A SUMMARY OF HYDRAULICS RELATED TO WELLS

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INTRODUCTION

This paper is designed to summarize very briefly the present state of knowledge regarding the more commonly used theories of hydraulics of wells. Engineers are often confronted with the problem of predicting the performance of a well from the point of view of water supply; and, somewhat less frequently, drainage. Among the questions that engineers may need to answer in regard to a well are:

1. What firm discharge may be expected?
2. What will be the drawdown and the power requirement?
3. How will the static level of the well change with time?
4. What interference may result between wells?
5. For drainage wells, how will the water table be affected?

In addition to being engineering structures for providing water supply or accomplishing drainage, wells may also constitute valuable instruments for underground exploration and investigation. Properly used they may provide such information as the water conductivity and storage coefficient of underground deposits. They may also furnish data on recharge, seepage from canals, and even ground water draft for consumptive use by crops. Workable well hydraulics theory is based on rather highly idealized conditions which do not really fully occur in nature. This means that such theory is useful in helping to understand the action of wells, but that quantitative results may need to be modified by engineering judgement. This is a condition with which the experienced engineer is entirely familiar, because it is common in all branches of engineering. Fortunately, in most cases, well hydraulics theory yields rather good estimates of what in fact may occur; however, the shortcomings of well theory need to be thoroughly understood so that rational allowances may be made in applying it. Among the principal deviations from idealized conditions occurring in nature are:

(1) Non-horizontal bedding of aquifer.
(2) Non-isotropic materials.
(3) Variation in thickness of aquifer stratum.
(4) A non-steady state, which may include a measure of recharge.

THEORETICAL TREATMENT

The earliest theoretical treatment of well hydraulics postulated a steady state of flow in a symmetrical hydraulic field. By applying the equation of Darcy to the resulting conditions of continuity, Dupuit (1863) obtained the earliest equations of this type. While valuable and still

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widely used, the Dupuit equation for a well in an unconfined aquifer did not take into account the important fact that the water surface outside of such a well invariably stands at a higher elevation than inside. Contrary to the Dupuit assumptions, many wells draw their supplies largely from storage, a factor which precludes a steady state. Rigorously speaking, a steady state cannot exist in a level hydraulic field of finite extent, the only way water can be removed from such a field is by depletion of storage. In spite of the foregoing shortcomings, the Dupuit type equations may be very useful for practical well engineering problems if properly applied.

Recognizing that in most cases, well flows originated principally from depletion of storage, Theis (1935) applied analogous heat flow equations to wells and developed the equations and procedure for analyzing non-steady state wells of constant discharge. This procedure has been widely and successfully used. It has the advantage that the effect of depletion on future piezometric elevations is predicted. It does not take into account the effects of recharge, however. For water supply wells, the volume of water originating directly from recharge is ordinarily relatively small compared to that originating directly from depletion of storage - especially for relatively short periods of pumping. This may not be true for wells designed primarily to provide drainage. In general, the non-steady state equations have yielded very excellent results, even when natural deviations from the idealized conditions assumed in their derivation are relatively great.

In an attempt to include quantitatively the effects of recharge or replenishment, Peterson and his colleagues (1950) suggested equations which combined the steady flow equations of Dupuit with considerations of continuity arising from recharge. These investigators were interested primarily in wells for land drainage - where recharge is a primary factor. While recharge considerations obviously cause aberrations from the ideal Dupuit state; these equations are believed to be very useful where recharge is a primary design consideration. They have not been extensively checked in the field.

Steady State Equations

In nature wells are almost invariably in a transient state. Even so, perturbations of the piezometric level usually rapidly decrease with time of pumping and distance from the well. If these perturbations are inappreciable, the error resulting from applying the steady state equations is very small.

Water Table Wells - Fig. 1 is a definition sketch of a water table well. At any distance r,

$$Q = 2\pi r y \frac{dy}{dr} ; \qquad (1)$$

which may be integrated to yield, between r_1 and r_2,

$$Q = \frac{\pi K\,(y_2^2 - y_1^2)}{2.3\,\log(r_2/r_1)} \qquad (2)$$

In Eqs 1 and 2, y is the water depth at radial distance r, y_1 and y_2 are depths at r_1 and r_2 respectively and K is the hydraulic conductivity. The principal point to remember in using Eq 2 is that the effect of curvilinearity of flow as the water table is drawn down toward the well is neglected, thus Eq 2 should not be applied in the region close to the well. It is valid for radial distances exceeding the order of perhaps $100r_w$, if r_w is the radius of the well. Near the well, the water table will be higher than predicted by Eq 2 and will stand at a higher elevation immediately outside of the casing than inside. If $r = r_w$ and $y = h_w$ then Eq 2 becomes

$$Q = \frac{\pi K\,(y_2^2 - h_w^2)}{2.3\,\log(r_2/r_w)} \qquad (3)$$

Inspite of the deviation of Eq 2 near the well, Eq 3 has been found to yield correct results for discharge if r is chosen relatively large. For this purpose, note that the water depth h_w inside the well, not outside, is used.

<u>Artesian Wells</u> - If the well exists in an artesian stratum of thickness m, then a similar procedure to the one used above yields

$$Q = \frac{2\pi K\,m(y_2 - y_1)}{2.3\,\log(r_2/r_1)} \qquad (4)$$

where y_2 and y_1 are now piezometric levels measured from the same horizontal datum. If r_1 equals the well radius r_w, and r_2 the radial distance r_2 to a point where perturbations in the piezometric level are insignificant, then

$$Q = \frac{2\pi m D}{2.3\,\log(r_e/r_w)} \qquad (5)$$

where D is the drawdown.
Since the flow in the artesian case is not curvilinear, Eq 4 and 5 are correct even near the well.

<u>Modifications to the Dupuit Equation</u> - In a practical water table well, the difference of elevatio [illegible] the water level inside and outside of the casing may be partly because of loss of hydraulic head through the casing. Even in a fully efficient water table well, however, some such difference of elevation would exist because of purely hydrodynamic reasons. Fig. 2, observed by Zee shows the radial flow pattern existing for a typical water table well.

Hansen (1949) attacked this problem. From dimensional considerations he postulated

$$Q/Kr_W^2 = F_1 \left(\frac{h_S}{r_W} , \frac{h_W}{r_W} \right) \quad . \tag{6}$$

In Eq 6, F_1 indicates a functional relationship and h_S is the water depth immediately outside of the casing. Hansen, using available data from model studies, was able to determine graphs of this function for a limited range of the variables. Since Hansen's original publication, much additional information has been made available, particularly from the studies of Yang who solved several cases using numerical methods; and of Zee, who used a combined electrical and membrane analogue model. All available data were used to produce Fig. 3. If two of the three parameters of Eq 6 are known, one may enter Fig. 3 to determine the third. Code and Peterson (1954) considered the results of a field test on a well near Mosca, Colorado, to determine the point marked A on Fig. 3. For this test, observed Q/Kr_W^2 was 288 and h_W/r_W was 13.33; entering Fig. 3 gives $h_S/r_W = 17.0$, which agrees well with the observed value of 17.1. This actual well was unusual in that the ideal conditions of a uniform aquifer and level initial water table were very closely realized.

Zee utilized several sources of data to develop generalized profiles of the free water surface near a water table well, Fig 4. These curves are referred to a datum arbitrarily chosen at $115r_W$. If the value of y at $115r_W$ (h_{115}) and any other value of y are known, the profile curve may be determined by entering Fig. 4. The profile curve may also be determined if Q/Kr_W^2 and either h_S, h_W or h_{115} are known by using Fig. 3 to obtain either the unknown value of h_S or h_{115} needed for entry into Fig. 4.

