. We distinguish the values from the restriction maps by using square brackets.

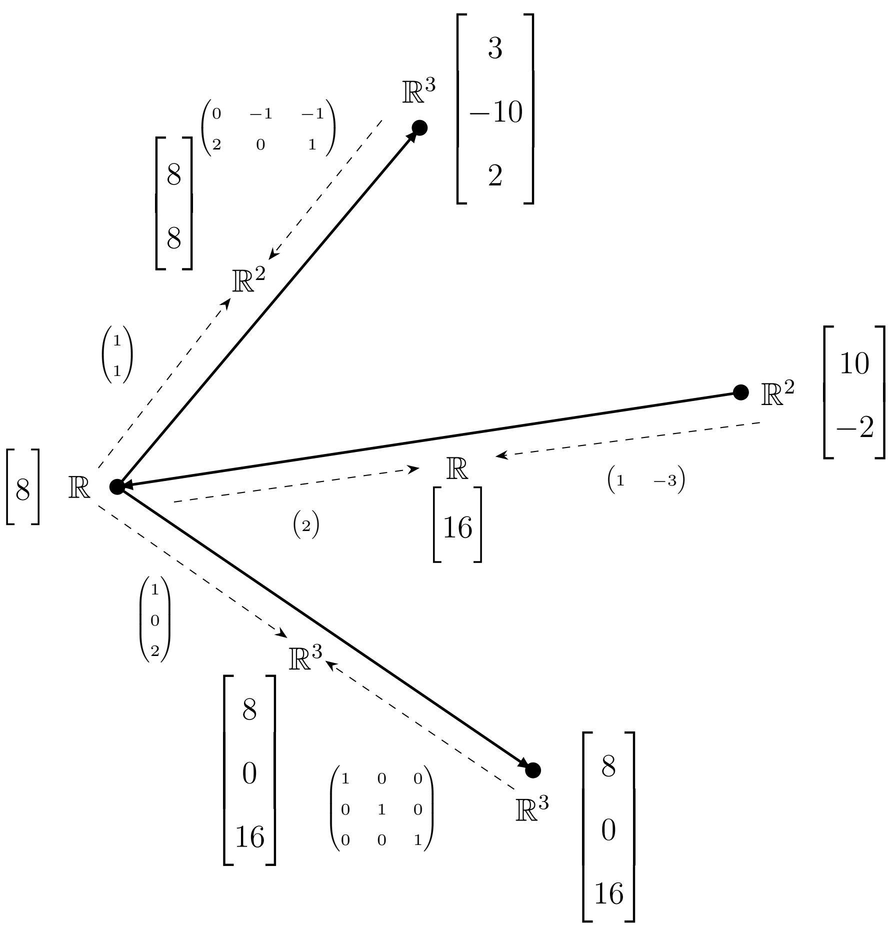


Figure 3.2: Global section: An assignment of elements such that the restriction maps are satisfied.

We can verify that figure 3.2 is indeed a global section by checking $\operatorname{vec} X$ is in the kernel.

$$\left(\begin{array}{ccc|c|cc|ccc} 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & -1 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \begin{bmatrix} 3 \\ -10 \\ 2 \\ 8 \\ 10 \\ -2 \\ 8 \\ 0 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can see the aforementioned subtracting of images when we observe the first two rows (i.e. the first block) of the matrix multiplication:

