Dissertation

Continued Exploration of Nearly Continuous Kakutani Equivalence

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Bethany Diane Springer

Department of Mathematics

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Doctoral Committee:

Advisor: Patrick Shipman
Co-Advisor: Andres del Junco

Richard Eykholt
Gerhard Danglemayr
Rachel Pries

Abstract

Continued Exploration of Nearly Continuous Kakutani Equivalence

Nearly continuous dynamical systems, a relatively new field of study, blends together topological dynamics and measurable dynamics/ergodic theory by asking that properties hold modulo sets both meager and of measure zero. In the measure theoretic category, two dynamical systems (X, T) and (Y, S) are called Kakutani equivalent if there exists measurable subsets $A \subset X$ and $B \subset Y$ such that the induced transformations T_A and S_B are measurably conjugate. We say that a set $A \subset X$ is nearly clopen if it is clopen in the relative topology of a dense G_δ subset of full measure. Nearly continuous Kakutani equivalence refines the measure-theoretic notion by requiring the sets A and B to be nearly clopen and T_A and S_B to be nearly continuously conjugate. If A and B have the same measure, then we say that the systems are nearly continuously evenly Kakutani equivalent. All irrational rotations of the circle and all odometers belong to the same equivalence class for nearly continuous even Kakutani equivalence. For our first main result, we prove that if A and B are nearly clopen subsets of the same measure of X and Y respectively, and if ϕ is a nearly continuous conjugacy between T_A and S_B, then ϕ extends to a nearly continuous orbit equivalence between T and S. We also prove that if $A \subset X$ and $B \subset Y$ are nearly clopen sets such that the measure of A is larger than the measure of B, and if T is a nearly uniquely ergodic transformation and T_A is nearly continuously conjugate to S_B, then there exists $B' \subset Y$ such that X is nearly continuously conjugate to $S_{B'}$. We then introduce the natural topological analog of rank one transformations, called strongly rank one transformations, and show that all strongly rank one transformations are nearly continuously evenly Kakutani equivalent to the class containing all adding machines. Our main result proves that all minimal isometries of compact metric spaces are nearly continuously evenly Kakutani equivalent to the binary odometer.

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Table of Contents

List of Figures

CHAPTER 1

Introduction

1.1. Nearly Continuous Dynamics

In the 1979 paper, *Almost Topological Dynamical Systems*, Manfred Denker and Michael Keane [8] set out to systematically lay the framework to study the interplay between the topological and metrical properties of dynamical systems, which was of recent interest in ergodic theory. In ergodic theory, one only asks that properties hold on a set of full measure, whereas, in topological dynamics, one requires properties to hold on the entire space. The authors gave a definition of an almost topological dynamical system on a compact metric space Ω with Borel probability measure m (with $\mathcal{F}$ as the completion of the Borel σ-field with respect to m) in the following manner: An almost topological dynamical system $(\Omega, \mathcal{F}, m)$ is given by a subset Ω_1 and and a map $T : \Omega_1 \to \Omega_1$ where $\Omega_1 \in \mathcal{F}$ is a residual subset of full measure, and T is a homeomorphism in the induced topology of Ω_1 which preserves the probability measure m restricted to Ω_1. In the authors' definition of almost topological or finitary conjugacy, between two almost topological dynamical systems T on $(\Omega, \mathcal{F}, m)$ and T' on $(\Omega', \mathcal{F}', m')$, with domains of definition Ω_1 and Ω_1', they ask for $\Omega_2 \subseteq \Omega_1$, $\Omega_2' \subseteq \Omega_1'$ such that $\phi : \Omega_2 \to \Omega_2'$ where $\Omega_2 \in \mathcal{F}$ and $\Omega_2' \in \mathcal{F}'$ are residual subsets of full measure in Ω and Ω', ϕ is continuous, surjective, and measure-preserving, and $\phi T = T'\phi$ on Ω_2. Denker and Keane's paper focuses on describing invariants of finitary isomorphism and demonstrates that finitary isomorphism is indeed strictly stronger than metrical isomorphism.

In the 2009 paper, *Measured Topological Orbit and Kakutani Equivalence* by A. del Junco, D. Rudolph, and B. Weiss, [5] the authors recast and add to the theory set forth by Denker and Keane [8]. The definition of a dynamical system shifts to requiring that X be a Polish probability space, and T be an ergodic measure preserving homeomorphism on X. Polish spaces provide the natural setting for this theory. If one starts with a compact metric space and then removes a meager set of measure zero, one creates a Polish space. The authors also change the terminology, replacing "almost topological" with "nearly continuous" to avoid confusion with the description that a function is almost continuous. We add to the basic results from [8] and [5]. Herein, we name the theory *nearly continuous dynamics*, and require properties to hold on sets considered large both in measure-theoretic and topological settings, i.e. on the complement of a meager set of measure zero.

1.2. Nearly Continuous Kakutani Equivalence

Kakutani equivalence between two ergodic dynamical systems T on X and S on Y, as introduced by S. Kakutani in the 1940s [14] asks for the existence of measurable sets $A \subset X$ and $B \subset Y$ of positive measure such that T_A and S_B are measurably conjugate. Dye [9] later defined orbit equivalence between two dynamical systems to be a measurable map $\phi : X \to Y$ which maps orbits in X to orbits in Y. Dye then proved that any two ergodic measure preserving transformations are orbit equivalent. A. del Junco and A. Şahin [6] generalized Dye's Theorem to the nearly continuous category in the following way: Given any two ergodic homeomorphisms T and S of Polish probability spaces X and Y, there exist full mueasure subsets $X_0 \subset X$ and $Y_0 \subset Y$ together with an orbit equivalence $\phi : X_0 \to Y_0$ such that ϕ is a measure preserving homeomorphism from X_0 to Y_0. We say that a set $A \subset X$ is nearly clopen if it is clopen in the relative topology of a dense G_δ subset of full measure.

For two ergodic homeomorphisms T and S of Polish probability spaces X and Y respectively, the authors of [5] defined even Kakutani equivalence in the nearly continuous category to be given by a nearly continuous orbit equivalence $\phi : X_0 \to Y_0$ where X_0 and Y_0 are G_δ subsets of full measure in X and Y respectively, and a nearly clopen set A of positive measure such that ϕ restricted to A is a conjugacy between T_A and $S_{\phi(A)}$. In this thesis, we show that if one starts with a nearly continuous conjugacy ϕ between induced transformations T_A and S_B on nearly clopen subsets $A \subset X$ and $B \subset Y$ of the same measure, then ϕ extends to a nearly continuous orbit equivalence between X and Y. This result demonstrates that in [5], the requirement for an orbit equivalence between the systems is superfluous to the definition of nearly continuous even Kakutani equivalence. An ergodic measure preserving homeomorphism T on a Polish probability space X is said to be nearly uniquely ergodic if the ergodic averages converge unformly to the integral on a G_δ subset of full measure for any function bounded and continuous on a G_δ subset of full measure. We prove that if X and Y are Polish probability spaces, if T is a nearly uniquely ergodic transformation of X and S is an ergodic measure preserving homeomorphism of Y, and if $A \subset X$ and $B \subset Y$ are nearly clopen subsets with $\mu_X(A) > \mu_Y(B)$, and T_A is nearly continuously conjugate to S_B, then S induces T on a nearly clopen subset.

In [5], the authors construct an example which demonstrates that nearly continuous Kakutani equivalence is strictly stronger than Kakutani equivalence. In the quest to understand the nature of nearly continuous Kakutani equivalence, Roychowdhury and Rudolph in [18] prove that all odometers are nearly continuously evenly Kakutani equivalent to the binary odometer. Dykstra and Rudolph in [10] prove that all irrational rotations are nearly continuously evenly Kakutani equivalent to the binary odometer. Our main result, that all minimal isometries of compact metric spaces are nearly continuously evenly Kakutani equivalent to the binary odometer, encompasses the results in [18] and [10] and answers the question posed in [10] about this more general class of systems.

1.3. Organization of Thesis

We organize the thesis as follows: Chapter Two introduces nearly continuous dynamical systems, nearly clopen sets, and various notions of near equivalence.

In Chapter Three we focus on the nearly continuous equivalents of skyscrapers, towers, and cutting and stacking. Towers provide a powerful tool in ergodic theory, and deserve attention in the nearly continuous category. We prove a nearly continuous version of Rokhlin's Tower lemma as well as nearly continuous version of the Alpern multi-tower lemma. Our attention turns to cutting and stacking as we show that any nearly continuous dynamical system is nearly continuously conjugate to a cutting and stacking of the unit interval. In Chapter Three we define a class of transformations called strongly rank one, the natural analog of rank one transformations from ergodic theory, and then show that any such transformation is nearly continuously conjugate to a rank one cutting and stacking of the unit interval.

In Chapter Four, we consider the problem of extending nearly continuous conjugacies on induced transformations in relation to nearly continuous Kakutani equivalence. Let T and S be ergodic measure-preserving homeomorphisms of Polish probability spaces X and Y respectively. If A and B are nearly clopen subsets of X and Y respectively such that A and B have the same positive measure, and if ϕ is a nearly continuous conjugacy between T_A and S_B then ϕ extends to a nearly continuous orbit equivalence between X and Y. We then explore the possibility of modifying and extending a nearly continuous conjugacy between induced transformations on nearly clopen sets $A \subset X$ and $B \subset Y$ where the measure of A is greater than that of B in order to find a nearly clopen subset $B' \subset Y$ such that T is nearly continuously conjugate to $S_{B'}$. We will see that this is not possible in general unless the appropriate system is nearly uniquely ergodic. We then apply this result to show that all strongly rank one transformations are nearly continuously evenly Kakutani equivalent to the binary odometer.

In Chapter Five, we introduce our version of the tower and template machinery in order to prove the main result that minimal isometries of compact metric spaces are nearly continuously evenly Kakutani equivalent to the binary odometer and hence to all other minimal isometries of compact metric spaces.

CHAPTER 2

Background

We consider a Polish space X which is a dense subset of a compact metric space, endowed with a non-atomic Borel probability measure μ of full support. Polish spaces are topological spaces homeomorphic to complete separable metric spaces. The topology τ is the Polish topology, and whenever we write (X, τ, μ) we mean precisely such a space, referred to as a *Polish probability space*. The proof of the following theorem may be found in [12]

Theorem 2.0.1. *Let X be a Polish space.*

(1) (P. Alexandrov) A G_δ subset of X is also Polish.

(2) If a subspace of X is Polish, then it is G_δ.

By analogy with measurable dynamics and ergodic theory, we will neglect meager null sets.

Definition 2.0.2. *A G_δ subset of full measure $X_0 \subseteq X$ is called a* ***nearly full*** *subset.*

Because the support of μ is X, any nearly full subset is dense in X.

2.1. Nearly Clopen Sets

Definition 2.1.1. *A set $C \subseteq X$ is said to be* ***nearly clopen*** *if there exists a nearly full subset $X_0 \subseteq X$ such that $C \cap X_0$ is clopen in the relative topology of X_0.*

Proposition 2.1.2. *Let (X, τ, μ) be a Polish probability space. The following are equivalent:*

(1) a set $C \subseteq X$ is nearly clopen;

(2) there exists an open set $A \subset X$ and a closed set $B \subset X$ such that $\mu(A\Delta C) = 0$ and $\mu(B\Delta C) = 0$;

(3) there exists an open set $A \subset X$ such that $\mu(A\Delta C) = 0$ and $\mu(\partial A) = 0$.

PROOF. First, we show that 1 and 2 are equivalent. Suppose that $C \subseteq X$ is nearly clopen. Then, there exists a nearly full set $X_0 \subseteq X$ such that $C \cap X_0$ is clopen in X_0. There exists an open set A such that $C \cap X_0 = A \cap X_0$. $A \triangle C \subset X_0^C$ which implies that $\mu(A \triangle C) = 0$ and C is within measure zero of an open set. The argument that C is within measure zero of a closed set B is essentially the same.

Next, suppose that $C \subseteq X$ is within measure zero of open and within measure zero of closed. Let A be an open set within measure zero of C and let B be a closed set within measure zero of C. As X has full support, $A \subseteq C \subseteq B$ and $\partial C \subset (B \cap A^c)$ has measure zero. Let $X_0 = X - (B \cap A^c)$. Then X_0 is a nearly full set with $A \cap X_0 = C \cap X_0$ and $B \cap X_0 = C \cap X_0$, and C is nearly clopen in X_0.

The proof that 2 and 3 are equivalent is similar to the proof of Lemma 2.7 from [5]. First assume there exists an $A \subset X$ and $B \subset X$ open and closed, respectively, such that $\mu(A\Delta C) = 0$ and $\mu(B\Delta C) = 0$. Then $A \cap B^c$ is open and $\mu(A \cap B^c) = 0$, which implies that $A \subseteq B$ and $\mu(\partial A) = 0$.

Next, suppose there exists an open set $A \subset X$ such that $\mu(A\Delta C) = 0$ and $\mu(\partial A) = 0$. Then $\bar{A}$ is a closed set such that $\mu(\bar{A}\Delta C) \leq \mu(A\Delta C) + \mu(\partial A) = 0$.

□

DEFINITION 2.1.3. *Let (X, τ, μ) be a Polish probability space. Two sets $A \subseteq X$ and $B \subseteq X$ are said to be **nearly equal** if there exists a nearly full set X_0 such that $A \cap X_0 = B \cap X_0$.*

We remark that if A is a nearly clopen set and B is nearly equal to A, then B is a nearly clopen subset with $\mu(A) = \mu(B)$. Near equality of nearly clopen sets is an equivalence relation.

Let $\mathcal{A} = \mathcal{A}(X)$ denote the set of equivalence classes of nearly equal clopen sets of a Polish probability space $(X, \tau\, u)$. It is straightforward to show that $\mathcal{A}$ is a Boolean algebra.

LEMMA 2.1.4. *([6] Theorem 2) Suppose (X, τ, μ) is a Polish probability space. There exists a nearly full subset $X_0 \subset X$, such that the topology restricted to X_0 has a countable base consisting of clopen sets.*

PROOF. Let d be a complete metric on X, and let $\{x_i\}$ be a countable, dense subset of X. Let $S(x_i, r) = \{y : d(x_i, y) = r\}$ be the metric sphere of radius r centered at x_i. As only countably many of the spheres $S(x_i, r)$ may have positive measure, find a countable, dense subset $R_{x_i} \subset \mathbb{R}^+$ for each x_i such that $\mu(S(x_i, r)) = 0$ for all $r \subset R_{x_i}$. Let

$$X_0 = \left(\bigcup_i \bigcup_{r \in R_{x_i}} S(x_i, r) \right)^c .$$

Then X_0 is a G_δ of full measure whose topology is generated by the balls $B(x_i, r) = \{y : d(x_i, y) < r\} \cap X_0$ for $r \in R_{x_i}$.

□

LEMMA 2.1.5. *([5] Lemma 2.8) Suppose (X, τ, μ) is a nonatomic Polish probability space, A is a measurable set, and $\epsilon > 0$. Then, there is a nearly clopen set C such that $\mu(C \Delta A) < \epsilon$.*

PROOF. Select an open set U and a closed set E with $E \subset A \subset U$ and $\mu(U) - \mu(E) < \epsilon$. As any Polish space may be metrized as a dense G_δ subset of a compact metric space, there exists a continuous function $f : X \to [0,1]$ which is 0 on E and 1 on $X \backslash U$. There must exist a value $t \in (0,1)$ so that $\mu(\{x | f(x) = t\} = 0$. The set $C = \{x | f(x) < t\}$ is open and is nearly equal to $C = \{x | f(x) \leq t\}$, i.e. C is an open set with boundary measure zero.

$\square$

LEMMA 2.1.6. *([5] Lemma 2.9) Suppose* (X, τ, μ) *is a nonatomic Polish probability space. For every* $\alpha \in [0,1]$ *there exists a nearly clopen set* $C \subseteq X$ *with* $\mu(C) = \alpha$.

PROOF. As the complement of any nearly clopen set is nearly clopen, we may assume w.l.o.g that $0 < \alpha \leq 1/2$. The balls $B_r(x)$ of radius r around x have measure $\mu(B_r(x))$ decreasing to zero and continuous at Lebesgue almost every r, so there are arbitrarily small open and almost closed balls around each point. Select a point $x_0 \in X$ and a radius $r_0 > 0$ with $0 < \mu(B_{r_0}(x_0)) < 1/2$ so that $B_{r_0}(x_0)$ is nearly clopen (i.e. its boundary has measure zero). Select a sequence of distinct points $\{x_i\} \to x_0$, all in $B_{r_0}(x_0)$ and radii $r_i > 0$ so that the $r_i \to 0$, all the balls $B_{r_i}(x_i)$ are nearly clopen, are in $B_{r_0}(x_0)$ and are disjoint and of positive measure. We construct our nearly clopen set of measure α as a union of nearly clopen subsets $C_0 \subset B_{r_0}(x_0)^c$ and $C_i \subset B_{r_i}(x_i)$. These sets will be disjoint. As each C_i will be nearly open, so is their union. As each is nearly closed, and the only nontrivial limit point for their union is x_0, the union is also nearly closed. To select the C_i, set $\alpha_i = \mu(B_{r_i}(x_i)) > 0$. Use the previous lemma on $B_{r_0}(x_0)^c$, which is a Polish space, first choose a subset A_0 with $\alpha - \frac{1}{2}\alpha_1 < \mu(A_0) < \alpha$ then use the lemma again to find a nearly clopen subset $C_0 \subset B_{r_0}(x_0)^c$ by approximating A_0 so well that we still have $\alpha - \alpha_1 < \mu(C_0) < \alpha$.

Suppose we have selected the subsets C_0,C_1,...,C_n so that $\alpha - \alpha_{n+1} < \sum_{i=0}^{n} \mu(C_i) < \alpha$. Set $\beta_{n+1} = \alpha - \sum_{i=0}^{n} \mu(C_i)$, the amount of measure still needed. As $\beta_{n+1} < \alpha_{n+1}$ we can select a subset $A_{n+1} \subset B_{r_{n+1}}(x_{n+1})$ with measure $\beta_{n+1} - \alpha_{n+2} < \mu(A_{n+1}) < \beta_{n+1}$. Now inside the nearly clopen set $B_{r_{n+1}}(x_{n+1})$ we can apply the above lemma and approximate A_{n+1} by a nearly clopen subset C_{n+1} so that we still have

$$\beta_{n+1} - \alpha_{n+2} = \alpha - \sum_{i=0}^{n} \mu(C_i) - \alpha_{n+2} < \mu(C_{n+1}) < \alpha - \sum_{i=0}^{n} \mu(C_i)$$

and hence $\alpha - \alpha_{n+2} < \sum_{i=0}^{n+1} \mu(C_i) < \alpha$, and the induction proceeds.

