DISSERTATION

HYPERGRAPHS AND THEIR ASSOCIATED LIE ALGEBRAS

Submitted by

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ABSTRACT

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When studying features in networks, communities, and general relations between objects, hypergraphs permit a more complex and accurate description of the underlying data. As hypergraphs admit higher valent relations between vertices, the set of all hypergraphs and their underlying features is infinite, as the number of vertices and the maximum valence of relations present in a hypergraph are both unbounded. In this dissertation, we present a new result which shows that there exists a finite characterization, utilizing the generators of simple Lie algebras, of global features present in a hypergraph. Furthermore, this characterization is implemented as an algorithm to identify specific configurations of relational structures which are present in a given hypergraph.

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Chapter 1

Introduction

In this dissertation, we explore a new spectral theory for hypergraphs. Hypergraphs organize the data of relations and interactions between objects into more complex forms and permit a more accurate representation than graphs alone. We defer our technical definitions to Chapters 2 and 3; for now, we say a hypergraph is a subset $\mathcal{H} \subset 2^V$. Our main goal is to explore how algebras act on hypergraphs with methods developed by [11] and [5]. To that end, we prove that there exist finitely many criteria to determine which of the infinitely many *classical geometries*—i.e., general, orthogonal, unitary, or symplectic—arise in the symmetries of hypergraphs. We state the main result of this dissertation as a corollary to Theorem 4.5.3.

Corollary 1.0.1. There exist finitely many finite predicates in finitely many variables that characterize all vertex sets V and hypergraphs $\mathcal{H} \subset 2^V$ for which classical geometries (in the natural and adjoint representations) act as symmetries.

In this Corollary, the predicates refer to statements of the form "a matrix is an element of the derivation algebra if and only if the following hyperedges exist in $\mathcal{H}$". In Chapter 4, we formalize these predicates. Additionally, we utilize the natural and adjoint representations in Theorem 4.3.3. Given a different choice of representation for the classical geometries, Theorem 4.5.3 would hold with a different finite bound, and Theorem 4.3.3 would be written with different matrices and hyperedges in each predicate.

Proving Theorem 4.5.3 involves the language of simple Lie algebras, which we develop in Chapter 2 and in Section 4.2. The main results of this dissertation, Theorems 4.3.3 and 4.5.3, require us to review tensors in Chapter 2 and hypergraphs in Chapter 3. In this chapter, we begin by providing context for the problems we address.

1.1 Case study: interactions in a high school

To provide an example of a hypergraph for study, we present a case study of interactions among students within a high school. The following dataset was retrieved from [2] and [16]. As detailed in [16], a subset of 327 students in a specialized 'classes préparatoires' in a French high school were given wearable badges that recorded physical proximity to other students in the study. For this dissertation, we study a random selection of these students. An edge is created between two students if they were within 1 to 1.5 meters of each other. Mutual interactions in a group, known as cliques, were identified by recording interactions within a 20-second interval. If multiple students were in proximity during that 20-second interval, a simplicial complex was created to record the interaction.

We focus on simplices with three vertices, which, for this study, are all 3-faces of the simplicial complexes in the dataset. These 3-faces can be graphed in a 3-way array. We place a value of 1 at position (i, j, k) if students i, j, and k appear in a 3-face in our dataset. We can then represent this as a point in $\mathbb{R}^3$ whose coordinates are positive integers. We select 80 students at random and graph their 3-cliques in the 3-way array shown in Figure 1.1. To the left is the full dataset; to the right is the resulting dataset after applying a new algorithm detailed in [11] to the 3-way array. The 3-way array is interpreted as a *tensor*, as formally defined in Chapter 2, and the resulting *stratification* of the tensor is shown in Figure 1.1.

For our purposes, stratification is a form of clustering high-dimensional data onto a lower-dimensional surface or structure. It is not immediately obvious that these tensor analysis techniques should yield information about hypergraphs. These techniques rely on linearity within the underlying data, and Chapter 3 discusses the justification for interpreting linearity within hypergraphs.

This is not a combinatorial optimization problem, nor a method that requires the training of an algorithm. This stratification uniquely identifies features that are already present in the dataset itself. Unlike machine learning methods, the features produced by this method are reproducible, require no training, and can be computed efficiently. This method is not simply identifying cliques,

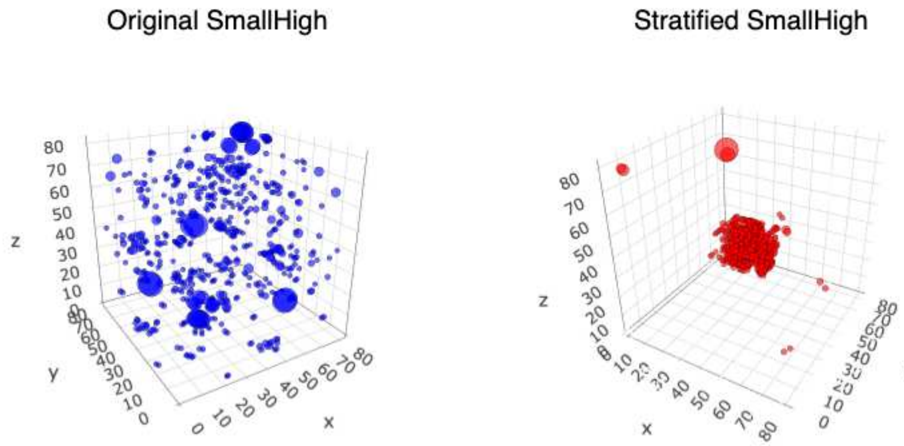


Figure 1.1: High School cliques shown on the left, higher order relations between students shown on the right

as they have already been identified to store our data. We are also not simply identifying connectivity, which can be found in the data by standard graph theoretic algorithms. This stratification measures a higher order of connectivity via linear relations.

For this stratification, a *Lie algebra* is computed, and its elements are used to calculate projections. If the data were truly random, we would expect this Lie algebra to be three-dimensional, as the cliques themselves are three-dimensional. However, in our calculations, the Lie algebra is at least ten-dimensional (the true dimension was not computed due to memory constraints). This indicates that our data has a richer underlying structure that can be identified using this algebra. For example, the central cluster in the stratification indicates a more connected subset of students that isn't readily observable in the original dataset. This discussion of dimensionality continues in Chapter 3. These calculations were the original motivation for our use of Lie algebras to study hypergraphs.

Alongside this stratification, we computed the barycentric adjacency tensors of this hypergraph, which are a variation on the barycentric subdivision of simplicial complexes. This allows us to subdivide the hypergraph into nested hyperedge relations. We develop these tensors in Chapters 2 and 3. For now, it suffices to know that these barycentric adjacency tensors count relations between hyperedges that are reminiscent of barycentric subdivisions. In Section 4.1.1 we revisit this case study with greater detail. This stratification algorithm and its associated theory had remained unexplored for hypergraphs, and yet it uncovers potential structure that we further study. In this dissertation, we explore how these algorithms uncover structure in our data and provide a more detailed description of what this structure means for our hypergraphs.

1.2 Outline

This dissertation is divided into three parts. Chapter 2 reviews the necessary algebra we use to formalize and study adjacency tensors. In Chapter 3, we generalize adjacency matrices to adjacency tensors and show how the latter can be used to study hypergraph features. Finally, in Chapter 4, we present the main results of this dissertation, demonstrating how the stratification algorithm identifies features in the hypergraph. Moreover, this identification establishes a bijection between the algebra's elements and the hypergraph's hyperedges. Theorem 4.3.3 covers hypergraphs in which each edge has exactly three vertices, and Theorem 4.5.3 shows that analogs of Theorem 4.3.3 only need to be computed for hypergraphs with edge sizes from four to ten. For all hypergraphs where each edge has ten or more vertices, the higher-dimensional analogs of our main results will have relations equivalent to those found for hypergraphs where each edge has exactly ten vertices.

1.3 The advantages of algebra

We close this chapter with a brief discussion of our motivation for using algebra to study hypergraphs. As inherently combinatorial objects, there are many questions regarding the structure of a hypergraph that can be answered with combinatorial algorithms and tools. These algorithms tend to be greedy, necessarily exploiting local features present in a structure. Therefore, the information found is very local. We can learn about minimal, maximal, or average vertex degrees, shortest paths between vertices, and cliques.

In contrast, algebraic methods provide more global perspective of data. We can ask about vertex transitivity or the presence of global symmetries in the data. This creates a fundamental difference in the kinds of questions we ask and the answers we give regarding hypergraphs in this dissertation.

The work of [18] and [15] defined eigenvalues and eigenvectors on a multilinear product, which has since been used to study eigenvalues of adjacency tensors. [9] utilizes a rescaled adjacency tensor to create an analog for spectral graph theory on hypergraphs, which was used to study bounds on average degrees and coloring of vertices. The Laplacian tensor, constructed by subtracting the rescaled adjacency tensor from the diagonal multiway array, has been used to study the general connectivity of a hypergraph [10] [14] [17] [1].

For this dissertation, we use an alternative spectral theory of tensors developed by [11] and [5]. As we see in Sections 3.2 and 3.3, this method of defining the spectral properties of tensors more closely aligns with studying matrices and tensors as algebraic objects with a multiplication, rather than as linear transformations on another space. This allows us to provide a new language for describing eigenvalues; this language is not constrained to studying the inputs of a tensor.

Chapter 2

Tensors

We begin by exploring how tensors may appear in real-world data and the justifications for using common tools to study them. The subsequent sections further explore linearity and the specific tools we use in this dissertation to study linear data.

2.1 Distinguishing tensors from grids of numbers

When evaluating data, it is tempting to use various programs and algorithms to extract actionable information. These tools face a great challenge. Barring an egregious error in your data, such as inputting a letter when only integers are expected, these tools will successfully calculate some values.

Consider the grids of numbers in Figures 2.1 and 2.2.

	Oil	Wheat	Electronics
Country A	50	-20	10
Country B	-30	40	-15
Country C	-20	-20	5

Figure 2.1: A matrix representing trade

	100m	5k	Half Marathon
Runner A	50	-20	10
Runner B	-30	40	-15
Runner C	-20	-20	5

Figure 2.2: A grid of numbers for runners

In Figure 2.1, we first state that each column represents a different traded good, each row a different country, and each entry the trade surplus or deficit in billions of dollars.

- Scaling a row corresponds to a country increasing either its imports or exports,
- adding rows corresponds to combining countries into one economic block,
- scaling columns corresponds to the price of the good changing, and
- adding columns corresponds to collecting different goods into one class (such as TVs and computers being added to electronics).

This makes Figure 2.1 a *matrix*. In Figure 2.2, the rows represent different runners, the columns represent different races, and the entries represent the difference between the runners' most recent time in that race vs their personal record. In this context, none of the linear operations we would apply to a matrix provides us with *actionable information*. In Chapter 3 we focus on showing that the adjacency tensors of hypergraphs provide actionable information after specific operations.

As this dissertation studies higher-valence tensors, we provide an example of how such tensors can arise. In Section 2.2, we provide more specific definitions.

Example 2.1.1. As an example of a higher valence tensor (beyond just a matrix), we can consider an extension to a standard linear programming problem. Suppose three factories (A, B, and C) produce goods that need to be shipped to three warehouses (X, Y, and Z). Each factory produces a high-quality and a low-quality version of its product. The cost of shipping from each factory to each warehouse, which includes the costs associated with each item, is given by Figure 2.3,

	X	Y	Z
A	4+6z	6+7z	8+10z
B	5+7z	4+5z	7+10z
C	6+11z	5+8z	6+9z

Figure 2.3: A supply chain tensor

where $a + bz$ is a flattening of a multiway array, where a is the cost of the low-quality product and b is the cost of the high-quality product. We call this multiway array a *tensor*. Each factory has a limited supply: A can supply 70 units, B can supply 50 units, and C can supply 30 units. Each warehouse has a demand: X needs 40 units, Y needs 60 units, and Z needs 50 units. For the purposes of shipping and storage, each factory and each warehouse consider low- and high-quality products identical. Therefore, the previous counts of units are the total sum of both the low-quality and high-quality products. Finally, the total number of high-quality goods must be more than twice the total number of low-quality goods.

Let x_{ij} denote the number of low-quality units shipped from factory i to warehouse j, and let y_{ij} denote the number of high-quality units shipped from factory i to warehouse j. The linear programming problem is presented in Figure 2.4. As with our example in Figure 2.1, we are justified in treating this data as linear.

$$\begin{array}{ll}
\text{Minimize} & 4x_{AX} + 6x_{AY} + 8x_{AZ} + 5x_{BX} + 4x_{BY} + 7x_{BZ} + 6x_{CX} + 5x_{CY} + 6x_{CZ} \\
 & +6y_{AX} + 7y_{AY} + 10y_{AZ} + 7y_{BX} + 5y_{BY} + 10y_{BZ} + 11y_{CX} + 8y_{CY} + 9y_{CZ} \\
\text{subject to} & x_{AX} + x_{AY} + x_{AZ} + y_{AX} + y_{AY} + y_{AZ} \leq 70 \\
 & x_{BX} + x_{BY} + x_{BZ} + x_{BX} + y_{BY} + y_{BZ} \leq 50 \\
 & x_{CX} + x_{CY} + x_{CZ} + y_{CX} + y_{CY} + y_{CZ} \leq 30 \\
 & x_{AX} + x_{BX} + x_{CX} + y_{AX} + y_{BX} + y_{CX} = 40 \\
 & x_{AY} + x_{BY} + x_{CY} + y_{AY} + y_{BY} + y_{CY} = 60 \\
 & x_{AZ} + x_{BZ} + x_{CZ} + y_{AZ} + y_{BZ} + y_{CZ} = 50 \\
 & \sum_{i,j} y_{ij} - 2x_{ij} > 0 \\
 & x_{ij} \geq 0 \quad \text{for all } i, j
\end{array}$$

Figure 2.4: A linear programming problem presented as a tensor

In Chapter 3, we discuss adjacency matrices of graphs and the additional work required to refer to them as *matrices*. The challenge we work around is that for a hypergraph, we represent nodes as vectors. Linearity is then difficult to show, as the addition of vectors would correspond to an 'addition of vertices', which has no canonical interpretation.

2.2 Formally defining tensors

As we have seen, simply having data is insufficient to say you are working with linear objects. This is our motivation for giving a purely algebraic definition of tensors. We follow notation and convention as seen in [5] and [11]. In this dissertation, $\mathbb{F}$ represents a field of characteristic zero, and vector spaces V are finite-dimensional. In practice, we choose $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Q}$ as our field, but for the purposes of this dissertation, we do not need to specify the field.

In this dissertation, we use $\bar{i}$ to represent set complements. When used with indices the notation $a_{\bar{i}}$ represents $\{a_j \mid j \neq i\}$.

Definition 2.2.1. Let V_i be $\mathbb{F}$ vector spaces. A *tensor* is a function

$$\langle t| : V_1 \times \cdots \times V_\ell \rightarrowtail V_0,$$
$$(v_1, \ldots, v_\ell) \mapsto \langle t|v_1, \ldots, v_\ell\rangle$$

where

$$\langle t|\alpha u_a + \beta v_a, v_{\bar{a}}\rangle = \alpha \langle t|u_a, v_{\bar{a}}\rangle + \beta \langle t|v_a, v_{\bar{a}}\rangle, \ \forall 1 \leq a \leq \ell, \ \forall u_a \in V_a, \ \forall v_i \in V_i,$$

i.e. the function is *multilinear*.

Throughout this dissertation, we use $\rightarrowtail$ to denote a multilinear map. The property of being multilinear can be understood in terms of partial evaluations.

Definition 2.2.2. A function

$$\langle t| : V_1 \times \cdots V_\ell \to V_0,$$

is $\mathbb{F}$ *multilinear* if, for all $(u_1, \ldots, u_\ell)$, the curried maps

$$d_j : \prod_{i \neq j} V_i \to (V_j \to V_0),$$
$$d_j(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_\ell) = \langle t|u_1, \ldots, u_{j-1}, \cdot, u_{j+1}, \ldots, u_\ell\rangle : V_j \to V_0$$

are linear maps.

Example 2.2.3. Consider

$$\langle t| : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrowtail \mathbb{R}^2,$$

$$\langle t|u,v\rangle \mapsto \left(u^T \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} v, u^T \begin{bmatrix} 7 & -3 & 3 \\ -2 & 2 & 4 \end{bmatrix} v \right).$$

Since matrix multiplication is linear, this map is linear in both $\mathbb{R}^2$ and $\mathbb{R}^3$, and thus it is a tensor.

Example 2.2.4. The inner product is a tensor. To see this, consider $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined via the standard inner product $(u, v) = u^T v$. Then

$$(\alpha u_1 + \beta u_2, v) = (\alpha u_1 + \beta u_2)^T v = (\alpha u_1^T + \beta u_2^T)v = \alpha u_1^T v + \beta u_2^T v = \alpha(u_1, v) + \beta(u_2, v),$$
$$(u, \alpha v_1 + \beta v_2) = u^T(\alpha v_1 + \beta v_2) = \alpha(u^T v_1) + \beta(u^T v_2) = \alpha(u, v_1) + \beta(u, v_2).$$

As the inner product is multilinear, it is indeed a tensor.

Often, we define objects as elements of the larger space they inhabit. In linear algebra, we define a vector as an element of a vector space. In the same vein, we define tensors as elements of a tensor space.

Definition 2.2.5. Fix a field $\mathbb{F}$, and finite-dimensional $\mathbb{F}$-vector spaces $V_0, V_1, \ldots, V_\ell$. A *tensor space* is a module T with a $\mathbb{F}$-linear injection $\langle\cdot| : T \hookrightarrow \hom_{\mathbb{F}}(V_1 \otimes \cdots \otimes V_\ell, V_0)$. A *tensor* t is an element of a tensor space T. $(V_0, \ldots, V_\ell)$ is the *frame* of the tensor space, and V_i is the *i-axis* of the tensor.

It is natural to talk about the *dimension* of a tensor. We must use a different term to avoid confusion with the vector space dimensions which are the axes of t.

Definition 2.2.6. Let t be a tensor, and let $(V_0, \ldots, V_\ell)$ be the frame of t. If $\dim(V_0) \geq 2$, then the *valence* of t is $l + 1$; otherwise, the valence of t is l.

Example 2.2.7. The tensor from example 2.2.3 has valence 3 and its frame is $(\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^2)$.

For the purposes of this dissertation, we make the following notational choices. As our vector spaces are finite-dimensional, we may write $V_i \cong \mathbb{F}^{d_i}$. Furthermore, the injection given by $\langle\cdot|$

allows us to interpret tensors t as multilinear functions

$$\langle t| : V_1 \times \cdots \times V_\ell \rightarrowtail V_0$$
$$\langle t|v\rangle := \langle t|v_1, \ldots, v_\ell\rangle\,, \quad v_i \in V_i$$

An advantage of this interpretation of tensors is that there is a clear distinction between tensors of valence less than or equal to 2 and those of valence greater than or equal to 3. We discuss this in Section 2.4. Our motivation for this discussion is complex inner product spaces as seen in Example 2.2.4.

2.3 Frobenius products and barycentric divisor tensors

In Chapter 3, we discuss a spectral theory for hypergraphs. In preparation for that discussion, we introduce a new operation on tensors which is reminiscent of the Frobenius product. For matrices, their powers are defined via dot products. Specifically, for a symmetric matrix A, $A^2 = AA^T$ records the dot products $a_i \cdot a_j$ for the rows of A. When viewed as a tensor, each e_i represents a slice of A along an axis. Therefore, when extending our analogy to tensors, we consider multiplication between slices of the tensor.

Definition 2.3.1. Let $\langle t| : V^l \rightarrowtail \mathbb{F}$ be a symmetric tensor. Let $A \subset ([n] \cup *)^m$. An A-*slice* of $\langle t|$, denoted by $\langle t_A|$, is

$$\langle t_A| : V^{l-m} \rightarrowtail \mathbb{F}$$
$$\langle t_A|u\rangle := \langle t|v_A, u\rangle$$
$$v_A := (e_{a_1}, e_{a_2}, \ldots, e_{a_m}),$$
$$a_i \in A[i]$$

Example 2.3.2. Let $\langle t| : \mathbb{R}^m \times \mathbb{R}^n \rightarrowtail \mathbb{F}$ be a matrix. The $A = \{2, 3\}$ slice is the slice along the second row and the third column, i.e., $A_{2,3}$. The $A = \{2, *\}$ slice is the slice along the second row, including every column, i.e., the second row $A_{2,*}$.

We can also view $\langle t_A|$ as the partial evaluation of $\langle t|$. As $\langle t|$ is symmetric, we do not need to keep track of specific axes. In the language of matrices, this is equivalent to row i of a symmetric matrix being equal to column i of the same matrix. We use an extension of the Frobenius inner product to define our barycentric tensors.

Definition 2.3.3. Let $A, B \in \mathbb{F}^{n_1 \times \cdots \times n_l}$ be two multiway arrays. Their *Frobenius inner product* is defined to be

$$\langle A, B\rangle_F := \sum_{a_1,\ldots,a_l=1}^{n_1,\ldots,n_l} A_{a_1,\ldots,a_l} B_{a_1,\ldots,a_l}.$$

For multiple $A_1, \ldots, A_m \in \mathbb{F}^{n_1 \times \cdots \times n_l}$, we can define their *Frobenius product*

$$\langle A_1, \ldots, A_m\rangle_F := \sum_{a_1,\ldots,a_l=1}^{n_1,\ldots,n_l} \prod_{i=1}^{m} (A_i)_{a_1,\ldots,a_l}$$

Utilizing this Frobenius product of tensors, we can define *barycentric tensors*. Note that these are used in Chapter 3 to discuss a variant of barycentric subdivisions.

Definition 2.3.4. Let $\Gamma : V \times \cdots \times V \rightarrowtail \mathbb{F}$ be symmetric. We define the associated *barycentric tensor* as

$$\langle t|^{(l,1)} : V^l \rightarrowtail \mathbb{F}$$
$$\langle t|^{(l,1)} |e_{a_1}, \ldots e_{a_l}\rangle = \sum_{i=1}^{n} \langle \langle t_{\{a_{\bar{i}}\}}| \rangle_F$$

In the 3-valent case, this simplifies to:

$$\langle t|^{(l,1)} |e_i, e_j, e_k\rangle = \langle \langle t_{i,j}| , \langle t_{i,k}| , \langle t_{j,k}| \rangle_F$$
$$= \sum_{l=1}^{n} \langle t|e_i, e_j, e_l\rangle \langle t|e_i, e_l, e_k\rangle \langle t|e_l, e_j, e_k\rangle .$$

Proposition 2.3.5. Let $\langle t| : \overbrace{V \times \cdots V}^{l} \rightarrowtail \mathbb{F}$ be a symmetric tensor. $\langle t|^{(l,1)}$ is a tensor.

