GENERAL STORAGE EQUATION FOR DIFFERENT FLOW REGULATION

by

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SYNOPSIS

The paper deals with the design of a reservoir when (i) the outflow is the same as the average inflow (ii) the outflow is varied with the inflow, (iii) the outflow varies with the amount of water storage in the reservoir and (iv) the outflow is varied both with the inflow and with amount of storage in the reservoir.

INTRODUCTION

Reservoir storage is usually calculated by the mass curve analysis introduced by W. Rippl[2] (1883). Hurst[3] (1951) from a numerical study of rainfall and runoff statistics gave an equation for determining the capacity of a reservoir in order to maintain a certain annual discharge in over year flow regulations.

The input into the reservoir varies from month to month, and from year to year. The planner must take into account the fact that, given the capacity of the reservoir and the operating rule which determines how the water is to be released, the actual utilization of the reservoir depends on uncertain factors

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1. Civ. Engr. "Energoprojekt", Beograd, Yugoslavia; manuscript is part of Ph. D. dissertation prepared by writer at Colorado State University, Fort Collins, Colorado.

2. Rippl, W., The Capacity of Storage Reservoirs for Water Supply, Proc. Inst. of Civil Engrs., Vol. 71; pp. 270-278, 1883.

3. Hurst, H. E., Long-term Storage Capacity of Reservoirs. Transactions of the American Society of Civil Engineers, Vol. 116; pp. 770-808, 1951.

and, therefore, must involve the theory of probability. Statistical analysis assumes that the existing record of streamflows is only one sample drawn from an infinite set of possible samples that could be observed from a population of streamflows.

A basic model studied in storage problems consists of one random and one deterministic component. In storage theory the input is taken either as random independent or dependent time series, and the output is a random dependent time series. Both can be represented in time t (continuous or discrete) by the equation for storage function S (t)

$$S(t) = I(t) - 0(t)\,. \tag{1}$$

I (t) is a random inflow function which feeds the storage functions, 0 (t) is an outflow function which depletes it. The outflow function 0 (t) may be either constant or may depend on S (t), I (t), or both.

The basic relation between inflow, outflow, and storage is expressed by the equation of continuity, which for an incompressible fluid such as water, may be written

$$\frac{dS}{dt} = Q_i(t) - Q_o(t) \tag{2}$$

where $Q_i(t)$ is the inflow, $Q_o(t)$ is the outflow, and dS/dt is the rate of change in time of storage volume.

Reservoir storage capacities are always finite. However, the theoretical concept of infinite storage is useful as the limiting case in treating stochastic problems in design of reservoirs. By this concept of infinite storage capacity, a reservoir is assumed to be capable of storing any water surplus as incurred by the difference of inflow and outflow, and to supply any deficit for the difference between outflow and inflow. The concept of infinite storage leads to the introduction of three basic and important variables into the stochastic analysis of storage problems: range, surplus and deficit.

The finite storage problem conceived as a stochastic process with two barriers, an upper barrier with the reservoir full, S_f, and a lower barrier with the empty reservoir. The initial storage content may be anywhere between 0 and S_f.

A. Infinite Storage

1. General storage equation and its solution

The following types of flow regulation by reservoirs are briefly discussed here:

(1) The outflow is constant ($Q_o = Q_a$), with Q_a = the average flow rate;

(2) The outflow is varied with the inflow;

(3) The outflow is varied with the volume of storage amount in the reservoir;

(4) The outflow is varied both with the inflow and with the storage amount in the reservoir;

The storage equations for these four cases are:

$$Q_i = Q_a + w_i(t) \tag{3}$$

Case (1)

$$Q_o = Q_a, \tag{4}$$

and

$$\frac{dS}{dt} = w_i(t) \tag{5}$$

with Q_i = inflows, Q_a = average inflow or outflow, Q_o = outflow, and $w_i(t)$ = departures of inflows from the mean inflow as they change with time.

Case (2)

$$Q_o = Q_a + w_o(t), \tag{6}$$

$$w_o(t) = \beta w_i(t), \tag{7}$$

and

$$\frac{dS}{dt} = (1 - \beta) w_i(t) \tag{8}$$

with $w_o(t)$ departures of outflow from the mean outflow, and β is a proportionality parameter.

Case (3)

$$Q_o = Q_a + w_o(t) \qquad (9)$$

$$w_o(t) = \alpha S , \qquad (10)$$

and

$$\frac{dS}{dt} = w_i(t) - \alpha S ; \qquad (11)$$

with α being the parameter.

Case (4)

$$Q_o = Q_a + w_o(t) \qquad (12)$$

$$w_o(t) = \beta w_i(t) + \alpha S , \qquad (13)$$

and

$$\frac{dS}{dt} = (1 - \beta) w_i(t) - \alpha S ; \qquad (14)$$

with α and β the parameters.

The general storage equation for all four cases is

$$\frac{dS}{dt} = q_i(t) - \alpha S . \qquad (15)$$

with $q_i(t)$ different for each case is as follows:

Case (1), $q_i(t) = w_i(t), \ \alpha = 0$;

Case (2), $q_i(t) = (1 - \beta) w_i(t), \ \alpha = 0, \ -\infty < \beta < \infty$;

Case (3), $q_i(t) = w_i(t), \ -\infty < \alpha < \infty$;

Case (4), $q_i(t) = (1 - \beta) w_i(t), \ -\infty < \alpha < \infty, \ -\infty < \beta < \infty$.

In equations (5), (8), (11) and (14) $w_i(t)$ is assumed to be a normal independent variable with mean zero and variance σ_1^2. The $q_i(t)$ in equation (15) is also a normal independent variable. For the cases (1) and (3) it has mean zero and variance $\sigma^2 = \sigma_1^2$, and for the cases (2) and (4) it has mean zero and variance $\sigma^2 = (1 - \beta)^2 \sigma_1^2$.

For the case (2) and for $\beta < 0$, there is an inverse regulation, that is, the release is smaller than the average inflow when inflow is greater than the average, and vice versa. For case (4) when the outflow is a function of both inflow and storage amount in the reservoir, the properties of storage amounts and the type of regulation depend on the relation between α and β.

Equation (15) has the form known in physics as Langevin's equation for the Brownian motion of a free particle.

As for the fluctuating part of the inflow $q_i(t)$ the following principle assumptions are made:

(i) The mean of $q_i(t)$ is zero, $\overline{q_i(t)} = 0$;

(ii) There is correlation between the values of $q_i(t)$ at different times t_1 and t_2 only when $|t_1 - t_2|$ is very small.

(iii) $q_i(t)$ varies extremely rapidly compared to the variation of S.

(iv) $q_i(t)$ has a Gaussian distribution with mean zero and variance σ^2.

The problem is to determine the probability that storage after the time t lies between S and S + dS, when $S = S_o$ at $t = 0$.

Langevin's equation has been solved by many authors, using different methods. One solution of this equation has been given by S. Chandrasekhar[4] for the Brownian motion of a free particle. His method is used here to obtain a solution of the general storage equation (15). This solution of the storage equation (15) has to be understood in the sense of specifying a probability density distribution $f(S, t; S_o)$. Physical circumstances of the problem require that $f(S, t; S_o)$ tends to the following distribution, which is independent of S_o as $t \longrightarrow \infty$,

$$f(S, t; S_o) = \frac{1}{(2\pi \sigma_s^2)^{1/2}} e^{-S^2/2\sigma_s^2} . \qquad (16)$$

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4. Wax, N., Selected papers on noise and stochastic processes. New York, Dover Publications, 1954, 337 p.