$$[(-10)(-1) + 2(-1)] + [8(-1)] = 0$$
$$[3(2) + 2(1)] + [8(-1)] = 0.$$

Now pasting these values onto figure 3.1,

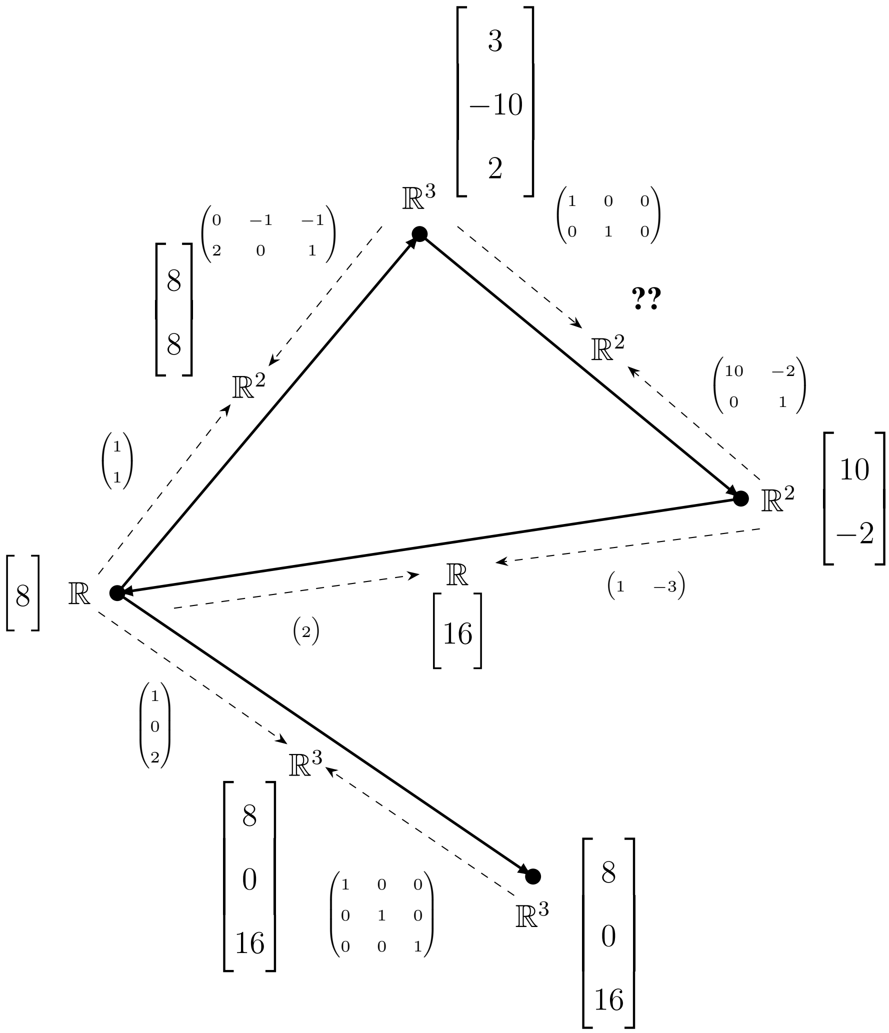


Figure 3.3: The failure of the local data to expand into global data.

Our choice of elements from before does not lead to a nontrivial global section of this more complex graph. That is, local data does not expand into global data. There is no element that satisfies both new restriction maps assigned to the edge we added back in. A proper global section for this graph was found using code and is shown in figure 3.4.

