THESIS

SYMMETRIC FUNCTIONS, SHIFTED TABLEAUX, AND A CLASS OF DISTINCT SCHUR Q-FUNCTIONS

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ABSTRACT

SYMMETRIC FUNCTIONS, SHIFTED TABLEAUX, AND A CLASS OF DISTINCT SCHUR Q-FUNCTIONS

The Schur Q-functions form a basis of the algebra Ω of symmetric functions generated by the odd-degree power sum basis p_d, and have ramifications in the projective representations of the symmetric group. So, as with ordinary Schur functions, it is relevant to consider the equality of skew Schur Q-functions $Q_{\lambda/\mu}$. This has been studied in 2008 by Barekat and van Willigenburg in the case when the shifted skew shape λ/μ is a ribbon. Building on this premise, we examine the case of near-ribbon shapes, formed by adding one box to a ribbon skew shape. We particularly consider frayed ribbons, that is, the near-ribbons whose shifted skew shape is not an ordinary skew shape. We conjecture with evidence that all Schur Q functions for frayed ribbon shapes are distinct up to antipodal reflection. We prove this conjecture for several infinite families of frayed ribbons, using a new approach via the "lattice walks" version of the shifted Littlewood-Richardson rule, discovered in 2018 by Gillespie, Levinson, and Purbhoo.

ACKNOWLEDGEMENTS

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Chapter 1

Introduction

The study of symmetric function theory lies in the intersection between pure combinatorics and its applications to other fields, namely algebraic geometry and representation theory. On one side, there are combinatorial definitions of the Schur functions, Schur Q- and P- functions, and others, that allow for the study of various constructions within the different algebras of symmetric functions. On the other side, these discoveries directly answer certain questions relation to decompositions in certain geometric objects and in the theory related to representations of the symmetric group.

Ordinary Schur functions arise in algebraic geometry through the cohomology ring of the Grassmannian. Here, multiplying Schur functions in the algebra of symmetric functions corresponds to multiplying cohomology classes in $H^*(\mathrm{Gr}(n,k))$, and this relation gives us insight into the intersection of Schubert varieties. The ties between Schur P- and Q-functions and algebraic geometry are similar: multiplication of Schur P-functions corresponds to multiplication in the cohomology ring of the orthogonal Grassmannian.

In representation theory, Schur functions occur as characters of finite-dimensional irreducible representations of the general linear group, and in representations of the symmetric group via the **Frobenius map**: Given a class function χ on the conjugacy classes of S_n, define

$$\mathrm{Frob}(\chi) = \frac{1}{n!}\sum_{\pi\in S_n}\chi(\pi)p_{c(\pi)}$$

where $p_{c(\pi)}$ is the power sum symmetric function for the partition given by the cycle type of π. When χ_λ is a character of an irreducible representation V_λ of S_n, we find that $\mathrm{Frob}(\chi_\lambda) = s_\lambda$, the Schur function for the partition λ. Further, if χ' is the character of the induced representation of $V_\lambda \otimes V_\mu$, then $\mathrm{Frob}(\chi') = s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu$, where $c^\nu_{\lambda,\mu}$ is the Littlewood-Richardson coefficient for the triple ν, λ, μ.

In regards to Schur P- and Q-functions, we consider the projective representations of the symmetric group, that is, homomorphisms from S_n to $\mathrm{PGL}_n(\mathbb{C})$. These correspond with non-projective representations of a double cover $\tilde{S_n}$ of S_n, and are called *spin representations*. The conjugacy classes of $\tilde{S_n}$ are indexed by odd partitions, and the irreducible representations are indexed by strict partitions. Given a class function χ on the conjugacy classes of $\tilde{S_n}$, define the **projective Frobenius map** to be

$$\mathrm{PFrob}(\chi) = \sum_{\mu \in OP} \frac{1}{z_\mu} 2^{\ell(\mu)/2} \chi(\mu) p_\mu$$

where z_μ is the size of the centralizer of any permutation of cycle-type μ. Then, for an irreducible character χ_λ corresponding to a strict partition λ, for some fixed k, we have $\mathrm{PFrob}(\chi_\lambda) = 2^{k/2} Q_\lambda$, where Q_λ is the Schur Q-function corresponding to the partition λ. For more on this connection, see [1].

These connections motivate the study of structure constants - coefficients arising from expressing a product of basis elements in terms of a given basis - in the algebra of symmetric functions, since in doing so, we answer questions about intersections or decompositions in algebraic geometry or representation theory. Since the same structure constants occur in the products of Schur P-functions and in the expression of a skew Schur Q-functions, we need only study the skew case of Schur Q-functions.

Originally stated by Schur [2], the Schur P- and Q-functions can be defined in multiple equivalent ways, including as the $t = -1$ evaluation of the Hall-Littlewood P- and Q-polynomials [3]. Another is in terms of semistandard shifted tableaux with entries from a “doubled" alphabet $1' < 1 < 2' < 2 < 3' < 3 < \ldots$. The latter combinatorial definition of Schur P- and Q-functions has opened opportunities for further understanding of the functions and their connections with other fields, as the theory of shifted tableaux has developed.

Sagan [4] and Worley [5] developed shifted versions of combinatorial tools used in non-shifted tableaux, including the Robinson-Schensted-Knuth bijection, the Knuth equivalence relations, and jeu de taquin sliding moves. Sagan further used these to prove Stanley’s conjecture that the straight-shape Schur P- and Q-functions are Schur-positive with regards to the original Schur basis s_λ.

Building on these, Haiman [6] developed the process of mixed insertion on shifted tableaux, answering several more open questions in this direction.

For Schur Q functions, the above combinatorial theory gave rise to a generalized definition of Schur Q functions $Q_{\lambda/\mu}(x_1, x_2, \ldots)$ for skew shifted shapes λ/μ. These are known to expand positively in terms of the straight shape Schur Q basis. The resulting *shifted Littlewood-Richardson coefficients* in this expansion have several known combinatorial rules [1, 7, 8], and in fact coincide with the structure coefficients that arise when multiplying two Schur P functions and expressing the result back in the Schur P basis.

In the study of skew Schur Q functions, the following natural problem remains largely open.

Question 1.0.1. When are two skew Schur Q functions equal to each other? More generally, when is their difference Schur Q positive (when expanded in the Schur Q basis)?

The natural analog of Question 1.0.1 has been studied more thoroughly for the unshifted case of ordinary Schur functions. In [9], van Willigenburg characterized the case when a skew Schur function is equal to a straight shape Schur function, finding that $s_{\lambda/\mu}$ and s_ν are equal only when λ/μ and ν are the same shape, or 180° rotations of each other. Billera, Thomas, and van Willigenburg [10] determined an exact condition for the equality of ribbon Schur functions. Reiner, Shaw, and van Willigenburg [11] expanded on this result, giving further conditions for equality for general shapes, and soon after, McNamara and van Willigenburg [12] gave a single composition operation that maintains Schur equality. Similar results for the problem of determining when the difference of two skew Schur functions is Schur positive were given in, for instance, [13–16].

In the case of Schur Q-functions, Salmasian [17] found exact criteria for when the Schur Q-function $Q_{\lambda/\mu}$ of a shifted skew shape was equal to that of a shifted straight shape function Q_ν. Barekat and van Willigenburg [18] investigated the problem of Schur Q-function equality in the case of ribbons, finding a construction via compositions that gives families of shapes with equal Schur Q-function, and conjecturing that it is a necessary and sufficient condition for equality. However, the remaining results from the ordinary Schur function case have not yet been replicated for Schur Q-functions.

In this thesis, we introduce the basics of symmetric function theory, present new results by the author and Gillespie from [19] on the open problem of determining when two skew Schur Q-functions are equal, and conjecture with evidence that these results extend to an infinite class of families of distinct skew Schur Q-functions.

In Chapter 2, we walk through the necessary background information on symmetric functions, including pertinent definitions related to partitions, which index symmetric functions, and describe several bases of the algebra of symmetric functions. We focus on the basis of Schur functions, which will prove a useful analogue to our primary object of study, the Schur Q-functions. We then give two equivalent definitions of the Schur Q-function Q_λ, one related to the Hall-Littlewood polynomial, and the other from a more combinatorial perspective.

In Chapter 3, we discuss the computational tools that are relevant for the study of the Schur and Schur Q- functions. In the case of Schur functions, the Littlewood-Richardson rule gives a combinatorial formula for the decomposition of a skew Schur functions into the basis of non-skew Schur functions. In the case of Schur Q-functions, we use a new lattice-walk analogue of the Littlewood-Richardson rule, discovered in 2018 by Gillespie, Levinson and Purbhoo. Further, we exhibit computational evidence towards our main conjecture, that is:

Conjecture 1.0.2. *If D and E are frayed ribbon shapes such that $Q_D = Q_E$, then we have either $D = E$ or $D = E^a$.*

In Chapter 4, we state and prove our main theorem, which provides an infinite class of skew shifted partitions with distinct Schur Q-function:

Theorem 1.0.3. *If D and E are frayed ribbons with $Q_D = Q_E$, then:*

- *D and E have the same number of turns;*
- *If D and E have no turn or one turn, then $D = E$ or $D = E^a$;*
- *If D has two turns and at most one square between the turns, then $D = E$ or $D = E^a$.*

Chapter 2

Background on Partitions and Symmetric Functions

2.1 Partitions

We begin with the tool that is used to index most symmetric functions: partitions. Two useful references for the theory of partitions are [20] and [21].

Definition 2.1.1. A **partition** of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of non-negative integers $\lambda_i \geq 0$ such that $\sum_{i=1}^{\infty} \lambda_i = n$ and $\lambda_i \geq \lambda_{i+1}$ for all i.

If λ is a partition of n, we write $\lambda \vdash n$, or $|\lambda| = n$. Note that, since n is finite, we must have some integer j such that $\lambda_i = 0$ for all $i \geq j$. In other words, only finitely many of the λ_i may be zero. If $i = k$ is the largest index for which λ_i is nonzero, we usually write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Here, k is the **length** of λ, denoted $\ell(\lambda)$. Each λ_i is called a **part** of λ. We set Par(n) to be the set of all partitions of n. For example, Par$(5) = \{(5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1)\}$.

A partition is called **strict** if $\lambda_i > \lambda_{i+1}$ for all i. For example, the partition $(4,3,3,2,1,1)$ of 14 is not strict, whereas the partition $(6,4,3,1)$ is.

Definition 2.1.2. Given a partition $\lambda \vdash n$, define the **Young diagram** of λ to be an array of n unit boxes in the plane, left-aligned, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on.

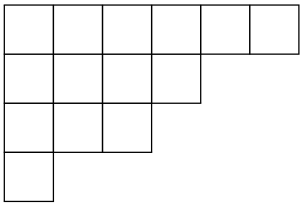


Figure 2.1: The Young diagram for the partition $(6,4,3,1)$.

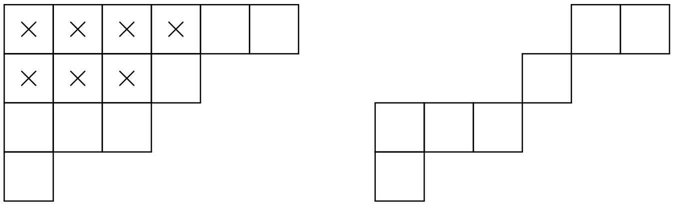


Figure 2.2: The skew diagram for the skew partition $(6,4,3,1)/(4,3)$

In English notation, the rows are drawn with λ_1 as the topmost row, with subsequent rows descending, while in French notation, λ_1 is the bottom-most row. In this thesis, we will use the former notation. We define the **transpose**, or conjugate, of a partition λ, denoted λ', to be the partition corresponding to the Young diagram of λ reflected across the northwest-southeast diagonal.

If λ and μ are partitions such that $\lambda_i \geq \mu_i$ for all i, then we write $\mu \subseteq \lambda$, and can define the **skew diagram** of the **skew partition** λ/μ to be the Young diagram of λ, with the boxes corresponding to the Young diagram of μ removed. Figure 2.2 shows the construction of such a skew diagram. When $\mu = \emptyset$ is the empty partition of 0, we consider $\lambda/\mu = \lambda$. Observe that the notion of the transpose $(\lambda/\mu)'$ of a skew diagram is well-defined.

Definition 2.1.3. If λ is a strict partition, define the **shifted (Young) diagram** of λ to be formed by shifting the i-th row of the Young diagram $i-1$ steps to the right.

As with non-shifted partitions, if λ and μ are strict partitions such that $\mu \subseteq \lambda$, we can define the **shifted skew diagram** of the skew partition λ/μ to be the shifted diagram of λ with the boxes corresponding to the shifted diagram of μ removed.

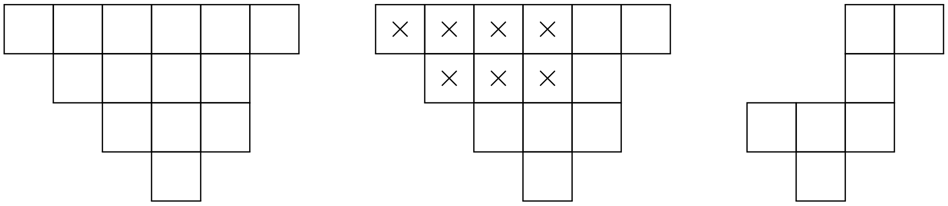


Figure 2.3: The shifted diagram for the partition $(6,4,3,1)$ and the shifted skew diagram for the skew partition $(6,4,3,1)/(4,3)$

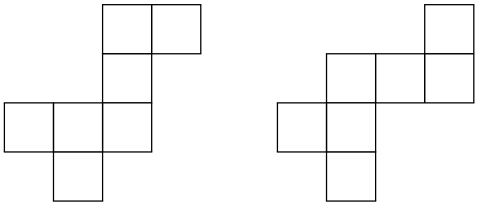


Figure 2.4: The shifted skew diagram for the skew partition $(6, 4, 3, 1)/(4, 3)$ and its antipodal reflection, $((6, 4, 3, 1)/(4, 3))^a = (6, 5, 2, 1)/(5, 2)$.