In a real well, the region near the well is influenced most by the curvilinear nature of the flow. This is more predominant near the top of the flow zone than near the bottom. At the extreme bottom flow line Dupuit's equation applies because there is no curvilinearity in this line. Fig. 5, due to Hansen, indicates the predominant flow zones near a water table well.[1]

Non-steady State Wells

Non-steady flow to an axially symmetrical well in a uniform aquifer must satisfy the differential equation

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t} \tag{7}$$

[1] Peterson, D. F., O. W. Israelson and V. E. Hansen, Hydraulics of Wells. Bul. 351, Utah Agricultural Experiment Station, 1950.

In Eq 7, S is the storage coefficient, the volume of water which will drain from a unit tributary area as the result of a unit decrease in piezometric head. In the artesian case, this yield results from compression of the aquifer and decompression of the water. In the water table case, S equals the specific yield. The transmissivity T equals Km. For the particular boundary conditions of constant discharge, the drawdown at any point distance r from the well at time t since the start of pumping is

$$s = \frac{Q}{4\pi T} \int_{\frac{r^2S}{4Tt}}^{\infty} \frac{e^{-u}}{u}\, du \quad . \tag{8}$$

Fig. 6 shows qualitatively the variation in the piezometric level with time and with radial distance. The above definite integral is known as the <u>exponential integral</u>, often written $Ei(u)$, where u is the finite limit. In well literature it may also appear as $W(u)$. Tables of $Ei(u)$ are published by the Smithsonian Institution[1] and elsewhere.

An explicit solution of Eq 8 is not possible. An implicit solution may be made by a matching process. Values of $W(u)$ are plotted as a function of u. For a specific test, s is plotted as a function of r^2/t to the same scale on transparent paper. By overlaying the latter curve on the former, the two may be matched and a specific match point chosen. Values of u, $W(u)$, s, and r^2/t are chosen for this point. Substitute s, Q and $W(u)$ in Eq 8 to solve for T. Remembering that $u = \frac{r^2S}{4Tt}$, S may be calculated.[2]

An approximate explicit solution of Eq 8 may be made since

$$Ei(u) = \frac{Q}{4\pi T}\left(-0.5772 - \ln u + u - \frac{u^2}{2\cdot 2!} + \frac{u^3}{3\cdot 3!} \;\ldots\ldots\right) . \tag{9}$$

For small values of u (small r, large t) only the first two terms of Eq 9 are significant and

$$s = \frac{Q}{4\pi T}\left(2.3 \log \frac{1}{u} - 0.5772\right) . \tag{10}$$

At a particular radius r and drawdowns s_1 and s_2 corresponding to times t_1 and t_2,

$$s_2 - s_1 = \frac{0.183\ Q}{T} \log t_2/t_1 \quad . \tag{11}$$

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[1] Smithsonian Institution. Physical Tables. Table 32. 1933.

[2] See Wesler and Brater, Hydrology. Wiley and Sons, 1949. p 231.

Equation 11 may be most readily solved by plotting s against t on semi-logarithmic paper as shown in Fig. 7. then if $s_2 - s_1$ be chosen for one logarithmic cycle, $\log t_2/t_1 = 1$. By extending the curve to the intercept, t_0, one may calculate S using

$$S = 2.25T\, t_0/r^2 \quad (12)$$

The foregoing theory applies to artesian wells. Prof. C. E. Jacob has shown that the non-steady state equations may also be applied to water table wells if the observed drawdown s is corrected by deducting $s^2/2m$. The thickness m is taken as the original depth of the unconfined water.

Replenishment Wells

If replenishment is adequate, an eventual steady state should theoretically develop. For certain purposes, i.e., the design of a relief drainage well system, average conditions over a replenishment cycle may be considered as an equivalent steady state. Considerable judgment must be applied in the interpretation of conclusions resulting from application of the replenishment equations, however.

The variables at the well, D, m, and r_w and the conductivity K, do not appear sufficient to determine the eventual steady state discharge from a well in symmetrical, isotropic aquifer. However, if one postulates an additional variable, the replenishment I, one concludes intuitively that these are adequate. This hypothesis may be expressed by the statement

$$Q = F_2\,(D,\ m,\ r_w,\ K,\ I) \quad (13)$$

If I be expressed in the units volume of replenishment per unit of time per unit of tributary area, dimensional considerations yield

$$\frac{Q}{Kr_w^2} = F_3\,(D/r_w,\ m/r_w,\ I/K) \quad (14)$$

Intercepting Artesian Wells - For this case, Q/Kr_w^2 may be divided by D/r_w and by m/r_w to yield

$$Q/KDm = F_4\,(D/r_w,\ m/r_w,\ I/K) \quad (15)$$

An approximate functional relationship F_4 may be developed as follows. At some effective radius of interception, r_e, $Q = 2r_eTi_n$; substituting the resulting value of r_e in Eq 5 gives

$$\frac{Q}{KDm} = \frac{2}{2.3 \operatorname{Log}\left[\frac{1}{2}\left(\frac{Q}{KDm}\right)\left(\frac{D/r_w}{i_n}\right)\right]} \quad (16)$$

Equation 16 cannot be explicitly solved for Q/KDm; however, its solution is plotted as Fig. 8.

Intercepting Water Table Well - Since the thickness of the transmitting stratum varies, h_w and h_e rather than m and D need to be considered for this case. Both m and D may be characterized in terms of h_w and h_e, thus D/r_w and m/r_w may be replaced by h_w/r_w and h_e/r_w. For this case also $I = Ki_n$. If the right side be divided by $(h_e/r_w)^2$, then

$$\frac{Q}{Kh_e^2} = F_5 \left(\frac{h_e}{r_w}, \frac{h_w}{h_e}, i_n \right) . \qquad (17)$$

At some effective radius r_e, $Q = 2r_eKh_e$. If the corresponding value of r_e and the values of $y = h_e$ when $r = r_e$ be substituted in Eq 3, one gets, for an approximate statement of the foregoing functional relationship,

$$\frac{Q}{Kh_e^2} = \frac{\pi \left[1 - \left(\frac{h_w}{h_e} \right)^2 \right]}{1.15 \log \left[\frac{1}{\pi} \left(\frac{Q}{Kh_e^2} \right) \left(\frac{h_e/r_w}{i_n} \right) \right]} . \qquad (18)$$

This implicit equation is plotted as Fig. 9.

Relief Water Table Wells - These are most frequently installed to remove excess water originating from vertical percolation. For this case, the replenishment I is equal to q_v, the volume percolating downward per unit of time per unit of area. This case is similar to that described by Eq 17 except that i_n needs to be replaced by q_v/K. Utilizing a procedure similar to that used in deriving the Dupuit equation[1] gives

$$Q = \frac{\pi K (h_e^2 - h_w^2)}{2.3 \log(r_e/r_w) - n/2} \qquad (19)$$

for a combined interception and relief well. In equation 19, n is the proportion of the total discharge originating from vertical percolation - the remainder originating from interception of horizontal flow. For the purely relief well, $n = 1$ and $Q = q_v \pi r_e^2$. Substituting the above characterized value of r_e into Eq 19 gives

[1] Peterson, D. F., O. W. Israelsen and V. E. Hansen, Hydraulics of Wells, Bulletin 251, Utah Agricultural Experiment Station. 1950.

$$\frac{Q}{Kh_e^2} \qquad \frac{\pi\left[1 - (h_w/h_e)^2\right]}{1.15 \log\left[\frac{1}{\pi}\left(\frac{Q}{Kh_e^2}\right)\left(\frac{h_e^2/S_y^2}{S_y/K}\right)\right] - \frac{1}{2}} \quad . \tag{20}$$

Eq 20 cannot be explicitly solved for the discharge parameter Q/Kh_e^2 ; however, the functional relationship expressed is plotted as Figs. 10a and 10b.

