DISSERTATION

# TWO-STEP CODING THEOREM IN THE NEARLY CONTINUOUS CATEGORY

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## ABSTRACT

## TWO-STEP CODING THEOREM IN THE NEARLY CONTINUOUS CATEGORY

In measurable dynamics, one studies the measurable properties of dynamical systems. A recent surge of interest has been to study dynamical systems which have both a measurable and a topological structure. A nearly continuous $\mathbb{Z}$-system consists of a Polish space $X$ with a non-atomic Borel probability measure $\mu$ and an ergodic measure-preserving homeomorphism $T$ on $X$. Let $f : X \to \mathbb{R}$ be a positive, nearly continuous function bounded away from 0 and $\infty$. This gives rise to a flow built over $T$ under the function $f$ in the nearly continuous category. Rudolph proved a representation theorem in the 1970's, showing that any measurable flow, where the function $f$ is only assumed to be measure-preserving on a measurable $\mathbb{Z}$-system, can be represented as a flow built under a function where the ceiling function takes only two values. We show that Rudolph's theorem holds in the nearly continuous category.

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DEDICATION

Daniel J. Rudolph

Late advisor and mentor

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# 1. INTRODUCTION

In the 1940's, Ambrose and Kakutani[1][2] showed that any measure-preserving $\mathbb{R}$-action on a Lebesgue probability space is measurably isomorphic to a flow built under a function. Rudolph[16] simplified this representation further for ergodic $\mathbb{R}$-actions with the Two-Step Coding Theorem. He showed that given the Ambrose-Kakutani representation of an ergodic $\mathbb{R}$-action, and any irrationally related $p, q \in \mathbb{R}^+$, there exists a representation where the ceiling function only takes values $p$ and $q$. Hence any measure-preserving ergodic $\mathbb{R}$-action can be represented as a flow built under a function where the ceiling function only takes two values.

The goal of this paper is to prove the Two-Step Coding Theorem in the *nearly continuous category*, where along with the measure theoretic properties, we also study the topological properties of the underlying systems. Keane and Smorodinsky [8, 9] were the first to investigate the interplay of measure and topology, and Denker and Keane [5] formalized the category in what they called *almost topological dynamical systems*. There thas been a recent spurge of interest in this field by the works of Hamachi and Keane [7] and followed by works of Roychowdhury [15], [14] and Rudolph [12], del Junco and Şahin [4] and Del Junco, Rudolph and Weiss [3] repalacing the term *almost topological* with *nearly continuous*. A nearly continuous (n.c.) dynamical system consists of a Polish space, i.e, a separable and completely metrizable space, equipped with a Borel probability measure. The group action on this space is nearly continuous, i.e., measure-preserving and continuous on an invariant $G_\delta$ subset of full measure, with respect to the induced topology.

Informally, our result is to show that given any irrationally related $p, q \in \mathbb{R}^+$, any ergodic n.c. flow built under a function can be represented as a n.c. flow built under a function where the ceiling function only takes values $p$ and $q$.

Note here that we are not saying any n.c. $\mathbb{R}$-action, or even one that is ergodic, can be represented as a flow built under a function. We do not yet know if this statement is true. It is an open question whether the Ambrose-Kakutani result is true in the n.c. category.

In what follows, we give formal definitions and state our result precisely.

**Definition 1.1.** $(X, \mu, T)$ is called a *nearly continuous (n.c) $\mathbb{Z}$-system*, whenever $X$ is a Polish space with a non-atomic Borel probability measure $\mu$ and $T : X \to X$ is an ergodic measure-preserving homeomorphism.

**Definition 1.2.** A function $f : X \to \mathbb{R}^+$ is called *nearly continuous (n.c)* if there exists a $G_\delta$-subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that $f$ is continuous on $X_0$ in the induced topology.

To define the n.c. flow built over $T$ and under the function $f$ we first assume that $f : X \to \mathbb{R}$ is a n.c function and uniformly bounded away from $0$. We define a space $\tilde{X}$ to be the of all points that lie under the graph of $f$, i.e.,

$$\tilde{X} = \{\tilde{x} = (x, s) : x \in X, 0 \le s < f(x)\}$$

./ with the identification that every point $(x, f(x))$ on the graph of $f$ is identified with the point $(Tx, 0)$. The topology on $\tilde{X}$ is given by the product of the topology of $X$ and the usual topology of $\mathbb{R}$. We let $\tilde{\mu}$ denote the completed product measure of $\mu$ on $X$ and the Lebesgue measure on $\mathbb{R}$. Without loss of generality (by rescaling $f$ if necessary) we assume $\tilde{\mu}(X) = 1$.

It is easy to check from definitions that $\tilde{X}$ is Polish space with Borel probability measure $\tilde{\mu}$. The n.c. flow built over $T$ and under $f$ is an $\mathbb{R}$-action denoted by $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ and defined by

$$\mathcal{U}_t(x, s) = (T^n x, s + t - f(x, n)) \tag{1.1}$$

where $n \in \mathbb{Z}$ such that $f(x, n) \le s + t < f(x, n + 1)$ and where $f(x, n)$ is given by

$$f(x,n) = \begin{cases} \sum_{i=0}^{n-1} f(T^i x) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \sum_{i=n}^{-1} f(T^i x) & \text{if } n < 0 \end{cases} . \tag{1.2}$$

The idea is that every point $(x, s)$ in the space flows vertically up at unit speed until it reaches the graph of $f$. The point $(x, f(x))$ is identified with the point $(T(x), 0)$ in the base and the flow continues upward as seen in Figure 1.1.

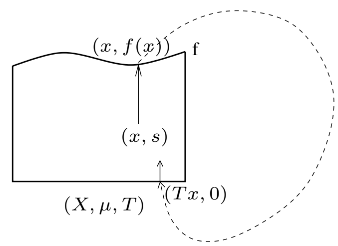


*Fig. 1.1:* Flow built over $T$ under $f$

**Definition 1.3.** Let $\tilde{X}$ be a Polish space with Borel probability measure $\tilde{\mu}$. An $\mathbb{R}$-action $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ on $\tilde{X}$ is called a *nearly continuous flow* if there exists a $G_\delta$-subset $X_0 \subseteq X$ with $\mu(X_0) = 1$ such that for each $t \in \mathbb{R}, \mathcal{U}_t$ is a measure-preserving homeomorphism of $X_0$ in the relative topology.

It is easy to check that $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$, as defined in (1.1), is an ergodic n.c. $\mathbb{R}$-action on $\tilde{X}$. Henceforth, we will always refer to $X$ as the discrete space and $\tilde{X}$ as the flow space.

**Definition 1.4.** Let $(\tilde{X}, \tilde{\mu})$ and $(\tilde{Y}, \tilde{\nu})$ be Polish probability spaces and let $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$, $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ be n.c. $\mathbb{R}$-actions on $\tilde{X}$ and $\tilde{Y}$ respectively. We say $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ is *nearly continuously conjugate* to $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ if there exist invariant $G_\delta$-subsets $\tilde{X}_0 \subseteq \tilde{X}$, $\tilde{Y}_0 \subseteq \tilde{Y}$ with $\tilde{\mu}(\tilde{X}_0) = \tilde{\nu}(\tilde{Y}_0) = 1$ and a measure-preserving homeomorphism $\phi : \tilde{X}_0 \to \tilde{Y}_0$ such that $\phi\mathcal{U}_t = \mathcal{V}_t\phi$ for all $t \in \mathbb{R}$.

The goal of this paper is to prove the following theorem:

**Theorem 1.5.** *Let $(X, \mu, T)$ be a $\mathbb{Z}$-system and let $f : \tilde{X} \to \mathbb{R}$ be a n.c. function such that there exist constants $c, c' \in \mathbb{R}$ satisfying $0 < c < f(x) < c' < \infty$ for all $x \in X$. Let $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ be an* *ergodic n.c. flow built over $T$ under $f$ and let $\alpha \in \mathbb{R}$ be any positive irrational. Then there exists a n.c. $\mathbb{Z}$-system $(Z, \nu, T_Z)$ and a n.c. function $g : Z \to \{1, 1+\alpha\}$, such that the n.c. flow built over $T_Z$ under $g$ is n.c. conjugate to $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$.*

The main idea in proving the two-step coding theorem in the measurable category is to be able to identify a measurable cross-section $Z$ of the flow space $\tilde{X}$ so that the orbit of a.e. point visits $Z$ precisely in time intervals 1 or $1+\alpha$. Rudolph achieved this by defining a map $\phi$ on $\tilde{X}$, so that for a.e. $\tilde{x} \in \tilde{X}$, $\phi(\tilde{x})$ is a tiling of $\mathbb{R}$ using only intervals of length $1$ and $1+\alpha$. The set of all points whose corresponding tiling has its origin located at the beginning of an interval, will form the set $Z$.

The map $\phi$ is the limit of inductively defined maps $\phi^i$, where each $\phi^i$ associates a partial orbit of a point to a partial tiling of $\mathbb{R}$. At stage $i+1$, the partial tilings are extended to cover longer partial orbits, in a manner that they match most of the previously defined partial tilings. As it is not possible to exactly fill in the gaps between the partial tilings with intervals of length 1 and $1+\alpha$, the definition of the partial tilings are changed in a measurable way and on a small measure set, with the measure going to zero, as the induction goes to infinity. Borel-Cantelli Lemma then says that the set of points where the definition of patches changes infinitely often is a set of measure zero. Hence a.e. point in $\tilde{X}$, has its orbit tiled by intervals of length 1 and $1+\alpha$.

We cannot mimic this argument in the n.c. category, as the Borel-Cantelli Lemma does not allow for any topological control on the set of points where there is convergence of the maps $\phi^i$, and thus there is no guarantee that the invariant set of full measure is a $G_\delta$, nor that the section $Z$ is one either. Instead, to control the topological structure of sets in our construction we use the *template machinery* of Rudolph et al., where the idea is to not define the maps $\phi^i$ explicitly, but rather at every stage, for a large set of points, define a set of choices for $\phi^i$ with the property that some subset of each set of choices will have the property that the gaps beteween them can be tiled as needed. In other words, the idea of the template machinery is to defer making a choice of tilings rather than to make adjustments to the choices that have already been made. This idea was introduced in [12] and used extensively in [13, 6] as the scaffolding necessary to control the topological structure of the constructions.

So far, in every example where the template machinery is used, there has been a natural way of defining towers in the underlying spaces. As $\tilde{X}$ does not have a natural tower structure, we need to set one up, which will let us use the template machinery. To do so, we first define a special sequence of tower decompositions of the discrete space $X$ in Chapter 2. These decompositions are a generalization of the *skeleta machinery* of Keane and Smorodinsky [8]. In Chapter 3 we use the special sequence and construct tower partitions of the flow space $\tilde{X}$. We give basic definitions of tilings in Chapter 4, and prove two lemmas that are crucial in the proof of our result.

The proof of Theorem 1.5 consists of two parts. In the first part we construct the maps $\phi^i$ that converge to a map $\phi$ from $\tilde{X}$ to the tiling space. In the second part we construct the $\mathbb{Z}$-system $(Z, \nu, T_Z)$ and the n.c. flow built over $T_Z$ and under a two-step function, and show that this flow is n.c. conjugate to $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$. In Chapter 5 we introduce the template machinery and define *stencils* and *templates*. We also give sufficient conditions that the stencils and the templates need to satisfy to guarantee the convergence of the maps $\phi^i$. In Chapter 6.3 we prove Theorem 1.5, where we first inductively construct the stencils and templates and then show that they satisfy the conditions from Chapter 5. In Chapter 6.3 we finish with the second part of the proof.

# 2. SKELETA DECOMPOSITION

Let $(X, \mu, T)$ be a n.c. $\mathbb{Z}$-system. We want to define a sequence of tower partitions of $X$ that satisfy certain specific properties. These tower partitions play the role of *skeleta machinery* of Keane and Smorodinsky[8, 10] in the proof of the finitary isomorphism theorem for Bernoulli shifts.

**Definition 2.1.** A *clopen tower* (or just *tower*) $\tau$ of height $h$ in $X$ is a sequence of pairwise disjoint clopen subsets $E_0, E_1, ..., E_{h-1}$ of $X$ such that $T^i(E_0) = E_i$ for $i = 0, 1, ..., h-1$. We call $E = E_0$ the base of the tower and the sets $E_i, i = 0, 1, ..., h-1$ the levels of the tower.

**Definition 2.2.** If $\tau$ is a tower of height $h$ with base $E$, then a sequence of clopen sets $C, TC, T^2C$, $..., T^{h-1}C$ is called a *column* of $\tau$ if $C \subset E$.

The special partitions referred to in the beginning of this chapter, will be denoted by $P^i, i \geq 0$ and will consist of countably many disjoint clopen towers, i.e., for all $i = 0, 1, 2, ...$, there exists a $J_i \subseteq Z$ such that

$$P^i = \{\tau^i_j : j \in J_i\}. \tag{2.1}$$

These partitions will be defined based on two parameters - an increasing sequence $\{N_i\}_{i=0}^{\infty}$ and an $L_0 \in \mathbb{N}$. The $N_i$'s play the role similar to that of *markers*, and $L_0$ plays a role similar to the length of *fillers* in the skeleta machinery.

**Definition 2.3.** Let $(X, \mu, T)$ be a n.c. $Z$-system, $\{N_i\}_{i=0}^{\infty}$ be a strictly increasing sequence of positive integers and $L_0 \in \mathbb{N}$. A sequence of tower partitions $\{P^i = \{\tau^i_j : j \in J_i\} : i = 0, 1, 2, ...\}$ of $X$ is called a *skeleta decomposition* of $(X, \mu, T)$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$ if there exists a clopen subset $A \subset X$ with $0 < \mu(A) < 1$ and the $\{P^i\}_{i=0}^{\infty}$ satisfy the following properties:

(i) for all $i = 0, 1, 2, ..$ and $j \in J_i$, any level of $\tau^i_j$ is either contained in $A$ or disjoint from $A$.

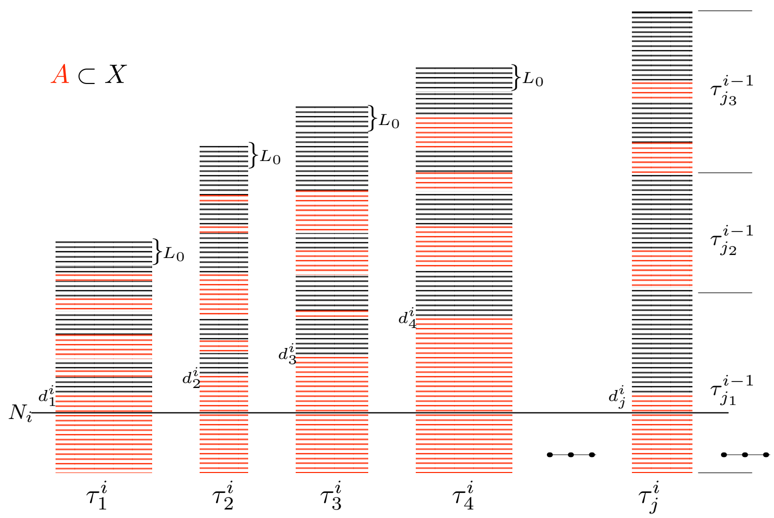


*Fig. 2.1:* Tower partition $P^i$ for $i \geq 1$

(ii) for all $i = 0, 1, 2, ..$ and $j \in J_i$, there exists $d_j^i \geq N_i$ such that

a. $\bigcup_{n=0}^{d_j^i-1} T^n(E_j^i) \subset A$

b. for any $m \in \{d_j^i, d_j^i + 1, ..., h_j^i - N_i\}$ we have $\bigcup_{n=m}^{m+N_i-1} T^n(E_j^i) \not\subset A$.

In other words, the initial $d_j^i$ levels are contained in $A$ and after that we do not see $N_i$ consecutive levels that are all in $A$. These initial levels $\bigcup_{n=0}^{d_j^i-1} T^n(E_j^i)$ form the *discrete collar* of $\tau_j^i$.

(iii) for all $i = 0, 1, 2, ..$ and $j \in J_i$, the top $L_0$ levels are disjoint from $A$ i.e., $\bigcup_{n=1}^{L_0} T^{h_j^i-n}(E_j^i) \cap A = \emptyset$.

(iv) for all $i \in \mathbb{N}$ and $j \in J_i$, the tower $\tau_j^i$ has a unique decomposition into columns from towers in $P^{i-1}$ i.e., there exists $k \in \mathbb{N}$ and $j_1, ..., j_k \in J_{i-1}$ such that

$$\tau_j^i = \bigcup_{m=1}^{k} \bigcup_{n=0}^{h_{j_m}^{i-1}-1} T^n C_m$$

where

a. $C_m \subset E_{j_m}^{i-1}$

b. $T^{h^{i-1}_{j_m}}(C_m) = C_{m+1}$ for all $m = 1, 2, ..., k-1$.

The sequence $\{\tau^{i-1}_{j_m} : j_m \in J_{i-1}, m = 1, 2, ..., k\} \subset P^{i-1}$, is called the *associated sequence of previous stage towers* for $\tau^i_j$, and for $m = 1, 2, ..., k$, the column $\cup_{n=0}^{h^{i-1}_{j_m}-1} T^n C_m$ is called the $m^{th}$ *sub-tower* of $\tau^i_j$.

Note that

$$d^i_j = d^{i-1}_{j_1} \quad \text{for all } i \geq 2. \tag{2.2}$$

Hence for a tower $\tau^i_j$, if there exists an $n > i$ such that $d^i_j \geq N_n$, then any column of $\tau^i_j$ that appears in a tower of $P^n$, can only appear within its bottom most sub-tower.

The following proposition lets us assume without loss of generality that every $(X, \mu, T)$ has a skeleta decomposition with respect to any given $\{N_i\}_{i=0}^{\infty}$ and $L_0 \in \mathbb{N}$.

**Proposition 2.4.** *Let $(X, \mu, T)$ be a n.c $\mathbb{Z}$-system. Given any increasing sequence $\{N_i\}_{i=0}^{\infty}$ and $L_0 \in \mathbb{N}$, there exists a $T$-invariant $G_\delta$-subset $X' \subseteq X$ with $\mu(X') = 1$ and a sequence of tower partitions $\{P^i = \{\tau^i_j : j \in J_i\}\}_{i=0}^{\infty}$ that give rise to a skeleta decomposition of $(X', \mu|_{X'}, T|_{X'})$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$.*

*Proof.* By [4], we know that there exists a $T$-invariant $G_\delta$-subset $X_1 \subseteq X$ with $\mu(\tilde{X}_1) = 1$ and $X_1$ has a countable base of clopen sets with respect to the induced topology. We first show that there exists a countably infinite clopen tower decomposition $Q = \{\tau_j : j \in J\}$ of the space $X_1$ such that if $E_j$ denotes the base and $h_j$ denotes the height of the tower $\tau_j$, then $h_j > N_0 + L_0$ and $h_j \to \infty$.

For each $n \in \mathbb{N}$, let $r_j = \frac{1}{2^j}$ so that $\sum_{j \in \mathbb{N}} r_j = 1$. By Lemma 2(b) in [4] applied to $X_1$ and $r_j$, there exist disjoint clopen subsets $B_j$ of $X_1$ such that $\mu(B_j) = r_j$. By Lemma 4(b) in [4] applied to these sets $B_j$ and $N_0 + L_0$, we then get clopen towers $\tau_j$ with height $j(N_0 + L_0)$. Let $J = \mathbb{N}$ and $Q = \{\tau_j : j \in J\}$.

Define sets $\Lambda_n \subset J$ based on the heights of the towers in $Q$, i.e., for $n \in \mathbb{N}$, define

$$\Lambda_n = \{j \in J : N_n + L_0 \leq h_j < N_{n+1} + L_0\}$$

We define the set $A$ to be the union of the bottom $N_n$ levels of each tower $\tau_j$, whenever $j \in \Lambda_n$, i.e.,

$$A = \bigcup_{n\in\mathbb{N}} \bigcup_{j\in\Lambda_n} \bigcup_{m=0}^{N_n-1} T^m(E_j).$$

Note that at least the top $L_0$ levels of any tower are disjoint from $A$, and therefore $0 < \mu|_{X_1}(A) < 1$. As $A$ and $A^c$ are both unions of clopen levels, they are both clopen in the topology of $X_1$. Also as $\{\tau_j : j \in J\}$ is a tower partition of $X_1$, any point $x \in X_1$ satisfies the following conditions:

C1 both the forward and the backward orbit of $x$ visits the set $A^c$ infinitely often.

C2 both the forward and the backward orbit of $x$ see at least $N_0$ consecutive occurrences of the set $A$.

To define the sequence of partitions $\{P^i\}_{i=0}^{\infty}$ that give rise to a skeleta decomposition with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$, we will need condition C2 to be true for all $N_i$, i=0,1,..., i.e.,

C2' for all $i = 0, 1, ...$, both the forward and the backward orbit of $x$ sees $N_i$ consecutive occurrences of the set $A$.