Notice that, instead of being left-aligned against a vertical axis, we can view a shifted diagram as being left-aligned against a "staircase" shape. Further, since λ is strict, we get that $(\lambda_i + (i - 1)) \geq (\lambda_{i+1} + (i))$, so the boundary of the shifted diagram includes the staircase, the top horizontal line, and a path from the staircase to the horizontal line consisting of steps $(1, 0)$ and $(0, 1)$. Further, in a shifted skew diagram λ/μ, since λ and μ both have this property, the diagram can be defined by two non-crossing paths with steps $(1, 0)$ and $(0, 1)$.

Unfortunately, the notion of the transpose of a partition does not extend well to shifted diagrams. If λ is a strict partition, the reflection of its shifted diagram across the northwest-southeast diagonal will not be a shifted diagram (or even a shifted skew diagram) when λ has at least two parts. Further, the transpose of the shifted skew diagram for λ/μ will not be a shifted skew diagram when λ/μ contains more than one box on the lower staircase. However, since the shifted shape of λ is uniquely determined by a path of $(1, 0)$ and $(0, 1)$ steps, we have a similar construction that does extend to shifted shapes.

Definition 2.1.4. Given a skew partition λ/μ of strict partitions λ and μ, define the **antipodal reflection** $(\lambda/\mu)^a$ to be the reflection of the shifted skew diagram of λ/μ across the northeast-southwest diagonal.

Since λ/μ is defined uniquely by a pair of non-crossing walks $w = w_1w_2 \dots w_k$ and $v = v_1v_2 \dots v_\ell$ of $(1, 0)$ and $(0, 1)$ steps, the antipodal reflection $(\lambda/\mu)^a$ is a shifted skew shape, with non-crossing walks given by $v' = v'_\ell v'_{\ell-1} \dots v'_1$ and $w' = w'_k w'_{k-1} \dots w'_1$, where $w'_i = (1, 1) - w_i$ and $v'_i = (1, 1) - v_i$. Thus, we get the following fact:

Remark 2.1.5. Given a shifted skew diagram λ/μ, the antipodal reflection $(\lambda/\mu)^a$ is also a shifted skew diagram.

In fact, the notion of the antipodal reflection does extend to non-shifted skew shapes, but we will see later that they behave very nicely in the shifted case for Schur Q-functions. In particular, if D are E are shifted skew diagrams such that $D = E^a$, then we have that $Q_D = Q_E$, that is, there is the same number of shifted semistandard fillings of D and E for each content.

In this thesis, we focus primarily on a specific subset of shifted skew diagrams, called frayed ribbons, which are a specific case of near-ribbon shapes, as the natural next step after the work done on ribbons in [18].

Definition 2.1.6. A **ribbon** is an ordinary skew shape that is connected and does not contain a 2×2 sub-diagram of boxes. We say a shifted skew shape is a ribbon if its underlying diagram is a ribbon as an ordinary skew shape. A **near-ribbon** is a connected non-ribbon shape for which it is possible to remove one square to form a ribbon. A **frayed ribbon** is a near ribbon containing two boxes on the staircase.

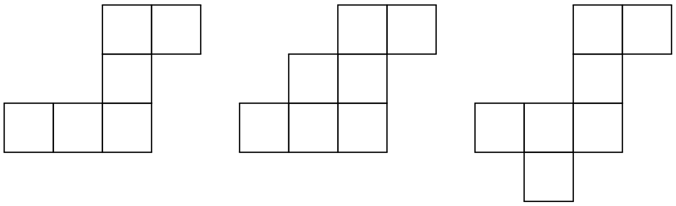


Figure 2.5: On the left, an example of a ribbon $(6, 4, 3)/(4, 3)$. In the middle, an example of a near-ribbon $(6, 4, 3)/(4, 2)$. On the right, an example of a frayed ribbon $(6, 4, 3, 1)/(4, 3)$.

2.2 Symmetric Functions

We defined partitions and their diagrams in order to index and study the set of symmetric functions, which we define here. A standard reference is [21]. First, however, we define a **weak composition** α of n to be a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of non-negative integers such that $\sum_{i=1}^{\infty} \alpha_i = n$.

Notably, compositions do not require that $\alpha_i \geq \alpha_{i+1}$, so a composition is a generalization of a partition.

Definition 2.2.1. A **homogeneous symmetric function** of degree n over a commutative ring R (with identity) is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha$$

where each c_α is in R, the index α ranges over all weak compositions of n, where we set the term $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots$. Further, we require that f is invariant under every permutation of the indices.

The set of all homogeneous symmetric functions of degree n over R, in the set of variables $X = \{x_1, x_2, \ldots\}$ is denoted $\Lambda_R^n(X)$. Then, for $a, b \in R$ and $f, g \in \Lambda_R^n(X)$, the formal power series $af + bg$ is also invariant under permutations of the indices, so $af + bg \in \Lambda_R^n(X)$, and so $\Lambda_R^n(X)$ is a module over R. We define the **algebra of symmetric functions** $\Lambda_R(X)$ to be $\Lambda_R^0(X) \oplus \Lambda_R^1(X) \oplus \Lambda_R^2(X) \oplus \cdots$. By the construction of an infinite direct sum of modules, an element $f \in \Lambda_R(X)$ has the form $f = f_0 + f_1 + f_2 + \ldots$, where $f_i \in \Lambda_R^i(X)$, and only finitely many f_i are nonzero.

Observe that, if $f \in \Lambda_R^n(X)$ and $g \in \Lambda_R^m(X)$, then $f \cdot g \in \Lambda_R^{n+m}$, so $\Lambda_R(X)$ is a graded algebra. Going forward, we usually consider symmetric functions over $R = \mathbb{Q}$, omit the subscript in this case, and omit the set of variables to write Λ^n for the module of functions of degree n over $\mathbb{Q}$, and Λ for the algebra of symmetric functions over $\mathbb{Q}$.

We can also define a notion of symmetric functions on a finite set of variables $\{x_1, \ldots, x_n\}$ by setting $x_{n+1} = x_{n+2} = \ldots = 0$ for a given n. In this case, a symmetric function is a polynomial in n indeterminants over $\mathbb{Q}$ that is invariant under the group action of S_n that permutes the variables. For example, in three variables, $f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$ is symmetric, since for all $\sigma \in S_3$, we have that $x_{\sigma(1)}x_{\sigma(2)}$ is a term of the sum. However, a function such as $g(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_1x_3$ is not symmetric, since g is missing terms $x_3^2, -x_1x_2$, and $-x_2x_3$. Because Λ^n is a module over $\mathbb{Q}$, we have that Λ^n is a vector space, so we can ask about its bases.

Definition 2.2.2. Given a partition λ of n, define the **monomial symmetric function** $m_\lambda(x) \in \Lambda^n$ by $m_\lambda(x) = \sum_\alpha x^\alpha$, where the index α ranges over all distinct permutations of the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$.

Here, a permutation σ is distinct if for $i < j$, $\lambda_i = \lambda_j$, then $\sigma(i) < \sigma(j)$. In other words, we don't want to include permutations that would produce monomial terms $x_i x_j$ and $x_j x_i$, since these are the same monomial. For example, $m_{(3,2,1,1)} = \sum x_i^3\, x_j^2\, x_k\, x_\ell$ where i, j, k, ℓ are distinct and without loss of generality, $k < \ell$.

For any homogeneous symmetric function $f(x) \in \Lambda^n$, by definition, $f(x) = \sum_\alpha c_\alpha x^\alpha$, and so we can write $f(x) = \sum_\lambda c_\lambda m_\lambda$, where the sum is over partitions λ of n. This gives us that the set $\{m_\lambda \,|\, \lambda \vdash n\}$ spans the vector space Λ^n. Further, if $m_\lambda = \sum_\mu c_\mu m_\mu$, then $c_\mu = 0$ for all $\mu \neq \lambda$, and thus the set of m_λ are linearly independent, and form a basis of Λ^n. We now examine a few other types of symmetric functions.

Definition 2.2.3. Define the **elementary symmetric function**

$$e_k(x) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

For a partition λ of n, define $e_\lambda = e_{\lambda_1}\, e_{\lambda_2}\, \cdots\, e_{\lambda_{\ell(\lambda)}}$.

For example, $e_{(3,2,1)} = (x_1x_2x_3 + x_1x_2x_4 + \ldots)(x_1x_2 + x_1x_3 + \ldots)(x_1 + x_2 + \ldots)$.

Definition 2.2.4. Define the **power sum symmetric function**

$$p_k(x) = \sum_i x_i^k$$

For a partition λ of n, define $p_\lambda = p_{\lambda_1}\, p_{\lambda_2}\, \cdots\, p_{\lambda_{\ell(\lambda)}}$.

For example, $p_{(3,2,1)} = (x_1^3 + x_2^3 + \ldots)(x_1^2 + x_2^2 + \ldots)(x_1 + x_2 + \ldots)$.

Definition 2.2.5. Define the **complete homogeneous symmetric function** to be

$$h_k(x) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} x_{i_1}\, x_{i_2}\, \cdots\, x_{i_k}$$

For a partition λ of n, define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}}$.

For example, $h_{(3,2,1)} = (x_1^3 + x_1^2 x_2 + x_1 x_2 x_3 + \ldots)(x_1^2 + x_1 x_2 + \ldots)(x_1 + x_2 + \ldots)$.

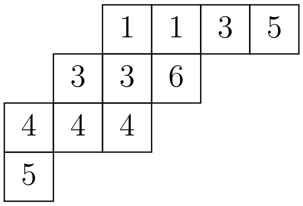
Since the above three sets of formal power series are defined as sums across all indices of certain terms, they are clearly symmetric functions. Further, they all also form bases of Λ^n:

Remark 2.2.6 ([21]). The sets $\{e_\lambda\}$, $\{p_\lambda\}$, and $\{h_\lambda\}$, where $\lambda \vdash n$, each form bases for Λ^n.

To define our final basis of Λ^n, we will need to build up the notion of a semistandard tableau.

A **Young tableau** T of shape λ/μ is a filling of the Young diagram for λ/μ with positive integers. We say T is **standard** if the entries strictly increase down columns and left-to-right across rows. We say T is **semistandard** if the entries strictly increase down columns, and weakly increase left-to-right across rows. The **content** of a Young tableau is the weak composition $c(T) = (c_1(T), c_2(T), \ldots)$, where $c_i(T)$ is the number of occurrences of i in T. Often, we will consider cases where the content of a tableau is a partition λ, in which case, the filling has λ_1 1's, λ_2 2's, and so on.

We define SSYT(λ/μ) to be the set of semistandard Young tableaux of shape λ/μ. Given $T \in$ SSYT(λ/μ), we define the monomial $x^T = x_1^{c_1(T)} x_2^{c_2(T)} \cdots$. We define the **reading word** of a semistandard Young tableaux to be the word formed by concatenating the rows of T from bottom to top. The **reading order** of the entries in the tableau is the total order given by the reading word. An example is shown in Figure 2.6.



$$\begin{array}{cccccc} & & 1 & 1 & 3 & 5 \\ & 3 & 3 & 6 & & \\ 4 & 4 & 4 & & & \\ 5 & & & & & \end{array}$$

Figure 2.6: A semistandard tableau T of shape $(6,4,3,1)/(2,1)$. Notice that the entries increase down columns and across rows, but repeats are allowed in rows. Then, the reading word of T is $5,4,4,4,3,3,6,1,1,3,5$, and the monomial $x^T = x_1^2 x_3^3 x_4^3 x_5^2 x_6$

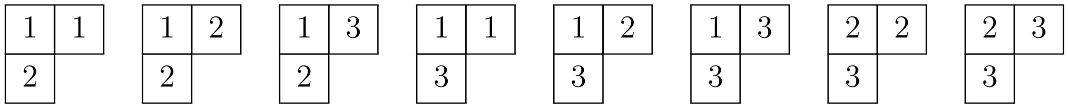
Definition 2.2.7. For a partition λ of n, we define the Schur function

$$s_\lambda(x) = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T$$

We extend this definition to skew partitions, and write

$$s_{\lambda/\mu}(x) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T.$$

For example, for the partition $\lambda = (2, 1)$, there are eight semistandard Young tableaux of shape $(2, 1)$ and with entries at most 3:



The corresponding Schur function is

$$s_{(2,1)}(x_1, x_2, x_3) = x_1^2\, x_2 + x_1\, x_2^2 + x_1^2\, x_3 + x_1\, x_3^2 + x_2^2\, x_3 + x_2\, x_3^2 + 2x_1\, x_2\, x_3.$$

From our combinatorial definition of the Schur functions, it is not obvious that they are symmetric functions. However, they are, which can be shown via a bijection between semistandard Young tableaux with the same shape, and content differing by an adjacent transposition of entries.

Proposition 2.2.8 ([21, Theorem 7.10.2])**.** Given a skew partition λ/μ of n, the Schur function $s_{\lambda/\mu}$ is a symmetric function. That is, $s_{\lambda/\mu}$ is invariant under permutations of the indices of the indeterminants.

From this, we get that if λ is a partition of n, then s_λ is a homogeneous symmetric function of degree n. Still, it is even less obvious that the Schur functions form a basis of Λ^n. We use a formulation called the Kostka numbers to prove this fact.

For a partition λ and weak composition α of n, define the **Kostka number** $K_{\lambda,\alpha}$ to be the number of semistandard Young tableaux of shape λ with content α. Then, by definition, $s_\lambda = \sum_\alpha K_{\lambda,\alpha} x^\alpha$. We can group the set of weak compositions by their representative partition, and write $s_\lambda = \sum_{|\mu|=n} K_{\lambda,\mu} m_\mu$. Hence, the Kostka numbers give the transition relations between the Schur functions and the monomial symmetric function basis.