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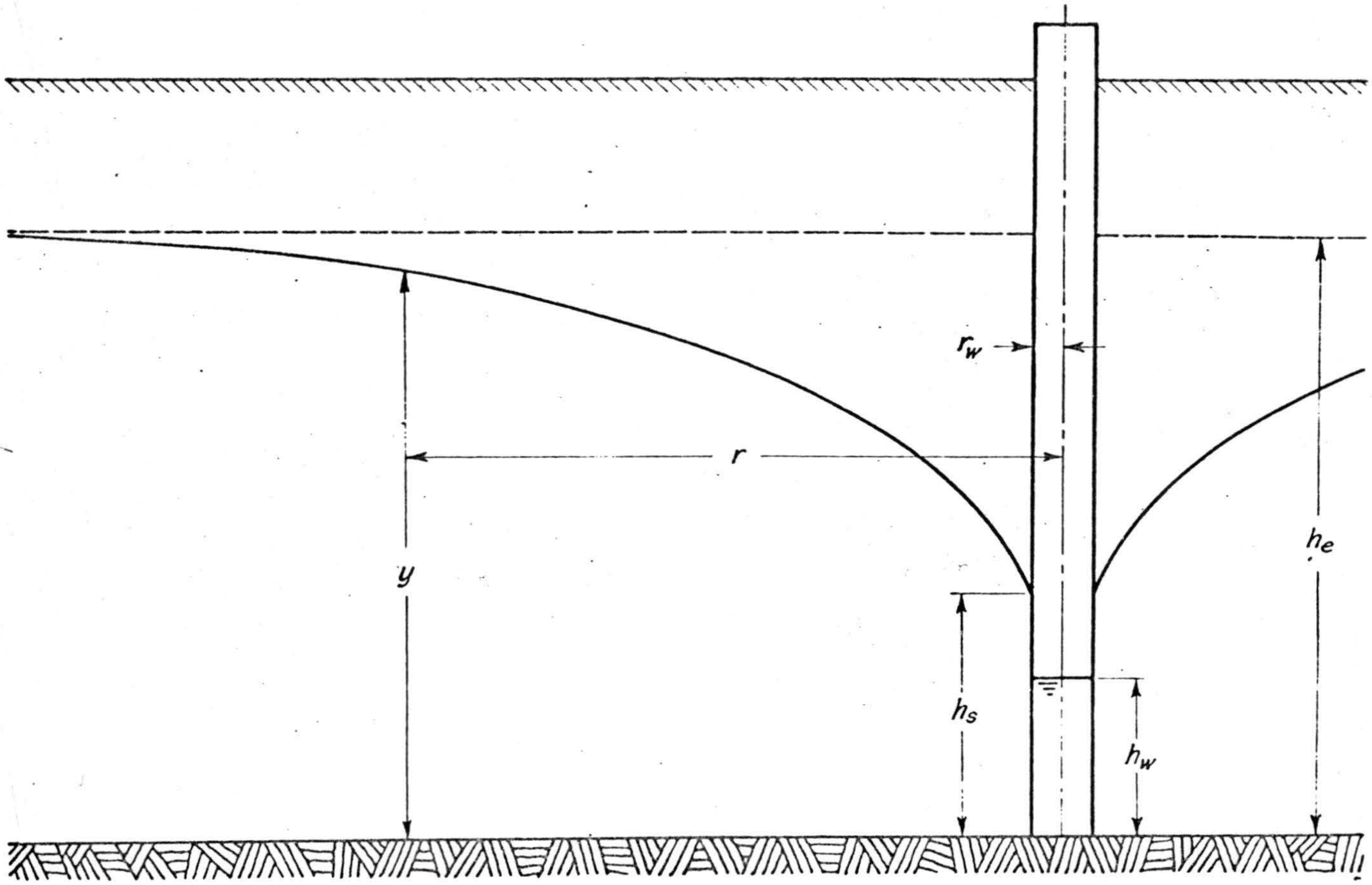