Proof. We show the proof for the three-valent case, as the proof for higher-valence tensors follows naturally. Let Γ be a symmetric 3-valent tensor. We first give a coordinate-free definition of $\Gamma^{(3)}$. We define the product

$$[t|u, v, w] := \langle \langle t|u, v, \cdot\rangle , \langle t|u, \cdot, w\rangle , \langle t|\cdot, v, w\rangle , \rangle_F$$

Here, each term in the Frobenius product is an element of V^*. As $V^* \cong V$, the Frobenius product is defined naturally. After a change of basis, we have

$$[\langle t| |Xu, Xv, Xw] = \langle \langle t|Zu, Xv, X(\cdot)\rangle , \langle t|Xu, X(\cdot), Xw\rangle , \langle t|X(\cdot), Xv, Xw\rangle , \rangle_F$$

We can now define an action of $\langle t|$ by a change of basis X by

$$[\langle t|^X |u, v, w] := \langle \langle t|Xu, Xv, X(\cdot)\rangle , \langle t|Xu, X(\cdot), Xw\rangle , \langle t|X(\cdot), Xv, Xw\rangle , \rangle_F$$

This shows that $(\langle t|^{(3,1)})^X = (\langle t|^X)^{(3,1)}$, i.e., that the barycentric tensor is invariant under change of basis. $\square$

As the barycentric tensor is itself a tensor, we can iterate this product.

Definition 2.3.6. The i^{th} *iterated barycentric tensor* of a tensor $\langle t|$ is defined to be

$$\langle t|^{(l,i)} := \langle \langle t|^{(l,i-1)} , \langle t| , \ldots , \langle t| \rangle_F$$

$$\langle t|^{(l,0)} := \langle t|$$

This allows us to create new tensors iteratively, and the Frobenius product provides a way to compare them. In Section 3.3, we continue this discussion on barycentric tensors as they relate to adjacency tensors. For now, we note that if $\langle t|$ were the adjacency matrix A of a graph, then $\langle t|^{(2,i)}$ would be equivalent to A^i, which counts the paths of length i between vertices. When we translate this to hypergraphs, we look at cliques and higher-order relations between them.

2.4 The centroid

We now return to our discussion on the valence of a tensor. To see where the distinction between 2-valent and 3-valent tensors lies, consider operators on complex inner product spaces V with the standard inner product $\langle x, y\rangle = u^*v$, where u^* is the conjugate transpose. When we apply an operator A to V, we obtain the following relation.

$$\langle Au, v\rangle = (uA)^*v = u^*(A^*v) = \langle u, A^*v\rangle.$$

where A^* is the Hermitian adjoint of A. An operator A is ***Hermitian*** if $A = A^*$. An equivalent definition for a Hermitian operator is $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in V$. If A and B are two Hermitian operators, then

$$\langle ABu, v\rangle = \langle Bu, Av\rangle = \langle u, BAv\rangle.$$

Unless $AB = BA$, AB is not Hermitian. This is why the space of Hermitian operators is a vector space rather than an associative algebra.

Now, we extend this analogy to tensors of valence three. Instead of the adjoint and Hermitian operators, we use the centroid, which can be thought of as a natural progression from adjoints.

Definition 2.4.1. The *centroid* of a tensor $\langle t| : V_1 \times \cdots \times V_l \rightarrowtail V_0$ is

$$\mathrm{Cen}(t) := \left\{ A_i \in \prod_{i=0}^{l} \mathrm{End}(V) \mid \langle t|A_a v_a, v_{\bar{a}}\rangle = A_0 \langle t|v\rangle \right\}$$

Example 2.4.2. Consider the $2 \times 2 \times 2$ tensor

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, \tag{2.1}$$

where c is any real number.

Our goal is to decompose the tensor into clearly defined blocks separated by zeros. We can attempt standard matrix operations to decompose this tensor. Viewed from front to back, the rows and columns are linearly independent, so there is no way to expose additional structure. A similar issue arises if we view the tensor from any perspective. As currently viewed, standard matrix operations would not expose additional block structures. To expose a clearer block structure, we first express this tensor as a multiplication table. If we take the ring of polynomials over the real numbers $\mathbb{R}[x]$, we can consider the front half of the tensor as the rule for the constants in polynomial multiplication, and the second half as the rule for the linear part. We may write:

*	1	x
1	1	x
x	x	c

.

In this format, we see that $x^2 = c$. As a result, this multiplication table represents multiplication in $\mathbb{R}[x]/(x^2 - c)$. The centroid of this tensor is generated by:

$$\left\{ (I, I, I), \left(\begin{bmatrix} 0 & c \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & c \\ 1 & 0 \end{bmatrix} \right) \right\}$$

This can be computed manually by solving for (A_0, A_1, A_2) that satisfy $A_0(u*v) = (A_1u)*v = u*(A_2v)$. A similar calculation is shown in Example 2.5.3. From here, we consider three cases depending on the value of c.

First, assume $c > 0$, and for the sake of calculation, let $c = 5$. Then $x^2-5 = (x-\sqrt{5})(x+\sqrt{5})$. This tells us that $\mathbb{R}[x]/(x^2-5) = \mathbb{R}[x]/(x+\sqrt{5}) \oplus \mathbb{R}[x]/(x-\sqrt{5})$. We have identity elements for $\mathbb{R}[x]/(x+\sqrt{5})$ and $\mathbb{R}[x]/(x-\sqrt{5})$ which we denote as e_1 and e_2, respectively. This ring has zero divisors and no nilpotent elements. The identity elements allow us to rewrite our multiplication table as:

$*$	e_1	e_2
e_1	e_1	0
e_2	0	e_2

.

This choice can also be seen in that $A = \begin{bmatrix} 0 & 5 \\ 1 & 0 \end{bmatrix}$ is a matrix representation of $\sqrt{5}$, as $A^2 = 5I$.

For our second case, assume $c = 0$. Then our multiplication table is:

$*$	1	x
1	1	x
x	x	0

.

In this case, our ring is $\mathbb{R}[x]/(x^2)$, which has nilpotent elements and zero divisors. Finally, consider $c < 0$, and for the sake of calculation, let $c = -5$.

$*$	1	x
1	1	x
x	x	-5

In this case, $\mathbb{R}[x]/(x^2+5) \cong \mathbb{R}(\sqrt{-5}) \cong \mathbb{C}$. Since $\mathbb{C}$ is a field, there are no zero divisors, so our tensor is indecomposable.

Now, assume we have $(A, A, A), (B, B, B) \in Cen(\langle t|)$ for a tensor $\langle t| : V \times V \rightarrowtail V$. Consider (AB, AB, AB). We have that

$$\langle t|ABv_1, v_2\rangle = A\,\langle t|Bv_1, v_2\rangle = A\,\langle t|v_1, Bv_2\rangle = A\,\langle t|v_1, ABv_2\rangle$$
$$\langle t|ABv_1, v_2\rangle = \langle t|v_1, ABv_2\rangle = AB\,\langle t|v_1, v_2\rangle$$

This shuffling is reminiscent of three-card monte or Towers of Hanoi. With Hermitian operators, we only have two positions to move around; we do not have control over their order. However, once we have a third location for the operator to move to, we can freely swap positions of the endomorphisms, which tells us that the centroid is an algebra.

2.5 Transverse operators

As we have seen, the centroid is an algebra that we can utilize to study tensors. We now want to see which other algebras can be used to study tensors. Similar to how we study actions on vector spaces rather than the vectors themselves, we discuss operators on tensor spaces.

Definition 2.5.1. Let t be a tensor in the tensor space T, with the frame $(V_0, \ldots, V_\ell)$. *Transverse operators* are elements ω of

$$\Omega := \prod_{i=0}^{\ell} \mathrm{End}(V_i),$$

where $\mathrm{End}(V_i) \cong M_{d_i}(\mathbb{F})$. $\omega = (\omega_0, \omega_1, \ldots, \omega_\ell) \in \Omega$ act on t via the action

$$\langle t^\omega|v\rangle = \omega_0\,\langle t|\omega v\rangle = \omega_0\,\langle t|\omega_1 v_1, \ldots, \omega_\ell v_\ell\rangle\,.$$

For a given tensor and operator, we can define a more complex polynomial relation.

Definition 2.5.2. Let t be a tensor and ω an operator on t. Let $p \in \mathbb{F}[X] := \mathbb{F}[x_0, \ldots, x_\ell]$, where $p(X) = \sum_e \lambda_e X^e$, $e : [[\ell]] \to \mathbb{N}_{\geq 0}$, and $X^e := \prod_{i=0}^{\ell} x_i^{e(i)}$. Then the action of $p(X)$ on t with a given

ω is

$$\langle t|\, p(\omega)\, |v\rangle := \sum_e \lambda_e \omega_0^{e(0)} \left\langle t \middle| \omega_1^{e(1)} v_1, \dots, \omega_\ell^{e(\ell)} v_\ell \right\rangle$$

Example 2.5.3. Recall that the centroid is defined as

$$\mathrm{Cen}(\langle t|) := \left\{ A_i \in \prod_{i=0}^{\ell} \mathrm{End}(V) \mid \langle t|A_a v_a, v_{\overline{a}}\rangle = A_0 \langle t|v\rangle \right\}$$

We may instead express this in the language of transverse operators using the polynomials $p_a(x) = x_0 - x_a$, and operators $(A_0, A_1, \dots, A_\ell)$. If we define

$$\begin{aligned} \langle t| : \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrowtail \mathbb{R}^2 \\ (u, v) &\mapsto \left(u^T \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} v, u^T \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix} v \right)^T, \\ C \langle t|u, v\rangle &= C \left(u^T \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} v, u^T \begin{bmatrix} 1 & 2 \\ -4 & 1 \end{bmatrix} v \right)^T, \end{aligned}$$

then $\mathrm{Cen}(\langle t|)$ is the **Z**-set of the tensor $\{\langle t|\}$ and the polynomials $\{(x_0 - x_1), (x_0 - x_2)\}$. We are solving for operators $\omega_1, \omega_2, \omega_0 \in M_2(\mathbb{R})$, which we denote by A, B, and C, respectively. After calculating the centroid relations on a basis, we obtain the following equations:

$$\begin{aligned} a_{11} + 4a_{21} - b_{11} - 2b_{21} &= 0, & a_{11} - 4a_{21} - b_{11} - 2b_{21} &= 0, \\ a_{11} + 4a_{21} - c_{11} - c_{12} &= 0, & a_{11} - 4a_{21} - c_{21} - c_{22} &= 0, \\ 2a_{11} + a_{21} - b_{12} - 2b_{22} &= 0, & 2a_{11} + a_{21} - b_{12} - 2b_{22} &= 0, \\ 2a_{11} + a_{21} - 2c_{11} - 2c_{12} &= 0, & 2a_{11} + a_{21} - 2c_{21} - 2c_{22} &= 0, \\ a_{12} + 4a_{22} - 4b_{11} - b_{21} &= 0, & a_{12} - 4a_{22} + 4b_{11} - b_{21} &= 0, \end{aligned}$$

$$a_{12} + 4a_{22} - 4c_{11} + 4c_{12} = 0, \qquad a_{12} - 4a_{22} - 4c_{21} + 4c_{22} = 0,$$

$$2a_{12} + a_{22} - 4b_{12} - b_{22} = 0, \qquad 2a_{12} + a_{22} + 4b_{12} - b_{22} = 0,$$

$$2a_{12} + a_{22} - c_{11} - c_{12} = 0, \qquad 2a_{12} + a_{22} - c_{21} - c_{22} = 0.$$

After solving this system of linear equations, we get as our solution:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} c_{21} + c_{22} & \frac{8}{9}c_{21} \\ 0 & -\frac{7}{9}c_{21} + c_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} -\frac{7}{9}c_{21} + c_{22} & 0 \\ \frac{8}{9}c_{21} & c_{21} + c_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} \frac{2}{9}c_{21} + c_{22} & \frac{7}{9}c_{21} \\ c_{21} & c_{22} \end{bmatrix}.$$

The centroid is then generated by:

$$(\omega_0, \omega_1, \omega_2)_I = (I_2, I_2, I_2)$$
$$(\omega_0, \omega_1, \omega_2)_\alpha = \left(\begin{bmatrix} 9 & 8 \\ 0 & -7 \end{bmatrix}, \begin{bmatrix} -7 & 0 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ 1 & 0 \end{bmatrix} \right)$$

Example 2.5.4. Let $\langle t| : \mathbb{F}^2 \times \mathbb{F}^3 \times \mathbb{F}^5 \rightarrowtail \mathbb{F}^4$. Set $\omega = (A, B, C, D) \in M_4(\mathbb{F}) \times M_2(\mathbb{F}) \times M_3(\mathbb{F}) \times M_5(\mathbb{F})$. Finally, let $p(X) = 2x_0^2 x_1 - 3x_2 x_3 + 5x_1^3 x_2$. Then

$$\langle t| \, p(\omega) \, |v\rangle = 2A^2 \langle t|Bv_1, v_2, v_3\rangle + (-3) \langle t|v_1, Cv_2, Dv_3\rangle + 5 \left\langle t \middle| B^3 v_1, Cv_2, v_3 \right\rangle$$

Example 2.5.5. Let $\langle t| : \mathbb{F}^3 \times \mathbb{F}^4 \rightarrowtail \mathbb{F}^2$ be defined by

$$\langle t|u,v\rangle = \left(u^T \begin{bmatrix} 7 & -3 & 10 & 0 \\ -8 & 5 & -2 & 9 \\ 4 & -10 & 6 & -7 \end{bmatrix} v, u^T \begin{bmatrix} -6 & 2 & 8 & -9 \\ 5 & -10 & 0 & 3 \\ -1 & 7 & -4 & 10 \end{bmatrix} v \right)^T$$

Set $\omega = (A, B, C) \in M_2(\mathbb{F}) \times M_3(\mathbb{F}) \times M_4(\mathbb{F})$, and let $p(X) = 5x_0^3 x_1 - 2x_1 x_2 + 7x_0 x_1^2 x_2^3$. Then

$$\langle t|\, p(\omega)\, |u,v\rangle =$$

$$5A^3 \left(u^T B^T \begin{bmatrix} 7 & -3 & 10 & 0 \\ -8 & 5 & -2 & 9 \\ 4 & -10 & 6 & -7 \end{bmatrix} v, u^T B^T \begin{bmatrix} -6 & 2 & 8 & -9 \\ 5 & -10 & 0 & 3 \\ -1 & 7 & -4 & 10 \end{bmatrix} v \right)^T$$

$$-2 \left(u^T B^T \begin{bmatrix} 7 & -3 & 10 & 0 \\ -8 & 5 & -2 & 9 \\ 4 & -10 & 6 & -7 \end{bmatrix} Cv, u^T B^T \begin{bmatrix} -6 & 2 & 8 & -9 \\ 5 & -10 & 0 & 3 \\ -1 & 7 & -4 & 10 \end{bmatrix} Cv \right)^T$$

$$+7A \left(u^T (B^2)^T \begin{bmatrix} 7 & -3 & 10 & 0 \\ -8 & 5 & -2 & 9 \\ 4 & -10 & 6 & -7 \end{bmatrix} C^3 v, u^T (B^2)^T \begin{bmatrix} -6 & 2 & 8 & -9 \\ 5 & -10 & 0 & 3 \\ -1 & 7 & -4 & 10 \end{bmatrix} C^3 v \right)^T$$

If we let

$$B = \begin{bmatrix} 2 & -3 & 1 \\ -1 & 3 & 0 \\ -2 & 1 & -3 \end{bmatrix},$$

$$C = \begin{bmatrix} -2 & 3 & 0 & 1 \\ 1 & -1 & 2 & -3 \\ 3 & 0 & -2 & 2 \\ -1 & 2 & -3 & 0 \end{bmatrix},$$

then we can compute

$$B^T \begin{bmatrix} 7 & -3 & 10 & 0 \\ -8 & 5 & -2 & 9 \\ 4 & -10 & 6 & -7 \end{bmatrix} C = \begin{bmatrix} 6 & 43 & -17 & 7 \\ -14 & -97 & 28 & -143 \\ -8 & 0 & 7 & -102 \end{bmatrix},$$

and so on.

2.5.1 Derivations

An important set of transverse operators is the set of derivations. The Lie algebras referenced in Section 1.1 arise from these derivations, and Theorems 4.3.3 and 4.5.3 utilize these derivations.

Definition 2.5.6. Let $\langle t| \mid V_\ell \times \cdots \times V_1 \rightarrowtail V_0$ be a tensor. Its *derivation algebra* $\mathrm{Der}(\langle t|)$ is

$$\mathrm{Der}(\langle t|) := \left\{ D \in \mathrm{End}(V_0) \times \prod_{i=1}^{\ell} \mathrm{End}(V_i) \mid \sum \langle t| D_\alpha v_\alpha, v_{\overline{\alpha}} \rangle = D_0 \langle t|v\rangle \right\}$$

Example 2.5.7. Given the tensor

$$\langle M| : \mathbb{R}^2 \times \mathbb{R}^3 \rightarrowtail \mathbb{R}^1$$
$$\langle u| M |v\rangle \mapsto u^T \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} v,$$

the derivation algebra $\mathcal{D}$ of Γ is the set of triples $(D_0, D_1, D_2) \in M_1(\mathbb{R}) \times M_2(\mathbb{R}) \times M_3(\mathbb{R})$ that satisfy the Leibniz rule on a basis, i.e.

$$D_0 \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} = D_1^T \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} D_2$$

As its name suggests, the derivation algebra is indeed an algebra. However, it is a non-associative algebra, namely a Lie algebra.

Definition 2.5.8. A *Lie Algebra* is an algebra $\mathfrak{g}$ whose multiplication satisfies the following conditions:

- $x^2 = 0$ (Alternating), and
- $(xy)z + (yz)x + (zx)y = 0$ (Jacobi Identity).

Example 2.5.9. Let $GL(n, \mathbb{C})$ be the vector space of invertible $n \times n$ matrices over $\mathbb{C}$. Define the product of two matrices X and Y via the commutator $[X, Y] = XY - YX$. $GL(n, \mathbb{C})$ with the commutator as its multiplication is a Lie algebra.

Proposition 2.5.10. Let $\langle t|$ be a tensor. $\mathrm{Der}(\langle t|)$ is a Lie Algebra.

Proof. We take the set of tuples of matrices $\mathrm{Der}(\langle t|)$, and define addition component-wise, scaling as distributive over addition, and the multiplication via component-wise commutators. It can then be verified that these operations are closed, and they satisfy the properties required for $\mathrm{Der}(\langle t|)$ to be a Lie algebra. □

As we do with algebras, we can define the category of Lie algebras. In this category, the objects are Lie algebras, and the morphisms are linear maps that respect the Lie product:

$$\phi([x, y]) = [\phi(x), \phi(y)].$$

These homomorphisms allow us to define ideals, which are the kernels of Lie algebra homomorphisms. Together, these allow us to define simple Lie algebras.

Definition 2.5.11. Let $\mathfrak{g}$ be a Lie algebra. $\mathfrak{g}$ is *simple* if it is not abelian and has no nontrivial ideals. $\mathfrak{g}$ is *semisimple* if it is the direct sum (as an algebra) of simple Lie algebras.

If $\mathfrak{g}$ is semisimple, the corresponding simple Lie algebras in its direct sum are ideals of $\mathfrak{g}$. Not all Lie algebras are known to occur as the Lie algebra of a tensor, but every simple Lie algebra is the derivation algebra of some tensor. We are interested in the simple Lie algebras that arise from hypergraphs. As we see in Chapter 4, and specifically Section 4.2.1, the fact that $\mathrm{Der}(\langle t|)$ is a Lie algebra allows us to make more informed choices for generators. It is from these generators that we construct the different cases in Theorem 4.3.3, and this is why we can check finitely many cases.

2.6 TIZ correspondence

In this section we discuss one of the main results of [11], which allows us to identify features in our data, as in Figure 1.1. We first fix a tensor space $T := \mathbb{F}^{d_0} \otimes \cdots \otimes \mathbb{F}^{d_l}$. We then have transverse operators $\Omega := M_{d_0}(\mathbb{F}) \times \cdots \times M_{d_l}(\mathbb{F})$ and the ring of polynomials $\mathbb{F}[X] := \mathbb{F}[x_0, \ldots, x_l]$. For a fixed set of tensors $S \subset T$ and operators $\Delta \subset \Omega$, define:

$$\mathbf{I}(S, \Delta) \left\{ p(X) := \sum_e \lambda_e X^e \;\middle|\; \forall t \in S,\ \forall \omega \in \Delta,\ \langle t|\, p(\omega) = 0 \right\}$$

where we say $\langle t|\, p(\omega) = 0$ if and only if $\langle t|\, p(\omega)\, |v\rangle = 0$ for all $v \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_l}$. Our goal is to look at the zero loci of $\mathbf{I}(S, \Delta)$. Next, with $P \subset \mathbb{F}[X]$ we define:

$$\mathbf{N}(P, \Delta) := \{t \in T \mid P \subset I(t, \Delta)\}.$$

This is the set of all tensors for which the given polynomials in P lie within $\mathbf{I}(\{t\}, \Delta)$. It can be thought of as tensors satisfying the relation $\langle t|\, p(\omega) = 0$. Finally, we define a set of transverse operators.

$$\mathbf{Z}(S, P) := \{\omega \in \Omega \mid P \subset \mathbf{I}(S, \omega)\}$$

Similarly to $\mathbf{N}(P, \Delta)$, $\mathbf{Z}(S, P)$ is the set of operators satisfying the relation $\langle t|\, p(\omega) = 0$. The following examples give further insight into these three sets.

Example 2.6.1. Let $l = 0$, $T = \mathbb{F}^{d_0}$, and $\Delta = \{\omega\}$. Then $\mathbf{I}(T, \Delta) = (\text{minpoly}_\omega(x) \subset \mathbb{F}[x])$, and the zero loci of $\mathbf{I}(T, \Delta)$ are the zeros of the ideal generated by the minimal polynomial of ω, i.e., the eigenvalues of ω.

Example 2.6.2. Let $l = 0$, $P = (x - \lambda)$, and fix $\omega \in M_{d_0}(\mathbb{F})$. Then $\mathbf{N}((x-\lambda), \omega) = \{t \in \mathbb{F}^{d_0} \mid \omega t = \lambda t\}$ is the eigenspace of ω.