For this, we will need to restrict the space $X_1$ further as follows: Let $B_1$ denote the set of all points in $X_1$ that never see $N_i$, $i = 1, 2, ...$, consecutive occurrences of the set $A$ in either their forward or backward orbits, i.e.,

$$\begin{aligned} B_1 &= \bigcup_{i\in\mathbb{N}} \{x \in X_1 : \forall n \geq 0,\ \exists\, 1 \leq m < N_i \text{ such that } T^{m+n} \notin A\} \\ &\quad \cup \bigcup_{i\in\mathbb{N}} \{x \in X_1 : \forall n \geq 0,\ \exists\, 1 \leq m < N_i \text{ such that } T^{-(m+n)} \notin A\} \\ &= \bigcup_{i\in\mathbb{N}} \bigcap_{n\geq 0} \bigcup_{m=1}^{N_i-1} T^{-(n+m)}(A^c) \cup \bigcup_{i\in\mathbb{N}} \bigcap_{n\geq 0} \bigcup_{m=1}^{N_i-1} T^{n+m}(A^c) \end{aligned}$$

and let

$$B_2 = \bigcup_{n \in \mathbb{Z}} T^n B_1.$$

Note that $B_2$ is a $T$-invariant subset of $X_1$ and hence by ergodicity of $T$ has measure 0. Let $X' = X_1 \setminus B_2$. Then $X'$ is a $T$-invariant $G_\delta$-subset of $X_1$ (and hence of $X$), with $\mu(X') = 1$ and satisfying conditions C1 and C2' for all $x \in X'$.

We will use induction to construct the sequence of partitions $\{P^i\}_{i=0}^{\infty}$. For $i = 0$, let $J_0 = J$ and $P^0 = \{\tau_j^0 = \tau_j : j \in J_0\}$. For each $j \in J_0$, define $d_j^0 = N_n$, whenever $j \in \Lambda_n$.

By construction of $A$ we know that for all $j \in J_0$, each level of $\tau_j^0$ is either contained in $A$ or disjoint from $A$, the bottom most $d_j^0 \geq N_0$ levels are all contained in $A$, and the top $h_j^0 - d_j^0 \geq L_0$ levels are all disjoint from $A$. Hence $P^0$ satisfies the conditions for skeleta decomposition for $N_0$ and $L_0$.

Now suppose that for some $i$, there exists $J_i \subseteq \mathbb{Z}$ such that $P^i = \{\tau_j^i : j \in J_i\}$ satisfies all conditions in the definition of skeleta decomposition of $X'$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$. Then every tower $\tau_j^i, j \in J_i$ has its first $d_j^i \geq N_i$ levels all contained in the set $A$. Let $K \subset J_i$ so that $j \in K \iff d_j^i \geq N_{i+1}$.

Fix a $j \in K$. We want to partition the base $E_j^i$ of the tower $\tau_j^i$ in a very specific way so that the orbits of all points in a partition set visit the same towers $\tau_{j_1}, \tau_{j_2}, ..., \tau_{j_{m-1}}$, with $j_1, j_2, ..., j_{m-1} \in K^c$, before first returning to a tower $\tau_{j_m}^i, j_m \in K$. For each $m \in \mathbb{N}, j_1, j_2, ..., j_{m-1} \in K^c$, and $j_m \in K$, define

$$\begin{aligned} E_j^i(j_0 = j, j_1, ..., j_m) &= \{x \in E_j^i : T^{h_{j_0}^i} x \in E_{j_1}^i, ..., T^{\sum_{k=0}^{m-1} h_{j_k}^i} x \in E_{j_m}\} \\ &= E_j^i \cap T^{-h_{j_0}^i}(E_{j_1}^i) \cap \cdots \cap T^{-\sum_{k=0}^{m-1} h_{j_k}^i}(E_{j_m}). \end{aligned}$$

As the $E_{j_k}^i, k = 0, ..., m$ are all clopen and $T$ is a homeomorphism, the set $E_j^i(j_0, ..., j_m)$ is clopen in $X'$. Let $Q^i_j = \{E^i_j(j_0, ..., j_m) \neq \emptyset : m \in \mathbb{N}, j_0 = j, j_1, ..., j_{m-1} \in K^c, j_m \in K\}$. It is straight forward to check that it is indeed a partition of $E^i_j$. For each $E^i_j(j_0, ..., j_m) \in Q^i_j$, let $\tau^i_j(j_0, ..., j_m)$ be the tower with base $E^i_j(j_0, ..., j_m)$ and height $\sum_{k=0}^{m-1} h^i_{j_k}$. Observe that for $n = 1, ..., m-1$, we have

$$T^{\sum_{k=0}^{n-1} h^i_{j_k}} E^i_j(j_0, ..., j_m) \subset E_{j_n}.$$

Therefore $\tau^i_j(j_0, ..., j_m)$ is made by stacking columns of towers $\tau^i_{j_0}, ..., \tau^i_{j_{m-1}}$ in $P^i$. Define

$$P^{i+1} = \{\tau^i_j(j_0, ..., j_m) : j \in K, E^i_j(j_0, ..., j_m) \in Q^i_j\}.$$

To see that $P^{i+1}$ is a tower partition of $X'$, let $x \in X'$ and first suppose that $x \in \tau^i_j$ for some $j \in K$, then $T^{-n}x \in E^i_j$ for some $0 \leq n < h^i_j$. Hence $x$ belongs to $T^n E^i_j(j_0, ..., j_m) \subset \tau^i_j(j_0, ..., j_m)$ for some tower in $P^{i+1}$.

Now suppose $x \in \tau^i_j$ for some $j \notin K$. As the orbit of $x$ sees $N_{i+1}$ occurrences of $A$ in both its forward and backward orbits, there exist $n_1, n_2$ (least) in $\mathbb{N}$, and $j', j'' \in K$ such that $T^{-n_1}x \in E^i_{j'}$ and $T^{n_2}x \in E^i_{j''}$. This implies that there exists $m \geq 1$, $j_1, ..., j_{m-1} \in K^c$ such that $x \in \tau^i_j(j_0 = j', j_1, ..., j_{m-1}, j_m = j'') \in P^{i+1}$.

All that remains to show is that $P^{i+1}$ satisfies the properties of skeleta decomposition for $X'$. For convenience, let $J_{i+1}$ enumerate the towers in $P^{i+1}$ and rename the towers in $P^{i+1}$ as

$$P^{i+1} = \{\tau^{i+1}_j : j \in J_{i+1}\},$$

where each tower $\tau^{i+1}_j = \tau^i_{j_0}(j_0, ..., j_m)$ for some $j_0 \in K$, has base $E^{i+1}_j = E^i_{j_0}(j_0, ..., j_m)$ and height $h^{i+1}_j = \sum_{k=0}^{m-1} h^i_{j_k}$.

Every level of $\tau^{i+1}_j$ is a subset of a level from some $\tau^i_{j_k}, k = 0, ..., m-1$, and therefore is either contained in $A$ or disjoint from A. The tower $\tau^i_{j_0}(j_0, ..., j_m)$ is made up of columns of towers $\tau^i_{j_k}$, $m = k, ..., m-1$ and hence we can define the sequence of previous stage towers for $\tau^{i+1}_j$ to be $\{\tau^i_{j_k} : k = 0, ..., m-1\}$. Clearly $d^{i+1}_j = d^i_{j_0} \geq N_{i+1}$ as $j_0 \in K$. Also, the top $L_0$ levels of $\tau^{i+1}_j$ are subsets of the top $L_0$ levels of $\tau^i_{j_m}$, and therefore are disjoint from $A$.

By induction, the sequence $\{P^i\}_{i=0}^{\infty}$ satisfies the properties of skeleta decomposition for $X'$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$.

$\square$

# 3. TOWERS IN THE FLOW SPACE

Recall that $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ is the n.c. flow built over $T$ under $f$ on the flow space $\tilde{X}$. Given a tower $\tau$ in the discrete space $X$ with base $E$ and height $h$ we define a corresponding tower $\tilde{\tau}$ in $\tilde{X}$ with respect to $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$. Informally, we obtain $\tilde{\tau}$ by "filling in" the flow times to get from one level to the other in $X$. More formally, let $\tilde{E}_0 = E \times \{0\} \subset \tilde{X}$ and define

$$\tilde{\tau} = \bigcup_{(x,0)\in\tilde{E}} \bigcup_{0\le t<f(x,h)} \mathcal{U}_t(x,0).$$

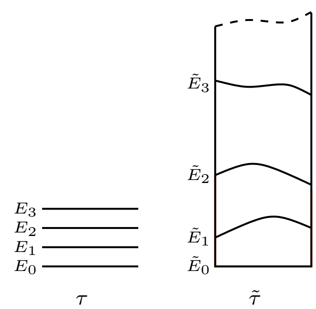


*Fig. 3.1:* Towers in $\tilde{X}$

Note that the levels $E_n$ of $\tau$ now correspond to levels $\tilde{E}_n = E_n \times \{0\}$ in $\tilde{\tau}$. The time it takes for a point $(x,0) \in \tilde{E}$ to flow to the set $\tilde{E}_n$ is $f(x,n)$ and the time any point in $\tilde{E}$ spends in the tower $\tilde{\tau}$ is $f(x,h)$, as defined in (1.2). As the function $f$ is not constant, these times vary for different points in $\tilde{E}$. Therefore it is not possible to define *the* height for the tower $\tilde{\tau}$. For future use, we will want to define the towers in $\tilde{X}$ in such a manner that the variation in the time it takes for two points in the base $\tilde{E} = E \times \{0\}$ to reach certain levels $\tilde{E}_n$, is controlled. We will do so by imposing specific conditions on the skeleta decomposition of space $X$.

Suppose $\{P^i\}_{i=0}^{\infty}$ is a skeleta decomposition of $X$. We define a corresponding sequence of tower partitions in $\tilde{X}$ by $\{\tilde{P}^i\}$ where

$$\tilde{P}^i = \{\tilde{\tau}^i_j : j \in J_i\} \text{ for } i = 0, 1, 2, ... \tag{3.1}$$

and for each $j \in J_i$, $\tilde{\tau}^i_j$ is the tower that corresponds to the tower $\tau^i_j$ in $X$. Suppose $\{\epsilon_i\}_{i\in\mathbb{N}}$ is any sequence decreasing to 0. We want to assume, without loss of generality, that for any $i \in N$ and $j \in J_i$, if $\tilde{\tau}^i_j$ is a tower in $\tilde{P}^i$ with base $\tilde{E}^i_j$ and its associated sequence of previous stage towers $\{\tilde{\tau}^{i-1}_{j_m} : j_m \in J_{i-1}, m = 1, 2, ..., k\}$, then the time it takes for any two points in the base $\tilde{E}^i_j$ to flow to the base of the $m^{th}$ sub-tower i.e., to $\tilde{E}^{i-1}_{j_m}$ for $m = 1, 2, ..., k$, is within $\epsilon_i$ of each other. The following proposition lets us assume so:

**Proposition 3.1.** *Let $(X, \mu, T)$ be a n.c. $\mathbb{Z}$-system and $f : X \to \mathbb{R}$ be a n.c. function such that there exist constants $c, c' \in \mathbb{R}$ satisfying $0 < c < f(x) < c' < \infty$ for all $x \in X$. Let $L_0 \in \mathbb{N}$, $\{N_i\}_{i=0}^\infty$ be an increasing sequence and $\{\epsilon_i\}_{i\in\mathbb{N}}$ be a sequence with $\epsilon_i$ decreasing to $0$. Then there exists a $T$-invariant $G_\delta$-subset $X' \subseteq X$ with $\mu(X') = 1$ and a skeleta decomposition $\{P^i = \{\tau^i_j : j \in J_i\}\}_{i=0}^\infty$ of $(X', \mu|_{X'}, T|_{X'})$ with respect to $\{N_i\}_{i=0}^\infty$ and $L_0$, such that the $P^i$ satisfy the following properties:*

*a. for any tower $\tau^0_j \in P^0$, and for all $x, y \in E^0_j$ we have*

$$|f(x, h^0_j) - f(y, h^0_j)| < \epsilon_0. \tag{3.2}$$

*b. for any tower $\tau^i_j \in P^i$, $i \in \mathbb{N}, j \in J_i$, with its associated sequence of previous stage towers $\{\tau^{i-1}_{j_m} : j_m \in J_{i-1}, m = 1, 2, ..., k\}$, and for all $x, y \in E^i_j$ and $m = 2, ..., k$, we have*

$$\left| f(x, h^{i-1}_{j_1} + \cdots h^{i-1}_{j_{m-1}}) - f(y, h^{i-1}_{j_1} + \cdots h^{i-1}_{j_{m-1}}) \right| < \frac{\epsilon_i}{2}. \tag{3.3}$$

*Proof.* By Proposition 2.1 in [3], there exists a $T$-invariant $G_\delta$-subset $X_1 \subset X$ with $\mu(X_1) = 1$, such that $f$ is continuous on $X_1$. By Proposition 2.4 applied to $X_1$, $\{N_i\}_{i=0}^\infty$ and $L_0$, we get a $T$-invariant $G_\delta$-subset of $X_2 \subset X_1$ with $\mu|_{X_1}(X_2) = \mu(X_2) = 1$ and a skeleta decomposition

$\bar{P}^i = \{\tau^i_j : j \in J_i\}, i = 0, 1, 2, ...\ X_2$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$.

For each $i = 0, 1, ...$ and each $\tau^i_j \in \bar{P}^i$, we will define a partition $Q^i_j = \{E^i_{j,n}\}$ of the base $E^i_j$, so that all points $x, y \in E^i_{j,n}$ satisfy (3.2) and (3.3). We will then restrict to an invariant $G_\delta$ subset $X' \subset X_2$, so that each $E^i_{j,n}$ is clopen in $X_2$ with respect to the induced topology. Then letting $\tau^i_{j,n}$ be the tower with base $E^i_{j,n}$ and height $h^i_j$, we get a sequence of refined tower partitions $P^i = \{\tau^i_{j,n} : E^i_{j,n} \in Q^i_j, j \in J_i\}$. Since we only partitioned individual towers in $\bar{P}^i$ to get the towers in $P^i$, the sequence $\{P^i\}_{i=0}^{\infty}$ will be a skeleta decomposition of $X'$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$ and in addition, will also satisfy (3.2) and (3.3).

Suppose $j \in J_0$ and $\tau^0_j \in \bar{P}^0$. Let

$$a = \inf_{x \in E^0_j} \{f(x, h^0_j)\}, \quad b = \sup_{x \in E^0_j} \{f(x, h^0_j)\}.$$

If $a + \epsilon_0/2 > b$, then (3.2) is true for all $x, y \in E^0_j$. Define $Q^0_j = \{E^0_j\}$ and $B^0_j = \emptyset$. If $a + \epsilon_0/2 \leq b$, then choose $r \in \mathbb{N}$ so that $a + r\epsilon_0/2 > b$ and define $Q^0_j = \{E^0_{j,n} : n = 1, ..., r\}$ where

$$\begin{aligned} E^0_{j,n} &= \{x \in E^0_j : f(x, h^0_j) \in (a + (n-1)\epsilon_0/2, a + n\epsilon_0/2)\} n = 1, ..., r\} \\ &= E^0_j \cap \left( \sum_{m=0}^{h^0_j - 1} f \circ T^m \right)^{-1} ((a + (n-1)\epsilon_0/2, a + n\epsilon_0/2)) . \end{aligned}$$

Since $f : X_2 \to \mathbb{R}$ and $T : X_2 \to X_2$ are both continuous and $E^0_j$ is clopen, each $E^0_{j,n}$ is open in $X_2$. Also, for each $x, y \in E^0_{j,n}$, (3.2) is now true. Define

$$B^0_j = \{x \in E^0_j : f(x, h^0_j) = a + (n-1)\epsilon_0/2, n = 1, ..., r\},$$

and note that $B^0_j$ is a closed set of measure $0$ in $X_2$.

Now let $i \geq 1$, $j \in J_i$ and $\tau^i_j$ be the tower from $\bar{P}^i$ with its associated sequence of previous stage towers $\{\tau^{i-1}_{j_m} : m = 1, 2, ..., k\}$. If $m = 1$, then $E^i_j$ trivially satisfies (3.3). In this case, let $Q^i_j = \{E - j^i\}$ and $B^i_j = \emptyset$.

If $m \geq 2$, then for each $x \in X_2$ and $m = 2, 3, ..., k$, let

$$F_m(x) = f(x, h^{i-1}_{j_1} + \cdots h^{i-1}_{j_{m-1}}).$$

As $f : X_2 \to \mathbb{R}$ and $T : X_2 \to X_2$ are both continuous, so is $F_m$. Note that for all $x \in E^i_j$, $F_m(x)$ is precisely the time it takes $x$ to reach the base of the $m^{th}$ sub-tower. For each $m = 2, ..., k$, define

$$e_m = \inf_{x \in E^i_j} \{F_m(x)\}, \quad d_m = \sup_{x \in E^i_j} \{F_m(x)\}.$$

If $e_m + \epsilon_i/2 > d_m$, then define $Q_m = \{E^i_j\}$. If $e_m + \epsilon_i/2 \leq d_m$, then choose $r_m \in \mathbb{N}$ so that $e_m + r_m\epsilon_i/2 > d_m$ and define $Q_m = \{E_{m,n} : n = 1, ..., r_m\}$ where

$$\begin{aligned} E_{m,n} &= \{x \in E^i_j : F_m \in (e_m + (n-1)\epsilon_i/2, e_m + n\epsilon_i/2)\} n = 1, ..., r_m\} \\ &= E^i_j \cap (F_m)^{-1} \left((e_m + (n-1)\epsilon_i/2, e_m + n\epsilon_i/2)\right). \end{aligned}$$

Since $F_m$ is continuous and $E^0_j$ is clopen, each $E^0_{j,n}$ is open in $X_2$. Let $Q^i_j = \vee^k_{m=2} Q_m$ and note that every $E \in Q^i_j$ is open in $X_2$, and every $x, y \in E$ satisfy (3.3). Let

$$B^i_j = \bigcup_{m=2}^{k} \{E^i_j \cap F^{-1}_m(\{e_m + (n-1)\epsilon_i/2\}) : n = 1, ..., r_m\},$$

so that $B^i_j$ is a closed set of measure in $X_2$. Finally let

$$X' = X_2 \setminus \bigcup_{n \in \mathbb{Z}} T^n(\cup^\infty_{i=0} \cup_{j \in J_i} B^i_j).$$

Then $X'$ is a $T$-invariant $G_\delta$-subset of $X$ with $\mu|_{X_2}(X') = \mu(X') = 1$, and restricted to $X'$, each

$Q^i_j$ is a clopen partition of $E^i_j$. For each $E^i_{j,n} \in Q^i_j$, let $\tau^i_{j,n}$ be the column of $\tau^i_j$, with base $E^i_{j,n}$ and height $h^i_j$. Let $P^i = \{\tau^i_{j,n} : E^i_{j,n} \in Q^i_j, j \in J_i\}$. It is easy to check that $\{P^i\}_{i=0}^{\infty}$ forms a sequence of clopen tower partitions of $X'$ and that the sequence $\{P^i\}_{i=0}^{\infty}$ is a skeleta decomposition of $X'$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$ satisfying (3.2) and (3.3) as desired.

$\square$

# 4. TILINGS

In this chapter, we introduce some definitions and notations about tilings which we use in this paper, and refer the reader to [11] for standard definitions and details. We also introduce two lemmas that will be used heavily in the proof of Theorem 1.5.

Any closed interval of $\mathbb{R}$ is called a *tile*. Let $\alpha \in \mathbb{R}$ be a fixed positive irrational. We will only consider tiles that have length 1 or $1+\alpha$, i.e., all tiles will be of the form $[b, b+1]$ or $[b, b+1+\alpha]$ for some $b \in \mathbb{R}$. If $[a, b]$ is a tile, call the location $a \in \mathbb{R}$ the *base point of the tile*, and $b \in \mathbb{R}$ the *end point of the tile*.

A *tiling* $\Gamma$ of $\mathbb{R}$ is a collection of tiles such that any two tiles have pairwise disjoint interiors and their union covers $\mathbb{R}$. Let $Y$ denote the space of all tilings of $\mathbb{R}$ by tiles of length $1$ and $1+\alpha$. For any $\Gamma \in Y$ and $t \in \mathbb{R}$, define the translation of $\Gamma$ by $t$, denoted by $S_t(\Gamma)$, to be the tiling obtained by shifting each tile of $\Gamma$ to the left by $t$ i.e.,

$$S_t(\Gamma) = \{D - t : D \in \Gamma\}.$$

A *patch* $\omega$ is a finite subset of a tiling $\Gamma \in Y$, such that the union of tiles in $\omega$ is connected. This union is called the *support of* $\omega$ and written $supp(\omega)$. If $\omega_1$ and $\omega_2$ are patches in $Y$ and $\omega_1 \subset \omega_2$ then $\omega_1$ is a called a *sub-patch* of $\omega_2$.