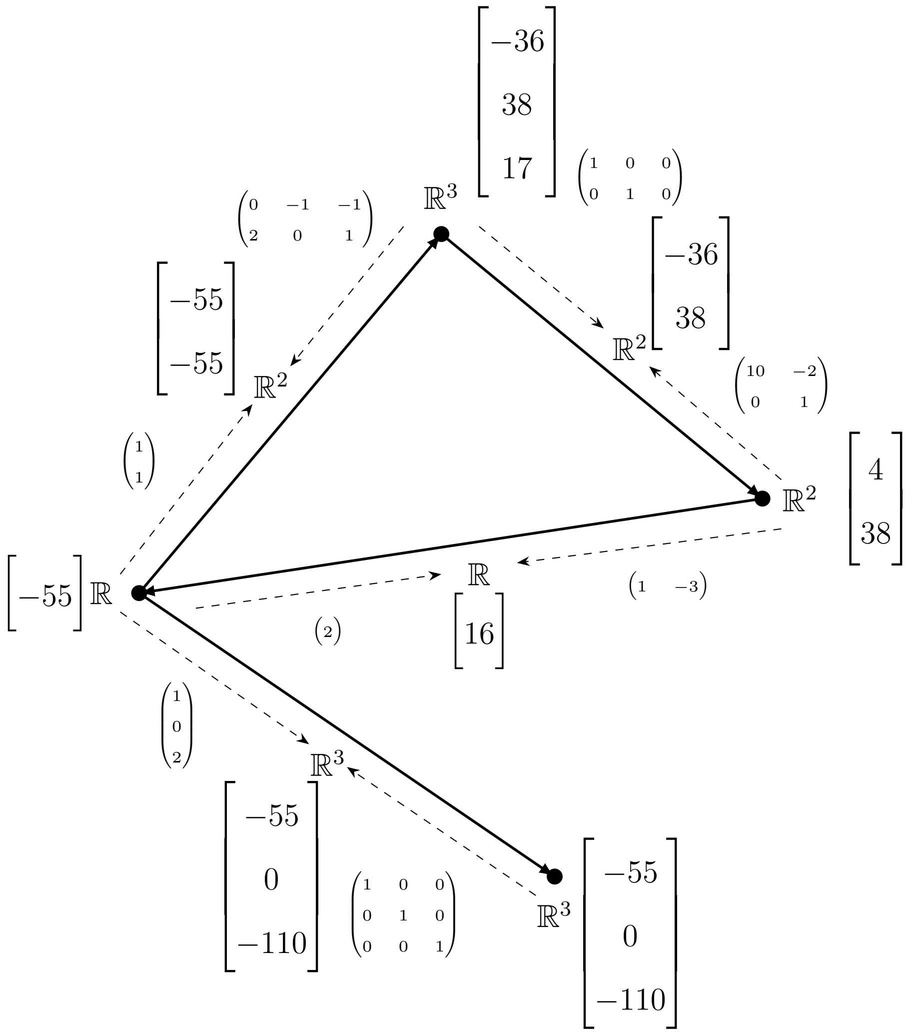


Figure 3.4: A global section of a graph with a cycle.

3.2 Cellular Sheaf Cohomology

Definition 14. *Given a sheaf $\mathcal{F}$ over a (compact) cell complex, the* ***cellular sheaf cohomology*** *is the cohomology of the cochain complex with groups*

$$C^i(X;\mathcal{F}) = \prod_{\dim \sigma = i} \mathcal{F}(\sigma)$$

and coboundary maps

$$\delta(\sigma) = \sum_{\sigma \trianglelefteq \tau} [\sigma : \tau] \mathcal{F}(\sigma \trianglelefteq \tau).$$

Note that δ *is naturally* $C^n \to C^{n+1}$ *since* $[\sigma : \tau] = 0$ *unless* σ *is one dimension lower than* τ *(*σ *is a face of* τ *with codimension 1).*

Example 4. *Continuing example 3,*

$$\delta = \left(\begin{array}{ccc|c|cc|ccc} 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 10 & -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & -1 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R} \oplus \mathbb{R}^3 \xrightarrow{\delta} \mathbb{R}^3 \oplus \mathbb{R} \oplus \mathbb{R}^2 \oplus \mathbb{R}^3$$

$$\operatorname{im}\delta = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ \frac{-832040}{930249} \\ \frac{416020}{930249} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{470832}{665857} \\ 0 \\ 0 \\ \frac{665857}{941664} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{477902}{799683} \\ \frac{-216361}{905103} \\ \frac{-246481}{515552} \\ \frac{-477902}{799683} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{12672}{137215} \\ \frac{12672}{137215} \\ \frac{162559}{880111} \\ \frac{-12672}{137215} \\ \frac{-540542}{836157} \\ \frac{270271}{836157} \\ 0 \\ \frac{540542}{836157} \end{bmatrix}, \begin{bmatrix} \frac{-345031}{946477} \\ \frac{-345031}{946477} \\ \frac{-690062}{946477} \\ \frac{345031}{946477} \\ \frac{-13940}{164151} \\ \frac{2040}{18239} \\ 0 \\ \frac{4080}{18239} \end{bmatrix}, \begin{bmatrix} \frac{-37527}{989888} \\ \frac{-37527}{989888} \\ \frac{-64007}{844189} \\ \frac{37527}{989888} \\ \frac{-698781}{921623} \\ \frac{-64007}{222155} \\ 0 \\ \frac{-128014}{222155} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{-832040}{930249} \\ 0 \\ \frac{416020}{930249} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\ker\delta = \operatorname{span}\left\{ \begin{bmatrix} -36 \\ 38 \\ 17 \\ -55 \\ 4 \\ 38 \\ -55 \\ 0 \\ -110 \end{bmatrix} \right\}$$

Thus $H^0 = \ker \delta = \mathbb{R}$ *and* $H^1 = \mathbb{R}^9/\mathbb{R}^8 = \mathbb{R}$.