This requirement on $f(S, t; S_o)$ conversely requires that $q_i(t)$ satisfy certain statistical conditions. According to the storage equation (15), the general solution is

$$S - S_o e^{-\alpha t} = e^{-\alpha t} \int_o^t e^{\alpha \xi} q_i(\xi)\, d\xi \qquad (17)$$

Consequently, the statistical properties of

$$S - S_o e^{-\alpha t} \qquad (18)$$

must be the same as those of

$$e^{-\alpha t} \int_o^t e^{\alpha \xi} q_i(\xi)\, d\xi . \qquad (19)$$

As $t \longrightarrow \infty$, expression (18) tends to S ; and the distribution of

$$\lim_{t \longrightarrow \infty} \left\{ e^{-\alpha t} \int_o^t e^{\alpha \xi} q_i(\xi)\, d\xi \right\} \qquad (20)$$

must be the distribution

$$\frac{1}{(2\pi \sigma_s^2)^{1/2}} e^{-S^2/2\sigma_s^2} . \qquad (21)$$

The right-hand side of equation (17) may be written as

$$e^{-\alpha t} \sum_j e^{\alpha j \Delta t} \int_{j\Delta t}^{(j+1)\Delta t} q_i(\xi)\, d\xi . \qquad (22)$$

Let

$$q(\Delta t) = \int_t^{t+\Delta t} q_i(\xi)\, d\xi . \qquad (23)$$

The physical meaning of $q(\Delta t)$ is that it represents the inflow into the storage reservoir during an interval Δt. Equation (17) then becomes

$$S - S_o e^{-\alpha t} = \sum_j e^{\alpha(j\Delta t - t)} q(\Delta t) \qquad (24)$$

with the condition that the quantity on the right-hand side tends to the distribution equation (21) as $t \longrightarrow \infty$. This further requires that the probability of occurrence of different values of $q(\Delta t)$ be governed by the distribution function

$$f\left[q(\Delta t)\right] = \frac{1}{(2\pi\sigma^2 \Delta t)^{1/2}} e^{-\left[q(\Delta t)\right]^2 / 2\Delta t \sigma^2} \qquad (25)$$

where

$$\sigma^2 = 2\alpha \sigma_s^2 \text{ or } \sigma_s^2 = \frac{\sigma^2}{2\alpha} . \qquad (26)$$

To prove this assertion we have to show that the distribution function $f(S, t; S_o)$ derived on the basis of equations (24) and (25) does in fact tend to the distribution equation (21) as $t \longrightarrow \infty$. Let

$$S = \int_o^t \theta_1(\xi) q_i(\xi) d\xi . \qquad (27)$$

Then, the probability distribution of S is given by

$$f(S) = \frac{1}{\left[2\pi\sigma^2 \int_o^t \theta_1^2(\xi) d\xi\right]^{1/2}} e^{-S^2/2\sigma^2 \int_o^t \theta_1^2(\xi) d\xi} . \qquad (28)$$

In order to prove this the interval (o, t) is first divided into a large number of subintervals of duration Δt, so that

$$S = \sum_j \theta_1 (j\Delta t) \int_{j\Delta t}^{(j+1)\Delta t} q_i (\xi)\, d\xi \; . \tag{29}$$

Using equation (24), S can be expressed in the form

$$S = \sum_j s_j \, , \tag{30}$$

where

$$s_j = \theta_1 (j\Delta t)\, q\, (\Delta t) \; . \tag{31}$$

According to equation (25), the probability distribution of s_j is given as

$$f(s_j) = \frac{1}{\left[2\pi \, \theta_1^2 (j\Delta t)\, \sigma^2 \, \Delta t\right]^{1/2}} \, e^{-s_j^2/2\theta_1^2 (j\Delta t)\sigma^2} \tag{32}$$

Hence,

$$f(S) = \frac{1}{\left[2\pi\sigma^2 \sum_j \theta_1^2 (j\Delta t)\, \Delta t\right]^{1/2}} \, e^{-S^2/2\sigma^2 \sum_j \theta_1^2 (j\Delta t)} \tag{33}$$

As

$$\sum_j \theta_1^2 (j\Delta t)\, \Delta t = \int_o^t \theta_1^2 (\xi)\, d\xi \, , \tag{34}$$

then

$$f(S) = \frac{1}{\left[2\pi\sigma^2 \int_o^t \theta_1^2 (\xi)\, d\xi\right]^{1/2}} \, e^{-S^2/2\sigma^2 \int_o^t \theta_1^2 (\xi)\, d\xi} \, . \tag{35}$$

Which proves equations (27) and (28).

The right-hand side of equation (17) may be expressed as

$$\int_0^t \theta_1(\xi)\, q_i(\xi)\, d\xi \tag{36}$$

with

$$\theta_1(\xi) = e^{\alpha(\xi - t)} . \tag{37}$$

With the foregoing definition of $\theta_1(\xi)$, equation (35) governs the probability distribution of

$$S - S_o e^{-\alpha t} . \tag{38}$$

Since,

$$\int_0^t \theta_1^2(\xi)\, d\xi = \int_0^t e^{2\alpha(\xi - t)} d\xi = \frac{1}{2\alpha}\left(1 - e^{-2\alpha t}\right), \tag{39}$$

and taking into account the relationship shown in equation (26) then

$$f(S, t; S_o) = \frac{1}{\left[2\pi\left(1-e^{-2\alpha t}\right)\sigma_s^2\right]^{1/2}}\, e^{-(S - S_o e^{-\alpha t})^2/2(1-e^{-2\alpha t})\sigma_s^2} . \tag{40}$$

Therefore, equation (40) converges to

$$f(S, t; S_o) = \frac{1}{(2\pi\sigma_s^2)^{1/2}}\, e^{-S^2/2\sigma_s^2} \tag{41}$$

for $t \longrightarrow \infty$.

This proves the assertion that with the statistical properties of $q(\Delta t)$ implied in equations (25) and (26), equation (24) leads to a distribution $f(S, t; S_o)$ which tends to be independent of S_o as $t \longrightarrow \infty$.

2. Range of the storage

Suppose we have a record $\left[q_k\right]$ of mutually independent random variables, (inflow minus outflow), with a common distribution $f(q)$.

Let

$$S_n = q_1 + q_2 + \ldots + q_n$$

and let

$$M_n = \max\left[0, S_1, S_2, \ldots, S_n\right]$$

$$m_n = \min\left[0, S_1, S_2, \ldots, S_n\right] \qquad (42)$$

The random variable M_n will be called the surplus of the cumulative sums S_n. The random variable m_n will be called the deficit of the cumulative sums S_n and the random variable

$$R_n = M_n - m_n \qquad (43)$$

will be called the range of the cumulative sums S_n, or the range of the storage.

The sums S_n are asymptotically normally distributed and, therefore, the asymptotic distribution of the range is independent of the function $f(q)$. The sum S_n can then be considered as the value at time $t = n$ of a continuously changing normal variable S_t. According to equation (40), with $S_0 = 0$, S_t is a normal variable with mean zero and variance $\sigma^2 (1 - e^{-2\alpha t})/2\alpha$. As $t \longrightarrow \infty$ S_t approaches a normal variable with mean zero and variance $\sigma^2/2\alpha$.

It should be noted that the term $(1 - e^{-2\alpha t})$ is larger than 0.99 for $t > 2.303/\alpha$.

3. Mean, variance and skewness of the range.

The expected value of the range is $E\left[R_n\right] = E\left[M_n - m_n\right] = E\left[M_n\right] - E\left[m_n\right]$. For $\alpha \neq 0$ it is very difficult to find the exact analytical expression for the mean, variance and skewness of the range. However, if one accepts the hypothesis that the mean and the variance of the range depend on the variance of S_t alone, for a given α, then the following expressions, tested by the writer, are valid:

$$E[R_n] = C_1 \sum_{t=1}^{n} \frac{\sqrt{Var[S_t]}}{t} ; \qquad (44)$$

and

$$Var[R_n] = C_2 \sum_{t=1}^{n} \frac{Var[S_t]}{t} . \qquad (45)$$

The constants C_1 and C_2 were found to be functions of α alone. Using the computer, for nine different values of α $(-0.04 \leq \alpha \leq 2.00)$ the expected value and variance of the range were calculated for n between two and fifty. From these results it is found that the variations of C_1 and C_2 may be approximated by the following expressions:

$$C_1 = \sqrt{\frac{2}{\pi}} \, (1 + 3\, \alpha^2 \, e^{-2\alpha}) \qquad (46)$$

and

$$C_2 = 4\,(\ln 2 - 2/\pi)\,(1 - 8\, e^{-20\alpha})\,\frac{(1 - e^{-2\alpha})}{2\alpha} . \qquad (47)$$