Note that if $t \in \mathbb{R}$ and $\omega$ is any patch in $Y$, then

$$supp(S_t\omega) = supp(\{D - t : D \in \omega\}) = supp(\omega) - t. \tag{4.1}$$

The topology on the tiling space $Y$ is based on the idea that two tilings are close if after a small translation they agree on a large interval around the origin. Let $\Gamma_1, \Gamma_2 \in Y$. The tiling metric $d$ is defined by

$$d(\Gamma_1, \Gamma_2) = \inf\{\{\frac{1}{\sqrt{2}}\} \cup \{0 < r < \frac{1}{\sqrt{2}} : \exists \text{ patches } \omega_i \in \Gamma_i, i = 1, 2, \text{ covering } (-\frac{1}{r}, \frac{1}{r})$$

$$\text{and } t \in (-r, r) \text{ such that } S_t\omega_1 = \omega_2\}\}$$

Given a patch $\omega$ with support $K \subset \mathbb{R}$ and $\epsilon > 0$, the cylinder set given by $\omega$ and $\epsilon$, denoted by $C(\omega, \epsilon)$, is defined by

$$C(\omega, \epsilon) = \{\Gamma \in Y : \exists \ t \in (-\epsilon, \epsilon) \text{ so that } S_t\Gamma = \omega \text{ on } K\}.$$

The cylinder sets are open, and form the basis for the topology on $Y$. By [11], $Y$ is compact and the translation $\{S_t\}_{t\in\mathbb{R}}$ is a continuous $\mathbb{R}$-action on $Y$. In this paper we will use special patches, called *grid patches*, which consist of a union of successive translates of the patch$\{[0, 1], [1, 2+\alpha]\}$.

**Definition 4.1.** A patch $\omega \in Y$ with $supp(\omega) = [a, b]$ is called a *grid patch* whenever $b = a + n(2 + \alpha)$ for some $n \in \mathbb{N}$ and $\omega$ is a concatenation of the patches $S_{-a-m(2+\alpha)}\{[0, 1], [1, 2 + \alpha]\}$ for $m = 0, 1, ..., n - 1$.

The following lemma from [16], will play an important role in our proof:

**Lemma 4.2.** *If $\alpha \in \mathbb{R}$ is irrational, then given any $\epsilon > 0$, there exists an $M \in \mathbb{N}$, such that for any $\gamma \in (-2 - \alpha, 2 + \alpha)$, there exist $u, v \in \mathbb{Z}$ with $|u| + |v| < M$ and*

$$|u + v(1 + \alpha) - \gamma| < \epsilon.$$

We will also need the following lemmas in the proof of Theorem 1.5. Suppose we have a $\gamma \in \mathbb{R}$ with $|\gamma| < 2 + \alpha$ and a patch in the tiling space, consisting of three sub-patches, the first and the third being grid patches. Suppose we want to rearrange the tiles in such a way that the middle patch remains the same, but is shifted by a distance which is approximately $\gamma$. Lemma 4.3 says that there is a way to do this, provided the grid patches have appropriate lengths. Lemma 4.4 is a one-sided version of Lemma 4.3. These lemmas are also the key lemmas used by Rudolph in [16], to prove the measurable version of Theorem 1.5.

**Lemma 4.3.** *Given any $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $\gamma \in (-2-\alpha, 2+\alpha)$, if $\omega$ is a patch in $Y$ with supp$(\omega) = [a,b]$ for some $a, b \in \mathbb{R}$ and there exist $p, q \in \mathbb{R}$, $a < p < q < b$ such that $\omega|_{[a,p]}$ and $\omega|_{[q,b]}$ are grid patches with*

$$p - a, b - q \geq M(2+\alpha) \tag{4.2}$$

*then there exists a patch $\omega' \in Y$ with supp$(\omega') = [a,b]$ such that $\omega|_{[p,q]} = S_t\omega'_{[p+t,q+t]}$ for some $|\gamma - t| < \epsilon$.*

*Proof.* Let $\epsilon > 0$. Then by Lemma 4.2, there exists $M \in \mathbb{N}$ such that for any $\gamma \in (-2-\alpha, 2+\alpha)$, there exist $u, v \in \mathbb{Z}$ with $|u| + |v| < M$ and $|u + v(1+\alpha) - \gamma| < \epsilon$. Let $t = u + v(1+\alpha)$, and let $\omega$ be a patch satisfying the hypothesis. Let $\omega_1 = \omega|_{[a,p]}$, $\omega_2 = \omega|_{[p,q]}$ and $\omega_3 = \omega|_{[q,b]}$.

Suppose $u > 0$ and $v < 0$. Equation (4.2) guarantees that we have at least $M$ tiles each of length 1 and $1+\alpha$ in the patches $\omega_1$ and $\omega_3$. As $|u|, |v| < M$, we can interchange $|u|$ tiles of length 1 in $\omega_3$ with $|v|$ tiles of length $1+\alpha$ from $\omega_1$. This will shift $\omega_2$ to the left by a distance of $|u| - |v|(1+\alpha) = u + v(1+\alpha) = t$.

Call this modified patch $\omega'$ and note that $\omega_2$ appears in $\omega'$ at location $p+t$, as desired. The other cases can be argued in a similar manner.

□

**Lemma 4.4.** *Given any $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that for all $\gamma \in (-2-\alpha, 2+\alpha)$, if $\omega$ is a patch in $Y$ with supp$(\omega) = [a,b]$ for some $a, b \in \mathbb{R}$ and there exists $p \in \mathbb{R}$, $a < p < b$*

*such that* $\omega|_{[a,p]}$ *is a grid patch with* $p - a \geq M(2+\alpha)$*, then there exists a patch* $\omega' \in Y$ *with* $supp(\omega') = [a, b+t]$ *such that* $\omega|_{[p,b]} = S_t\omega'_{[p+t,b+t]}$ *for some* $|\gamma - t| < \epsilon$*.*

*Proof.* Let $\omega$ be a patch in $Y$ satisfying the hypothesis. Let $\theta$ be the concatenation of $\omega$ and a grid patch with support $[b, b+M(2+\alpha)]$, where $M$ is obtained by applying Lemma 4.2 to $\epsilon$.

Apply Lemma 4.3 to $\theta$ to get a patch $\theta'$ and $t \in \mathbb{R}$, $|t-\gamma| < \epsilon$. Let $\omega'$ be the restriction of $\theta'$ to the interval $[a, b+t]$.

□

# 5. STENCILS AND TEMPLATES

In this chapter, we introduce the template machinery to define the maps $\phi^i$, as discussed in Chapter 1. Recall that $(X, \mu, T)$ is a $\mathbb{Z}$-system and $f$ is a n.c function on $X$. The n.c. flow built over $T_Z$ under $f$ is given by $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$. We want to define maps $\phi^i$ so that they converge to give a near conjugacy $\phi$ between $\tilde{X}$ and the tiling space $Y$. The maps $\phi^i$ will be defined as point-to set maps, using the towers from $\{\tilde{P}^i\}$, corresponding to a skeleta decomposition $\{P^i\}$ of the discrete space $X$. To every tower $\tilde{\tau}^i_j \in \tilde{P}^i$, we will associate a patch $\lambda^i_j$, and for points $\tilde{x} \in \tilde{\tau}^i_j$, we will define $\phi^i(\tilde{x})$ to be a cylinder set given by $\lambda^i_j$. To understand this better, we define *stencils* and *templates*, and ask the reader to refer to Figure 5.1 for a geometrical interpretation.

**Definition 5.1.** Let $\tau$ be a tower in $X$ with base $E$ and height $h$. Let $\tilde{\tau}$ be the corresponding tower in $\tilde{X}$ with base $\tilde{E}$. An $\epsilon$-*stencil* for $\tilde{\tau}$ is a 3-tuple $(\lambda, G, \phi)$ where $\lambda$ is a patch in the tiling space $Y$ with $supp(\lambda) \subsetneq [0, \inf_{(x,0)\in\tilde{E}} f(x,h)]$, $G \subset \tilde{\tau}$ is of the form $G = \cup_{s\in(p,q)}\mathcal{U}_s(\tilde{E})$ with $(p,q) \subset supp(\lambda)$ and $\phi$ is a point-to-set map defined on $G$ by $\phi(\tilde{x}) = C(S_s\lambda, \epsilon)$ whenever $\tilde{x} \in \mathcal{U}_s(\tilde{E})$.

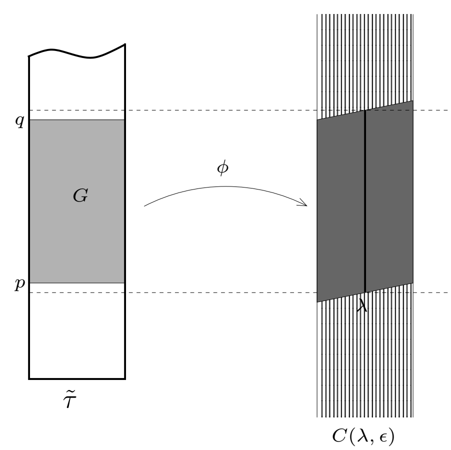


*Fig. 5.1:* $\epsilon$-stencil $(\lambda, G, \phi)$ for a tower $\tilde{\tau}$

**Definition 5.2.** Let $P$ be a tower partition of $(X, \mu, T)$ and $\tilde{P}$ denote the corresponding towers in $\tilde{X}$. Let $(\lambda_j, G_j, \phi_j)$ be $\epsilon$-stencils for towers $\tilde{\tau}_j \in P, j \in J$. A *template for* $\tilde{P}$ is a 2-tuple $(G, \phi)$ where $G = \cup_{j\in J}G_j$ and $\phi$ is a point-to-set map defined on $G$ by $\phi(\tilde{x}) = \phi_j(\tilde{x})$ whenever $\tilde{x} \in G_j$.

To define the map $\phi$, we will first define $\epsilon_i$-stencils $(\lambda_j^i, G_j^i, \phi_j^i)$ for each tower in $\tilde{\tau}_j^i \in \tilde{P}^i$. Using these stencils, we will then define templates $(G^i, \phi^i)$ for each $\tilde{P}^i$. The following proposition gives sufficient conditions to guarantee that the template maps $\phi^i$ converge to a near conjugacy $\phi : \tilde{X} \to Y$, and also that the cross-section $Z$ consisting of all points whose corresponding tiling has its origin located at the base point of a tile, is indeed a $G_\delta$-subset of $\tilde{X}$.

**Proposition 5.3.** *Let $(X, \mu, T)$ be a n.c. $\mathbb{Z}$-action, $f : X \to \mathbb{R}^+$ be continuous and $\{\epsilon_i\}_{i\in\mathbb{N}}$ be a decreasing sequence of reals with $\epsilon_i$ decreasing to 0. Let Y denote the space of all tilings of $\mathbb{R}$ by tiles of length 1 and $1 + \alpha$. Let $\{P^i\}_{i\in\mathbb{N}}$ be a sequence of tower partitions of $X$, with $P^i = \{\tau_j^i : j \in J_i\}, i = 1, 2, ...$ Let $(\tilde{X}, \mu, \{\mathcal{U}_t^f\}_{t\in\mathbb{R}})$ denote the n.c. flow built under the function and let $\tilde{P}^i = \{\tilde{\tau}_j^i : j \in J_i\}, i = 1, 2, ...$ denote the corresponding tower partitions of $\tilde{X}$. Suppose for each $i \geq 1$ and $j \in J_i$, there exist $\epsilon_i$-stencils $(\lambda_j^i, G_j^i, \phi_j^i)$ for $\tilde{\tau}_j^i$ and their corresponding templates $(G^i, \phi^i)$ for $\tilde{P}^i$ satisfying:*

*(a) for any $\tilde{x} \in \tilde{X}$ and $t_1, t_2 \in \mathbb{R}$, there exists an $i \in \mathbb{N}$ and $j \in J_i$ such that the partial orbit $\cup_{s\in[t_1,t_2]}\mathcal{U}_s\tilde{x}$ is contained in $G_j^i$.*

*(b) $G^1 \subset G^2 \subset ...$, and*

*(c) for any $i \in \mathbb{N}$ and $\tilde{x} \in G^i$, $\phi^i(x) \supset \phi^{i+1}(\tilde{x})$.*

*Then there exists a map $\phi : \tilde{X} \to Y$ such that for all $\tilde{x} \in \tilde{X}$ and $t \in \mathbb{R}$ we have*

$$\phi(\mathcal{U}_t\tilde{x}) = S_t(\phi\tilde{x}).$$

*Furthermore, for each $i \in \mathbb{N}$ and $j \in J_i$, if $G_j^i = \cup_{s\in(p_j^i,q_j^i)}\mathcal{U}_s\tilde{E}_j^1$ with $(p_j^i, q_j^i) \subset supp(\lambda_j^i)$, then let*

$$Z_j^i = \{\bigcup_{\eta\in(p_j^i+2\epsilon_i,q_j^i-2\epsilon_i)}\ \bigcup_{|\eta-s|<\epsilon_i} \mathcal{U}_s\tilde{E}_j^i : \eta \text{ is the basepoint of a tile in } \lambda_j^i\}$$

*and $Z^i = \cup_{j\in J_i} Z_j^i$. Suppose for each $i \in \mathbb{N}$, the sets $Z^i$ satisfy*

*(d) $G^i \cap \overline{Z^{i+1}} \subset Z^i$ and*

*(e)* $Z^1 \cap Z^i \neq \emptyset$.

*Then the set of all points $\tilde{x} \in \tilde{X}$ such that the origin is located at the base point of a tile in $\phi(\tilde{x})$, forms a non-empty $G_\delta$-subset of $\tilde{X}$.*

*Proof:* For any $\tilde{x} \in \tilde{X}$, by (a) and (b), there exists an $n(\tilde{x}) \in \mathbb{N}$ such that $\tilde{x} \in G^i$ for all $i \geq n(\tilde{x})$. Hence, using (c), we get that the sequence $\{\phi^i(\tilde{x})\}_{i \geq n(\tilde{x})}$ forms a decreasing sequence of nested cylinder sets. Since each $\phi^i(\tilde{x}) = \phi^i_{j(i)}(\tilde{x}) = C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i)$ for some $j(i) \in J_i$ and $t(i) \in \mathbb{R}$, such that $\tilde{x} \in \mathcal{U}_{t(i)}\tilde{E}^i_{j(i)}$, and $Y$ is compact, we have

$$\bigcap_{i \geq n(\tilde{x})} \phi^i(\tilde{x}) = \bigcap_{i \geq n(\tilde{x})} C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i) \supset \bigcap_{i \geq n(\tilde{x})} \overline{C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i/2)} \neq \emptyset, \tag{5.1}$$

as $Y$ is compact. By (a) and (b), there also exists a strictly increasing sequence $\{i_k\}_{k \geq 1}$ such that $i_1 \geq n(\tilde{x})$ and $\cup_{s \in [-k,k]}\mathcal{U}_s\tilde{x} \in G^{i_k}_{j(i_k)}$. As $\tilde{x}$ is at height $t(i_k)$ in the tower $\tilde{\tau}^{i_k}_{j(i_k)}$, we then have $[t(i_k) - k, t(i_k) + k] \subset (p^{i_k}_{j(i_k)}, q^{i_k}_{j(i_k)})$ which by definition is a subset of $supp(\lambda^{i_k}_{j(i_k)})$. This implies that $[-k, k] \subset supp(\lambda^{i_k}_{j(i_k)}) - t(i_k) = supp(S_{t(i_k)}\lambda^{i_k}_{j(i_k)})$. Therefore

$$\lim_{k \to \infty} supp(S_{t(i_k)}\lambda^{i_k}_{t(i_k)}) = \mathbb{R},$$

which in turn implies that

$$\lim_{i \to \infty} supp(S_{t(i)}\lambda^{i}_{t(i)}) = \mathbb{R}.$$

Using this and the fact that $\epsilon_i \to 0$ as $i \to \infty$, we get

$$\bigcap_{i \geq n(\tilde{x})} \phi^i(\tilde{x}) = \bigcap_{i \geq n(\tilde{x})} C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i) \tag{5.2}$$

is at most a singleton. By (5.1) and (5.2), the map $\phi : \tilde{X} \to Y$ defined by

$$\phi(\tilde{x}) = \bigcap_{i \geq n(\tilde{x})} \phi^i(\tilde{x})$$

is then well-defined on $\tilde{X}$. Also note that for any $n \geq n(\tilde{x})$, we have $\phi(\tilde{x}) \in \cap_{i\geq n}\phi^i(\tilde{x})$ and by the same arguments as in (5.1) and (5.2), is a singleton. Therefore

$$\phi(\tilde{x}) = \cap_{i\geq n}\phi^i(\tilde{x}) \quad \text{for any } n \geq n(\tilde{x}). \tag{5.3}$$

Now suppose that $\tilde{x} \in \tilde{X}$ and $t \in \mathbb{R}$. We will show that $\phi(\mathcal{U}_t\tilde{x}) = S_t(\phi\tilde{x})$. By (a) and (b), there exists an $n \geq \max\{n(\tilde{x}), n(\mathcal{U}_t\tilde{x})\}$ such that $\tilde{x}, \mathcal{U}_t\tilde{x} \in G^i_{j(i)}$ for all $i \geq n, j(i) \in J_i$. Then $\phi^i(\tilde{x}) = \phi^i_{j(i)}(\tilde{x}) = C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i)$, where $t(i) \in \mathbb{R}$ such that $\tilde{x} \in \mathcal{U}_{t(i)}\tilde{E}^i_{j(i)}$. This implies that $\mathcal{U}_t\tilde{x} \in \mathcal{U}_{t+t(i)}\tilde{E}^i_{j(i)}$ and hence

$$\phi^i(\mathcal{U}_t\tilde{x}) = \phi^i_{j(i)}(\mathcal{U}_t\tilde{x}) = C(S_{t+t(i)}\lambda^i_{j(i)}, \epsilon_i) = S_tC(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i) = S_t\phi^i(\tilde{x}).$$

Using (5.3), we then have

$$\phi(\mathcal{U}_t\tilde{x}) = \bigcap_{i\geq n}\phi^i(\mathcal{U}_t\tilde{x}) = \bigcap_{i\geq n}S_t\phi^i(x) = S_t\bigcap_{i\geq n}\phi^i(x) = S_t\phi(\tilde{x}).$$

Let $Z$ denote the set of all points $\tilde{x} \in \tilde{X}$ such that the origin origin is located at the base point of a tile in $\phi(\tilde{x})$. To show $Z$ is $G_\delta$, first note that $E^i_j$ is clopen in $X$ for any $i \in N$ and $j \in J_i$, and therefore $\cup_{|s-\eta|<\epsilon_i}\mathcal{U}_s\tilde{E}$ is open in $\tilde{X}$. As a result all $Z^i_j$, and hence $Z^i$, are open subsets of $\tilde{X}$. We will show that $Z = \cap_{n\geq 1}\cup_{i\geq n} Z^i$, and hence will be a $G_\delta$-subset of $\tilde{X}$.

Suppose $\tilde{x} \in Z$. By definition, there exists $n(\tilde{x}) \in \mathbb{N}$ such that $\phi(\tilde{x}) = \cap_{i\geq n(\tilde{x})}\phi^i(\tilde{x})$. Hence for each $i \geq n(\tilde{x})$, there exists a $j(i) \in J_i$ and $t(i) \in \mathbb{R}$ such that $\tilde{x} \in \mathcal{U}_{t(i)}\tilde{E}^i_{j(i)} \subset G^i_{j(i)}$ and $\phi^i(\tilde{x}) = \phi^i_{j(i)}(\tilde{x}) = C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i)$. Since $\tilde{x} \in Z$, the origin origin is located at the base point of a tile in $\phi(\tilde{x})$. As $\phi(\tilde{x}) \in C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i)$ there exists $\eta(i) \in (t(i) - \epsilon_i, t(i) + \epsilon_i)$ such that $\eta$ is the base point of a tile. This implies $\tilde{x} \in \mathcal{U}_{t(i)}\tilde{E}^i_{j(i)}$ and $|t(i) - \eta(i)| < \epsilon_i$, and therefore $\tilde{x} \in Z^i_{j(i)} \subset Z^i$. Hence $\tilde{x} \in \cap_{i\geq n(\tilde{x})}Z^i \subset \cap_{n\geq 1}\cup_{i\geq n} Z^i$.