Ghrist [4] asserts that the theory of sheaves is a structure for the "collation of data parametrized by a space," the ability to fuse local data into global data through restriction maps and the requirements on them. Later he says interpreting sheaves as a coefficient system for cohomology is "illustrative." We find it especially illustrative in cases when the different components of our sheaves represent a real life problem, such as embedding graphs.

Chapter 4

Constant Sheaf and its Cohomology

We can find the cohomology of a graph and the cohomology of a sheaf on a graph. The constant sheaf is a sheaf that bridges this divide - it has the choice of restriction maps and abelian groups that lead us to the ordinary cohomology of the space.

Definition 15. *For any vector space V, the* ***constant sheaf*** *valued in V is the sheaf where $\mathcal{F}(\sigma) = V$ for all σ and $\mathcal{F}(v_i \trianglelefteq e_i) = id$ for all $v_i \trianglelefteq e_i$.*

From the definition of cellular sheaf cohomology, we can see that if we choose $V = \mathbb{R}^d$, then $\mathcal{F}(\sigma \trianglelefteq \tau) = I_d$ and our sheaf coboundary map is nearly the regular coboundary map:

$$\begin{aligned}\delta(\sigma) &= \sum_{\sigma \trianglelefteq \tau} [\sigma : \tau] \mathcal{F}(\sigma \trianglelefteq \tau) \\ &= \sum_{\sigma \trianglelefteq \tau} [\sigma : \tau] I_d \\ &= (\partial^T \otimes I_d).\end{aligned}$$

The fact that the cohomology of the constant sheaf is the same as regular cohomology is a well known result. Ghrist [4] says this casually on page 184 of his book *Elementary Applied Topology*:

> "note that this (sheaf cohomology) aligns with the definition of cellular cohomology in the case of the constant sheaf."

When we compare the coboundary matrix with the sheaf coboundary matrix, we see that not much has changed, we have simply gone from $\partial^T \to \partial^T \otimes I_d$.

Example 5. *If we want to embed our graph G from Chapter 3 into $\mathbb{R}^3$, our sheaf coboundary map is*

$$(\partial^T \otimes I_3) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \otimes I_3.$$

If the position of our vertices is specified by

$$\operatorname{vec} X = \begin{pmatrix} -2 & 3 & 7 & 9 & -4 & 3 & -2 & -2 & 6 & -7 & 1 & 1 \end{pmatrix}^T,$$

then $(\partial^T \otimes I_3) \operatorname{vec} X$ *is computed by*

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \\ 7 \\ 9 \\ -4 \\ 3 \\ -2 \\ -2 \\ 6 \\ -7 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -11 \\ 7 \\ 10 \\ 0 \\ -5 \\ -1 \\ 11 \\ -2 \\ -3 \\ -16 \\ 5 \\ -2 \end{pmatrix}.$$

Note that the matrix $(\partial^T \otimes I_d)$ *subtracts the appropriate vertex positions component-wise to find the displacement vectors.*

Because we are dealing with $(\partial^T \otimes I_d)$*, we know the kernel and image will follow Lemma 1. Thus,*

$$\begin{aligned}
\ker(\partial^T \otimes I_3) &= \ker \partial^T \otimes \mathbb{R}^3 \\
\operatorname{im}(\partial^T \otimes I_3) &= \operatorname{im} \partial^T \otimes \mathbb{R}^3 \\
H^0 &= \mathbb{R} \otimes \mathbb{R}^3 \cong \mathbb{R}^3 \\
H^1 &= \mathbb{R} \otimes \mathbb{R}^3 \cong \mathbb{R}^3 \\
\dim H^0 &= 3 = 3 \cdot (1 \textit{ component}) \\
\dim H^1 &= 3 = 3 \cdot (1 \textit{ cycle}).
\end{aligned}$$

Up to this point, we have stated the results from [1, 2] in a different framework but have not introduced any new ideas. The next chapter introduces our new sheaf inspired by the constant sheaf's relation to ordinary cohomology.