Assuming that $[q_k]$ is a normal variable with mean zero and variance unity, the following equations for the expected value and variance of the range are obtained:

$$E[R_n] = \frac{(1 + 3\alpha^2 e^{-2\alpha})}{\sqrt{\alpha\, \pi}} \sum_{t=1}^{n} \frac{\sqrt{1 - e^{-2\alpha\, t}}}{t} ; \qquad (48)$$

and

$$Var[R_n] = (\ln 2 - 2/\pi)\,(1 - 8\alpha\, e^{-20\alpha})\,\left(\frac{1 - e^{-2\alpha}}{\alpha^2}\right) \sum_{t=1}^{n} \frac{1 - e^{-2\alpha\, t}}{t} . \qquad (49)$$

As α tends to zero, the expected value of range is the same as found by Anis and Lloyd[5] and the variance has the same form as derived by Feller[6], i, e.,

$$E\left[R_n\right] = \sqrt{\frac{2}{\pi}} \sum_{t=1}^{n} \frac{1}{\sqrt{t}} \; ; \qquad (50)$$

and

$$Var\left[R_n\right] = 4\,(\ln 2 - 2/\pi)\,n \, . \qquad (51)$$

In the case α tends to infinity the expected value and variance of the range converges to zero.

Figure 1 shows the mean range for n values between zero and 50, and α from -0.04 through 2.00. The points represent the results obtained by the Monte Carlo method and the full lines represent the theoretical values of equation (48).

Calculations obtained from the computer give the same results for the expected value at the range as equation (48), except when both $\alpha < o$, and n is large. When n increases, the number of subsamples of size n as used by the computer decreases. The sampling errors may, therefore, be the reason for the above differences encountered. However, for any practical calculation, these differences may be considered as negligible.

Figure 2 shows the variance of the range for n between zero and 50, and for the same α's that have been used for the computation of the mean range. The mean and the variance of the range decreases rapidly with an increase of α, but the variance of outflow increases with α as

$$Var\ Q_0 = 0.5\alpha\,(1 - e^{-2\alpha t})\,\sigma^2 \; . \qquad (52)$$

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5. Anis, A. A., and Lloyd, E. H., On The Range of Partial Sums of a Finite Number of Independent Normal Variates. Biometrika, Vol. 40; pp. 35-42, 1953.

6. Feller, W., The Asymptotic Distributions of The Range of Sums of Independent Random Variables. Annals of Mathematical Statistics, Vol. 22; pp. 427-432, 1951.

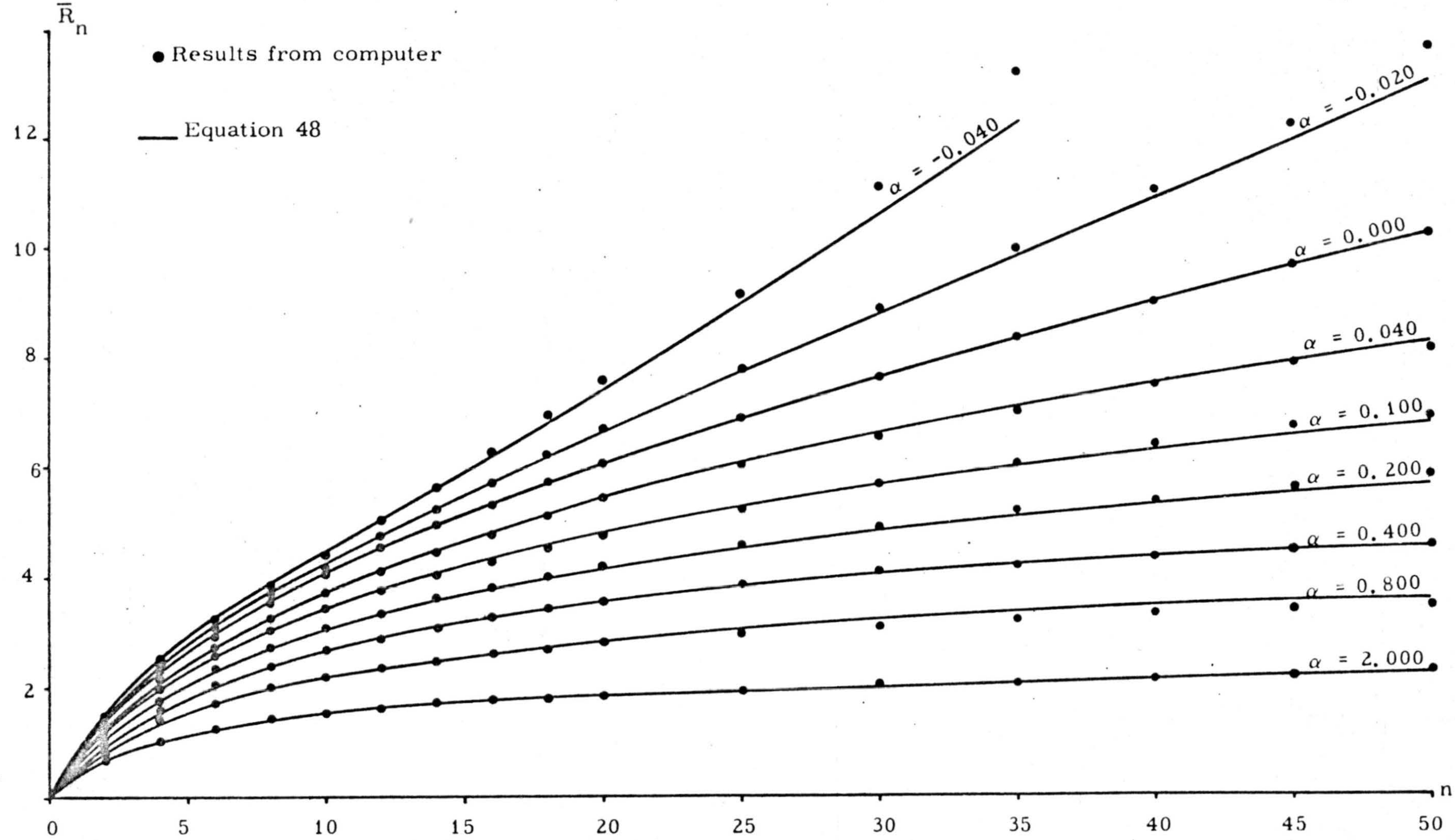

FIG. 1 - THE MEAN RANGE $\overline{R}_n$ FOR INFINITE STORAGE, AND AS A FUNCTION OF n and α.