□

LEMMA 2.1.7. *([5] Lemma 2.10) Suppose (X, τ, μ) is a nonatomic Polish probability space, $A \subset X$ is measurable, and $\epsilon > 0$. There exists a nearly clopen set C such that $\mu(C) = \mu(A)$ and $\mu(C \Delta A) < \epsilon$.*

This follows from Lemma 2.1.6

A nearly clopen partition $\mathcal{P}$ of a Polish probability space (X, τ, μ) is a set of pairwise disjoint nearly clopen sets whose union $\bigcup_{p \in \mathcal{P}} p = X$ up to an F_σ of measure zero. Given a measurable set $C \subset X$, and metric d on X, the diameter of the set is given by

$$\operatorname{diam}(C) = \sup_{x,y \in C} d(x, y).$$

Given a nearly clopen partition $\mathcal{P}$ of X,

$$\operatorname{diam}\mathcal{P} = \sup_{p \in \mathcal{P}} \operatorname{diam}(p).$$

LEMMA 2.1.8. *Every Polish probability space (X, τ, μ) has a refining sequence of nearly clopen partitions, $\mathcal{P}_i$, which form a base for the topology of a nearly full subset $X_0 \subseteq X$.*

PROOF. For the set R_{x_i} defined as in Lemma 2.1.4, assign an ordering $\{r_1 > r_2 > r_3 > \ldots\}$. Define $\mathcal{P}_i$ to be the partition which arises from the disjointification of the following union of sets

$$\bigcup_{k=1}^{i}\bigcup_{j=1}^{i} B(x_j, r_k) \cup \left(\bigcup_{k=1}^{i}\bigcup_{j=1}^{i} B(x_j, r_k)\right)^c.$$

This gives a refining sequence of partitions which is a base for the topology of X_0 as defined the the proof of 2.1.4. Note that $\lim_{i\to\infty} \operatorname{diam}(\mathcal{P}_i) = 0$.

□

LEMMA 2.1.9. *If $\mathcal{P}$ is a countable nearly clopen partition of (X, τ, μ), and $\mathcal{Q}$ is a countable collection of partition elements of $\mathcal{P}$, then the union of all $q \in \mathcal{Q}$ is nearly clopen.*

PROOF. Let X_0 be the nearly full subset for which $\mathcal{P}$ is a partition consisting of clopen sets. Then $\bigcup_{q\in\mathcal{Q}} q$ is open in X_0. As $\bigcup_{p\in\mathcal{P}\setminus\mathcal{Q}} p$ is open in X_0, its complement $\bigcup_{q\in\mathcal{Q}} q$ is closed in X_0 and hence nearly clopen.

□

LEMMA 2.1.10. *([6] Lemma 2)*

(a) If $A \subset X$ is a nearly clopen set and $\epsilon > 0$, then there exists a partition of A into nearly clopen sets $A_1, A_2, \ldots, A_k$ such that $\mu(A_i) < \epsilon$ for $1 \leq i \leq k$. Thus, any nearly open set may be written as a countable disjoint union of nearly clopen sets of measure less than ϵ.

(b) Let $A \subset X$ be a nearly clopen set with $\mu(A) = a$. Let $\{a_i\}_{i\in I}$ for $I \subset \mathbb{N}$ be a list of real numbers such that $\sum_{i\in I} a_i = a$. There exists a nearly clopen partition of A into sets A_i for $i \in I$ such that $\mu(A_i) = a_i$.

PROOF. For (a), let $A \subset X$ be nearly clopen. There exists a countable basis $\{B_i\}_{i\in\mathbb{N}}$ of nearly clopen balls with $\mu(B_i) < \epsilon$. Define $A_1 = A \cap B_1$, and inductively

$$A_n = \left(A \setminus \left(\bigcup_{k=1}^{n-1} A_k \right) \cap B_n \right),$$

to give a partition of A with $\mu(A_n) \leq B_n < \epsilon$. If only finitely many A_i are nonempty, we are done. Otherwise, as $\sum_{i=1}^{\infty} \mu(A_n) = \mu(A)$, there exists an $N \in \mathbb{N}$ such that $\sum_{i=1}^{N} \mu(A_n) > \mu(A) - \epsilon$. We then replace A_n for $n > N$ with $\bigcup_{i>N} A_i$, which is nearly clopen by Lemma 2.1.9.

For (b), we construct a sequence of sets $A_1^n, A_2^n, \ldots$ of disjoint clopen sets such that for each i, A_i^n increases to the set A_i. Suppose that for a given n, we have that $a_i - \frac{1}{n} \leq \mu(A_i^n) \leq a_i$. By (a) we may partition the open set $C := A \setminus \bigcup_i A_i^n$ into nearly clopen subsets $C_1, C_2, \ldots$ of measure less than ϵ. If ϵ is sufficiently small, then by taking finite unions of the sets C_i we may find disjoint nearly clopen subsets $B_1, B_2, \ldots$ of C, all but finitely many empty, such that if we set $A_i^{n+1} = A_i^n \bigcup_i B_i$ then we have

$$a_i - \frac{1}{n+1} < \mu(A_i^{n+1}) < a_i.$$

$\square$

2.2. MAPS AND TRANSFORMATIONS

In topological dynamics, one asks for maps to be homeomorphisms on the entire space, whilst in measurable dynamics, one only asks that a map be measurable on a set of measure one. We blend the two notions by asking maps to be homeomorphisms on a nearly full set.

Definition 2.2.1. *Let $\phi : X \to Y$ be a transformation of Polish probability spaces (X, τ, μ) and (Y, τ, ν). If there exists nearly full subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that $\phi : X_0 \to Y_0$ is a homeomorphism, then ϕ is a* ***near homeomorphism*** *and the spaces are called* ***nearly homeomorphic****.*

Observe that any two Polish probability spaces are homeomorphic. We use the abbreviation n.c. for nearly continuous or nearly continuously.

Definition 2.2.2. *Let (X, τ, μ) be a Polish probability space. We call (X, τ, μ, T) such that T is an ergodic, measure-preserving homeomorphism of X a* ***nearly continuous dynamical system****.*

In metrical dynamics, if two functions agree on all but a null set, then we consider those functions to be equivalent. Naturally then, in nearly continuous dynamics, two functions are considered equivalent if they agree on all but a meager null set. Likewise, nearly clopen sets are equivalent if they agree on all but a meager null set. A T-invariant nearly full subset of X is called a carrier. The intersection of two carriers is a carrier. In proofs to follow in later chapters, it is often necessary to restrict T to a carrier $X_0 \subseteq X$. X_0 is then again a Polish space, $T : X_0 \to X_0$ is a homeomorphism, and studying (X_0, τ, μ, T) is equivalent to studying (X, τ, μ, T).

Nearly clopen sets play an important role in the study of induced transformations. Let (X, τ, μ, T) be a nearly continuous dynamical system, and $A \subseteq X$ be nearly clopen. There exists a nearly full set $X' \subset X$ such that $A \cap X'$ is clopen in X'. Let $X_1 = \bigcap_{i=-\infty}^{\infty} T^{-i}(X')$, giving a carrier of T such that $A_1 = A \cap X_1$ is clopen in X_1. A_1 is a nearly full subset of A.

Then, define the return time function $r_{A_1} : A_1 \to \mathbb{N}$ by

$$r_{A_1}(x) = \inf\{r > 0 : (T|_{X_1})^r(x) \in A_1\}.$$

LEMMA 2.2.3. *Let (X, τ, μ, T) be a n.c. dynamical system and let $A \subset X$ be nearly clopen. Then, the return time function to A is nearly continuous and T_A is nearly a homeomorphism on A.*

PROOF. Define X_1 be a carrier of T such that $A_1 = A \cap X_1$ is clopen in $X_!$. Define

$$B_r = \{x \in A_1 : r_{A_1}(x) = r\}.$$

Each B_r is clopen in X_1 and $r_{A_1}(x)$ is constant on each B_r and thus nearly continuous on A_1. For $x \in B_r$, $r_{A_1}(x) = r$ so that $T_{A_1} = T^{r_{A_1}}$ on B_r. Thus, the induced map is a function defined piecewise by homeomorphisms on each set in a clopen decomposition of X_1.

□

Given a n.c. dynamical system (X, τ, μ, T) and a finite nearly clopen partition $\mathcal{P}$ consisting of n-elements, there is a natural map $f_{\mathcal{P}} : X \to \{1, 2, \ldots, n\}^{\mathbb{Z}}$ given by $(f_{\mathcal{P}}x)_k = j$ if and only if $T^k(x)$ is in the jth partition element of $\mathcal{P}$. We call the sequence $f_{\mathcal{P}}x$ the $T, \mathcal{P}$-name, or just the $\mathcal{P}$-name of x. The initial block of length k,

$$(f_{\mathcal{P}}x)_0, (f_{\mathcal{P}}x)_1, \ldots, (f_{\mathcal{P}}x)_{k-1},$$

is called the $\mathcal{P}$-name of length k.

An important characterization of ergodic systems comes from Birkhoff's point-wise ergodic theorem:

PROPOSITION 2.2.4. *If T is an ergodic, measure preserving transformation, then*

$$\lim_{N\to\infty}\sum_{n=0}^{N-1} f(T^n(x)) = \int_X f d\mu$$

for μ-a.e.x and any Lebesgue integrable function f.

Later, we shall introduce a stronger notion called nearly unique ergodicity. We now turn our attention to the various notions of equivalence between two nearly continuous dynamical systems.

2.3. Nearly Continuous Equivalences

DEFINITION 2.3.1. *Two systems (X, τ, μ, T) and (Y, τ, ν, S) are said to be* ***nearly continuously conjugate*** *if there exists carriers $X_0 \subset X$ and $Y_0 \subset Y$ and a homeomorphism $\phi : X_0 \to Y_0$ such that $\phi \circ T|_{X_0} = S|_{Y_0} \circ \phi$.*

THEOREM 2.3.2. *([8] Theorem 3) Any nearly continuous dynamical system (X, τ, μ, T) is nearly continuously conjugate to a homeomorphism of a compact metric space.*

PROOF. Let $\widetilde{X}$ be a compact metric space such that X is a dense G_δ subset of $\widetilde{X}$. Let $\tilde{d}$ denote the metric on $\widetilde{X}$. We may assume it is bounded as, for any metric, there exists an equivalent metric which is bounded. Define a new metric d on X by setting

$$d(x, y) = \sum_{k\in\mathbb{Z}} 2^{-|k|}\tilde{d}(T^k x, T^k y).$$

Note that $\tilde{d} \leq d$. As T is continuous on X, if $x_n, x \in X_0$ with $\lim_{n\to\infty}\tilde{d}(x_n, x) = 0$, then $\lim_{n\to\infty}\tilde{d}(T^k x_n, T^k x) = 0$, and $\lim_{n\to\infty} d(x_n, x) = 0$. The two metrics define the same topology. Also for x, $y \in X$,

$$\frac{1}{2}d(x,y) \leq d(Tx,Ty) \leq 2d(x,y)$$

showing T and T^{-1} are uniformly continuous with respect to d. Let $\epsilon > 0$. Select k_0 so that

$$\sum_{|k|\geq k_0} 2^{-|k|}\text{diam}X < \epsilon/2. \tag{2.3.1}$$

As $\tilde{d}$ is a totally bounded metric, the metric $\sum_{|k|<k_0} 2^{-|k|}\tilde{d}(T^k x, T^k y)$ trivially satisfies the total boundedness condition as the finite sum of totally bounded metrics. So, there exists $x_1, \dots, x_n$ with $x_i \in X$ so that

$$X \subset \bigcup_{i=1}^{n} \left\{ y : \sum_{|k|<k_0} 2^{-|k|}\tilde{d}(T^k x_i, T^k y) < \epsilon/2 \right\}. \tag{2.3.2}$$

By 2.3.1 and 2.3.2,

$$X = \bigcup_{i=1}^{n} \{y : d(x_i, y) < \epsilon\},$$

showing that d is a totally bounded metric. Let X' be the completion of X with respect to d. X' is a compact metric space, and as $T : X \to X$ is uniformly continuous, it can be extended to a homeomorphism $T' : X' \to X'$. Also, as $\tilde{d} \leq d$, there is a continuous map $\phi : X' \to X$ extending the identity map on X. $\phi^{-1}(X)$ is a dense G_δ subset of X'. To finish, define $\mu'(A) = 0$ for $A \subset X' \backslash X$ and $\mu'(A) = \mu(A)$ for $A \subset X$.

□

The following theorem due to Zhuravlev [20] plays a critical role in several results throughout the paper, and we include the proof for completeness. Note that if we drop the restriction that the map be defined on nearly full subsets, then this is a case of Von Neumann's theorem that every homomorphism of measure algebras arises from a point homomorphism mod 0, which may be found as Theorem 4.7 in [16] and Theorem 9.5.1 in [2].

THEOREM 2.3.3. *Let (X, τ, μ) and (Y, τ, ν) be two Polish probability spaces, and let $\mathcal{A}$ and $\mathcal{B}$ denote Boolean algebras of equivalence classes of nearly clopen sets for X and Y, respectively. If $\Phi : \mathcal{A} \to \mathcal{B}$ is a measure preserving isomorphism, then Φ is induced by a nearly continuous point map, i.e. there exist nearly full subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ and a measure preserving map $\phi : X_0 \to Y_0$ and $\phi(A) = \Phi(A)$ for each $A \in \mathcal{A}$.*

PROOF. The forward direction is obvious. Let Φ be meet the hypothesis of the theorem. Let $\epsilon_i = 2^{-i}$. First build refining sequences of partitions $\mathcal{P}_i$ and $\mathcal{Q}_i$ of finitely many nearly clopen sets in the following manner: Let $\mathcal{P}_1$ be a nearly clopen partition of X into finitely many atoms of size less than ϵ_1. Define $\mathcal{Q}_1 = \Phi(\mathcal{P}_1)$. Choose $\mathcal{Q}_2$ a (finite) refinement of $\mathcal{Q}_1$ so that $\nu(q) < \epsilon_2$ for $q \in \mathcal{Q}_2$. Define $\mathcal{P}_2 = \Phi^{-1}(\mathcal{Q}_2)$. As Φ is measure preserving, $\mathcal{P}_2$ has elements of size less than ϵ_2 and refines $\mathcal{P}_1$. Continue inductively. If $\mathcal{P}_i$ is a nearly clopen partition of X of diameter less than ϵ_i, then $\mathcal{Q}_i$ is a nearly clopen partition of Y with $\text{diam}\mathcal{Q}_i < \epsilon_i$. Select a $\mathcal{Q}_{i+1}$ which is finite and refines $\mathcal{Q}_i$ and has diameter less than ϵ_{i+1}. Then, $\mathcal{P}_{i+1} = \Phi^{-1}(\mathcal{Q}_{i+1})$ refines $\mathcal{P}_i$ and $\text{diam}\mathcal{P}_{i+1} < \epsilon_{i+1}$.

Let X_0 and Y_0 be invariant dense G_δ subsets of full measure in X and Y such that $\mathcal{P}_i$ is clopen in X_0 for all i and $\mathcal{Q}_i$ is clopen in Y_0 for all i. As the diameters go to zero, the sequences of partitions generate the restricted topologies.

Consider the intersection $\cap_{i\in\mathbb{N}} p_i$ of a nested sequence of atoms with $p_i \in \mathcal{P}_i$ and $p_{i+1} \subset p_i$. As $\epsilon_i \searrow 0$ each $\cap_{i\in\mathbb{N}} p_i$ is either empty, or, contains a single point.

Let

$$Z = \{(p_1, p_2, \ldots) : p_i \in \mathcal{P}_i \text{ and } p_{i+1} \subset p_i\}$$

be the space of sequences of symbols for the partitions where each p_{i+1} is contained in p_i. Z is a subset of the full product space of symbols $p_i \in \mathcal{P}_i$. A cylinder set of length k $x_1, x_2, \ldots, x_k$ is the set of sequences $\{(p_1, p_2, p_3, \ldots) : x_i = p_i \text{ for } i = 1, \ldots, k\}$. Each cylinder set $p_1 p_2 \ldots p_k$ of length k has measure $\mu_k(p_1 p_2 \ldots p_k) = \mu(p_k)$ where p_k denotes the partition element of $\mathcal{P}_k$. Let μ^* be the measure induced by the μ_k on the cylinders, and let the topology be generated by these cylinder sets, coinciding with the product space topology. Theorem 7.9 of Stromberg, 1994, *Probability for Analysts* [19] asserts that the product space of all possible sequences $\{p_1, p_2, p_3, \ldots\}$ is Polish as a countable product of Polish spaces. Z is a closed subset of the full product space. Thus, Z is Polish.

Define

$$\eta : X_0 \to Z, \text{ by } \eta(x) = (p_1, p_2, p_3, \ldots)$$

where p_i is the atom of $\mathcal{P}_i$ containing x. Note that $\mu^* = \eta_* \mu = \mu \circ \eta^{-1}$. We now show $\eta : X_0 \to \eta(X_0)$ is a homeomorphism onto its image. It is one-to-one as $\cap_{i \in \mathbb{N}} p_i$ is either empty, or, contains a single point. The pull back of any cylinder, $\eta^{-1}(p_1, p_2, \ldots, p_k) = p_k$, for some clopen $p_k \in \mathcal{P}_k$. (Any open set is the countable union of cylinder sets.) Hence η is continuous. Likewise, the clopen sets of each $\mathcal{P}_i$ form a basis for the topology of X_0, and $\eta(p_k) = p_1, p_2, \ldots, p_k$ for each $p_k \in \mathcal{P}_k$, showing η^{-1} is continuous. Thus $\eta(X_0)$ is the homeomorphic image of a Polish space and also Polish, meaning again G_δ.

$\eta(X_0)$ is dense in Z: Consider again the basis for the topology of Z consisting of the cylinder sets and look at $p_1 p_2 \ldots p_k \cap \eta(X_0)$ for each cylinder set, and note that the intersection is non-empty. Then $\eta(X_0)$ has full measure under μ^* by property of μ^* being generated by the measures on the cylinder sets.