Fig. 1 Definition sketch for water table well

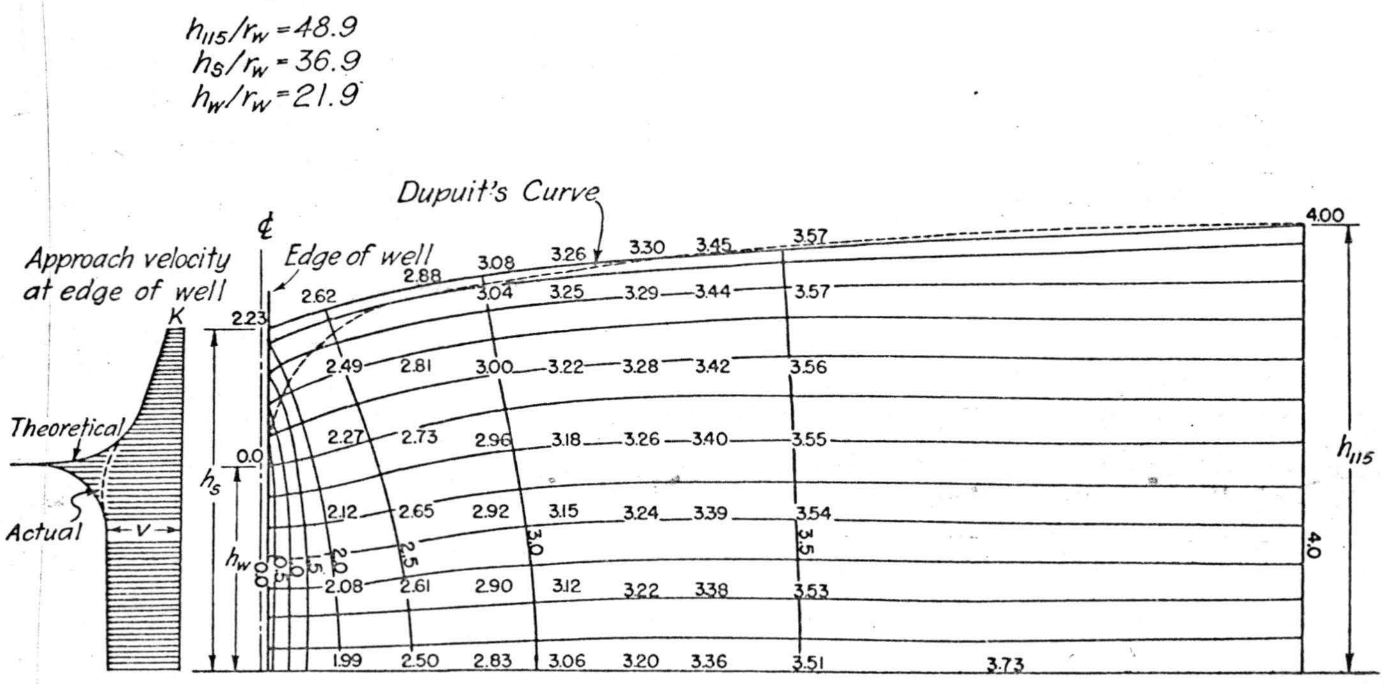

Fig. 2 Typical flow pattern to water table well

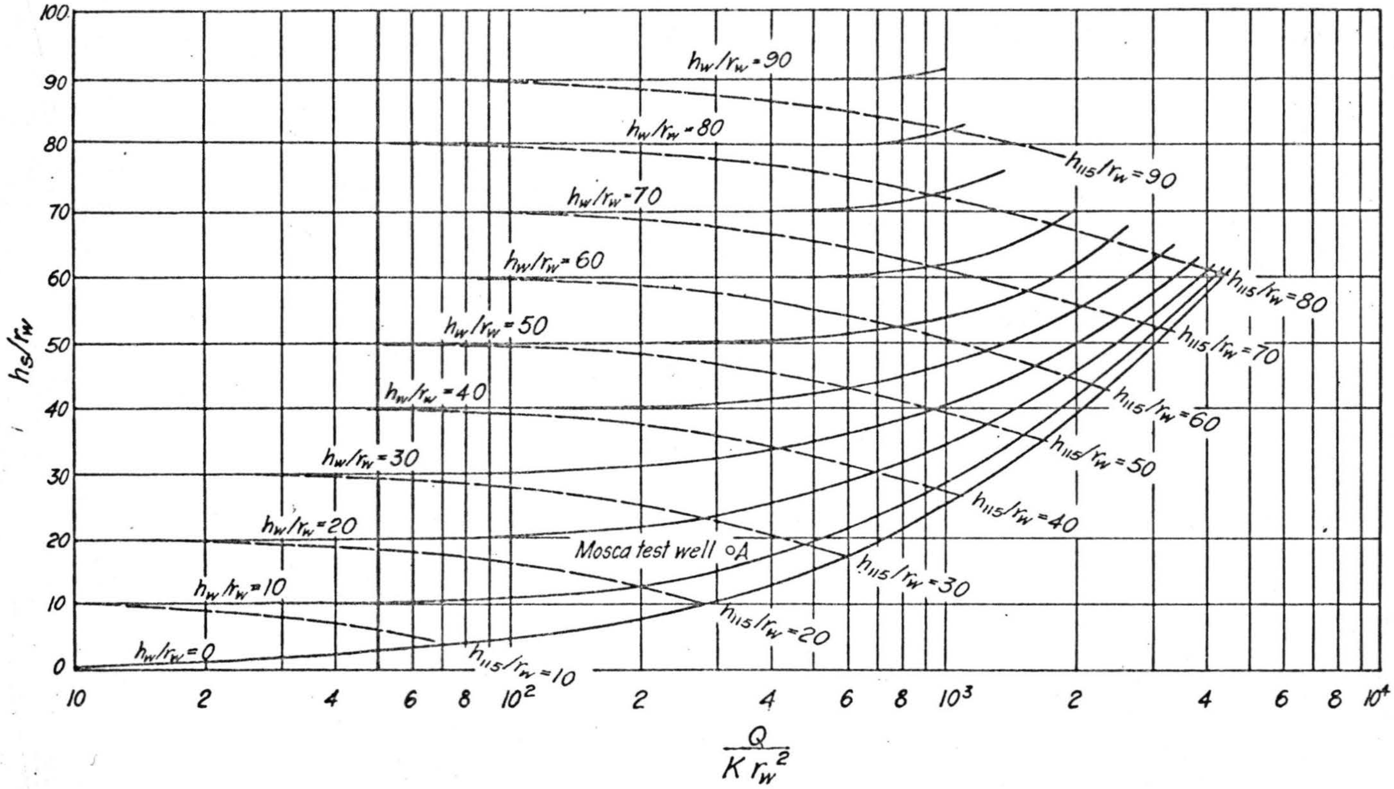

Fig. 3 Relationship of seepage face and discharge parameters

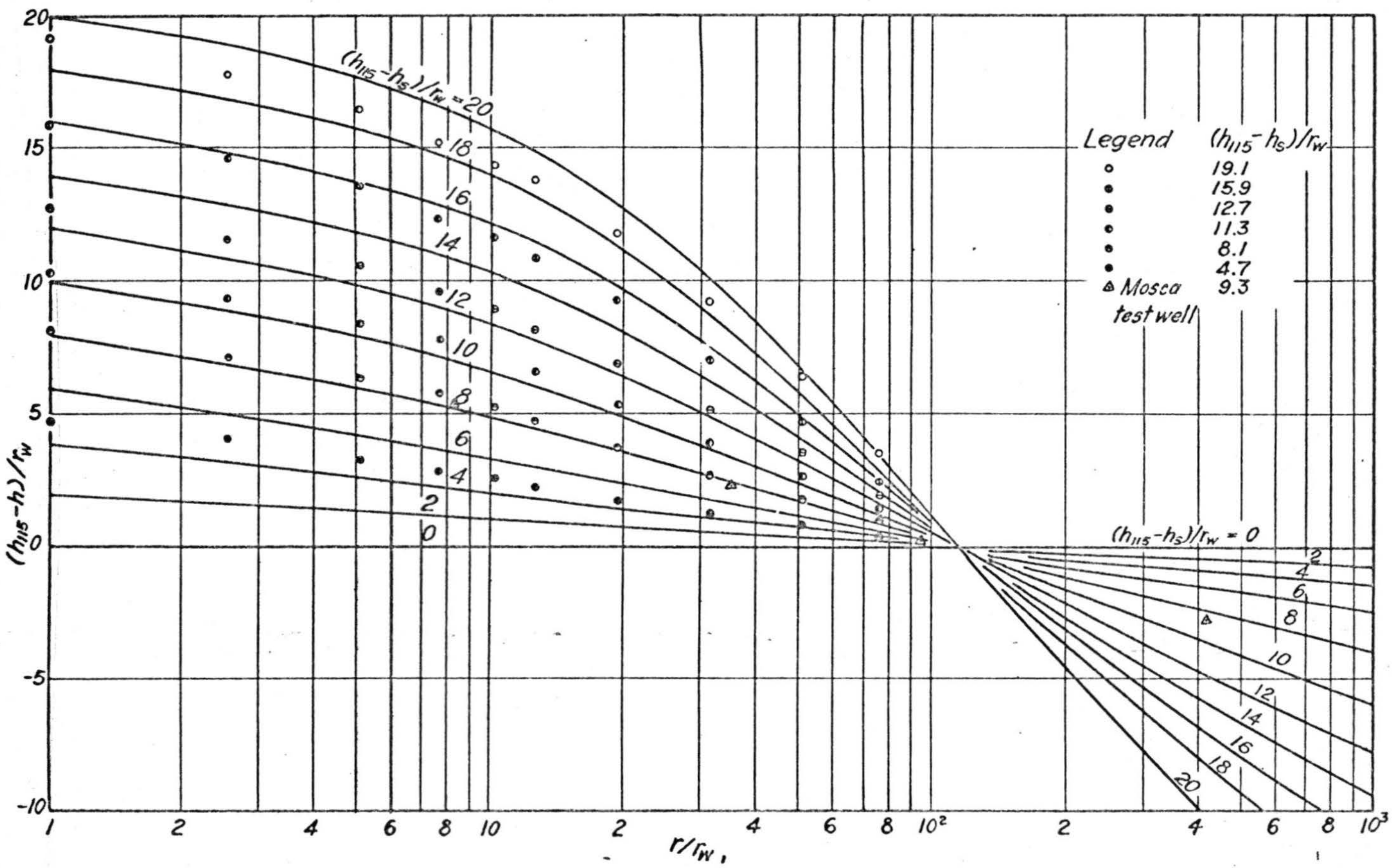

Fig. 4 Generalized drawdown curves for water table well.