Now suppose $\tilde{x} \in \cap_{n\geq 1} \cup_{i\geq n} Z^i$. Then there exists a strictly increasing sequence $\{n_k\}$ such that $\tilde{x} \in Z^{n_k}$ for all $k \geq 1$. Note that for any $i \in \mathbb{N}$, and $j \in J_i$, $E^i_j$ is clopen in $X$, and therefore

$$\overline{Z^i} = \bigcup_{j\in J_i} \bigcup_{\eta\in(p^i_j+2\epsilon_i, q^i_j-2\epsilon_i)} \bigcup_{|s-\eta|\leq\epsilon_i} \mathcal{U}_s\tilde{E} \subsetneq G^i.$$

Hence for each $k \geq 1$, we have $\overline{Z^{n_k}} \subset G^{n_k}$. Now $\tilde{x} \in Z^{n_k} \cap Z^{n_{k+1}}$ and $G^{n_k} \subset G^{n_k+r}$ for all $r \geq 1$, implies $\tilde{x} \in G^{n_{k+1}-1} \cap Z^{n_{k+1}}$. Using (d) we then get $\tilde{x} \in Z^{n_{k+1}-1}$. By the same argument, using (d) repeatedly, we get $\tilde{x} \in Z^{n_{k+1}-2}, ..., Z^{n_1+1}$. This is true for all $k \geq 1$ and therefore $\tilde{x} \in Z^i$ for all $i \geq n_1$. Hence for all $i \geq n_1$, there exists $j(i) \in J_i$ and $\eta(i) \in \mathbb{R}$ such that $\eta(i)$ is the base point of a tile in $\lambda^i_{j(i)}$ and $\tilde{x} \in \mathcal{U}_{t(i)}\tilde{E}^i_j$ for some $|t(i) - \eta(i)| < \epsilon_i$, and

$$\phi^i(\tilde{x}) = \phi^i_{j(i)}(\tilde{x}) = C(S_{t(i)}\lambda^i_{j(i)}, \epsilon_i) \subset C(S_{\eta(i)}\lambda^i_{j(i)}, 2\epsilon_i).$$

In other words, for every tiling in $\phi^i(\tilde{x})$, the origin is located within $2\epsilon_i$ of the base point of a tile. Since $\epsilon_i \to 0$ as $i \to \infty$ and $\phi(\tilde{x}) \in \phi^i(\tilde{x})$, we see that the origin is located at the base point of a tile in $\phi(\tilde{x})$. Therefore $\tilde{x} \in Z$. All that remains to show now is that $Z \neq \emptyset$. For each $i \geq 1$, let $K^i = Z^1 \cap Z^i$. By (e) $K^i \neq \emptyset$. and $K^{i+1} = Z^1 \cap Z^{i+1} = \cap^{i+1}_{n=1} Z^n \supset Z^1 \cap Z^i = K^i$. Therefore $\{\overline{K^i}\}$ is a nested decreasing sequence of closed sets in $\tilde{X}$ and have a non-empty intersection as $\tilde{X}$ is complete.

Using (d) and the fact that $\overline{Z^1} \subset G^1 \subset G^i$, we get

$$\overline{K^{i+1}} \subset \overline{Z^1} \cap \overline{Z^{i+1}} \subset G^i \cap \overline{Z^{i+1}} \subset Z^i.$$

Therefore

$$Z = \bigcup_{n\geq 1} \bigcap_{i\geq n} Z^i \supset \bigcup_{n\geq 1} \bigcap_{i\geq n} \overline{K^{i+1}} \neq \emptyset.$$

□

# 6. PROOF OF THEOREM 1.5

We are now ready to give a proof of Theorem 1.5. We will first define the sequences $\{N_i\}_{i=0}^{\infty}$, $\{\epsilon_i\}_{i=1}^{\infty}$ and an $L_0 \in \mathbb{N}$, and then define a skeleta partition of $\{\tilde{P}^i\}_{i=0}^{\infty}$ of $\tilde{X}$ with respect to $\{N_i\}_{i=0}^{\infty}$ and $L_0$. Then in Section 6.1, we will do an inductive construction using the sequence $\{\tilde{P}^i\}_{i=0}^{\infty}$, so that at the end of stage $i$, we would have associated to each tower $\tilde{\tau}_j^i \in \tilde{P}_i^i$, an $\epsilon_i$-stencil $(\lambda_j^i, G_j^i, \phi_j^i)$. In Section 6.2, we will show that for all $i \geq 1, j \in J_i$, the $\epsilon_i$-stencils $(\lambda_j^i, G_j^i, \phi_j^i)$ and their corresponding templates $(G^i, \phi^i)$ satisfy the hypothesis of Proposition 5.3. In the last Section, we will define the $\mathbb{Z}$-system $(Z, \nu, T_Z)$ and the function $g : Z \to \{1, 1+\alpha\}$, and conclude the proof of Theorem 1.5 by showing that the n.c. flow built over $T_Z$ under $g$ is n.c. conjugate to the flow $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ on $\tilde{X}$.

*Proof of Theorem 1.5:* Let $(X, \mu, T)$ be a n.c. $\mathbb{Z}$-system and $f : \tilde{X} \to \mathbb{R}$ be a n.c. function such that there exist constants $c, c' \in \mathbb{R}$ satisfying $0 < c < f(x) < c' < \infty$ for all $x \in X$. Let $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ be the n.c. flow built over $T$ under $f$, and let $\alpha \in \mathbb{R}$ be a positive irrational. Define $L_0 \in \mathbb{N}$ such that

$$cL_0 > 4(2+\alpha). \tag{6.1}$$

Define a sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ so that

$$\sum_{i=0}^{\infty} \epsilon_i < \frac{1}{3} \tag{6.2}$$

and

$$\epsilon_{i+1} < \frac{\epsilon_i}{4}, \quad \text{for all } i \in \mathbb{N}. \tag{6.3}$$

Apply Lemma 4.2 to the sequence $\{\epsilon_i\}_{i\in\mathbb{N}}$ to get $\{M_i\}_{i\in\mathbb{N}}$ with the property that for all $i \in \mathbb{N}$ and $\gamma \in (-2-\alpha, 2+\alpha)$, there exist $u_i, v_i \in \mathbb{Z}$ such that

$$|u_i| + |v_i| < M_i \quad \text{and} \quad |u_i + v_i(1+\alpha) - \gamma| < \epsilon_{i+1}. \tag{6.4}$$

Without loss of generality, we can assume

$$M_{i+1} > M_i + 2, \text{ for all } i = 0, 1, 2, ... \tag{6.5}$$

Define the sequence $\{N_i\}_{i=0}^{\infty}$ so that for all $i \in \mathbb{N}$, we have $N_i > N_{i-1}$ and

$$cN_i \geq 6\left(1 + \sum_{n=0}^{i} M_n\right)(2+\alpha). \tag{6.6}$$

Apply Proposition 3.1 to $\{\mathbb{N}_i\}_{i=0}^{\infty}$, $L_0 \in \mathbb{N}$ and $\{\epsilon_i\}_{i\in\mathbb{N}}$ to obtain a $T$-invariant $G_\delta$-subset $X' \subset X$ with $\mu(X) = 1$ so that there exists a clopen set $A \subset X'$ and a skeleta decomposition $\{P^i\}_{i=0}^{\infty}$ of $X'$ satisfying equations (3.2) and (3.3). In the end, we will show that the flow $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ restricted to the flow space $\tilde{X}'$ with the $\mathbb{Z}$-system $(X', \mu|_{X'}, T|_{X'})$ as its base, is continuously conjugate to the two-step flow built under the function $g$. Therefore, without loss of generality, we will assume that $(X, \mu, T)$ itself is such that there exists a clopen set $A \subset X$ and the sequence of tower partitions $\{P^i\}_{i=0}^{\infty}$ is a skeleta decomposition of $X$ satisfying equations (3.2) and (3.3).

Let $\{\tilde{P}^i\}_{i=0}^{\infty}$ to be the sequence of tower partitions of the flow space $\tilde{X}$, corresponding to the skeleta decomposition of the discrete space $X$.

We want to define a *flow collar* for each tower $\tilde{\tau}_j^i \in \tilde{P}^i$. The purpose for doing so is to leave ourselves enough room to be able to construct $\epsilon_{i+1}$-stencils at stage $i+1$, from the $\epsilon_i$-stencils at stage $i$. The height of the flow collar of a tower will help control the size of the gaps between the supports of patches in the stencils at any given stage. We will use Lemmas 4.3 and 4.4 heavily to build the patches of the stencils. Recall from those lemmas

that to be able to shift a patch $\omega$ by a distance within $\epsilon_i$ of a $\gamma \in (-2-\alpha, 2+\alpha)$, we need to be able to concatenate $\omega$ on both sides by grid patches of lengths at least $M_{i-1}(2+\alpha)$, and hence the gaps between shorter patches needs to be at least $2M_{i-1}(2+\alpha)$. If we want to shift a patch independently by a distance within $\epsilon_n$ of $\gamma_n \in (-2-\alpha, 2+\alpha)$, for all $n = 0, 1, ..., i$, we will need the gaps to be at least $2\sum_{k=0}^{i-1} M_k(2+\alpha)$ long. For technical reasons, we demand that the gaps, and hence the heights of the flow collar of towers, be at least $6\sum_{k=0}^{i-1} M_k(2+\alpha)$.

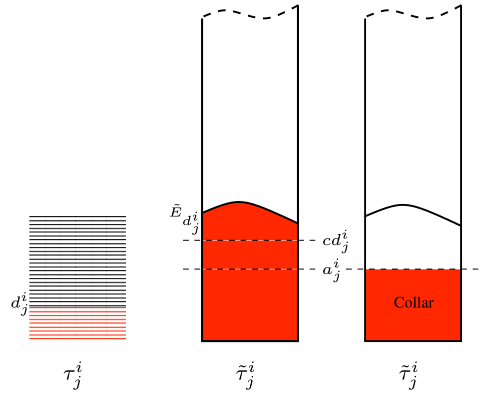


*Fig. 6.1:* Collar of $\tilde{\tau}_j^i$

Also, recall that for $i \in \mathbb{N}, j \in J_i$, $d_j^i$ is the number of levels in the discrete collar of the corresponding tower $\tau_j^i$ in $X$. We want to define the height of the flow collar of $\tilde{\tau}_j^i$ in such a manner that if $\tilde{x}$ belongs to the flow collar of $\tilde{\tau}_j^i$ and has the form $\tilde{x} = (x, 0)$ for some $x \in X$, then $x$ should belong to the discrete collar of $\tau_j^i$ in $X$. This means that we want the height of the flow collar of $\tilde{\tau}_j^i$ to be at most $cd_j^i$, as for any $x \in X$, we have $f(x) \geq c$.

Therefore we want to define the height of the flow collar of $\tilde{\tau}_j^i$ such that it is in between $6\sum_{k=0}^{i-1} M_k(2+\alpha)$ and $cd_j^i$. Let $a_j^i$ denote the height of the flow collar of $\tilde{\tau}_j^i$ and define

$$a_j^i = \left(1 + 6\sum_{k=0}^{n} M_k\right)(2+\alpha) \tag{6.7}$$

where $n \geq i$ is such that $N_n \leq d_j^i < N_{n+1}$. By (6.6) and the fact that $N_n \leq d_j^i$, we get

$$6\sum_{k=0}^{i-1} M_k(2+\alpha) \leq a_j^i \leq cN_n < cd_j^i. \tag{6.8}$$

as desired. For ease of notation, let

$$r_j^i = 1 + \sum_{k=1}^{n} M_k, \tag{6.9}$$

and rewrite

$$a_j^i = 6r_j^i(2+\alpha). \tag{6.10}$$

Also, for $i \geq 1$, as $d_j^i = d_{j_1}^{i-1}$, the number of levels in the discrete collar of $\tau_j^i$ in $X$ is the same as the number of levels in the discrete collar of its first sub-tower. This gives us the same relationship for the heights of the flow collars

$$a_j^i = a_{j_1}^{i-1} \quad \text{for all } i \geq 1, \tag{6.11}$$

i.e., the height of the flow collar of any tower $\tilde{\tau}_j^i$ is the same as that of its first sub-tower. Also recall from the proof of Proposition 3.1, that if $\{\tilde{\tau}_{j_m}^{i-1} : m = 1, ..., k\}$ is the associated sequence of previous stage towers for $\tilde{\tau}_j^i \in \tilde{P}^i$, then $F_m(x)$ is the time it takes for a point $(x,0) \in \tilde{E}_j^i$ to reach the base of the $m^{th}$ sub-tower and

$$e_m = \inf_{(x,0)\in \tilde{E}_j^i} \{F_m(x)\}. \tag{6.12}$$

By (3.3), we know that for any $\tilde{x} = (x,0) \in \tilde{E}_j^i$,

$$|e_m - F_m(\tilde{x})| \leq \frac{\epsilon_i}{2} < \epsilon_i. \tag{6.13}$$

Also note that since $a_{j_1}^{i-1} < F_2(x)$ and $\epsilon_i < 2+\alpha$, using (6.11) and (6.13), we have

$$a_j^i < e_m + a_{j_m}^{i-1} \quad \text{for all } m \geq 2. \tag{6.14}$$

## 6.1 *Constructing the $\epsilon_i$-stencils.*

We now describe an inductive construction in $\tilde{X}$, using the skeleta decomposition $\{\tilde{P}^i\}_{i=0}^{\infty}$. At the end of stage $i$, we will associate to every tower $\tilde{\tau}_j^i \in \tilde{P}_i^i$, its $(\epsilon_i)$-stencil $(\lambda_j^i, G_j^i, \phi_j^i)$. We will also associate to this stencil two numbers $u_j^i$ and $v_j^i$, which will help construct the $\epsilon_{i+1}$-stencils at stage $i+1$.

Before starting the induction, we define patches $\lambda_j^0$ for each tower $\tilde{\tau}_j^0 \in \tilde{P}^0$ at stage 0. Let $\tilde{\tau}_j^0$ be a tower from $P^0$. Let $b_j^0$ denote the largest multiple of $2+\alpha$ so that $b_j^0$ is at least $(2+\alpha)$ smaller than $f(x, h_j^0)$ for any $(x,0)$ in the base $E_j^0$, i.e.,

$$b_j^0 + (2+\alpha) < \inf_{(x,0)\in \tilde{E}_j^0} f(x, h_j^0) < b_j^0 + 2(2+\alpha). \tag{6.15}$$

Since $|f(x,h_j^0) - f(y,h_j^0)| < \epsilon_0$ for all $(x,0),(y,0) \in \tilde{E}_j^0$, we then have

$$f(x,h_j^0) - 2(2+\alpha) < b_j^0 < f(x,h_j^0) - (2+\alpha) + \epsilon_0. \tag{6.16}$$

As $f(x) > c$ for all $x \in X$, and the fact that the tower $\tau_j^0$ in $X$ has at least $d_j^0 + L_0$ levels, we know that the time each point $(x,0) \in \tilde{E}_j^0$ spends in $\tilde{\tau}_j^1$ is at least $c(d_j^0 + L_0)$. Hence using (6.16), we get $b_j^0 + 2(2+\alpha) > c(d_j^0 + L_0)$ and by choice of $a_j^0$ and $L_0$ in (6.10) and (6.1), we get $b_j^0 + 2(2+\alpha) \geq a_j^0 + 4(2+\alpha)$. Therefore

$$b_j^0 > a_j^0 + 2(2+\alpha) \quad \text{for all } (x,0) \in E_j^0. \tag{6.17}$$

Let $\lambda_j^0$ be the grid patch of length $l_j^0 = b_j^0 - a_j^0$ with $supp(\lambda_j^0) = [a_j^0, b_j^0]$. It follows from (6.16) and (6.17) that $supp(\lambda_j^0) \subset [0, \inf_{(x,0)\in\tilde{E}_j^0} f(x,h_j^0)]$.

**<u>Stage 1</u>**: Let $\tilde{\tau}_j^1 \in \tilde{P}^1$, and let $\{\tilde{\tau}_{j_1}^0, \tilde{\tau}_{j_2}^0, ...\tilde{\tau}_{j_k}^0\}$ be its associated sequence of previous stage towers. We will first construct the patch $\lambda_j^1$ and define $u_j^1, v_j^1$ and then define the $\epsilon_1$-stencil $(\lambda_j^1, G_j^1, \phi_j^1)$.

If $k=1$ then $\tilde{\tau}_j^1$ consists of a column from a single tower $\tilde{\tau}_{j_1}^0$. In this case, let $b_j^1 = b_{j_1}^0$ and note that $a_j^1 = a_{j_1}^0$. As $h_j^1 = h_{j_1}^0$, by (6.17) we have $b_j^1 < f(x,h_j^1)$ for all $(x,0) \in \tilde{E}_j^1 \subset E_{j_1}^0$. Let $\lambda_j^1 = \lambda_{j_1}^0$ and note that $supp(\lambda_j^1) = [a_j^1, b_j^1] \subsetneq [0, \inf_{(x,0)\in\tilde{E}_j^1} f(x,h_j^1)]$. Define $u_j^1 = v_j^1 = 0$.

Suppose $k \geq 2$. For $m = 2, 3, ..., k$, recall that $e_m$ as defined as in equation (6.12), is the the approximate entry time of a point in $\tilde{E}_j^1$ to the $m^{th}$ sub-tower $\tilde{\tau}_{j_m}^0$. Then $e_m + a_{j_m}^0$ is the approximate height where the collar of the $m^{th}$ sub-tower ends in $\tilde{\tau}_j^1$. Ideally, we want to define $\lambda_j^1$ in such a way that when we look at the sub-patch covering the interval $[e_m + a_{j_m}^0, e_m + a_{j_m}^0 + l_{j_m}^0]$, it matches the patch $\lambda_{j_m}^0$ of the corresponding sub-tower, as seen in Figure 6.2. The natural thing to do would be to place the the $\lambda_{j_m}^0$ as sub-patches covering the intervals $[e_m + a_{j_m}^0, e_m + a_{j_m}^0 + l_{j_m}^0]$, and fill the gaps between the sub-patches by intervals of length 1 and $1 + \alpha$. Note that $e_1 = 0$ and $a_{j_1}^0$ is a multiple of $2 + \alpha$, and therefore there is no problem placing the patch $\lambda_{j_1}^0$ as a sub-patch of $\lambda_j^1$, starting at the desired location $a_j^1$. But for $m = 2, ..., k$, as $e_m$ (and hence $e_m + a_j^0$) are not necessarily linear combinations of 1 and $1 + \alpha$, it would not possible to tile all the gaps as desired.

We will define a patch $\lambda_j^1$ which does not quite achieve the goal described above. Instead it will satisfy the property that for each $m = 2, ..., k$, the sub-patch $\lambda_{j_m}^0$ will appear at a location within $\epsilon_1$ of the desired location $e_m + a_{j_m}^0$.

We construct $\lambda_j^1$ by modifying a grid patch. For $m = 2, ..., k$, choose $\beta_m \in \mathbb{N}$ so that $\beta_m(2+\alpha)$ is the closest $2 + \alpha$ multiple to $e_m + a_{j_m}^0$ as seen in Figure 6.2, i.e.,

$$|e_m + a_{j_m}^0 - \beta_m(2 + \alpha)| \leq \frac{1}{2}(2 + \alpha) \tag{6.18}$$

and let $\hat{b}_j^1 = \beta_k(2 + \alpha) + l_{j_k}^0$. Note that since $l_{j_k}^0$ is a multiple of $2 + \alpha$, so is $\hat{b}_j^1$. Let $\hat{\lambda}_j^1$ be the grid patch with support $[a_j^1, \hat{b}_j^1]$.

As each $\lambda_{j_m}^0$ is a grid patch, the idea is to think of $\lambda_{j_m}^0$ as being the sub-patch of $\hat{\lambda}_j^1$ covering the interval $[\beta_m(2 + \alpha), \beta_m(2 + \alpha) + l_{j_m}^0]$. This sub-patch will have to be moved if it were to appear in the perfect location, i.e., beginning at location $e_m + a_{j_m}^0$. For $m = 2, ..., k$, let $|\gamma_m|$ denote the distance to be shifted, i.e.,

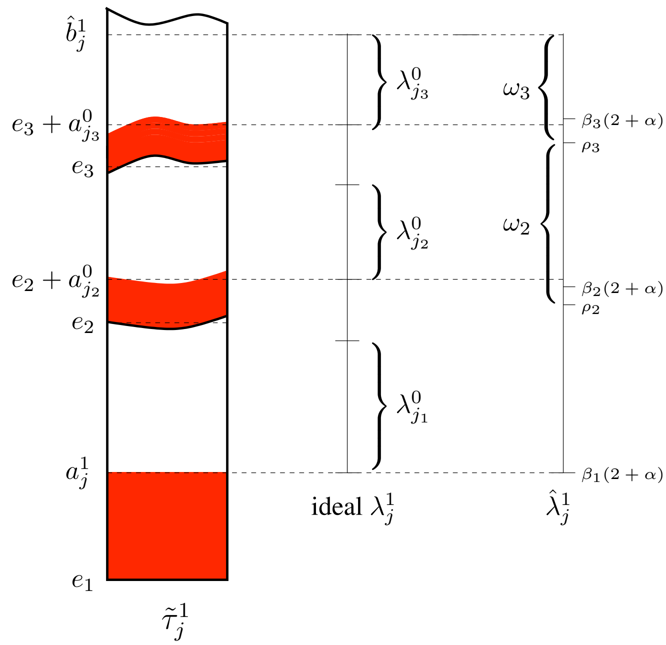


*Fig. 6.2:* A flow tower, indicating the approximate locations of sub-towers, the approximation of these locations by multiples of $2+\alpha$, and the patches $\omega_m$

$$\gamma_m = e_m + a_{j_m}^0 - \beta_m(2+\alpha).$$

Then from (6.18), we have $|\gamma_m| \leq (2+\alpha)/2$.