By definition of the sequence of partitions $\mathcal{Q}_i$, and following the same argument as above, we define

$$\zeta : Y_0 \to Z \text{ by } \zeta(y) = (q_1, q_2, q_3, \ldots)$$

s.t. q_i is the atom of $\mathcal{Q}_i$ containing y. The measure $\nu^* = \zeta_*(\nu) = \mu^*$. $\zeta(Y_0)$ is a dense G_δ subset of Z. Z is a complete metric space, hence Baire, thus $Z^* = \eta(X_0) \cap \zeta(Y_0)$ is dense in Z and also G_δ with $\mu^*(Z^*) = 1$. Define

$$\varphi = \zeta^{-1} \circ \eta : X^* \to Y^*$$

with $X^* = \eta^{-1}(Z^*)$ and $Y^* = \zeta^{-1}(Z^*)$. φ is a measure preserving homeomorphism as the composition of homeomorphism. The image $\varphi(X^*) = \zeta^{-1}(Z^*)$ is dense G_δ.

□

Corollary 2.3.4. *Two nearly continuous dynamical systems (X, τ, μ, T) and (Y, τ, ν, S) are nearly continuously conjugate iff there exists a measure preserving isomorphism Φ of Boolean algebras of equivalence classes of nearly clopen sets $\mathcal{A} = \mathcal{A}(X)$ and $\mathcal{B} = \mathcal{A}(Y)$ such that $\Phi(T(A)) = S(\Phi(A))$ for all $A \in \mathcal{A}$.*

Proof. The forward direction is obvious. For the backwards direction, by Theorem 2.3.3, Φ is induced by a nearly continuous point map ϕ such that for any $A \in \mathcal{A}$, $\phi(A) = \Phi(A)$. It only remains to show that $\phi(T(x)) = S(\phi(x))$ on a nearly full set. Let X_0 be the nearly full set on which ϕ is defined, and let $x \in X_0$. Then, there exists a nested sequence of nearly clopen sets $A_1 \supset A_2 \supset A_3 \supset \cdots$ such that $x \in A_i$ for all $i \in \mathbb{N}$, implying $T(x) \in T(A_i)$

for all $i \in \mathbb{N}$. For each A_i, $\Phi(T(A_i)) = S(\Phi(A_i))$

$$\Rightarrow \quad \bigcap_{i=1}^{\infty} \Phi(T(A_i)) = \bigcap_{i=1}^{\infty} S(\Phi(A_i))$$

$$\Rightarrow \quad \Phi\left(T\left(\bigcap_{i=1}^{\infty} A_i\right)\right) = S\left(\Phi\left(\bigcap_{i=1}^{\infty} A_i\right)\right)$$

$$\Rightarrow \quad \phi(T(x)) = S(\phi(x)) \text{ for each } x \in X_0.$$

□

DEFINITION 2.3.5. *For a nearly continuous dynamical system $T : X \to X$, the **orbit** of a point $x \in X$ is*

$$Orb_T(x) = \{T^i(x) : i \in \mathbb{Z}\}.$$

DEFINITION 2.3.6. *Two nearly continuous dynamical systems (X, τ, μ, T) and (Y, τ, ν, S) are **nearly continuously orbit equivalent** if there exists carriers $X_0 \subseteq X$ and $Y_0 \subseteq Y$, a homeomorphism $\phi : X_0 \to Y_0$, and continuous maps $p : X_0 \to \mathbb{Z}$ and $q : Y_0 \to \mathbb{Z}$ such that $\phi \circ T^p = S^q \circ \phi$ and $\phi(Orb_T(x)) = Orb_S(\phi(x))$ for $x \in X_0$.*

In the nearly continuous category, del Junco and Şahin [6] proved the following analogue to Dye's theorem:

THEOREM 2.3.7. *([6]) Suppose (X, τ, μ, T) (Y, τ, ν, S) are nearly continuous dynamical systems. Then, the systems are nearly continuously orbit equivalent.*

Prior to the introduction of orbit equivalence, S. Kakutani [14] called two dynamical systems (X, μ, T) and (Y, ν, S) Kakutani equivalent if there exist measurable subsets A and B of X and Y respectively such that T_A is measurably conjugate to S_B. If $\mu(A) = \nu(B)$, the systems are said to be evenly Kakutani equivalent.

Definition 2.3.8. *Two nearly continuous dynamical systems* (X, τ, μ, T) *and* (Y, τ, ν, S) *are* ***nearly continuously Kakutani equivalent*** *if there exist nearly clopen sets* $A \subseteq X$ *and* $B \subseteq Y$ *with* $\mu(A) > 0$ *and* $\nu(B) > 0$ *such that* T_A *and* S_B *are nearly continuously conjugate.*

If $\mu(A) = \nu(B)$*, then the systems are* ***nearly continuously evenly Kakutani equivalent*** *.*

In chapter 4, we are going to be extending n.c. even Kakutani equivalence to an orbit equivalence.

CHAPTER 3

Towers and Cutting and Stacking

We begin with defining the equivalent of skyscrapers and towers in the nearly continuous category, developing versions of Rohklin towers and a multi-tower Alpern lemma. The chapter describes a procedure for constructing nearly continuous dynamical systems first through an abstract cutting and stacking using nearly clopen sets, and then shows that any n.c. dynamical system is nearly continuously conjugate to a cutting and stacking of the unit interval. We give a definition for the class of strongly rank one transformations and end by showing that this class is equivalent to a cutting and stacking which consists of one tower at each stage. The objects in this chapter play a lage role in the results of later chapters.

3.1. Towers

Definition 3.1.1. *Given a n.c. dynamical system* (X, τ, μ, T) *and a nearly clopen set* $A \subset X$*, the* ***skyscraper*** *over* A *is the ordered list of sets*

$$A,\ T(A)\backslash A,\ T^2(A)\backslash(A \cup T(A)), \dots, T^k(A)\backslash \bigcup_{i=0}^{k-1} T^i(A), \dots$$

We refer to these sets as levels of the skyscraper, and A *is called the base.*

The levels are commonly visualized as intervals, with A on the bottom, $T(A)\backslash A$ above, etc. The transformation T moves points in a level to the level above. If the image of a point is not in the next level, then the image appears in the base. For an $x \in A$, T moves the point x up the skyscraper to height $r_A(x)$, then to the point $T_A(x)$. See Figure 3.1

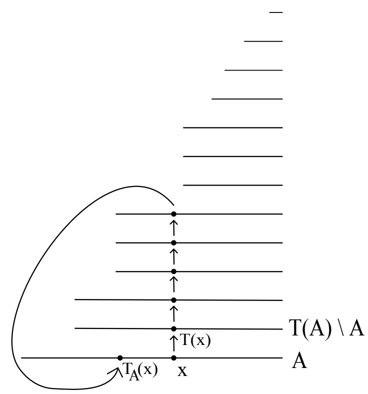

FIGURE 3.1. The skyscraper over the set A

DEFINITION 3.1.2. *Given a n.c dynamical system* (X, τ, μ, T) *and a nearly clopen set* A*, the ordered list of sets*

$$A, T(A), T^2(A), \ldots, T^{n-1}(A)$$

such that the sets are pairwise disjoint is called the ***nearly clopen tower*** *of height* n *over* A*. The sets are called levels of the tower and* A *is called the base. See Figure 3.2.*

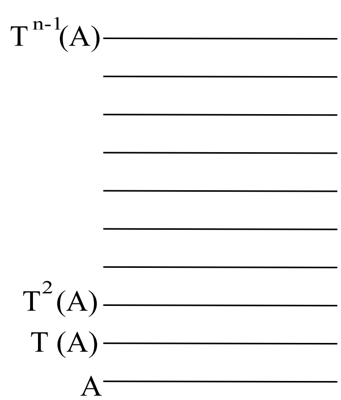

FIGURE 3.2. A tower of height n over A.

Note that the tower does not necessarily cover all of X.

Definition 3.1.3. *Given a skyscraper over a nearly clopen set A, and a nearly clopen set $B \subset A$ such that $r_A(x) = r$ for all $x \in B$, we call the ordered list of sets*

$$B, T(B), \ldots, T^{r-1}(B)$$

*a **column** of the skyscraper. B is called the base.*

A column is a tower in itself. See Figure 3.3.

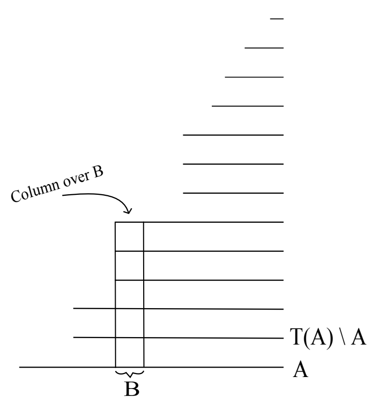

Figure 3.3. A column over set B of a skyscraper over set A.

Lemma 3.1.4. *Let (X, τ, μ, T) be a nearly continuous dynamical system. Given a positive integer N, there exists a nearly clopen set $B \subset X$ such that the skyscraper over B has height at least N.*

Proof. Select a nearly clopen set $A \subset X$ such that $0 < \mu(A) < \frac{1}{N}$. Decompose A into nearly clopen sets A_i such that $r_A(x) = i$ for all $x \in A_i$. Note that the images of all A_i are nearly clopen. The collection of sets $\{T^j(A_i) : \text{for } i = 1, 2, \ldots, \text{ and } j = 0 \ldots i-1\}$ gives a nearly clopen partition of X. Let $B = \bigcup_{i=N}^{\infty} A_i$. By Lemma 2.1.9, B is a nearly clopen set such that the skyscraper over B has height at least N. □

The next result is the equivalent of Rokhlin's Tower lemma in the nearly continuous category.

Proposition 3.1.5. *Let (X, τ, μ, T) be a n.c. dynamical system. For every $\epsilon > 0$ and $N \in \mathbb{N}$, there exists a nearly clopen tower of height N which covers all but ϵ of X.*

Proof. By Lemma 3.1.4, we may select a nearly clopen set A such that the skyscraper over A has height at least $\frac{N-1}{\epsilon}$. Decompose A into nearly clopen sets A_i such that $r_A(x) = i$ for each $x \in A_i$. Fix an i. Note that for an $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the integer part of x. Define

$$q(i) = \begin{cases} \lfloor \frac{i}{N} \rfloor \text{ if } \frac{i}{N} \text{ is not an integer, and} \\ \frac{i}{N} - 1 \text{ if } \frac{i}{N} \text{ is an integer.} \end{cases}$$

Define $B = \bigcup_{i=N}^{\infty} \bigcup_{k=0}^{q(i)} T^{kN}(A_i)$. Note that B is nearly clopen, and $T^j(B) \neq T^l(B)$ for $j \neq l$ and $0 \leq j, l \leq N-1$ by construction. The only levels not included in the tower over B are at most the top $N-1$ levels in each column. Consider again the single column over the set A_i. The topmost $N-1$ levels occupy a fraction $\frac{N-1}{i}$ of the column, and A has height at least $\frac{N-1}{\epsilon}$ implying $\frac{N-1}{i} < \epsilon$ so that $\mu(\bigcup_{i=0}^{N-1} T^i B) > 1 - \epsilon$.

□

Definition 3.1.6. *Given an interval*

$$I = [m, M-1]$$

of $\mathbb{N}$, and a finite set of integers $J \subset \mathbb{N}$, we say that the set of intervals of $\mathbb{N}$ whose lengths come from J ***tile*** *I if I may be written as a disjoint union of intervals whose lengths come* *from J. We call an ordered list of intervals $[m, m+j_1-1]$, $[m+j_1, m+j_1+j_2-1]$,...,$[M-j_k, M]$ with $j_i \in J$ a **tiling** of I. The sub-intervals of length j_i are called **tiles**.*

Our next result is equivalent of Alpern's Multi-Tower lemma in the nearly continuous category.

LEMMA 3.1.7. *Let (X, τ, μ, T) be a nearly continuous dynamical system. For every list of integers $n_1, \ldots, n_k$ with $k \geq 2$ and gcd $\{n_i\} = 1$, and a probability vector $(p_1, \ldots, p_k)$, there exists a decomposition of X into k disjoint, nearly clopen towers such that the i^{th} tower has height n_i and mass p_i.*

PROOF. Given a set of numbers $n_1, \ldots, n_k$ with $k \geq 2$ such that $gcd\{n_i\} = 1$, there exists a positive integer N such that all $m \geq N$ may be written as a positive linear combination of the n_i. Let $P = \Pi_{i=1}^{k} n_i$.

Select a nearly clopen set A such that the skyscraper over A has height at least $\frac{N+P}{\min_i\{p_i\}}$. Let $A_j = \{x \in A : r_A(x) = j\}$, giving a nearly clopen decomposition of the skyscraper into columns over the sets A_j with distinct heights.

Write $j = lP + m$ with $N \leq m < N + P$. The integers in the interval $[0, lP + m - 1]$ of $\mathbb{N}$ index the levels of the column over A_j by height. Tile $[0, lP - 1]$ using l tiles of length P. Tile $[lp, lp + m - 1]$ using tiles of length n_1,...,n_k. If a tile has length n_r, then label the level indexed by the integer at the beginning of the tile with $\tilde{n}_r$. It a tile has length P, then label the level indexed by the integer at the beginning of the tile with $\tilde{P}$. Repeat for each column. For $r = 1, \ldots, k$, and $r = P$, define $\tilde{B}_r$ to be the union of levels labeled by $\tilde{n}_r$. There are now $k + 1$ nearly clopen sets $\tilde{B}_1$,...,$\tilde{B}_k$ and $\tilde{B}_P$ which are bases of disjoint towers of heights

$n_1, \ldots, n_k$ and P. As $N \leq m \leq N+P$,

$$\frac{m}{\min_i\{p_i\}} < \frac{N+P}{\min_i\{p_i\}} < j = Pl + m$$

so that $\frac{m}{Pl+m} < \min\{p_i\}$ and

$$\mu(\bigcup_{i=0}^{n_r-1} T^i(\tilde{B}_r)) < p_r \text{ for each } r = 1, \ldots, k.$$

Let $q_r = p_r - \mu(\bigcup_{i=0}^{n_r-1} T^i(\tilde{B}_r)) < p_r$ for each $r = 1, \ldots, k$. Decompose $\tilde{B}_P$ into nearly clopen sets $C_1, \ldots C_k$ such that $\mu(C_r) = \frac{q_r}{P}$. For $r = 1, \ldots, k$, define

$$B_r = \tilde{B}_r \cup \bigcup_{i=0}^{\frac{P}{n_r}-1} T^{in_r} C_r.$$

Then, the tower over B_r has height n_r and mass p_r.

□

3.2. Cutting and Stacking

Cutting and Stacking provides a way for ergodic theorists to visualize transformations and to create examples of transformations with specific properties. For instance, Chacon [3] used cutting and stacking in constructing an example of a transformation which is weak mixing but not mixing (though the category argument [13] predates Chacon's example). And, cutting and stacking helped Jack Feldman build an example of a transformation which is not Loosely Bernoulli but has zero entropy. Cutting and stacking representations played a large role in proving that all odometers are nceKe [18] and that all irrational rotations of the circle are nceKe [10]. In this section, we focus on a description of cutting and stacking, some results related to cutting and stacking, and defining strongly rank one transformations.

By a cutting and stacking construction, we mean the following:

A stack $\mathcal{S}$ of height h and width w is a collection of h intervals of length w thought of as being placed one above the other with a transformation implicitly defined on the union of the first $h-1$ levels by translating each level to the level above. Denote by $|\mathcal{S}|$ the union of the levels of the stack $\mathcal{S}$. To cut a stack $\mathcal{S}$ into b columns (which may also be viewed as stacks) of smaller width w_i with $w = \Sigma_{i=1}^{b} w_i$ is to divide each level of $\mathcal{S}$ into b intervals of length w_1, w_2,...,w_b (reading left to right) and form b separate columns. A spacer is an extra interval of length w_i.

A cutting and stacking construction starts with a finite collection of disjoint stacks, cuts each into narrower columns, and forms new stacks of greater height by concatenating columns of the same width, one on top of the other, perhaps inserting spacers on top of columns before we concatenate. This is repeated infinitely often. At each stage, we extend the transformation to levels no longer at the top of a stack. If the total lengths of intervals is finite and the widths tend to zero, this defines an ergodic transformation of a finite interval and preserves Lebesgue measure.

P. Arnoux, D. Ornstein, and B. Weiss prove in [1] that any aperiodic measure preserving transformation may be realized by a cutting and stacking construction with spacers. We improve their result by removing the need for spacers.

Theorem 3.2.1. *Any nearly continuous dynamical system (X, τ, μ, T) is nearly continuously conjugate to a cutting and stacking of the unit interval, without spacers.*

Proof. Given $\epsilon_i > 0$ with $\epsilon_i \searrow 0$, let $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ be a refining sequence of finite nearly clopen partitions of X such that $\vee_{i=1}^{\infty}\mathcal{P}_i$ provides a base for the topology of a nearly full set $X' \subseteq X$, and for each partition, diam $(\mathcal{P}_i) = \max\{\mu(p_i) : p_i \in \mathcal{P}_i\} < \epsilon_i$.

Let $n_i = 10^i$. Using Lemma 3.1.7, we select a nearly clopen set B such that the multitower τ_1 over B has heights n_1 or $n_1 + 1$. Place the $\mathcal{P}_1$ names on τ_1, and purify into maximal $\mathcal{P}_1$-pure columns: partition B into maximal sets $B_1, \ldots, B_j$ so that the levels of the columns above each B_i lie entirely inside a single parition element of $\mathcal{P}_1$. Let $\mathcal{Q} = \{B_i\}$ be the corresponding partition of B. Define $\mathcal{S}_1$ to be the first finite set of stacks, the set of maximal $\mathcal{P}_1$-pure columns.

Using Lemma 3.1.7 for the system (B, τ, μ_B, T_B), select a nearly clopen set $C \subset B$ such that the multitower over C has height n_2 or $n_2 + 1$. Let τ_2' be the multitower over C for the system (B, τ, μ_B, T_B), and let τ_2 be the multitower over C for the system (X, τ, μ, T). Purify τ_2 using T and $\mathcal{P}_2$ to form the second set of stacks $\mathcal{S}_2$, i.e. the set of $\mathcal{P}_2$-pure columns of τ_2. We now show how $\mathcal{S}_2$ arises from a cutting and stacking of $\mathcal{S}_1$.