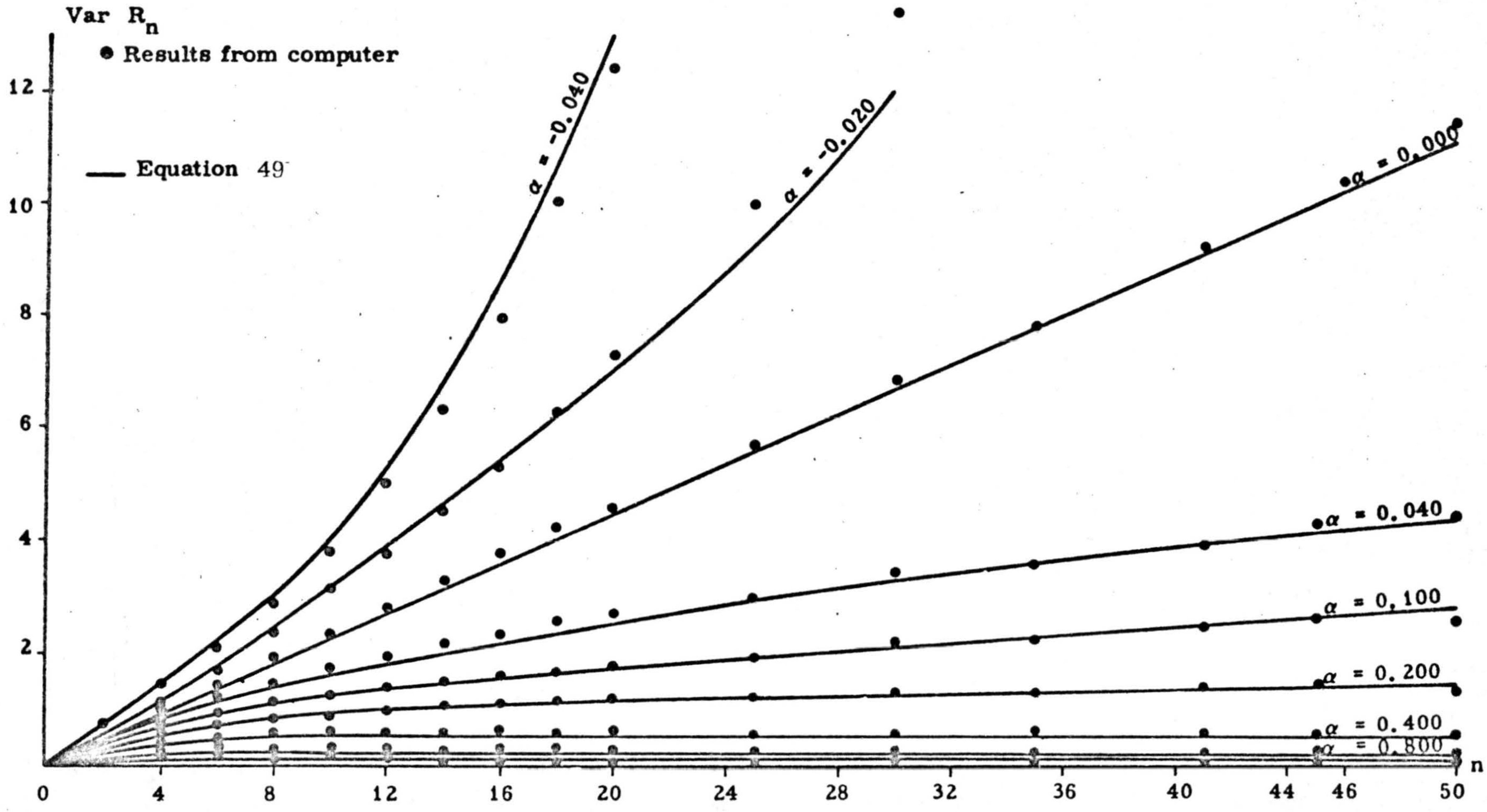

FIG. 2 - THE VARIANCE OF RANGE, Var R_n FOR INFINITE STORAGE AS FUNCTION OF n AND α.

The correlation coefficient between Q_0, t_1 and Q_0, t_2 or between S_{t_1} and S_{t_2} is defined by

$$\rho (Q_0, t_1, Q_0, t_2) = \rho (S_{t_1}, S_{t_2}) = e^{-\alpha (t_2 - t_1)}; \; t_2 > t_1, \qquad (53)$$

for $\alpha > 0$ and $t \to \infty$.

For $\alpha < 0$, the mean and the variance of range increase rapidly, which is to be expected because when the storage of water in reservoir is largest, the outflow is smallest and vice versa. To obtain an equation for the skewness of the range no hypothesis is available. However, the skewness of the range for n between zero and 50, and for various values α are obtained by the Monte Carlo method and are plotted in figure 3.

4. Correlation coefficient between M_n and m_n

The expression for the correlation coefficient between M_n and m_n is derived from the general equation for the variance of R_n,

$$\mathrm{Var}[R_n] = \mathrm{Var}[M_n] + \mathrm{Var}[m_n] - 2\,\mathrm{Cov}[M_n, m_n], \qquad (54)$$

$$\mathrm{Cov}[M_n, m_n] = \rho (M_n, m_n) \sqrt{\mathrm{Var}[M_n]} \sqrt{\mathrm{Var}[m_n]}, \qquad (55)$$

$$\mathrm{Var}[m_n] = \mathrm{Var}[M_n], \qquad (56)$$

and

$$\rho (M_n, m_n) = 1 - \frac{\mathrm{Var}[R_n]}{2\,\mathrm{Var}[M_n]}. \qquad (57)$$

For n large, and $\alpha = 0$, $\mathrm{Var}[R_n] = 0.2181n$ (Feller), and $\mathrm{Var}\, M_n = n\left(1 - \frac{2}{\pi}\right) - \frac{2 + \sqrt{2}}{\pi} \sqrt{n}$ (Anis). For this case the correlation coefficient between M_n and m_n becomes:

$$\rho (M_n, m_n) = 1 - \frac{0.2181}{2(1 - 2/\pi)} = 0.700. \qquad (58)$$

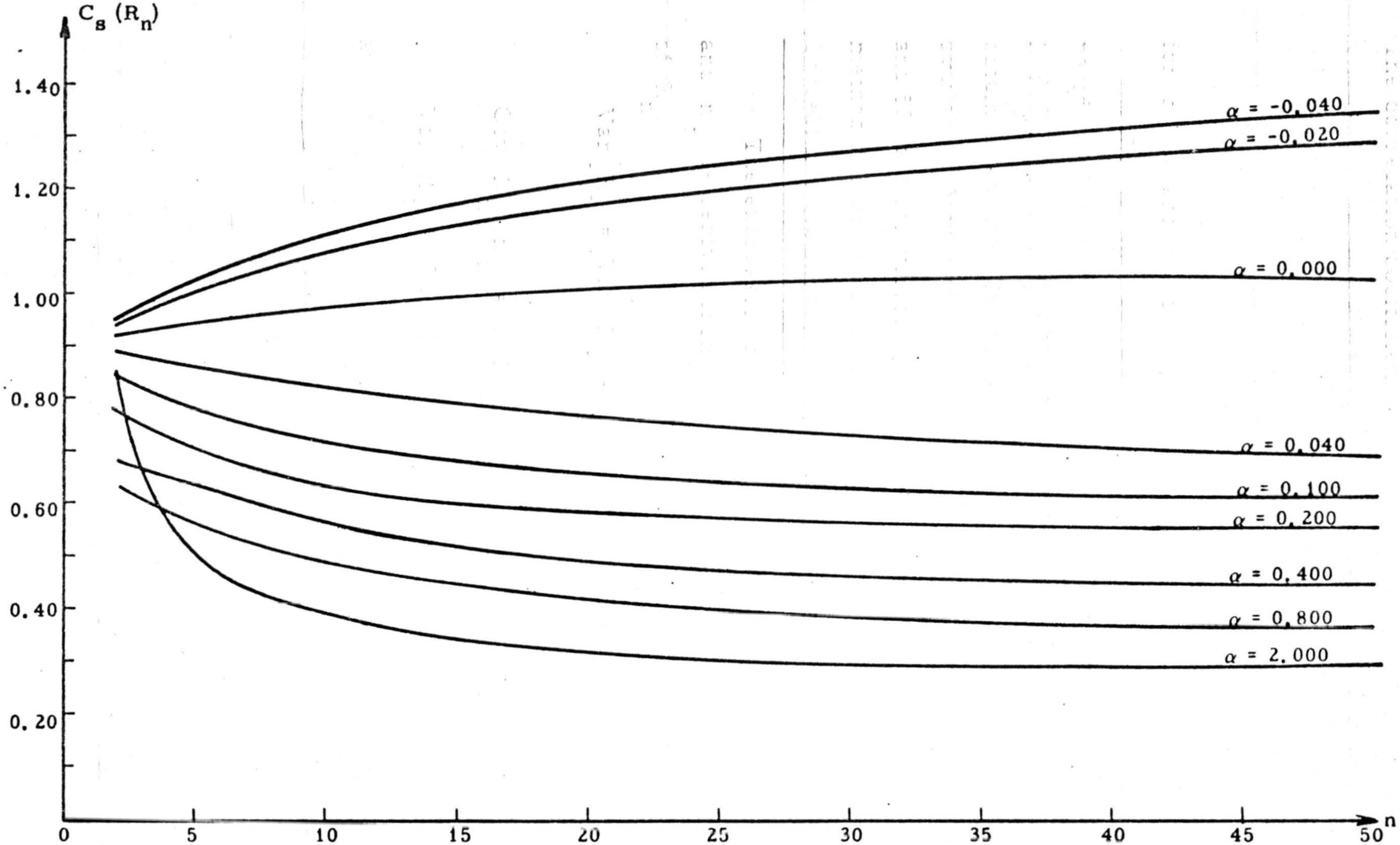

Fig. 3 The Skewness of Range, $C_s(R_n)$ as a Function of n and α.