Proposition 2.2.9 ([21], Prop 7.10.5)**.** Suppose that λ and μ are partitions of n and $K_{\lambda,\mu} \neq 0$. Then, under the dominance order $\leq_D$ for partitions of n, we get $\mu \leq_D \lambda$.

This theorem shows that the matrix formed by the Kostka numbers, which give the transition relation between $\{m_\lambda\}$ and $\{s_\mu\}$, is upper-triangular and hence invertible. So the matrix $K = (K_{\lambda,\mu})$ is a change of base matrix, and hence the set $\{s_\lambda : \lambda \vdash n\}$ is a basis of Λ^n.

When we multiply Schur functions, we can express the result in the Schur basis via the following formula:

$$s_\mu s_\nu = \sum_{|\lambda|=|\mu|+|\nu|} c^\lambda_{\mu,\nu}\, s_\lambda.$$

The terms $c^\lambda_{\mu,\nu}$ are called the *Littlewood-Richardson coefficients*, and we will present a formula for calculating them in Chapter 3.

2.3 Hall-Littlewood Polynomials and Schur Q-Functions

One construction that connects the Schur, complete homogeneous, and elementary symmetric functions is the Hall-Littlewood polynomial:

Definition 2.3.1. The **Hall-Littlewood polynomial** is defined as

$$P_\lambda(x\,;\,t) = \sum_{\sigma \in S_n/S_n^\lambda} \sigma\left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} \prod_{\lambda_i \geq \lambda_j} \frac{x_i - t\,x_j}{x_i - x_j} \right).$$

Note that P_λ is a polynomial in the variables $x_1, ..., x_n$ and t. The connection between P_λ and the other symmetric functions is given in the following statements:

Proposition 2.3.2 ([3, Chapter 3])**.** Each of the following hold: $P_\lambda(x\,;\,0) = s_\lambda(x)$, $P_\lambda(x\,;\,1) = m_\lambda(x)$, and $P_{(1,\ldots,1)}(x\,;\,1) = e_r(x)$ where $(1,\ldots,1)$ has length r.

There are a few other functions that arise in the study of the Hall-Littlewood polynomials. We define the function $q_r(x\,;\,t) = (1-t)P_{(r)}(x\,;\,t)$ and $Q_\lambda(x\,;\,t) = b_\lambda(t)P_\lambda(x\,;\,t)$ where $b_\lambda(t) = \prod_{i\geq 1}(1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)})$ and $m_i(\lambda)$ is the number of parts of λ with size i. These definitions are fairly abstract, and a deep understanding is not needed for the rest of this paper. However, we note that $Q_{(r)}(x\,;\,t) = q_r(x\,;\,t)$.

The Schur Q-functions occur in the $t = -1$ specialization of the study of Hall-Littlewood polynomials. Henceforth, we will write $Q_\lambda(x) = Q_\lambda(x;-1)$, $P_\lambda(x) = P_\lambda(x;-1)$, and $q_r(x) = q_r(x;-1)$.

We say that the generating function of a combinatorial object C_k is a function $C(t)$ such that the coefficient of t^k is C_k. For example, the generating function for the elementary symmetric function $e_k(x)$ is $E(t) = \sum_k e_k(x)\,t^k = \prod_i(1+tx_i)$, since the coefficient on t^k is produced by considering any choice of k distinct indeterminants x_i, which is how we formed $e_k(x)$. The generating function for the complete homogeneous symmetric function $h_k(x)$ is $H(t) = \sum_k h_k(x)\,t^k = \prod_i \frac{1}{1-tx_i}$.

Proposition 2.3.3 ([3, Chapter 3.2])**.** The generating function for $q_k(x)$ is

$$Q(t) = \sum_k q_k(x)\,t^k = \prod_i \frac{1+tx_i}{1-tx_i} = E(t)\cdot H(t)$$

Macdonald also proves the following fact: If $n = 2m$ is even, then q_n is in the span of $\{q_1,\, q_3,\, \ldots,\, q_{2m-1}\}$, meaning q_n is generated by the odd-degree q_r. We define the following submodule of symmetric functions:

$$\Omega_{\mathbb{Q}}(X) = \mathbb{Q}[q_1, q_2, q_3, \ldots] = \mathbb{Q}[q_r\ :\ r \text{ odd}].$$

As with $\Lambda_{\mathbb{Q}}$, we usually omit the subscript when the base ring is $\mathbb{Q}$ and write Ω as the subalgebra of Λ generated by the odd-degree q_r.

Proposition 2.3.4 ([3, Chapter 3.8])**.** We can express the odd-degree power sum symmetric functions p_r in terms of the set of q_k, and vice versa

As a result, we get that $\Omega = \mathbb{Q}[p_r : r \text{ odd}]$. Using the fact that there are the same number of strict partitions of n and partitions of n into odd parts, one can also show that the set $\{q_\lambda : \lambda \text{ is strict}\}$ forms a basis of Ω. This fact is used in proving the following statement, which gives us the transition relation between the Q_λ and q_μ.

Proposition 2.3.5 ([3, Chapter 3.8])**.** If λ is strict, we get

$$Q_\lambda = q_\lambda + \sum_{\mu >_D \lambda} a_{\lambda\mu} q_\mu$$

for some $a_{\lambda\mu} \in \mathbb{Z}$, and where the sum ranges over strict μ that dominate λ.

Exactly as with the Schur functions and the monomial basis, this tells us that the transition matrix between the Q_λ and q_μ over strict partitions is triangular, and even further, has all 1's on the diagonal. So this transition is invertible, and thus $\{Q_\lambda : \lambda \text{ is strict}\}$ is a basis of Ω.

There is a combinatorial definition of the Schur Q-functions that follows from the theory of Hall-Littlewood polynomials. To do so, we must define an analogue of semistandard Young tableau on shifted shapes.

Definition 2.3.6. Let $A = 1' < 1 < 2' < 2 < 3' < 3 < \ldots$ be an ordered alphabet of symbols. Given a shifted skew partition λ/μ, a **shifted semistandard Young tableau of shape** λ/μ is a filling of the shifted skew diagram with entries from A such that the following hold:

(a) Rows are weakly increasing left to right, and for each i, only one i' can appear in each row.

(b) Columns are weakly increasing top to bottom, and for each i, only one unprimed i can appear in each column.

We write $\text{ShSSYT}(\lambda/\mu)$ to be the set of all shifted semistandard Young tableaux of shape λ/μ. As a shorthand convention, we write i^* to denote an entry that is either i or i'. Given a tableau T, we define the monomial associated to T as $x^T = x_1^{c_1^*(T)} x_2^{c_2^*(T)} \cdots$, where $c_i^*(T)$ is the number

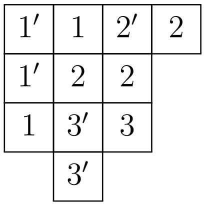


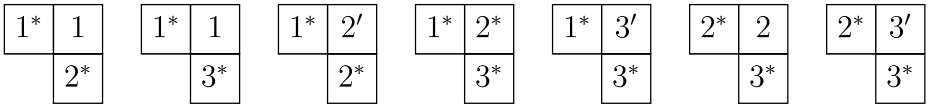
Figure 2.7: A shifted semistandard Young tableau of shape $(6, 4, 3, 1)/(2, 1)$. The reading word of T is $3', 1, 3', 3, 1', 2, 2, 1', 1, 2', 2$, and the associated monomial is $x^T = x_1^4\, x_2^4\, x_3^3$.

of times i^* appears in T. As with the non-shifted case, the **content** of T is the weak composition $c^*(T) = (c_1^*(T),\, c_2^*(T),\, \ldots)$, and we will often consider the case when the content is a partition ν.

Definition 2.3.7. For a shifted skew partition λ/μ, we define the **(skew) Schur Q-function**

$$Q_{\lambda/\mu}(x) = \sum_{T \in \mathrm{ShSSYT}(\lambda/\mu)} x^T$$

For example, for the shifted partition $\lambda = (2, 1)$, there are 32 shifted semistandard Young tableaux with shape $(2, 1)$ and entries at most 3:



Note that each i^* denotes an entry that could be i or i', and so each diagram above corresponds to 2^k shifted semistandard Young tableaux, if k is the number of starred entries. The corresponding Schur Q-function is

$$Q_{(2,1)} = 4x_1^2\, x_2 + 4x_1^2\, x_3 + 4x_1\, x_2^2 + 8x_1\, x_2\, x_3 + 4x_1\, x_3^2 + 4x_2^2\, x_3 + 4x_2\, x_3^2.$$

As an analogue to the Kostka number for regular Schur functions, if we define $K'_{\lambda,\mu}$ to be the number of shifted Young tableaux of shape λ with content μ, then $Q_\lambda = \sum_{|\mu|=n} K'_{\lambda,\mu} m_\mu$. Since the set of $\{Q_\nu\}$ with ν strict form a basis of Ω, we turn our attention to the decomposition of a skew Schur Q-function $Q_{\lambda/\mu}$ in terms of the non-skew Schur Q-functions.

When multiplying Schur P-functions, or decomposing a skew Schur Q-function into the Schur P- and Q- bases, we get an analogue of the Littlewood-Richardson coefficients:

$$P_\mu P_\nu = \sum_{|\lambda|=|\mu|+|\nu|} f^\lambda_{\mu,\nu} P_\lambda \qquad\qquad Q_{\lambda/\mu} = \sum_{|\nu|=|\lambda|-|\mu|} f^\lambda_{\mu,\nu} Q_\nu$$

The terms $f^\lambda_{\mu,\nu}$ are called the *shifted Littlewood-Richardson coefficients*, and we will explore them more in Chapter 3.

Recall that a near-ribbon is a shifted skew shape for which one box can be removed such that the remaining shape is a ribbon. We find that the Schur Q-functions corresponding to near-ribbons form a distinct class of functions, which we will show in Chapter 4.

Proposition 2.3.8 (Corollary 4.0.3). *Suppose that D is a near-ribbon. Then if $Q_D = Q_E$ for some shifted skew shape E, E must also be a near-ribbon.*

Further, given a Schur Q-function Q_D for a shifted skew diagram D, we can always find another diagram E such that $Q_D = Q_E$, namely when $E = D^a$.

Proposition 2.3.9 ([22, Theorem 6.13]). *Let D be a shifted skew shape. Then $Q_D = Q_{D^a}$.*

This result does not extend back to ordinary Schur functions. For example, given the non-shifted shape $(3, 1)$, the antipodal reflection is $(2, 2, 2)/(1, 1)$. However, using the Kostka number calculation, we find that the corresponding Schur functions are

$$s_{(3,1)} = m_{(3,1)} + m_{(2,2)} + 2m_{(2,1,1)} + 3m_{(1,1,1,1)} \text{ and } s_{(2,2,2)/(1,1)} = m_{(2,1,1)} + 3m_{(1,1,1)}.$$

This example is the first indication that the Schur Q-functions have a different structure than the ordinary Schur functions, and we further examine these differences in the next section.

Chapter 3

Computational Tools

3.1 The Littlewood-Richardson Rule

Since the Schur functions $\{s_\lambda\}$ form a basis of the graded algebra Λ, we can explore their structure constants - coefficients that arise when expressing the product of basis elements back in terms of the basis. In the case of Schur functions, these structure constants are called Littlewood-Richardson coefficients.

Definition 3.1.1. The **Littlewood-Richardson coefficient** $c^\lambda_{\mu,\nu}$ is the structure constant that arises in the multiplication of Schur functions, that is,

$$s_\mu s_\nu = \sum_{|\lambda|=|\mu|+|\nu|} c^\lambda_{\mu,\nu}\, s_\lambda$$

These coefficients also arise the expansion of a skew Schur functions in terms of the non-skew Schur functions:

Proposition 3.1.2. If λ/μ is a skew shape, then we have

$$s_{\lambda/\mu} = \sum_{|\nu|=|\lambda|-|\mu|} c^\lambda_{\mu,\nu}\, s_\nu$$

In order to compute these coefficients, we define a specific type of integer sequence.

Definition 3.1.3. A **ballot sequence** is a sequence $a_1\, a_2\, \ldots\, a_n$ of positive integers such that for any initial string $a_1\, a_2\, \ldots\, a_j$, the number of i's is greater than or equal to the number of $(i+1)$'s for all i. A tableau is **ballot** if its reverse reading word (the reading word read backwards) is ballot.

For example, the sequence $1, 1, 2, 3, 2, 1, 3, 1, 2, 2$ is ballot since each initial string (such as $1, 1, 2, 3, 1$) contains at least as many 1's as 2's, and at least as many 2's as 3's. Note that this means, for instance, that the sequence must start with a 1.

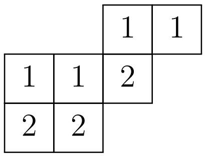
Proposition 3.1.4 ([21, Theorem A1.3.4]). The Littlewood-Richardson coefficient $c^{\lambda}_{\mu,\nu}$ is equal to the number of semistandard Young tableaux of shape λ/μ with content ν that are ballot.

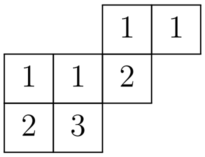
There are multiple statements of the Littlewood-Richardson rule, all related to counting the number of tableaux with certain properties, but we include this one as the most natural analogue with the shifted rule we use when counting skew shifted tableaux.

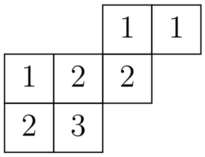
When considering the reading word of a semistandard tableau T in non-reversed order, one can restate the above rule as follows: For each pair $(i, i+1)$, construct a walk on $\mathbb{Z}_{\geq 0}$ corresponding to T as follows: Take the subset of the $i, i+1$ entries of the reading word of T, and begin the walk at 0. For each $i+1$, take a $(+1)$ step, and for each i, take a (-1) step if the walk is not currently at 0. Then T has a ballot reverse reading word if and only if the above walk ends at 0.