Purify τ_2' using T_B into maximal $\mathcal{Q}$-pure columns. The collection of all levels from the $\mathcal{Q}$-pure columns gives a refinement $\mathcal{Q}'$ of $\mathcal{Q}$. By creating a column over each $q \in \mathcal{Q}'$, we cut $\mathcal{S}_1$ into narrower columns. If for p and q in $\mathcal{Q}'$, $p = T_B(q)$, stack the column of $\mathcal{S}_1$ over p on top of the column of $\mathcal{S}_1$ over q. We may further cut the columns just created by cutting $\mathcal{P}_2$-pure columns without changing the stacking. Thus, $\mathcal{S}_2$ comes from a cutting and stacking of $\mathcal{S}_1$. Continue the process of cutting and stacking indefinitely, following the above steps to select $\mathcal{S}_i$ and show how it arises from a cutting and stacking of $\mathcal{S}_{i-1}$

On the unit interval, mimic the process of cutting and stacking in the natural way. For each column of $\mathcal{S}_1$, select sub-intervals of $[0, 1]$ whose lengths are the measures of the levels of the column, place the intervals accordingly, and define S_1 to be upward translation on all but the tops of the columns. To form the next stack for $[0,1]$, follow the prescribed cutting and stacking of $\mathcal{S}_1$ to form $\mathcal{S}_2$, and define S_2 to be upward translation on all but the tops of the columns. Continue indefinitely, and define S to be the limit of the S_i. As the heights of the stacks go to infinity, and the widths of the stacks go to zero, S is defined on a nearly full subset of the space.

The topology for the cutting and stacking of $[0,1]$ is generated by the levels of the stacks. Let Φ be the identification of the levels from the cutting and stacking for T on X and the intervals of the cutting and stacking of $[0,1]$. Φ is an isomorphism between Boolean algebras of nearly clopen sets. Thus, by Theorem 2.3.3, Φ comes from a near homeomorphism $\varphi : X \to [0,1]$. From the construction, it is apparent that φ preserves orbits and their order, φ is defined on a G_δ of full measure, φ preserves topologies, and is nearly continuous. Hence, there exists $\varphi : X \to [0,1]$ s.t. (X, τ, μ, T) and $([0,1], \tau, \nu, S)$ are n.c. conjugate where S is a cutting and stacking of $[0,1]$.

□

3.3. Nearly Continuous Speed-Ups

Given a nearly continuous dynamical system (X, τ, μ, T), we call S a nearly continuous speed-up of T if there exists a nearly continuous map $p : X \to \mathbb{N}$ such that $S = T^p$.

Proposition 3.3.1. *Let (X, τ, μ, T) be a nearly continuous dynamical system. Given two nearly clopen sets A and B in X such that $\mu(A) = \mu(B)$, there exists a nearly continuous speed-up S of T such that $S(A) = B$.*

Proof. Let A and B be nearly clopen subsets of X with $\mu(A) = \mu(B)$. Let

$$k_1 := \min\{k > 0 : \mu(T^k(A) \cap B) > 0\}.$$

Let $B_1 := T^{k_1}(A) \cap B$ and $A_1 := T^{-k_1}(B_1)$. Select each

$$k_{i+1} = \min\{k > k_i : \mu(T^k(A \setminus \cup_{j=1}^{i} A_j) \cap (B \setminus \cup_{j=1}^{i} B_j)) > 0.$$

Let

$$\begin{aligned} B_{i+1} &= T^{k_{i+1}}(A \setminus \cup_{j=1}^{i} A_j) \cap (B \setminus \cup_{j=1}^{i} B_j) \text{ and} \\ A_{i+1} &= T^{-k_{i+1}}(B_{i+1}). \end{aligned}$$

By ergodicity, $B = \bigcup_{i=1}^{\infty} B_i$. Define $p : X \to \mathbb{N}$ by $p(x) = k_i$ for $x \in A_i$ and $p(x) = 0$ for $x \in X \setminus A$. Note that all A_i and B_i are nearly clopen and that p is a piecewise function on nearly clopen subsets of A where each piece of the function is composed of homeomorphisms T^k. Thus, p(x) is nearly continuous and $S = T^p$ is a nearly continuous speed-up of T such that $S(A) = B$.

□

In the ergodic setting, Arnoux, Ornstein and Weis [1] proved that any aperiodic measure-preserving transformation is a speed-up of any other aperiodic measure-preserving transformation. We prove a similar statement in the nearly continuous setting.

Theorem 3.3.2. *Let (X, τ, μ, T) and (Y, τ, ν, S) be two nearly continuous dynamical systems. There exists a nearly continuous speed-up $\bar{S}$ of T such that $\bar{S} : X \to X$ is nearly a homeomorphism and $(X, \tau, \mu, \bar{S})$ is nearly continuously conjugate to (Y, τ, ν, S).*

Proof. Our proof of the theorem relies on Proposition 3.3.1 as well as a nearly continuous cutting and stacking representation of (Y, τ, ν, S).

Let $\mathcal{S}_i$ be a cutting and stacking representation of (Y, τ, ν, S) such that the heights of the stacks go to infinity. Our goal is to mimic the cutting and stacking representation $\mathcal{S}_i$ inside the system (X, τ, μ, T) in order to define a sequence S_i of speed-ups of T on the stacks which mimic S and whose limit $\bar{S}$ is conjugate to S. Note that $\mathcal{S}_1$ is a collection of columns, and let A denote the set consisting of the bases of the distinct columns in $\mathcal{S}_1$. For each $B \in A$ such that B is the base of a column of $\mathcal{S}_1$, let $R(B) = r_A(B) = \min\{r > 0 : T^r(x) \in A \text{ for all} x \in B\}$. For each $B \in A$, select $R(B)$ pairwise-disjoint nearly clopen subsets of X of size $\nu(B)$ (one set for each level in the column of $\mathcal{S}_1$ over B). Label the sets $\widetilde{B}$, $\widetilde{S(B)}$, $\widetilde{S^2(B)}$,...,$\widetilde{S^{R-1}(B)}$. By Proposition 3.3.1, there exists a speed-up S_1 of T so that $S_1(\widetilde{B}) = \widetilde{S(B)}$, $S_1(\widetilde{S(B)}) = \widetilde{S^2(B)}$,..., $S_1(\widetilde{S^{R-2}(B)} = \widetilde{S^{R-1}(B)}$ and S_1 is the identity on $\widetilde{S^{R-1}(B)}$. After creating a column in X for each column in $\mathcal{S}_1$, $\widetilde{\mathcal{S}_1}$ is the resulting first stack in the sequence of stacks we are creating in X, and S_1 translates all but the top-most levels of $\widetilde{\mathcal{S}_1}$ "upward".

$\mathcal{S}_2$ is a cutting and stacking of $\mathcal{S}_1$. For each $B \in A$ which is the base of a column of $\mathcal{S}_1$, cutting $\mathcal{S}_1$ to form $\mathcal{S}_2$ cuts B into smaller sets. For each of these smaller sets $b \subset B$ (and for each B which is the base of a column of $\mathcal{S}_1$), select a nearly clopen set $\tilde{b} \subset \tilde{B}$ of measure $\nu(b)$. Using the $\tilde{b}$ and the images of $\tilde{b}$ under S_1, cut $\tilde{\mathcal{S}}_1$ into narrower columns. If, in cutting and stacking from $\mathcal{S}_1$ to $\mathcal{S}_2$, the column of $\mathcal{S}_1$ over the set c follows the column of $\mathcal{S}_1$ over the set b, then stack the column of $\widetilde{\mathcal{S}_1}$ over the set $\tilde{c}$ on top of the column of $\widetilde{\mathcal{S}_1}$ over $\tilde{b}$. Using Proposition 3.3.1, "extend" S_1 in the following way to define S_2, another nearly continuous speed-up of T: For a level $\tilde{a}$ of a column of $\mathcal{S}_2$ which did not come from the top of a column of $\mathcal{S}_1$, define $S_2(\tilde{a}) = S_1(\tilde{a})$. For a level $S_1^{r-1}(\tilde{b})$ which was at the top of the column of $\mathcal{S}_2$ of height r over $\tilde{b}$ and such that the column of $\tilde{\mathcal{S}}_1$ over $\tilde{c}$ was stacked upon the column of $\tilde{\mathcal{S}}_1$ over $\tilde{b}$, define $S_2(S_1^{r-1}(\tilde{b})) = \tilde{c}$. $\widetilde{\mathcal{S}_2}$ is the resulting second stack in the construction, and S_2 gives "upward" translation on all but the top-most levels. Continue to mimic the cutting and stacking $\mathcal{S}_i$ to form the sequence of stacks $\widetilde{\mathcal{S}}_i$, defining the S_i by extending the S_{i-1}.

Define $\bar{S}$ as the limit of the S_i. $\bar{S}$ is defined on a nearly full subset and is obviously a nearly-continuous speed-up of T which is invertible. The levels of the stacks $\mathcal{S}_i$ generate a Boolean Algebra $\mathcal{B} = \mathcal{A}(Y)$ of equivalence classes of nearly clopen sets of (Y, τ, ν), and the levels of the stacks $\widetilde{\mathcal{S}}_i$ generate a Boolean algebra $\mathcal{A} = \mathcal{A}(X)$ of equivalence classes of nearly clopen sets of (X, τ, μ). Let $\Phi(x)$ be the map between $\mathcal{A}$ and $\mathcal{B}$ which arises as a result of the identification between the levels of $\mathcal{S}_i$ and the levels of $\widetilde{\mathcal{S}}_i$. As Φ is an isomorphism, by Theorem 2.3.3, there exists a near homeomorphism ϕ such that $\bar{S}$ is n.c. conjugate to S.

□

3.4. Strongly Rank One

We define a class of transformations which are the natural analogue of rank one transformations from ergodic theory. We call a cutting and stacking of the unit interval consisting of only one stack at each stage (and a leftover spacer) *interval rank one.* To describe an interval rank one transformation, one only needs to designate the number of columns into which one must cut the stage i stack $\mathcal{S}_i$ to form the stage $i+1$ stack, and the number of spacers to be placed above each column before stacking. We let $c(i)$ denote the number of columns of equal width into which to cut the stage i stack. We let $s(i, j)$ denote the number of spacers to place above column j, reading left to right, of the stage i stack. The new columns are then stacked to produce the stage $i+1$ stack $\mathcal{S}_{i+1}$.

The next series of figures shows the process of cutting and stacking for a generic strongly rank one transformation.

FIGURE 3.4. A tower and the left-over interval at some stage of an interval rank one construction.

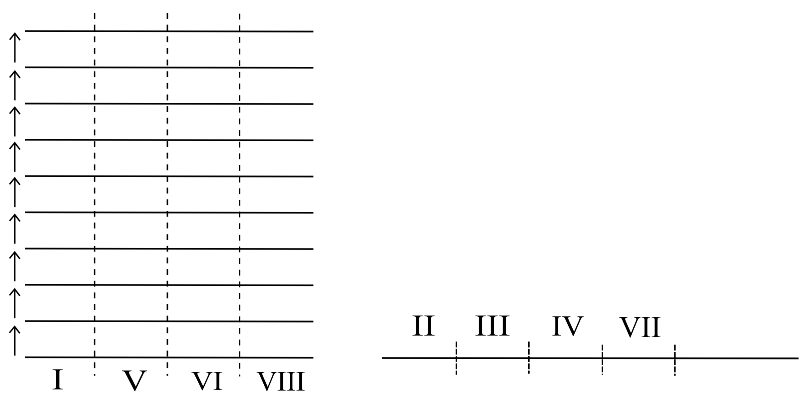

FIGURE 3.5. The tower and left-over interval after being cut.

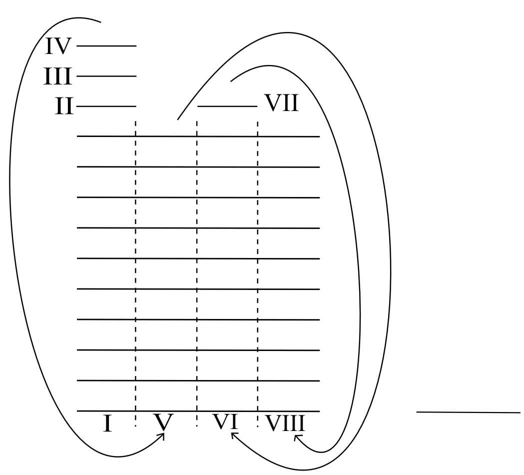

FIGURE 3.6. The arrows demonstrate how to stack the columns and spacers.

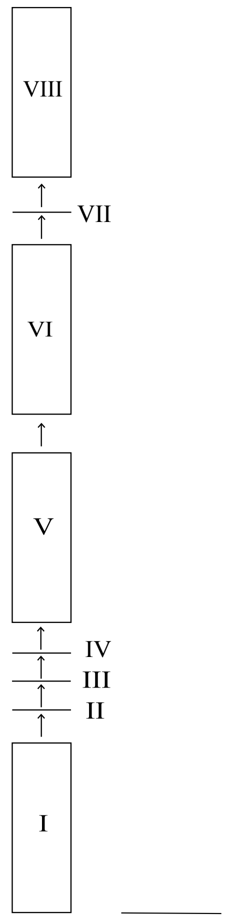

FIGURE 3.7. The transformation after stacking the columns and spacers with upward translation defined.

Recall that any tower with nearly clopen levels yields a natural partition of the space into nearly clopen sets which are the levels of the tower. Given two towers $\mathcal{T}_1$ and $\mathcal{T}_2$, we say that $\mathcal{T}_2$ refines $\mathcal{T}_1$ if the partition associated with $\mathcal{T}_2$ is a refinement of the partition associated with $\mathcal{T}_1$, i.e. any level in $\mathcal{T}_1$ can be written as a union of levels of $\mathcal{T}_2$.

DEFINITION 3.4.1. *A system is* ***strongly rank one*** *if there exists a refining sequence of nearly clopen Rohklin towers $\mathcal{T}_i$ whose levels provide a base for the topology of a nearly full subset with the heights of the $\mathcal{T}_i$ increasing to infinity.*

PROPOSITION 3.4.2. *A nearly continuous dynamical system (X, τ, μ, T) is strongly rank one if and only if it is nearly continuously conjugate to an interval rank one transformation.*

Proof. The backwards direction is obvious. For the forward direction, suppose (X, τ, μ, T) is strongly rank one. Consider a refining sequence $\{\mathcal{T}_i\}_{i=1}^{\infty}$ of Rohklin towers. Let B be the base of tower $\mathcal{T}_1$. Let n_i be the height of $\mathcal{T}_i$, and let $\mathcal{P}_i^j$ for $1 \leq j \leq n_i$ be the levels of the tower as well as the name of the partition element given by the level. At stage i, $X = \bigcup_{j=1}^{n_i+1} \mathcal{P}_i^j$ where $\mathcal{P}_i^{n_1+1} = X \setminus \bigcup_{j=1}^{n_i} \mathcal{P}_i^j$ represents the set from which we cut spacers and final partition element with measure $\mu(\mathcal{P}_i^{n_1+1}) = \epsilon_i$. As we move from $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$, each $\mathcal{P}_i^j = \bigcup_{k \in I_i^j} \mathcal{P}_{i+1}^k$ for some collection of indices $I_i^j \subset \mathbb{N}$ and $1 \leq j \leq n_i$.

To define a cutting and stacking of intervals producing a transformation which is n.c. conjugate to (X, τ, μ, T), observe the transition from tower $\mathcal{T}_i$ to $\mathcal{T}_{i+1}$. Let $c(i) = |I_i^1| = \frac{\mu(\mathcal{P}_i^1)}{\mu(\mathcal{P}_{i+1}^1)}$. Starting from the base of $\mathcal{T}_{i+1}$, read off the names of levels according to the partition $\mathcal{P}_i$ given by $\mathcal{T}_i$. We see full columns from the tower $\mathcal{T}_i$ possibly with the name of the spacer set, $\mathcal{P}_i^{n_i+1}$, inserted between the columns. Let $s(i,j)$ be the number of $\mathcal{P}_i^{n_1+1}$ which occur after the j^{th} column of $\mathcal{T}_i$ reading up the tower $\mathcal{T}_{i+1}$.

To mimic the cutting and stacking of the sequence $\mathcal{T}_i$ to build the cutting and stacking construction with the unit interval, begin with n_1 sub-intervals of width $w_1 = \mu(B)$ of $[0,1]$ to form $\mathcal{S}_1$. Given the stage i stack, cut it into $c(i)$ columns of equal width, and place $s(i,j)$ spacers above column j of width $w_i = \frac{w_1}{\Pi_{k=1}^{i} c(k)}$. Then, stack the columns, right on top of left, to form $\mathcal{S}_{i+1}$. Let S be the transformation which arises in the limit of the S_i where $S_i(\mathcal{P}_i^j) = \mathcal{P}_i^{j+1}$ for $1 \leq j < n_i$ as $i \to \infty$.

Let Φ be the map which arises naturally in the identification of levels of the towers $\mathcal{T}_i$ with the sub-intervals of $[0,1]$ which form the levels of the stacks $\mathcal{S}_i$. This is obviously a measure preserving isomorphism between the Boolean algebra of nearly clopen sets generated by the levels from the towers for (X, T) and the Boolean algebra of intervals generated by the levels of the cutting and stacking for (Y, S). By Theorem 2.3.3 and Corollary 2.3.4, Φ is induced by a nearly continuous point map ϕ and ϕ is a nearly continuous conjugacy between T and S.

□

3.5. Classical Examples of Interval Rank One Transformations

We give two classical examples to illustrate the process of cutting and stacking the unit interval. We start with a description of cutting and stacking for the binary odometer, which is important to the proof in Chapter 5. Take the unit interval, $[0, 1]$, and cut it in half, forming the two intervals $[0, 1/2]$ and $[1/2, 1]$. Now, stack the right half of the interval on top of the left half, and define the map at this stage stage to be upward translation from $[0, 1/2]$ to $[1/2, 1]$.

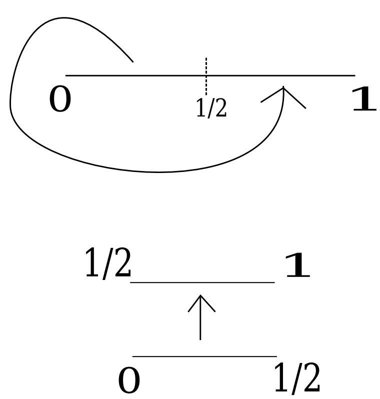

Figure 3.8. The first stage of cutting and stacking for the binary odometer.

To move from one stage to the next, cut the stack in half and stack the right side upon the left. The levels of the stacks/towers generate the topology, and the final map is defined as the limit of the upward translations at each stage. For a generic odometer or adding machine, one only has one stack at each stage that is cut into narrower columns of equal width that are then stacked one-upon-the other from right to left. The same number of cuts need not be used at each stage.