Example 2.6.3. Let $l = 0$, $T = \mathbb{F}^{d_0}$, and $P = (p(x))$. Then $\mathbf{Z}(S, P)$ is the set of matrices ω for which $p(\omega) = 0$, i.e., the matrices which have at least one eigenvalue that is a root of $p(x)$.

The following is the main result of [11]:

Theorem 2.6.4 (First-Maglione-Wilson). Let $P \subset \mathbb{F}[X]$, $S \subset \mathbb{F}^{d_0} \otimes \cdots \otimes \mathbb{F}^{d_l}$, and $\Delta \subset \Omega$. Then $\mathbf{I}(S, \Delta)$ is an ideal of $\mathbb{F}[X]$, $\mathbf{N}(P, \Delta)$ is a subspace of T, and $\mathbf{Z}(S, P)$ is a closed variety of Ω. Furthermore, the following Galois connection holds:

$$S \subset \mathbf{N}(P, \Delta) \iff P \subset \mathbf{I}(S, \Delta) \iff \Delta \subset \mathbf{Z}(S, P)$$

2.6.1 Densors

From the language of Theorem 2.6.4, we have an equivalent definition of the derivation algebra. Recall Definition 2.5.6, which says that for a tensor $\langle t|$, the derivations are tuples of endomorphisms D such that $\sum \langle t|D_\alpha v_\alpha, v_{\overline{\alpha}}\rangle = D_0 \langle t|v\rangle$. We may now view this in the context of TIZ.

By construction, $\delta \in Der(\langle t|)$ if and only if

$$\delta_0 \langle t|v\rangle - \sum_{a=1}^{l} \langle t|\delta_a v_a, v_{\overline{a}}\rangle = 0$$

Let $\langle t|$ be a tensor, and set $p = x_0 - \sum_{i=1}^{l} x_i \in \mathbb{F}[X]$. The derivations of $\langle t|$, denoted by $\mathrm{Der}(\langle t|)$, are the set $\mathbf{Z}(\langle t|, p)$.

Proposition 2.6.5. Let $\langle t|$ be a tensor. Then $\mathrm{Der}(\langle t|) = Z(\langle t|, d)$, where $d = x_1 + \cdots + x_l - x_0$.

Given this proposition, by the TIZ correspondence, there exists a corresponding $N(x_1 + \cdots + x_l - x_0, \mathrm{Der}(\langle t|))$.

Definition 2.6.6. Given a tensor $\langle t| \in T$, we define the *densor space* of $\langle t|$, denoted by $(\!|t|\!)$, as

$$(\!|t|\!) := N(x_1 + \cdots + x_l - x_0, \operatorname{Der}(\langle t|))$$

equivalently,

$$(\!|t|\!) := \{\langle s| \in T \mid \operatorname{Der}(\langle t|) \subset \operatorname{Der}(\langle s|)\}$$

Example 2.6.7. Using the same tensor $\langle t|$ as Example 2.5.3, we compute its densor space. We first compute the derivation algebra. These are operators (A, B, C) such that for all $u \in \mathbb{R}^2$, $v \in \mathbb{R}^2$,

$$-A \langle t|u, v\rangle + \langle t|Bu, v\rangle + \langle t|u, Cv\rangle = 0$$

After a manual calculation similar to Example 2.5.3, we have that the generators of the derivation algebra are

$$\delta_1 = \left(\begin{bmatrix} 2 & 7 \\ 9 & 0 \end{bmatrix}, \begin{bmatrix} 9 & 8 \\ 0 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$
$$\delta_2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$
$$\delta_3 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right),$$
$$\delta_4 = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} \right).$$

We note that δ_2 represents $\langle t|\alpha u, v\rangle = \alpha \langle t|u, v\rangle$, and δ_3 represents $\langle t|\alpha u, v\rangle = \langle t|u, \alpha v\rangle$, which are the multilinear conditions. To compute the densor space we find all tensors $\langle s|$ such that

$$\langle s| \, d(\delta_1) \, |u, v\rangle = 0, \quad \langle s| \, d(\delta_2) \, |u, v\rangle = 0,$$
$$\langle s| \, d(\delta_3) \, |u, v\rangle = 0, \quad \langle s| \, d(\delta_4) \, |u, v\rangle = 0,$$

As δ_2 and δ_3 arise from $\langle t|$ being multilinear, they hold trivially on any choice of a tensor $\langle s| \in T$. Therefore, we only need to focus on δ_1 and δ_4. A generic tensor $\langle s| \in T$ is of the form

$$\langle s| : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrowtail \mathbb{R}^2,$$
$$\langle s|u, v\rangle = \left(u^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} v, u^T \begin{bmatrix} e & f \\ g & h \end{bmatrix} v \right)^T$$

We again perform calculations similar to those used to compute the derivations and the centroid, resulting in 16 equations in 8 variables. After solving this system of linear equations, we have two generators for the densor space:

$$\langle s_1| := \langle s_1|u, v\rangle = \left(u^T \begin{bmatrix} 7 & 14 \\ 0 & 7 \end{bmatrix} v, u^T \begin{bmatrix} 7 & 14 \\ 8 & 7 \end{bmatrix} v \right)^T,$$
$$\langle s_2| := \langle s_2|u, v\rangle = \left(u^T \begin{bmatrix} 9 & 18 \\ 8 & 9 \end{bmatrix} v, u^T \begin{bmatrix} 9 & 18 \\ 0 & 9 \end{bmatrix} v \right)^T.$$

We can see that $\langle t| = \frac{1}{2}(\langle s_2| - \langle s_1|)$. This shows that this subspace of our tensor space indeed contains our original tensor. Thus, $(\!|t|\!) = \langle \langle s_1| \, , \langle s_2| \rangle$.

By the TIZ correspondence, $(\!|t|\!)$ is closed. When we refer to the stratification algorithm, this stratification comes from sampling the densor space of a tensor. The densor space can be thought of as a neighborhood of the tensor, and when we sample from it, we find the generic largest orbit.

These orbits arise from specific actions of the derivations in $\mathrm{Der}(\langle t|)$. As we look for more samples, we need to create more polynomials p to create our **N**-sets. This is where we define chisels.

2.6.2 Chisels

When constructing the densor space, we are allowed to make choices for our polynomial p. Each choice of p, called a *chisel*, provides a new space that we may sample from. We use these chisels to decompose our tensors.

Definition 2.6.8. Let $\langle t| : V_1 \times \cdots \times V_l \rightarrowtail \mathbb{F}$ be a tensor. A *sparsity pattern* is a decomposition of the V_i such that:

- $V_i = \bigoplus W_{j,i}$, where $W_{j,i}$ is a $\mathbb{F}$ vector space,
- After relabeling, $\left\langle t \middle| W_{1,1}, W_{2,1}, W_{\overline{12},a} \right\rangle = 0$ for all a.

The images in Figure 1.1 are generated by this chisel algorithm to find a sparsity pattern as developed in [4]. Currently, the main implementation of this algorithm, as described in [6], selects random elements from the **N**-set's orbits and computes the corresponding null patterns in polynomial time. With Theorem 4.3.3, we have a more systematic and informative way of selecting derivations.

2.7 Symmetry in tensors

As this dissertation studies tensors arising from hypergraphs, these adjacency tensors possess symmetries not present in all tensors. In this section, we formally define these symmetries. First, we briefly discuss covariance and contravariance. In general, a tensor is expressed as

$$\begin{aligned} \langle t| : V_1 \times \cdots \times V_\ell &\rightarrowtail V_0, \\ V_i &\cong \mathbb{F}^{d_i}. \end{aligned}$$

We construct an isomorphism from $\langle t|$ to $(t|$, where

$$\begin{gathered}(t| : V_0 \times V_1 \times \cdots \times V_\ell \rightarrowtail \mathbb{F}, \\ (t|u^0, u_1, \ldots, u_\ell) := u^0 \langle t|u_1, \ldots, u_\ell\rangle, \\ u^0 \in V_0^*, \quad u_i \in V_i,\end{gathered}$$

where V_0^* is the dual space of V_0. We first view this isomorphism in an example.

Example 2.7.1. Let

$$\begin{aligned}\langle t| : \mathbb{R}^3 &\to \mathbb{R}^2 \\ v &\mapsto \begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} v\end{aligned}$$

As these are finite-dimensional vector spaces, there is a natural isomorphism $\mathbb{R}^3 \cong (\mathbb{R}^3)^*$ via the transpose and matrix multiplication. We can then define $(t|$ as:

$$\begin{aligned}(t| : \mathbb{R}^2 \times \mathbb{R}^3 &\rightarrowtail \mathbb{R} \\ (t|u, v) &\mapsto u \langle t|v\rangle = u^T \left(\begin{bmatrix} 1 & 2 & 3 \\ -2 & 1 & 4 \end{bmatrix} v \right)\end{aligned}$$

Figure 2.5 shows diagrams of the transformations $Y_i : V_i \to U_i$ we apply to our tensors.

In Figure 2.5, $\hat{Y}_0$ is the contravariant transformation to Y_0. In this dissertation, our transformations are isomorphisms, and all vector spaces are finite-dimensional, in which case we can express $(t|$ without the dual space V_0^*, as shown in Figure 2.6.

The isomorphism between $\langle t|$ and $\langle s|$ in general is made through partial evaluations and the isomorphism $\mathbb{F}^n \cong (\mathbb{F}^n)^*$. Furthermore, this isomorphism allows us to choose between $\langle t|$ and $\langle s|$.

Definition 2.7.2. A tensor $\langle t| : V_1 \times \cdots \times V_\ell \rightarrowtail \mathbb{F}$ is *symmetric* if

1. $V_i = V_j$ for all i, j, and

$$
\begin{array}{ccccccccc}
V_1 & \times & V_2 \times & \cdots & \times & V_\ell & \xrightarrow{\langle t|} & V_0 \\
\downarrow{\scriptstyle Y_1} & & \downarrow{\scriptstyle Y_2} & & & \downarrow{\scriptstyle Y_l} & & \downarrow{\scriptstyle Y_0} \\
U_1 & \times & U_2 \times & \cdots & \times & U_l & \longrightarrow & U_0
\end{array}
$$

$$
\begin{array}{cccccccccc}
V_0^* \times & V_1 & \times & V_2 \times & \cdots & \times & V_\ell & \xrightarrow{(t|} & \mathbb{F} \\
\uparrow{\scriptstyle \hat{Y}_0} & \downarrow{\scriptstyle Y_1} & & \downarrow{\scriptstyle Y_2} & & & \downarrow{\scriptstyle Y_l} & & \| \\
U_0^* \times & U_1 & \times & U_2 \times & \cdots & \times & U_l & \longrightarrow & \mathbb{F}
\end{array}
\quad .
$$

Figure 2.5: Transformations on tensors which highlight covariance and contravariance

$$
\begin{array}{cccccccccc}
V_0^* \times & V_1 & \times & V_2 \times & \cdots & \times & V_\ell & \xrightarrow{(t|} & \mathbb{F} \\
\downarrow{\scriptstyle Y_0} & \downarrow{\scriptstyle Y_1} & & \downarrow{\scriptstyle Y_2} & & & \downarrow{\scriptstyle Y_l} & & \| \\
U_0^* \times & U_1 & \times & U_2 \times & \cdots & \times & U_l & \longrightarrow & \mathbb{F}
\end{array},
$$

Figure 2.6: (t| expressed without the dual space

2. $\langle t|v\rangle = \langle t|v_{\sigma(1)}, \dots, v_{\sigma(l)}\rangle$ for all σ in the symmetric group S_l.

In other words, a symmetric tensor is one in which all inputs come from the same vector space, and permuting the inputs does not change the final calculation. This is an analog of symmetric matrices and is important for studying adjacency tensors, since we do not assume an ordering of vertices.

Example 2.7.3. Let $\langle t| : V_1 \times V_2 \rightarrowtail \mathbb{F}$. Then $\langle t|$ is symmetric as a tensor if and only if it is symmetric as a matrix.

When a tensor is defined via a higher-valence array, we obtain an equivalent definition.

Definition 2.7.4. Let $\langle t| : \overbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}^{\ell} \to \mathbb{F}$ be a tensor. The *Gram array* $A \in \mathbb{F}^{\overbrace{n \times \cdots \times n}^{\ell}}$ for $\langle t|$ is a multiway array, where

$$A(x_1, x_2, \dots, x_\ell) := \langle t|e_{x_1}, \dots, e_{x_\ell}\rangle .$$

The Gram array is a generalization of a multiplication table. In this instance, the multiplication may involve more than two elements.

Example 2.7.5. Let X, Y, and Z be finite enumerated lists of nonzero vectors. We can construct a parallelepiped by choosing one vector from each list. Given $u_i \in X$, $v_j \in Y$, and $w_k \in Z$, the volume of the parallelepiped defined by u v and w is given by $u \cdot (v \times w)$. We then construct a 3-way array $A \in \mathbb{R}_{\geq 0}^{n_1 \times n_2 \times n_3}$, where $A(i, j, k)$ is equal to the volume of the parallelepiped defined by the vectors u_i, u_j, and u_k.

Definition 2.7.6. Let $A : \overbrace{[n] \times \cdots \times [n]}^{\ell} \to \mathbb{F}$ be the ℓ-valent Gram array for the tensor $\langle t|$. Then $\langle t|$ is *symmetric* if

$$A(x_1, x_2, \ldots, x_\ell) = A(x_{\sigma(1)}, \ldots, x_{\sigma(\ell)})$$

for all permutations σ in S_ℓ.

If $\langle t|$ is an inner product, this definition coincides with that of a Gram matrix. Furthermore, in this dissertation, we discuss non-associative and non-commutative products. This motivates our explicit definition of symmetric tensors with respect to permutations of the associated Gram array.

Given a symmetric tensor $\langle t| : V \times \cdots \times V \rightarrowtail \mathbb{F}$, we refer to the following related tensor as *symmetric*:

$$\langle s| : V^{\ell-1} \rightarrowtail V^*$$
$$\langle s|v_1, \ldots, v_{\ell-1}\rangle := \langle t|v_1, \ldots, v_{\ell-1}, \cdot\rangle\,,$$

where $\langle s|$ is the partial evaluation of $\langle t|$ with the first $\ell - 1$ inputs $(v_1, \ldots, v_{l-1})$. As before, $V^* \cong V$, so we may write $\langle s| : V^{\ell-1} \rightarrowtail V$. We refer to $\langle t|$ and $\langle s|$ as representing the same data.

Another instance of symmetry is found in the operators acting on a tensor.

Definition 2.7.7. Let $\omega \in \prod_{i=0}^{\ell} \mathrm{End}(V)$ be a transverse operator on a symmetric tensor $\langle t| : V^\ell \rightarrowtail V$. ω is a *symmetric operator* if $\omega_i = \omega_j$ for all $0 \leq i, j \leq \ell$.

When $\omega_0 \in \mathrm{End}(V)$, the symmetric operator ω applies a change of basis to the vector space V on which $\langle t|$ acts. These symmetric operators define the cases of Theorem 4.3.3.

Chapter 3

Hypergraphs

In this chapter, we formally define hypergraphs and establish a spectral theory to study adjacency matrices and tensors. From here on, we also make a notational change to our tensors and tensor evaluations. We write

$$\Gamma(u, v, w) := \langle t|u, v, w\rangle \,.$$

This is to more closely align our notation with that of an inner product (u, v) or typical function evaluations $f(x, y)$.

3.1 What is a hypergraph

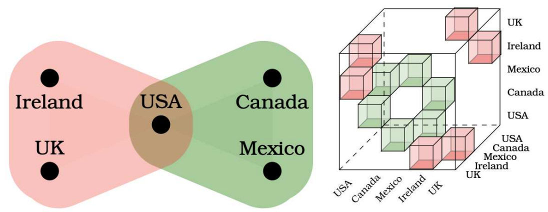
In this section, we establish the notation we use to describe hypergraphs. This follows a notation similar to [3].

Definition 3.1.1. A *hyperedge* $h \subset V$ is a subset of vertices. A *hypergraph* $\mathcal{H}$ is a collection of hyperedges.

Example 3.1.2. A graph is an example of a hypergraph in which all hyperedges have either two elements (edges) or one element (an isolated vertex).

Example 3.1.2 highlights an important distinction present in this dissertation. We define a hypergraph $\mathcal{H}$ to be a subset of 2^V. Crucially, the elements of $\mathcal{H}$ are subsets of V *without* repetition. For a graph an edge with one element is not a loop edge, as this could be represented as a subset $\{x, x\}$. As such we only consider graphs without loops, and hypergraphs without "loops".

Example 3.1.3. Trade agreements provide another example of hypergraphs. In Figure 3.1 each hyperedge represents a trade agreement, while each vertex represents a distinct nation.



	Red	Green
USA	1	1
Canada	0	1
Ireland	1	0
UK	1	0
Mexico	0	1

Figure 3.1: Trade agreement hypergraph [13]

We provide three different visualizations of this hypergraph. First, we have a pictorial representation in which each filled-in shape (here, triangles) represents a trade agreement, and each vertex corresponds different nation. We then represent this data as an adjacency multiway array. Finally, we use an incidence matrix, where each row represents a nation, and each column a trade agreement. We record which nations are included in each agreement by placing a value of 1 in the appropriate cells.

In Example 3.1.3, we chose to represent a hypergraph using a multiway array. However, it is not always possible to do this. Consider the hypergraph with the incidence matrix

	Edge 1	Edge 2
Vertex A	1	0
Vertex B	1	
Vertex C	1	1
Vertex D	0	1

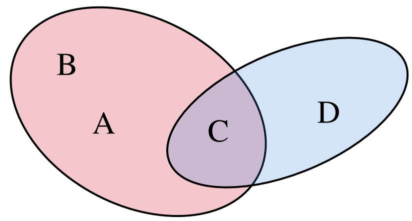


Figure 3.2: Example non uniform hypergraph

This hypergraph is shown in Figure 3.2. When constructing a multiway array to encode hypergraph data, if not all hyperedges have the same number of vertices, we cannot record all hyperedges. For this reason, we work with *uniform* hypergraphs.

Definition 3.1.4. Let $\mathcal{H}$ be a hypergraph. $\mathcal{H}$ is ℓ*-uniform* if $|h| = \ell$ for all $h \in \mathcal{H}$.

Proposition 3.1.5. There exists a bijection between the set of all hypergraphs and the set of all labeled uniform hypergraphs with at most two labels.

Proof. Let $\mathcal{H}$ be an arbitrary hypergraph. Define $k = \max_{h \in \mathcal{H}} |h|$. For each hyperedge e which contains $j < k$ vertices, adjoin to it $k - j$ new vertices $(w_e)_1, \ldots, (w_e)_{k-j}$, so that

$$\{v_1, v_2, \ldots, v_k\} \mapsto \{v_1, v_2, \ldots, v_k, (w_e)_1, (w_e)_2, \ldots, (w_e)_{k-j}\}.$$

For each hyperedge, we use distinct $w_i's$ to construct our new hypergraph. For our labeling of $\overline{\mathcal{H}}$, we label our original vertices A and the newly adjoined vertices B.

For any labeled uniform hypergraph with two labels, we choose one of the labels and, for each hyperedge, remove all vertices with that label. The resulting object is an unlabeled hypergraph. □

A different approach is to add a single point at infinity to 'complete' each hyperedge, as was done by [19] when embedding the hypergraph into a lower-dimensional space to determine the closeness of communities. In this dissertation, we assume that multiple distinct points are added so that we do not inadvertently introduce further relationships among hyperedges beyond those in the original hypergraph. With Lemma 3.1.5, we can regard all hypergraphs as uniform, provided we retain the information of which vertices were added after applying the bijection.

We can now formally define the *adjacency tensor*.

Definition 3.1.6. For a given ℓ-uniform hypergraph $(V, \mathcal{H})$, where $V = \{v_1, \ldots, v_n\}$, the associated *adjacency tensor* is the multilinear map

$$\Gamma : \overbrace{\mathbb{F}^n \times \cdots \times \mathbb{F}^n}^{\ell} \to \mathbb{F},$$
$$\Gamma(e_{i_1}, e_{i_2}, \ldots, e_{i_\ell}) := \begin{cases} 1 & \{v_{i_1}, v_{i_2}, \ldots, v_{i_\ell}\} \in \mathcal{H} \\ 0 & else \end{cases}.$$

If $\mathcal{H}$ is not ℓ-uniform, we define Γ_ℓ as the adjacency tensor to $\mathcal{H}_\ell := \{e \in \mathcal{H} \mid |e| = \ell\}$. Furthermore, given a fixed labeling of the vertices of V as $\{1, 2, \ldots, n\}$, we can express Γ as

$$\Gamma' : \overbrace{V \times \cdots \times V}^{\ell} \to \mathbb{F},$$
$$\Gamma'(i_1, i_2, \ldots, i_\ell) := \Gamma(e_{i_1}, e_{i_2}, \ldots, e_{i_\ell}).$$

via an embedding of $V \to \mathbb{F}^n$ given by $i \mapsto e_i$.

Currently, the adjacency tensor of a graph is simply an array. More work is needed before we can call this object a tensor. In Section 3.2, we develop this justification for both adjacency matrices and adjacency tensors.

Finally, to prove Theorem 4.5.3, we discuss hypergraph contractions and inflations.

Definition 3.1.7. Let $\mathcal{H}$ be a hypergraph on a vertex set V, and let $X \subset V$. The *contraction* of $\mathcal{H}$ along X, denoted by $\mathcal{H} - X$, is the hypergraph:

$$\mathcal{H} - X := \{E - X \mid E \in \mathcal{H}\}.$$

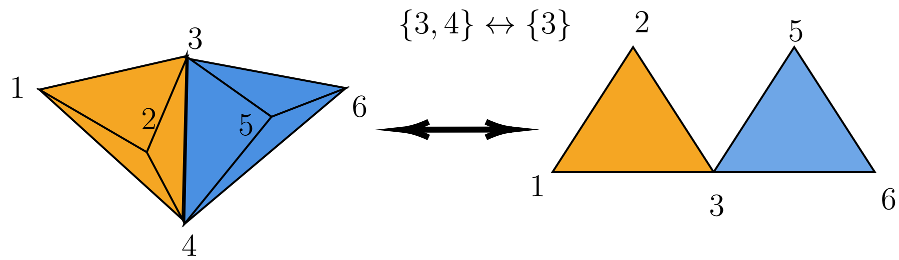


Figure 3.3: Contracting hyperedges in a hypergraph along the vertex labeled "4"

We note that $\mathcal{H} - X$ may not be uniform. Now, similarly, an *inflation* of $\mathcal{H}$ along X on hyperedges E, denoted by $(\mathcal{H} + X)_{\{E\}}$, is the hypergraph

$$(\mathcal{H} + X)_{\{E\}} := \{h \in \mathcal{H} \mid h \notin E\} \cup \{h \cup X \mid h \in E\}.$$

The process of contraction contracts all hyperedges incident to the specified vertices, while inflation is specific to hyperedges and the inclusion of new vertices in those hyperedges. Figure 3.3 shows the process of contracting.