This value is verified by using the large amount of random numbers simulated on the digital computer.

For $\alpha \neq 0$ the correlation coefficient between M_n and m_n was obtained by using only the random numbers and it was shown to be less than 0.700. If $\alpha \to \pm \infty$ the correlation coefficient approaches zero.

Figure 4 shows the correlation coefficients $\rho(M_n, m_n)$ for various values n and α obtained by the Monte Carlo method.

5. Mean, variance and skewness of surplus (M_n) or deficit (m_n)

The mean surplus (M_n) or the mean deficit (m_n) is

$$E[M_n] = E[m_n] = \frac{1}{2} E[R_n] \tag{59}$$

Figure 1 and table 1 show the mean range $(E[R_n])$ as a function of n and α.

The variance of surplus or deficit is defined by

$$\mathrm{Var}[M_n] = \mathrm{Var}[m_n] = \frac{\mathrm{Var}[R_n]}{2(1-\rho[M_n, m_n])} . \tag{60}$$

The variance of range is given in table 2, and the correlation coefficient between surplus (M_n) and deficit (m_n) is given in table 3. Using the data from these two tables and equation (60), the variance for surplus (M_n) or deficit (m_n) can be calculated.

The skewness of surplus (M_n) or deficit (m_n) for n between zero and 50 and for various values α are plotted in figure 5. For $0 \leq \alpha \leq 0.200$ the skewness decreases with an increase of n. For $\alpha < 0$ and $\alpha > 0.200$ the skewness decreases with an increase of n only for small values of n, and increases for big values of n. These results are obtained by the Monte Carlo method.

The probability of $M_n = 0$ or $m_n = 0$ for $1 \leq n \leq 50$ is shown in figure 6 as a function of α and n. It can be concluded from these results that the probability of $M_n = 0$ (or $m_n = 0$) decreases rapidly with an increase of α and n. Results were obtained from computer.

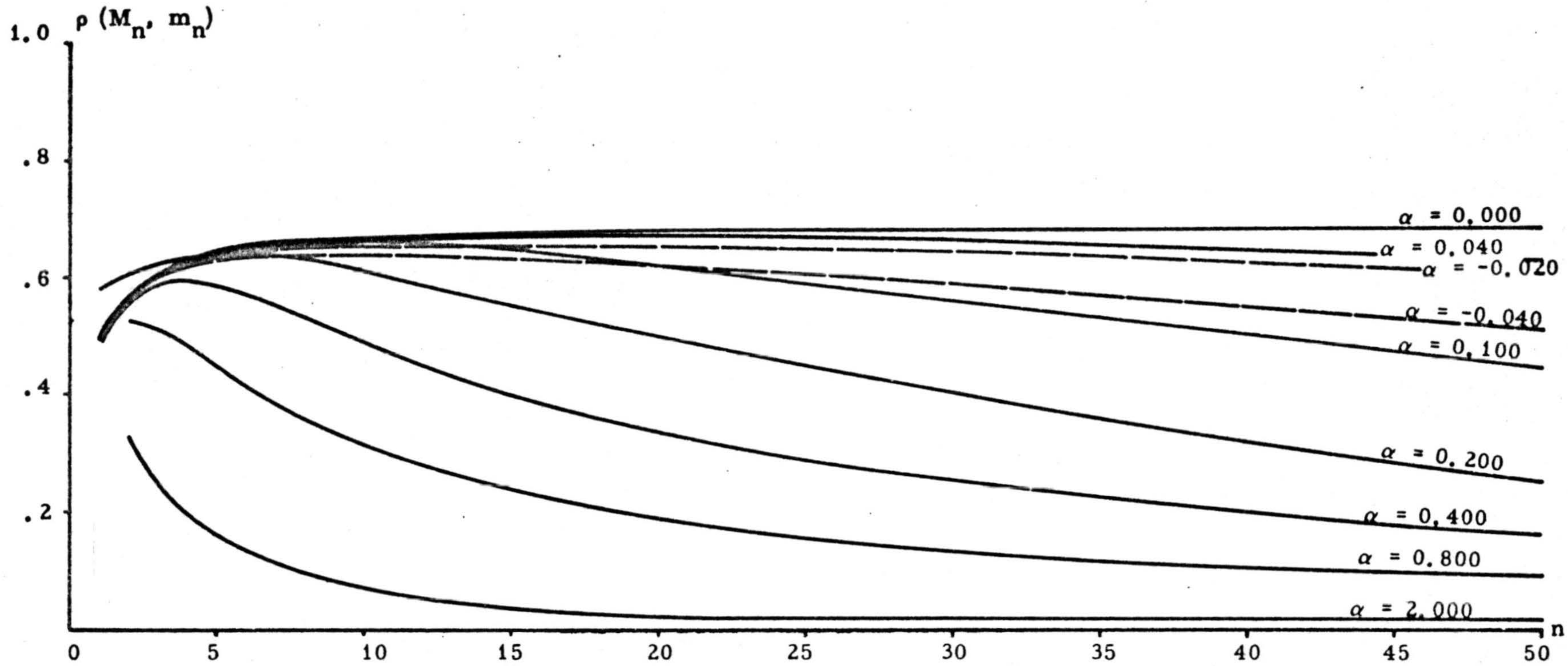

FIG. 4 - CORRELATION COEFFICIENTS BETWEEN M_n AND m_n, FOR INFINITE STORAGE FOR VARIOUS VALUES OF α AS FUNCTION OF n.

TABLE 1

MEAN OF RANGE ($\bar{R}_n/\sigma$), OBTAINED BY COMPUTER

n/α	-0.040	-0.020	0.000	0.040	0.100	0.200	0.400	0.800	2.000
2	1.402	1.382	1.363	1.325	1.274	1.200	1.082	0.923	0.680
3	1.894	1.858	1.823	1.760	1.678	1.564	1.398	1.191	0.887
4	2.327	2.271	2.221	2.129	2.013	1.862	1.654	1.410	1.045
5	2.727	2.650	2.581	2.458	2.308	2.120	1.876	1.595	1.168
6	3.096	2.995	2.904	2.749	2.566	2.346	2.070	1.751	1.267
7	3.459	3.328	3.213	3.021	2.802	2.551	2.242	1.883	1.347
8	3.784	3.623	3.483	3.257	3.009	2.733	2.397	2.003	1.421
9	4.134	3.931	3.760	3.490	3.204	2.898	2.534	2.103	1.478
10	4.461	4.221	4.022	3.714	3.398	3.065	2.666	2.195	1.533
12	5.065	4.737	4.477	4.094	3.719	3.340	2.889	2.354	1.623
14	5.682	5.246	4.917	4.452	4.025	3.606	3.096	2.489	1.696
16	6.312	5.746	5.332	4.780	4.292	3.824	3.259	2.597	1.756
18	6.934	6.217	5.716	5.079	4.545	4.032	3.406	2.691	1.808
20	7.595	6.687	6.080	5.348	4.764	4.205	3.523	2.763	1.851
25	9.269	7.787	6.898	5.949	5.264	4.610	3.807	2.945	1.948
30	11.152	8.897	7.673	6.513	5.711	4.936	4.018	3.078	2.023
35	13.372	9.951	8.320	6.947	6.071	5.218	4.204	3.195	2.086
40	15.784	11.042	9.007	7.428	6.430	5.457	4.348	3.281	2.137
45	19.000	12.267	9.642	7.814	6.718	5.648	4.477	3.365	2.183
50	23.402	13.636	10.191	8.145	6.984	5.834	4.589	3.435	2.221