Chapter 5

Constructing a New Sheaf

5.1 Defining

As a polymer model, we may want to restrict some vertices to be pinned to certain subspaces of our vector space. We construct a different sheaf related to such embeddings whose cochain structure captures our compatibility condition. We will call this sheaf the ***Regular Subspace Sheaf***. We use the word "regular" to emphasize that the subspaces must truly be subspaces and go through the origin, not just any surface, in anticipation of future work.

Definition 16. *The vector spaces of the **Regular Subspace Sheaf** assigned to each cell are as such:*

- *Vertices not restricted to a subspace are assigned $\mathbb{R}^d$ where d is the dimension of the general space we are embedding into. We call these unrestricted vertices **free vertices**.*
- ***Restricted vertices** are assigned $\mathbb{R}^{d_i}$ where d_i is the dimension of the subspace it lies in.*
- *Edges are assigned $\mathbb{R}^d$.*

The restriction maps of the Regular Subspace sheaf assigned are as such:

- *Restriction maps from a free vertex to an edge are assigned I_d.*
- *Restriction maps coming out of restricted vertex are assigned $Q_i : \mathbb{R}^{d_i} \to \mathbb{R}^d$ where Q_i's columns consist of orthonormal basis vectors for the subspace the vertex is restricted to.*

Then the sheaf coboundary map is $N : C^0 \to C^1$ to be $N = (\partial^T \otimes I_d)\,\mathrm{diag}(Q_1, \dots Q_{|V|})$.

The reason behind the definition of our map N lies in the difference between N and the earlier map $(\partial^T \otimes I_d)$ from the cochain complex and constant sheaf. In each of these linear maps, the ∂^T contributes to subtracting one vertex position from another. In our new sheaf, the vertex positions in C^0 are hidden under one more layer in the parametrization of the subspace they lie in; this is remedied by multiplying by the diagonal matrix full of the orthonormal bases needed to go from the subspace to the general space.

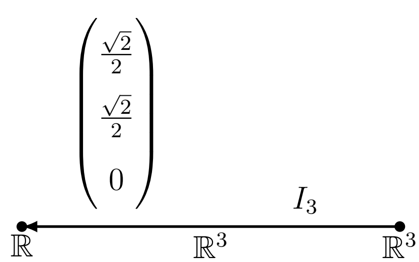
Looking at the structure of N, the output in general is,

$$(\partial^T \otimes I_d)\,\mathrm{diag}(Q_1, \dots Q_{|V|})\,\mathrm{vec}\, X = \begin{bmatrix} Q_i v_i - Q_j v_j \\ \vdots \end{bmatrix}$$

where $Q_i v_i$ and $Q_j v_j$ are the position vectors of the vertices since Q_i is the parametrization of the subspace v_i must lie in. So like before, N is finding edge displacements by subtracting vertex positions.

Back in Chapter 2, we went through the different statements of X and W being compatible. We had $X\partial = W$ where $X \in \mathrm{Mat}_{d\times|V|}$ and applied the vec functor to land on a map whose input was $\mathrm{vec}\, X$. For our map N, the input is $\mathrm{vec}\, X$ as well but the vector does not come from a true matrix since the dimension of the vertex positions need not be the same so each "column vector" would not necessarily have the same length, making an array that is not a matrix. To remedy this, we will notate the input of N as $\mathrm{vec}\, X = \mathrm{vec}\{x_1, \dots, x_{|E|}\}$ where each x_i is the parameterization of the position vector within its space or subspace. We see this used in the next example.