To move the sub-patch $\hat{\lambda}_j^1|_{[\beta_m(2+\alpha),\beta_m(2+\alpha)+l_{j_m}^0]}$ by $\gamma_m$, we will use Lemmas 4.3 and 4.4, and therefore need these sub-patches to be preceded and followed by grid patches of appropriate lengths. Recall that all towers in $\tilde{P}^1$ have flow collar heights $a_j^1 = 6r_j^1(2+\alpha)$ where $r_j^1 = 1 + M_0 + M_1$. For $m = 2, ..., k$, define

$$\rho_m = (\beta_m - 3r_{j_m}^0)(2+\alpha),$$

and let $\omega_m$ denote the sub-patches of $\hat{\lambda}_j^1$ covering the intervals $[\rho_m, \rho_{m+1}]$ for $m = 2, ..., k-1$ and $[\rho_m, \hat{b}_j^1]$ for $m = k$, as shown in Figure 6.2. Geometrically, the sub-patch $\omega_m$ is the patch $\lambda_{j_m}^1$ concatenated before and after with grid patches $\hat{\lambda}_j^1|_{[\rho_m,\beta_m(2+\alpha)]}$ and $\hat{\lambda}_j^1|_{[\beta_m(2+\alpha)+l_{j_m}^0,\rho_{m+1}]}$ that cover half of the flow collars of the $m^{th}$ and the $m+1^{th}$ sub-towers respectively. Since these grid patches have supports about $3r^i_{j_m}(2+\alpha)$ and $3r^i_{j_{m+1}}(2+\alpha)$ respectively, and since $r^i_{j_m}, r^i_{j_{m+1}} > M_0$, the supports are longer than $M_0(2+\alpha)$. Therefore the patch $\omega_m$ satisfies the hypothesis of Lemma 4.3. Apply Lemma 4.3 to $\omega_m$ with $\gamma_m$ and $\epsilon_1$, to get a patch $\omega'_m$ with

$$supp(\omega'_m) = supp(\omega_m) = [\rho_m, \rho_{m+1}] \tag{6.19}$$

and such that the sub-patch $\hat{\lambda}^1_j|_{[\beta_m(2+\alpha), \beta_m(2+\alpha)+l^0_{j_m}]}$ is shifted to a location beginning within $\epsilon_1$ of $e_m + a^0_{j_m}$ as seen in Figure 6.3.

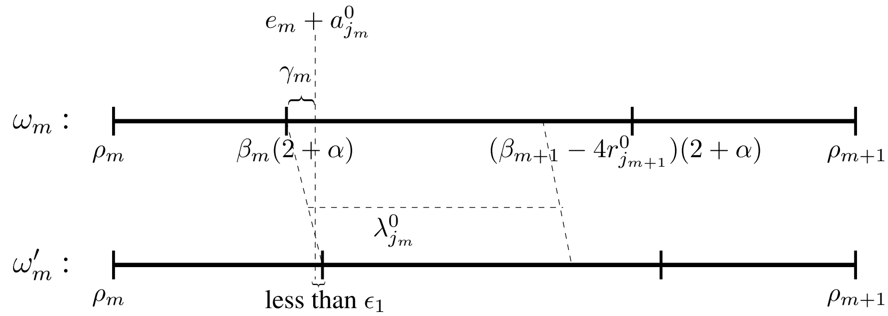


*Fig. 6.3:* $\omega_m$ and $\omega'_m$

Similarly, for $m = k$, the grid patch $\hat{\lambda}^1_j|_{[\rho_k, \beta_k(2+\alpha)]}$ acts as the grid patch preceding $\hat{\lambda}^1_j|_{[\beta_m(2+\alpha), \hat{b}^1_j]}$ and is longer than $M_0(2+\alpha)$. Therefore $\omega_k$ satisfies the hypothesis of Lemma 4.4. Apply Lemma 4.4 to $\omega_k$ with $\gamma_k$ and $\epsilon_1$, to get a patch $\omega'_k$ with $supp(\omega'_k) = [\rho_k, \hat{b}^i_j + t]$, where $t = u + v(1+\alpha)$ for some $u, v \in \mathbb{Z}$ with $|u| + |v| < M_0$ and $|t - \gamma_k| < \epsilon_1$, and such that the sub-patch $\hat{\lambda}^1_j|_{[\beta_k(2+\alpha), \hat{b}^i_j]}$ is now shifted to a location beginning within $\epsilon_1$ of $e_k + a^0_{j_k}$. Note here that by using Lemma 4.4, we pretended that we had a grid patch after $\hat{\lambda}^1_j|_{[\beta_m(2+\alpha), \hat{b}^1_j]}$of length longer than $M_0(2+\alpha)$. The $u$ and $v$ keep track of the number of intervals interchanged, so that in later stages, when this tower $\tilde{\tau}^1_j$ appears as a sub-tower within a tower in $P^i$ for $i \geq 2$, we will be able to reconcile this interchange of tiles using a part of the flow collar that appears above it.

Also note that the support of $\omega'_k$ starts at the same location as the support of $\omega_k$. By (6.19), we know that $supp(\omega_m) = supp(\omega'_m)$ for all $m = 2, ..., k-1$. Therefore, we can construct a new patch $\lambda^1_j$ on the interval $[a^1_j, \hat{b}^1_j + t]$, by replacing all sub-patches $\omega_m$ with $\omega'_m$ for all $m = 1, ..., k$.

We want to show that the support of this new patch $\lambda_j^1$ is strictly contained in $[0, \inf_{(x,0)\in\tilde{E}_j^1} f(x, h_j^1)]$. Geometrically speaking, $supp(\lambda_j^1)$ is the same as $supp(\hat{\lambda}_j^1)$ up to $\rho_k$. The only difference comes from the part where we shifted $\lambda_{j_k}^0$ to begin at the location $\beta_k(2+\alpha)+t$, instead of beginning at $\beta_k(2+\alpha)$. As $\beta_k(2+\alpha)$ is within $(2+\alpha)/2$ of $e_k + a_{j_k}^0$ and $\beta_k(2+\alpha)+t$ is within $\epsilon_1$ of $e_k + a_{j_k}^0$, we have moved closer to the desired location $e_k + a_{j_k}^0$. Now, for any point $\tilde{x} = (x, 0) \in \tilde{E}_j^1$, the part of its orbit from time $F_k(x)$ to $f(y, h_{j_k}^0)$, where $(y, 0) = \mathcal{U}_{F_k(x)}\tilde{x} \in \tilde{E}_{j_k}^0$, is the part that belongs to the $k^{th}$ sub-tower. By (6.13), we know that $|F_k(x) - e_k| < \epsilon_1$. Therefore the difference between where the patch $\lambda_{j_k}^0$ begins in $\lambda_j^1$, i.e., $\beta_k(2+\alpha)+t$, and where we expect it to begin in relation to the partial orbit of $\tilde{x}$ i.e., at location $F_k(x) + a_{j_k}^0$, is at most $2\epsilon_1$. Hence

$$|F_k(x) + a_{j_k}^0 - \beta_k(2+\alpha) - t| < 2\epsilon_1. \tag{6.20}$$

Let $b_j^1$ denote the end point of $\lambda_j^1$, i.e.,

$$b_j^1 = \beta_k(2+\alpha) + t + (b_{j_k}^0 - a_{j_k}^0).$$

Then (6.20) implies $b_j^1 < F_k(x) + 2\epsilon_1 + b_{j_k}^0$. By (6.16) we know $b_{j_k}^0 < f(y, h_{j_k}^0) - (2+\alpha) + \epsilon_0$. Using this and the fact that $F_k(x) + f(y, h_{j_k}^0) = f(x, h_j^1)$ we get

$$b_j^1 < f(x, h_j^1) - (2+\alpha) + 2(\epsilon_0 + \epsilon_1). \tag{6.21}$$

To show that $a_j^1 < b_j^1$, recall from (6.14) that $a_j^1 < e_k + a_{j_k}^0$ as $k \geq 2$. Since $e_k + a_{j_k}^0$ is within $\epsilon_1$ of $\beta_k(2+\alpha)+t$, we get $a_j^1 < \beta_k(2+\alpha) + t + \epsilon_1$. Therefore using (6.17), we get

$$b_j^1 - a_j^1 > b_{j_k}^0 - a_{j_k}^0 - \epsilon_1 > 2(2+\alpha) - \epsilon_1.$$

Therefore $supp(\lambda_j^1) = [a_j^1, b_j^1] \subsetneq [0, \inf_{(x,0)\in\tilde{E}_j^1} f(x, h_j^1)$.

For future reference, we also compute a lower bound for $b^1_j$. By (6.20) again, we have $b^1_j > F_k(x) - 2\epsilon_1 + b^0_{j_k}$. Using (6.16), we then get

$$b^1_j > F_k(x) - 2\epsilon_1 + f(y, h^0_{j_k}) - 2(2+\alpha) = f(x, h^1_j) - 2(2+\alpha) - 2\epsilon_1.$$

Let $l^1_j = b^1_j - a^1_j$ denote the length of $supp(\lambda^1_j)$. Define $u^1_j = u$ and $v^1_j = v$ Note that

$$l^1_j - u^1_j - v^1_j(1+\alpha) = l^1_j - t = \hat{b}^1_j$$

and hence is a multiple of $2+\alpha$. Also note that

$$|u^1_j| + |v^1_j| < M_0.$$

Define

$$p^1_j = a^1_j + \frac{1}{2} \quad q^1_j = b^1_j - \frac{1}{2} + \epsilon_0,$$

and let

$$G^1_j = \bigcup_{p^1_j < s < q^1_j} \mathcal{U}_s \tilde{E}^1_j.$$

For all $\tilde{x} \in \mathcal{U}_s \tilde{E} \subset G^1_j$, define the map $\phi^1_j(\tilde{x}) = C(S_s\lambda^1_j, \epsilon_1)$. Clearly, $(\lambda^1_j, G^1_j, \phi^1_j)$ forms an $\epsilon_1$-stencil for $\tilde{\tau}^1_j$.

**Stage i+1**: Now suppose for $i \in \mathbb{N}$, every tower $\tilde{\tau}^i_j \in \tilde{P}^i$ is assigned an $\epsilon_i$-stencil $(\lambda^i_j, G^i_j, \phi^i_j)$ with $supp(\lambda^i_j) = [a^i_j, b^i_j] \subsetneq [0, \inf_{(x,0)\in\tilde{E}^i_j} f(x, h^i_j)$ and such that

$$f(x, h^i_j) - 2(2+\alpha) - 2\sum_{n=1}^{i} \epsilon_n < b^i_j < f(x, h^i_j) - (2+\alpha) + 2\sum_{n=0}^{i} \epsilon_n. \tag{6.22}$$

Let $l^i_j = b^i_j - a^i_j$ satisfy

$$l_j^i > 2(2+\alpha) - \sum_{n=1}^{i} \epsilon_n. \tag{6.23}$$

Suppose $G_j^i = \bigcup_{p_j^i < s < q_j^i} \mathcal{U}_s \tilde{E}_j^i$ where

$$p_j^i = a_j^i + 1/2 \quad \text{and} \quad q_j^i = b_j^i - 1/2 + \sum_{n=0}^{i-1} \epsilon_n. \tag{6.24}$$

Also suppose that $u_j^i, v_j^i$ are such that

$$|u_j^i| + |v_j^i| < \sum_{m=0}^{i-1} M_m \tag{6.25}$$

and

$$l_j^i - u_j^i - v_j^i(1+\alpha) \text{ is a multiple of } (2+\alpha). \tag{6.26}$$

Fix a $\tilde{\tau}_j^{i+1} \in \tilde{P}^{i+1}$. Let $\{\tilde{\tau}_{j_1}^i, \tilde{\tau}_{j_2}^i, ... \tilde{\tau}_{j_k}^i\}$ be its associated sequence of previous stage towers. Again we first construct the patch $\lambda_j^{i+1}$ and define $u_j^{i+1}, v_j^{i+1}$ and then define the $\epsilon_{i+1}$-stencil for $\tilde{\tau}_j^{i+1}$.

If $k = 1$ define $\lambda_j^{i+1} = \lambda_{j_1}^i$ and $b_j^{i+1} = b_{j_1}^i$. Note that $a_j^{i+1} = a_{j_1}^i$, and as $h_j^{i+1} = h_{j_1}^i$, we have (6.22) is satisfied. Therefore $supp(\lambda_j^{i+1}) = [a_j^{i+1}, b_j^{i+1}] \subsetneq [0, \inf_{(x,0)\in \tilde{E}_j^{i+1}} f(x, h_j^{i+1})]$. Also define $u_j^{i+1} = u_{j_1}^i, v_j^{i+1} = v_{j_1}^i$ and $l_j^{i+1} = l_{j_1}^i$. It is clear that (6.23), (6.25) and (6.26) are also true.

Suppose $k \geq 2$. We will follow what we did in Stage 1 almost exactly with the exception that the patch $\hat{\lambda}_j^{i+1}$ will not be a grid patch to begin with. The patches $\lambda_{j_m}^i$, corresponding to the sub-towers $\tilde{\tau}_{j_m}^i$, for $m = 1, ..., k$ from the previous stage are not grid patches any more. We will construct the patch $\hat{\lambda}_j^{i+1}$ by placing $\lambda_{j_m}^i$ as sub-patches beginning at the nearest $2+\alpha$ multiple of the final desired location, and fill in the gaps with patches constructed using $u_j^i, v_j^i$ and grid patches of appropriate lenghts. Having once constructed $\hat{\lambda}_j^{i+1}$, the rest of the construction will be the same as in Stage 1.

Recall from (6.12) that $e_m$ is the approximate entry time of a point in $E_j^{i+1}$ to the $m^{th}$ sub-tower $\tilde{\tau}_{j_m}^i$. For each $m = 2, 3, ..., k$, choose $\beta_m \in \mathbb{N}$, to be the closest $2+\alpha$ multiple of $e_m + a_{j_m}^i$ i.e.,

$$|e_m + a_{j_m}^i - \beta_m(2+\alpha)| \leq \frac{1}{2}(2+\alpha) \tag{6.27}$$

and let $\beta_1(2+\alpha) = a_j^{i+1} = a_{j_1}^i$. Let $\mathcal{M} = \sum_{m=0}^{i} M_m$ and for $m = 1, ..., k-1$ let $s_m = \mathcal{M} - u_{j_m}^i + (\mathcal{M} - v_{j_m}^i)(1+\alpha)$. The role of $s_m$ is to help define a patch $\theta_m$ that reconciles the interchange of tiles that took place in the top sub-tower of $\tilde{\tau}_{j_m}^i$ at Stage $i$. To construct $\hat{\lambda}_j^{i+1}$, we will concatenate the sub-patches $\lambda_{j_m}^i$ with the patches $\theta_m$, and fill in the gaps with grid patches, as shown in Figure 6.4.

For $m = 1, ..., k-1$, define $\theta_m$ to be a patch consisting $\mathcal{M} - u_{j_m}^i$ tiles of length 1 followed by $\mathcal{M} - v_{j_m}^i$ tiles of length $1+\alpha$ and covering the interval $[\beta_m(2+\alpha) + l_{j_m}^i, \beta_m(2+\alpha) + l_{j_m}^i + s_m]$. By hypotheses we have $|u_{j_m}^i| + |v_{j_m}^i| < \mathcal{M}$, and therefore $|supp(\theta_m)| = s_m < 2\mathcal{M}(2+\alpha)$. We claim that $\beta_m(2+\alpha) + l_{j_m}^i + s_m$ is a multiple of $2+\alpha$ and is smaller than $(\beta_{m+1} - 4r_{j_{m+1}})(2+\alpha)$. By (6.26), $l_j^i - u_j^i - v_j^i(1+\alpha)$ is a multiple of $(2+\alpha)$ and therefore

$$\beta_m(2+\alpha) + l_{j_m}^i + s_m = \beta_m(2+\alpha) + 2\mathcal{M} + (l_j^i - u_j^i - v_j^i(1+\alpha))$$

is also a multiple of $2+\alpha$.

We now show $\beta_m(2+\alpha) + l_{j_m}^i + s_m < (\beta_{m+1} - 4r_{j_{m+1}})(2+\alpha)$. Using (6.27) and the fact that for any $(x, 0) \in \tilde{E}_j^{i+1}$, $e_m$ approximates $F_m(x)$ within $\epsilon_{i+1}$, we get

$$(\beta_{m+1} - \beta_m)(2+\alpha) > F_{m+1}(x) - F_m(x) + a_{j_{m+1}}^i - a_{j_m}^i - (2+\alpha) - 2\epsilon_{i+1}.$$

Since $F_{m+1}(x) - F_m(x) = f(y, h_{j_m}^i)$ for $(y, 0) = \mathcal{U}_{F_m(x,0)}$, which by (6.22), is greater than $b_{j_m}^i$, we in turn get

$$(\beta_{m+1}-\beta_m)(2+\alpha) > b^i_{j_m} + a^i_{j_{m+1}} - a^i_{j_m} - (2+\alpha) - 2\epsilon_{i+1}. \tag{6.28}$$

Now by definition, we have $r^i_{j_{m+1}} = 1+\mathcal{M}$, and $a^i_{j_{m+1}} = 6r^i_{j_{m+1}}(2+\alpha)$. Since $s_m < 2\mathcal{M}(2+\alpha) < 2r^{i-1}_{j_{m_1}}(2+\alpha) - 2(2+\alpha)$, we get

$$a^i_{j_{m+1}} > 4r^i_{j_{m+1}}(2+\alpha) + s_m + 2(2+\alpha).$$

Substituting in (6.28), we get $(\beta_{m+1}-\beta_m)(2+\alpha) > b^i_{j_m} - a^i_{j_m} + 4r_{j_{m+1}} + s_m$ and hence our claim.

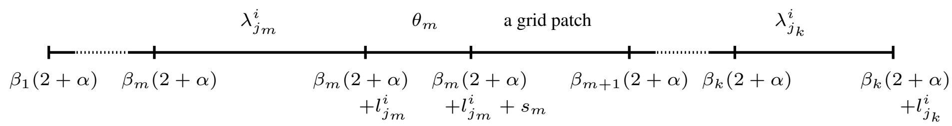

*Fig. 6.4:* The patch $\hat{\lambda}^{i+1}_j$

Let $\hat{b}^{i+1}_j = \beta_k(2+\alpha) + l^i_{j_k}$. We define $\hat{\lambda}^{i+1}_j$ on on the interval $[a^{i+1}_j, \hat{b}^{i+1}_j]$, using the patches patches $\lambda_{j_m}, \theta_m$ and grid patches. For $m = 1, 2, ..., k$, define $\hat{\lambda}^{i+1}_j$ to be the patch $\lambda^i_{j_m}$ on the interval $[\beta_m(2+\alpha), \beta_m(2+\alpha) + l^i_{j_m}]$, i.e.,

$$\hat{\lambda}^{i+1}_j|_{[\beta_m(2+\alpha), \beta_m(2+\alpha)+l^i_{j_m}]} = S_{a^i_{j_m} - \beta_m(2+\alpha)} \lambda^i_{j_m}.$$

For $m = 1, 2, ..., k-1$, define $\hat{\lambda}^{i+1}_j$ to be the patch $\theta_m$ on the intervals $[\beta_m(2+\alpha) + l^i_{j_m}, \beta_m(2+\alpha) + l^i_{j_m} + s_m]$, i.e.,

$$\hat{\lambda}^{i+1}_j|_{[\beta_m(2+\alpha)+l^i_{j_m}, \beta_m(2+\alpha)+l^i_{j_m}+s_m]} = S_{-\beta_m(2+\alpha)-l^i_{j_m}} \theta_m, \tag{6.29}$$

As $\beta_m(2+\alpha) + l_{j_m} + s_m$ is a multiple of $2+\alpha$, fill the remaining gaps, i.e the intervals $[\beta_m(2+\alpha) + l^i_{j_m} + s_m, \beta_{m+1}(2+\alpha)]$ for $m = 2, ..., k-1$, with grid patches of appropriate lengths.