3.2 Spectral theory of hypergraphs

In this section, we justify why we can call the multiway array that records the adjacency of a hypergraph a *tensor*. We begin with the adjacency matrix of a graph, as it is a well-studied and commonly known example. We assume that all graphs are undirected and thus their adjacency matrix is symmetric.

3.2.1 Adjacency matrices

Given a graph $G = (V, E)$ with n vertices, we construct the adjacency matrix $A \in \mathbb{C}^{n \times n}$ as follows:

$$(A)_{ij} := \begin{cases} 1 & (i, j) \in E \\ 0 & else \end{cases}$$

If A were a matrix, we would be justified in applying matrix operations to it. However, we do not apply such operations, as row and column operations do not hold meaning with respect to the underlying graph. What remains is taking powers of the adjacency matrix. A^2 records the number of walks of length 2 from i to j, given by $(A^2)_{ij}$. In general, $(A^k)_{ij}$ is the number of walks of length k from i to j. In this sense, it is not row operations but rather dot products that inform us about the graph. For two square matrices, $(AB^T)_{ij} = a_i \cdot b_j$, where a_i is row i of A, b_j is row j of B, and $\cdot$ is the standard dot product. For the adjacency matrix of a graph,

$$(A^2)_{ij} = (AA^T)_{ij} = a_i \cdot a_j.$$

Returning to row i of A, this has a nonzero value at position k if and only if $(i, k) \in E$. This means that for the dot product $a_i \cdot a_j = \sum_{k=1}^{n} (a_i)_k (a_j)_k$, nonzero terms only appear when both $(i, k) \in E$ and $(j, k) \in E$, i.e., vertices i and j share a neighbor. Since the number of walks between vertices i and j is exactly equal to the number of shared neighbors between i and j, this validates our interpretation of A^2 as counting the number of length-2 walks between vertices. Continuing with this change of perspective, A^k can be seen as utilizing dot products to count these 'neighbor of a neighbor' or 'friend of a friend' relations between vertices (where repetitions are allowed).

We now perform a similar computation for adjacency tensors. We first define a typical inner product for 1-valent slices of an adjacency tensor and then see how this extends to higher-valent slices. Let Γ be the adjacency tensor for $\mathcal{H} = \{\{1, 2, 3\}, \{3, 4, 5\}\}$. Recall Definitions 2.3.3 and 2.3.4 for the Frobenius product. Consider the two slices $\Gamma_{1,2}$ and $\Gamma_{4,5}$. When viewed as multiway arrays, $\Gamma_{1,2} \cong (0, 0, 1, 0, 0, 0)$ and $\Gamma_{3,4} \cong (0, 0, 1, 0, 0, 0)$. The Frobenius inner product of these two slices is equal to 1, which can be restated as 'there is exactly one vertex which is in a hyperedge with both $\{1, 2\}$ and $\{4, 5\}$.' This is our first indication that we are justified in calling adjacency multiway arrays *tensors*.

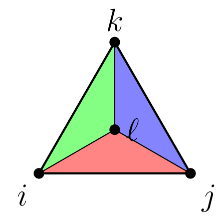


Figure 3.4: A reduced barycentric subdivision

3.3 Spectral reduced barycenters

We continue our study of the spectral properties of adjacency tensors from Section 3.2 and recall Definition 2.3.4 for the barycentric tensor. When the tensor arises as an adjacency tensor of a hypergraph, we refer to the *barycentric adjacency tensor*. As we have named these barycentric, we show what geometric feature they count. Let Γ be the adjacency tensor of a 3-uniform hypergraph. If we consider each term $\Gamma^{(3,1)}_{ijk}$, we get a nonzero value if and only if $\{i,j,\ell\}, \{i,\ell,k\}, \{\ell,j,k\} \in \mathcal{H}$ for some $\ell \in V$. This is analogous to the barycentric subdivision of the simplex $\{i,j,k\}$. We create a fourth point ℓ at the barycenter of the face $\{i,j,k\}$, and create three new simplices that contain ℓ as one of its points. This is not a true barycentric subdivision, however, as we do not subdivide $\{i,j\}$, $\{i,k\}$, and $\{j,k\}$. We refer to this as a *reduced barycentric subdivision*. This subdivision appears in Figure 3.4.

Another key distinction is that $\{i,j,k\}$ itself does not necessarily serve as a hyperedge in $\mathcal{H}$ for this construction. This allows the barycentric adjacency tensor to find relations that may be missed in a traditional barycentric subdivision. From here, $(\Gamma^{(\ell,1)})_{a_1 \ldots a_\ell}$ counts the number of vertices which can be used to identify a barycentric subdivision of the simplex $\{a_1, \ldots, a_\ell\}$. By Lemma 2.3.5, this is a tensor.

As with characteristic polynomials of a matrix, we can define a characteristic polynomial for these barycentric tensors.

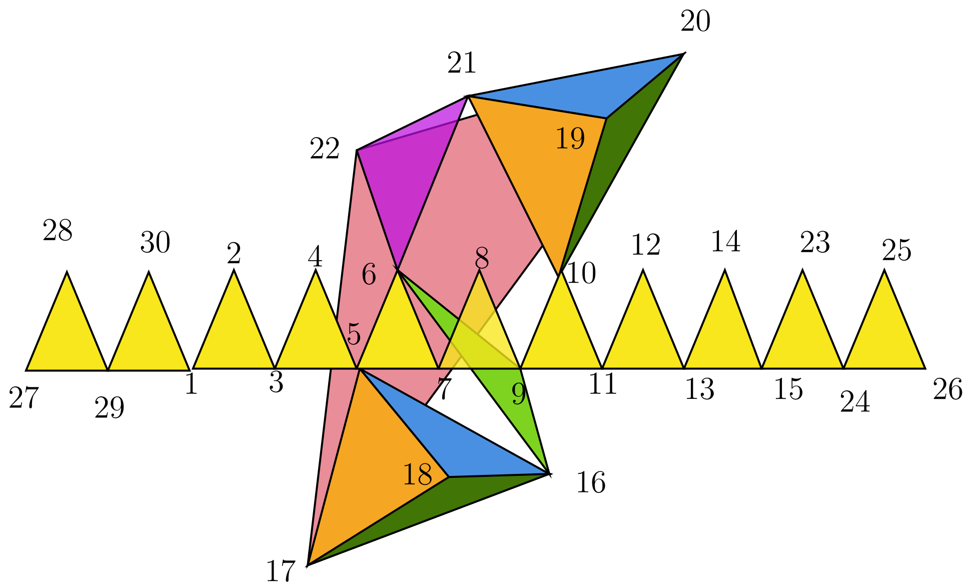


Figure 3.5: A hypergraph with structure

Theorem 3.3.1. Let Γ be the adjacency tensor of an l-uniform hypergraph $\mathcal{H}$ and let $\Gamma^{(\ell,i)}$ be the i^{th} iterated barycentric tensor of Γ. Then there exists a unique minimal finite k such that

$$\alpha_0\Gamma + \sum_{i=1}^{k} \alpha_i \Gamma^{(\ell,i)} = 0$$

We refer to this as the *barycentric characteristic polynomial*.

Proof. Let $\Gamma^{(\ell,i)}$ be the i^{th} iterated barycentric tensor of Γ. Each $\Gamma^{(\ell,i)}$ and Γ is an element of the tensor space $T = \overbrace{\mathbb{F}^n \otimes \cdots \otimes \mathbb{F}^n}^{\ell}$. T is finite dimensional with a basis $\{e_{i_1} \otimes \cdots e_{i_\ell}\}$. Once we calculate $\Gamma^{(\ell,n^\ell)}$, we obtain a list of $n^\ell + 1$ elements in an n^ℓ-dimensional space, and thus a linear relationship exists among the set of tensors. □

Example 3.3.2. Consider the 3-uniform hypergraph in Figure 3.5.

After we compute the barycentric subdivision, we obtain a tensor which we interpret as an adjacency tensor. Figure 3.6 shows the corresponding hypergraph.

The second barycentric adjacency tensor matches the first, so we get the linear relation $\Gamma^{(\ell,2)} - \Gamma^{(\ell,1)} = 0$; i.e., $X^2 - X = 0$.

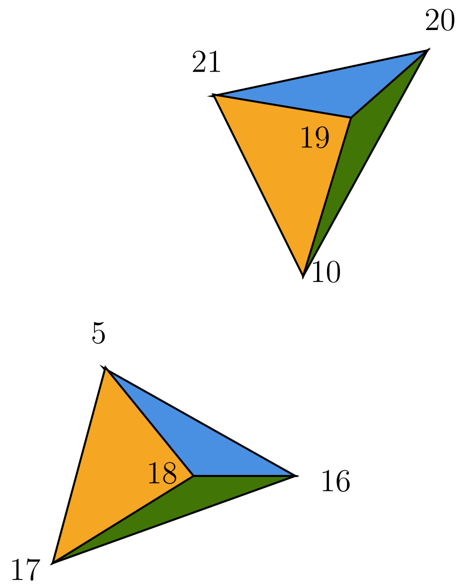


Figure 3.6: First barycentric adjacency tensor

Example 3.3.3. We consider a series of iterated reduced barycentric subdivisions as shown in Figure 3.7:

The barycentric characteristic polynomials for these reduced barycentric subdivisions are:

$$X,\ X^2,\ X^3,\ X^4, \ldots$$

The barycentric characteristic polynomials count the number of iterations of taking the reduced barycentric subdivisions of the hyperedge $\{i, j, k\}$.

Example 3.3.4. We define a set of hypergraphs as seen in Figure 3.8

where the hyperedges are $\{1, 2, 4\}$, $\{2, 3, 4\}$, $\{1, 4, 5\}$, $\{3, 4, 5\}$, and in general $\{1, i, i + 1\}$ and $\{3, i, i + 1\}$ until $\{1, 3, n + 3\}$. We start from the hypergraph with three hyperedges $\{1, 2, 4\}$, $\{2, 3, 4\}$, $\{1, 4, 5\}$, and add more nested hyperedges; the barycentric characteristic polynomials are

$$X^3 - X$$

$$X^5 - 3X^3 + 2X$$

$$X^7 - 4X^5 + 4X^3 - X$$

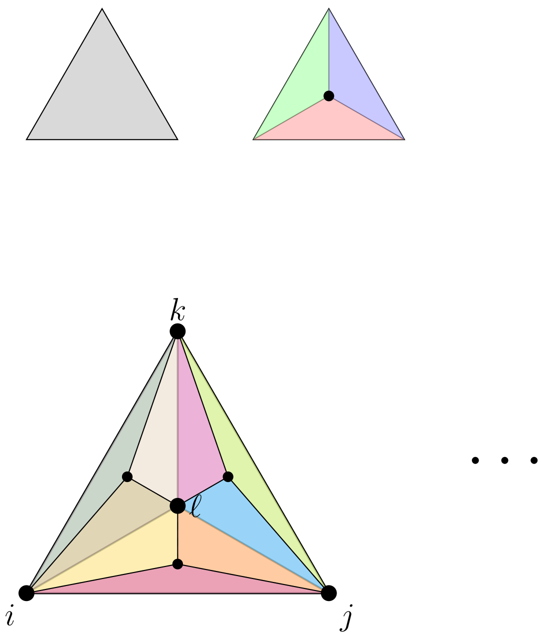


Figure 3.7: Iterated reduced barycentric subdivisions

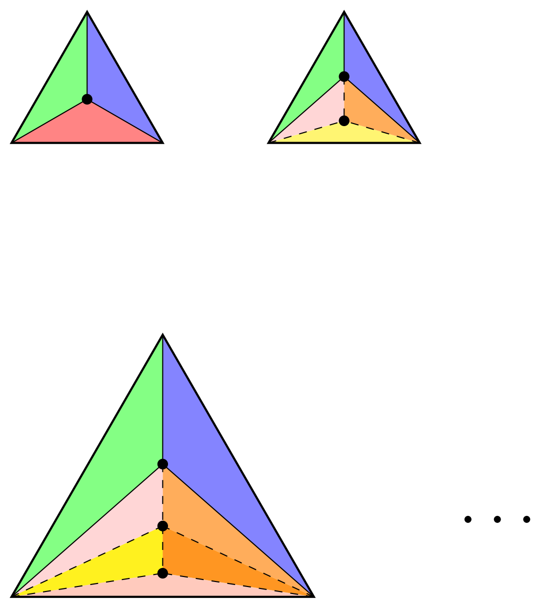


Figure 3.8: Hypergraphs that have nested barycentric subdivision

$$X^5 - 4X^3 + 3X$$

$$X^9 - 6X^7 + 11X^5 - 7X^3 + X$$

$$X^9 - 7X^7 + 16X^5 - 14X^3 + 4X$$

$$X^9 - 7X^7 + 15X^5 - 10X^3 + X$$

$$X^{11} - 9X^9 + 29X^7 - 41X^5 + 25X^3 - 5X$$

and so on.

We note that this linear relation is reminiscent of a characteristic polynomial. For the adjacency matrix of a graph A, Theorem 3.3.1 refers simply to the characteristic polynomial, where $\Gamma^{(2,i)}$ is A^i. Our interpretation of A^i is simplified by the fact that matrices are 2-valent. A^i is the composition of linear transformations, so we can clearly define $A^i u = \lambda u$ to define eigenvalues, and it can be shown that these eigenvalues are the roots of the characteristic polynomial. There is an immediate challenge once the valence of our tensor is greater than 2. A higher valence tensor $\langle t|$ must have either at least two inputs or at least two outputs. From here, there is no longer a canonical choice for an iteration. The work of [18] and [15] made the choice to consider $\langle t|u, u, \ldots, u\rangle = \lambda^\ell u$ to define eigenvalues; however, this restricts us to considering our tensors as functions with a single input $F(u) := \langle t|u, u, \ldots, u\rangle$. The goal of this dissertation is to consider adjacency tensors as the algebraic objects they are, and these additional features lead us to future avenues of research.

We end this chapter by providing additional context on why we use the derivation algebra to study hypergraphs. As shown by [4] and implemented in [6], the simple subalgebras of the derivation algebra can be used in a clustering algorithm. The derivation algebra can be embedded into a sufficiently large matrix algebra $M_p(\mathbb{R})$, and the Jordan normal form of the space corresponds to the semisimple decomposition [12]. Previous work shows how these simple subalgebras, specifically the diagonal elements, can be used to define quotients of a hypergraph [7]. In this dissertation, we consider derivations which we think of as upper-triangular. In this sense, we have nilpotent elements and thus recursive algorithms on the derivations terminate, as seen with the barycentric adjacency tensor. In Chapter 4, we directly compare generic forms of symmetric derivations and the configurations of vertices in a hypergraph. This is similar in form to the work of [7], although in this dissertation, we consider non-diagonal matrices.

Chapter 4

Derivations for hypergraph features

In this chapter, we prove Theorem 4.3.3. This theory builds on the work of [11] [5] [4] on derivation algebras. While this dissertation has focused on 3-uniform hypergraphs, we provide a template for extending this theory to ℓ-uniform hypergraphs via Theorem 4.5.3.

4.1 Motivating derivations

We return to the derivation algebra to provide an alternative perspective on its role. Consider the tensor

$$\Gamma = \left[\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{\{1\}}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\{x\}} \right].$$

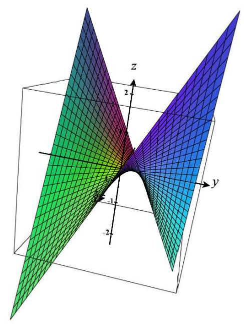
The labels 1 and x are used to track the slices of our $2 \times 2 \times 2$ tensor. This information can be stored in a multiplication table.

$*$	1	x
1	1	x
x	x	-1

This translation happened by viewing 1 and x as the vectors $(1, 0)$ and $(0, 1)$. This multiplication table expresses $\mathbb{R}[x]/(x^2+1) = \mathbb{C}$. This tells us that our tensor encoded the complex numbers, as we deduced by viewing it as a multiplication table. This motivates our study of products. We can then look at the simplest product. Take

$$z = f(x, y) = xy.$$

As a first step, we would graph this function.



We can then study the curvature of this function. This is done by computing derivatives. Since we have a product, we need to work with the product rule for derivatives:

$$\frac{d}{dx}(f(x)g(x)) = \left(\frac{d}{dx}f(x)\right)g(x) + f(x)\left(\frac{d}{dx}g(x)\right)$$

Let us now rewrite this rule. Define $f(x)g(x) := \Gamma(f(x), g(x))$, and $D := \frac{d}{dx}$. Then the product rule is

$$D\Gamma(f(x), g(x)) = \Gamma(Df(x), g(x)) + \Gamma(f(x), Dg(x)),$$

which aligns with the definition of a derivation:

$$\left\{D \in \mathrm{End}(V_0) \times \prod_{i=1}^{\ell} \mathrm{End}(V_i) \mid \sum \langle \Gamma \mid D_\alpha v_\alpha, v_{\overline{\alpha}}\rangle = D_0\langle \Gamma \mid v\rangle\right\}.$$

In practice, the derivation algebra is computed by solving large linear systems. We have $\prod_{i=0}^{\ell}(\dim(V_i))^2$ variables and a system of $\prod_{i=1}^{\ell}\dim(V_i)$ equations. We can give a lower bound for the dimension of the derivation algebra.

Lemma 4.1.1. Let $\langle t| : V_1 \times \cdots \times V_\ell \rightarrowtail \mathbb{F}$ be a tensor. Then $\dim(\mathrm{Der}(\langle t|)) \geq l$.

Proof. We show that there exist at least ℓ linearly independent derivations of $\langle t|$. As $\langle t|$ is multilinear, we have that

$$\langle t|\alpha u_a, u_{\overline{a}}\rangle = \alpha \langle t|u\rangle .$$

This can be rewritten as

$$\langle t|(\alpha I)u_a, u_{\overline{a}}\rangle = \alpha \langle t|u\rangle .$$

This allows us to generate ℓ derivations

$$\begin{aligned}&(\alpha, \alpha I, 0, \dots, 0),\\ &(\alpha, 0, \alpha I, \dots, 0),\\ &\vdots\\ &(\alpha, 0, \dots, \alpha I).\end{aligned}$$

These are linearly independent, and as such, the dimension of $\mathrm{Der}(\langle t|)$ is at least ℓ. □

In Section 4.2, we review the theory of decomposing Lie algebras in support of Theorem 4.3.3.

4.1.1 Revisiting our case study

Before beginning the proof of Theorem 4.3.3, we revisit the case study introduced in Section 1.1. Recall that the vertices of our hypergraph were students, and hyperedges were created by constructing cliques of students that were near one another during a 20-second time period. This information was stored in an adjacency tensor Γ, and the derivation algebra $\mathrm{Der}(\Gamma)$ was then computed. As detailed in Section 4.1, we solve a system of linear equations to obtain the full derivation algebra. In implementing this chiseling algorithm, a random element $(D_0, D_1, D_2, D_3) \in \mathrm{Der}(\Gamma)$ is used. The projections are as follows:

$$V \to D_1V, \quad V \to D_2V, \quad V \to D_2V, \quad \mathbb{F} \to D_0\mathbb{F}.$$

Once this is done, we reconstruct the tensor as $\hat{\Gamma} : D_1V \times D_2V \times D_3V \rightarrowtail D_0\mathbb{F}$. This new tensor $\hat{\Gamma}$ is the second image in Figure 1.1. The minimum dimension of the derivation algebra is three, and for a random 3-valent tensor its derivation algebra is close to three-dimensional. The fact that the derivation algebra is much larger than expected tells us that there are additional symmetries present in the adjacency tensor, which are being detected by the endomorphisms in the derivation algebra.

4.2 Generating Lie algebras

As our goal is to use the Lie algebra of derivations arising from the adjacency tensor of a hypergraph, we take a moment to study these derivations as Lie algebras. To this end, we establish a background on Lie algebras. This allows us to focus our study on matrices with support of at most three. Recall that the support of a matrix is the number of nonzero values in the matrix.

To begin, we assert that we consider our derivation algebras with respect to the natural basis. By construction, we map the vertices of our hypergraph to the standard basis vectors. From here, when we construct our adjacency tensor, we do so to act on tuples of these standard basis vectors. Finally, the derivation algebra is constructed as tuples of matrices that act on these standard basis vectors. While further work may be done to study other representations of the Lie algebra of derivations of the adjacency tensor, that is beyond the scope of this dissertation. By focusing on the natural and adjoint representations, we obtain useful constraints on the basis elements.

Lemma 4.2.1 (Chevalley [8])**.** Let $\mathfrak{L}$ be a finite dimensional simple Lie algebra. There exists a basis $\mathcal{U}$ of $\mathfrak{L}$ such that for all $u \in \mathcal{U}$, the support of u is less than or equal to three.

The proof of this lemma is modeled from [12]. Furthermore, the representations of the simple Lie algebras in Theorem 4.2.1 are in the natural or adjoint representation. As referenced in 1.0.1, different representations of simple Lie algebras have different bounds for the support of their generators. Finally, for the purposes of this proof we express our vectors as row vectors, so that our products are xA as opposed to Ax.

Proof. We consider the generators of the simple Lie algebras.