Example 6. *Observe the graph consisting of a single edge with coboundary matrix* $\partial^T = \begin{bmatrix} 1 & -1 \end{bmatrix}$. *Let the embedding space be* $\mathbb{R}^3$ *and let the first vertex be restricted to the subspace parametrized by* $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$.



Then,

$$N = (\begin{bmatrix} 1 & -1 \end{bmatrix} \otimes I_3) \operatorname{diag}(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, I_3) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \operatorname{diag}(\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, I_3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

If our restricted vertex is parametrized by $x_1 = [2]$ *and the free vertex is parametrized by* $x_2 = \begin{bmatrix} -3 & 1 & 2 \end{bmatrix}^T$, $\operatorname{vec} X = \{x_1, x_2\} = \begin{bmatrix} 2 & -3 & 1 & 2 \end{bmatrix}^T$. *Then,*

$$N \begin{bmatrix} 2 \\ -3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} + 3 \\ \sqrt{2} - 1 \\ -2 \end{bmatrix}.$$

We can verify that our resulting vector is indeed the edge between the two points. Our parametrization leads us to the coordinates $[\sqrt{2}, \sqrt{2}, 0]^T$ *for the first vertex and we subtract* $[-3, 1, 2]^T$ *to find the edge displacement.*

The way N was built is a very complex way to write down something simple, as a lot of math can be. For more complicated graphs, we rely on the organization the coboundary matrix provides to subtract the relevant vertices, like how this example knew which order to subtract the vertices. In the end, N is a fancy way of subtracting a bunch of vectors.

5.2 Properties of the Regular Subspace Sheaf

Our goal now is to study more of the coboundary map for our **Regular Subspace Sheaf**, parallel to the results from Cantarella et al. [2].

Theorem 1. $\operatorname{im} N = \{\operatorname{vec}\begin{bmatrix} w_1 & \dots & w_{|E|} \end{bmatrix}\} : \forall w_i,$ *adjacent vertices* $x_i \in \operatorname{im} Q_i\}$

Since N is focused on the natural way of defining an embedding once you have decided the vertex positions, $\operatorname{im} N$, while it's elements are $W \in C^1$, is defined by how the vertex embeddings behave.

Theorem 2. *The kernel of the Regular Subspace sheaf coboundary map consists of vector assignments where all vertices are assigned a vector* $v \in \bigcap Q_i$. *Otherwise said,*

$$\ker N = \{\operatorname{vec}\{x_i, \dots, x_{|E|}\} : x_i = x \in \bigcap Q_i \textit{ if } v_i \textit{ is free, and } x_i = Q_i^T x \textit{ if } v_i \textit{ is a restricted vertex}\}.$$

Proof. As we have seen from [4], the study of H^0 is a study of the global sections of the sheaf. We start by observing the restriction map coming out of a free vertex. Give this free vertex the value x. Since the associated restriction map is the identity, the value at the adjacent edges is $I_d x = x$ as well. Any adjacent vertex which is free must be assigned a vector u solving the equation $I_d u = x$; that is, $u = x$. For adjacent vertices which are restricted, their assigned vector u most solve $Q_i u = x$. If x is not in the image of Q_i, this has no solution. On the other hand, if $x \in \operatorname{im} Q_i$, then $Q_i^T x$ is the unique solution, since $Q_i Q_i^T x = x$. Thus x must be in the intersection of the Q_i. □

We expect the cohomology groups of the Regular Subspace Sheaf to still be indicative of the graph's structure since the input and output of N is the same as $(\partial^T \otimes I_d)$ by starting with vertex placements and naturally finding the edge displacements. Unpacking the meaning behind the cohomology groups of our Regular Subspace Sheaf is generally complex due to the different dimensions of the subspaces and involves several subtleties.