We can now use Lemmas 4.3 and 4.4 to modify $\hat{\lambda}^{i+1}_j$ to $\lambda^{i+1}_j$ so that for each $m = 2, 3, ..., k$.

the sub-patch $\lambda^i_{j_m}$ appears at a location within $\epsilon_{i+1}$ of $e_m + a^i_{j_m}$ within $\lambda^{i+1}_j$. For $m = 2, ..., k$ define

$$\gamma_m = e_m + a^i_{j_m} - \beta_m(2+\alpha)$$

and

$$\rho_m = (\beta_m - 3r^i_{j_m})(2+\alpha).$$

By (6.27), $|\gamma_m| \leq (2+\alpha)/2$. Let $\omega_m$ denote the sub-patches covering the intervals $[\rho_m, \rho_{m+1}]$ for $m = 2, ..., k-1$ and $[\rho_m, \hat{b}^i_j]$ for $m = k$. Note that the sub-paches $\hat{\lambda}^{i+1}_j|_{[\rho_m, \beta_m(2+\alpha)]}$ and $\hat{\lambda}^{i+1}_j|_{[\beta_m(2+\alpha)+l^i_{j_m}+s_m, \rho_{m+1}]}$ act as the grid patches preceding and succeeding the patch $\hat{\lambda}^{i+1}_j|_{[\beta_m(2+\alpha), \beta_m(2+\alpha)+l^i_{j_m}+s_m]}$ in $\omega_m$. Since $r^i_{j_m}, r^i_{j_{m+1}} \geq \mathcal{M} > M_i$ each of the above two grid patches have length at least $M_i(2+\alpha)$. Therefore for $m = 2, ..., k-1$, $\omega_m$ satisfy the hypothesis of Lemma 4.3. Apply Lemma 4.3 to $\omega_m$ and with $\gamma_m$ and $\epsilon_{i+1}$ to get a patch $\omega'_m$ with

$$supp(\omega'_m) = supp(\omega_m) = [\rho_m, \rho_{m+1}] \tag{6.30}$$

and note that the sub-patch $\hat{\lambda}^{i+1}_j|_{[\beta_m(2+\alpha), \beta_m(2+\alpha)+l^i_{j_m}]}$ is shifted to the location beginning $\beta_m(2+\alpha) + t_m$ with $|e_m + a^i_{j_m} - \beta_m(2+\alpha) - t_m| < \epsilon_{i+1}$.

For $m = k$, in a similar fashion, $\omega_k$ satisfies the hypothesis of Lemma 4.4. Applying Lemma 4.4 to $\omega_k$ with $\gamma_k$ and $\epsilon_{i+1}$, we get a patch $\omega'_k$ with $supp(\omega'_k) = [\rho_k, \hat{b}^{i+1}_j + t]$, where $t = u + v(1+\alpha)$ for some $u, v \in \mathbb{Z}$, with $|u| + |v| < M_i$ and $|t - \gamma_k| < \epsilon_{i+1}$. Note that the sub-patch $\hat{\lambda}^{i+1}_j|_{[\beta_k(2+\alpha), \hat{b}^{i+1}_j]}$ is now shifted to the location the location beginning $\beta_m(2+\alpha) - t_k$ with $|e_m + a^i_{j_k} - \beta_m(2+\alpha) - t_k| < \epsilon_{i+1}$.

Also note that the support of $\omega'_k$ starts at the same location as the support of $\omega_k$. By (6.30), we know that $supp(\omega_m) = supp(\omega'_m)$ for all $m = 2, ..., k-1$. Therefore, we can construct a new patch $\lambda^1_j$ on the interval $[a^1_j, \hat{b}^1_j + t]$, by replacing all sub-patches $\omega_m$ with $\omega'_m$ for all $m = 2, ..., k$. We then have

$$\lambda_j^{i+1} = S_{a_{j_m}^i - \beta_m(2+\alpha) - t_m} \lambda_{j_m}^i \tag{6.31}$$

on the interval $[\beta_m(2+\alpha) + t_m, \beta_m(2+\alpha) + t_m + l_{j_m}^i]$ with

$$|e_m + a_{j_m}^i - \beta_m(2+\alpha) - t_m| < \epsilon_{i+1} \tag{6.32}$$

Define

$$b_j^{i+1} = \beta_k(2+\alpha) + t_k + (b_{j_k}^i - a_{j_k}^1). \tag{6.33}$$

Using the same arguments as in Stage 1, it is easily shown that

$$f(x, h_j^{i+1}) - 2(2+\alpha) - 2\sum_{n=1}^{i+1} \epsilon_n < b_j^{i+1} < f(x, h_j^{i+1}) - (2+\alpha) + 2\sum_{n=0}^{i+1} \epsilon_n \tag{6.34}$$

for all $(x, 0) \in \tilde{E}_j^{i+1}$ and

$$l_j^{i+1} = b_j^{i+1} - a_j^{i+1} > 2(2+\alpha) - \sum_{n=0}^{i+1} \epsilon_n. \tag{6.35}$$

Therefore $supp(\lambda_j^{i+1}) = [a_j^{i+1}, b_j^{i+1}] \subsetneq [0, \inf_{(x,0)\in\tilde{E}_j^{i+1}} f(x, h_j^{i+1})]$. Let $u_j^{i+1} = u_{j_k}^i + u, v_j^{i+1} = v_{j_k}^i + v$. Note that

$$|u_j^{i+1}| + |v_j^{i+1}| = |u_{j_k}^i| + |v_j^{i+1}| + |u| + |v| < \sum_{m=0}^{i} M_m.$$

Using (6.33) and the fact that $t_k = u + v(1+\alpha)$, we get

$$l_j^{i+1} - u_j^{i+1} - v_j^{i+1}(1+\alpha) = \beta_k(2+\alpha) + l_j^i - u_j^i - v_j^i(1+\alpha)$$

and hence $l_j^{i+1} - u_j^{i+1} - v_j^{i+1}(1+\alpha)$ is a multiple of $2+\alpha$. Define

$$G_j^{i+1} = \bigcup_{p_j^{i+1} < t < q_j^{i+1}} \mathcal{U}_t \tilde{E}_j^{i+1}$$

where

$$p_j^{i+1} = a_j^{i+1} + \frac{1}{2} \quad \text{and} \quad q_j^{i+1} = b_j^{i+1} - \frac{1}{2} + \sum_{n=0}^{i} \epsilon_n. \tag{6.36}$$

For all $\tilde{x} \in \mathcal{U}_s \tilde{E} \subset G_j^{i+1}$, define the map $\phi_j^{i+1}(\tilde{x}) = C(S_s \lambda_j^{i+1}, \epsilon_{i+1})$. It is then clear that $(\lambda_j^{i+1}, G_j^{i+1}, \phi_j^{i+1})$ forms an $\epsilon_{i+1}$-stencil for $\tilde{\tau}_j^{i+1}$.

## 6.2 The $\epsilon_i$-stencils for $\tilde{\tau}_j^i \in \tilde{P}^i$ satisfy the hypothesis of Proposition 5.3.

Recall that there exists a clopen set $A \subset X$ and a skeleta decomposition $\{P^i\}_{i=0}^{\infty}$ of $X$ with respect to $L_0$ and $\{N_i\}_{i=0}^{\infty}$, as defined in (6.1) and (6.6). Also recall that the sequence $\{\tilde{P}^i\}_{i=0}^{\infty}$ denotes the corresponding sequence of tower partitions for the flow space $\tilde{X}$. For each $i \geq 1$ and $\tilde{\tau}_j^i \in \tilde{P}^i$, $(\lambda_j^i, G_j^i, \phi_j^i)$ are the $\epsilon_i$-stencils for $\tilde{\tau}_j^i$ from Section 6.1. Let $(G^i, \phi^i)$ denote the corresponding templates, so that $G^i = \cup_{j \in J_i} G_j^i$ and $\phi^i(\tilde{x}) = \phi_j^i(\tilde{x})$ whenever $\tilde{x} \in G_j^i$. To show that the sets $G_j^i$ satisfy condition (a) of Proposition 5.3 in the flow space $\tilde{X}$, we first show that a similar condition holds true in the discrete space $X$.

**Lemma 6.1.** *Let $x \in X$ and $n_1 \leq n_2 \in \mathbb{Z}$. Then there exists $i \in \mathbb{N}$ and $\tau_j^i \in P^i$ such that $T^n x \in \tau_j^i$ for all $n_1 \leq n \leq n_2$, and away from the discrete collar of $\tau_j^i$ i.e., if $T^{n_2 - m} x \in E_j^i$, then $m \geq d_j^i$.*

*Proof.* Choose $m_1, m_2 \in \mathbb{Z}$ so that $m_1 \leq n_1 \leq n_2 \leq m_2$ and $T^{m_1} x, T^{m_2} x \notin A$. Choose $i \in \mathbb{N}$ so that $N_i > m_2 - m_1 + 1$. Then there exists $j \in J_i$ such that both $T^{m_1} x$ and $T^{m_2} x$ belong to the tower $\tau_j^i$. If not, $T^{m_1} x$ and $T^{m_2} x$ belong to different towers, and as the bottom most $N_i$ levels of every tower in $P^i$ are contained in the set $A$, there exist at least $N_i$ occurrences of $A$ between $T^{m_1} x$ and $T^{m_2} x$. This is a contradiction as $N_i > m_2 - m_1 + 1$.

Therefore $T^{m_1}x$ and $T^{m_2}x$ belong to the same tower $\tau^i_j$ which implies

$$T^n x \in \tau^i_j \text{ for all } m_1 \leq n \leq m_2.$$

Also, as $T^{m_1}x \notin A$, and the lower $d^i_j$ levels of $\tau^i_j$ are all contained in $A$, we have $T^{m_2}x$ is not in the first $d^i_j$ levels of the tower. This implies that if $T^{m_2-m}x \in E^i_j$ then $m \geq d^i_j$. Therefore the result is true for $m_1, m_2$, and follows for $n_1, n_2$.

$\square$

**Lemma 6.2.** *For any $\tilde{x} \in \tilde{X}$ and any $t_1 < t_2 \in \mathbb{R}$, there exists an $i \in N$ and $j \in J_i$ such that $\{\mathcal{U}_t\tilde{x} : t_1 \leq t \leq t_2\} \subset G^i_j$.*

*Proof.* Let $\tilde{x} = (x, s) \in \tilde{X}$ for $x \in X$ and $0 \leq s < f(x)$. It suffices to show that there exists an $i \in \mathbb{N}$ and $j \in J_i$ such that $\mathcal{U}_{t_1}\tilde{x}$ is at a height greater than $p^i_j$ and $\mathcal{U}_{t_2}\tilde{x}$ is at a height smaller than $q^i_j$ in $\tilde{\tau}^i_j$.

First choose $n_1 \leq n_2 \in \mathbb{Z}$ so that

$$\mathcal{U}_{t_1}\tilde{x} = (T^{n_1}x, s_1) \text{ for some } 0 \leq s_1 < f(T^{n_1}x)$$

and

$$\mathcal{U}_{t_2}\tilde{x} = (T^{n_2-1}x, s_2) \text{ for some } 0 \leq s_2 < f(T^{n_2-1}x).$$

Recall that the ceiling function $f$ satisfies $f(x) > c$ for all $x \in X$. Now choose $k \in \mathbb{N}$ so that $ck > 5(2+\alpha)$ and choose $m_1, m_2 \in \mathbb{Z}$ so that $m_1 < n_1 - k \leq n_2 + k < m_2$. By Lemma 6.1 applied to $m_1, m_2$, there exist $i \in \mathbb{N}$ and $j \in J_i$ so that $T^{m_1}x, T^{m_2}x \in \tau^i_j$, and away from the collar of the tower, i.e., $d^i_j < m_1 < m_2 < h^i_j$.

Therefore in the tower $\tilde{\tau}^i_j$, we have $\mathcal{U}_{t_1}\tilde{x} = (T^{n_1}x, s_1)$ is at height greater than or equal to $c(m_1+k)+s_1 > c(d^i_j+k)+s_1$. As $cd^i_j \geq a^i_j$ and $a^i_j + 1/2 = p^i_j$, we get $\mathcal{U}_{t_1}\tilde{x}$ is at a height greater than $a^i_j + 5(2+\alpha) + s_1 > p^i_j$ in $\tilde{\tau}^i_j$.

Also, $\mathcal{U}_{t_2}\tilde{x} = (T^{n_2-1}x, s_2)$ is at height at least $c(m_2 - n_2) > ck > 5(2+\alpha)$ below $(T^{m_2}x, 0)$ in the tower $\tilde{\tau}^i_j$. In other words, if $\mathcal{U}_{t_2}(\tilde{x}) = (y,t)$ for some $(y,0) \in E^i_j$, then $t + 5(2+\alpha) < cm_2 < f(y, h^i_j)$. By definition, $q^i_j = b^i_j - 1/2 + \sum_{n=0}^{i-1} \epsilon_n$, and by (6.34), we know $b^i_j > f(y, h^i_j) - 2(2+\alpha) - 2\sum_{n=0}^{i} \epsilon_n$. Therefore $q^i_j > f(y, h^i_j) - 3(2+\alpha) > t + 2(2+\alpha)$. Hence $\mathcal{U}_{t_1}\tilde{x}, \mathcal{U}_{t_2}\tilde{x} \in G^i_j$.

□

The following lemma shows that the sets $G^i, i \in \mathbb{N}$ satisfy condition (b) of Proposition 5.3.

**Lemma 6.3.** *For all $i \in \mathbb{N}$, $G^i \subset G^{i+1}$.*

*Proof.* Suppose $\tilde{x} \in G^i$ for some $i \in \mathbb{N}$. Then there exists $r \in J_i$ such that $\tilde{x} \in \tilde{\tau}^i_r$ and $\tilde{x} \in \mathcal{U}_{s_1}\tilde{E}^i_r$ for some $p^i_r < s_1 < q^i_r$. As $\tilde{P}^{i+1}$ is a tower partition of the flow space $\tilde{X}$, there exists $j \in J_{i+1}$ and $s_2 \geq 0$ such that $\tilde{x} \in \tilde{\tau}^{i+1}_j$, and $\tilde{x} \in \mathcal{U}_{s_2}\tilde{E}^{i+1}_j$ for some $s_2 \geq 0$. To show that $\tilde{x} \in G^{i+1}$, it suffices to show that $p^{i+1}_j < s_2 < q^{i+1}_j$.

Now $\tilde{x} \in \tilde{\tau}^i_r \cap \tilde{\tau}^{i+1}_j$ implies that $\tilde{\tau}^i_r \in \{\tilde{\tau}^i_{j_m} : m = 1, ..., k\}$, the associated sequence of previous stage towers of $\tilde{\tau}^{i+1}_j$, i.e., $r = j_m$ for some $m = 1, 2, ..., k$. Hence either $s_2 = s_1$ if $m = 1$ or $s_2 = s_1 + F_m(y)$ if $m > 1$ for some $(y, 0) \in \tilde{E}^{i+1}_j$. If $m = 1$, then $p^{i+1}_j = p_{j_1} < s_1 = s_2$. If $m \neq 1$, then $F_m(y) > p^{i+1}_j$ and therefore $s_2 > p^{i+1}_j$.

It remains to show that $s_2 < q^{i+1}_j$. Suppose not i.e., $q^{i+1} \leq s_2$. This implies that $\tilde{x}$ belongs to the top sub-tower of $\tilde{\tau}^{i+1}_j$, i.e., $s_2 = s_1 + F_k(y)$. As $|F_k(y) - e_k| < \epsilon_{i+1}$ and $s_1 < q^i_{j_k}$, we get $q^{i+1}_j \leq s_2 < q^i_{j_k} + e_m - \epsilon_{i+1}$. Using (6.36) and replacing $q^i_{j_k}$ and $q^{i+1}_j$ in terms of $b^i_{j_k}$ and $b^{i+1}_j$, we get $b^{i+1}_j + \epsilon_i < b^i_{j_k} + e_k + \epsilon_{i+1}$, or equivalently $\epsilon_i < e_k + b_{j_k} - b^{i+1}_j + \epsilon_{i+1}$.

By definition of $b^{i+1}_j$ from (6.33), $b^{i+1}_j = \beta_k(2+\alpha) + (b^i_{j_k} - a^i_{j_k}) + t_k$, and by (6.32), we know $|e_k + a^i_{j_k} - \beta_k(2+\alpha) - t_k| < \epsilon_{i+1}$. Therefore we get

$$\epsilon_i < e_k + a^i_{j_k} - \beta_k(2+\alpha) - t_k + \epsilon_{i+1} < 2\epsilon_{i+1}$$

which is a contradiction as $4\epsilon_{i+1} < \epsilon_i$ for all $i$.

Hence $s_2 < q_j^{i+1}$.

□

We next show that the template maps $\phi^i, i \in \mathbb{N}$, are consistent from stage $i$ to $i+1$.

**Lemma 6.4.** *If* $\tilde{x} \in G^i$*, then* $\phi^i(\tilde{x}) \supset \phi^{i+1}(\tilde{x})$.

*Proof.* Let $\tilde{x} \in G^i \subset G^{i+1}$, then $\tilde{x} \in \ \tilde{\tau}_j^{i+1} \cap \tilde{\tau}_{j_m}^i$, where $\tilde{\tau}_{j_m}^i$ is a sub-tower of $\tilde{\tau}_j^{i+1}$. Therefore we can write $\tilde{x}$ in two ways:

$$\tilde{x} = \mathcal{U}_{s_1}\tilde{y}_1 \text{for some } \tilde{y}_1 = (y_1, 0) \in \tilde{E}_{j_m}^i,$$

and

$$\tilde{x} = \mathcal{U}_{s_2}\tilde{y}_2 \text{for some } \tilde{y}_2 = (y_2, 0) \in \tilde{E}_j^{i+1}$$

where $s_2 = s_1 + F_m(y_2)$. This implies

$$\phi_{j_m}^i \tilde{x} = C(S_{s_1}\lambda_{j_m}^i, \epsilon_i)$$

and

$$\phi_j^{i+1}\tilde{x} = C(S_{s_2}\lambda_j^{i+1}, \epsilon_{i+1}).$$

We need to show that $C(S_{s_1}\lambda_{j_m}^i, \epsilon_i) \supset C(S_{s_2}\lambda_j^{i+1}, \epsilon_{i+1})$. Let $\Gamma \in C(S_{s_2}\lambda_j^{i+1}, \epsilon_{i+1})$. Then there exists $r \in (-\epsilon_{i+1}, \epsilon_{i+1})$ such that

$$S_r\Gamma = S_{s_2}\lambda_j^{i+1}$$

on

$$supp(S_{s_2}\lambda_j^{i+1}) = supp(\lambda_j^{i+1}) - s_2 = [a_j^{i+1} - s_2, b_j^{i+1} - s_2].$$

If $m = 1$, then $s_1 = s_2$, $a_j^{i+1} = a_{j_1}^i$ and $\lambda_j^{i+1} = \lambda_{j_1}^i$ on $supp(\lambda_{j_1}^i)$. This means

$$S_r\Gamma = S_{s_1}\lambda_{j_1}^i \text{ on } supp(S_{s_1}\lambda_{j_1}^i),$$

and $|r| < \epsilon_{i+1} < \epsilon_i$. Hence $\Gamma \in C(S_{s_1}\lambda_{j_m}^i, \epsilon_i)$ as desired.

Suppose $m > 1$. Then by equation (6.31) we know that

$$\lambda_j^{i+1} = S_{a_{j_m}^i - \beta_m(2+\alpha) - t_m}\lambda_{j_m}^i$$

on the interval $[\beta_m(2+\alpha) + t_m, \beta_m(2+\alpha) + t + l_{j_m}^i]$ and for some $|t_m| < \epsilon_{i+1}$. Therefore

$$S_r\Gamma = S_{s_2}\lambda_j^{i+1} = S_{s_2 + a_{j_m}^i - \beta_m(2+\alpha) - t_m}\lambda_{j_m}^i$$

on the interval $[\beta_m(2+\alpha) + t_m - s_2, \beta_m(2+\alpha) + t_m + l_{j_m}^i - s_2]$.

By (6.32) and (6.13), we know $|F_m(y_2) - e_m| < \epsilon_{i+1}$ and $|e_m + a_{j_m}^i - \beta_m(2+\alpha) - t_m| < \epsilon_{i+1}$. Hence $|F_m(y_2) + a_{j_m}^i - \beta_m(2+\alpha) - t_m| < 2\epsilon_{i+1}$. Since $s_2 - s_1 = F_m(y_2)$, we get

$$|(s_2 + a_{j_m}^i - \beta_m(2+\alpha) - t_m) - s_1| = |F_m(y_2) + a_{j_m}^i - \beta_m(2+\alpha) - t_m| < 2\epsilon_{i+1}.$$

We can then write $s_2 + a_{j_m}^i - \beta_m(2+\alpha) - t_m = s_1 + r'$ for some $|r'| < 2\epsilon_{i+1}$, and $\beta_m(2+\alpha) + t_m - s_2 = a_{j_m}^i - s_1 - r'$. This implies

$$S_r\Gamma = S_{s_1+r'}\lambda_{j_m}^i \text{ on } [a_{j_m}^i - s_1 - r', a_{j_m}^i - s_1 - r' + l_{j_m}^i] = [a_{j_m}^i - s_1 - r', b_{j_m}^i - s_1 - r'],$$

which in turn implies

$$S_{r-r'}\Gamma = S_{s_1}\lambda_{j_m}^i \text{ on } [a_{j_m}^i - s_1, b_{j_m}^i - s_1] = supp(S_{s_1}\lambda_{j_m}^i).$$

As $|r - r'| < 3\epsilon_{i+1} < \epsilon_i$, we get $\Gamma \in C(S_{s_1}\lambda^i_{j_m}, \epsilon_i)$.