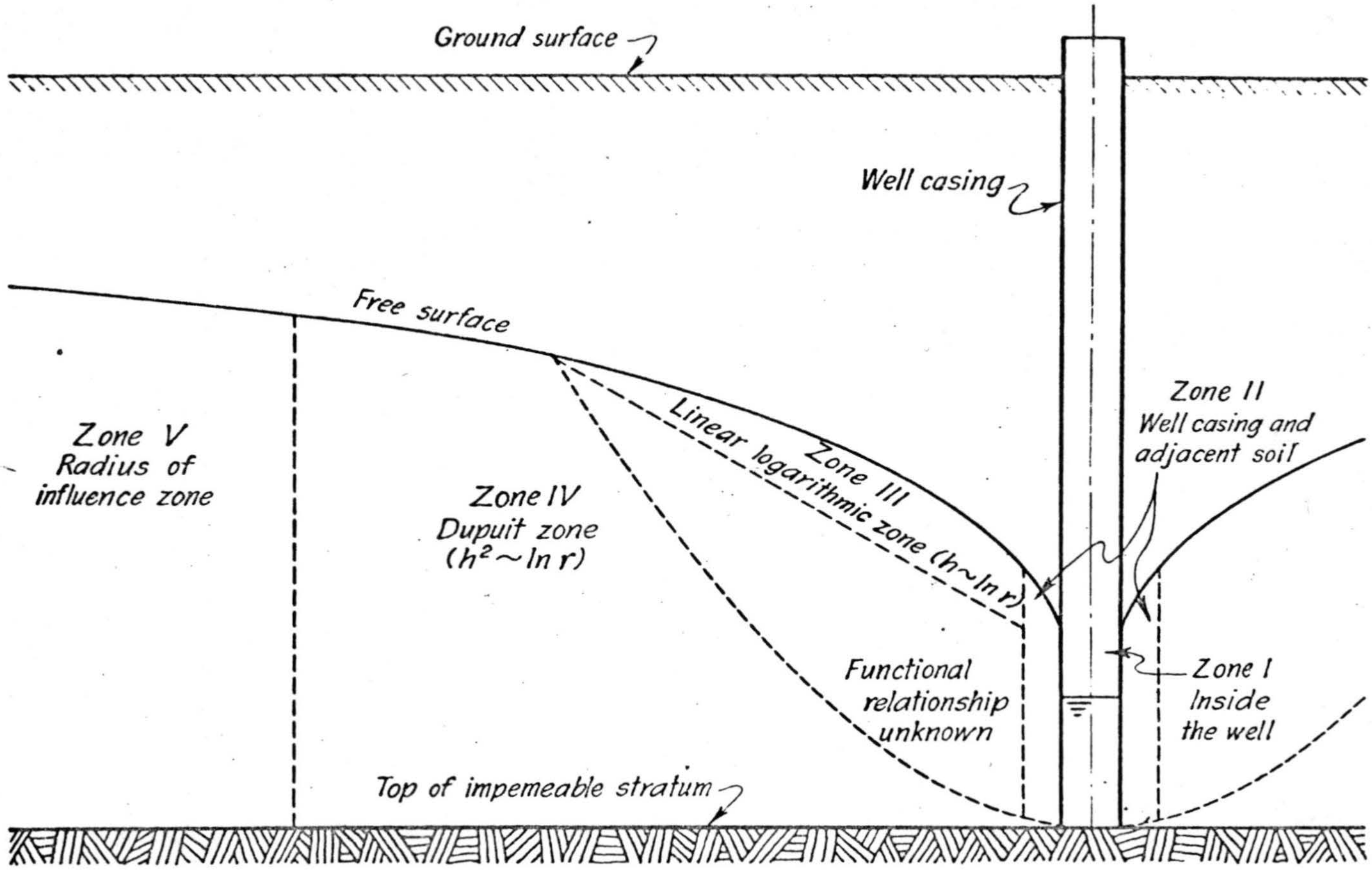

Fig. 5 Hydrodynamic zones in field of water table well

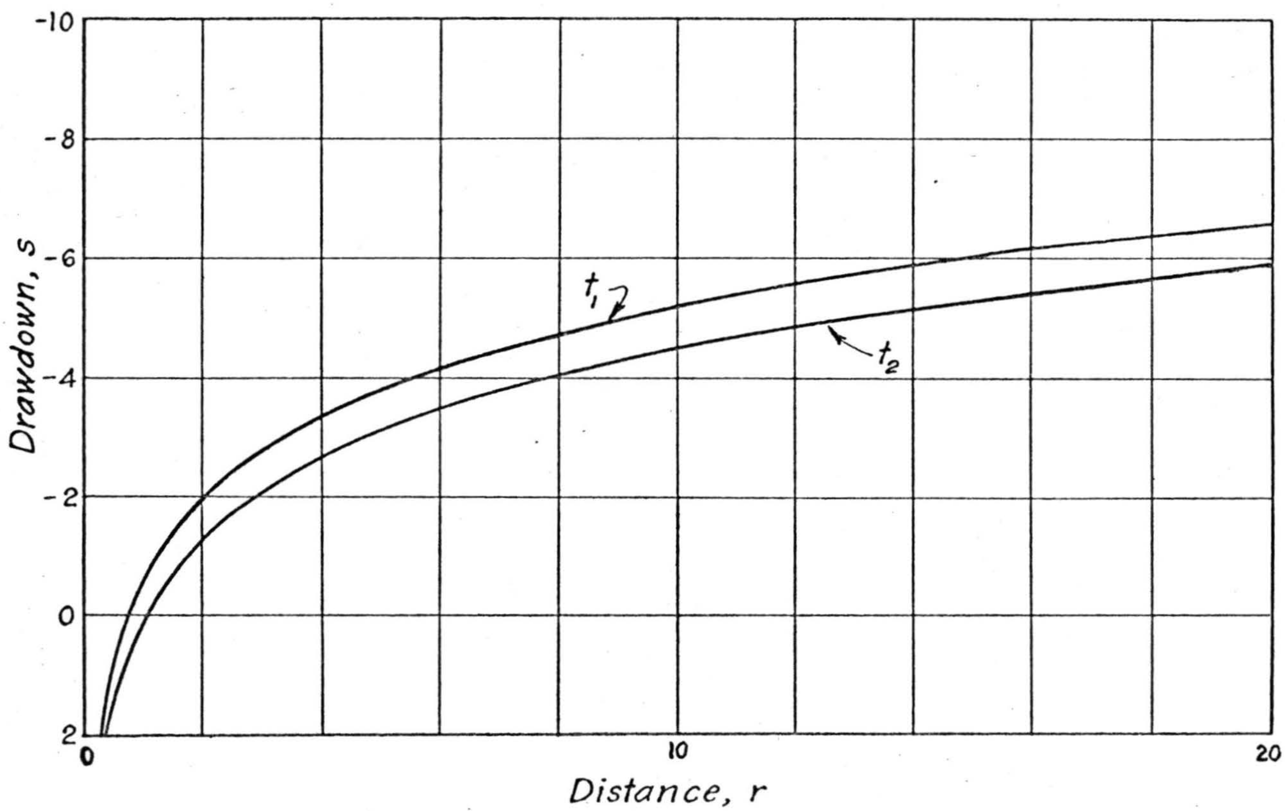

Fig. 6 Effect of time and distance on drawdown for a non-steady state well