-A_n, or $\mathfrak{sl}_n$. The elements of $\mathfrak{sl}_n$ can be identified with $n \times n$ matrices of trace zero. We denote the standard basis of $n \times n$ matrices by E_{ij}, where

$$E_{ij}E_{km} = \delta_{jk}E_{im},$$
$$\sum E_{ii} = I_n.$$

We can express a basis for $\mathfrak{sl}_n$ as

$$H_k = E_{kk} - E_{nn}, \quad k < n,$$
$$E_{ij}, \quad i \neq j,\ 1 \leq i, j \leq n.$$

-B_n, or $\mathfrak{so}_{2n+1}$. Let $\mathcal{V}$ be an $2n + 1 = m$ dimensional vector space, with a non-degenerate symmetric bilinear form (x, y). Let $\mathfrak{L}$ be the Lie algebra of matrices A that are skew-symmetric in relation to (x, y), i.e. $(xA, y) = -(x, yA)$. Let $\{u_1, \ldots, u_m\}$ be a basis for $\mathcal{V}$, and denote the Gram matrix by

$$S = (\sigma_{ij}),$$
$$(u_i, u_j) = \sigma_{ij}.$$

A matrix A is an element of $\mathfrak{L}$ if and only if $(u_iA, u_j) = -(u_i, Au_j)$ for all $1 \leq i, j, m$. We can express u_iA by

$$u_iA = \sum_k \alpha_{ik}u_k,$$

which allows us to rewrite these conditions as

$$\sum_k \alpha_{ik}\sigma_k j = -\sum_k \sigma_{ij}\alpha_{jk},$$
$$XS = -SX^T, \tag{4.1}$$
$$X = (\alpha_{ij}).$$

As this form is of maximal Wit index, we can choose a basis so that

$$S = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & \sigma I_n \\ 0 & \sigma I_n & 0 \end{pmatrix},$$

Where σ is some nonzero element of $\mathbb{F}$. We can rescale S so that

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix},$$

We can then partition X so that

$$X = \begin{pmatrix} \alpha_{11} & v_1 & v_2 \\ w_1 & Y_{11} & Y_{12} \\ w_2 & Y_{21} & Y_{22} \end{pmatrix},$$

Where $Y_{ij} \in \mathbb{F}^{n\times n}$, and $v_1, v_2, w_1^T, w_2^T \in \mathbb{F}^{1\times n}$. Then equation 4.1 holds if and only if

$$w_1 = -v_2^T, \tag{4.2}$$
$$w_2 = -v_1^T, \tag{4.3}$$
$$Y_{22} = -Y_{11}^T, \tag{4.4}$$

$$Y_{12}^T = -Y_{12}, \tag{4.5}$$

$$Y_{21}^T = -Y_{21}, \tag{4.6}$$

$$\alpha_{11} = 0. \tag{4.7}$$

From these conditions, we get the following set of basis elements for $\mathfrak{L}$,

- Equation 4.4 gives us the elements $H_i = E_{i+1,i+1} - E_{i+n+1,i+n+1}$ and $F_{i,-j} = E_{j+1,i+1} - E_{i+n+1,j+n+1}$ for $i \neq j$,
- Equation 4.6 gives us the elements $F_{i,j} = E_{i+n+1,j+1} - E_{j+n+1,i+1}$ for $i < j$,
- Equation 4.5 gives us the elements $F_{-i,-j} = E_{j+1,i+n+1} - E_{i+1,j+n+1}$ for $i < j$,
- Equation 4.3 Gives us the elements $F_i = E_{1,i+1} - E_{i+n+1,1}$, and
- Equation 4.2 gives us the elements $F_{-i} = E_{i+1,1} - E_{1,i+n+1}$,

for $1 \leq i, j \leq n$.

-C_n, or $\mathfrak{sp}_{2n}$. Let $\mathcal{V}$ have dimension $2n = m$, and let (x, y) be a non-degenerate skew-symmetric bilinear form on $\mathcal{V}$. Let $\mathfrak{L}$ be the Lie algebra of matrices that are skew-symmetric relative to (x, y). Choose a basis $\{u_1, \ldots, u_m\}$ for $\mathcal{V}$ so that the Gram matrix $Q = ((u_i, u_j))$ is

$$Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We can identify $\mathfrak{L}$ with the Lie algebra of matrices $A \in \mathbb{F}^{m \times m}$ such that $AQ = -QA^T$, which implies that

$$A = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \tag{4.8}$$

$$X_{ij} \in \mathbb{F}^n \tag{4.9}$$

$$X_{22} = -X_{11}^T, \tag{4.10}$$

$$X_{12}^T = X_{12}, \tag{4.11}$$

$$X_{21}^T = X_{21}. \tag{4.12}$$

These equations allow us to define the following basis:

- from Equation 4.10 we get the elements $H_i = E_{ii} - E_{n+1,n+i}$ and $F_{j,-i} = E_{ij} - E_{j+n,i+n}$ where $i \neq j$,
- from Equation 4.11 we get the elements $F_{-i,-j} = E_{i,j+n} + E_{j,i+n}$ where $i < j$ and $F_{-2i} = E_{i,i+n}$, and
- from Equation 4.12 we get $F_{i,j} = E_{i+n,j} + E_{j+n,i}$ where $i < j$ and $F_{2i} = E_{i+n,i}$,

where $1 \leq i, j \leq n$.

-D_n, or $\mathfrak{so}_{2n}$. Let $\mathcal{V}$ be $2n = m$ dimensional with a symmetric non-degenerate bilinear form (x, y) of maximal Witt index. Choose a basis $\{u_i\}$ such that the Gram matrix T has the form

$$T = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

and $\mathfrak{L}$ is expressed as the Lie algebra of matrices A that satisfy $AT = -TA^T$. These are matrices of the form

$$A = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$$

$$X_{ij} \in \mathbb{F}^n$$

$$X_{22} = -X_{11}^T,$$

$$X_{12}^T = -X_{12},$$

$$X_{21}^T = -X_{21}.$$

These equations allow us to define the following basis:

- $H_i = E_{i,i} - E_{n+i,n+i}$,
- $F_{i,-j} = E_{j,i} - E_{i+n,j+n}$, for $i \neq j$,
- $F_{i,j} = E_{i+n,j} - E_{j+n,i}$ for $i < j$, and
- $F_{-i,-j} = E_{j,i+n} - E_{i,j+n}$ for $i < j$,

where $1 \leq i, j \leq n$.

For the sake of brevity, we do not include the proofs for the five exceptional Lie algebras. Empirically, these Lie algebras have not appeared in the derivations we have computed. Future work may study these cases to determine the likelihood that their elements occur as derivations of adjacency tensors for hypergraphs. □

4.3 Derivations and their associated hypergraph structure

For the purposes of this section, the hypergraph $\mathcal{H}$ is connected, 3-uniform on m vertices, all derivations are of the form $(0, D, D, D)$, and the derivation algebra is in the natural or adjoint representation. This allows us to embed our derivations into $M_m(\mathbb{F})$ and utilize the results of Section 4.2. With this embedding, we use D alone to refer to our derivations. Furthermore, if $\mathcal{H}$ is not 3-uniform, we can either use Lemma 3.1.5 or consider $\mathcal{H}_3$. The proof for Theorem 4.3.3 is split into two parts. Subsection 4.3.1 proves the case of matrices D with support less than three, while Subsection 4.3.2 proves the case of matrices D with support exactly three.

4.3.1 Support less than three

We first consider all possible matrices of support less than three. We see which of these matrices can be used to generate derivations of the form $(0, D, D, D)$, and what hypergraph structures they uncover. Images of the described hypergraph structures are presented in Theorem 4.3.3.

Lemma 4.3.1. Let $D \in \mathrm{Der}(\mathcal{H})$, where $\mathcal{H}$ is 3-uniform and connected on m vertices. Let $1 \leq x, y \leq m$ be arbitrary, and let a, b, c, d, x, and y be distinct. If the support of D is less than 3, then the following holds:

1. The support of D is not equal to 1,

2. D is of the form $\alpha E_{ij} + \beta E_{kl}$ for some $1 \leq \ell \leq m$, where $\alpha + \beta = 0$.

3. $D = \alpha E_{ab} + \beta E_{cb}$ if and only if

   - $\{a, b, c\}, \{a, b, x\} \notin \mathcal{H}$ and
   - $(\{a, x, y\} \in \mathcal{H} \iff \{c, x, y\} \in \mathcal{H})$.

4. $D = \alpha E_{ab} + \beta E_{cd}$ if and only if

   - $\{a, x, y\} \in \mathcal{H}$ if and only if $x = d$ and $\{b, c, y\} \in \mathcal{H}$
   - $\{c, x, y\} \in \mathcal{H}$ if and only if $x = b$ and $\{a, d, y\} \in \mathcal{H}$.

5. $D = \alpha E_{aa} + \beta E_{ba}$ if and only if

   - $\{a, b, x\} \notin \mathcal{H}$ and
   - $\{a, x, y\} \in \mathcal{H} \iff \{b, x, y\} \in \mathcal{H}$.

6. $D = \alpha E_{aa} + \beta E_{bb}$ if and only if for every $x, y \notin \{a, b\}$,

   - $\{a, x, y\} \notin \mathcal{H}$,
   - $\{b, x, y\} \notin \mathcal{H}$.

Proof. As everything is finite-dimensional, this lemma can be proven by exhausting all possible calculations of the derivation equations. The proofs for 1 and 2 follow from explicit calculations shown. Recall that D is a derivation if and only if it satisfies the following relation on all $u, v, w \in V$

$$\Gamma(Du, v, w) + \Gamma(u, Dv, w) + \Gamma(u, v, Dw) = 0.$$

This can be further simplified by asserting that the relation holds with respect to a basis denoted by e_i. Let $1 \leq a, b, c, d, x, y \leq m$ be distinct. Figure 4.1 lists all matrices with support equal to 1 or 2.

$$\begin{array}{ccc} \alpha E_{ab}, & \alpha E_{aa}, & \alpha E_{ab} + \beta E_{ac}, \\ \alpha E_{ab} + \beta E_{cb}, & \alpha E_{ab} + \beta E_{cd}, & \alpha E_{ab} + \beta E_{ba}, \\ \alpha E_{ab} + \beta E_{ca}, & \alpha E_{aa} + \beta E_{ab}, & \alpha E_{aa} + \beta E_{ba}, \\ \alpha E_{aa} + \beta E_{bc}, & \alpha E_{aa} + \beta E_{bb}. & \end{array}$$

Figure 4.1: Matrices of support equal to one or two

The action of αE_{ab} is as follows:

$$\alpha E_{ab} e_i = \begin{cases} \alpha e_a & i = b \\ 0 & else \end{cases}.$$

Since e_i is associated to i we evaluate our tensor on (i, j, k) instead of (e_i, e_j, e_k). All possible combinations of basis elements to check are as follows:

(a,b,c)	(a,b,d)	(a,c,d)	(b,c,d)	(a,b,x)	(a,c,x)
(a,d,x)	(b,c,x)	(b,d,x)	(c,d,x)	(a,x,y)	(b,x,y)
(c,x,y)	(d,x,y)	(a,a,b)	(a,a,c)	(a,a,d)	(b,b,a)
(b,b,c)	(b,b,d)	(c,c,a)	(c,c,b)	(c,c,d)	(d,d,a)
(d,d,b)	(d,d,c)	(a,a,x)	(b,b,x)	(c,c,x)	(d,d,x)

Figure 4.2: Potentially non-trivial triples of basis elements

Certain properties of triples of basis elements (u, v, w) may allow them to hold trivially in the calculations for derivations. For example, (u, u, u) would always hold, as $\Gamma(v, u, u) = 0$. We illustrate several representative calculations; the remaining cases follow similarly.

First, we consider the matrix $D = \alpha E_{ab}$, and thus the triple $\overline{D} = (0, D, D, D)$. This is a derivation of Γ if and only if

$$\Gamma(\alpha E_{ab} i, j, k) + \Gamma(i, \alpha E_{ab} j, k) + \Gamma(i, j, \alpha E_{ab} k) = 0$$

for all $1 \leq i, j, k \leq m$. First, we know that if none of i, j, k equals b, then this relation holds trivially. Therefore, we can consider the case where at least one of i, j, k equals b. Furthermore, if all three equal b, then $\Gamma(\alpha E_{ab} b, b, b) = \alpha\Gamma(a, b, b) = 0$, and likewise for the other two terms. All other calculations are as follows:

$$\begin{aligned}
(b, x, y) :\ & \Gamma(\alpha E_{ab} b, x, y) + \Gamma(b, \alpha E_{ab} x, y) + \Gamma(b, x, \alpha E_{ab} y) \\
& = \alpha\Gamma(a, x, y) \\
(b, b, x) :\ & \Gamma(\alpha E_{ab} b, b, x) + \Gamma(b, \alpha E_{ab} b, x) + \Gamma(b, b, \alpha E_{ab} x) \\
& = 2\alpha\Gamma(a, b, y) \\
(a, b, x) :\ & \Gamma(\alpha E_{ab} a, b, x) + \Gamma(a, \alpha E_{ab} b, x) + \Gamma(a, b, \alpha E_{ab} x) \\
& = \alpha\Gamma(a, a, x) \\
(a, b, b) :\ & \Gamma(\alpha E_{ab} a, b, b) + \Gamma(a, \alpha E_{ab} b, b) + \Gamma(a, b, \alpha E_{ab} b) \\
& = 2\alpha\Gamma(a, a, b) \\
(a, a, b) :\ & \Gamma(\alpha E_{ab} a, a, b) + \Gamma(a, \alpha E_{ab} a, b) + \Gamma(a, b, \alpha E_{ab} b) \\
& = \alpha\Gamma(a, a, a)
\end{aligned}$$

Now, $\overline{D}$ is a derivation if and only if all the preceding terms are zero. $\Gamma(a, a, b) = 0$ and $\Gamma(a, a, a) = 0$ by definition. $\Gamma(a, x, y) = 0$ if and only if $\{a, x, y\} \notin \mathcal{H}$, and $\Gamma(a, b, x) = 0$ if and only if $\{a, b, x\} \notin \mathcal{H}$. These two conditions are equivalent to saying that $(0, D, D, D)$ is a derivation if and only if a is not a vertex in any hyperedge in $\mathcal{H}$, which contradicts the assumption that $\mathcal{H}$ is 3-uniform and connected. Therefore, $(0, \alpha E_{ab}, \alpha E_{ab}, \alpha E_{ab})$ cannot be a derivation of Γ. The proof for 1 follows from performing this calculation αE_{aa}.

We now prove 3, which considers matrices $D = \alpha E_{ab} + \beta E_{cb}$. This is a derivation if and only if

$$\Gamma((\alpha E_{ab} + \beta E_{cb})i, j, k) + \Gamma(i, (\alpha E_{ab} + \beta E_{cb})j, k) + \Gamma(i, j, (\alpha E_{ab} + \beta E_{cb})k) = 0.$$

for all $1 \leq i, j, k \leq m$. The calculations are as follows:

$$\begin{aligned}
(b, x, y):&\ \alpha\Gamma(a, x, y) + \beta\Gamma(c, x, y)\\
(a, b, x):&\ \alpha\Gamma(a, a, x) + \beta\Gamma(a, c, x)\\
&= \beta\Gamma(a, c, x)\\
(b, c, x):&\ \alpha\Gamma(a, c, x) + \beta\Gamma(c, c, x)\\
&= \alpha\Gamma(a, c, x)\\
(a, b, c):&\ \alpha\Gamma(a, a, c) + \beta\Gamma(a, c, c) = 0\\
(b, b, x):&\ 2\alpha\Gamma(a, b, x) + 2\beta\Gamma(b, c, x)\\
(a, b, b):&\ 2\beta\Gamma(a, b, c)\\
(b, b, c):&\ 2\alpha\Gamma(a, b, c)
\end{aligned}$$

$\overline{D}$ is a derivation if and only if all the preceding terms are equal to zero. $\Gamma(a, b, c) = 0$ and $\beta\Gamma(a, c, x) = 0$ if and only if $\{a, c, x\} \notin \mathcal{H}$ for all x, including $x = b$, i.e., there is no hyperedge $e \in \mathcal{H}$ that contains both a and c as a vertex. Furthermore, we have the relations $\alpha\Gamma(a, x, y) + \beta\Gamma(c, x, y) = 0$ and $2\alpha\Gamma(a, b, x) + 2\beta\Gamma(b, c, x) = 0$. If $\alpha + \beta \neq 0$, then both relations require that $\Gamma(a, x, y) = 0$ and $\Gamma(a, b, x) = 0$, which holds if and only if a is not a vertex in any hyperedge in $\mathcal{H}$, a contradiction. Therefore, we must have that $\alpha + \beta = 0$. It follows that $\alpha\Gamma(a, x, y) + \beta\Gamma(c, x, y) = 0$ if and only if $\Gamma(a, x, y) = \Gamma(c, x, y)$, and likewise $2\alpha\Gamma(a, b, x) + 2\beta\Gamma(b, c, x) = 0$ if and only if $\Gamma(a, b, x) = \Gamma(b, c, x)$. This holds if and only if for all $1 \leq x, y \leq m$, where x may equal b,

$$\{a, x, y\} \in \mathcal{H} \iff \{c, x, y\} \in \mathcal{H}$$

holds. This concludes the proof for 3. In this proof, we have part of the proof for 2. The rest of the proof for 2 follows from the proofs for 4-6, in which the relation $\alpha + \beta \neq 0$ implies a vertex is isolated. The proofs for 4-6 follow the same format as those for 1 and 3. □

Example 4.3.2. Consider the hypergraph given by the following hyperedges:

$$\{1,2,3\} \quad \{1,4,5\} \quad \{1,3,8\} \quad \{2,3,9\} \quad \{2,6,7\} \quad \{3,8,9\} \quad \{6,7,8\} \quad \{6,7,9\}$$

Figure 4.3: Example hypergraph

The adjacency tensor of this hypergraph has the following derivations:

$$D = E_{1,1} - E_{9,1},$$

$$F = E_{2b} - E_{8b}, \quad 1 \leq b \leq 9, \quad b \neq 2, 8.$$

4.3.2 Support less than or equal to three

Now that we have characterized all matrices of support equal to one and two, we begin characterizing all matrices with support equal to three. Following this, we have a complete characterization of derivations and their associated hypergraph structure. To begin, there are 40 generic matrices of support equal to 3, listed in Figure 4.4, and 79 possible inputs listed in Figure 4.5.

With this information, we can say what each matrix present in the derivation algebra tells us about the hypergraph. For simplicity, we again assume that all hypergraphs are connected and 3-uniform. Much of the work presented in this section is similar in practice to the proof of Proposition 4.3.1. We separate our cases by the numbered matrix forms listed in Figure 4.4.

Theorem 4.3.3. Let Γ be the adjacency tensor 3-uniform connected hypergraph $\mathcal{H}$, and let $D \in \mathrm{Der}(\Gamma)$ be such that the support of D is less than or equal to three. Then

1. $D = \alpha E_{ab} + \beta E_{cb}$ if and only if

    - $\{a, b, c\}, \{a, b, x\} \notin \mathcal{H}$ and

Case	Matrix Forms	
1. All in one row	$M1 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{ad}$	$M2 = \alpha E_{aa} + \beta E_{ab} + \gamma E_{ac}$
2. All in one column	$M3 = \alpha E_{ab} + \beta E_{cb} + \gamma E_{db}$	$M4 = \alpha E_{aa} + \beta E_{ba} + \gamma E_{ca}$
3. Two in one row, one elsewhere	$M5 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{de}$	$M22 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{da}$
	$M6 = \alpha E_{aa} + \beta E_{ab} + \gamma E_{cd}$	$M23 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{bd}$
	$M7 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{dd}$	$M24 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{ba}$
	$M8 = \alpha E_{aa} + \beta E_{ab} + \gamma E_{cc}$	$M25 = \alpha E_{ab} + \beta E_{aa} + \gamma E_{bc}$
4. Two in one column, one elsewhere	$M9 = \alpha E_{ab} + \beta E_{cb} + \gamma E_{de}$	$M26 = \alpha E_{ab} + \beta E_{cb} + \gamma E_{da}$
	$M10 = \alpha E_{aa} + \beta E_{ba} + \gamma E_{cd}$	$M27 = \alpha E_{ab} + \beta E_{cb} + \gamma E_{bd}$
	$M11 = \alpha E_{ab} + \beta E_{cb} + \gamma E_{dd}$	$M28 = \alpha E_{ab} + \beta E_{cb} + \gamma E_{ba}$
	$M12 = \alpha E_{aa} + \beta E_{ba} + \gamma E_{cc}$	$M29 = \alpha E_{aa} + \beta E_{ba} + \gamma E_{cb}$
5. L-shape	$M13 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{db}$	$M17 = \alpha E_{ab} + \beta E_{aa} + \gamma E_{ca}$
	$M14 = \alpha E_{aa} + \beta E_{ab} + \gamma E_{cb}$	$M30 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{cb}$
	$M15 = \alpha E_{ab} + \beta E_{ac} + \gamma E_{bb}$	$M31 = \alpha E_{aa} + \beta E_{ab} + \gamma E_{ba}$
	$M16 = \alpha E_{ab} + \beta E_{aa} + \gamma E_{bb}$	
6. All rows and columns distinct	$M18 = \alpha E_{ab} + \beta E_{cd} + \gamma E_{ef}$	$M35 = \alpha E_{ab} + \beta E_{cd} + \gamma E_{ea}$
	$M19 = \alpha E_{aa} + \beta E_{bc} + \gamma E_{de}$	$M36 = \alpha E_{ab} + \beta E_{cd} + \gamma E_{ba}$
	$M20 = \alpha E_{aa} + \beta E_{bb} + \gamma E_{cd}$	$M37 = \alpha E_{ab} + \beta E_{cd} + \gamma E_{da}$
	$M21 = \alpha E_{aa} + \beta E_{bb} + \gamma E_{cc}$	$M38 = \alpha E_{ab} + \beta E_{bc} + \gamma E_{ca}$
	$M32 = \alpha E_{aa} + \beta E_{bc} + \gamma E_{cd}$	$M39 = \alpha E_{ab} + \beta E_{bc} + \gamma E_{da}$
	$M33 = \alpha E_{aa} + \beta E_{bc} + \gamma E_{cb}$	$M40 = \alpha E_{ab} + \beta E_{bc} + \gamma E_{cd}$
	$M34 = \alpha E_{ab} + \beta E_{cd} + \gamma E_{be}$	

Figure 4.4: All matrices of support equal to 3

(a,a,b)	(a,a,c)	(a,a,d)	(a,a,e)	(a,a,f)	(b,b,a)
(b,b,c)	(b,b,d)	(b,b,e)	(b,b,f)	(c,c,a)	(c,c,b)
(c,c,d)	(c,c,e)	(c,c,f)	(d,d,a)	(d,d,b)	(d,d,c)
(d,d,d)	(d,d,e)	(d,d,f)	(e,e,a)	(e,e,b)	(e,e,c)
(e,e,d)	(e,e,e)	(e,e,f)	(f,f,a)	(f,f,b)	(f,f,c)
(f,f,d)	(f,f,e)	(c,e,f)	(d,e,f)	(c,d,e)	(c,d,f)
(a,b,c)	(a,b,d)	(a,b,e)	(a,b,f)	(a,c,d)	(a,c,e)
(a,c,f)	(a,d,e)	(a,d,f)	(a,e,f)	(b,c,d)	(b,c,e)
(b,c,f)	(b,d,e)	(b,d,f)	(b,e,f)	(d,e,x)	(d,f,x)
(a,a,x)	(b,b,x)	(c,c,x)	(d,d,x)	(e,e,x)	(f,f,x)
(a,b,x)	(a,c,x)	(a,d,x)	(a,e,x)	(a,f,x)	(b,c,x)
(b,d,x)	(b,e,x)	(b,f,x)	(c,d,x)	(c,e,x)	(c,f,x)
(e,f,x)	(a,x,y)	(b,x,y)	(c,x,y)	(d,x,y)	(e,x,y)
(f,x,y)					