$\square$

For each $i \geq 1$ and $j \in J_i$, let

$$Z^i_j = \{ \bigcup_{\eta \in (p^i_j + 2\epsilon_i, q^i_j - 2\epsilon_i)} \bigcup_{|\eta - s| < \epsilon_i} \mathcal{U}_s \tilde{E}^i_j : \eta \text{ is the basepoint of a tile in } \lambda^i_j \}$$

and let $Z^i = \cup_{j \in J_i} Z^i_j$. The following lemmas shows that the sets $Z^i$ satisfy conditions (d) of Proposition 5.3.

**Lemma 6.5.** *$G^i \cap \overline{Z^{i+1}} \subset Z^i$ for all $i \in \mathbb{N}$.*

*Proof.* Let $\tilde{x} \in G^i \cap \overline{Z^{i+1}}$. As $\tilde{P}^i$ and $\tilde{P}^{i+1}$ are tower partitions of $\tilde{X}$, there exist $j \in J_{i+1}$ and $j_m \in J_i$ such that $\tilde{x} \in \tilde{\tau}^{i+1}_j \cap \tilde{\tau}^i_{j_m}$, where $\tilde{\tau}^i_{j_m}$ denotes the $m^{th}$ sub-tower of $\tilde{\tau}^{i+1}_j$. Then we can write $\tilde{x}$ in two ways

$$\tilde{x} = \mathcal{U}_{s_1}\tilde{y}_1, \text{ for some } \tilde{y}_1 = (y_1, 0) \in \tilde{E}^i_{j_m}$$

and

$$\tilde{x} = \mathcal{U}_{s_2}\tilde{y}_2, \text{ for some } \tilde{y}_2 = (y_2, 0) \in \tilde{E}^{i+1}_j$$

where $s_2 = s_1 + F_m(y_2)$.

As $\tilde{x} \in G^i$, $p^i_{j_m} < s_1 < q^i_{j_m}$ and since $\tilde{x}$ also belongs to $\overline{Z^{i+1}}$, there exists $\eta' \in \mathbb{R}$ such that $\eta'$ is the basepoint of a tile in $\lambda^{i+1}_j$ and $\tilde{x} \in \mathcal{U}_{s_2}\tilde{E}^{i+1}_j$ with $|\eta' - s_2| < \epsilon_{i+1}$. By (6.31), we have

$$\lambda^{i+1}_j = S_{a^i_{j_m} - \beta_m(2+\alpha) + t_m}\lambda^i_{j_m}$$

on the interval $[\beta_m(2+\alpha) + t_m, \beta_m(2+\alpha) + t_m + l^i_{j_m}]$. Therefore

$$\eta = \eta' + a^i_{j_m} - \beta_m(2+\alpha) + t_m$$

is the base point of a tile in $\lambda^i_{j_m}$ with $supp(\lambda^i_{j_m}) = [a^i_{j_m}, b^i_{j_m}]$. Hence, by (6.13) and (6.32), we get $|\eta' - s_1 - e_m| < 2\epsilon_{i+1}$ and hence $|s_1 - \eta| \leq 3\epsilon_{i+1} < \epsilon_i$.

Now $p^i_{j_m} < s_1 < q^i_{j_m}$ implies that $\eta \in (p^i_{j_m} - \epsilon_i, q^i_{j_m} + \epsilon_i)$. Note that $p^i_{j_m} = a^i_{j_m} + 1/2$ and $a^i_{j_m}$ is the base point of the first tile in the patch $\lambda^i_{j_m}$. Since all tiles have length at least 1, and $\sum_{n=0}^{\infty} \epsilon_n < 1/6$, we will never see the base point of a tile in the interval $(a^i_{j_m} + 1/2 - \epsilon_i, a^i_{j_m} + 1/2 + 2\epsilon_i) = (p^i_{j_m} - \epsilon_i, p^i_{j_m} + 2\epsilon_i)$. Similarly, as $b^i_{j_m}$ is the end point of the last tile in $\lambda^i_{j_m}$, we will never see the base point of a tile in the interval $(b^i_{j_m} - 1/2 + \sum_{n=1}^{i-1} \epsilon_n - 2\epsilon_i, b^i_{j_m} - 1/2 + \sum_{n=1}^{i-1} \epsilon_n + \epsilon_i) = (q^i_{j_m} - 2\epsilon_i, q^i_{j_m} + \epsilon_i)$. Therefore all base points in $(p^i_{j_m} - \epsilon_i, q^i_{j_m} + \epsilon_i)$ are actually contained in $(p^i_{j_m} + 2\epsilon_i, q^i_{j_m} - 2\epsilon_i)$. This means that

$$\tilde{x} = \mathcal{U}_{s_1}(\tilde{y}_1), \text{ for some } \tilde{y}_1 \in \tilde{E}^i_{j_m} \text{ and } |\eta - s_1| < \epsilon_i$$

and $\eta$ is the base point of a tile in $\lambda^i_{j_m}$ such that $\eta \in (p^i_{j_m} + 2\epsilon_i, q^i_{j_m} - 2\epsilon_i)$. Therefore $\tilde{x} \in Z^i_{j_m} \subset Z^i$.

$\square$

We introduce a notation here to describe the difference in heights of two points in the same tower from any given $\tilde{P}^i$. Let $\tilde{x}, \tilde{y} \in \tilde{\tau}^i_j$ for some $i \geq 1$ and $j \in J_i$, and suppose they located at heights $t$ and $s$ respectively i.e., $\tilde{x} \in \mathcal{U}_s \tilde{E}^i_j$ and $\tilde{y} \in \mathcal{U}_t \tilde{E}^i_j$. Then we define

$$|\tilde{x}, \tilde{y}|_i = |s - t|.$$

In the following lemma, we show that if two points $\tilde{x}, \tilde{y}$ are close in a tower in $\tilde{P}^{i+1}$, with $\tilde{x} \in Z^i$ and $\tilde{y} \in Z^{i+1}$, then $\tilde{y} \in Z^i$.

**Lemma 6.6.** *For $i \geq 1$ and $j \in J_{i+1}$, let $\tilde{\tau}^{i+i}_j \in \tilde{P}^{i+1}$ and let $\tilde{\tau}^i_{j_m}$ be its $m^{th}$ sub-tower. Suppose $x \in Z^i_{j_m} \cap \tilde{\tau}^{i+1}_j$ and $\tilde{y} \in \tilde{\tau}^{i+1}_j$ with $|\tilde{x}, \tilde{y}|_{i+1} < 2\epsilon_{i+1}$. Then $\tilde{y} \in Z^i_{j_m}$ and $|\tilde{x}, \tilde{y}|_i < 2\epsilon_i$.*

*Proof.* Since $x \in \tilde{\tau}^i_{j_m} \cap \tilde{\tau}^{i+1}_j$, we have $\tilde{x} = \mathcal{U}_{s_1}(x_1, 0)$ for some $(x_1, 0) \in \tilde{E}^{i+1}_j$ and $\tilde{x} = \mathcal{U}_{s_2}(x_2, 0)$

where $(x_2, 0) = \mathcal{U}_{F_m(x_1)}(x_1, 0) \in \tilde{E}^i_{j_m}$ with $s_2 = s_1 = F_m(x_i)$. As $\tilde{x} \in Z^i_{j_m}$, there exists an $\eta$ such that $\eta$ is the base point of a tile in$\lambda^i_{j_m}$ and $|s_2 - \eta| < \epsilon_i$.

As $\tilde{y} \in \tilde{\tau}^{i+1}_j$, we know $\tilde{y} = \mathcal{U}_{r_1}(y_1, 0)$ for some $(y_1, 0) \in \tilde{E}^{i+1}_j$ and as $|\tilde{x}, \tilde{y}|_{i+1} < 2\epsilon_{i+1}$, we have $|s_1 - r_1| < 2\epsilon_{i+1}$. Let $(y_2, 0) = \mathcal{U}_{F_m(y_1)}(y_1, 0) \in \tilde{E}^i_{j_m}$ and $r_2 = r_1 - F_m(y_1)$. By Proposition 3.1, we know $|F_m(x_1) - F_m(y_1)| < \epsilon_{i+1}$. Therefore we get

$$|r_2 - \eta| = |r_1 - F_m(y_1) - \eta| < |s_1 - F_m(x_1) - \eta| + 3\epsilon_{i+1} = |s_2 - \eta| + 3\epsilon_{i+1} < 4\epsilon_{i+1}.$$

As $\epsilon_i > 4\epsilon_{i+1}$, we get $|r_2 - \eta| < \epsilon_i$ which implies that $\tilde{y} \in \cup_{|s-\eta|<\epsilon_i}\mathcal{U}_s|E^i_{j_m} \subset Z_{j^i_m}$ and $|\tilde{x}, \tilde{y}|_i = |s_2 - r_2| < |s_2 - \eta| -| r_2 - \eta| < 2\epsilon_i$.

□

**Corollary 6.7.** *Suppose $\tilde{x} \in Z^n$ for all $n \leq i$, and $\tilde{y} \in Z^i$ such that $|\tilde{x}, \tilde{y}|_i < 2\epsilon_i$. Then $\tilde{y} \in Z^n$ for all $n \leq i$.*

*Proof.* Let $\tilde{x} \in Z^n$ for all $n \leq i$ and $\tilde{y} \in Z^i$ with $|\tilde{x}, \tilde{y}|_i < 2\epsilon_i$. Then by Lemma 6.6 applied to $\tilde{x} \in Z^{i-1}$, we know $\tilde{y} \in Z^{i-1}$ with $|\tilde{x}, \tilde{y}|_{i-1} < 2\epsilon_{i-1}$. Applying Lemma 6.6 successively, we get $\tilde{y} \in Z^n$ for all $n = i-1, i-2, ..., 1$.

□

**Lemma 6.8.** *For all $i \geq 1$, $Z^1 \cap Z^i \neq \emptyset$.*

*Proof.* We prove the result by induction on $i$. As $Z^1 \neq \emptyset$, the result is true for $i = 1$. Suppose the result is true for some $i = n$, i.e., $Z^1 \cap Z^n \neq \emptyset$. We will show that $Z^1 \cap Z^{n+1} \neq \emptyset$.

Let $\tilde{x} \in Z^1 \cap Z^n$. By skeleta decomposition, there exists a $j \in J_{n+1}, j_m \in J_n$, such that $\tilde{x} \in \tilde{\tau}^{n+1}_j \cap \tilde{\tau}^n_{j_m}$ and $\tilde{\tau}^n_{j_m}$ is the $m^{th}$ sub-tower of $\tilde{\tau}^{n+1}_j$. Hence we can write $\tilde{x} = \mathcal{U}_{s_1}(x_1, 0)$ for some $(x_1, 0) \in \tilde{E}^{n+1}_j$ and $\tilde{x} = \mathcal{U}_{s_2}(x_2, 0)$ where $(x_2, 0) = \mathcal{U}_{F_m(x_1)}(x_1, 0) \in \tilde{E}^n_{j_m}$ with $s_2 = s_1 = F_m(x_i)$.

As $\tilde{x} \in Z^n_{j_m}$, there exists an $\eta$ such that $\eta$ is the base point of a tile in $\lambda^n_{j_m}$ and $|s_2 - \eta| < \epsilon_n$. By (6.31), we know $\lambda^{n+1}_j = S_{a^n_{j_m} - \beta_m(2+\alpha) - t_m} \lambda^n_{j_m}$ on the interval $[\beta_m(2+\alpha)+t_m, \beta_m(2+\alpha)+t_m+l^n_{j_m}]$. for some $|e_m + a^n_{j_m} - \beta_m(2+\alpha) - t_m| < \epsilon_{i+1}$. Hence $\eta' = \eta - a^n_{j_m} - \beta_m(2+\alpha) - t_m$ is the base pooint of a tile in $\lambda^{n+1}_j$.

Choose a $(y_1, 0) \in \tilde{E}^{n+1}_j$ and an $r_1 \in \mathbb{R}$ such that $|r_1 - \eta'| < \epsilon_{n+1}$ and let $\tilde{y} = \mathcal{U}_{r_1}(y_1, 0)$ in $\tilde{\tau}^{n+1}_j$. Then $\tilde{y} \in Z^{n+1}_j \subset Z^{n+1}$. We will show that $\tilde{y} \in Z^1$.

Let $(y_2, 0) = \mathcal{U}_{F_m(y_1)}(y_1, 0) \in \tilde{E}^i_{j_m}$ and $r_2 = r_1 - F_m(y_1)$. By (6.13), we know $|e_m - f_m(y_1)| < \epsilon_{n+1}$, and therefore

$$|r_2 - \eta| = |r_1 - e_m - \eta' - a^n_{j_m} + \beta_m(2+\alpha) + t_m| + \epsilon_{n+1} < |r_1 - \eta| + 2\epsilon_{n+1}.$$

As $|r_1 = \eta| < \epsilon_{n+1}$, we have $|r_2 - \eta| < 3\epsilon_{n+1} < \epsilon_n$. This implies $\tilde{y} \in \mathbb{Z}^i_{j_m}$ and as $|r_2 - \eta| < \epsilon_n$, we get $|r_2 - s_2| < 2\epsilon_n$. By Corollary 6.7, we get $\tilde{y} \in Z^k$ for all $k = 1, 2, ..., n$. Hence $\tilde{y} \in Z^1 \cap Z^{n+1} \neq \emptyset$.

$\square$

We have now proved that for all $i \in \mathbb{N}$, the $\epsilon_i$-stencils $(\lambda^i_j, G^i_j, \phi^i_j)$ and the templates $(G^i, \phi^i)$ satisfy the hypothesis of Proposition 5.3.

## *6.3 The two-step n.c. flow*

By Proposition 5.3, we get a map $\phi : \tilde{X} \to Y$ so that the set $Z$, consisting of all points $\tilde{x} \in \tilde{X}$ such that the origin is located at the base point of a tile in $\phi(\tilde{x})$, forms a non-empty $G_\delta$-subset of $\tilde{X}$. Let $Z$ inherit the subspace topology from $\tilde{X}$. As $\tilde{X}$ is Polish, $Z$ forms a Polish space with respect to the subspace topology. In this section, we will define a homeomorphism $T_Z : Z \to Z$ and a continuous map $g : Z \to \{1, 1+\alpha\}$. We will then show that the topological space $\hat{Z}$ consisting of the points under the graph of $g$ over $(Z, T_Z)$, with every point $(\tilde{z}, g(\tilde{z}))$ identified with $(T_Z\tilde{z}, 0)$, and endowed with the product topology of $Z$ and $\mathbb{R}$, is homeomorphic to $\tilde{X}$. In the end, we will show that there exists a Borel measure $\nu$ on $Z$ such that the flow built under the function $g$ over $(Z, \nu, T_z)$ is n.c. conjugate to $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ on $\tilde{X}$.

First note that every point $\tilde{z} \in Z$ has the property that the origin is located at the base point of a tile in $\phi(\tilde{x})$. As all tiles in $Y$ have length either 1 or $1+\alpha$, we get either $\mathcal{U}_1\tilde{z} \in Z$ or $\mathcal{U}_{1+\alpha}\tilde{z} \in Z$, depending on the length of the tile that begins at the origin of $\phi(\tilde{z})$. And similarly, either $\mathcal{U}_{-1}\tilde{z} \in Z$ or $\mathcal{U}_{-1-\alpha}\tilde{z} \in Z$ depending on the length of the tile that ends at the origin. We use this property to define four sets $Z_i, i = \pm 1, \pm 2$ as follows:

$$\begin{aligned} Z_1 = &\quad \{\tilde{z} \in Z : \mathcal{U}_1\tilde{z} \in Z\} &&= Z \cap \mathcal{U}_{-1}Z \\ Z_2 = &\quad \{\tilde{z} \in Z : \mathcal{U}_{1+\alpha}\tilde{z} \in Z\} &&= Z \cap \mathcal{U}_{-1-\alpha}Z \\ Z_{-1} = &\quad \{\tilde{z} \in Z : \mathcal{U}_{-1}\tilde{z} \in Z\} &&= Z \cap \mathcal{U}_{1}Z \\ Z_{-2} = &\quad \{\tilde{z} \in Z : \mathcal{U}_{-1-\alpha}\tilde{z} \in Z\} &&= Z \cap \mathcal{U}_{1+\alpha}Z \end{aligned} \tag{6.37}$$

Note that $Z = Z_1 \cup Z_2 = Z_{-1} \cup Z_{-2}$.

**Lemma 6.1.** *The sets $Z_i, i = \pm 1, \pm 2$ are open subsets of $Z$.*

*Proof.* We only show that $Z_1$ is open in $Z$. The argument for openness of the remaining sets is similar. Let $\tilde{z} \in Z_1$. Then the origin is located at the base point of a tile of length 1 in $\phi(\tilde{z})$. By Lemma 6.2, there exists an $i \in \mathbb{N}$ and a $j \in J_i$ such that the partial orbit $\{\mathcal{U}_s\tilde{z} : s \in (-2-\alpha, 2+\alpha)\}$ is contained in $G^i_j$ of the tower $\tilde{\tau}^i_j$. Therefore $\tilde{z} = \mathcal{U}_t\tilde{x}$ for some $\tilde{x} \in \tilde{E}^i_j$ and $t \in (p^i_j+2+\alpha, q^i_j-2-\alpha)$.

As $\tilde{z} \in Z$, there exists an $t_0 \in (t-\epsilon_i, t+\epsilon_i)$ so that the patch $\lambda^i_j$ has the base point of a tile of length 1 located at $t_0$. Now let $U = \cup_{s\in(t_0-2\epsilon_i, t_0+2\epsilon_i)}$. Then $U$ is open in $\tilde{X}$ and $\tilde{z} \in \mathcal{U}$. Also, for every point $\tilde{y} \in U \cap Z$, the origin is also located at the base point of a tile of length 1 in $\phi(\tilde{y})$. Hence $U \cap Z \subset Z_1$, implying $Z_1$ is open in $Z$. $\square$

Define a map $T_Z : Z \to Z$ based on the time of first return to the set $Z$, with respect to the flow $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$, as follows:

$$T_Z\tilde{z} = \begin{cases} \mathcal{U}_1\tilde{z} & \text{if } \tilde{z} \in Z_1 \\ \mathcal{U}_{1+\alpha}\tilde{z} & \text{if } \tilde{z} \in Z_2 \end{cases} \tag{6.38}$$

**Lemma 6.2.** *$T_z$ is a homeomorphism on $Z$.*

*Proof.* It is clear from definition that $T_Z$ is well-defined on $Z$. To see that $T_Z$ is 1-to-1, suppose there exist $\tilde{z}_1, \tilde{z}_2 \in Z$ such that $T_Z\tilde{z}_1 = T_Z\tilde{z}_2$. Without loss of generality, assume $\tilde{z}_1 \in Z_1$. Then $\tau_Z\tilde{z}_1 = \mathcal{U}_1 z_1$.

By definition, $T_Z\tilde{z}_2$ is either $\mathcal{U}_1\tilde{z}_2$ or $\mathcal{U}_{1+\alpha}\tilde{z}_2$. If $T_Z\tilde{z}_2 = \mathcal{U}_{1+\alpha}\tilde{z}_2$, then $\mathcal{U}_1 z_1 = \mathcal{U}_{1+\alpha}\tilde{z}_2$, which implies $z_1 = U_\alpha\tilde{z}_2$, and hence $\phi(\tilde{z}_1) = \phi(\mathcal{U}_\alpha\tilde{z}_2) = S_\alpha\phi(\tilde{z}_2)$. Now both $\tilde{z}_1$ and $\tilde{z}_2$ are in $Z$, therefore both $\phi(\tilde{z}_1)$ and $\phi(\tilde{z}_2)$ have base points of some tiles at their origins, which means $\phi(\tilde{z}_1)$ cannot be the same as $S_\alpha\phi(\tilde{z}_2)$, as there are no tiles or patches of length $\alpha$ in the tiling space $Y$. Therefore $T_Z\tilde{z}_2 = \mathcal{U}_1\tilde{z}_2 = \mathcal{U}_1 z_1$, which implies $\tilde{z}_1 = \tilde{z}_2$ as $\mathcal{U}_1$ is a homeomorphism of $\tilde{X}$.