TABLE 2

VARIANCE OF RANGE ($\sigma^2[R_n]/\sigma^2$) OBTAINED BY COMPUTER

n/α	-0.040	-0.020	0.000	0.040	0.100	0.200	0.400	0.800	2.000
2	0.648	0.623	0.600	0.556	0.500	0.427	0.331	0.236	0.148
3	0.923	0.869	0.819	0.734	0.633	0.514	0.382	0.275	0.172
4	1.232	1.135	1.050	0.910	0.755	0.590	0.429	0.308	0.179
5	1.558	1.404	1.273	1.067	0.853	0.647	0.464	0.327	0.177
6	1.912	1.687	1.501	1.221	0.950	0.709	0.506	0.346	0.173
7	2.370	2.040	1.777	1.398	1.054	0.767	0.536	0.349	0.168
8	2.683	2.259	1.934	1.485	1.106	0.810	0.567	0.358	0.163
9	3.288	2.689	2.247	1.664	1.197	0.858	0.588	0.360	0.160
10	3.716	2.970	2.439	1.766	1.256	0.896	0.607	0.357	0.153
12	4.849	3.682	2.910	2.012	1.397	0.990	0.645	0.357	0.146
14	6.194	4.439	3.360	2.209	1.503	1.060	0.664	0.350	0.140
16	7.767	5.269	3.817	2.386	1.587	1.104	0.671	0.344	0.133
18	9.935	6.280	4.320	2.585	1.715	1.169	0.681	0.338	0.131
20	12.359	7.331	4.812	2.759	1.807	1.210	0.682	0.334	0.128
25	20.291	10.001	5.790	3.003	1.931	1.257	0.675	0.320	0.121
30	32.423	13.496	6.982	3.453	2.220	1.342	0.668	0.304	0.114
35	51.970	17.549	7.798	3.530	2.260	1.346	0.655	0.296	0.107
40	86.768	24.183	9.426	3.971	2.438	1.417	0.675	0.298	0.105
45	134.872	31.127	10.686	4.387	2.631	1.417	0.635	0.280	0.101
50	211.738	39.442	11.550	4.404	2.540	1.356	0.620	0.267	0.096

TABLE 3

CORRELATION COEFFICIENT BETWEEN UPPER SUM OF RANGE (M_n)

AND LOWER SUM OF RANGE (m_n), ρ (M_n, m_n)

n/α	-0.040	-0.020	0.000	0.040	0.100	0.200	0.400	0.800	2.000
2	0.567	0.570	0.570	0.575	0.575	0.615	0.568	0.528	0.330
3	0.605	0.607	0.615	0.612	0.616	0.614	0.595	0.516	0.250
4	0.621	0.626	0.636	0.634	0.635	0.630	0.596	0.480	0.195
5	0.633	0.636	0.642	0.646	0.650	0.640	0.580	0.450	0.160
6	0.633	0.641	0.648	0.654	0.657	0.640	0.570	0.410	0.135
7	0.635	0.654	0.650	0.657	0.658	0.640	0.553	0.392	0.118
8	0.645	0.650	0.664	0.667	0.662	0.630	0.526	0.356	0.095
9	0.640	0.654	0.660	0.669	0.666	0.627	0.511	0.335	0.080
10	0.640	0.656	0.665	0.670	0.664	0.614	0.494	0.315	0.080
12	0.640	0.657	0.662	0.670	0.654	0.586	0.446	0.272	0.060
14	0.635	0.656	0.670	0.673	0.648	0.560	0.410	0.247	0.040
16	0.635	0.655	0.670	0.676	0.641	0.544	0.380	0.222	0.040
18	0.632	0.654	0.672	0.675	0.630	0.518	0.360	0.207	0.035
20	0.617	0.662	0.674	0.678	0.632	0.500	0.342	0.200	0.025
25	0.603	0.646	0.682	0.680	0.605	0.450	0.285	0.156	0.005
30	0.588	0.646	0.680	0.665	0.550	0.400	0.260	0.146	0.020
35	0.573	0.646	0.694	0.662	0.548	0.375	0.235	0.134	0.030
40	0.536	0.620	0.678	0.648	0.500	0.320	0.181	0.096	0.020
45	0.522	0.610	0.675	0.622	0.455	0.300	0.190	0.100	0.020
50	0.522	0.614	0.692	0.635	0.455	0.300	0.185	0.108	0.040

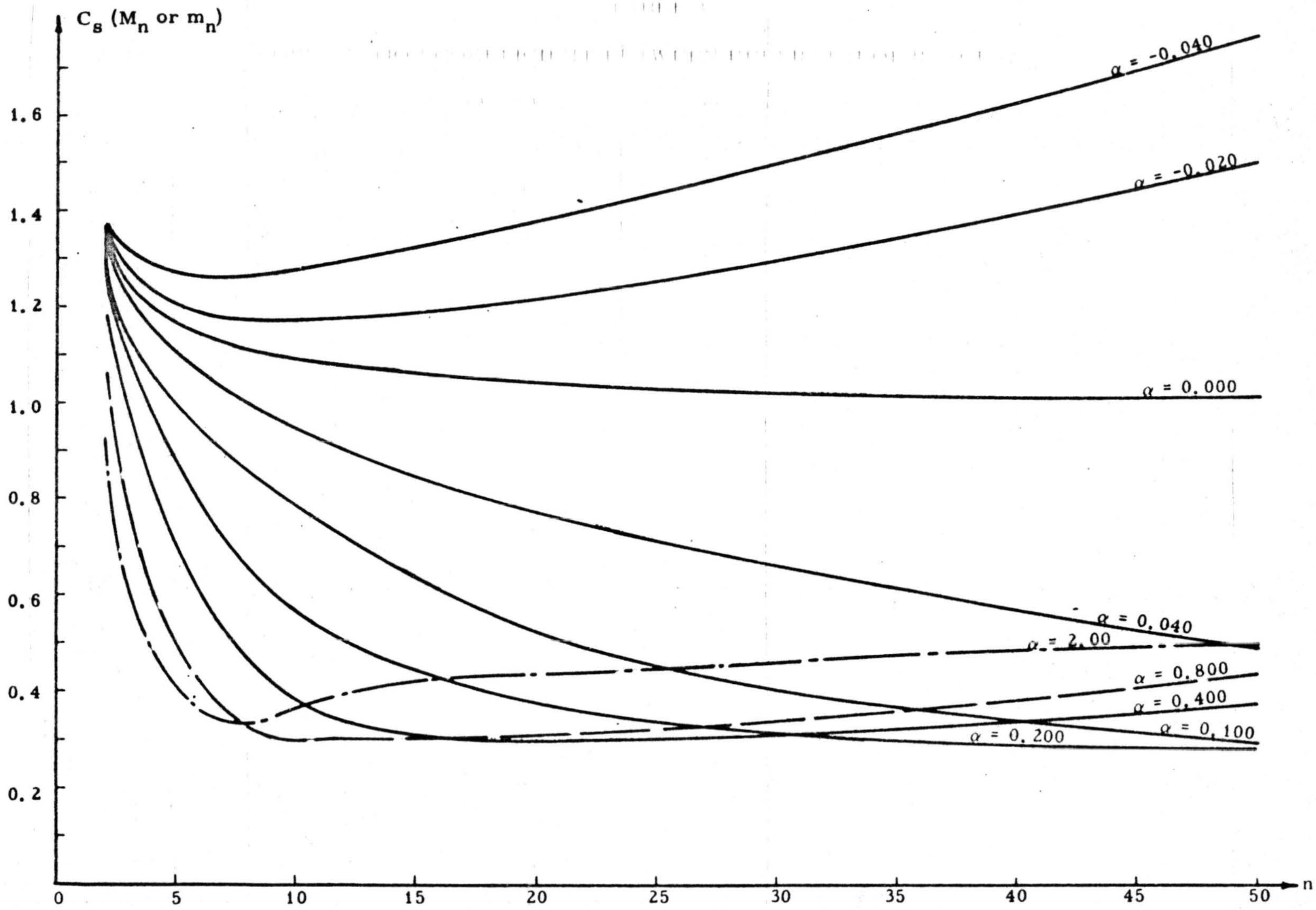

Fig. 5 The Skewness of Maximum Surplus (M_n) or Deficit (m_n) as a Function of n and α.