Figure 4.5: All inputs for matrices of support equal to three

- $(\{a, x, y\} \in \mathcal{H} \iff \{c, x, y\} \in \mathcal{H})$,

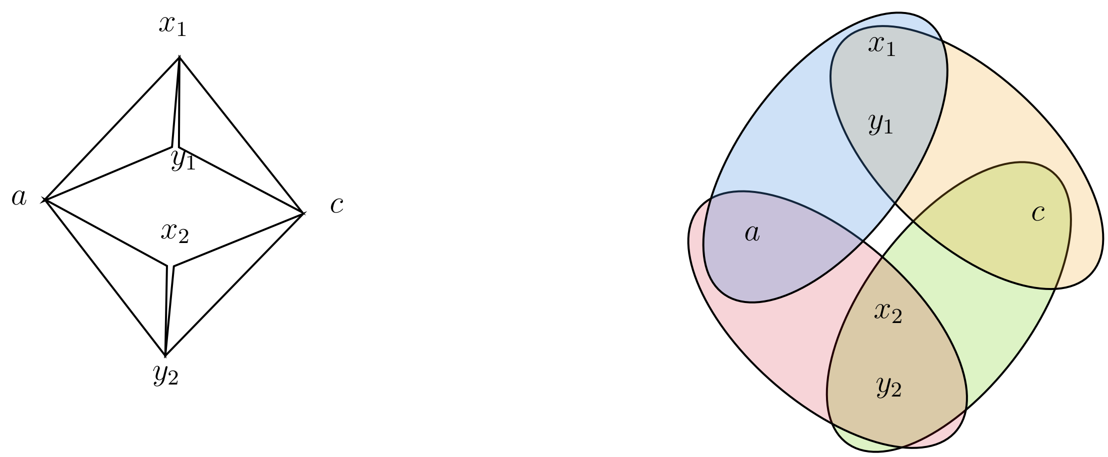


Figure 4.6: Theorem 4.3.3: Case 1

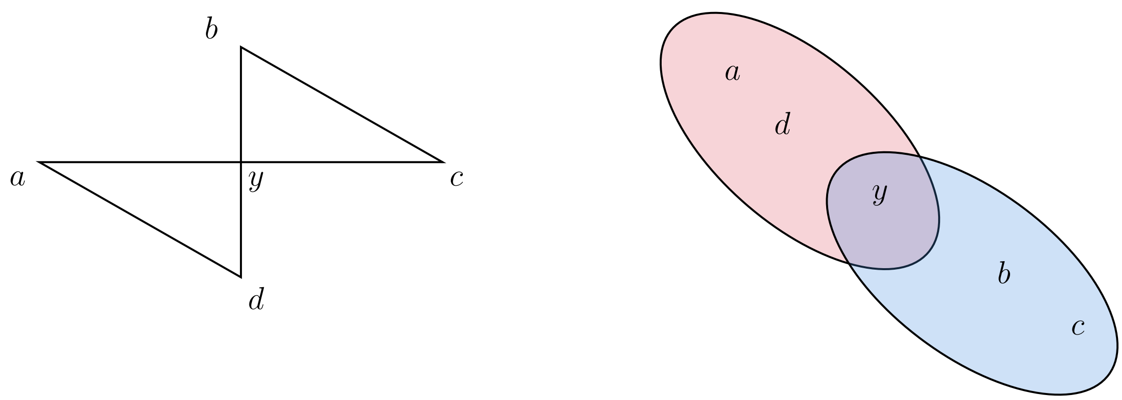


Figure 4.7: Theorem 4.3.3: Case 2

2. $D = \alpha E_{ab} + \beta E_{cd}$ if and only if

   - $\{a, x, y\} \in \mathcal{H}$ if and only if $x = d$ and $\{b, c, y\} \in \mathcal{H}$
   - $\{c, x, y\} \in \mathcal{H}$ if and only if $x = b$ and $\{a, d, y\} \in \mathcal{H}$,

3. $D = \alpha E_{aa} + \beta E_{ba}$ if and only if

   - $\{a, b, x\} \notin \mathcal{H}$, and
   - $\{a, x, y\} \in \mathcal{H} \iff \{b, x, y\} \in \mathcal{H}$,

4. $D = \alpha E_{aa} + \beta E_{bb}$ if and only if for every $x, y \notin \{a, b\}$,

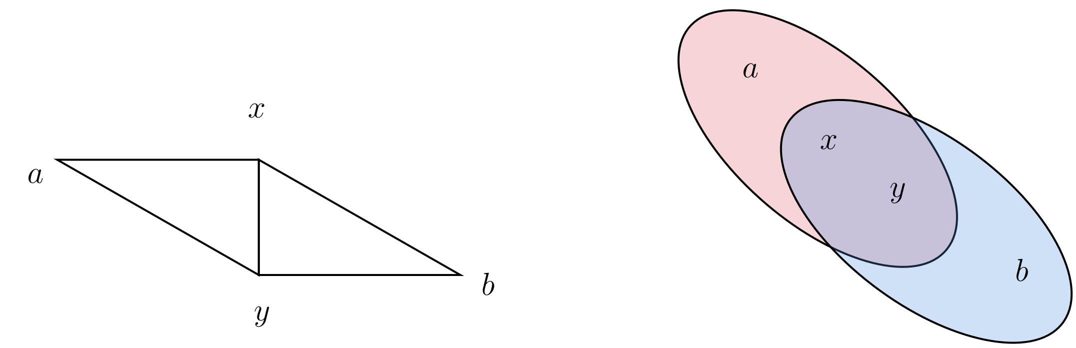


Figure 4.8: Theorem 4.3.3: Case 3

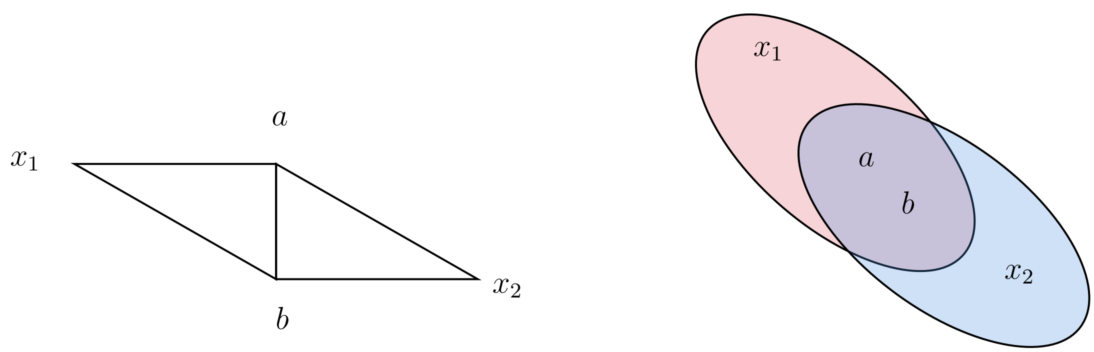


Figure 4.9: Theorem 4.3.3: Case 4

- $\{a, x, y\} \notin \mathcal{H}$, and
- $\{b, x, y\} \notin \mathcal{H}$,

5. $D = \alpha E_{aa} + \beta E_{ab} + \gamma E_{cc}$ if and only if
   - $\alpha + \gamma = 0$,
   - $a \in e \in \mathcal{H}$ implies that $e = \{a, c, x\}$ for some $x \in V - \{a, c\}$, and
   - $c \in e \in \mathcal{H}$ implies that $e = \{a, c, x\}$ for some $x \in V - \{a, c\}$

6. $D = \alpha E_{aa} + \beta E_{bb} + \gamma E_{cd}$ if and only if
   - $\alpha + \beta = 0$,
   - $a \in e \in \mathcal{H}$ implies that $e = \{a, b, x\}$ for some $x \in V - \{a, b\}$, and

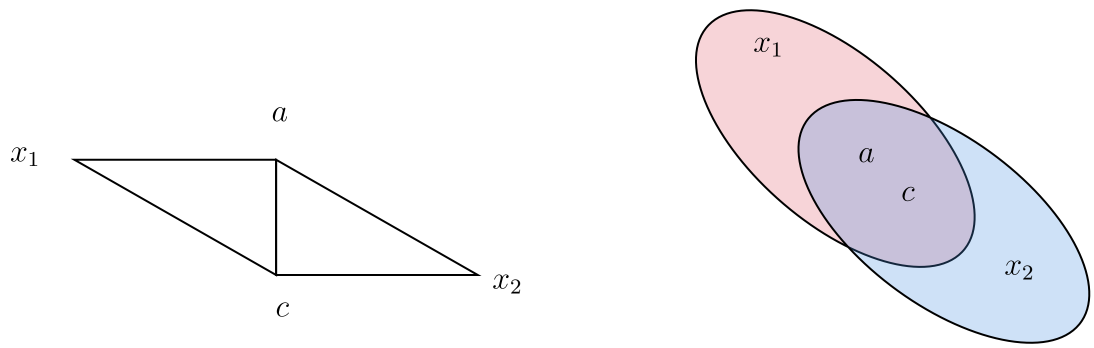


Figure 4.10: Theorem 4.3.3: Case 5

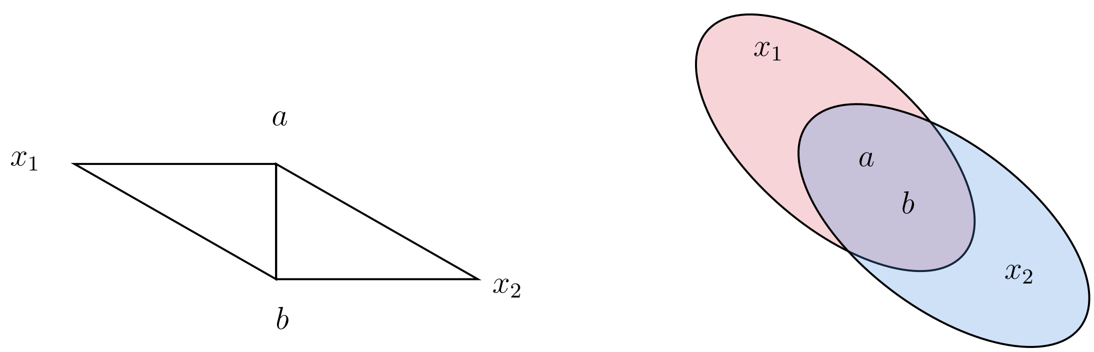


Figure 4.11: Theorem 4.3.3: Case 6

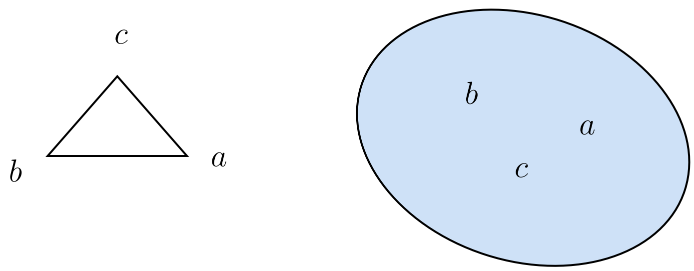


Figure 4.12: Theorem 4.3.3: Case 7

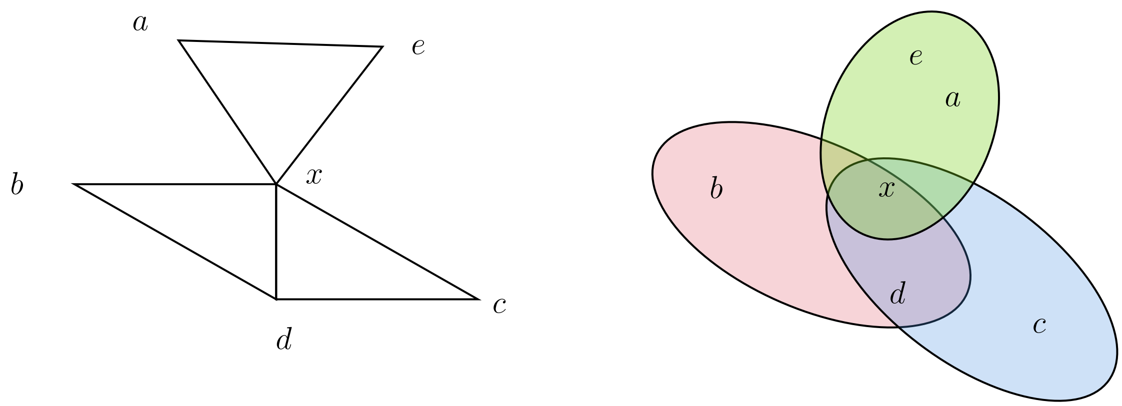


Figure 4.13: Theorem 4.3.3: Case 8

- $b \in e \in \mathcal{H}$ implies that $e = \{a, b, x\}$ for some $x \in V - \{a, b\}$

7. $D = \alpha E_{aa} + \beta E_{bb} + \gamma E_{cc}$ where $(\alpha + \beta + \gamma) = 0$ if and only if

   - $\mathcal{H} = \{\{a, b, c\}\}$
   - $(\alpha + \beta) \neq 0$, $(\alpha + \gamma) \neq 0$, and $(\beta + \gamma) \neq 0$,

8. $D = \alpha E_{ab} + \beta E_{ac} + \gamma E_{de}$ if and only if

   - $\{a, e, x\}, \{b, d, x\}, \{c, d, x\} \in \mathcal{H}$ for some $x \in V - \{a, e, \}$,
   - $\{b, c, d\}, \{d, x, y\} \notin \mathcal{H}$ for all $x, y \in \mathcal{H}$,
   - $\alpha + \gamma = 0$, and
   - $\beta + \gamma = 0$

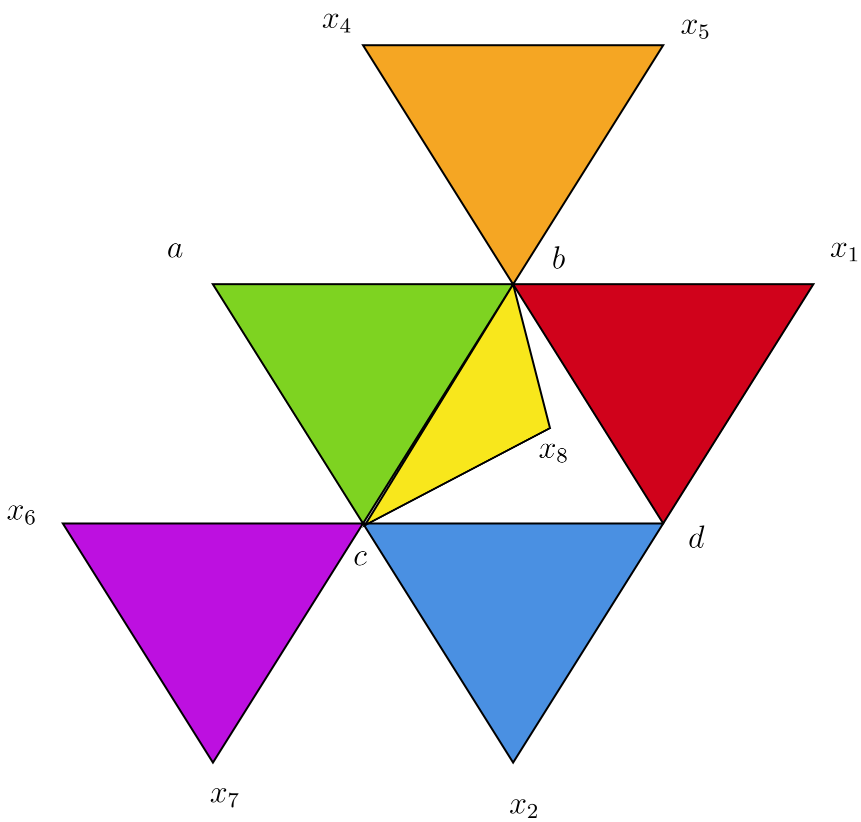


Figure 4.14: Theorem 4.3.3: Case 9

9. $D = \alpha E_{ab} + \beta E_{ac} + \gamma E_{db}$ if and only if

    - $\alpha + \gamma = 0$,
    - $\beta + \gamma = 0$,
    - $\{a, b, c\} \in \mathcal{H}$ for $x \notin \{c, d\}$ if and only if $\{b, d, x\} \in \mathcal{H}$,
    - $\{c, d, x\} \in \mathcal{H}$ for $x \notin \{a, b\}$ if and only if $\{a, b, c\} \in \mathcal{H}$, and
    - All other hyperedges in $\mathcal{H}$ are of the form $\{b, c, x\}$, $\{b, x, y\}$, $\{c, x, y\}$ and $\{x, y, z\}$.

10. $D = \alpha E_{ab} + \beta E_{cb} + \gamma E_{db}$ if and only if

    - $\{a, x, y\}, \{c, x, y\}, \{d, x, y\} \in \mathcal{H}$,
    - $\{a, c, x\} \notin \mathcal{H}$, $\{a, d, x\} \notin \mathcal{H}$, $\{c, d, x\} \notin \mathcal{H}$ for $x \in \mathcal{H} - \{a, c, d\}$,
    - $\{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\} \notin \mathcal{H}$, and

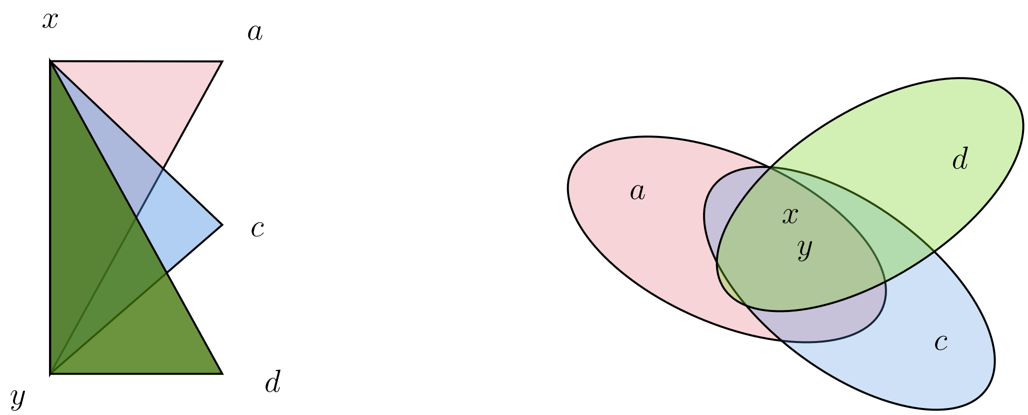


Figure 4.15: Theorem 4.3.3: Case 10

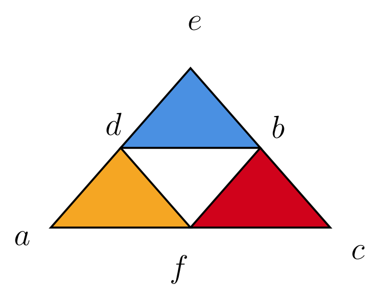


Figure 4.16: Theorem 4.3.3: Case 11

- $\alpha + \beta + \gamma = 0$.

11. $D = \alpha E_{ab} + \beta E_{cd} + \gamma E_{ef}$, where $\alpha + \beta + \gamma \neq 0$, if and only if

    - $\mathcal{H} = \{\{a, d, f\}, \{b, c, f\}, \{b, d, e\}\}$

12. $D = \alpha E_{aa} + \beta E_{ba} + \gamma E_{ca}$ if and only if

    - $\{a, b, c\}, \{a, b, x\}, \{a, c, x\}, \{b, c, x\} \notin \mathcal{H}$ for $x \in V - \{a, b, c\}$,
    - $\{a, x, y\}, \{b, x, y\}, \{c, x, y\} \in \mathcal{H}$ for $x, y \in V - \{a, b, c\}$, and
    - $(\alpha + \beta + \gamma) = 0$.