To show that $T_Z$ is a homeomorphism of $Z$, it suffices to show that given any open $C \subset Z$, both $T_Z^{-1}C$ and $T_Z C$ are open in $Z$. Fix an open subset $C$ in $Z$. Note that

$$\begin{aligned} T_Z^{-1}C &= (\mathcal{U}_{-1}C \cap Z_1) \cup (\mathcal{U}_{-1-\alpha}C \cap Z_2) \\ \text{and } T_Z C &= (\mathcal{U}_1 C \cap Z_{-1}) \cup (\mathcal{U}_{1+\alpha}C \cap Z_{-2}) \end{aligned}$$

where $Z_i, i = \pm 1, \pm 2$, are as defined in (6.37). We first show that $\mathcal{U}_{-1}C \cap Z_1$ is open in $Z$.

$C$ is open in $Z$ implies that there exists an open $V \subset \tilde{X}$ such that $V \cap Z = C$. Therefore

$$\mathcal{U}_{-1}C \cap Z_1 = \mathcal{U}_{-1}(V \cap Z) \cap Z_1 = \mathcal{U}_{-1}V \cap \mathcal{U}_{-1}Z \cap Z_1 = \mathcal{U}_{-1}V \cap Z_1$$

as $\mathcal{U}_{-1}Z \cap Z_1 = Z_1$. As $\mathcal{U}_{-1}V$ is open in $\tilde{X}$, we have $\mathcal{U}_{-1}V \cap Z$ is open in $Z$. Also, $Z_1$ is open in $Z$. Therefore $(\mathcal{U}_{-1}V \cap Z) \cap Z_1 = \mathcal{U}_{-1}V \cap Z_1$ is open in $Z$, which means $U_{-1}C \cap Z_1$ is an open subset of $Z$.

Using similar arguments as above, we can show that the sets $\mathcal{U}_{-1-\alpha}C \cap Z_2, \mathcal{U}_1 C \cap Z_{-1}$ and $\mathcal{U}_{1+\alpha}C \cap Z_{-2}$ are also open in $Z$. Therefore $T_Z$ is a homeomorphism of $Z$.

□

Next we define a function $g : Z \to \{1, 1+\alpha\}$ to be the first return time function on $Z$, given by

$$g(\tilde{z}) = \begin{cases} 1 & \text{if } \tilde{z} \in Z_1 \\ 1+\alpha & \text{if } \tilde{z} \in Z_2 \end{cases}$$

**Lemma 6.3.** *$g$ is continuous on $Z$.*

*Proof.* $g$ is continuous as both $g^{-1}(\{1\}) = Z_1$ and $g^{-1}(\{1+\alpha\}) = Z_2$, are open in $Z$.

□

Note that, using the definition of $g$, we can write $T_Z$ in terms of $g$, so that

$$T_Z \tilde{z} = \mathcal{U}_{g(\tilde{z})} \tilde{z} \quad \text{for all } \tilde{z} \in Z. \tag{6.39}$$

Now consider the product space $Z \times \mathbb{R}$ with the product topology. In this space identify every point of the form $(\tilde{z}, g(\tilde{z}))$ with the point $(T_Z \tilde{z}, 0)$. With this identification, we let $\hat{Z}$ to be the part of $Z \times \mathbb{R}$ that lie under the graph of $g$ i.e.,

$$\hat{Z} = \{(\tilde{z}, p) : \tilde{z} \in Z, 0 \leq p < g(\tilde{z})\}$$

It is easy to check from definitions that $\hat{Z}$ forms a Polish space with respect to the identification topology. Also, for ease of notation, let $g(\tilde{z}, n)$ denote the time it takes for a point $\tilde{z} \in Z$ to return $n$ times to $Z$, under the map $T_Z$, i.e.,

$$g(\tilde{z},n) = \begin{cases} \sum_{i=0}^{n-1} g(T_Z^i \tilde{z}) & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ \sum_{i=n}^{-1} g(T_Z^i \tilde{z}) & \text{if } n < 0 \end{cases} \tag{6.40}$$

Then, using the identification $(\tilde{z}, g(\tilde{z})) = (T_Z\tilde{z}, 0)$, we get that if $t \in \mathbb{R}$, then

$$(\tilde{z}, p+t) = (T_Z^n \tilde{z}, p+t-g(\tilde{z},n)) \tag{6.41}$$

where $n \in \mathbb{Z}$ such that $g(\tilde{z},n) \leq p+t < g(\tilde{z},n+1)$.

We want to show that the spaces $\tilde{X}$ and $\hat{Z}$ are topologically the same. To understand the sameness, observe that for every $\tilde{x} \in \tilde{X}$, by definition, $\phi(\tilde{x})$ is a tiling consisting of tiles of length 1 and $1+\alpha$. Therefore, there exists a unique $p$ such that the origin shows up at a distance $p$ from the base point of a tile in $\phi(\tilde{x})$, where $0 < p < 1$ if the origin of $\phi(\tilde{x})$ is strictly inside a tile of length 1, $0 < p < 1+\alpha$ if the origin of $\phi(\tilde{x})$ is strictly inside a tile of length $1+\alpha$ or $p = 0$ if the origin is located at the base point of a tile. In other words, there exists a unique $p$ such that $S_{-p}\phi(\tilde{x})$ has its origin at the base point of a tile, and $0 \leq p <$ the length of this tile. As $S_{-p}\phi(\tilde{x}) = \phi(\mathcal{U}_{-p}\tilde{x})$, this means that there exists a unique $p$ such that $\mathcal{U}_{-p}\tilde{x} = \tilde{z}$ for some $\tilde{z} \in Z$ and $0 \leq p < g(\tilde{z})$. Here $g(z)$ is the time it takes for $\tilde{z}$ to return to $Z$ which is in fact, the length of the tile containing $\tilde{z}$. Therefore, every point $\tilde{x} \in \tilde{X}$ can be uniquely represented as

$$\tilde{x} = \mathcal{U}_p\tilde{z}, \text{ for some } \tilde{z} \in Z \text{ and } 0 \leq p < g(\tilde{z}). \tag{6.42}$$

Using the above representation, define a map $\psi : \tilde{X} \to \hat{Z}$ by

$$\psi(\tilde{x}) = (\tilde{z}, p). \tag{6.43}$$

The uniqueness of the representation implies that $\psi$ is a well-defined bijection between $\tilde{X}$ and $\hat{Z}$.

**Lemma 6.4.** *$\tilde{X}$ and $\hat{Z}$ are homeomorphic via the map $\psi$.*

*Proof.* We first show $\psi$ is continuous. Suppose $A$ is an open subset of $\hat{Z}$ and $(\tilde{z}, p) \in A$. Let $\tilde{x} = \psi^{-1}(\tilde{z}, p)$, i.e., $\tilde{x} = \mathcal{U}_p \tilde{z}$ in $\tilde{X}$.

As $A$ is open in $\mathbb{Z}$, there exists an open $C \in Z$ and $a < p < b \in \mathbb{R}$ such that $(\tilde{z}, p) \in C \times (a, b) \subset A$. Choose $n \in \mathbb{N}$ such that $4\epsilon_n < \min\{p - a, b - p\}$. By Lemmas 6.2 and 6.3, there exists an $i \geq n$ and $j \in J_i$ such that $\tilde{z} \in G^i_j$. As $\tilde{z} \in Z$, there exists a $t_0 \in \mathbb{R}$ such that $t_0$ is the base point of a tile in the patch $\lambda^i_j$ and $\tilde{z} \in R = \cup_{|s-t_0|<\epsilon_i} \mathcal{U}_s E^i_j$.

As $C$ is open in $Z$, there exists an open set $U \in \tilde{X}$ so that $C = U \cap Z$. This implies that $\tilde{z} \in U \cap R$. Let $V = \mathcal{U}_p(U \cap R)$. Then $V$ is open in $\tilde{X}$ and $\tilde{x} = \mathcal{U}_p \tilde{z} \in V$. We will show that $\psi(V) \subset A$.

Let $\tilde{y} \in V$. Then $\mathcal{U}_{-p}\tilde{y} \in U \cap R$, and we can write $\mathcal{U}_{-p}\tilde{y} = \mathcal{U}_q \tilde{h}$ for some $\tilde{h} \in (U \cap R) \cap Z$ and $|q| < 2\epsilon_i$. As $U \cap Z = C$ and $\epsilon_i < \epsilon_n$, we get $\tilde{y} = \mathcal{U}_{p+q}\tilde{h}$ for some $\tilde{h} \in C$ and $p + q \in (p - 2\epsilon_n, p + 2\epsilon_n) \subset (a, b)$. Hence, $\psi(\tilde{y}) \in C \times (a, b) \subset A$ and $\psi$ is continuous.

To show that the inverse map $\psi^{-1}$ is also continuous, let $U$ be an open subset of $\tilde{X}$. It suffices to show that for every $\tilde{x} \in U$, there exists an open $V \in \hat{Z}$ such that $\psi(x) \in V \subset \psi(U)$. Fix an $\tilde{x} \in U$. Then $\psi(\tilde{x}) = (\tilde{z}, p)$, where $x$ has the unique representation $\tilde{x} = \mathcal{U}_p \tilde{z}$ or some $\tilde{z} \in \mathbb{Z}$ and $0 \leq p < g(\tilde{z})$.

Since $U$ is open and $\tilde{x} \in U$, there exists an open $A \subset X$ and $a, b \in \mathbb{R}$ such that $\tilde{x} = (x, t) \in A \times (a, b) \subset U$. Choose $n \in \mathbb{N}$ such that $4\epsilon_n < \min\{t - a, b - t\}$. By Lemmas 6.2 and 6.3, there exists an $i \geq n$ and $j \in J_i$ such that $\mathcal{U}_{-p}\tilde{x} \in G^i_j$. As $\mathcal{U}_{-p}\tilde{x} = \tilde{z} \in Z$, there exists a $t_0 \in \mathbb{R}$ such that

$t_0$ is the base point of a tile in the patch $\lambda_j^i$ and $\tilde{z} \in R = \cup_{|s-t_0|<\epsilon_i} \mathcal{U}_s E_j^i$.

Let $C = (R \cap A \times (t-p-\epsilon_i, t-p+\epsilon_i)) \cap Z$. Note that $C$ is open in $Z$ and $\tilde{z} \in C$. Let $V = C \times (p-\epsilon_i, p+\epsilon_i)$. Then $V$ is open in $\hat{Z}$ and $\psi(\tilde{x}) = (\tilde{z}, p) \in V$. It remains to show that $V \subset \psi(U)$.

Let $(h,q) \in V$. Then $h \in C \subset A \times (t-p-\epsilon_i, t-p+\epsilon_i)$ and $|p-q| < \epsilon_i$. Therefore we have $\mathcal{U}_q \tilde{h} \in A \times (t-(p-q)-\epsilon_i, t-(p-q)+\epsilon_i) \subset A \times (t-2\epsilon_i, t+2\epsilon_i) \subset A \times (a,b) \subset U$. This implies $(h,q) = \psi(\mathcal{U}_q \tilde{h}) \in \psi(U)$, and hence $\psi^{-1}$ is continuous.

$\square$

Next, we define a measure $\tilde{\nu}$ and a $\sigma$-algebra of measurable sets $\mathcal{G}$ on $\hat{Z}$, to be the respective push forwards of the measure $\tilde{\mu}$ and the $\sigma$-algebra of measurable sets $\mathcal{F}$ on $\tilde{X}$, i.e.,

$$\mathcal{G} = \{A \subset \tilde{X} : \psi^{-1}(A) \in \mathcal{F}\} \text{ and}$$

$$\tilde{\nu}(A) = \tilde{\mu}(\psi^{-1}A) \text{ whenever } A \in \mathcal{G}.$$

Using $\psi$, we also define a flow $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ on $\hat{Z}$ by

$$\mathcal{V}_t = \psi \circ \mathcal{U}_t \circ \psi^{-1} \quad \text{for all } t \in \mathbb{R}. \tag{6.44}$$

Then $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ is measurable with respect to $\tilde{\nu}$ on $\hat{Z}$ and is measurably conjugate to $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$. It is a straight forward computation to check that $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ satisfies the definition of the flow built over $T_Z$ under the function $g$, i.e., for all $(\tilde{z}, p) \in \hat{Z}$,

$$\mathcal{V}_t(\tilde{z}, p) = (T_Z^n \tilde{z}, p + t - g(\tilde{z}, n)) \tag{6.45}$$

where $n \in \mathbb{Z}$ such that $g(\tilde{z}, n) \le p + t < g(\tilde{z}, n+1)$ and $g(\tilde{z}, n)$ is as defined in (6.40).

At this point we have defined *a* measure on $\hat{Z}$ that makes the flows $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$ and $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ n.c. conjugate. We still need to show that $Z$ has a Borel measure $\nu$ and with respect to the product of $\nu$ and Lebesgue measure on $\hat{Z}$, the flow built over $T_Z$ under $g$ is in fact n.c. conjugate to $\{\mathcal{U}_t\}_{t\in\mathbb{R}}$. We will use the following representation theorem of Ambrose to show the existence of such a measure $\nu$ on $Z$.

**Theorem 6.5** (Ambrose[1])**.** *Let $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ be as defined in (6.45) on $\hat{Z}$, with a measure $\tilde{\nu}$ on $\hat{Z}$. If $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ is a measurable flow and if the functions $\tilde{F}$ and $\tilde{G}$ defined by*

$$\tilde{F}(\tilde{z},p) = g(\tilde{z}), \quad \tilde{G}(\tilde{z},p) = p \tag{6.46}$$

*for all $(\tilde{z},p) \in \hat{Z}$, are both $\tilde{\nu}$-measurable, then there exists a measure $\nu$ on $Z$ for which $g$ is a measurable function and $T_Z$ is a measure-preserving transformation and such that $\tilde{\nu}$ is the completed direct product measure of $\nu$ on $Z$ with Lebesgue measure on the vertical axis.*

**Proposition 6.6.** *Let $(\hat{Z}, \tilde{\nu}, \{\mathcal{V}_t\}_{t\in\mathbb{R}})$ be as defined above. Then there exists a Borel measure $\nu$ on $Z$ such that $\{\mathcal{V}_t\}_{t\in\mathbb{R}})$ forms the flow built over $T_Z$ under the function $g$.*

*Proof.* We will first show that the functions $\tilde{F}$ and $\tilde{G}$ as defined in (6.46) are $\tilde{\nu}$-measurable on $\hat{Z}$, and obtain a measure $\nu$ on $Z$ so that $\tilde{\nu} = \nu \times$Lebesgue on $\hat{Z}$. We will then show that $\nu$ is Borel and $T_Z$ is ergodic with respect to $\nu$. This will imply that $(Z, \nu, T_Z)$ is a n.c. $\mathbb{Z}$-system and since $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ satisfies (6.45), and $g$ is continuous on $Z$, $\{\mathcal{V}_t\}_{t\in\mathbb{R}}$ will indeed be the flow built over $T_Z$ under $g$.

Set $F = \tilde{F} \circ \psi$ and $G = \tilde{G} \circ \psi$ on $\tilde{X}$. To show that $\tilde{F}$ and $\tilde{G}$ are $\tilde{\nu}$-measurable, it suffices to show that $F$ and $G$ are $\tilde{\nu}$ measurable, as $\tilde{\nu}$ is the push-forard measure of $\tilde{\mu}$. Let $B \subset \tilde{X}$ be a fattening the set $Z$ defined by,

$$B = \bigcup_{t\in[0,\frac{1}{2}]} \mathcal{U}_t Z.$$

Since $Z = \cap_{n\in\mathbb{N}} \cup_{i\geq n} Z^i$, where each $Z^i = \cup_{j\in J_i} \cup_{m=1}^{n(i,j)} \cup_{|t-\eta_m|<\epsilon_i} \mathcal{U}_t \tilde{E}^i_j$, we get

$$B = \bigcap_{n\in\mathbb{N}} \bigcup_{i\geq n} \bigcup_{j\in J_i} \bigcup_{m=1}^{n(i,j)} \bigcup_{t=\eta_m-\epsilon_i}^{\frac{1}{2}+\eta_m+\epsilon_i} \mathcal{U}_t \tilde{E}^i_j,$$

and hence is measurable in $\tilde{X}$. Now

$$\{\tilde{x} : F(\tilde{x}) = 1\} = \bigcup_{t\in[0,\frac{1}{2})\cap\mathbb{Q}} \mathcal{U}_t(B \cap \mathcal{U}_1 B)$$

and

$$\{\tilde{x} : F(\tilde{x}) = 1+\alpha\} = \bigcup_{t\in[0,\frac{1}{2})\cap\mathbb{Q}} \mathcal{U}_t(B \cap \mathcal{U}_{1+\alpha} B)$$

and therefore $F$ is measurable on $\tilde{X}$. Similarly, $G$ is also measurable as

$$\{\tilde{x} : G(\tilde{x}) = r\} = \begin{cases} \bigcap_{t\in[0,\frac{1}{2}-r]\cap\mathbb{Q}} \mathcal{U}_t B & \text{if } r \leq \frac{1}{2} \\ \bigcap_{t\in[r-\frac{1}{2},r]\cap\mathbb{Q}} \mathcal{U}_t B & \text{if } r \geq \frac{1}{2} \end{cases}$$

Hence $\tilde{F}$ and $\tilde{G}$ are $\tilde{\nu}$-measurable in $\hat{Z}$, and by Theorem 6.5, there exista a measure $\nu$ on $Z$ for which $g$ is a measurable function and $T_Z$ is a measure-preserving transformation and $\tilde{\nu} = \nu\times$Lebesgue measure. The measure $\nu$ and the $\sigma$ algebra $\mathcal{B}$ on $Z$ are defined as follows:

$$\mathcal{B} = \{C \subset Z : \{(\tilde{z}, p) : \tilde{z} \in C, 0 \leq p < g(\tilde{z})\} \in \mathcal{G}\}, \tag{6.47}$$

and for all $C \in \mathcal{B}$,

$$\nu(C) = 2\bar{\nu}(C \times (0, 1/2))$$

We next show that $\nu$ is a Borel measure on $Z$. To see this, let $C$ be an open subset of $Z$. Let

$C_1 = C \cap Z_1$ and $C_2 = C \cap Z_2$ be open subsets of $Z$ so that $C = C_1 \cup C_2$. Define

$$\bar{C}_1 = \cup_{s \in (0,1)} \mathcal{U}_s C_1 \quad \bar{C}_2 = \cup_{s \in (0,1+\alpha)} \mathcal{U}_s C_2.$$

By (6.47), to show that $C$ is measurable in $Z$, is equivalent to showing $\bar{C}_1 \cup C_1$ and $\bar{C}_2 \cup C_2$ belong to $\mathcal{G}$. Note that $Z_1 = Z \cap \mathcal{U}_{-1} Z$ and $Z_2 = Z \cap \mathcal{U}_{-1-\alpha} Z$. Since $Z$ is a $G_\delta$ Subset of $\tilde{X}$, $Z_1, Z_2$ are measurable in $\tilde{X}$. Hence $C_1, C_2$ are measurable in $\tilde{X}$.

To show is that $\bar{C}_1, \bar{C}_2 \in \mathcal{G}$, note that $\psi(\bar{C}_1) = C_1 \times (0,1)$ and $\psi(\bar{C}_2) = C_2 \times (0, 1+\alpha)$, are open with respect to the product topology on $\hat{Z}$. Since $\hat{Z}$ is homeomorphic to $\tilde{X}$, $\bar{C}_1$ and $\bar{C}_2$, are open in $\tilde{X}$, and therefore measurable in $\tilde{X}$. Hence $C$ is a measurable subset of $\mathbb{Z}$ and $\nu$ is Borel.

All that remains to show is that $T_Z$ is ergodic with respect to $\nu$. Suppose not, then there exists a $T_Z$-invariant $A \subset Z$ with $0 < \nu(A) < 1$. Define $U \subset \tilde{X}$ to be the set

$$U = \bigcup_{s \in [0,1)} \mathcal{U}_s (A \cap Z_1) \quad \cup \bigcup_{s \in [0,1+\alpha)} \mathcal{U}_s (A \cap Z_2).$$

Then $0 < \tilde{\mu}(U) < 1$ and for any $t \in \mathbb{R}$, we have $\mathcal{U}_t U = U$. This is a contradiction as $\mathcal{U}_t$ is ergodic on $\tilde{X}$. Hence $T_Z$ is ergodic on $Z$ and as a result, $\{\mathcal{V}_t\}_{t \in \mathbb{R}})$ is indeed the flow built over $T_Z$ under the function $g$. □

**Corollary 6.7.** $(\tilde{X}, \tilde{\mu}, \{\mathcal{U}_t\}_{t \in \mathbb{R}})$ *is continuously conjugate to* $(\hat{Z}, \tilde{\nu}, \{\mathcal{V}_t\}_{t \in \mathbb{R}})$

*Proof.* By Proposition 6.6, $\{\mathcal{V}_t\}_{t \in \mathbb{R}}$ is the flow built over $T_Z$ under $g$. By (6.44) and Proposition 6.4, the two flows are continuously conjugate.

□

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