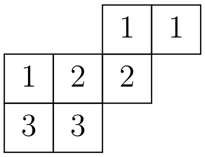
In the earlier example, suppose that $2, 2, 1, 3, 1, 2, 3, 2, 1, 1$ is the reading word of a tableau T. The $(1, 2)$-subword is $2, 2, 1, 1, 2, 2, 1, 1$. Then the integer walk has the form $0, 1, 2, 1, 0, 1, 2, 1, 0$, which ends at 0. The $(3, 2)$-subword is $2, 2, 3, 2, 3, 2$ with integer walk $0, 0, 0, 1, 0, 1, 0$, which also ends at 0. Therefore the reverse reading word is ballot.

The Littlewood-Richardson rule gives us a much nicer way of decomposing a product of Schur functions, or a skew Schur function, in terms of the original basis than by multiplying and directly computing the decomposition. For example, there are 4 ballot tableaux of shape $(4, 3, 2)/(2)$, listed below:









The contents of the above tableaux are $(4, 3)$, $(4, 2, 1)$, $(3, 3, 1)$, and $(3, 2, 2)$, and therefore we can write $s_{(4,3,2)/(2)} = s_{(4,3)} + s_{(4,2,1)} + s_{(3,3,1)} + s_{(3,3,2)}$.

This example helps us build up an intuition for what properties might arise when counting these shapes. For example, since a ballot word must begin with a 1, we know that the last entry of a ballot tableau T is a 1. Since T is semistandard, the preceding entries in the row must also be 1's. The next row must contain only 1 and 2 entries, since if there were a 3 in the second row, there couldn't be a 2 following it while maintaining that the tableau is semistandard. Both of these properties, in fact, will translate to the shifted case, and help us build up the theory of decomposing frayed ribbon shapes.

In the case of Schur Q-functions, we will find there is a similar criteria on shifted tableaux that allow us to compute decompositions of skew Schur Q-functions into the non-skew basis.

3.2 The Shifted Littlewood-Richardson Rule

Recall that a semistandard filling of a shifted skew diagram λ/μ is a filling with entries from the doubled alphabet $1' < 1 < 2' < 2 < 3' < 3 < \dots$ such that rows and columns are increasing left to right and top to bottom, repeats of primed entries are only allowed in columns, and repeats of non-primed entries are only allowed in rows.

Since the Schur P-functions form a basis of Ω, the subalgebra of Λ generated by the odd-degree power sum symmetric functions, as in the case of ordinary Schur functions, we can look at the structure constants that arise from multiplying Schur P-functions.

Definition 3.2.1. The **shifted Littlewood-Richardson coefficient** $f^{\lambda}_{\mu,\nu}$ is the structure constant in the multiplication of Schur P-functions, that is,

$$P_\mu \, P_\nu = \sum_{|\lambda|=|\mu|+|\nu|} f^{\lambda}_{\mu,\nu} P_\lambda$$

These coefficients also appear in the expansion of a skew Schur Q-function in terms of the non-skew Schur Q-function basis of Ω:

Proposition 3.2.2. If λ/μ is a shifted skew shape, then we have

$$Q_{\lambda/\mu} = \sum_{|\nu|=|\lambda|-|\mu|} f^{\lambda}_{\mu,\nu} \, Q_\nu$$

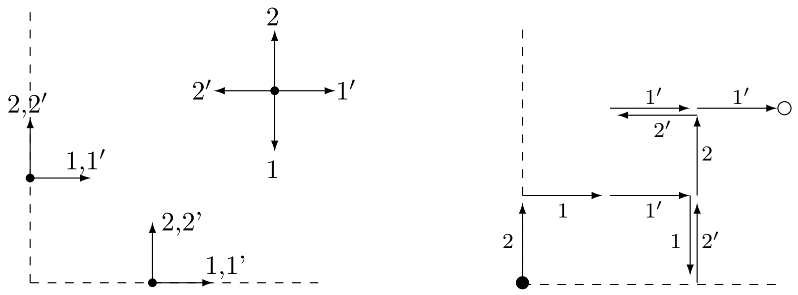


Figure 3.1: At left, the directions assigned to the letters $1', 1, 2', 2$ in the lattice walk of a word, depending on whether or not the step starts on an axis. At right, the walk for $w = 211'12'22'1'1'$

Once again, there are several combinatorial rules that can be used to compute the shifted Littlewood-Richardson coefficients, and we focus primarily on the one established in [8] via lattice walks. We recall some definitions from [8].

Definition 3.2.3. Let w be a word in the alphabet $\{1', 1, 2', 2\}$. The **(1,2)-walk** of the word w is a lattice walk in the first quadrant using one of the four orthogonal unit steps for each letter of the word, reading left to right:

$$\rightarrow = (1,0) \qquad \uparrow = (0,1) \qquad \leftarrow = (-1,0) \qquad \downarrow = (0,-1)$$

The walk starts at the origin $(0,0)$, and at the k-th step, we read the letter w_k and draw the next step of the walk according to Figure 3.1, with two cases depending on whether or not the walk is currently on at least on of the x- or y-axes. In particular, any 2 is an up arrow, any $1'$ is a right arrow, a 1 is either right if on an axis or down if not, and a $2'$ is either up if on an axis or left if not. We will generally label each step of the walk by the letter w_i, so as to represent both the word and its walk on the same diagram.

If w is a word in the alphabet $\{i', i, (i+1), (i+1)'\}$, we similarly define the $(i, i+1)$-walk of w by replacing 1^* with i^* and 2^* with $(i+1)^*$ in the above definition.

If w is a word in the alphabet $\{1', 1, 2', 2, 3', 3, \ldots\}$, we define the $(i, i+1)$-walk of w to be the $(i, i+1)$-walk of the subword of w consisting of only the i and $i+1$ entries. An example of the walks corresponding to the word $w = 212'231'3'1'121'11$ is shown in Figure 3.2.

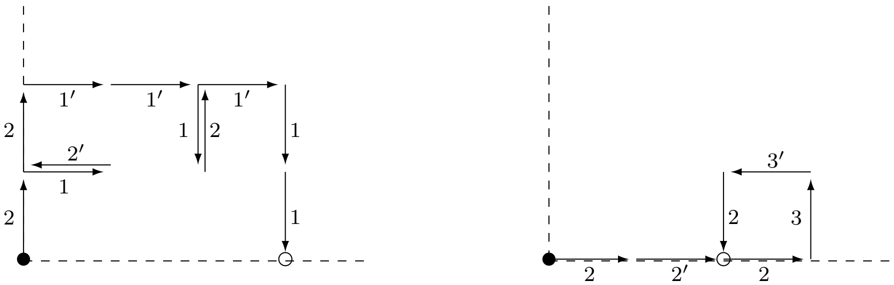
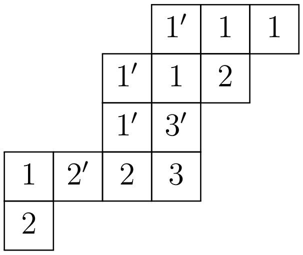


Figure 3.2: The $(1, 2)$- and $(2, 3)$-walks of the word $w = 212'231'3'1'121'11$. The $(1, 2)$-subword is $212'21'1'121'11$ and the $(2, 3)$-subword is $22'233'2$. Since both walks end on the x-axis, w is ballot.

Definition 3.2.4. A shifted skew Young tableau T is **ballot** if the $(i, i + 1)$-walk of the reading word of T ends on the x-axis for each i. T is in **canonical form** if for each i, the first i^* in the reading word of T is unprimed.

Example 3.2.5. The tableau below is a ballot tableau. Note that it is in canonical form; the first 1, the first 2, and the first 3 in reading order are all unprimed. Its reading word is $212'231'3'1'121'11$, which is the ballot word from Figure 3.2.

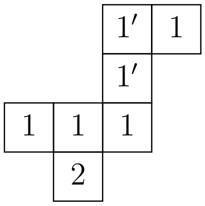


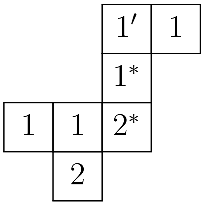
Shifted ballot tableaux can also be referred to as *shifted Littlewood-Richardson tableaux*, because they enumerate the shifted Littlewood-Richardson coefficients. We now have what we need to state the shifted Littlewood-Richardson rule.

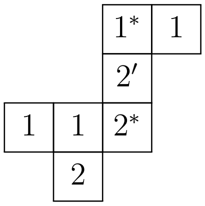
Theorem 3.2.6 ([8, Theorem 1.5])**.** *The shifted Littlewood-Richardson coefficient $f^{\lambda}_{\mu\nu}$ is equal to the number of shifted ballot tableaux of shape λ/μ, content ν, and in canonical form.*

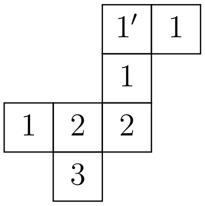
As in the case of ordinary Schur functions, the ballot condition together with the semistandard condition greatly restrict the number of tableaux of a certain shape, often to a point of being able to compute by hand. Intuitively, one should think of the definition of a ballot word for shifted tableaux similar to the non-shifted ballot case: Roughly speaking, after each $(i+1)^*$ entry, we need to have some unused i^* letter in the word to bring the walk back towards the x-axis. We will explore these properties in full in Chapter 4.

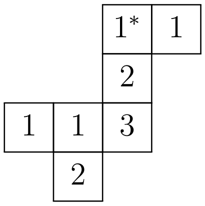
For example, there are 12 ballot tableaux of shifted skew shape $(6,4,3,1)/(4,3)$ in canonical form:











The contents of the above tableaux are $(6,1)$, $(5,2)$, $(4,3)$, $(4,2,1)$, and $(4,2,1)$, and therefore the corresponding Schur Q function is $Q_{(6,4,3,1)/(4,3)} = Q_{(6,1)} + 4Q_{(5,2)} + 4Q_{(4,3)} + 3Q_{(4,2,1)}$.

3.3 Computational Results

To identify patterns and confirm our main conjecture in the case of some small finite sets of shifted skew diagrams, we use Python code developed by the author, Maria Gillespie, and Jake Levinson, which can be found at https://www.mathematicalgemstones.com/maria/shiftedtabsWordsPy3.py. The following example shows how to calculate the decomposition of a skew Schur Q-function in the non-skew basis.

```
from shiftedtabsWordsPy3 import *


for pair in rflap_shapes(6):
    print pair[0], "skew", pair[1], ":"
    print SkewQ(pair[0],pair[1])
    print ""
```

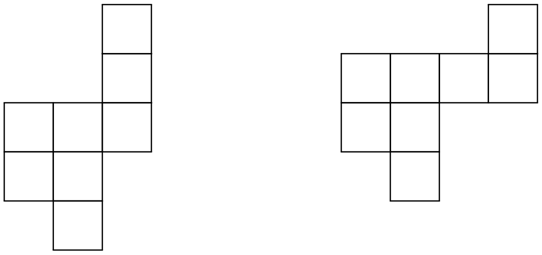
Here, `rflap_shapes(n)` computes the set of frayed ribbons with n boxes, as a pair (λ, μ) corresponding to the shifted skew shape λ/μ. Then, `SkewQ(pair[0],pair[1])` calculates the expansion of $Q_{\lambda/\mu}$ as a sum of Q_ν. The output of the above code is:

```
[5, 4, 3, 2, 1] skew [4, 3, 2] :

HLQ[5, 1]


[5, 4, 3, 1] skew [4, 3] :

HLQ[3, 2, 1] + 2*HLQ[4, 2] + HLQ[5, 1]


[5, 4, 2, 1] skew [4, 2] :

HLQ[3, 2, 1] + 4*HLQ[4, 2] + HLQ[5, 1]


[5, 4, 1] skew [4] :

2*HLQ[4, 2] + HLQ[5, 1]


[5, 3, 2, 1] skew [3, 2] :

2*HLQ[4, 2] + HLQ[5, 1]


[5, 3, 1] skew [3] :

HLQ[3, 2, 1] + 4*HLQ[4, 2] + HLQ[5, 1]


[5, 2, 1] skew [2] :

HLQ[3, 2, 1] + 2*HLQ[4, 2] + HLQ[5, 1]


[5, 1] skew [] :

HLQ[5, 1]
```

In the output, `HLQ` refers to the Schur Q-function. For example, this computation tells us that $Q_{(5,4,3,1)/(4,3)} = Q_{(3,2,1)} + 2Q_{(4,2)} + Q_{(5,1)}$. Using the tools in this code, we computed the following results.

First, we note that it is natural to ask whether the property that a frayed ribbon has two boxes on the staircase is sufficient to distinguish Schur Q-functions. The following example shows that this generalization does not hold.

Example 3.3.1. There exists a non-antipodal pair of shapes having at least two boxes on the staircase which have equal Schur Q functions (but are not frayed ribbons). The smallest such pair has size 8 and is shown below.



Their Schur-Q expansions are

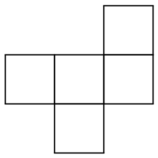
$$Q_{(6,5,4,2,1)/(5,4,1)} = Q_{(6,5,2,1)/(5,1)} = Q_{(6,2)} + 2Q_{(5,3)} + 2Q_{(5,2,1)} + 2Q_{(4,3,1)}.$$

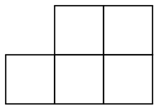
Next, we find that the set of frayed ribbons do not form a distinct class of Schur Q-functions when compared to other near-ribbon shapes.