The canonical cutting and stacking description for Chacon's map uses spacers. Start with the unit interval, and divide the unit interval into two segments, $[0, 2/3]$, the entirety of which is included in the tower at each stage of construction, and the left-over interval $[2/3, 1]$ from which we cut spacers. First, cut $[0, 2/3]$ into three subintervals of equal width, $[0, 2/9]$, $[2/9, 4/9]$, and $[4/9, 2/3]$. Also, cut the first spacer $[2/3, 8/9]$ from the left-over interval. Now, place the spacer above the middle section, $[2/9, 4/9]$, creating three narrower stacks/columns of heights one, two, and one. Stack the middle column upon the left-most, and stack the right-most upon the middle, forming an stack of four intervals, and define the map at this first stage to be upward translation so that $[0, 2/9] \to [2/9, 4/9] \to [2/3, 8/9] \to [4/9, 2/3]$.

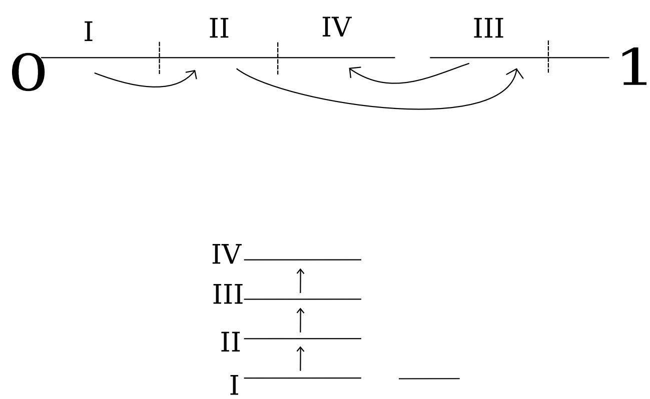

FIGURE 3.9. The first stage of cutting and stacking for Chacon's map.

Again, this process continues indefinitely, cutting into thirds, adding a spacer of equal width above the middle column, and stacking the middle column upon the left and the right upon the middle. The transformation at stage i is simply upward translation on all but the top level. Again, the levels of the stacks/towers generate the topology, and the final transformation is defined as the limit the stage i transformation.

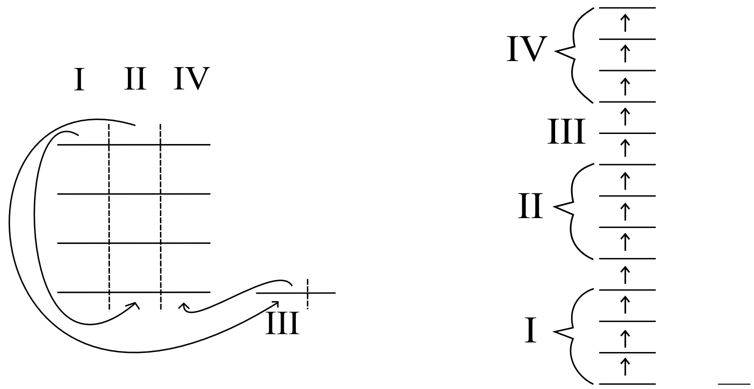

FIGURE 3.10. The second stage of cutting and stacking for Chacon's map.

CHAPTER 4

Extending Nearly Continuous Conjugacies

In the 1940s, S. Kakutani [14] introduced a new definition of equivalence between ergodic measure preserving transformations of finite measure spaces relevant to studying the relationships between measurable cross sections of measurable flows. Two ergodic measure preserving transformations S and T were called Kakutani equivalent if they had isomorphic induced maps, or equivalently, if S and T could be realized as induced transformations of a third system.

In 1984, A. del Junco and D. Rudolph extended the definition of even Kakutani equivalence to ergodic $\mathbb{Z}^n$ actions in [7], first by taking Katok cross-sections of a flow, and second by determining the existence of an orbit preserving injection with an extra asymptotic linearity condition. In one dimension, this reduced to the classical theory of Kakutani equivalence with an addition of an orbit equivalence between the two systems. When A. del Junco, D. Rudolph, and B. Weiss explored Kakutani equivalence in the context of what they named measured topological dynamical systems [5], this element of orbit equivalence persisted in the definition. The authors defined even Kakutani equivalence to be an orbit equivalence on sets of full measure which is a conjugacy when restricted to induced transformations of nearly clopen subsets of the same measure. In Chapter Two, we gave a slightly different definition for nearly continuous even Kakutani equivalence, asking only for the conjugacy between induced transformations on nearly clopen subsets of the same measure. The first aim of this chapter is to demonstrate that our definition of nearly continuous even Kakutani equivalence is equivalent to the definition given by del Junco, Rudolph and Weiss.

We will then turn our attention to nearly continuous dynamical systems which admit a conjugacy between induced maps on nearly clopen sets of different measure. In order to extend the conjugacy to its fullest extent, we require that the systems be nearly uniquely ergodic, an invariant of nearly continuous conjugacy introduced in [8] and further discussed in [5]. Rudolph and Roychowdhury [18] proved that all adding machines are nearly continuously even Kakutani equivalent, and Rudolph and Dykstra [10] proved that all irrational rotations of the circle belong to the class of adding machines. We apply these results to show that the class of strongly rank one transformations belongs to the equivalence class of irrational rotations.

4.1. Extending Nearly Continuous Even Kakutani Equivalence

Theorem 4.1.1. *Let (X, τ, μ, T) and (Y, τ, ν, S) be two nearly continuous dynamical systems. If $A \subset X$ and $B \subset Y$ are nearly clopen sets with $\mu(A) = \nu(B)$ such that ϕ is a n.c. conjugacy between T_A and S_B, then ϕ extends to a n.c. orbit equivalence between T and S.*

As $A \subset X$ and $B \subset Y$ are nearly clopen sets, there exists carriers $X_0 \subseteq X$ and $Y_0 \subseteq Y$ of T and S respectively such that $A_0 = A \cap X_0$ and $B_0 = B \cap Y_0$ are clopen in X and Y respectively, and $\phi : A_0 \to B_0$ is a homeomorphism and $\phi \circ T_{A_0} = S_{B_0} \circ \phi$. The system (X, τ, μ, T) is equivalent to (X_0, τ, μ, T), so to simplify notation, and without loss of generality, we let $X = X_0$, $A = A_0$, $Y = Y_0$, and $B = B_0$. Our goal is to extend ϕ to a carrier of T in such a way so as to establish an orbit equivalence, $\hat{\phi}$, while (nearly) preserving the conjugacy between the induced maps. We do so by defining a point map constructed piecewise on a clopen decomposition of X (based on return times to A and B), where $\hat{\phi}$ is a composition of the homeomorphisms T, T_A, ϕ, and S on each subset of the decomposition. To cast light on how the extension is defined, we begin with a description of the machinery used to create $\hat{\phi}$.

4.1.2. Describing Positive and Negative Fibers. We give a visualization or image of the machinery which corresponds to the definition of the point map. Construct the nearly clopen skyscraper of T over the set A. The height of the tower over a point $x \in A$ is

$$r_A(x) := \min\{n > 0 : T^n(x) \in A\}.$$

For any $x \in X$, define

$$h(x) = \min\{h \geq 0 : T^{-h}(x) \in A\}.$$

Let $\tilde{x} = T^{-h(x)}(x)$. The fiber of the tower containing the point x is the list of points

$$\left\{ \begin{array}{c} T^{r_A(\tilde{x})-1}(\tilde{x}) \\ \vdots \\ x \\ \vdots \\ T(\tilde{x}) \\ \tilde{x} \end{array} \right\}.$$

We refer to fibers for points in X as *positive fibers*. To the right of the fiber containing x, list the fiber above $T_A(\tilde{x})$. Continue listing the positive fibers so that to the right of each base point $\tilde{x} \in A$, one sees the fiber above $T_A(\tilde{x})$, and, to the left of $\tilde{x} \in A$ one sees the fiber above $T_A^{-1}(\tilde{x})$. This ordered list of fibers gives what we call the positive frame for the point x.

For each point x in A, consider $\phi(x) \in B$. For each of these points, we extract a fiber of the skyscraper built via S over the set B. Define for $\tilde{y} \in B$,

$$r_B(\tilde{y}) := \min\{n > 0 : S^n(\tilde{y}) \in B\}.$$

Simply arrange the fibers from the skyscraper over B in the same manner as for the fibers above points in A, with the fiber for $S_B(\tilde{y})$ listed immediately to the right of the fiber for $\tilde{y} \in B$, and the fiber for $S_B^{-1}(\tilde{y})$ listed immediately to the left of the fiber for $\tilde{y} \in B$. Now, "flip" the fibers over the points in B so the point at the top of each list is in B. We refer to fibers for points in Y as negative fibers.

For the last step in this visualization, line up the base of the positive frame (points in A) across from the base of the negative frame (points in B) via the conjugacy. One traverses the diagram in the following manner: applying the induced map T_A to the base of the positive frame shifts the points to the right along the base, and applying the induced map S_B to the top of the lower frame shifts the points to the left across the negative frame. Apply T to move away from the base in the positive fibers and apply S to move away from the base in the negative fibers. Cross between the positive frame and the negative frame by applying ϕ or ϕ^{-1}. In Fig. 4.1, dots represent points in A^C and B^C.

We create an orbit equivalence by developing a method for pairing up points from the positive fibers with points in the negative fibers. This method pairs up items in the positive fibers with items in the negative fibers directly across until either the positive fiber is neutral (to all points in the positive fiber, a map has been assigned) or the negative fiber is neutral

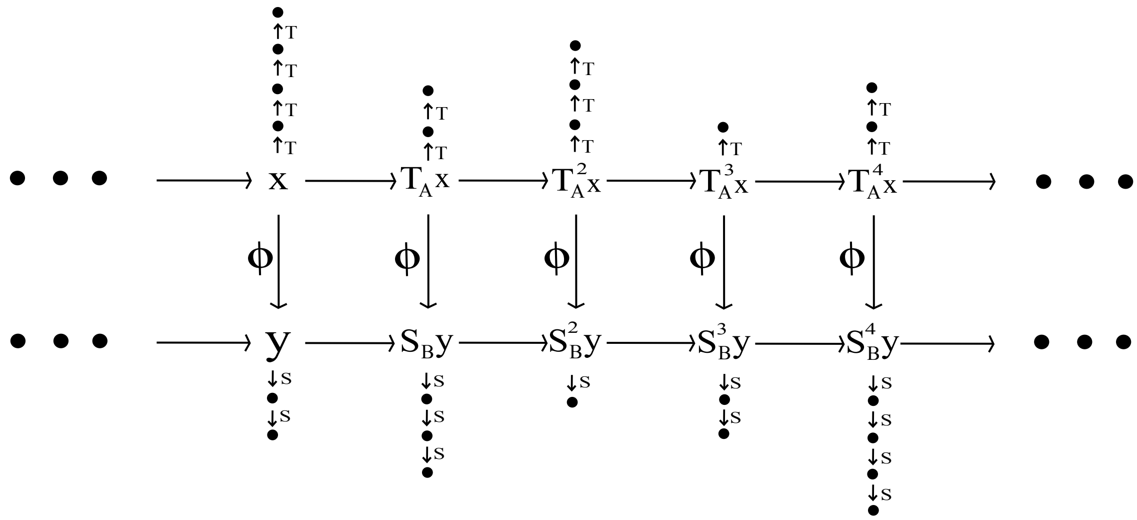

FIGURE 4.1. A segment of the full frame used for defining the extension of the conjugacy.

(each point of the negative fiber is the image of a point in X). Then, it merely shifts the upper frame one to the right and pairs up as much of the remainder of the positive fibers with the remainder of the negative fibers, shifts again and pairs again, etc. Here, we merely "shift" the positive frame across the negative frame by utilizing the induced map on the base of the frames. We remark that this method is similar to the filling-scheme used by Chacon and Ornstein in [4].

A point $x \in A^C$ maps to a point $y \in B^C$ if, after a shift, the fiber across from x still has unused space to receive the point. A point in a positive fiber cannot be mapped to a negative fiber if, after a shift, the fiber across from it is already neutral.

Define

$$\theta_k^n(x) = \sum_{i=k}^{n} r_{A_0}(T_{A_0}^i \circ T^{-h(x)}(x))$$

the sum of the lengths of the k^{th} fiber through the n^{th} fiber to the right of the positive fiber containing x, and

$$\psi_k^n(x) = \sum_{i=k}^{n} r_{B_0}(S_{B_0}^i \circ \phi \circ T^{-h(x)}(x)),$$

the sum of the lengths of the k^{th} fiber through the n^{th} fiber to the right of the negative fiber below the positive fiber containing x.

Let $n(x) = 0$ if $h(x) \leq r_{B_0}(\phi \circ T^{-h(x)}(x))$. Otherwise, let

$$n(x) = \min\{n > 0 : h(x) + \theta_1^n(x) \leq \psi_0^n(x)\}.$$

In essence, this is the smallest number of shifts required for there to be a point available in a negative fiber for the point x in a positive fiber. Let

$$d(x) := h(x) + \theta_1^{n(x)}(x) - \psi_0^{n(x)-1}(x).$$

Define $\hat{\phi}$ as:

$$\hat{\phi}(x) = \begin{cases} \phi(x) & \text{for} \quad x \in A \\ S^{d(x)} \circ \phi \circ T_A^{n(x)} \circ T^{-h(x)}(x) & \text{for} \quad x \notin A \end{cases}$$

In a similar manner, we define $\hat{\phi}^{-1}$.

$$D(y) = \min\{D > 0 : S^{-D}(y) \in B\}$$

Let

$$\gamma_k^n(y) = \sum_{i=k}^{n} r_B(S_B^{-i} \circ S^{-D(y)}(y)),$$

the sum of the lengths of the k^{th} through the n^{th} fibers to the left of the negative fiber holding y, and let

$$\Psi_k^n(y) = \sum_{i=k}^{n} r_A(T_A^{-i} \circ \phi^{-1} \circ S^{-D(y)}(y)),$$

the sum of the lengths of the k^{th} through the n^{th} positive fibers to the left of the positive fiber across from the negative fiber containing y. The inverse map pairs a point in a negative fiber with a point in a positive fiber by sliding the lower frame backwards along the upper frame, in reverse fashion to what we described before. The transformation is defined at the point y when, after a certain number of shifts, the positive fiber across from y has available space. If $D(y) \leq r_A(\phi^{-1} \circ S^{-D(y)}(y))$, then $m(y) = 0$. Otherwise,

$$m(y) = \min\{m > 0 : D(y) + \gamma_1^m(y) \leq \Psi_0^m(y).\}$$

$$H(y) = D(y) + \gamma_1^m(y) - \Psi_0^m(y).$$

$$\hat{\phi}^{-1}(y) = \begin{cases} \phi^{-1}(y) & \text{for} \quad y \in B \\ T^{H(y)} \circ \phi^{-1} \circ S_B^{-m(y)} \circ S^{-D(y)}(y) & \text{for} \quad y \notin B \end{cases}$$

LEMMA 4.1.3. *$\hat{\phi}^{-1}$ is the inverse of $\hat{\phi}$.*

The diagram of the $\hat{\phi}^{-1}$ construction is just the diagram of the $\hat{\phi}$ construction rotated 180°. We pair points starting at the base of the frames, and then work our way outward, shifting when necessary. Naturally, $D(\hat{\phi}(x)) = d(x)$, the same number of shifts are required. Then, it is easily follows that $H(\hat{\phi}(x)) = h(x)$.

$\hat{\phi}$ and $\hat{\phi}^{-1}$ need to be well-defined on nearly full sets. In order for $\hat{\phi}$ to be well-defined at a point, the values $h(x)$, $n(x)$, and $d(x)$ must exist and be finite. We may assume that $h(x)$ (and $D(y)$) are finite. Otherwise, we remove the entire orbit of x from X (or y from Y). To see that $n(x)$ (and $m(y)$) are finite on a subset of full measure, we begin with the following lemma:

LEMMA 4.1.4. *Suppose G is an ergodic measure-preserving transformation $T : X \to X$, and f is integrable such that $\int f(x) = 0$. Then, for almost every $x \in X$, there exists $n > 0$ such that $f(x) + f(T(x)) + f(T^2(x)) + \cdots + f(T^{n-1}(x)) \leq 0$.*

PROOF. Suppose not, that there exists $E \subseteq X$ with $\mu(E) > 0$, such that for all $x \in E$ and all $n > 0$, $\sum_{i=0}^{n-1} T^i(f(x)) > 0$.

Let $x \in E$. For a fixed value, n, calculate how many times the orbit of x has returned to E. Let $m(n) = \sum_{i=0}^{n-1} \chi_E(T^i(x))$.

Define the function $f_E(x) = \sum_{i=0}^{r_E(x)-1} f(T^i(x))$. By assumption, f_E is a positive function over E and T_E is ergodic $\Rightarrow$

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} f_E(T_E^i(x)) \to \int_E f_E d\mu_E > 0.$$

Also,

$$\sum_{i=0}^{m(n)-1} f_E(T_E^i(x)) = \sum_{i=0}^{n-1} f(T^i(x))$$

whenever $\chi_E(T^n(x)) = 1$.

For a subsequence n_k of n such that $\chi_E(T^{n_k}(x)) = 1$,

$$\lim_{n\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} f(T^i(x)) =$$

$$
\begin{aligned}
&= \lim_{n_k\to\infty} \frac{1}{n_k} \sum_{i=0}^{m(n_k)-1} f_E(T_E^i(x)) \\
&= \lim_{n_k\to\infty} \frac{1}{m(n_k)} \sum_{i=0}^{n_k-1} \chi_E(T^i(x)) \cdot \frac{1}{n_k} \sum_{i=0}^{m(m_k)-1} f_E(T_E^i(x)) \\
&= \lim_{n\to\infty} \frac{1}{n_k} \sum_{i=0}^{n_k-1} \chi_E(T^i(x)) \cdot \lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} f_E(T_E^i(x)) \\
&= \mu(E) \int_E f_E d\mu_E > 0
\end{aligned}
\tag{4.1.1}
$$

As $\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x))$ converges, any subsequence converges to the same value, implying that $\int f > 0$, a contradiction. □

LEMMA 4.1.5. *The value $n(x)$ is finite on a set of full measure.*

PROOF. Let $X = A$, identified with B, and $T = T_A \cong S_B$. And let $f(x) = r_A(x) - r_B(\phi(x))$. As $\mu(A) = \nu(B)$, $\int f(x) = 0$. Thus by Lemma 4.1.4, $n(x)$ is finite on a set of full measure. Fix n and let A_n be the set of points such that $\sum_{i=0}^{n-1} f(T^i(x)) \leq 0$ for the first time. As $f(x)$ takes only values in $\mathbb{Z}$, the set of points so that $\sum_{i=0}^{n-1} f(T^i(x)) \leq 0$ is nearly clopen. Unioning A_n over n gives a G_δ subset.