Example 7. *Continuing with Example 6 where*

$$N = \begin{bmatrix} \frac{\sqrt{2}}{2} & -1 & 0 & 0 \\ \frac{\sqrt{2}}{2} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \tag{5.1}$$

We can see that our last 3 columns, the columns relating to the 1st, 2nd, and 3rd dimensional component of our second vertex is a basis for $\mathbb{R}^3$ *and so the 1st column is dependent. Thus* $\operatorname{im} N = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ *and so* $H^1(N) = \mathbb{R}^4/\mathbb{R}^3 \cong \mathbb{R}$*. This gives us no insight unless we go deeper into the calculations. A similar situation happens with* $\ker N$*. Solving the system of linear equations, we find that* $\ker N = \operatorname{span}\left\{ \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ *and so* $H^0(N) = \mathbb{R}$*, which is only coincidentally the same as* $H^0(G, \mathbb{R})$*.*

Regarding the kernel, since our first vertex is restricted to the subspace parameterized by $\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$*, it is not surprising that our second vertex must have the same coordinates in the 1st and 2nd dimensions, making itself lie in* Q_1 *as expected from Theorem 2. It is also not surprising that the parametrization of our first vertex must be multiplied by* $\sqrt{2}$ *since*

$$\sqrt{2} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

5.3 Pseudoinverse of the Regular Subspace Sheaf

As with our previous map $(\partial^T \otimes I_d)$, the more interesting problem is making a proper embedding from a set of edge displacements. Like before, N^+ will give a pseudoinverse. We can indirectly study the image and kernel of N^+ with the following linear algebra facts:

Proposition 2.

$$\ker N^+ = (\operatorname{im} N)^\perp$$
$$\operatorname{im} N^+ = (\ker N)^\perp$$

Proposition 3. *With N^* denoting conjugate transpose of N,*

$$\ker N^+ = \ker N^*$$
$$\operatorname{im} N^+ = \operatorname{im} N^*$$

Using these results, the next theorem provides an understanding of how the kernel of the pseudoinverse behaves.

Theorem 3. *The kernel of N^+ consists of W such that each edge displacement in any particular cycle are equal (the edges not in a cycle get 0). Additionally, W such that for any vertex, the sum of adjacent edge displacements is perpendicular to Q_i. Algebraically,*

$$Q_i^T \left(\sum_{f=a(v_i)} w_f - \sum_{g=z(v_i)} w_g \right) = 0$$

Proof. Since our matrix has real components, Theorem 3 implies $\ker N^+ = \ker N^T$. Note

$$N^T = \operatorname{diag}(Q_i^T, \ldots, Q_{|V|}^T)(\partial \otimes I_d).$$

We are looking for $W \in C^1$ such that $N^T W = 0$. Since N^T is the product of two matrices, W has two opportunities to be in the kernel; If $W \in \ker(\partial \otimes I_d)$, then $W \in \ker N^T$. Since $\ker(\partial \otimes I_d) \cong \ker \partial \otimes \mathbb{R}^d$, we can observe known elements of $\ker \partial$. We know $\ker \partial$ is generated by the cycles in the graph. Viewing ∂ as a matrix, the input vector, say σ, must be that the entries corresponding to edges in a particular cycle all have the same value and edges not in a cycle are assigned 0. For our embedding, this means that all w_i corresponding to an edge in a particular cycle must be equal and all w_i not in a cycle must be the zero vector. Of course, these edge displacements will not form a proper embedding. In a sense, this is the “farthest away" an embedding can be from being a proper embedding.

If $W \notin \ker(\partial \otimes I_d)$ but $W \in \ker N^+$, then it must be that since

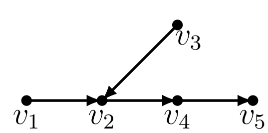
$$\begin{bmatrix} \sum_{f=a(v_1)} w_f - \sum_{g=z(v_1)} w_g \\ \vdots \\ \sum_{f=a(v_{|V|})} w_f - \sum_{g=z(v_{|V|})} w_g \end{bmatrix}$$

is the generic output of $(\partial \otimes I_d)$, $\forall v_i$,

$$Q_i^T \left(\sum_{f=a(v_i)} w_f - \sum_{g=z(v_i)} w_g \right) = 0.$$

Interpreting this geometrically, the sum of the vectors adjacent to a vertex v_i must be orthogonal to Q_i. □