13. $D = \alpha E_{ab} + \beta E_{cb} + \gamma E_{de}$, where $\alpha + \beta \neq 0$, if and only if

    - $\{a, b, d\}, \{b, c, d\}, \{a, b, x\},\ \{b, c, x\},\ \{a, b, e\},$
    $\{b, c, e\},\ \{a, x, y\},\ \{c, x, y\},\ \{a, c, e\} \notin \mathcal{H}$,

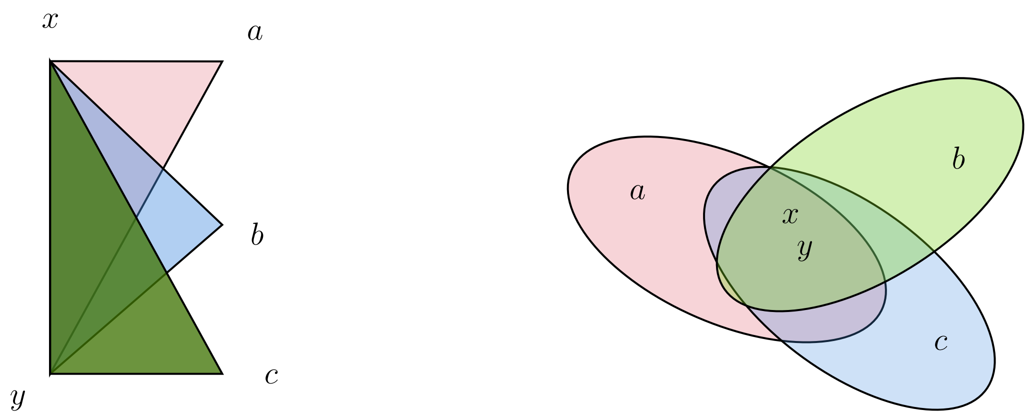


Figure 4.17: Theorem 4.3.3: Case 12

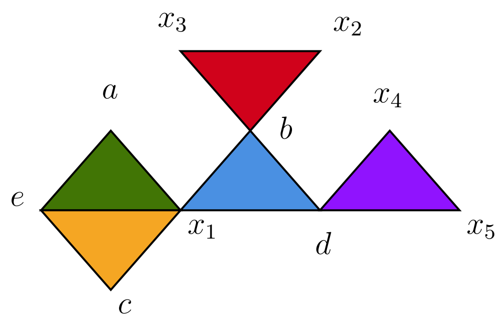


Figure 4.18: Theorem 4.3.3: Case 13

- $\{b, d, x\}, \{a, e, x\}, \{c, e, x\} \in \mathcal{H}$, and
- $\alpha + \beta + \gamma = 0$

14. $D = \alpha E_{ab} + \beta E_{cb} + \gamma E_{de}$, where $\alpha + \beta = 0$, $\{a, b, d\} \notin \mathcal{H}$, $\alpha + \gamma \neq 0$, and $\beta + \gamma = 0$, if and only if
    - $\{b, d, x\}, \{c, e, x\} \in \mathcal{H}$,
    - $\{a, e, x\} \notin \mathcal{H}$,

15. $D = \alpha E_{ab} + \beta E_{cb} + \gamma E_{de}$, where $\alpha + \beta = 0$, $\{a, b, d\} \notin \mathcal{H}$, $\alpha + \gamma = 0$, and $\beta + \gamma \neq 0$, if and only if
    - $\{b, d, x\}, \{a, e, x\} \in \mathcal{H}$,
    - $\{c, e, x\} \notin \mathcal{H}$,

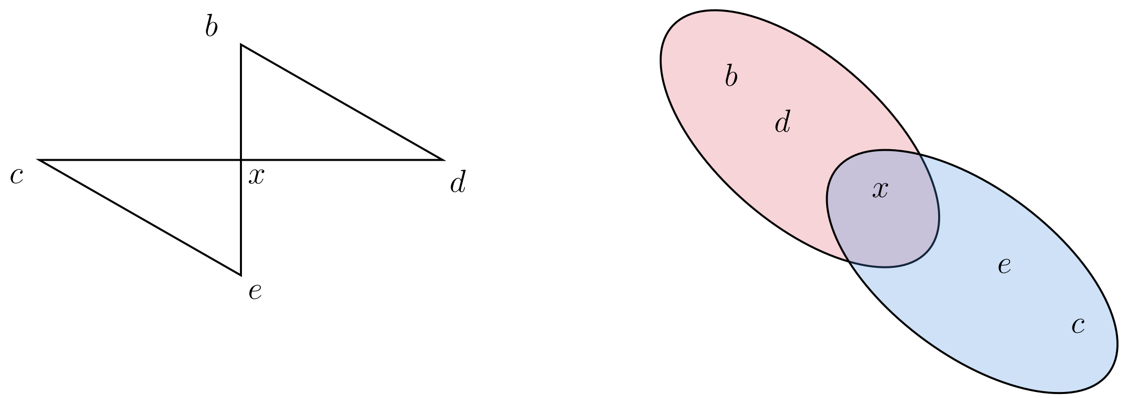


Figure 4.19: Theorem 4.3.3: Case 14

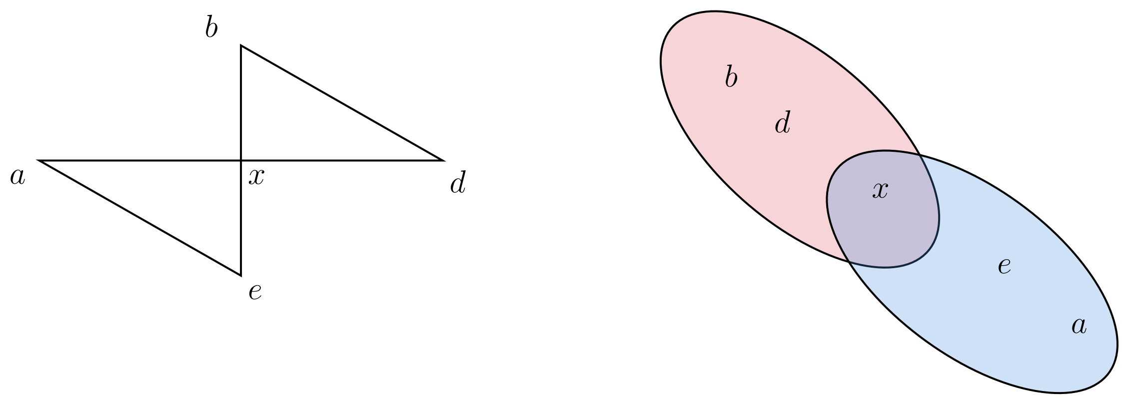


Figure 4.20: Theorem 4.3.3: Case 15

16. The support of D is not equal to 1,

17. If D is of the form $\alpha E_{ij} + \beta E_{kl}$ for some $1 \leq i, j, k, l \leq m$, then $\alpha + \beta = 0$,

Proof. The first four cases are outlined in Lemma 4.3.1. For all matrices with support equal to three, we refer to Figure 4.4 for the cases we must check. For each case, we outline either why it can never be a derivation or what hyperedges must exist for it to be a derivation. Before we evaluate our cases, we note a common contradiction that may occur. All the derivations of support equal to three are of the form $\alpha E_{ab} + \beta E_{cd} + \gamma E_{ef}$, where α, β, and γ are all nonzero. If in our calculations we obtain the implication that $(\alpha + \beta) = (\alpha + \gamma) = (\beta + \gamma) = 0$, then at least one of α, β, or γ must be zero, which is a contradiction.

Cases M1, M2, M6, M7, M10, M11, M12, M14, M16, M17, M22, M23, M24, M26, M27, and M31: These derivations exist if and only if there is an isolated vertex. To see why this is the case, we look at the calculations for $M1$, as the rest follow similarly. The derivation equation must hold for the basis elements (b,b,c). By inputting these entries into our derivation equation, we get

$$2\alpha\Gamma(a, b, c).$$

Therefore, $(0, M1, M1, M1)$ is a derivation only if $\Gamma(a, b, c) = 0$. As Γ is an adjacency tensor, this holds if and only if $\{a, b, c\} \notin \mathcal{H}$. Continuing this with different triples (i, j, k), we eventually get that $M1$ is a derivation if and only if a is an isolated vertex, which violates our assumption that $\mathcal{H}$ is 3-uniform and connected.

Case M8: Such a derivation exists if and only if $\alpha + \gamma = 0$ and a and c are only in hyperedges of the form $\{a, c, x\}$, for arbitrary vertices $x \in V \setminus \{a, c\}$.

Case M20: Such a derivation would exist if and only $\alpha + \beta = 0$, and for all hyperedges with either a or b as elements, such hyperedges are of the form $\{a, b, x\}$ for some $x \in V - \{a, b\}$.

Case M21: $M21$ is a derivation only if the following hold:

$$(\alpha + \beta + \gamma)\Gamma(a, b, c) = 0,$$

$$(\alpha+\beta)\Gamma(a,b,x) = 0,$$

$$(\alpha+\gamma)\Gamma(a,c,x) = 0,$$

$$(\beta+\gamma)\Gamma(b,c,x) = 0,$$

$$\Gamma(a,x,y) = 0,$$

$$\Gamma(b,x,y) = 0,$$

$$\Gamma(c,x,y) = 0,$$

First, we can say definitively that any derivation of the form of $M21$ would arise from a hypergraph where the vertices a, b, and c are only in hyperedges containing at least two of the three. We now go through the cases of what the sum of the coefficients may equal.

Case M21 (a): If $(\alpha+\beta+\gamma) \neq 0$, and any of $(\alpha+\beta)$, $(\alpha+\gamma)$, or $(\beta+\gamma)$ is not equal to zero, then there would be an isolated vertex, which is a contradiction. Therefore, $(\alpha+\beta+\gamma) \neq 0$ implies that $(\alpha+\beta) = (\alpha+\gamma) = (\beta+\gamma) = 0$, which implies that one of α, β, or γ equals zero, a contradiction.

Case M21 (b): $(\alpha+\beta+\gamma) = 0$, in which case $\{a,b,c\}$ may be a hyperedge. From here, we have eight cases. Due to symmetry, we only need to consider two.

i If any of $(\alpha+\beta)$, $(\alpha+\gamma)$, or $(\beta+\gamma) = 0$, then one of α, β, or γ equals zero, a contradiction to $M21$ having support equal to three.

ii If $(\alpha+\beta) \neq 0$, $(\alpha+\gamma) \neq 0$, and $(\beta+\gamma) \neq 0$, then this would imply that $\{a,b,c\}$ is the only hyperedge that could contain a, b, or c. For this not to contradict the assumption that $\mathcal{H}$ is connected, we must have that $\mathcal{H} = \{\{a,b,c\}\}$. Furthermore, this implies that

$$\Gamma(Da,b,c) + \Gamma(a,Db,c) + \Gamma(a,b,Dc) = 0$$

and M21 is exactly

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

Case M5: $\{a, e, x\} \in \mathcal{H}$ for some $x \in V$, as otherwise a would be isolated. This then implies that $\{a, e, x\} \in \mathcal{H}$ if and only if

- $\{b, d, x\}, \{c, d, x\} \in \mathcal{H}$,
- $\alpha + \gamma = 0$,
- $\beta + \gamma = 0$, and
- $\{b, c, d\}, \{d, x, y\} \notin \mathcal{H}$

all hold.

Case M13: If $\alpha + \gamma \neq 0$, then $M13$ is a derivation only if $\Gamma(a, b, x) = \Gamma(b, d, x) = 0$, which holds if and only if a is an isolated vertex, a contradiction. Therefore, $\alpha + \gamma = 0$. Similarly, $\beta + \gamma = 0$. From here, we have that $M13$ is a derivation if and only if for $x, y, z \notin \{a, b, c, d\}$,

- $\{a, b, x\} \in \mathcal{H}$ for $x \notin \{c, d\}$, if and only if $\{b, d, x\} \in \mathcal{H}$,
- $\{c, d, x\} \in \mathcal{H}$ for $x \notin \{a, b\}$ if and only if $\{a, b, x\} \in \mathcal{H}$, and
- All other hyperedges are of the form $\{b, c, x\}$, $\{b, x, y\}$, $\{c, x, y\}$, and $\{x, y, z\}$

all hold.

Cases M15 and M19: $M15$ and $M19$ are derivations only if a is an isolated vertex, a contradiction.

Case M18: From relations such as $\beta\Gamma(a, c, f) + \gamma\Gamma(a, d, e)$, we have that $M18$ is a derivation only if

$$\{a, c, f\} \in \mathcal{H} \iff \{a, d, e\} \in \mathcal{H} \iff \{b, c, e\} \in \mathcal{H}.$$

If $\{a, c, f\} \in \mathcal{H}$, then $\alpha + \gamma = 0$, $\beta + \gamma = 0$, and $\alpha + \beta = 0$, implying that one of α, β, or γ is zero, a contradiction. Therefore,

$$\{a, c, f\} \notin \mathcal{H}, \quad \{a, d, e\} \notin \mathcal{H}, \quad \{b, c, e\} \notin \mathcal{H}.$$

We now look at the relation $\alpha\Gamma(a, d, f) + \beta\Gamma(b, c, f) + \gamma\Gamma(b, d, e)$. We first assume that $\alpha + \beta + \gamma \neq 0$.

Case M18(a):

i If $\{a, d, f\}, \{b, c, f\}, \{b, d, e\} \in \mathcal{H}$, then $\alpha\Gamma(a, d, f) + \beta\Gamma(b, c, f) + \gamma\Gamma(b, d, e) \neq 0$, which would mean that $M18$ is not a derivation.

ii If only one of $\{a, d, f\}, \{b, c, f\}, \{b, d, e\}$ is an element of $\mathcal{H}$, we would have $\alpha = 0$, $\beta = 0$, or $\gamma = 0$, a contradiction to $M18$ having support equal to three.

iii Assume only two of $\{a, d, f\}, \{b, c, f\}, \{b, d, e\}$ are hyperedges in $\mathcal{H}$. Without loss of generality, assume $\{a, d, f\} \notin \mathcal{H}$. This implies that $\beta + \gamma = 0$, which further implies that $\{c, f, x\} \in \mathcal{H} \iff \{d, e, x\} \in \mathcal{H}$.

We now have two more subcases to consider.

Case M18(a)iii.i: Assume that $\alpha + \beta = 0$. This then implies that $\alpha + \gamma \neq 0$, $\{a, d, x\} \in \mathcal{H} \iff \{b, c, x\} \in \mathcal{H}$, and that $\{a, f, x\}, \{b, e, x\} \notin \mathcal{H}$.

Case M18(a)iii.ii: Assume that $\alpha + \beta \neq 0$ and $\alpha + \gamma \neq 0$. This then implies that

$$\{a, d, x\}, \{b, c, x\}, \{a, f, x\}, \{b, e, x\} \notin \mathcal{H}.$$

However, this implies that a is an isolated vertex, a contradiction.

Case M18(b): We now assume that $\alpha + \beta + \gamma = 0$. The remaining expressions to consider are $\alpha\Gamma(a, d, x) + \beta\Gamma(b, c, x)$, $\alpha\Gamma(a, f, x) + \gamma\Gamma(b, e, x)$, and $\beta\Gamma(c, f, x) + \gamma\Gamma(d, e, x)$. Without loss of generality, assume that $\{a, d, x\} \in \mathcal{H}$. This would imply that $\alpha + \beta = 0$, and hence that $\gamma = 0$, a contradiction. Therefore, none of $\{a, d, x\}$, $\{b, c, x\}$, $\{a, f, x\}$, $\{b, e, x\}$, $\{c, f, x\}$, or $\{d, e, x\}$ may be hyperedges. Therefore, $\{a, d, f\}$, $\{b, c, f\}$, and $\{b, d, e\}$ are hyperedges that must exist, as otherwise we would have an isolated vertex. Now, for $\mathcal{H}$ to be connected, there can only be these six vertices.

Case M3: First, we have that $\{a, c, d\} \notin \mathcal{H}$. We have equations $\alpha\Gamma(a, b, c) + \Gamma(a, b, d) = 0$, $\alpha\Gamma(a, b, c) + \gamma\Gamma(b, c, d) = 0$ and $\alpha\Gamma(a, b, d) + \beta\Gamma(b, c, d) = 0$. As α, β, and γ are all nonzero, we can rescale these to be $\frac{\alpha^2}{\gamma}\Gamma(a, b, c) + \alpha\Gamma(a, b, d) = 0$, $\frac{\alpha\beta}{\gamma}\Gamma(a, b, c) + \beta\Gamma(b, c, d) = 0$ and $\alpha\Gamma(a, b, d)+\beta\Gamma(b, c, d) = 0$. This then implies that $\{a, b, c\} \notin \mathcal{H}$, $\{a, b, d\} \notin \mathcal{H}$, and $\{b, c, d\} \notin \mathcal{H}$. There are two main subcases.

Case M3 (a): If $\alpha + \beta = \alpha + \gamma = \beta + \gamma = 0$, then at least one of α, β, or γ is zero, a contradiction.

Case M3 (b): If any of $\alpha + \beta$, $\alpha + \gamma$, or $\beta + \gamma$ i nonzero, this implies that

$$\{a, c, x\} \notin \mathcal{H}, \quad \{a, d, x\} \notin \mathcal{H}, \quad \{c, d, x\} \notin \mathcal{H}, \quad x \in \mathcal{H} - \{a, c, d\}.$$

Without loss of generality, assume that $\alpha + \beta \neq 0$. We consider the equation $\alpha\Gamma(a, x, y) + \beta\Gamma(c, x, y) + \gamma\Gamma(d, x, y) = 0$. $\Gamma(a, x, y) = 0$ if and only if $\{a, x, y\} \notin \mathcal{H}$, which would imply that a is an isolated vertex. Similarly, we must have that $\Gamma(c, x, y) = 1$ and $\Gamma(d, x, y) = 1$. This then implies that $\{a, x, y\}, \{c, x, y\}, \{d, x, y\} \in \mathcal{H}$, and $\alpha + \beta + \gamma = 0$.

Case M4: First, $\{a, b, c\} \notin \mathcal{H}$. with a similar calculation to **Case M3**, we also have that $\{a, b, x\} \notin \mathcal{H}$, $\{a, c, x\} \notin \mathcal{H}$, and $\{b, c, x\}\ /\mathcal{H}$. Our last equation to consider in this case is $\alpha\Gamma(a, x, y) + \beta\Gamma(b, x, y) + \gamma\Gamma(c, x, y) = 0$. For none of a, b, or c to be isolated vertices, we must have $\{a, x, y\}, \{b, x, y, \}, \{c, x, y\} \in \mathcal{H}$. This implies that $(\alpha + \beta + \gamma) = 0$.

Case M9: We have that

$$\{a, b, c\},\ \{a, d, e\},\ \{b, d, e\},\ \{c, d, e\},\ \{d, e, x\},$$
$$\{a, c, d\},\ \{a, c, x\},\ \{a, d, x\},\ \{c, d, x\},\ \{d, x, y\} \notin \mathcal{H}.$$

This implies that at least one of $\{a, d, b\}$, $\{b, c, d\}$, or $\{b, d, x\}$ is a hyperedge. Furthermore, there is no restriction on $\{b, e, x\}$. Therefore, if it appears that b or e may be an isolated vertex, we can assume that $\{b, e, x\}$ exists to avoid a contradiction. We have two subcases.

Case M9(a): If $\alpha + \beta \neq 0$, this further implies that $\{a, b, d\}, \{b, c, d\} \notin \mathcal{H}$. This then implies that $\{b, d, x\} \in \mathcal{H}$, and that

$$\{a, b, x\},\ \{b, c, x\},\ \{a, b, e\},\ \{b, c, e\},\ \{a, x, y\},\ \{c, x, y\},\ \{a, c, e\} \notin \mathcal{H}.$$

This implies that $\{a, e, x\} \in \mathcal{H}$, as otherwise a would be an isolated vertex. Similarly, $\{c, e, x\} \in \mathcal{H}$. Together, this tells us that $\alpha + \beta + \gamma = 0$.

Case M9(b): If $\alpha + \beta = 0$, then M9 is a derivation only if the following hold

$$\{a, b, d\} \in \mathcal{H} \iff \{b, c, d\} \in \mathcal{H}, \quad \{a, b, e\} \in \mathcal{H} \iff \{b, c, e\} \in \mathcal{H},$$
$$\{a, b, x\} \in \mathcal{H} \iff \{b, c, x\} \in \mathcal{H}, \quad \{a, x, y\} \in \mathcal{H} \iff \{c, x, y\} \in \mathcal{H},$$
$$\alpha\Gamma(a, e, x) + \beta\Gamma(c, e, x) + \gamma\Gamma(b, d, x) = 0.$$

There are now multiple subcases to consider, each determined by which hyperedges exist. We first note the following implications.

- If $\{b, c, d\} \in \mathcal{H}$ and $\alpha + \gamma \neq 0$, then the first assumption implies that $\{b, c, d\} \in \mathcal{H}$, while the second implies that $\{b, c, d\} \in \mathcal{H}$, a contradiction.

- If $\{a, b, d\} \in \mathcal{H}$ and $\beta + \gamma \neq 0$, then the first assumption implies that $\{a, b, d\} \in \mathcal{H}$, while the second implies that $\{a, b, d\} \in \mathcal{H}$, a contradiction.

Case M9(b)i: If $\alpha + \gamma = 0$, and that $\beta + \gamma = 0$, then $\gamma = 0$, a contradiction.

Case M9(b)ii: Assume that $\{a, b, d\} \notin \mathcal{H}$, $\alpha + \gamma \neq 0$, and that $\beta + \gamma = 0$. Then

$$(\{b, d, x\}, \{c, e, x\} \in \mathcal{H}) \wedge (\{a, e, x\} \notin \mathcal{H}).$$

Case M9(b)iii: Assume that $\{a, b, d\} \notin \mathcal{H}$, $\alpha + \gamma = 0$, and that $\beta + \gamma \neq 0$. Then

$$(\{b, d, x\}, \{a, e, x\} \in \mathcal{H}) \wedge \{c, e, x\} \notin \mathcal{H}.$$

Case M9(b)iv: Assume that $\{a, b, d\} \notin \mathcal{H}$, $\{a, e, x\}, \{c, e, x\} \in \mathcal{H}$, $\alpha + \gamma \neq 0$, and that $\beta + \gamma \neq 0$. Then $\{b, d, x\} \in \mathcal{H}$, and

$$\alpha\Gamma(a, e, x) + \beta\Gamma(c, e, x) + \gamma\Gamma(b, d, x) = \alpha + \beta + \gamma = \gamma,$$

which is equal to zero only if $\gamma = 0$, a contradiction to $M9$ having support equal to zero.

Case M9(b)v: Assume that$\{a, b, d\} \notin \mathcal{H}$, $\{a, e, x\} \notin \mathcal{H}$, $\{c, e, x\} \in \mathcal{H}$, $\alpha + \gamma \neq 0$, and that $\beta + \gamma \neq 0$. Then

$$\alpha\Gamma(a, e, x) + \beta\Gamma(c, e, x) + \gamma\Gamma(b, d, x) = \beta + \gamma \neq 0,$$

A contradiction to $M19$ being a derivation.