□

The proof that $\hat{\phi}^{-1}$ is defined on a set of full measure is essentially identical. We now define the sets $\hat{A}$ and $\hat{B}$ upon which $\hat{\phi}$ and $\hat{\phi}^{-1}$ are well-defined. First, let $A_{n,h,d}$ be the set of points for which $\hat{\phi} = S^d \circ \phi \circ T_A^n \circ T^{-h}$ for fixed n, d, and h. $A_{n,h,d}$ may tediously be written as a finite union of finite intersections of images of A and B via the homeomorphisms S, ϕ, and T. Thus $A_{n,h,d}$ is clopen in X. Let $B_{m,D,H} = \{y \in Y : \hat{\phi}^{-1}(y) = T^H \circ \phi^{-1} \circ S_B^{-m} \circ S^{-D}(y)\}$ so $B_{m,D,H} = \hat{\phi}(A_{m,H,D})$. Define

$$\hat{A} = \bigcup_{n=0}^{\infty} \bigcup_{h=0}^{\infty} \bigcup_{d=0}^{\infty} A_{n,h,d}$$

and

$$\hat{B} = \bigcup_{m=0}^{\infty} \bigcup_{D=0}^{\infty} \bigcup_{H=0}^{\infty} B_{m,D,H}.$$

LEMMA 4.1.6. *$\hat{A}$ and $\hat{B}$ are nearly full sets.*

PROOF. $\hat{A}$ and $\hat{B}$ are both open in X as they are the countable union of clopen sets in X, and of full measure by the previous lemma. □

LEMMA 4.1.7. *$\hat{\phi}$ is continuous on $\hat{A}$ and $\hat{\phi}^{-1}$ is continuous on $\hat{B}$.*

PROOF. Decompose $\hat{A}$ into the countable union of clopen sets $A_{n,h,d}$. On each of these clopen sets, $\hat{\phi}$ is the composition $S^d \circ \phi \circ T_A^n \circ T^{-h}$ of functions continuous in the relative topology of X. Decomposing $\hat{B}$ in a similar manner shows that $\hat{\phi}^{-1}$ is also continuous in the relative topology of $\hat{B}$. □

LEMMA 4.1.8. *$\hat{\phi}$ is measure preserving on $\hat{A}$ and $\hat{\phi}^{-1}$ is measure preserving on $\hat{B}$.*

PROOF. Identical to the proof of Lemma 4.1.7. □

We define the final carriers. Let

$$X^* = \bigcap_{i=-\infty}^{\infty} T^{-i}(\hat{A}) \cap \hat{\phi}^{-1}\left(\bigcap_{i=-\infty}^{\infty} S^{-i}(\hat{B})\right)$$

and

$$Y^* = \hat{\phi}(X^*).$$

LEMMA 4.1.9. *The sets X^* and Y^* on which $\hat{\phi}$ and $\hat{\phi}^{-1}$ are defined are G_δ sets of full measure.*

Proof. Note that $\hat{A}$ and $\hat{B}$ are both G_δ sets of measure 1 relative to X and Y.

□

In conclusion of the proof of Theorem 4.1.1, we have the existence of carriers $X^* \subseteq X$, $Y^* \subseteq Y$, and a nearly continuous orbit equivalence $\hat{\phi} : X^* \to Y^*$.

4.2. Extensions of Non-Even Kakutani Equivalence

So far, we only considered dynamical systems with an established n.c. conjugacy on nearly clopen subsets of the same relative measure. We explore the possibility of extending a conjugacy where $\mu(A) \neq \nu(B)$ by examining the case, w.l.o.g., $\mu(A) > \nu(B)$. To highlight a difference between Kakutani equivalence in the measure-theoretic sense and nearly continuous Kakutani equivalence, we give a theorem which holds in the measure-theoretic category, but fails to hold in the nearly continuous category. We show first that if (X, μ, T) and (Y, ν, S) are ergodic measure-preserving transformations on Polish probability spaces with $A \subset X$ and $B \subset Y$ such that T_A is measurably conjugate to S_B, then, T is measurably conjugate to $S_{B'}$ for some $B' \subset Y$. We then we introduce and discuss nearly unique ergodicity, an invariant of nearly continuous conjugacy, and construct an example of two nearly continuous dynamical systems which are nearly continuously (non-even) Kakutani equivalent, and for which one cannot find a set B' such that T and $S_{B'}$ are n.c. conjugate. After the example, we show that with the added condition that the systems are nearly uniquely ergodic, it is possible to obtain an analogous result in the nearly continuous category.

4.2.1. The Measure-Theoretic Category.

Theorem 4.2.2. *Let (X, μ, T) and (Y, ν, S) be two metrical dynamical systems. If $A \subset X$ and $B \subset Y$ are measurable sets such that T_A is measurably conjugate to S_B, and $\mu(A) > \nu(B)$, there exists $B' \subset Y$ such that T and $S_{B'}$ are measurably conjugate.*

Proof. For $x \in A$, let $r_A(x) := \min\{n > 0 : T^n(x) \in A\}$ denote the return time function. We build a tower $\hat{X}$ over A using the return time as the height and define a map $\hat{T}$ on this tower:

$$\hat{X} = \{(x,i) : 0 \leq i < r_A(x), x \in A\}$$

$$\hat{T}(x,i) := \begin{cases} (x, i+1) & for \quad 0 \leq i < r_A(x) \\ (T_A(x), 0) & for \quad i = r_A(x) \end{cases} .$$

Note that T is isomorphic to $\hat{T}$. For $y \in B$, let $r_B(y) := \min\left\{h > 0 : S^h(y) \in B\right\}$ be the return time function to B. Build the following tower $\hat{Y}$ over B and function $\hat{S}$ on the tower:

$$\hat{Y} := \{(y,i) : 0 \leq i < r_B(y), y \in B\}$$

$$\hat{S}(y,i) := \begin{cases} (y, i+1) & for \quad 0 \leq i < r_B(y) \\ (S_B(y), 0) & for \quad i = r_B(y) \end{cases} .$$

As $T_A \cong S_B$, $\hat{T}_A \cong \hat{S}_B$. Suppose that ϕ gives this conjugacy. Identify $x \in A$ with $\phi(x) \in B$ and rename this set C. Define a new tower over the set C. Let $r_C(x) = \max\{r_A(x), r_B(\phi(x))\}$ for $x \in C$. Let:

$$\hat{Z} = \{(x,i) : 0 \leq i < r_C(x), x \in C\},$$

$$\hat{R}(x,i) = \begin{cases} (x, i+1) & for \quad 0 \le i < r_C(x) \\ (T_A(x), 0) & for \quad i = r_C(x) \end{cases}.$$

Let $\hat{\mu} = \frac{\mu \times dn}{\int r_C d\mu}$ so that $\hat{\mu}(\hat{Z}) = 1$. As $\mu(A) > \nu(B)$ and $\mu(X) = \nu(Y) = 1$,

$$\hat{\mu}(\hat{X}) < \hat{\mu}(\hat{Y})$$

in $\hat{Z}$.

Lemma 1.3 of [15] gives that if T is an ergodic, measure preserving transformation on X and A and B are measurable subsets of X with $\mu(A) < \mu(B)$, then T_A is isomorphic to an induced transformation of T_B. We have that $\hat{\mu}(\hat{X}) < \hat{\mu}(\hat{Y})$, so $\hat{R}_{\hat{X}}$ is isomorphic to an induced transformation of $\hat{R}_{\hat{Y}}$. Note, $\hat{R}_{\hat{X}} \cong \hat{T}$ and $\hat{R}_{\hat{Y}} \cong \hat{S}$. By this lemma, there exists a set $B' \subset \hat{Y}$ so that $\hat{T}$ is isomorphic to $\hat{S}$ induced on B'. As $T \cong \hat{T}$ and $S_{B'} \cong \hat{S}_{B'}$, we obtain $T \cong S_{B'}$.

□

4.2.3. Nearly Unique Ergodicity. Denker and Keane in [8] introduced strict ergodicity, as an invariant of nearly continuous conjugacy. Later, the authors of [5] gave a slightly different, but completely equivalent, definition of strict ergodicity, which they rename nearly unique ergodicity. See [5] and [8] for discussion beyond what we include here.

Definition 4.2.4. *A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ on a Polish probability space (X, τ, μ) is said to converge* ***nearly uniformly*** *if there exists a nearly full subset $X_0 \subseteq X$ on which the functions converge uniformly.*

Definition 4.2.5. *Suppose* (X, τ, μ, T) *is a nearly continuous dynamical system. If the ergodic averages of* f

$$A_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(T^i)$$

converge nearly uniformly to $\int f d\mu$ *for all nearly bounded and nearly continuous functions* $f \in L_1(\mu)$*, we say* T *is* ***nearly uniquely ergodic****.*

We point out that our definition differs slightly from that of [5] in that we only require f to be nearly continuous and nearly bounded. Note that one could always restrict the dynamics to a carrier $X_0 \subseteq X$ so that f is continuous and bounded in X_0, and that studying the dynamics of (X_0, τ, μ, T) is equivalent to studying (X, τ, μ, T).

As 1_B is a nearly continuous function for all nearly clopen sets B, nearly unique ergodicity implies nearly uniform convergence of $A_n(1_B)$.

Lemma 4.2.6. *([5] Lemma 5.3) Suppose that* (X, τ, μ, T) *is a nearly continuous dynamical system and that for any nearly clopen set* B*,* $A_n(1_B)$ *converges nearly uniformly to* $\mu(B)$*. Then, the system is nearly uniquely ergodic.*

Proof. Note that the set of L_1-functions for which $A_n(f)$ converge nearly uniformly is closed under finite linear combinations and uniform limits. For any nearly bounded and nearly continuous function f and value $a \in \mathbb{R}$, $\{x : f(x) > a\}$ is nearly open and $\{x : f(x) \geq a\}$ is nearly closed. Further, for all but a countable collection of values a, $\{x : f(x) = a\}$ is a set of measure zero. Hence, for all but a countable collection of values $a < b$, $\{x : a \leq f(x) < b\}$ is nearly clopen. It follows easily that f can be approximated uniformly by finite linear combinations of characteristic functions of nearly clopen sets. $\square$

Proposition 4.2.7. *If (X, τ, μ, T) and (Y, τ, ν, S) are two nearly continuous dynamical systems such that (X, τ, μ, T) is nearly uniquely ergodic and is nearly continuously conjugate to (Y, τ, ν, S), then (Y, τ, ν, S) is nearly uniquely ergodic.*

This follows from the definition.

Theorem 4.2.8. *([5] Theorem 5.2) Suppose that for (X, τ, μ, T), the ergodic averages for any bounded continuous function converge nearly uniformly. Suppose $A \subset X$ is a nearly clopen set, and $f : X \to \mathbb{R}$ is a bounded continuous function. Then the ergodic averages for $f 1_A$ converge nearly uniformly to $\int_A f d\mu$.*

Proof. First notice that if the ergodic averages for g converge nearly unifromly and $g' = g$ a.s., then the ergodic averages for g' also converge nearly uniformly. As those functions for which the ergodic averages converge nearly uniformly form a linear space and contain the constants, it is enough to prove the result for $f \geq 0$.

Suppose U is an open set. Then $f 1_U$ is the increasing limit of nonnegative, bounded continuous functions. As A is almost open, there exists X_0, with $\mu(X_0) = 1$ and for every $\epsilon > 0$ there exists N so that for all $x \in X_0$ and $n > N$,

$$A_N(f 1_A)(x) \geq \int_A f d\mu - \epsilon.$$

Assume $0 \leq f \leq B$ and apply the same reasoning to $(B - f)1_A$ and 1_{A^c}, noting that A^c is almost open, to conclude that off a further set of measure zero, once n is large enough,

$$A_n((B - f)1_A)(x) \geq \int_A (B - f) d\mu - \epsilon$$

and

$$A_n(B1_{A^c})(x) \geq B\mu(A^c) - \epsilon.$$

But then

$$A_n(B1_A)(x) - A_n(f1_a)(x) + A_n(B1_A^c)(x) = B - A_n(f1_A)(x) \geq B - \int_A f d\mu - 2\epsilon$$

which says that

$$A_n(f1_A)(x) \leq \int_A f d\mu + 2\epsilon$$

completing the proof. □

A homeomorphism T of a compact metric space is called *uniquely ergodic* if there exists a unique T-invariant Borel probability measure. Equivalently, (Proposition 2.8 of Petersen [16]) (X, μ, T) is uniquely ergodic if the ergodic averages converge uniformly to a constant on X for each continuous function, or, if for each continuous function f, some subsequence of $A_n(f)$ converges pointwise to a constant on X.

THEOREM 4.2.9. *If (X, μ, T) is a uniquely ergodic transformation on a compact metric space, then it is nearly uniquely ergodic.*

PROOF. If (X, μ, T) is a uniquely ergodic transformation of a compact metric space, then the ergodic averages of f converge uniformly for continuous functions f. Let $f = 1$ for all $x \in X$. Then, by Theorem 4.2.8, the ergodic averages of 1_A converge nearly uniformly to $\mu(A)$. By Lemma 4.2.6, the system is nearly uniquely ergodic.

□

If a system (X, τ, μ, T) is nearly continuously conjugate to a uniquely ergodic system, then it is nearly uniquely ergodic. The converse is true as well, and we arrive at the following:

THEOREM 4.2.10. *A nearly continuous dynamical system (X, τ, μ, T) is nearly uniquely ergodic if and only if it is nearly continuously conjugate to a uniquely ergodic homeomorphism on a compact metric space.*

The proof follows from Theorem 4.2.9 and from Theorem 2.3.2 which shows that any nearly continuous dynamical system is nearly continuously conjugate to a homeomorphism of a compact metric space.

THEOREM 4.2.11. *([5], Theorem 5.4) Suppose (X, τ, μ, T) is a nearly continuous dynamical system which is nearly uniquely ergodic, and that $A \subset X$ is nearly clopen with $\mu(A) > 0$. Then, T_A is nearly uniquely ergodic.*

PROOF. First, note that we lose nothing by assuming that A is open and hence Polish. Select a nearly clopen set $B \subset A$. Let $A'_n(1_B)$ be the ergodic averages of 1_B under the action of T_A. We need to show that $A'_n(1_B)$ converge nearly uniformly on A. For each $x \in A$, let $r_n(x)$ be the n-th return time of x to A under the action of T. Hence

$$A'_n(1_B) = \frac{1}{n} \sum_{i=0}^{r_n(x)-1} 1_B(T^i(x)) = \left(\frac{r_n(x)}{n}\right)\left(\frac{1}{r_n(x)}\right) \sum_{i=0}^{r_n(x)-1} 1_B(T^i(x)).$$

Now

$$\frac{n}{r_n(x)} = \frac{1}{r_n(x)} \sum_{j=0}^{r_n(x)-1} 1_A(T^j(x)).$$

This converges nearly uniformly to $\mu(A)$ in $r_n(x)$. We need it to be uniformly in n. Nearly unique erqodicity of T though now implies that off a set of measure zero, once n is large enough $r_n(x) > \frac{n}{2\mu(A)}$ and in particular the value $r_n(x)$ tends to infinity uniformly off this set of measure zero. This then implies that $\frac{n}{r_n(x)}$ converges nearly uniformly to $\mu(A)$. This in turn implies that

$$\frac{1}{r_n(x)} \sum_{i=0}^{r_n(x)-1} 1_B(T^i(x)) \to \mu(B)$$

nearly uniformly. Hence, $A'_n(1_B)$ converges nearly uniformly to $\frac{\mu(B)}{\mu(A)}$. □

4.2.12. Example. Suppose that (Z, τ, μ, R) is a nearly uniquely ergodic dynamical system. Build two towers $\hat{S}$ for the transformation S and $\hat{T}$ for the transformation T over Z such that the return time function to R is unbounded for $\hat{S}$ and bounded for $\hat{T}$. Suppose the $\hat{S}$ tower has less total mass than the $\hat{T}$ tower. S and T are n.c. Kakutani equivalent as $S_Z = T_Z = R$. Z has more mass in $\hat{S}$ than it does in $\hat{T}$. But, as the return times to Z for S are unbounded, S is not nearly uniquely ergodic, and T cannot induce S.

Our example now follows easily. Let X be the unit interval with the endpoints identified, and let T be irrational rotation of the interval representation of the unit circle so that $T(x) = x + \alpha (\text{mod } 1)$. Let Z be the sub-interval $(0, \frac{1}{3})$, and let $R = T_Z$. T is isomorphic to the tower for T over Z. We build another tower $\hat{S}$ over Z so that the return time function to Z for this tower is given by

$$r(x) = \sum_{n=1}^{\infty} n \chi_{(3^{-(n+1)}, 3^{-n}]}(x).$$

Note that $\lim_{x \to 0} r(x) = \infty$. Let

$$S(x,i) = \begin{cases} (x, i+1) & \text{for} \quad i < r(x) \\ (R(x), 0) & \text{for} \quad i = r(x) \end{cases}$$

and $\hat{S}$ be the tower over Z for S. S is not nearly uniquely ergodic, and the tower for S over Z only has mass $\frac{2}{3}$. Then $\mu_{\hat{S}}(Z) > \mu_X(Z)$, but T cannot induce S.

4.2.13. The Nearly Continuous Category. If we remove the obvious obstacle, as highlighted by the above example, we are able to extend the n.c. conjugacy between T_A and S_B.

Theorem 4.2.14. *Let (X, τ, μ, T) and (Y, τ, ν, S) be two nearly continuous dynamical systems such that $A \subset X$ and $B \subset Y$ are nearly clopen with $\mu(A) > \nu(B)$ and T_A n.c conjugate to S_B. If (X, τ, μ, T) is nearly uniquely ergodic, then there exists a nearly clopen set $\bar{B} \subset Y$ such that $S_{\bar{B}}$ is nearly continuously conjugate to T on X.*

Proof. Let X_0 and Y_0 be carriers for T and S, respectively, such that $A_0 = A \cap X_0$ and $B_0 = B \cap Y_0$ are clopen in their relative topologies, and let $\phi : A_0 \to B_0$ be the homeomorphism giving the conjugacy between T_{A_0} and S_{B_0}. As before, we may and do assume w.l.o.g. that $X_0 = X$, $A_0 = A$, $Y_0 = Y$, and $B_0 = B$. The idea for the proof is to select a subset $A' \subseteq A$ such that the return times for points in A' are smaller than the return times for points in $\phi(A')$

Let $\epsilon > 0$ be given such that $\epsilon < \frac{1}{4}(\frac{1}{\nu(B)} - \frac{1}{\mu(A)})$. By Theorem 4.2.11, as (X, τ, μ, T) is nearly uniquely ergodic and A is a nearly clopen set with of positive measure, T_A is nearly uniquely ergodic. We use the nearly unique ergodicity of T_A to find a set with "nice" return times. For $x \in A$, let

$$r_A(x) = \min\{r > 0 : T^r(x) \in A\}.$$

Note that $r_A(x)$ is bounded. There exists an $N_1 = N_1(\epsilon)$ such that

$$\left| \frac{1}{n} \sum_{i=1}^{n} r_A \left(T_A^i \right)(x) - \int_A r_A d\mu_A \right| < \epsilon \tag{4.2.1}$$

for all $n > N_1$ and nearly all x.