Example 3.3.2. We have

$$Q_{(4,3,1)/(3)} = Q_{(4,3)/(2)} = Q_{(4,1)} + Q_{(3,2)},$$

and the shapes $(4,3,1)/(3)$ and $(4,3)/(2)$ are a frayed ribbon and near-ribbon respectively:



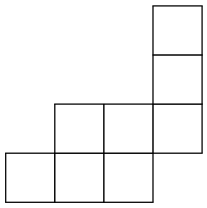


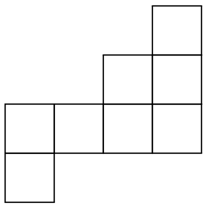
Finally, we find that in general, non-frayed near-ribbon shapes can have nontrivial equality. In particular, there exist pairs of near-ribbons that are not frayed in which equality holds, and their Schur Q functions are not equal by their shapes being antipodal, transposed, or antipodal transposed. An example is given below.

Example 3.3.3. We have

$$Q_{(7,6,5,3)/(6,5,2)} = Q_{(7,6,5,1)/(6,4,1)} = 3Q_{(4,3,1)} + 3Q_{(5,2,1)} + 5Q_{(5,3)} + 4Q_{(6,2)} + Q_{(7,1)},$$

and the shapes $(7,6,5,3)/(6,5,2)$ and $(7,6,5,1)/(6,4,1)$ are non-frayed near-ribbons that are not equivalent under any combination of the antipodal and transpose operations:





These examples motivate the study of frayed ribbons as a particular class of near-ribbons, since their inequality cannot be presupposed.

To end this sections, we note that we have computationally verified the following conjecture for all frayed ribbons with at most 11 boxes:

Conjecture 3.3.4. *If D and E are frayed ribbon shapes such that $Q_D = Q_E$, then we have either $D = E$ or $D = E^a$.*

In Chapter 4, we will show that there exist infinite families of frayed ribbons that adhere to this conjecture.

Chapter 4

Results on Skew Schur Q-Function Equality

We first show how to combinatorially distinguish Schur Q functions from each other via the leading monomial in lexicographic order.

Definition 4.0.1. For a shifted skew shape λ/μ, define its **greedy filling** to be the labeling formed by:

- First placing 1s in the maximal 'inner strip' of ribbons, by placing them in every square that is either in the top row or that shares a corner or edge with the inner shape μ,
- Then placing 2s maximally to form an inner strip of ribbons in the remaining empty squares, namely, in every square sharing a corner or edge with one of the 1s,
- Then placing 3s maximally in the same way in the remaining squares, and so on.

The **greedy monomial** of the shape is the monomial

$$2^r x_1^{m_1} x_2^{m_2} \cdots$$

where r is the total number of maximal, connected, uniformly-labeled ribbons formed by the greedy filling, and m_i is the number of squares labeled by i for each i. (See Figure 4.1.)

Proposition 4.0.2. *Suppose shifted skew shapes D and E have different greedy monomials. Then $Q_D \neq Q_E$.*

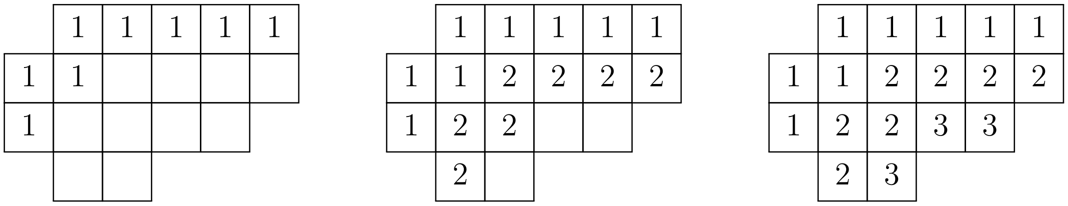


Figure 4.1: Forming the greedy filling of a shifted skew shape. The greedy monomial is $2^4 x_1^8 x_2^7 x_3^3$.

Proof. Consider the lexicographic ordering of monomials (based on their sequence of exponents in the order $x_1, x_2, x_3, \ldots$). We claim that the leading term of Q_D with respect to this ordering is the same as the greedy monomial of D.

Note that for any label i in the greedy filling and any connected component R of the corresponding disjoint union of ribbons, there are exactly two ways to prime or unprime each of the letters i in the ribbon R so as to make the ribbon semistandard. In particular, the lower left entry of the ribbon may be either primed or unprimed, and the rest of the entries are determined. An example corresponding to Figure 4.1 is shown:

	$1'$	1	1	1	1
$1'$	1	$2'$	2	2	2
1^*	$2'$	2	3^*	3	
	2^*	3^*			

where we recall that the $*$'s indicate the numbers that may be either primed or unprimed. This forms a shifted semistandard Young tableau whose contribution to the Schur Q function is the monomial $x_1^{m_1}x_2^{m_2}\cdots$, and since there are two choices for every connected ribbon in the decomposition, this monomial has a coefficient of 2^r in Q_D. This completes the proof. $\square$

As a corollary, we find that near-ribbon shapes have distinct Schur Q functions from all other types of shapes.

Corollary 4.0.3. *Suppose that D is a near-ribbon. Then if $Q_D = Q_E$ for some shifted skew shape E, E must also be a near-ribbon.*

Proof. If D has size n, since it is a near-ribbon its greedy monomial is $4x_1^{n-1}x_2$. Since $Q_D = Q_E$, it follows from Proposition 4.0.2 that $4x_1^{n-1}x_2$ is the greedy monomial of E, so E must decompose into one connected ribbon of 1s and a single square containing a 2 (since the coefficient of $4 = 2^2$ means there are two connected ribbons formed by the greedy filling). Therefore the shape E can be formed by adding a box to a ribbon, which is either a ribbon itself, a ribbon plus a disconnected box, or a near-ribbon. In the former two cases, the greedy monomial for E is $2x_1^n$ or $4x_1^n$ which is a contradiction. Therefore E is a near-ribbon. $\square$

4.1 Distinguishing Skew Shapes by Number of Turns

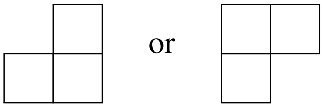
We now restate our main result, Theorem 1.0.3, here for the reader's convenience.

Theorem 1.0.3. *If D and E are frayed ribbons with $Q_D = Q_E$, then:*

- *D and E have the same number of turns;*
- *If D and E have no turn or one turn, then $D = E$ or $D = E^a$;*
- *If D has two turns and at most one square between the turns, then $D = E$ or $D = E^a$.*

We define the notion of a turn in a shifted skew shape as follows.

Definition 4.1.1. A **turn** in a frayed ribbon is a sub-diagram of the following form



that does not contain any boxes on the staircase. We call these two types of turns **outer turns** and **inner turns** respectively.

In order to prove the first statement of Theorem 1.0.3, we need the following result, which is a formalization of an observation made in Section 3.2.

Lemma 4.1.2. *The top row of any Littlewood-Richardson tableau has only 1^* entries, and in fact has at most one $1'$.*

Proof. By the ballot condition on the $(1, 2)$-walk, the last 1^* or 2^* in the word must be a 1^*, since no 2^* arrow can ever end on the x axis. Similarly the last 2^* in any ballot reading word comes after the last 3^*, and so on, meaning that the last letter in reading order is 1^*. Thus, by the semistandard condition, the entire top row consists of 1^* entries, with at most one $1'$ at the start of the row. $\square$

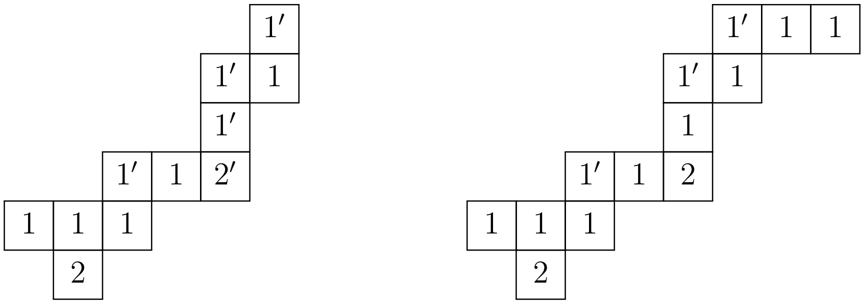


Figure 4.2: At left, a ballot tableau of a frayed ribbon with three outer turns and two inner turns, that is, $t_0 = 3$ and $t_1 = 2$ in the notation of the proof of Proposition 4.1.3. At right, a ballot tableau of a frayed ribbon with $t_0 = t_1 = 3$.

Proposition 4.1.3. *Suppose D is a frayed ribbon shape of size n with k turns. Then the coefficient of $Q_{(n-2,2)}$ in the expansion of Q_D is $2k$.*

Proof. We assume without loss of generality, by Proposition 2.3.9, that the second-to-bottom row of D has more than two squares. First, note that in a ballot tableau of shape D with content $(n-2,2)$, the entry in the bottom row cannot be 1^* by semistandardness, and cannot be $2'$ by the canonical form condition, so it must be a 2. Moreover, the other 2^* must be either in the corner of one of the outer turns or at the end of the top row, or else there would be a 1^* to its right or below it, contradicting semistandardness. But the 2^* cannot be at the end of the top row by Lemma 4.1.2. Thus the second 2^* must be in the corner of an outer turn.

Now, if t_0 is the number of outer turns and t_1 is the number of inner turns, we have $t_0 + t_1 = k$ and either $t_0 = t_1$ (if the topmost turn is an inner turn) or $t_0 = t_1 + 1$ (if the topmost turn is an outer turn). See Figure 4.2 for each such case.

For each outer turn of D, there are exactly four semistandard fillings containing a 2^* in the corner of that turn; in particular, the 2^* may be either 2 or $2'$, and the square above it may be either 1 or $1'$. We now check which of these are ballot. The reading word is of the form

$$211(1^* \cdots 1^*)2^*1^*(1^* \cdots 1^*)$$

where the strings $(1^* \cdots 1^*)$ in parentheses have a mix of primed and unprimed 1 entries that are uniquely determined by the shape of D. Just before the second 2^*, the walk is on the x axis, and the 2^* lifts it to a position just above the x axis. In order for the walk to return to the x axis in the end, it is necessary and sufficient that an unprimed 1 appears after the 2^*.

This is guaranteed to happen by semistandardness if there is an inner turn after the outer turn containing 2^*, but if not, then the only way it is guaranteed is if the 1^* just following the 2^* is unprimed (since all other 1^*s in the final column must be primed by semistandardness).

Thus, if $t_0 = t_1$ then we have an inner turn after all outer turns, and so each of the t_0 outer corners contributes four ballot tableaux. So the coefficient is

$$4t_0 = 2(t_0 + t_1) = 2k.$$

If instead $t_0 = t_1 + 1$, then the $t_0 - 1$ lowest outer corners contribute four ballot tableaux, but the topmost outer corner contributes only two since the 1^* above it must be unprimed. Thus we have a coefficient of

$$4(t_0 - 1) + 2 = 2t_0 + 2t_0 - 2 = 2t_0 + 2(t_1 + 1) - 2 = 2(t_0 + t_1) = 2k.$$

Therefore, in all cases, the coefficient is $2k$ as desired. □

As a corollary, we obtain the first statement of Theorem 1.0.3.

Corollary 4.1.4. *Let D and E be frayed ribbon shapes for which $Q_D = Q_E$. Then D and E have the same number of turns.*

4.2 Frayed Ribbons with One Turn

In the case of one turn, we build up a set of properties of ballot tableaux for these shapes with the following lemmas. For each, we assume that T is a Littlewood-Richardson tableau, meaning that T is ballot and in canonical form.

Lemma 4.2.1. *In T, let k^* be the largest entry of the rightmost column. Then for all $1 \leq i < k$, at least one i^* appears in the column, including exactly one unprimed i.*

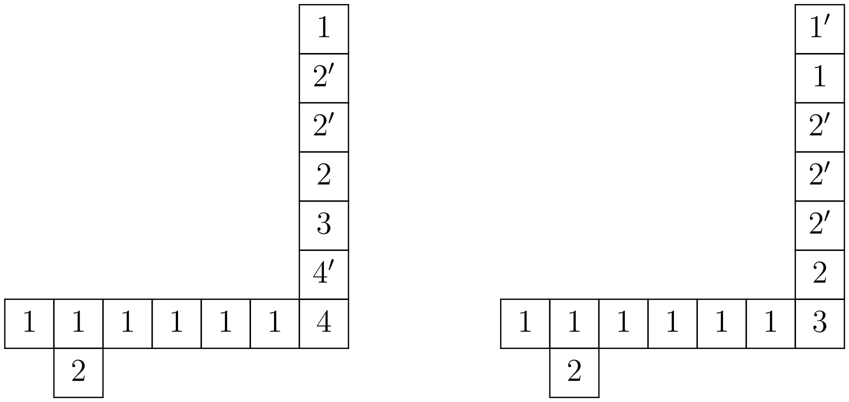


Figure 4.3: At left, an example of a tableau T that satisfies the conditions of statements of Lemmas 4.2.1, 4.2.2, and 4.2.3 At right, a tableau T that also satisfies Lemmas 4.2.4, 4.2.5, and 4.2.6.

.

Proof. First, since T is semistandard, the entries of the column are weakly increasing from top to bottom. In particular, the k^* entries in this column form a consecutive string of the reading word, then the $(k-1)^*$ entries are consecutive after them, and so on.

Since T is ballot, the $(k-1)/k$-walk returns to the x-axis. Just after reading the string of k^*'s in the column, the lattice walk cannot be on the x-axis (since it never can be after a k^* step), and the only arrow that can move the walk downwards is an unprimed $k-1$. Thus there is an unprimed $k-1$ in the column. This holds for all pairs $i-1$ and i for $1 < i \leq k$, so the column must contain nonempty strings of each i for $1 \leq i \leq k$, with an unprimed entry guaranteed for $1 \leq i < k$. Since T is semistandard, there is at most one unprimed i in the column for each i as well (with all the i' entries occurring above it). $\square$

Lemma 4.2.2. *In T, the long row has entries $1, \ldots, 1, k^*$ for some k.*

Proof. As in the previous lemma, let k^* be the entry in the bottom right corner. First, assume for contradiction that one of the entries, other than the rightmost entry, is i^* for $1 < i < k$. Then the $(i-1)/i$-walk reaches a point above the x-axis just after this i^*. Since T is semistandard, there is no $(i-1)$ in the reading word until after the unprimed i in the column guaranteed by Lemma 4.2.1. After the unprimed i, the walk is at least two steps above the x-axis, but by Lemma 4.2.1, there is only one more instance of an unprimed $i-1$, so the walk does not return to the x-axis. This contradicts that T was ballot, and thus every entry in the long row is either 1^* or k^*.