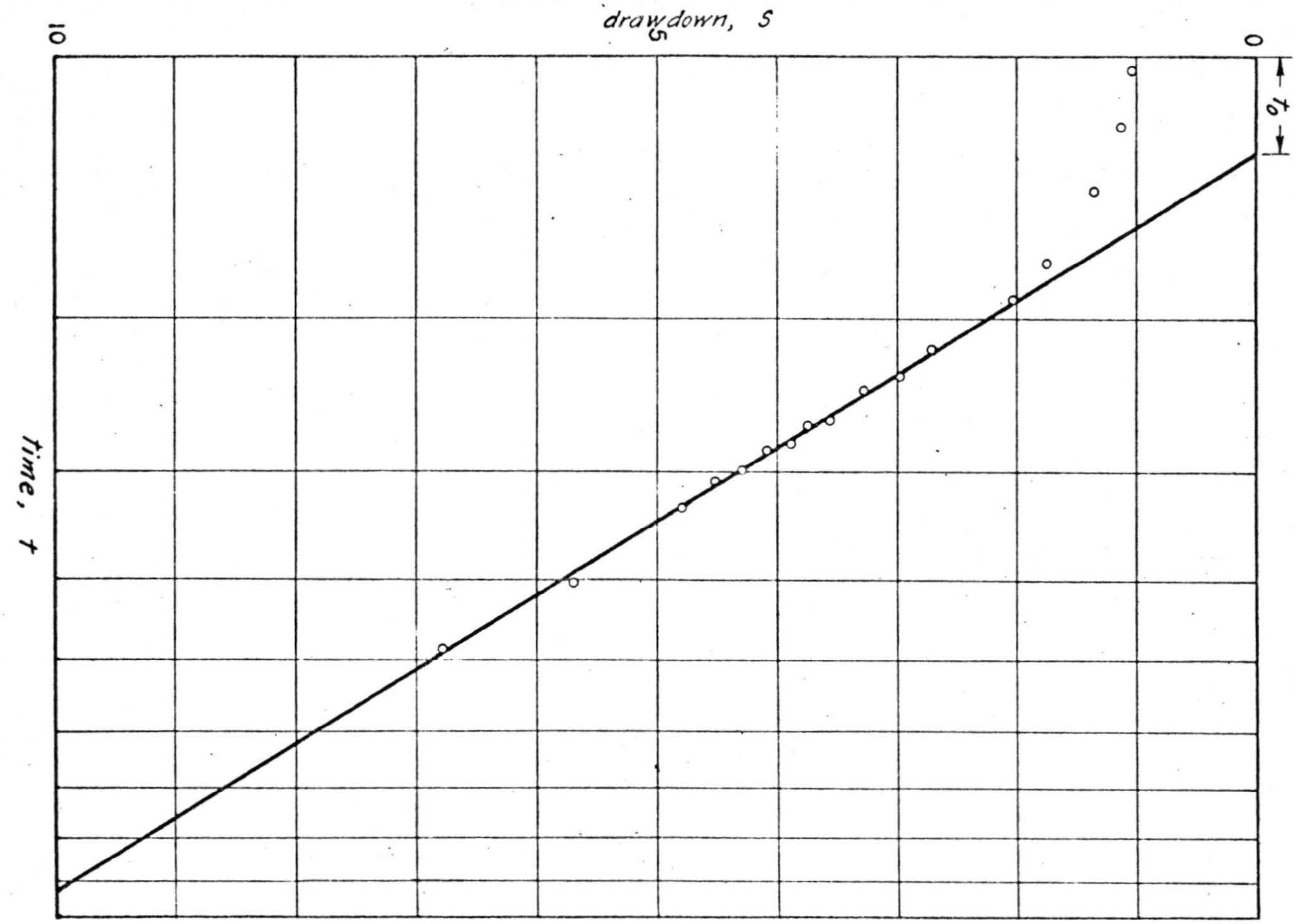

Fig. 7 Determination of t_o for approximate non-steady flow equation

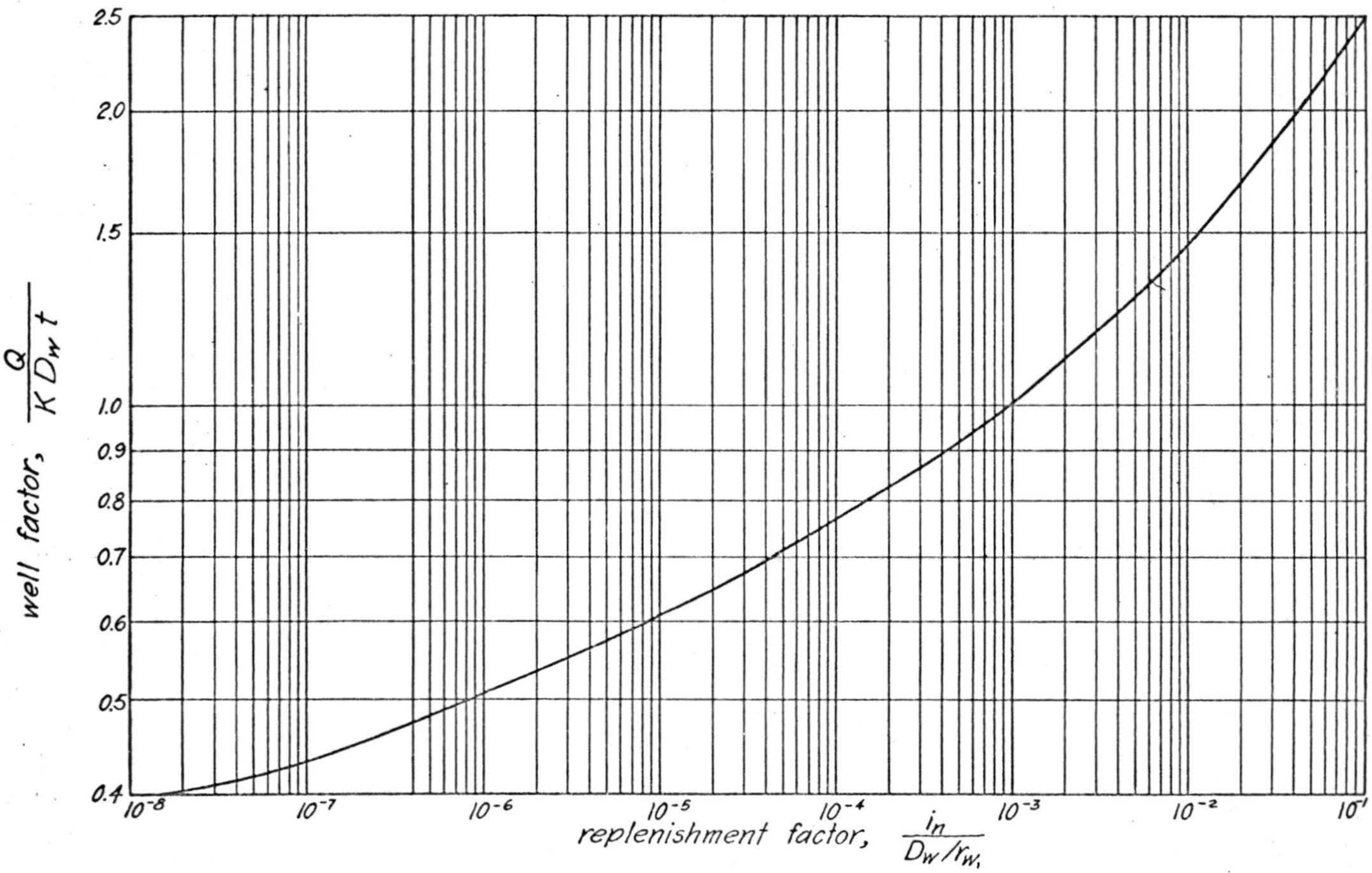

Fig. 8 Relationship of replenishment and discharge factors for an intercepting artesian well