Example 8. *This example is to illustrate a case where $W \in \ker N^T$ but $W \notin \ker(\partial \otimes I_d)$. Let the general space for our embedding be $\mathbb{R}^2$. We will work with this graph,*

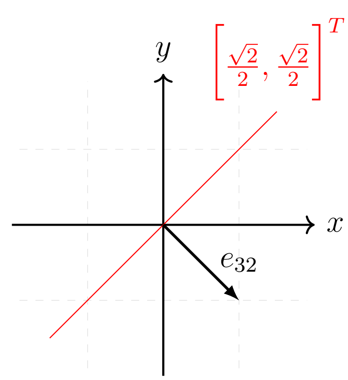


Let v_2, v_3 be restricted to the subspace parametrized by $\left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^T$. Thus

$$N^T = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using Python,

$$\ker N^T = \mathrm{span}\begin{pmatrix} 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 \end{pmatrix}^T$$



We can see that the sum of the vectors adjacent to v_2 (which is at the origin) is perpendicular to the subspace containing v_2.

Theorem 4. $\mathrm{im}\, N^+ = \{\mathrm{vec}(x_1, \ldots, x_{|E|}) : \mathrm{proj}_{\cap Q_i} \sum x_i = 0\}$

Proof. If $(x_1, \dots x_{|E|}) \in (\ker N)^\perp = \operatorname{im} N^+$,

$$\begin{aligned} &\operatorname{vec}(x_1, \dots, x_{|E|}) \cdot \operatorname{vec}(x_1, \dots, x_{|E|}) = 0 \quad \forall x \in \bigcap Q_i \\ \iff &x_1 x_1 + x_2 x_2 + \cdots + x_{|E|} x_{|E|} = 0 \quad \forall x \in \bigcap Q_i. \end{aligned}$$

As mentioned in the proof of Theorem 2, $x_i = v$ if the ith vertex is free and $x_i = Q_i^T v$ if the ith vertex is a restricted vertex. Continuing with our equivalences without loss of generality,

$$\begin{aligned} &\iff x_1 v + \cdots + x_{|E|} Q_{|E|}^T v = 0 \\ &\iff (x_1 + \cdots + x_{|E|} Q_{|E|}^T) v = 0. \end{aligned}$$

With an orthonormal basis $e_1, \dots, e_k$ for P,

$$\begin{aligned} \operatorname{proj}_{\cap Q_i}(x_1 + \cdots + x_{|E|} Q_{|E|}^T) &= \sum_{i=1}^{k} ((x_1 + \cdots + x_{|E|} Q_{|E|}^T) e_i) e_i \\ &= \sum_{i=1}^{k} 0 e_i \\ &= 0. \end{aligned}$$

Thus $\operatorname{im} N^+ = \{\operatorname{vec}(x_1, \dots, x_{|E|}) : \operatorname{proj}_{\cap Q_i} \sum x_i = 0\}$. □

Interpreting $\operatorname{im} N^+$ geometrically, reminiscent of the result on the center of mass in Lemma 2, the sum of the projections of all vertices onto $\cap Q_i$ is centered at the origin.

In conclusion, the analysis of the image and kernel of the pseudoinverse of the regular subspace sheaf has revealed insights into its structure. These findings not only deepen our understanding of the sheaf's behavior but also lay the groundwork for further exploration into its applications and potential future use.

Chapter 6

Future Work

Our sheaf handles restricting vertices to linear subspaces. More generally, it would be useful to handle restrictions to arbitrary affine subspaces—for example, different vertices restricted to parallel planes—whether this results in a completely new sheaf or a generalization of the one presented in this paper. With the sheaf described in this paper and any new ones found in the future, we hope to continue comparing and contrasting the results with the results from Cantarella et al. [2].

For example, having established a framework for working with random graph embeddings with vertices restricted to subspaces, the obvious next step is to explore the statistics of such embeddings, either by proving theorems about the distribution of quantities of physical significance like radius of gyration or by developing sampling algorithms to explore these distributions empirically.

The theory of sheaves is becoming more rich; we would also love to see if any more sheaf theory can help model physical phenomena occurring with polymers.

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