Case M9(b)vi: Assume that $\{a, b, d\} \notin \mathcal{H}$, $\{a, e, x\} \in \mathcal{H}$, $\{c, e, x\} \notin \mathcal{H}$, $\alpha + \gamma \neq 0$, and that $\beta + \gamma \neq 0$. Then

$$\alpha\Gamma(a, e, x) + \beta\Gamma(c, e, x) + \gamma\Gamma(b, d, x) = \alpha + \gamma \neq 0,$$

Case M9(b)vii: Assume that $\{a, b, d\} \notin \mathcal{H}$, $\{a, e, x\}, \{c, e, x\} \notin \mathcal{H}$, $\alpha + \gamma \neq 0$, and that $\beta + \gamma \neq 0$. Then

$$\alpha\Gamma(a, e, x) + \beta\Gamma(c, e, x) + \gamma\Gamma(b, d, x) = \gamma \neq 0,$$

Now, whether any of $\{a, b, e\}$, $\{a, b, x\}$, or $\{a, x, y\}$ are elements of $\mathcal{H}$ does not lead to further implications beyond the existence (or lack thereof) of hyperedges, and thus does not lead to additional subcases. We can therefore simplify and rename our cases for $M9$ as follows:

Case M9(a): If $\alpha + \beta \neq 0$, then

$$\{a,b,d\},\{b,c,d\},\{a,b,x\},\ \{b,c,x\},\ \{a,b,e\},\ \{b,c,e\},\ \{a,x,y\},\ \{c,x,y\},\ \{a,c,e\} \notin \mathcal{H}$$
$$\{b,d,x\},\{a,e,x\},\{c,e,x\} \in \mathcal{H}$$
$$\alpha + \beta + \gamma = 0$$

Case M9(b): If $\alpha + \beta = 0$, $\{a,b,d\} \notin \mathcal{H}$, $\alpha + \gamma \neq 0$, and $\beta + \gamma = 0$, then

$$\{b,d,x\},\{c,e,x\} \in \mathcal{H}$$
$$\{a,e,x\} \notin \mathcal{H}$$

Case M9(c): If $\alpha + \beta = 0$, $\{a,b,d\} \notin \mathcal{H}$, $\alpha + \gamma = 0$, and $\beta + \gamma \neq 0$, then

$$\{b,d,x\},\{a,e,x\} \in \mathcal{H}$$
$$\{c,e,x\} \notin \mathcal{H}$$

For the remaining matrices, we omit detailed proofs, as they follow analogous arguments to those already presented. Moreover, matrices with support greater than three need not be considered. By Lemma 4.2.1, the matrices discussed above generate the full Lie algebra of derivations. □

4.4 Returning to high school

With Theorem 4.3.3, we can give a more accurate analysis of our High School data set. The nontrivial derivations arose due to specific configurations of hyperedges in the hypergraph. From here, it is simply a matter of testing which specific derivations exist, and by Theorem 4.3.3, we identify what relations exist between vertices. To begin, we select two additional random samples of 80 students. We chose more students to increase the likelihood that multiple examples of each case in Theorem 4.3.3 arise. Our current implementation of the algorithm requires us to manually check all possible matrices of the forms listed in each case, and due to memory constraints, we only check cases 1 and 2 of the main theorem against these hypergraphs.

4.4.1 Class A

In Figure 4.21, we see the first of our two additional samplings. In this instance, the stratification lies on a surface, with a more central cluster as we saw before in Figure 1.1.

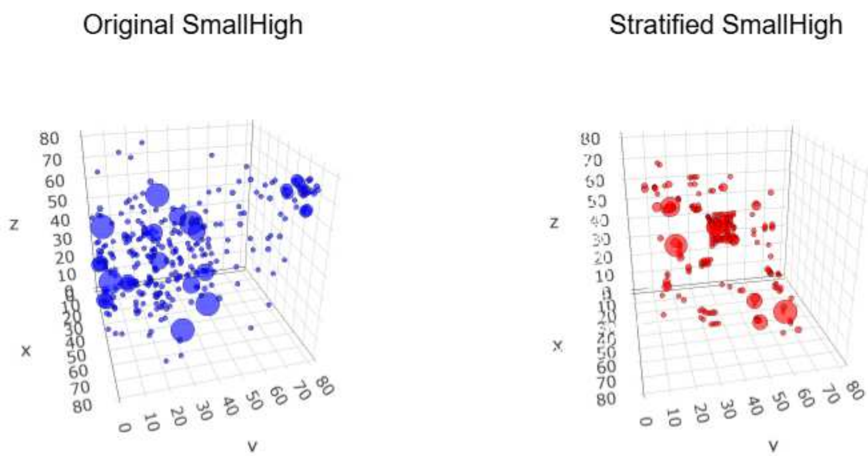


Figure 4.21: Our second sampling of the high school students, along with their stratification

In this stratification, 10 derivations were found. When applying the results of Theorem 4.3.3, no examples of Case 1 were found, but five examples of Case 2 were found. After relabeling the vertices, they were:

$$E_{3,4} - E_{13,41}, \quad E_{12,17} - E_{45,33}, \quad E_{15,9} - E_{37,26},$$
$$E_{18,7} - E_{45,23}, \quad E_{34,31} - E_{36,24}.$$

If we consider $E_{12,17} - E_{45,33}$, this indicates that the hypergraph contains the substructures $\{12, 33, x\}$ and $\{17, 45, x\}$. Indeed, we have the hyperedges $\{7, 12, 33\}$ and $\{7, 17, 45\}$, and no other such hyperedges exist in the hypergraph. Furthermore, $E_{18,7} - E_{45,23}$ informed us that we had hyperedges $\{18, 23, x\}$ and $\{7, 45, x\}$, the hypergraph, we have the hyperedges $\{17, 18, 23\}$ and $\{7, 45, 17\}$. This indicates that the hypergraph has the structure shown in Figure 4.22.

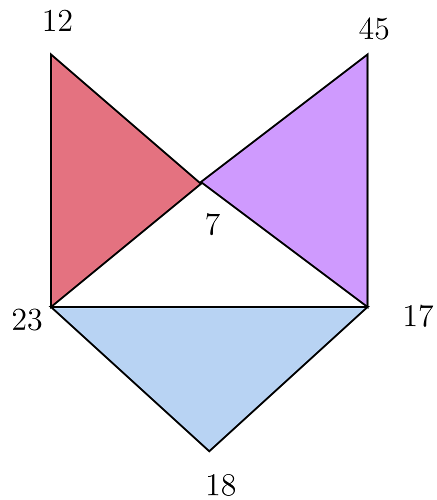


Figure 4.22: Sub-hypergraph present in our second sampling of the high school students

4.4.2 Class B

Figure 4.23 shows our third and final stratification of a sample of high school students. As before, the stratification shows a central cluster lying on a surface.

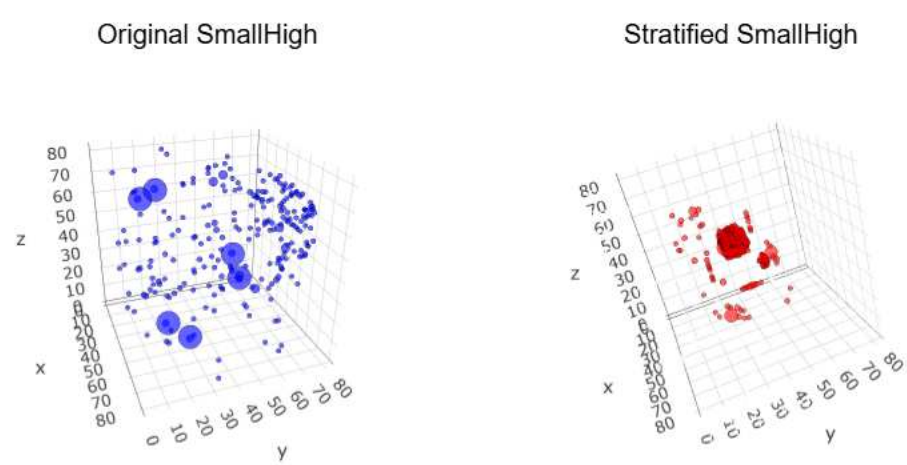


Figure 4.23: Our third sampling of the high school students, along with their stratification

In this class, all 35 of the derivations of Case 1, after relabeling, were in the form $E_{10,x} - E_{25,x}$. Indeed, $\{10, 23, 35\}$ and $\{25, 23, 35\}$ are the only hyperedges containing 10 and 23, respectively. For case 2, four examples were found. After relabeling, they are:

$$E_{1,22} - E_{31,27}, \quad E_{2,5} - E_{19,35}, \quad E_{2,23} - E_{10,12}, \quad E_{2,23} - E_{25,12}.$$

From $E_{1,22} - E_{31,27}$, we find the hyperedges $\{1, 2, 7, 29\}$ and $\{22, 29, 31\}$, and no other vertices exist such that $\{1, 27, x\}$ and $\{22, 31, x\}$ are hyperedges. From $E_{2,5} - E_{19,35}$, we find the hyperedges $\{2, 12, 35\}$ and $\{5, 12, 19\}$. $E_{2,23} - E_{10,12}$ being a derivation implies that $\{10, 23, x\}$ is a hyperedge if and only if $\{2, 12, x\}$ is a hyperedge. We have already seen that $\{10, 23, 25\}$ is the only hyperedge containing both 10 and 23, so $\{2, 12, 35\}$ must be a hyperedge. Similarly, $\{2, 12, 35\}$ is a hyperedge. Figure 4.24 shows the configuration of these hyperedges.

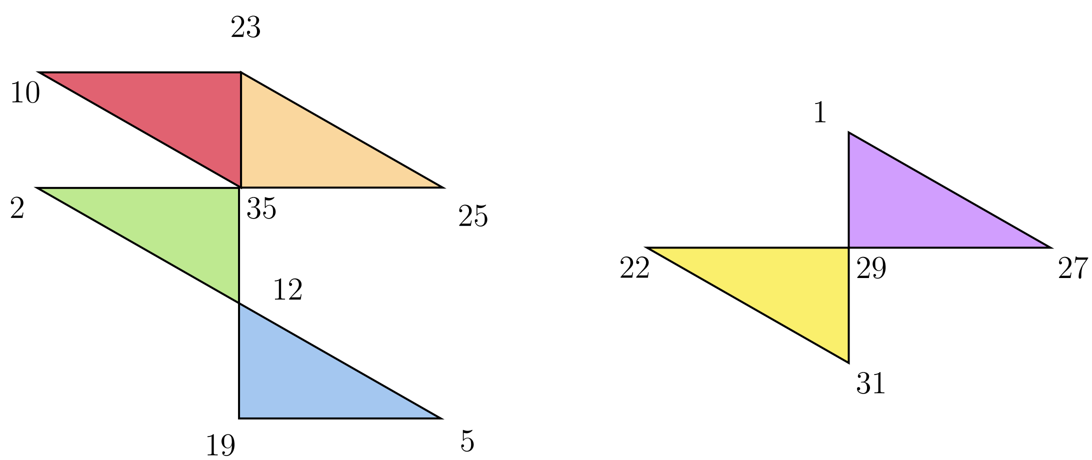


Figure 4.24: Sub-hypergraph present in our third sampling of the high school students

While more relations can be uncovered using Theorem 4.3.3, any further calculations would still respect these hyperedge configurations.

4.5 Hypergraphs which are ℓ-uniform

We end this dissertation with our main result, as well as an indication of the future work needed in this emerging field of the spectral theory of hypergraphs. While we have focused on 3-uniform hypergraphs, this theory extends to all uniform hypergraphs. The only additional work needed is to compute the equations that must be satisfied for a low-support matrix to be a derivation for an arbitrary ℓ-uniform hypergraph under study. While this is potentially an unbounded problem, as there are infinitely many hypergraphs and infinitely many ℓ, Theorem 4.5.3 shows that there are only finitely many equations to compute and analyze in the format Theorem 4.3.3.

To begin, we review first-order logic and the terms that are used in the statement of the theorem.

Definition 4.5.1. A *predicate* in set theory is a sentence using $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$, $\implies$, together with variables.

Example 4.5.2. In Theorem 4.3.3, the logical sentence

$D = \alpha E_{ab} + \beta E_{cb}$ only if

- $\{a, b, c\}, \{a, b, x\} \notin \mathcal{H}$ and
- $(\{a, x, y\} \in \mathcal{H} \iff \{c, x, y\} \in \mathcal{H})$,

is a predicate. Furthermore, the predicates referenced in Theorem 4.5.3 have this structure. The predicates of Theorem 4.3.3 were constructed by computing

$$\Gamma(Da, b, c) + \Gamma(a, Db, c) + \Gamma(a, b, Dc),$$
$$D = \alpha E_{qr} + \beta E_{st} + \gamma E_{xy}.$$

These predicates are indexed by a generalized matrix form as listed in Figure 4.4 and by a tuple of basis elements, as listed in Figure 4.5. The generating predicate for this fixed matrix and input is the sentence "$D \in \mathrm{Der}(\Gamma)$ only if $\Gamma(Da, b, c) + \Gamma(a, Db, c) + \Gamma(a, b, Dc) = 0$". We get the "if and only if" in our predicates by combining the predicates over all basis element tuple indices, as "$D \in \mathrm{Der}(\Gamma)$ if only if $\Gamma(Da, b, c) + \Gamma(a, Db, c) + \Gamma(a, b, Dc) = 0$ for all $a, b, c \in \mathcal{H}$". The proofs of each case in Theorem 4.3.3 provide a hypergraph interpretation for each predicate. In the proof for Theorem 4.3.3, tautological sentences occurred. These are predicates that hold trivially. For example, for the generic matrix $M2 = \alpha E_{aa} + \beta E_{ab} + \gamma E_{ac}$, if we construct the predicate indexed by $M1$ and the tuple (a, a, x), then we get the sentence "$D \in Der(\Gamma)$ if and only if

$$\Gamma(Da, a, x) + \Gamma(a, Da, x) + \Gamma(a, a, Dx) = \Gamma((\alpha E_{aa} + \beta E_{ab} + \gamma E_{ac})a, a, x)$$
$$+\Gamma(a, (\alpha E_{aa} + \beta E_{ab} + \gamma E_{ac})a, x) + \Gamma(a, a, (\alpha E_{aa} + \beta E_{ab} + \gamma E_{ac})x)$$
$$= \Gamma(\alpha a, a, x) + \Gamma(a, \beta a, x) + \Gamma(a, a, 0)$$

equals zero". As Γ is the adjacency matrix of a hypergraph, $\Gamma(a, a, x)$ is equal to zero, and as a tensor $\Gamma(a, a, 0)$ also equals zero. Furthermore, the variable in this predicate is x, where x is simply some vertex that isn't labeled a, b, or c. This predicate holds true trivially, and as such, is a tautological sentence. This can be thought of as not providing us with actionable information about the hypergraph's structure.

For Theorem 4.5.3, we obtain our bounds of c and d via the natural and adjoint representations of simple Lie algebras as described in Section 4.2. For another representation, the bound for the support of the generators may be another natural number. In that case, the analog of this theorem and its proof still hold, but with different bounds.

Theorem 4.5.3. Fix $\alpha, \beta, \gamma \in \mathbb{C}$. For all $\ell \in \mathbb{N}$, and for all vertex sets V, there exists a hypergraph $\mathcal{H} \subset 2^V$ on n vertices, and there exists index tuples $(q, r), (s, t), (x, y) \in V \times V$, such that $\alpha E_{qr} + \beta E_{st} + \gamma E_{xy} \in \mathrm{Der}_\ell(\mathcal{H})$ if and only if $\exists c \in \mathbb{N}$, $\exists d \in \mathbb{N}$, $\exists$ first order non-tautological predicates $P_1, \ldots, P_c$ in $x_1, \ldots, x_d$ variables such that for all $v_* = \{v_1, \ldots, v_d\} \subset V$, $P_1(v_*) \wedge \cdots \wedge P_c(v_*)$ hold.

Proof. Fix $\alpha, \beta, \gamma \in \mathbb{C}$. Let ℓ be arbitrary, let $\mathcal{H}$ be an arbitrary hypergraph on $n = |V|$ vertices, and let $D = \alpha E_{qr} + \beta E_{st} + \gamma E_{xy}$ such that $D \in \mathrm{Der}_\ell(\Gamma)$. We explicitly construct the first-order predicates, and in this construction we provide bounds for c and d. As $D \in \mathrm{Der}_\ell(\Gamma)$, by definition

$$\sum_{i=1}^{\ell} \Gamma(Du_i, u_{\bar{i}}) = 0$$

for all choices of $\{u_i\}_{i=1}^{\ell}$, where u_i is a standard basis vector of $\mathbb{F}^n$. Each choice of $\{u_i\}_{i=1}^{\ell}$ generates a predicate on ℓ variables. We say that the equation holds trivially for a choice of $\{u_i\}_{i=1}^{\ell}$ if $\Gamma(Du_i, u_{\bar{i}}) = 0$ for all i. This is equivalent to stating that the predicate generated by this choice of basis vectors is a tautological sentence. Fix an arbitrary $\{u_i\}_{i=1}^{\ell}$. We consider the first m summands of this derivation equation.

$$\begin{aligned}
0 = \sum_{i=1}^{\ell} \Gamma(Du_i, u_{\bar{i}}) &= \sum_{i=1}^{\ell} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}}) \\
= \sum_{i=1}^{m} &\Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}}) \\
+ \sum_{i=m+1}^{\ell} &\Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}}).
\end{aligned}$$

Since Γ is symmetric, we can choose m small enough and permute our ordering of $\{u_i\}_{i=1}^{\ell}$ so that $\{u_i\}_{i=1}^{m}$ is exactly the set $A = \{e_p \in \{u_i\}_{i=1}^{\ell} \mid p \in \{q, r, s, t, x, y\}\}$, which is to say the elements of $\{u_i\}_{i=m+1}^{\ell}$ are exactly the elements whose index are not in $\{q, r, s, t, x, y\}$. We denote this complement as A^C. Finally, for $v \in \{u_i\}_{i=1}^{\ell}$, $\overline{v}$ refers to the complement of v in $\{u_i\}_{i=1}^{\ell} \setminus \{v\}$, where this complement respects repetitions, i.e., $\overline{v}$ in the set $\{u, v, v\}$ is $\{u, v\}$. We can then rewrite our derivation equation as

$$\begin{aligned}
0 = &\sum_{v \in A} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overline{v}) \\
&+ \sum_{v \in A^c} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overline{v}).
\end{aligned}$$

Since the index of $v \in A^C$ does not have values of r, t, or y, $(\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v = 0$. Therefore, the above equation simplifies to

$$0 = \sum_{v \in A} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overline{v}).$$

We now show how large m can be so that this equation is a non-tautological predicate, i.e., does not hold trivially. We once again consider $v \in A$. Without loss of generality, let R be the set of $v \in A$ such that $v = e_r$, R^C the complement in A of R, and $k = |R|$. Then, after reordering,

$$\begin{aligned}
&\sum_{v \in A} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overline{v}) \\
=&\sum_{v \in R} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})e_r, \overbrace{e_r, \ldots, e_r}^{k-1}, \overline{v}) \\
+&\sum_{v \in R^C} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overbrace{e_r, \ldots, e_r}^{k}, \overline{v})
\end{aligned}$$

Now, if $k \geq 3$, then both terms are equal to zero, and so the derivation equation is trivially satisfied. Therefore, $|R| < 3$. We do the same for the indices t and y. This tells us that there are at most 6 elements in A whose indices are r, t, or y. We now examine the indices q, s, and x to determine other restrictions on the elements of A. Without loss of generality, let Q be the set of $v \in A$ such that $v = e_q$, Q^C the complement in A of Q, and $k = |Q|$. Then, after reordering,

$$\begin{aligned}
&\sum_{v \in A} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overline{v}) \\
=&\sum_{v \in Q} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})e_q, \overbrace{e_q, \ldots, e_q}^{k-1}, \overline{v}) \\
+&\sum_{v \in Q^C} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overbrace{e_q, \ldots, e_q}^{k}, \overline{v}) \\
=&\sum_{v \in Q^C} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overbrace{e_q, \ldots, e_q}^{k}, \overline{v})
\end{aligned}$$

Now, if $k \geq 2$, then $\Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})v, \overbrace{e_q, \ldots, e_q}^{k}, \overline{v}) = 0$ trivially. Therefore, $|Q| \leq 1$. The same holds for the indices s and x. Therefore, there are at most three elements in A which have q, s, or x as their index.

Together, A can have no more than nine elements, which means that m is bounded above by nine. This means that for $\ell \geq 9$, after our reordering, the terms of $\sum_{i=10}^{\ell} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}})$ always equal zero.

The final stage of this proof requires a hypergraph contraction. We consider

$$\sum_{i=1}^{10} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}}) = 0.$$

The set $\{u_i\}_{i=10}^{\ell}$ is a set of *complementary variables*. These variables are not in the set $\{q, r, s, t, x, y\}$, and they are never acted on by D. We apply a hypergraph contraction along the vertices $\{u_i\}_{i=11}^{\ell}$. If we take these basis vectors and construct a hypergraph contraction so that only $\{u_i\}_{i=1}^{10}$ remain, then the predicate $\sum_{i=1}^{10} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}}) = 0$ can be written as the predicate $\sum_{i=1}^{10} \Gamma'((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{10}) = 0$. This is a predicate for the 10-uniform hypergraphs, and it is a predicate on ten variables. This gives us our bound of $d = 10$. Moreover, all the predicates of the ℓ-uniform hypergraphs are of the form $\sum_{i=1}^{10} \Gamma((\alpha E_{qr} + \beta E_{st} + \gamma E_{xy})u_i, u_{\bar{i}}) = 0$, and so all such predicates can be contracted to a predicate of the 10-uniform hypergraphs. There are therefore c such predicates, where c is the number of predicates for the 10-uniform hypergraphs. The value of c has yet to be computed, but it is indeed bounded. There are only finitely many matrices of support less than or equal to three and finitely many choices of ten basis vectors with the condition that at most 9 of the basis vectors have indices in the set $\{q, r, s, t, x, y\}$, and the remaining basis vectors are denoted as complementary variables. The exact value of c can be found by replicating the proof of Theorem 4.3.3. □

Corollary 4.5.4. Let $\ell \geq 10$, and let Γ be an ℓ-uniform hypergraph. Then the representations of $\mathrm{Der}(\Gamma)$ are characterized by the representations of $\mathrm{Der}(\Omega)$ for some 10-uniform hypergraph Ω.

Proof. By Theorem 4.5.3, $\mathrm{Der}(\Gamma)$ is equal to $\mathrm{Der}(\Omega)$ for some 10-uniform hypergraph Ω. Therefore, the representations of $\mathrm{Der}(\Omega)$ are exactly the representations for $\mathrm{Der}(\Gamma)$. □

With this theorem, we see that all possible derivations D with support less than or equal to three, as well as the hypergraph structures they imply, can be determined for any ℓ-uniform hypergraph. The work necessary to understand the remaining seven cases is beyond the scope of this dissertation, but will be studied in future work. In practice, these proofs follow the proof of Theorem 4.3.3. This can be implemented as a proof by exhaustion in a standard programming language, as Theorems 4.3.3 and 4.5.3 show that there are only finitely many non-tautological predicates to check. Furthermore, the proof of Theorem 4.3.3 provides a framework for relating each predicate to a specific configuration of hyperedges. For higher valence, these relations take different forms, but the method of uncovering the hyperedge configuration implied by the predicate will be similar to Theorem 4.3.3. Finally, the current implementation of the algorithms presented in this dissertation is not optimized. The packages in [6] are efficient but have not yet been optimized to take advantage of the symmetry and sparsity present in adjacency tensors. The results of Theorems 4.3.3 and 4.5.3 will also need to be implemented in [6] as well as other software used to study hypergraphs.

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