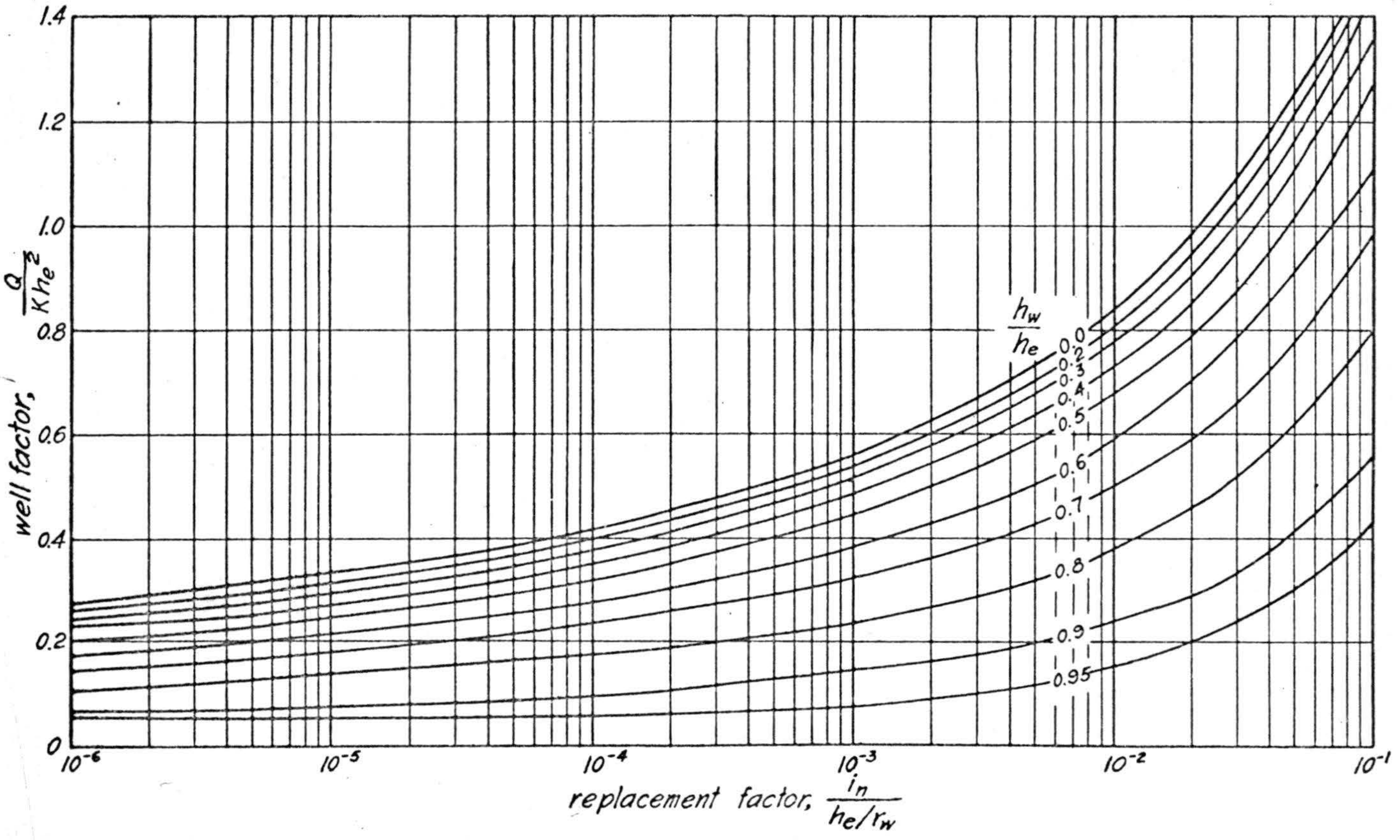

Fig. 9 Relationship of replenishment and discharge factors for an intercepting water table well

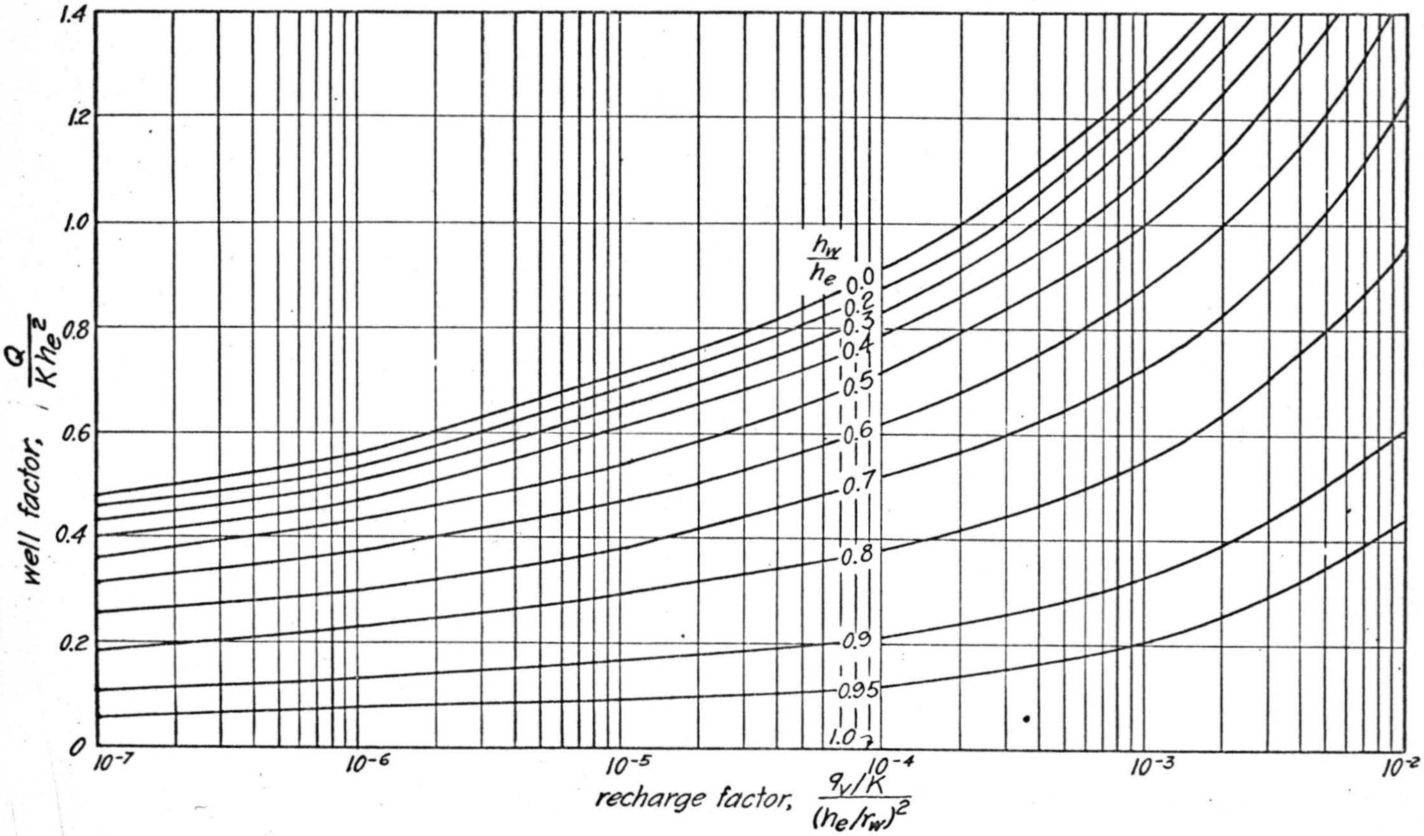

Fig. 10a Relationship of replenishment and discharge factors for a relief water table well