If some entry other than the bottom right entry is a k^*, then since T is semistandard, the bottom right entry is an unprimed k. Then, after the bottom right entry of k, the $(k-1)/k$-walk is at least two steps above the x-axis, and there is only one unprimed $k-1$, and thus one step down, in the remainder of the walk. Therefore the lattice walk for the subword of $(k-1)^*$ and k^* entries does not return to the x-axis, contradicting the ballotness of T.

Thus every entry in the long row, other than the bottom right entry, is 1^*. Since T is in canonical form, the first 1 must be unprimed, and so by semistandardness the long row has entries $1, \ldots, 1, k^*$. $\square$

Lemma 4.2.3. *In T, the entry in the square in the bottom row is a* 2*.*

Proof. Since T is ballot and hence in canonical form, any entry in the square in the bottom row must be unprimed, since it is first in reading order. If the entry were a 1, then there would be a vertical adjacency of two unprimed 1s by Lemma 4.2.2, contradicting that T is semistandard.

Suppose that the entry in the bottom row is i for $i > 2$. Then in the lattice walk for the subword of $(i-1)^*$ and i^* entries, by Lemma 4.2.2, there is no $(i-1)^*$ entry between the entry in the bottom row and the i^* entries in the column, so the lattice walk begins with two up steps. Then by Lemma 4.2.1, there is at most one more down step in the lattice walk from the single unprimed $i-1$, and so the walk does not return to the x-axis, contradicting that T is ballot. Thus the entry in the bottom row is a 2. $\square$

See the first diagram in Figure 4.3 for an example of a tableau that satisfies the three lemma statements above.

Lemma 4.2.4. *In T, the entry in the corner of the outer turn is at most 3.*

Proof. Suppose the entry in the lower right corner is i^* for $i > 3$, and consider the $(i-1)/i$-walk. Since $i > 3$, there is no instance of $(i-1)^*$ or i^* in the word before the entry from the lower right corner by Lemmas 4.2.2 and 4.2.3. Since T is ballot, the first i is unprimed, so the lower right entry is i. Then the $(i-1)/i$-subword has the form $i, i', \ldots, i', i-1, i-1', \ldots, i-1'$, with k copies of i^* for some k. However, after the string of i^*s, the walk has taken k steps up the y-axis, followed by a right step for the $i-1$ entry, and right steps for the remaining $i-1'$ entries. Thus the walk does not return to the x-axis, contradicting that T is ballot, and hence the lower right entry is at most 3. ☐

Lemma 4.2.5. *In T, there are at least as many 1s in the long row as there are 2^*s in T.*

Proof. Consider the subword of T containing 1^*s and 2^*s. From Lemma 4.2.3, we have that the bottommost square contains a 2, and from Lemma 4.2.2, the remaining 2^* entries are in the rightmost column. Suppose T contains more 2^* entries than 1^* entries in the second row, and say the number of 1^* entries in the second row is j. Then, following the initial 2 and subsequent string of 1 entries of length j, the lattice walk is at $(j-1, 0)$ (since the second 1 is a down step). Then there is a 2 and at least $j-1$ entries of value $2'$ following the initial string of 1s, which bring the path left to the y-axis. Then since only the first of the remaining 1^*s can be unprimed, the lattice walk contains no further down steps, and does not return to the x-axis. Thus there are at least as many 1^*s in the second row as 2^*s in T. ☐

Lemma 4.2.6. *In T, there is at most one 3^*, and if there is one, then it is unprimed and occurs in the corner of the outer turn.*

Proof. Suppose there are at least two 3^* entries. By Lemmas 4.2.2 and 4.2.3, the set of 3^*s must form a vertical strip in the right column. By Lemma 4.2.4, and since T is semistandard, the entry in the lower right corner must be a 3. Then the subword corresponding to entries of 2^* or 3^* has the form $2, 3, 3', \ldots, 3', 2, 2', \ldots, 2'$ for some number of $3'$ entries. If there is at least one $3'$, then after the steps for the 2 and the 3, the corresponding lattice walk is on the y-axis, so the next 2 gives a right step, and the following $2'$ entries also give right steps. Hence the walk does not return to the x-axis, contradicting that T is ballot. Therefore, there can be at most one 3^* in the filling, and it must be a 3 in the lower right corner. ☐

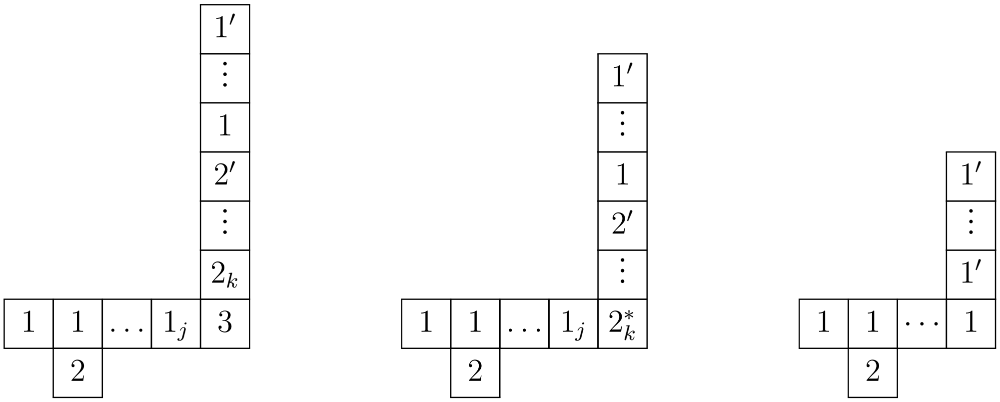


Figure 4.4: The forms that any ballot tableau of a frayed ribbon with one turn must take, from Lemmas 4.2.1 through 4.2.6, where if k is the number of 2^* entries in the long column and j is the number of 1 entries in the long row, we have $k < j$.

We now show the above lemmas completely characterize the ballot tableaux of frayed ribbon shapes with one turn.

Proposition 4.2.7. *A tableau T of a frayed ribbon shape with one (outer) turn is ballot if and only if it satisfies all of the conditions of Lemmas 4.2.1 through 4.2.6. In particular, they take one of the three forms shown in Figure 4.4.*

Proof. The forward implication is given by Lemmas 4.2.1 through 4.2.6. Now, suppose a tableau T satisfies the conditions of these six lemmas. Then it has one of the three forms shown in Figure 4.4 depending on whether more than one 2^* occurs, and whether any 3^* occurs or not (since if it occurs there is exactly one and it appears unprimed in the outer turn). Clearly, T is in canonical form. If T contains only one 2, then the word is ballot since the 2 is followed by at least two unprimed 1 entries.

In the other cases, the $(1, 2)$-walk has the form $2, 1, \ldots, 1, 2, 2', \ldots, 2', 1, 1', \ldots, 1'$, with at least as many 1s in the initial string than 2^*s in the second string. This condition gives that the walk is at $(x, 1)$ for $x \geq 1$, after the final $2'$, and so the remaining 1 returns the walk to the x-axis. If the filling contains a 3, then the $(2, 3)$-walk has the form $2, 3, 2, 2', \ldots, 2'$, which also ends on the x-axis. Hence each tableau is ballot. □

Define the **column height** of a frayed ribbon with one turn to be the number of boxes in the long column above the corner of the outer turn (not including the corner box of the turn itself). Proposition 4.2.7 leads to the following explicit Schur Q expansion for one turn frayed ribbons.

Corollary 4.2.8. *Let D be a frayed ribbon with one turn of size n and column height h. Then*

$$Q_D = Q_{(n-1,1)} + 2\sum_{i=2}^{m_1(h)} Q_{(n-i,i)} + \sum_{i=2}^{m_2(h)} Q_{(n-i,i,1)}$$

where $m_1(h) = \min(h+1, n-h-2)$ and $m_2(h) = \min(h, n-h-2)$.

Proof. The first and second terms correspond to the tableaux of the form on the right of Figure 4.4 (where for $i \geq 2$ the bottom 2 in the long column may be either primed or unprimed, hence the coefficient of 2), and the third term corresponds to the tableaux of the form on the left of Figure 4.4, which must have an unprimed 2 just above the 3 and hence have a coefficient of 1. $\square$

Theorem 4.2.9. *Suppose D and E are frayed ribbons with one turn of size n, where D has column height h and E has column height ℓ for $h \neq \ell$. Then $Q_D \neq Q_E$.*

Proof. Suppose $Q_D = Q_E$. Then by Corollary 4.2.8, we must have $m_1(h) = m_1(\ell)$ and $m_2(h) = m_2(\ell)$. The former equation states that $\min(h+1, n-h-2) = \min(\ell+1, n-\ell-2)$. Since $h \neq \ell$, it follows that either $h+1 = n-\ell-2$ or $n-h-2 = \ell+1$, which are in fact equivalent statements, and so they both hold.

From $m_2(h) = m_2(\ell)$, we have $\min(h, n-h-2) = \min(\ell, n-\ell-2)$, and so by the same reasoning we have $h = n-\ell-2$, which contradicts our equality above. Hence $Q_D \neq Q_E$. $\square$

4.3 Frayed Ribbons with Two Turns and Column Height 0

This section, along with the next two, will prove the third statement of Theorem 1.0.3 in three parts. As a first step, we will show that all frayed ribbons with two turns and no boxes between the turns have distinct Schur Q functions (up to antipodal reflection).

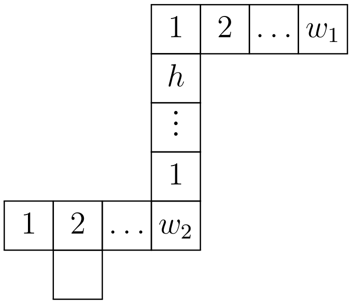


Figure 4.5: A general picture of a frayed ribbon with 2 turns and parameters h, w_1, and w_2. Note that, if $D = \nu/\mu$, then $w_1 = \nu_1 - \mu_1$ and $w_2 = \nu_{\ell(\mu)+1}$. Further, $n = w_1 + h + w_2 + 1$ Without loss of generality, by Proposition 2.3.9, we can assume that $w_2 \geq 3$.

Choosing only the representative of each antipodal pair with second-to-last row having length at least three, we see that any such shape is uniquely determined by three parameters, as illustrated in Figure 4.5:

- The **top width** w_1, defined as the number of squares in the top row,
- The **column height** h, defined as the number of squares in the vertical column between the two long rows, and
- The **bottom width** w_2, defined as the number of squares in the row second from the bottom.

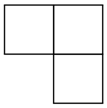
We now prove two technical lemmas that explicitly calculate a set of Littlewood-Richardson coefficients that arise in the two-turn height 0 case.

Lemma 4.3.1. *Let $D = \nu/\mu$ be a frayed ribbon of size n with two turns and parameters w_1, h, w_2 where $h = 0$. Then for any k such that $(n-k, k)$ is a shifted partition, the coefficient $f^{\nu}_{\mu,(n-k,k)}$ of $Q_{(n-k,k)}$ in the expansion of Q_D is given by:*

$$f^{\nu}_{\mu,(n-k,k)} = \begin{cases} 4 & \text{if } 2 \leq k \leq \min(w_1+1, w_2) - 1 \\ 2 & \text{if } k = \min(w_1+1, w_2) \\ 0 & \text{if } k \geq \min(w_1+1, w_2) + 1 \end{cases} .$$

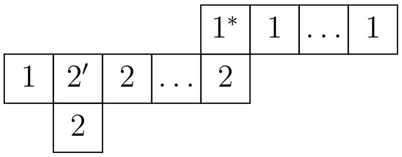
In particular, when $w_1 + 1 \neq w_2$ *there is a unique value of* k *for which* $f^{\nu}_{\mu,(n-k,k)} = 2$*. When* $w_1 + 1 = w_2$*, since* $(w_1 + 1, w_2)$ *is not a shifted partition, there is no value of* k *for which* $f^{\nu}_{\mu,(n-k,k)} = 2$*.*

Proof. **Case 1.** Suppose $w_1+1 \geq w_2$, so that $\min(w_1+1, w_2) = w_2$. We count the ballot tableaux of shape D with content $(n-k, k)$ for each k. If $k \geq w_2 + 1$, then since there must be only 1^* entries in the top row in any ballot tableau by Lemma 4.1.2, the remaining $w_2 + 1$ boxes must contain 2^* entries. However, there is no semistandard filling of the three boxes



with only 2^* entries, hence there is no ballot tableau of D with content $(n-k, k)$. So the coefficient is 0 in this case.

If $k = w_2$, we first note that $n = w_1 + w_2 + 1$, so the content $(n-k, k) = (w_1 + 1, w_2)$ is not a valid shifted partition (and hence the second case of the theorem not appearing when $w_1+1 = w_2$). So we may assume in this subcase that $w_1+1 > w_2$. Then there are only two semistandard fillings with this content in canonical form whose top row is all 1^* entries:



where the first 1 in the final row can be primed or unprimed. We claim that these fillings are in fact both ballot as well. Indeed, in the lattice walk for this filling, after reading the second row, the walk is at $(0, w_2 - 1)$. Since $w_1 + 1 > w_2$, we have $w_1 \geq w_2$, so after the 1^*, there are at least $w_2 - 1$ entries with value 1 in the top row. This brings the walk back to the x-axis, so the fillings are ballot. It follows that $f^{\nu}_{\mu,(n-k,k)} = 2$ when $(n-k, k)$ is a valid content in this case.