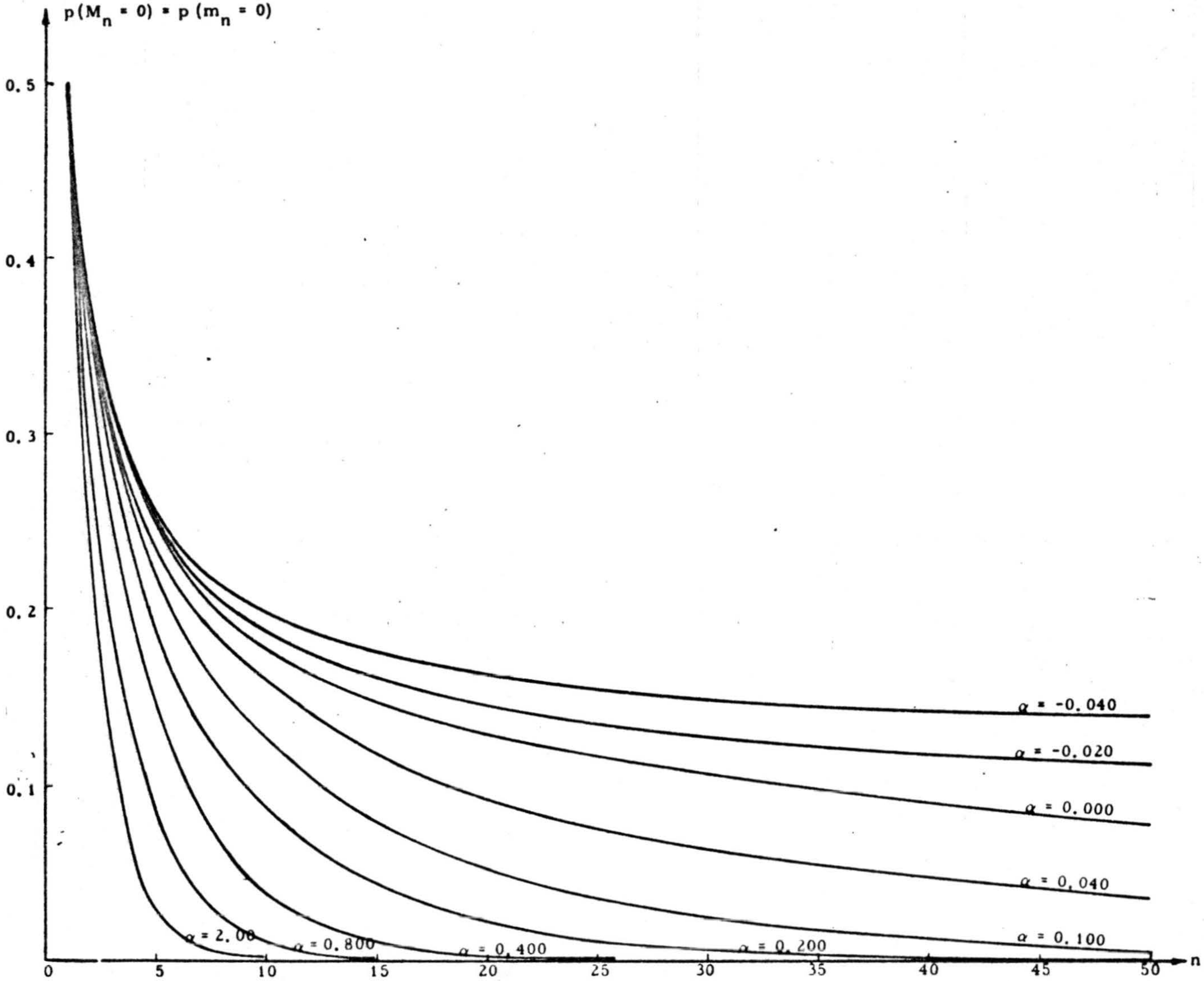

Fig. 6 Probability Density Function of $p(M_n = 0)$ or $p(m_n = 0)$, for $-0.04 \leq \alpha \leq 2.00$, and $1 \leq n \leq 50$.

B. Finite Storage

The problem of finite storage is analogous to the problem of random walk with two absorbing barriers. Theoretical solution of this problem is very complicated, and the equations are very long. The writer gives here only the final results obtained by the theory of random walk with two absorbing barriers and by the Monte Carlo method. The data used for the Monte Carlo simulation method on a digital computer are 100, 000 random numbers of a normal independent variable with mean zero and variance unity. The subsamples of length n used are for $n = 2, 4, 6, 8, 10, 15, 20, 30, 40$ and 50. By using the Monte Carlo method the mean water surplus and the mean water deficiency for the finite reservoir capacity are calculated for subsamples of size n. As the initial storage amount in the reservoir, S_0, is assumed to be equal to $S_f/2$, and as the distribution of the basic normal independent variable is symmetrical, the water spillover (surplus) is equal to the water deficiency. Whenever the word "surplus" is used in this text it refers also to the water deficiency.

1. Probability of surplus or deficiency for given finite reservoir capacity and given α.

Figure 7 shows the mean surplus as a function of the finite reservoir capacity S_f for various values of α. Two types of points are obtained: one from the computer and the other by the theory of random walk with two absorbing barriers. As the theory of random walk with two absorbing barriers is complicated to use for practical purposes, a simplified expression for the mean surplus is derived as a function of S_f and α. The boundary conditions set for this equation are: (1) For $S_f = 0$, the mean surplus is 0.50; and (2) for the mean surplus approaching zero, the reservoir capacity should tend to infinity. On the basis of the data obtained by the theory of random walk with two absorbing barriers and by the Monte Carlo method and the given boundary conditions, the following equation is fitted to the data:

$$\bar{s}_n = \frac{1}{2} \exp \left[- \left(\frac{1 + 2\alpha}{2} \right)^2 \frac{S_f}{\sigma} \right] \qquad (61)$$