As T_A is n.c. conjugate to S_B, by Proposition 4.2.7, S_B is nearly uniquely ergodic on B. For $y \in B$, let

$$r_B(y) = \min \left\{ r > 0 : S^r(y) \in B \right\}.$$

Note that the ergodic averages for $r_B(y)$ converge to $\int_B r_B(y) d\nu_B = \frac{1}{\nu(B)}$, but may not converge nearly uniformly as $r_B(y)$ may not be bounded. For an $M \in \mathbb{N}$, define $f : B \to \mathbb{N}$ by $f = \min\{r_B(y), M\}$. f is a bounded and nearly continuous function. Select M so that

$$\int_B f d\nu_B > \frac{1}{\nu(B)} - \frac{\epsilon}{2}.$$

There exists $N_2(\epsilon)$ s.t.

$$\left| \frac{1}{n} \sum_{k=0}^{n-1} f(S_B^k(y)) - \int_B f(y) d\nu_B \right| < \frac{\epsilon}{2}$$

for all $n > N_2$ and nearly all y. Then

$$\begin{array}{rrcl} & \int_B f(y) d\nu_B - \frac{\epsilon}{2} & < & \frac{1}{n} \sum_{k=0}^{n-1} f(S_B^k(y)) \\ \Rightarrow & \frac{1}{\nu(B)} - \frac{\epsilon}{2} - \frac{\epsilon}{2} & < & \frac{1}{n} \sum_{k=0}^{n-1} f(S_B^k(y)) \\ \Rightarrow & \frac{1}{\nu(B)} - \epsilon & < & \frac{1}{n} \sum_{k=0}^{n-1} f(S_B^k(y)) \\ \Rightarrow & \frac{1}{\nu(B)} - \epsilon & < & \frac{1}{n} \sum_{k=0}^{n-1} r_B(S_B^k(y)) \end{array} \tag{4.2.2}$$

Let $N = \max\{N_1, N_2, \frac{\mu(A)}{\mu(A)-\nu(B)}\}$. Use Theorem 3.1.7 to select a nearly clopen set $A' \subset A$ such that the return time to A' via T_A is either $2N$ or $2N+1$. Then

$$\sum_{k=0}^{2N-1} r_A(T_A^k(x)) \leq r_{A'}(x) \leq \sum_{k=0}^{2N} r_A(T_A^k(x)).$$

Let $B' = \phi(A')$. As T_A and S_B are conjugate, B' also has $2N$ and $2N+1$ as the only return times, and

$$\sum_{k=0}^{2N-1} r_B(S_B^k(y)) \leq r_{B'}(y) \leq \sum_{k=0}^{2N} r_B(S_B^k(y)).$$

It only remains to check that the tower over B' is taller than the tower over A'.

By 4.2.1,

$$\begin{aligned} \tfrac{1}{2N+1} \textstyle\sum_{i=0}^{2N} r_A(T_A^i(x)) &< \tfrac{1}{\mu(A)} + \epsilon \\ \Rightarrow r_{A'} &< (2N+1)(\tfrac{1}{\mu(A)} + \epsilon). \end{aligned}$$

By 4.2.2,

$$\begin{aligned} \tfrac{1}{2N} \textstyle\sum_{i=0}^{2N} r_B(S_B^i(y)) &\geq \tfrac{1}{\nu(B)} - \epsilon \\ \Rightarrow r_{B'} &> 2N(\tfrac{1}{\nu(B)} - \epsilon). \end{aligned}$$

We then need

$$\begin{aligned} (2N+1)(\tfrac{1}{\mu(B)} + \epsilon) &< 2N(\tfrac{1}{\nu(B)} - \epsilon) \\ \Rightarrow (2N+1)\left(\tfrac{3}{4}\tfrac{1}{\mu(A)} + \tfrac{1}{4}\tfrac{1}{\nu(B)}\right) &< 2N\left(\tfrac{3}{4}\tfrac{1}{\nu(B)} + \tfrac{1}{4}\tfrac{1}{\mu(A)}\right) \\ \Rightarrow \tfrac{3}{4}\tfrac{1}{\mu(A)} + \tfrac{1}{4}\tfrac{1}{\nu(B)} &< N\left(\tfrac{1}{\nu(B)} - \tfrac{1}{\mu(A)}\right) \end{aligned}$$

which holds as $N > \frac{\mu(A)}{\mu(A)-\nu(B)}$.

Let

$$h(x) = min\{h \geq 0 : T^{-h}(x) \in A'\}.$$

Define

$$\hat{\phi}(x) := S^{h(x)} \circ \phi \circ T^{-h(x)}(x) \text{ for } x \in X.$$

Let

$$\bar{B} = \hat{\phi}(X).$$

It is easy to see that $\bar{B}$ is nearly clopen and that $S_{\bar{B}}$ is nearly continuously conjugate to T. □

4.3. Application to Strongly Rank One Transformations

PROPOSITION 4.3.1. *Suppose that (Y, τ, μ_Y, S) and (X, τ, μ_X, T) are two n.c. dynamical systems, and that T is nearly uniquely ergodic. Suppose there exists a nearly clopen set $B \subset Y$ with $\mu_Y(B) > 0$ such that S_B and T are n.c. evenly Kakutani equivalent. Then there exists a nearly clopen set $D \subset Y$ such that S_D is n.c. conjugate to T and $\mu_Y(D) = \mu_Y(B)$.*

PROOF. As S_B and T are n.c. evenly Kakutani equivalent, there exists $F \subset B$ and $C \subset X$ such that ϕ is a n.c. conjugacy between $(S_B)_F = S_F$ and T_C. Both T_C and S_F are nearly uniquely ergodic and $\mu_X(C) = \mu_B(F)$. Let $A \subset B$ be a nearly clopen set of positive measure. There exists k such that $\mu_Y(T^k F \cap A) \neq 0$. Let $A' = T^k F \cap A$ and $C' = \phi^{-1}(F \cap T^{-k} A)$. Now, $\mu_Z(C') < \mu_A(A')$ so there exists a nearly clopen set $\widetilde{C} \subset X$ such that $T_{\widetilde{C}}$ is n.c. conjugate to $(S_B)_A$. Note that

$$\mu_Z(\widetilde{C}) = \frac{\mu_Z(C')}{\mu_A(A')} = \frac{\mu_Z(C')}{\mu_B(A')} \mu_B(A) = \mu_B(A)$$

as S_B and T are n.c. evenly Kakutani equivalent.

As $\mu_X(\widetilde{C}) = \mu_B(A)$, $\mu_X(\widetilde{C}) > \mu_Y(A)$, there exists a nearly clopen set $D \subset Y$ such that T is n.c. conjugate to S_D. Then

$$\mu_Y(D) = \frac{\mu_Y(A)}{\mu_X(\widetilde{C})} = \frac{\mu_Y(A)}{\mu_B(A)} = \mu_Y(A)\frac{\mu_Y(B)}{\mu_Y(A)} = \mu_Y(B).$$

□

Recall from [18] and [10] that all odometers and all irrational rotations of the circle are nearly continuously evenly Kakutani equivlanent to the binary odometer.

Lemma 4.3.2. *Let (Y, τ, ν, S) be any irrational rotation of the unit circle or adding machine. Given any irrational value α, $0 < \alpha < 1$, there exists a nearly clopen set $D \subset Y$ with $\nu(D) = \alpha$ such that S_D is nearly continuously conjugate to an irrational rotation.*

Proof. Let (X, τ, μ, T) be the irrational rotation of the unit circle by α. Let A be the interval $[0, \alpha]$, and induce on this set. One sees that this induced transformation may be represented by the interval exchange where $[0, 1-n\alpha]$ maps to $[(n+1)\alpha-1, \alpha]$ and $[1-n\alpha, \alpha]$ maps to $[0, (n+1)\alpha - 1]$ where $n = \lfloor \frac{1}{\alpha} \rfloor$, the fractional part of $\frac{1}{\alpha}$. Thus, T_A is an irrational rotation.

As S and T are n.c. evenly Kakutani equivalent, there exists nearly clopen subsets $B \subset X$ and $C \subset Y$ such that T_B and S_C are n.c. conjugate via ϕ. There exists k such that $\mu(T^k(B) \cap A) \neq 0$. Let $A' = T^k(B) \cap A$ and $C' = \phi^{-1}(B \cap T^{-k}A)$. $T_{A'}$ is n.c. conjugate to $S_{B'}$. As $\mu_A(A') > \nu(C)$, and T_A is nearly uniquely ergodic, there exists $D \subset Y$, nearly clopen, such that T_A is nearly continuously conjugate to S_D. Then $\nu(D) = \frac{\nu(C')}{\mu_A(A')} = \frac{\nu(C')}{\mu(A')}\mu(A) = \mu(A)$. □

Next, consider the case where α is rational.

Lemma 4.3.3. *Given an irrational rotation of the unit circle (or adding machine)* (Y, τ, ν, S), *for any rational value* α, $0 < \alpha < 1$, *there exists a nearly clopen set* D *with* $\nu(D) = \alpha$ *such that* S_D *is nearly continuously conjugate to an adding machine.*

Proof. Write α as $\frac{p}{q}$ where $p, q \in \mathbb{N}$. Let (X, τ, μ, T) be the adding machine produced by cutting the stage i stack/tower into q columns of equal width before stacking. Let the set A be the first p levels of the first tower of height q. The induced map T_A on A is n.c. conjugate to the adding machine produced by cutting the unit interval into p sub-intervals of equal width to create the first stack, then cutting the subsequent stacks into q columns of equal width.

Following the lines of the proof for the case where α is irrational, we have the existence of a nearly clopen subset $D \subset Y$ such that S_D and T_A are n.c. conjugate and $\nu(D) = \alpha$.

□

In the measure-theoretic category, given a dynamical system (X, μ, T), and any measurable subset $A \subset X$, T_A is evenly Kakutani equivalent to T. For the n.c. even Kakutani equivalence class of transformations containing irrational rotations of the unit circle, we obtain a similar result.

Corollary 4.3.3.1. *Let* (X, τ, μ, T) *be any irrational rotation of the unit circle or adding machine. If* $A \subset X$ *is any nearly clopen set, then* T_A *is n.c. evenly Kakutani equivalent to* T.

Proof. Let $\mu(A) = \alpha$. By Lemma 4.3.2 and Lemma 4.3.3, there exists a nearly clopen set $B \subset X$ with $\mu(B) = \alpha$ such that T_B is n.c. conjugate to an irrational rotation of the circle or an adding machine. For some $k \in \mathbb{Z}$, $\mu(T^k(A) \cap B) \neq 0$. Let $B' = T^k(A) \cap B$ and

$A' = A \cap T^{-k}(B)$. $T_{B'}$ and $T_{A'}$ are n.c. conjugate via T^k, $\mu_A(A') = \mu_B(B')$ so T_B and T_A are n.c. evenly Kakutani equivalent. □

Naturally, we wish to discover other systems which belong to this equivalence class.

Corollary 4.3.3.2. *All strongly rank one transformations belong to the nearly continuous even Kakutani equivalence class of irrational rotations.*

Proof. A cutting and stacking is defined by the number of columns of equal width into which we cut a stack and the number of spacers placed above each column during each stage of construction. At stage i, cut the stack into $c(i)$ columns of equal width and place $s(i,j)$ spacers above column j for $j = 1, \ldots, c(i)$. Moving to stage $i+1$, stack the new columns which include the spacers by placing columns on top of the column directly to the left, forming a single stack.

Let (X, τ, μ, T) be a strongly rank one transformation. Let I be the very first interval which we cut into $c(1)$ segments. I is nearly clopen, and inducing T_I on I ignores the addition of all spacers. T_I is thus n.c. conjugate to an adding machine.

Suppose $\mu(I) = \alpha$. Let (Y, τ, ν, S) be an adding machine. Whether α is irrational or rational by Lemma 4.3.2 and Lemma 4.3.3, there exists a nearly clopen set $J \subset Y$ with $\nu(J) = \alpha$ such that S_J is n.c. conjugate to an adding machine or an irrational rotation of the circle. As T_I is n.c. evenly Kakutani equivalent to S_J and $\nu(J) = \mu(I)$, T is n.c. evenly Kakutani equivalent to S.

□

The following figures provide an example of inducing on the very first interval, and not the interval from which we cut the spacers, for Chacon's map.

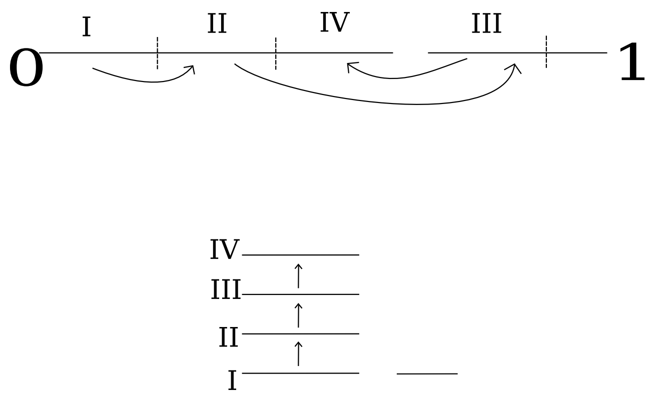

FIGURE 4.2. The first stage of cutting and stacking for Chacon's map.

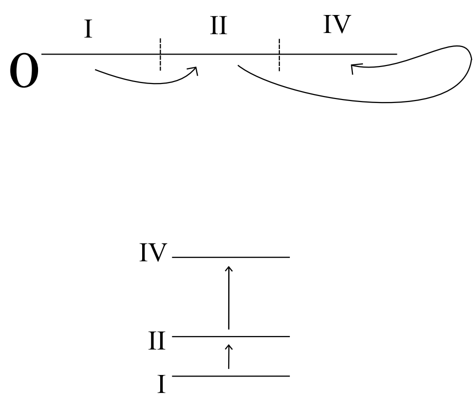

FIGURE 4.3. The first stage of cutting and stacking for the induced transformation, ignoring spacers.

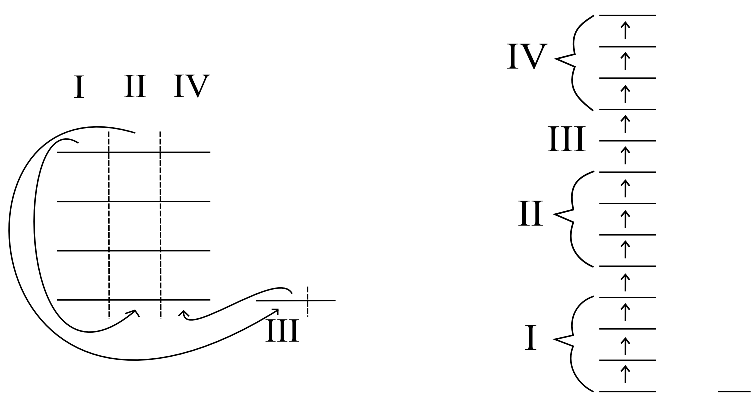

FIGURE 4.4. The second stage of cutting and stacking for Chacon's map.

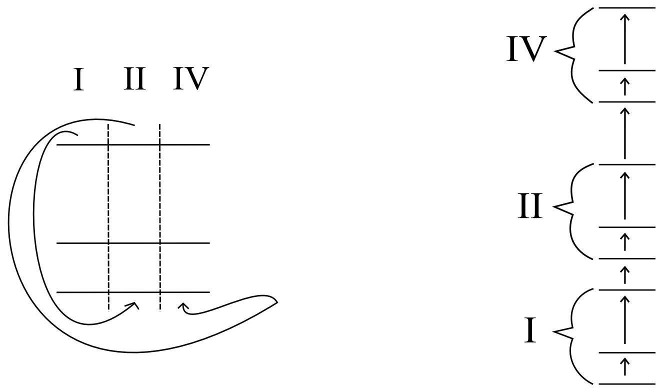

FIGURE 4.5. The second stage of cutting and stacking for the induced transformation, ignoring spacers.

CHAPTER 5

Minimal Isomorphisms of Compact Metric Spaces

While examples exist showing that nearly continuous even Kakutani equivalence is strictly stronger than even Kakutani equivalence, such examples, like the one given in Chapter 4, tend to be contrived. As all zero-entropy loosely Bernoulli systems are Kakutani equivalent, they provide an interesting test-bed for attempting to discover a more natural example. As mentioned, Roychowdhury and Rudolph [18] proved that all adding machines are nearly continuously Kakutani equivalent. Dykstra and Rudolph [10] then added irrational rotations of a circle to this class. The machinery employed in [18] and [10] relied heavily upon the existence of a classical sequence of towers, essentially cutting and stacking constructions, for both odometers and irrational rotations. Generic minimal isometries of compact metric spaces lack a classical sequence of towers, and showing that all minimal isometries of compact metric spaces are nceKe to the class containing irrational rotations of the circle provides a natural, and interesting, next step within the theory of nearly continuous dynamics. Within this chapter, we further modify the tower and template machinery in order to prove that minimal isometries of compact metric spaces are all nceKe to the binary odometer.

5.1. Preliminaries

5.1.1. Minimal Isometries of Compact Metric Spaces. X is a compact metric space with metric ρ inducing the topology τ. T represents a minimal isometry on X, meaning that the orbit of every point is dense and $\rho(x, y) = \rho(Tx, Ty)$. It is a well known fact that minimal isometries of compact metric spaces have a unique, T-invariant measure. Let μ

represent the T-invariant, uniquely ergodic measure. Simple examples of minimal isometries of compact metric spaces would be irrational rotations of tori.

5.1.2. Binary Odometer. Recall the classical description of cutting and stacking the unit interval to represent the binary odometer. Y is the unit interval. The first step of construction for the binary odometer consists of cutting the interval in half and stacking the right side on top of the left, and the stage 1 transformations translates the bottom level straight upward to the top level. The full transformation is found as the limit of cutting the stage i tower in half vertically, stacking the right side on top of the left, and then translating all but the top level directly to the level above. We use the representation of the binary odometer as a cutting and stacking in the machinery of our proof.