Finally, suppose $2 \leq k \leq w_1 + 1$. Then there are four valid fillings:

$$\begin{array}{ccccccccccc}
 & & & & & & & 1^* & 1 & \dots & 1 \\
1 & 1 & \dots & 1 & 2^* & 2 & \dots & 2 & & & \\
 & 2 & & & & & & & & &
\end{array}$$

where the first 2 in the second row and 1 in the top row can be primed or unprimed. Since $k \leq w_1+1$, there are at most $w_1 \leq w_2$ entries with value 2^* in the second row, and w_2 entries with value 1 in the final row, so the corresponding lattice walk returns to the x-axis. Therefore $f^{\nu}_{\mu,(n-k,k)} = 4$ in this case.

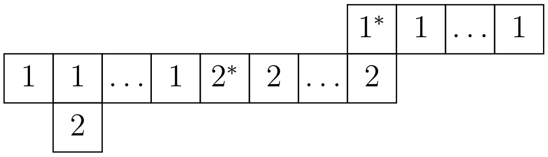
Case 2. Suppose that $w_1 + 1 < w_2$, so that $\min(w_1 + 1, w_2) = w_1 + 1$ and $w_1 + 1 \neq w_2$. If $k \geq w_1 + 2$, then for a filling of D to be ballot and semistandard we need all 1^* entries in the top row, the bottommost square contains a 2, and the remaining $w_1 + 1$ of the 2^* entries occur in the second row. After reading past these 2^* entries (all of which but the first must be 2 by semistandardness), the walk must be at height $w_1 + 1$, But only w_1 of the 1^* entries follow it in the top row. Thus the lattice walk cannot return to the x-axis, and so there are no ballot tableaux. Hence in this case $f^{\nu}_{\mu,(n-k,k)} = 0$.

If $k = w_1 + 1$, then since $w_1 + 1 < w_2$ is a strict inequality, we find four semistandard fillings of valid content $(w_2, w_1 + 1)$ in canonical form whose top row contains all 1^* entries:

$$\begin{array}{cccccccccc}
 & & & & & & & 1^* & \dots & 1 \\
1 & 1 & \dots & 1 & 2^* & 2 & \dots & 2 & & \\
 & 2 & & & & & & & &
\end{array}$$

We will show that only two of these are ballot, namely when the top 1^* is unprimed. Indeed, if x is the number of 1 entries in the second row, then since there are $k - 1 = w_1$ entries with value 2^* in the second row, the lattice walk is at $(x - 1, w_1)$ after reading this row. Then the remaining w_1 entries on the top row return the walk to the x-axis if and only if the 1^* is unprimed so that its arrow points down. Hence there are two valid fillings in this case, and $f^{\nu}_{\mu,(n-k,k)} = 2$.

Finally, if $2 \leq k \leq w_1$, then any ballot tableau has at most $w_1 - 1$ entries with value 2^* in the second row, and we find four ballot tableaux by the lattice walk reasoning above:



Hence there are four ballot tableaux in this case and $f^{\nu}_{\mu,(n-k,k)} = 4$. ☐

Lemma 4.3.2. *Let $D = \nu/\mu$ be a frayed ribbon of size n with two turns and parameters w_1, h, w_2 where $h = 0$. Then for any k such that $(n-k,k)$ is a shifted partition, the coefficient $f^{\nu}_{\mu,(n-k-1,k,1)}$ of $Q_{(n-k-1,k,1)}$ in the expansion of Q_D is given by:*

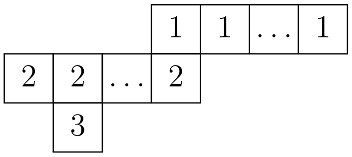
$$f^{\nu}_{\mu,(n-k-1,k,1)} = \begin{cases} 2 \text{ \textit{if} } k \leq \min(w_1, w_2) - 1 \\ 1 \text{ \textit{if} } k = \min(w_1, w_2) \\ 0 \text{ \textit{if} } k \geq \min(w_1, w_2) + 1 \end{cases}$$

In particular, when $w_1 \neq w_2$ there is a unique value of k for which $f^{\nu}_{\mu,(n-k-1,k,1)} = 1$. When $w_1 = w_2$, since $(w_1, w_2, 1)$ is not a shifted partition, there is no value of k for which $f^{\nu}_{\mu,(n-k-1,k,1)} = 1$.

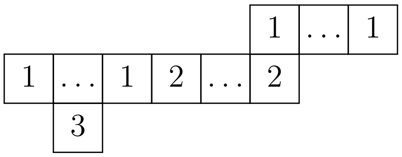
Proof. Note that in any ballot tableau with a 3, since the final row must consist of only 1^* entries, we must have the 3 entry in the bottommost square. Then, if the content is $(n-k-1,k,1)$, all k of the 2^*s are unprimed and in the second row by ballotness, canonical form, and semistandardness. By semistandardness, the top row is of the form $1^*1\cdots1$, and so there are at most two ballot tableaux with this content for any k.

First suppose that $k \geq \min(w_1, w_2) + 1$, so $k \geq w_1 + 1$ or $k \geq w_2 + 1$. In the first case, the $(1,2)$-walk is at height $k = w_1 + 1$ at the end of the second row, and the top row $1^*111\cdots1$ of length w_1 does not contain enough entries to bring the walk back to the x-axis, so there are no ballot tableaux. In the second case, there are more 2^* entries than there are boxes in the second row, so there are no ballot tableaux in this case and we have $f^{\nu}_{\mu,(n-k-1,k,1)} = 0$.

Next, suppose $k = \min(w_1, w_2)$. If $w_1 = w_2$, then $(n-k-1, k, 1) = (w_1, w_2, 1)$ does not have valid content, so we may assume $w_1 \neq w_2$. If $\min(w_1, w_2) = w_2$, then $k = w_2 < w_1$, so there is one ballot tableau:

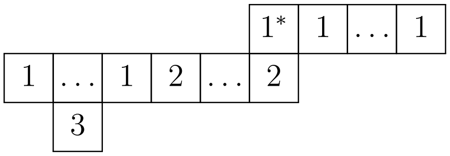


(Note that the first 1 in the top row must be unprimed by the canonical form condition in this case, and it is easily checked that the ballot walk condition holds.) If instead $\min(w_1, w_2) = w_1$, since $k = w_1 < w_2$, there is again one ballot tableau:



Indeed, after the second row, the $(1,2)$-walk is at (x, w_1) for some $x \geq 1$, and so the w_1 entries $1^*111\cdots1$ in the top row return the walk to the x-axis only when the first 1 is unprimed. Thus in this case we have $f^{\nu}_{\mu,(n-k-1,k,1)} = 1$.

Finally, suppose $k \leq \min(w_1, w_2) - 1$, so $k \leq w_1 - 1$ and $k \leq w_2 - 1$. We find two fillings:



Since $k \leq w_2 - 1$, the second row consists of a string of at least one 1 entry, followed by k 2s, so the $(1,2)-$lattice walk is at (x, k) for $x \geq 1$ and $k \leq w_1 - 1$. Then the w_1 1^* entries in the last row return the walk to the x-axis, regardless of whether the first 1 is primed or unprimed. Thus there are 2 ballot tableaux of this content, and $f^{\nu}_{\mu,(n-k-1,k,1)} = 2$. □

We can now prove the main result of this section.

Theorem 4.3.3. *Suppose D and E are frayed ribbon shapes of size n with 2 turns and parameters w_1, h, w_2 and w_1', h', w_2' respectively, where $h = h' = 0$ and $w_2, w_2' \geq 3$. Then if $w_1 \neq w_1'$, we have $Q_D \neq Q_E$.*

Proof. Since D and E both have height 0, we have $w_2 = n - 1 - w_1$ and $w_2' = n - 1 - w_1'$. Thus, since $w_1 \neq w_1'$ by assumption, we also have $w_2 \neq w_2'$.

Case 1. Suppose $\min(w_1, w_2) = w_1$ and $\min(w_1', w_2') = w_1'$. Then by Lemma 4.3.2 the unique (or nonexistent) value of k at which the coefficient $f^{\nu}_{\mu,(n-k,k)}$ is 1 is either different for D and E, or exists for one of D and E and does not exist for the other. Hence $Q_D \neq Q_E$ in this case.

Case 2. Suppose $\min(w_1, w_2) = w_2$ and $\min(w_1', w_2') = w_2'$. Then since $w_2 \neq w_2'$, the same proof as in Case 1 shows $Q_D \neq Q_E$.

Case 3. Suppose that one of the minima has subscript 1 and the other 2; without loss of generality suppose $\min(w_1, w_2) = w_1$ and $\min(w_1', w_2') = w_2'$. If $w_1 \neq w_2'$ then we are done as before; otherwise, suppose $w_1 = w_2'$. Then $\min(w_1' + 1, w_2') = w_2'$ and $\min(w_1 + 1, w_2)$ is either $w_1 + 1$ or w_2. But $w_1 + 1 \neq w_2'$ since $w_1 = w_2'$, and $w_2 \neq w_2'$, so in either case we have

$$\min(w_1 + 1, w_2) \neq \min(w_1' + 1, w_2').$$

So, by Lemma 4.3.2, the unique (or nonexistent) value of k for which the coefficient $f^{\nu}_{\mu,(n-k-1,k,1)}$ is 1 is either different for D and E, or exists for one of D or E and does not exist for the other. Hence $Q_D \neq Q_E$. ☐

4.4 Frayed Ribbons with Two Turns and Column Height 1

We now show that all Schur Q functions of frayed ribbons with column height 1 are distinct from one another. We use the same notation w_1, h, w_2 established at the start of Section 4.3, and note that throughout this section we will have $h = 1$.

Lemma 4.4.1. *Let $D = \nu/\mu$ be a two-turn frayed ribbon of size n with parameters w_1, h, w_2 where $h = 1$. Then for any k such that $(n-k, k)$ is a shifted partition, the coefficient $f^{\nu}_{\mu,(n-k,k)}$ of $Q_{(n-k,k)}$*

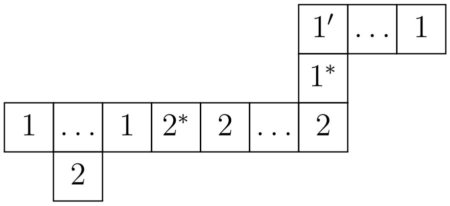
in the expansion of Q_D is given by:

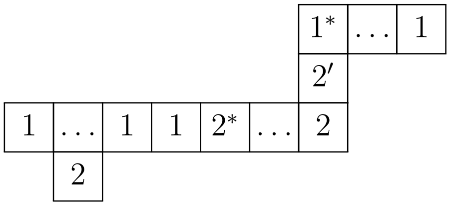
$$f^{\nu}_{\mu,(n-k,k)} = \begin{cases} 8 \textit{ if } 3 \leq k \leq \min(w_1, w_2 - 1) \\ 6 \textit{ if } k = \min(w_1, w_2 - 1) + 1 \textit{ and } w_1 \neq w_2 - 1 \\ 4 \textit{ if } k = \min(w_1, w_2 - 1) + 1 \textit{ and } w_1 = w_2 - 1 \\ 2 \textit{ if } k = \min(w_1, w_2 - 1) + 2 \\ 0 \textit{ if } k \geq \min(w_1, w_2 - 1) + 3 \end{cases}$$

In particular, there is a unique value of k for which $f^{\nu}_{\mu,(n-k,k)}$ is either 4 or 6.

Proof. In order for a filling with content $(n-k,k)$ to be semistandard and ballot, the upper row must consist of only 1^* entries (Lemma 4.1.2), and the bottommost square must contain a 2.

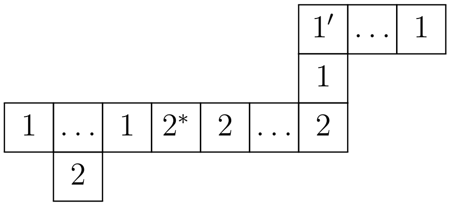
First, suppose that $3 \leq k \leq \min(w_1, w_2 - 1)$. Since $k \geq 3$, there are 8 possible fillings:

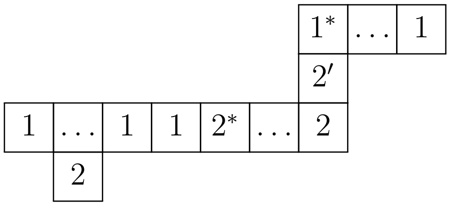




Since $k \leq w_2 - 1$, the third row contains at least two 1 entries, so after the string of 2^* entries in the reading word, the $(1,2)$-walk is at either $(x, k-1)$ or $(x-1, k-2)$ for $x \geq 1$, according to the two cases shown above. Then, since $k \leq w_1$, there are at at least $k-1$ entries with value 1 to return the walk to the x-axis. Thus each of these fillings is ballot.

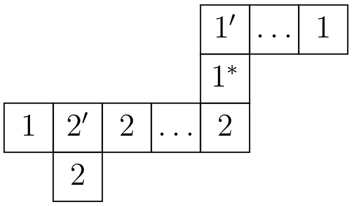
Next, suppose that $k = \min(w_1, w_2 - 1) + 1$. Note that since $w_1 \geq 2$ and $w_2 \geq 3$ by the definition of the two-turn shape, we have $k \geq 3$ again. If $w_1 \neq w_2 - 1$, and $k = w_1 + 1$, we claim there are only 6 ballot tableaux, as the type on the left above must have an unprimed 1:

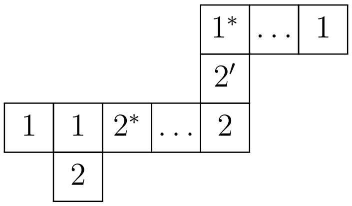




Indeed, since $k < w_2$, we get $k \leq w_2 - 1$, so there are at least two 1 entries in the third row in the left hand diagram above, and at least three in the right diagram. Thus the $(1,2)$-walk is at either $(x, k-1)$ for some $x \geq 1$ for the left hand diagram, or $(x-1, k-2)$ for some $x \geq 2$ respectively after the string of 2^* entries. Since $k = w_1 + 1$, then if the string of 2^* entries ends in the third row, the following 1 must be unprimed so that there are $k-1$ unprimed 1 entries to return the subwalk to the x-axis.