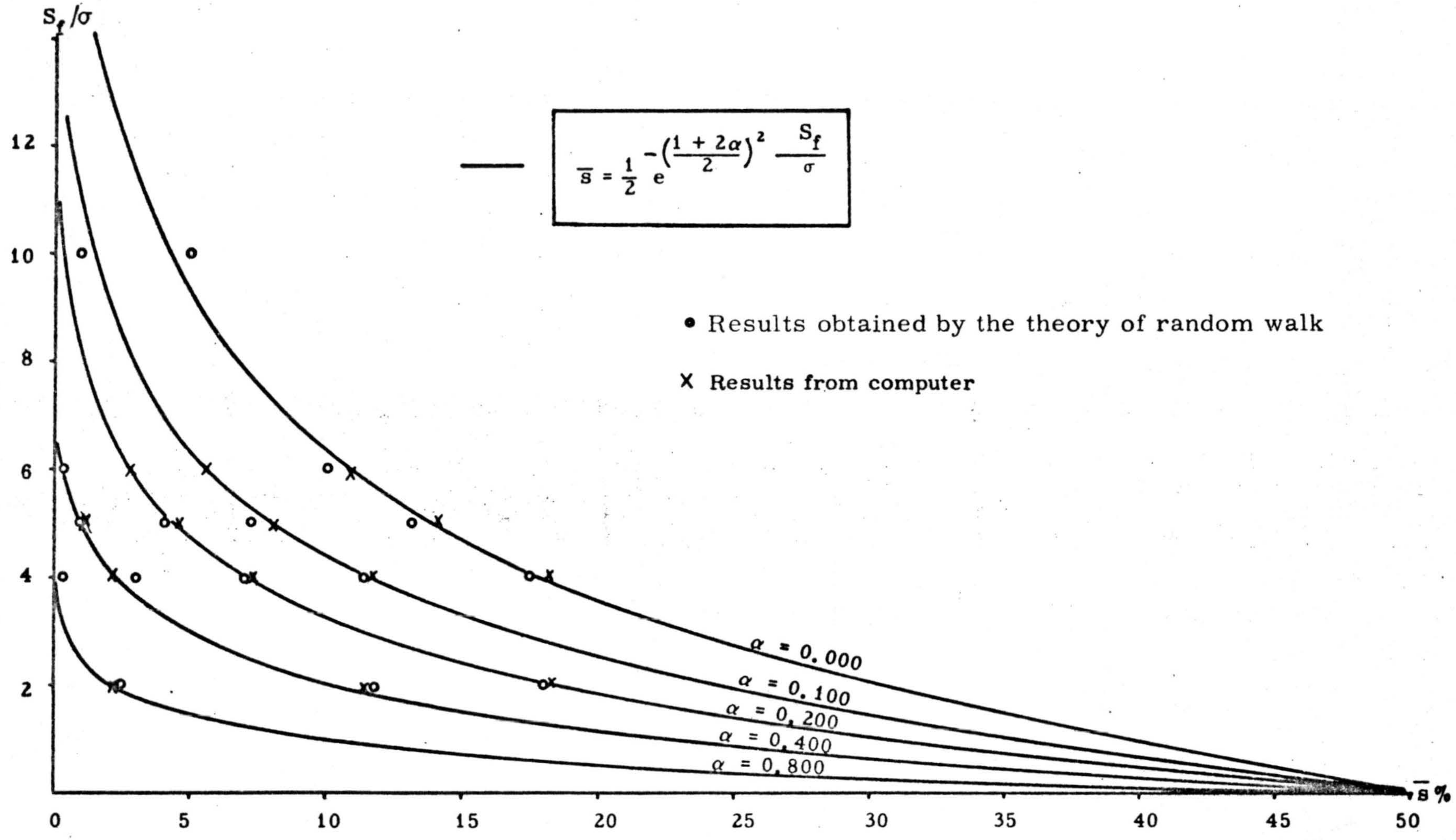

FIG. 7 - THE MEAN SURPLUS OR DEFICIENCY FOR FINITE STORAGE S_f, DURING THE TIME t, FOR $t \geq 25$

with $\bar{s}_n$ denoting the mean surplus as $t \geq 25$. Lines of equation (61) are plotted in figure 7 and they do not differ very much from the results obtained either by the Monte Carlo method or by the theory of random walk with two absorbing barriers.

2. General equation for the probability density function of storage

By considering the results derived by the theory of random walk with two absorbing barriers and by the Monte Carlo method, the following general equations for the probability density function of storage during the time t for any value of α, and any finite reservoir capacity S_f, if $t \geq 25$ are

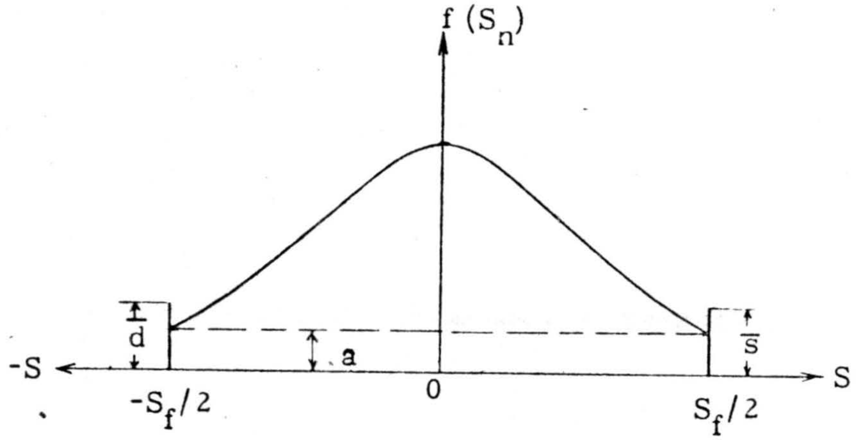

Figure 8 - Probability density function of storage, during the time t, for $t \geq 25$.

$$f(S_n) = a + p_e(S_n) ; \; -S_f/2 \leq S_n \leq S_f/2 \text{ and } a = \text{constant}. \qquad (62)$$
$$= 0 \; ; \text{elsewhere.}$$

$$p_e(S_n) = \frac{1}{\sigma}\sqrt{\frac{\alpha}{\pi}}\,\left(e^{-\alpha |S|^2/\sigma^2} - e^{-\alpha(S_f - |S|)^2/\sigma^2} + \right. \qquad (63)$$
$$\left. + e^{-\alpha(S_f + |S|)^2/\sigma^2} - e^{-(2S_f - |S|)^2/\sigma^2} + \ldots\right), -\frac{S_f}{2} \leq S_n \leq \frac{S_f}{2}$$
$$= 0; \text{elsewhere.}$$

From the probability theory the following expression must be satisfied:

$$a\,S_f + 2\bar{s} + \int_{-S_f/2}^{S_f/2} p_e\,(S)\,ds = 1\,, \qquad (64)$$

with $\bar{s} = \bar{d}$ (the surplus being equal to the deficiency).

Putting the expression for the surplus of equation (61) into equation (64) the following equation for the constant a is obtained:

$$a = \frac{1 - e^{\frac{-(1+2\alpha)^2}{4}\frac{S_f}{\sigma}} - \int_{-S_f/2}^{S_f/2} p_e\,(S_n)\,dS_n}{S_f}\,. \qquad (65)$$

C. Conclusions

The following conclusions are based on the analysis of storage problems when the outflow is dependent on the storage in a reservoir:

1. Results presented are applicable to all four cases of flow regulation, if the proper variance for the inflow is used.

2. The agreement between data obtained by the Monte Carlo simulation method on a digital computer and the theory derived in this study indicates that the hypotheses involved in the development of the theory are valid.

3. The probability density functions of storage at the time t become stationary for $t \geq 2.3/\alpha$.

4. For $\alpha \neq 0$, it is practically impossible to derive a general theoretical expression for the moments of range distribution even for very small n such as 2 or 3 .

5. Equations (48) and (49) for the mean range and the variance of range, respectively, are derived under the hypothesis that the mean range and the variance of range depend only on the variance of the storage at the time t for given α . A good agreement exists between the results obtained by the Monte Carlo method and by equation (48) and (49) . As α tends to zero the mean range tends to the same expression as it is found by Anis and Lloyd; and the variance tends to the same form as derived by Feller.

6. Equation (61) is a general expression for the computation of the mean surplus for $t \geq 25$ for any value of the finite reservoir capacity and for any value of α .

7. Equation (62) is a general expression for the probability density functions of the storage during the time t for any values of α and any finite reservoir capacity as $t \geq 25$.

8. The analysis is based on independent inflows into the reservoir. If, however, successive values of inflow are correlated, it is possible

to revise the equation to include the effect of the linkage by using the correction factor for the standard deviation of inflow as derived by Langbein[7].

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7. Langbein, W. B., Queuing theory and water storage. Journal of the Hydraulics Division, Proceedings of the American Society of Civil Engineers, Vol. 83, No. HY 5, paper 1811, 24 p.; 1958.

Appendix - Notation

The following symbols have been adopted for use in this paper:

a = constant for probability density function of finite storage capacity;

C = constant;

C_s = skewness;

d = diferential;

dS/dt = the rate of change in time of storage volume;

$E[M_n]$ = mean of surplus of the cumulative sums;

$E[m_n]$ = mean of deficit of the cumulative sums;

$E[R_n]$ = mean of range;

f = function;

I = inflow;

j = number of time intervals;

Lim = limit;

M_n = surplus of the cumulative sums;

m_n = deficit of the cumulative sums;

n = number of discrete time intervals;

0 = output;

p = probability density function;

Q_a = average inflow or outflow;

Q_i = inflow;

Q_0 = outflow;

q_i = departures of inflows from the mean inflow multiplied by parameter $(1 - \beta)$ or one;

R_n = range of the cumulative sums;

$\bar{R}_n$ = mean of range;

S = storage;

S_f = finite storage capacity;

S_n = cumulative sums;

S_o = oringinal storage;

$\overline{s}$	= average surplus;
t	= time;
$\mathrm{Var}\ R_n$	= variance of range;
w_i	= departures of inflows from the mean inflow;
w_0	= departures of outflows from the mean outflow;
α	= proportionality parameter;
β	= proportionality parameter;
Δt	= time interval;
θ_1	= function;
ξ	= time exis;
ρ	= correlation coefficient;
σ^2	= variance of q_i ;
σ_1^2	= variance of w_i ;