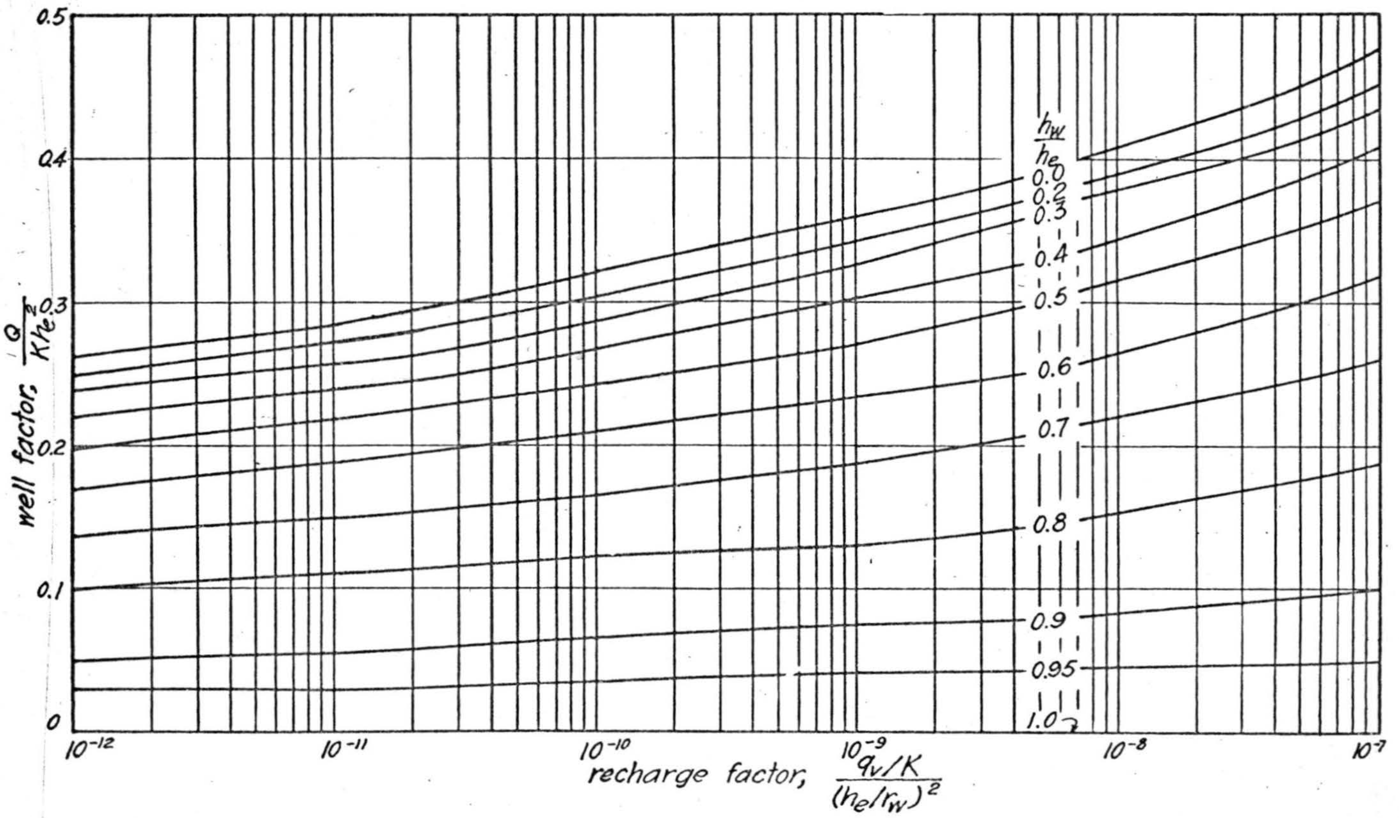

Fig. 10b Relationship of replenishment and discharge factors for a relief water table well

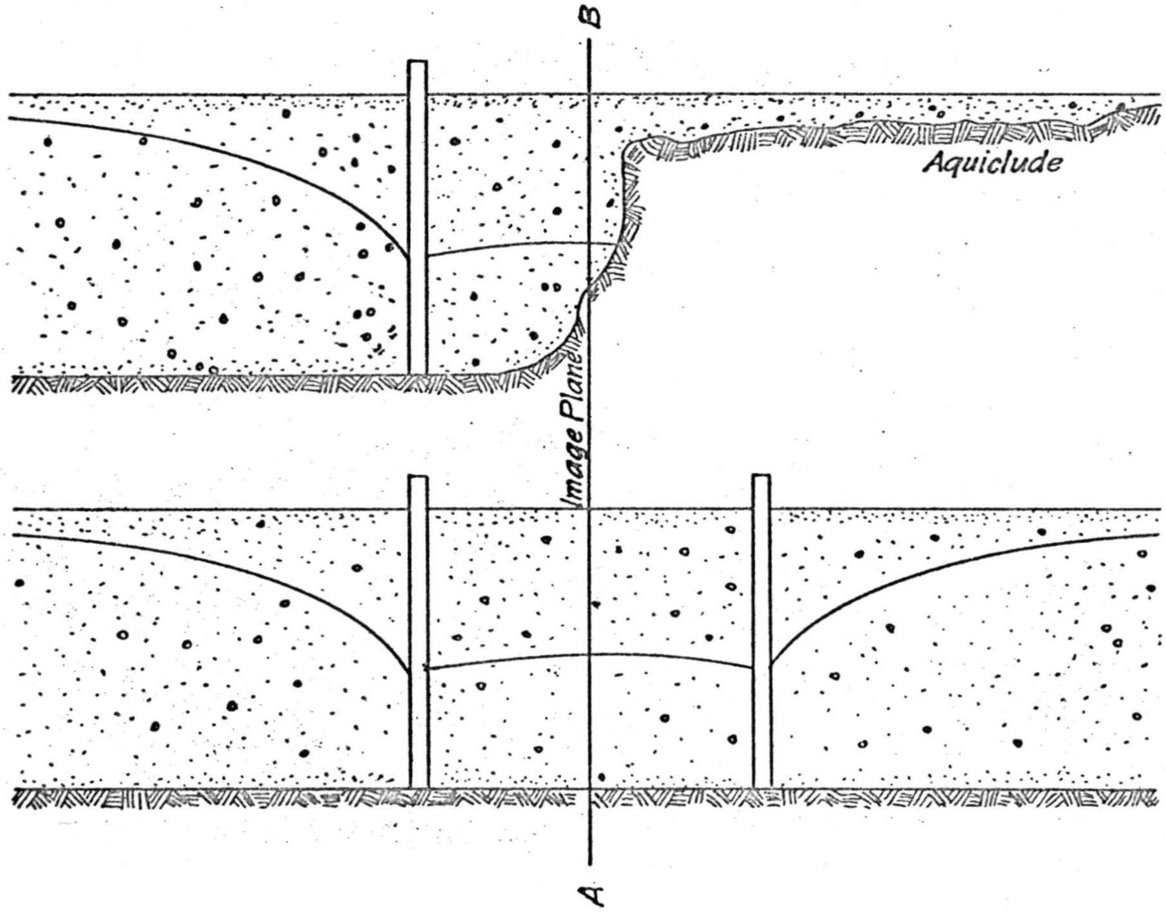

Fig. 11 Method of images applied to a well near an aquiclude.