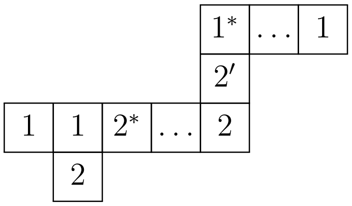
If $k = w_2 = \min(w_1, w_2 - 1) + 1$ with $w_1 \neq w_2 - 1$, then we claim there are again 6 ballot tableaux:





Indeed, in the two possible forms of tableaux above, after the string of 2^* entries, the walk is at $(0, k-1)$ or $(0, k-2)$ respectively. It is followed by a 1^* entry that moves the walk one step to the right and then at least $w_1 - 1 \geq k - 1$ entries with value 1 to return the walk to the x axis (since $k < w_1 + 1$). Furthermore, since $k = w_2$, if the single box in the second row contains a 1^*, then the first 2^* entry in the third row is directly above the 2 in the bottommost box, and so it must be primed for the tableau to be semistandard. Hence there are exactly 6 ballot tableaux.

If $k = w_1 + 1 = w_2$, then we claim the only 4 ballot tableaux are of the form:

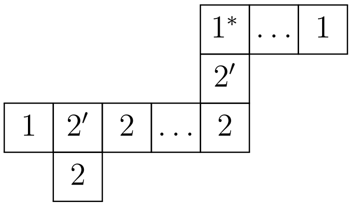


We first show there cannot be a 1^* in the single box in the second row. If so, since $k = w_2$ the first 2^* in the third row must be primed and is above the bottommost 2. Then the walk of the word is at $(0, k-1)$ following the string of 2^* entries. Since $k = w_1 + 1$, the remaining string $1^*, 1', 1, \ldots, 1$ of length $w_1 + 1$ does not return the walk to the x-axis. Thus there is a $2'$ in the second row. In this case, there are two 1 entries in the third row, so the subwalk is at $(0, k-2)$ after the string of 2^*

entries. It follows that the remaining single 1^* and $w_1 - 1 = k - 2$ more entries with value 1 return the walk to the x-axis. Hence there are exactly 4 ballot tableaux in this case.

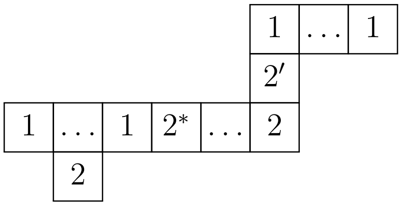
Now suppose that $k = \min(w_1, w_2 - 1) + 2$. Note that, since $n = w_1 + w_2 + 2$, if $|w_1 - (w_2 - 1)| \leq 1$, the content $(n - k, k)$ is not valid. So $|w_1 - (w_2 - 1)| \geq 2$.

If $k = w_2 + 1$, then there are only 2 possible ballot tableaux:



To see that these are both ballot, after the string of 2^* entries, the $(1, 2)$-walk is at $(0, k - 1)$, and the next 1^* moves the walk right one step to $(1, k - 1)$. From above, we get $k - 1 = w_2 \leq w_1 - 1$, so the remaining string of at least $k - 1$ entries of value 1 returns the walk to the x-axis.

If instead $k = w_1 + 2$, then there are also 2 ballot tableaux:



In particular, by the inequality $|w_1 - (w_2 - 1)| \geq 2$, since $w_1 < w_2 - 1$ in this case we have $k - 1 = w_1 + 1 \leq w_2 - 2$, and so there are at least two 1 entries in the third row. Thus, if the single box in the second row contains a 1^*, then after the string of 2^* entries, the walk is at $(x, k - 1)$ for some $x \geq 1$. Then there are at most $w_1 = k - 2$ unprimed 1 entries remaining, so the walk does not return to the x-axis. It follows that the second row must contain a $2'$. In this case, after the string of 2^* entries, the walk is at $(x - 1, k - 2)$. Then the remaining string $1^* 1 \cdots 1$ of length $w_1 = k - 2$ returns the walk to the x-axis only if the first 1 is unprimed. So there are two ballot tableaux with this content.

Finally, suppose that $k \geq \min(w_1, w_2 - 1) + 3$. If $k \geq w_1 + 3$, then the final string of 2^* entries has length at least $w_1 + 2$, but there are at most $w_1 + 1$ entries of value 1^* remaining, so the walk does not return to the x-axis. If $k \geq w_2 + 2$, then there are no semistandard fillings of D, since the top row and first box of the third row must contain 1^* entries. In either case, there are no ballot tableaux with this content. $\square$

Lemma 4.4.2. *Let $D = \nu/\mu$ be a two-turn frayed ribbon of size n with parameters w_1, h, w_2 where $h = 1$. Then for any k such that $(n-k-2, k, 2)$ is a shifted partition, the coefficient $f^{\nu}_{\mu,(n-k-2,k,2)}$ of $Q_{(n-k-2,k,2)}$ in the expansion of Q_D is given by:*

$$f^{\nu}_{\mu,(n-k-2,k,2)} = \begin{cases} 4 \text{ if } k \leq \min(w_1, w_2) - 1 \\ 2 \text{ if } k = \min(w_1, w_2) \\ 0 \text{ if } k \geq \min(w_1, w_2) + 1 \end{cases}$$

In particular, when $w_1 \neq w_2$ there is a unique value of k for which $f^{\nu}_{\mu,(n-k-2,k,2)} = 2$. When $w_1 = w_2$, since $(w_1, w_2, 2)$ is not a shifted partition, there is no value of k for which $f^{\nu}_{\mu,(n-k-2,k,2)} = 2$.

Proof. For a filling with content $(n-k-2, k, 2)$ to be semistandard and ballot, the first row must consist only of 1^* entries (Lemma 4.1.2), the single box in the second row must contain a 2 for the $(2,3)$-walk to end on the x axis, the third row has reading word $1 \cdots 12 \cdots 23^*$, and the fourth row must contain a 3. Further, we must have $k \geq 3$, so that λ is a strict partition.

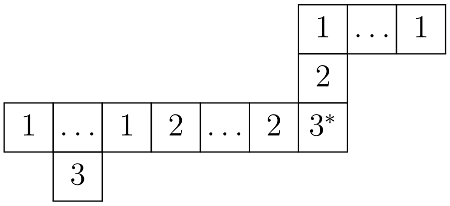
First, suppose that $k \leq \min(w_1, w_2) - 1$. There are 4 semistandard tableaux satisfying the conditions above:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} &&&&&& 1^* & \dots & 1 \\ &&&&&& 2 && \\ 1 & \dots & 1 & 2 & \dots & 2 & 3^* && \\ & 3 &&&&&&& \end{array}$$

We show that these four tableaux are all ballot. Since $k \geq 3$, the $(2,3)$-walk returns to the x-axis. Since $k \leq w_2 - 1$, there is at least one 1 entry in the third row, so following the final 2 in the reading word, the $(1,2)$-walk is at (x, k) for some $x \geq 1$. Since $k \leq w_1 - 1$, there are at least k

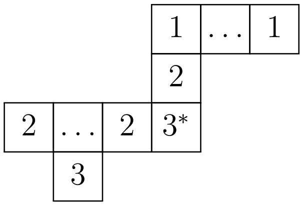
unprimed 1 entries in the first row, so the $(1,2)$-walk returns to the x-axis. Hence there are 4 ballot tableaux with this content.

Next, suppose that $3 \leq k = \min(w_1, w_2)$. If $w_1 = w_2$, then $n - k - 2 = n - w_1 - 2 = w_2 = k$, so $(n-k-2, k, 2)$ is not a valid content. If $k = w_1 < w_2$, then we get two fillings:



Since $k < w_2$, there is at least one 1 entry in the third row, so after the 2 in the second row, the $(1,2)$-walk is at (x,k) for some $x \geq 1$. Then, since $k = w_1$, the top row must consist of only unprimed 1 entries for the subwalk to return to the x-axis. Thus there are 2 valid fillings.

If instead $k = w_2 < w_1$, then there are two such fillings:



Since $k = w_2$, the third row contains no 1 entries, so the first 1 entry in the reading word is in the first row, and must be unprimed for the tableau to be in canonical form. Following the string of 2 entries, the $(1,2)$-walk is at $(0,k)$. Since $k < w_1$, there are enough 1 entries following the 2 entries to return the walk to the x-axis. Therefore there are 2 valid fillings in this case.

Finally, suppose that $k \geq \min(w_1, w_2) + 1$. If $\min(w_1, w_2) = w_1$ so that $k \geq w_1 + 1$, then after the last 2, the $(1,2)$-walk is at (x,k) for some $x \geq 0$. Then the remaining $w_1 < k$ entries of 1 do not return the walk to the x-axis. If $\min(w_2, w_2) = w_2$ so that $k \geq w_2 + 1$, then since the 3 entries must be in the third and fourth rows, there is a 2 entry in the top row, which cannot happen by Lemma 4.1.2. So there are no valid fillings with this content. $\square$

We can now prove distinctness for height 1 frayed ribbons with two turns.

Theorem 4.4.3. *If D and E are frayed ribbon shapes of size n with 2 turns, column height 1, and different first row width w_1, then $Q_D \neq Q_E$.*

Proof. Suppose that $Q_D = Q_E$. Let $w_1, h = 1, w_2$ be D's parameters and $w_1', h = 1, w_2'$ be E's parameters. Then by Lemma 4.4.2, among the terms of the form $Q_{(n-k-2,k,2)}$ in Q_D's expansion, there is either a unique coefficient of 2 or none. If there is a unique one then it occurs at $k = \min(w_1, w_2)$ and similarly for Q_E, so since $w_1 \neq w_1'$ we must have $w_1 = w_2'$ (and $w_2 = w_1'$).

But then, by Lemma 4.4.1, there is a unique value of k for which the coefficient of $Q_{(n-k,k)}$ in Q_D is either 4 or 6, namely, $k = \min(w_1, w_2 - 1) + 1$, and similarly for E. Thus $w_1 = w_2' - 1$, a contradiction. $\square$

4.5 Distinguishing Height 0 from Height 1

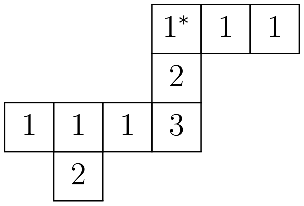
In the previous two sections we showed that all height 0 two-turn frayed ribbons have distinct Schur Q functions, and that all height 1 such shapes also have distinct Schur Q functions. We now show that these two classes are also all distinct from each other. To do so, we in fact prove a more general statement - that any shape with height 0 actually has a Schur Q function that is distinct from all other frayed ribbons.

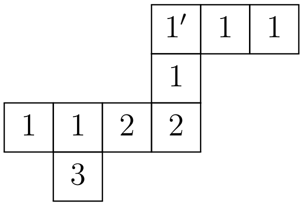
Proposition 4.5.1. *If D is a frayed ribbon shape of size n with two turns and column height 0, then $Q_D \neq Q_E$ for any frayed ribbon shape $E \neq D$, other than its antipodal D^a.*

Proof. We already know that if E does not have two turns or if it has two turns and column height 0, $Q_D \neq Q_E$ from Theorem 4.3.3 and Corollary 4.1.4. Thus it suffices to consider the case in which E has 2 turns and column height greater than 0. We may assume without loss of generality that E's second-to-bottom row has at least three boxes by Proposition 2.3.9, and simply show that $Q_D \neq Q_E$ since E will not be the antipodal shape D^a under this assumption.

Consider the coefficient of $Q_{(n-3,2,1)}$ in the straight shape Schur Q expansion of both Q_D and Q_E. For D, this coefficient is either 1 or 2 by Lemma 4.3.2.

For E, on the other hand, we show that this coefficient is at least 3. Indeed, there are always two Littlewood-Richardson tableaux of the form shown below at left, in which the 3 is in the corner, one 2 is in the bottommost square, and the other 2 is above the 3 (and unprimed so that the $(2,3)$ subword is ballot). The 1 at the top of the column can be either primed or unprimed and the $(1,2)$ word will still be ballot, so this gives two possibilities. Then, there is also always at least one of the form shown below at right, in which a 3 is in the bottommost square and the 2's are in the next to last row; if the 1 in the column is unprimed then this guarantees that the $(1,2)$ word is ballot since there is at least one 1 before the 2s and at least two 1s after them to bring the walk back down to the x-axis.





Thus the coefficient of $Q_{(n-3),2,1}$ in Q_D is at most 2, and its coefficient in Q_E is at least 3, and therefore $Q_D \neq Q_E$ as desired. $\square$

From this proof and Theorems 4.3.3 and 4.4.3, we can conclude that the frayed ribbon shapes with two turns and height either 0 or 1 give another collection of distinct skew Schur Q-functions:

Corollary 4.5.2. *The skew Schur Q functions Q_D, where D ranges over all two-turn frayed ribbon shapes of height* 0 *or* 1*, are all distinct.*

Combining the results from the last three sections, we conclude by restating our central theorem, which gives infinite families of Schur Q-functions that are not equal.

Theorem 1.0.3. *If D and E are frayed ribbons with $Q_D = Q_E$, then:*

- *D and E have the same number of turns;*
- *If D and E have no turn or one turn, then $D = E$ or $D = E^a$;*
- *If D has two turns and at most one square between the turns, then $D = E$ or $D = E^a$.*

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