5.1.3. Partial Interval Maps. Given two intervals $I,\ J \subset \mathbb{Z}$, of length n and m respectively, and regarding intervals of the same length as equivalent, we write without loss of generality $I = [0, 1, \ldots, n-1]$, $J = [0, 1, \ldots, m-1]$. Let $f : I \to J$ be a bijection between two subsets of these intervals. We call the ordered triple $\hat{f} := [I, J, f]$ a partial interval bijection, which we abbreviate with pib. Two partial interval bijections are equivalent if one is a translate of the other.

Suppose $\hat{f}_i = [I_i, J_i, f_i]$ for $i = 1, 2$, and I_i and J_i start at 0. Let $t = |I_1|$ and $s = |J_1|$. We define the concatenation of the two partial interval bijections as

$$\hat{f}_1|\hat{f}_2 = [I_1 \cup (I_2 + t), J_1 \cup (J_2 + s), f]$$

where

$$f(j) = \begin{cases} f_1(j) & \text{for} \quad j \in I_1 \\ f_2(j-t) + s & \text{for} \quad j \in I_2 + t \end{cases}$$

Note that the concatenation of two pibs is also a pib.

For $\hat{f} = [I, J, f]$, define

$$\hat{f}^{-1} = [J, I, f^{-1}].$$

If the domain of f does not equal I, then we may extend $\hat{f}$ to a map $\hat{g} = [I, J, g]$ where $g|_{\text{domain}(f)} = f$. We say $\hat{g}$ extends $\hat{f}$.

5.2. Stages of the Construction

The construction proceeds by induction.

Requirement 5.2.1. *Let $\tilde{\epsilon}_n > 0$ be such that $\sum_{i=1}^{\infty} \tilde{\epsilon}_n < \frac{1}{2}$.*

5.3. The Base Case

Select a tower Q_0 of height h_0 from the classical cutting and stacking construction for the binary odometer such that $h_0 > 2$.

Select a point in X and label this point 0.

Lemma 5.3.1. *Given any value $N > 0$, there exists an $\eta = \eta(N) > 0$ such that the first return time to $\mathcal{B}_\eta(0) = \{x \in X : \rho(x, 0) < \eta\}$ is at least N for all $x \in \mathcal{B}_\eta(0)$.*

Proof. Suppose not. Then, there exists an $N > 0$ such that for every $\eta > 0$, $T^N(\mathcal{B}_\eta(0)) \cap \mathcal{B}_\eta(0) \neq \emptyset$. Letting $\eta \to 0$ implies that $T^N(0) = 0$, which contradicts that T is minimal. $\square$

By Lemma 5.3.1, we may select $\eta_1 > 0$ such that the skyscraper over $\mathcal{B}_{\eta_1}(0)$ (whose height is given by the first return time to $\mathcal{B}_{\eta_1}(0)$) has height at least $\frac{3h_0}{\tilde{\epsilon}_1}$ and such that $\mu(\partial \mathcal{B}_{\eta_1}(0)) = 0$. Decompose $\mathcal{B}_{\eta_1}(0)$ into the minimum number of nearly clopen sets such that the first return time to $\mathcal{B}_{\eta_1}(0)$ is constant on each set. As minimal isometries of compact metric spaces are uniquely ergodic, there are only finitely many first return times, and hence finitely many sets in this decomposition. Using each set as the base of a column in the skyscraper, decompose the skyscraper into nearly clopen columns of distinct heights. We shall refer to these columns as towers. Label the towers in non-decreasing order of height as $\mathcal{P}_1^1, \ldots, \mathcal{P}_1^w$. We shall refer to $\mathcal{P}_1$, the set of the stage 1 towers $\{\mathcal{P}_1^1, \ldots, \mathcal{P}_1^w\}$, as the stage 1 multi-tower.

A $\mathcal{Q}_0$ *template*, denoted ω_1, is an initial segment of the orbit of $0 \in Y$ of length mh_0 for $m \in \mathbb{N}$. For a tower $\mathcal{P}_1^*$ of height t_1^*, define $\hat{\Omega}_1^*$ as the set of five templates with the following prescribed lengths:

$$|\omega_1^{*,1}| < |\omega_1^{*,2}| < t_1^* \leq |\omega_1^{*,3}| < |\omega_1^{*,4}| < |\omega_1^{*,5}| \text{ and}$$
$$|\omega_1^{*,k}| = |\omega_1^{*,k-1}| + h_0 \text{ for } k = 2, 3, 4, 5.$$

Let

$$\Omega_1^* = \bigcup_{k=1}^{5} \omega_1^{*,k}$$

be the $\mathcal{Q}_0$-template set for $\mathcal{P}_1^*$ and

$$\Omega_1 = \bigcup_{*=1}^{w} \Omega_1^*$$

be the $\mathcal{Q}_0$-template set for $\mathcal{P}_1$.

Define the projection $\xi : \Omega_1 \to \mathcal{Q}_0$ by $\xi(S^k 0) = S^k(base(\mathcal{Q}_0))$ or $\xi(y) = d_0(y)$ where $d_0(y)$ is the level of $\mathcal{Q}_0$ which contains y.

Define the first set of partial interval bijections $\phi_1^{*,k}$ for $* \in \{1, \ldots, w\}$ and $k \in \{1, \ldots, 5\}$ from the indices of the levels of $\mathcal{P}_1^*$ to the indices of points in $\omega_1^{*,k}$ by

$$\phi_1^{*,k} = \Big[[0, t_1^* - 1], [0, |\omega_1^{*,k}|], f\Big]$$

where $f(i) = i$ for $i = 0, \ldots, \lfloor \frac{1}{2}(t_1^* - 1) \rfloor$ and $f(i)$ is not yet defined for $i = \lfloor \frac{1}{2}(t_1^* - 1) \rfloor + 1, \ldots, t_1^* - 1$. $\phi_1^{*,k}$ maps the indices of the levels of the bottom half of each tower to the indices of the approximate bottom half of each pib.

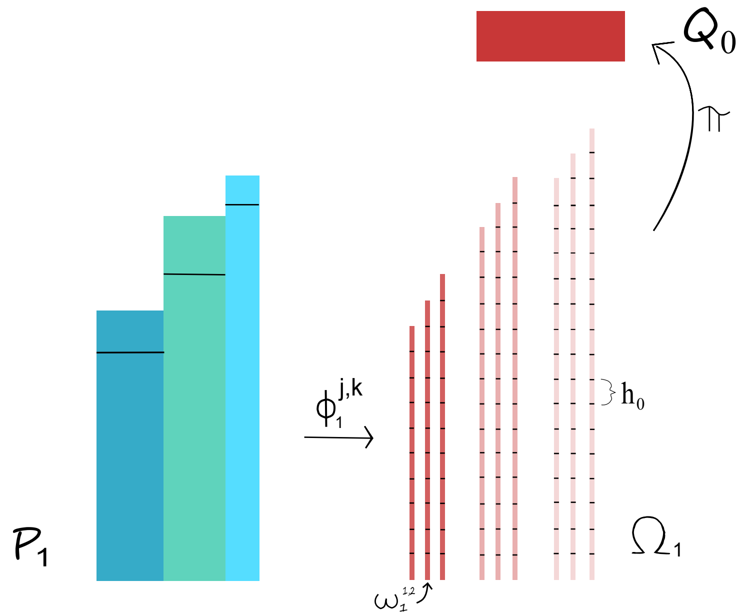

FIGURE 5.1. The first stage using the tower and template machinery.

5.4. The Inductive Steps

Assume that the stage $2n-1$ tower for the binary odometer was selected from the classical sequence of cutting and stacking, and call it $\mathcal{Q}_{2n-1}$. The height of $\mathcal{Q}_{2n-1}$ is h_{2n-1}. For the point $0 \in X$, let $\mathcal{B}_{\eta_{2n}}(0)$ be the base of the multi-tower $\mathcal{P}_{2n}$. Assume that $\mu(\partial \mathcal{B}_{\eta_{2n}}(0)) = 0$. The towers of $\mathcal{P}_{2n}$ were listed in non-decreasing order of height. Let $\mathcal{P}_{2n}^*$ be the $*$th tower of the list, and let the height of tower $\mathcal{P}_{2n}^*$ be t_{2n}^*. Suppose there are w towers and let base$(\mathcal{P}_{2n}^*)$ denote the base of tower $\mathcal{P}_{2n}^*$.

For tower $\mathcal{P}^*_{2n}$, assume we defined the template set, $\hat{\Omega}^*_{2n}$, as the set of five templates whose lengths satisfied:

$$|\omega_n^{*,1}| < |\omega_n^{*,2}| < t^*_{2n} \leq |\omega_n^{*,3}| < |\omega_n^{*,4}| < |\omega_n^{*,5}| \text{ and}$$
$$|\omega_{2n}^{*,k}| = |\omega_{2n}^{*,k-1}| + h_{2n-1} \text{ for } k = 2, 3, 4, 5.$$

Note that

$$|\omega_n^{*,1}| \quad < \quad t^*_{2n} - (h_{2n-1} + 1) \tag{5.4.1}$$

$$|\omega_{2n}^{*,5}| \quad \geq \quad t^*_{2n} + 2h_{2n-1} \tag{5.4.2}$$

We defined

$$\Omega^*_{2n} = \bigcup_{k=1}^{5} \omega_{2n}^{*,k},$$

the Q_{2n-1}-template set for $\mathcal{P}^*_{2n}$ and

$$\Omega_{2n} = \bigcup_{*=1}^{w} \Omega^*_{2n}$$

the Q_{2n-1} template set for $\mathcal{P}_{2n}$.

Assume that the set of partial interval bijections $\phi_{2n}^{*,k}$ have been defined from at least a fraction $(1 - \tilde{\epsilon}_{2n})$ of the indices of the levels of the $\mathcal{P}^*_{2n}$ to at least a fraction $(1 - \tilde{\epsilon}_{2n})$ of the indices of the points in the templates $\omega_{2n}^{*,k}$ for $* = 1, \ldots, w$ and $k = 1, \ldots, 5$.

We defined $\xi : \Omega_{2n} \to \mathcal{Q}_{2n-1}$ by $\xi(S^k 0) = S^k(\text{base}(\mathcal{Q}_{2n-1}))$ or $\xi(y) = d_{2n-1}(y)$ where $d_{2n-1}(y)$ is the level of $\mathcal{Q}_{2n-1}$ which contains y.

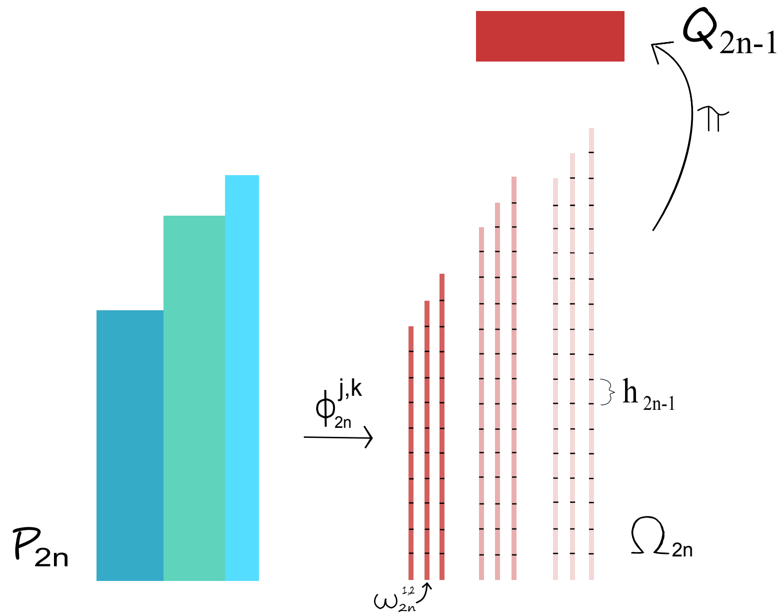

FIGURE 5.2. Stage $2n$ of the tower and template machinery.

5.4.1. STAGE 2N+1. Define

$$S_{2n} = \bigcup_{*=1}^{w} \bigcup_{i=0}^{t_{2n}^{*}-1} T^{i}(\partial \mathrm{base}(\mathcal{P}_{2n}^{*})).$$

Select $\epsilon_{2n} > 0$ such that

$$\mu(\{x : \rho(x, S_{2n}) < \epsilon_{2n}\}) < \frac{1}{2}\tilde{\epsilon}_{2n}$$

and define

$$S_{2n}^{\epsilon_{2n}} = \{x : \rho(x, S_{2n}) < \epsilon_{2n}\}.$$

LEMMA 5.4.2. *There exists P such that for all $p > P$ and μ-a.e.$x \in X$,*

$$\frac{1}{p}\sum_{i=0}^{p-1} \chi_{S_{2n}^{\epsilon_{2n}}}(T^{i}x) < \tilde{\epsilon}_{2n}.$$

Proof. By the unique ergodicity of minimal isometries of compact metric spaces, given $\frac{1}{4}\tilde{\epsilon}_{2n}$, there exists a value $p > P$ such that

$$\left| \frac{1}{p} \sum_{i=0}^{p-1} \chi_{S_{2n}^{\epsilon_{2n}}}(T^i x) - \mu(S_{2n}^{\epsilon_{2n}}) \right| < \frac{1}{4}\tilde{\epsilon}_{2n}.$$

□

Define $\{\mathcal{A}_R(x)\}$ to be the sequence of return times to $\mathcal{B}_{\epsilon_{2n}}(0)$, so $\mathcal{A}_1(x) = r_{\mathcal{B}_{\epsilon_{2n}}(0)}(x)$ the first return time, $\mathcal{A}_2(x) = r_{\mathcal{B}_{\epsilon_{2n}}(0)}(T^{\mathcal{A}_1(x)}(x))$, the second return time, etc. Define $\kappa = \max\{k : \mathcal{A}_1(x) = k \text{ for } x \in \mathcal{B}_{\epsilon_{2n}}(0)\}$. Let h_{2n} be the height of the tallest tower of $\mathcal{P}_{2n}$.

Requirement 5.4.3. *Select a tower $\mathcal{Q}_{2n+1}$ with height h_{2n+1} from the classical sequence of cutting and stacking for the binary odometer such that h_{2n+1} satisfies:*

(1) $h_{2n+1} > P + 3\kappa$ where P is as in Lemma 5.4.2

(2) $h_{2n+1} > \frac{h_{2n}\kappa}{1-\tilde{\epsilon}_{2n+1}}$,

(3) $h_{2n+1} > \frac{3\kappa}{\tilde{\epsilon}_{2n+1}h_{2n-1}}\left(h_{2n} + h_{2n-1}^2\right)$

(4) $h_{2n+1} > \frac{2\kappa h_{2n}+h_{2n-1}}{\tilde{\epsilon}_{2n+1}}$.

(5) $h_{2n+1} > \frac{1}{\tilde{\epsilon}_{2n+1}h_{2n-1}}\left(4\kappa h_{2n} + 4\kappa h_{2n-1} + (h_{2n-1})^2\right)$

(6) $h_{2n+1} > \frac{1}{\tilde{\epsilon}_{2n+1}h_{2n-1}}\left((\kappa-1)h_{2n} + (h_{2n-1})^2 - (\kappa-1)h_{2n-1}\right)$

Let $|\mathcal{P}_{2n}^k|$ be the union of the levels of tower $\mathcal{P}_{2n}^k$. Let $\alpha_{2n} = \{|\mathcal{P}_{2n}^1|, |\mathcal{P}_{2n}^2|, \ldots, |\mathcal{P}_{2n}^w|\}$, a finite partition of X. Recall the existence of a natural map from X to $\{1, 2, \ldots, w\}^{\mathbb{Z}}$ from Section 2.2 giving rise to α_{2n}-names of orbits of points in X. Let $\aleph$ be the set of α_{2n}-names of length $h_{2n+1}+2\kappa$ for forward orbits of points in $\mathcal{B}_{\epsilon_{2n}}(0)$. For each distinct name $\beta \in \aleph$, select a single point $x \in \mathcal{B}_{\epsilon_{2n}}(0)$ whose α_{2n}-name of length $h_{2n+1} + 2\kappa$ is β. Select 0 to represent the names in $\aleph$ with the same α_{2n}-name as the orbit of 0. We call these points representative points. Let $\mathcal{R}$ denote the set of representative points. For an $x \in \mathcal{R}$, a $\mathcal{T}_{2n+1}$-template for x, τ^x_{2n+1}, is an initial segment of the orbit of x such that $|\tau^x_{2n+1}| = \mathcal{A}_R(x)$ for some R. $\mathcal{T}^x_{2n+1}$, the set of Q_{2n+1}-*templates* for $x \in \mathcal{R}$ is the set of $\mathcal{T}_{2n+1}$ templates such that the shortest template $\tau^{x,1}_{2n+1}$ satisfies

$$|\tau^{x,1}_{2n+1}| = \max\{h : h_{2n+1} - h > \kappa \text{ and } T^h(x) \in \mathcal{B}_{\epsilon_{2n}}(0)\},$$

the longest template $\tau^{x,m(x)}_{2n+1}$ satisfies

$$|\tau^{x,m(x)}_{2n+1}| = \min\{h : h - h_{2n+1} > \kappa \text{ and } T^h(x) \in \mathcal{B}_{\epsilon_{2n}}(0)\},$$

(where $m(x)$ denotes the number of returns to $\mathcal{B}_{\epsilon_{2n}}(0)$ for $T^i(x)$ from $i = |\tau^{x,1}_{2n+1}|$ to $i = \min\{h : h - h_{2n+1} > \kappa \text{ and } T^h(x) \in \mathcal{B}_{\epsilon_{2n}}(0)\}$), and the templates of intermediate length $\tau^{x,2}_{2n+1}$,...,$\tau^{x,m(x)-1}_{2n+1}$ satisfy

$$|\tau^{x,l}_{2n+1}| = |\tau^{x,l-1}_{2n+1}| + \mathcal{A}_1(T^{|\tau^{x,l-1}_{2n+1}|}(x)) \text{ for } l = 2, \ldots, m(x) - 1.$$

Define

$$\mathcal{T}^x_{2n+1} = \bigcup_{l=1}^{m(x)} \tau^{x,l}_{2n+1},$$

the set of templates for a single point $x \in \mathcal{R}$. We call the set of templates for 0 the master templates. We call the sets of templates for $x \in \mathcal{R}$, $x \neq 0$, the representative templates. Define

$$\mathcal{T}_{2n+1} = \bigcup_{x \in \mathcal{R}} \mathcal{T}^x_{2n+1},$$

the full set of templates for the $2n+1$ stage of construction. Define the projection $\xi : \mathcal{T}_{2n+1} \to \mathcal{P}_{2n}$ by $\xi(x) = a_{2n}(x)$ where $a_{2n}(x)$ is the level of $\mathcal{P}_{2n}$ containing x.

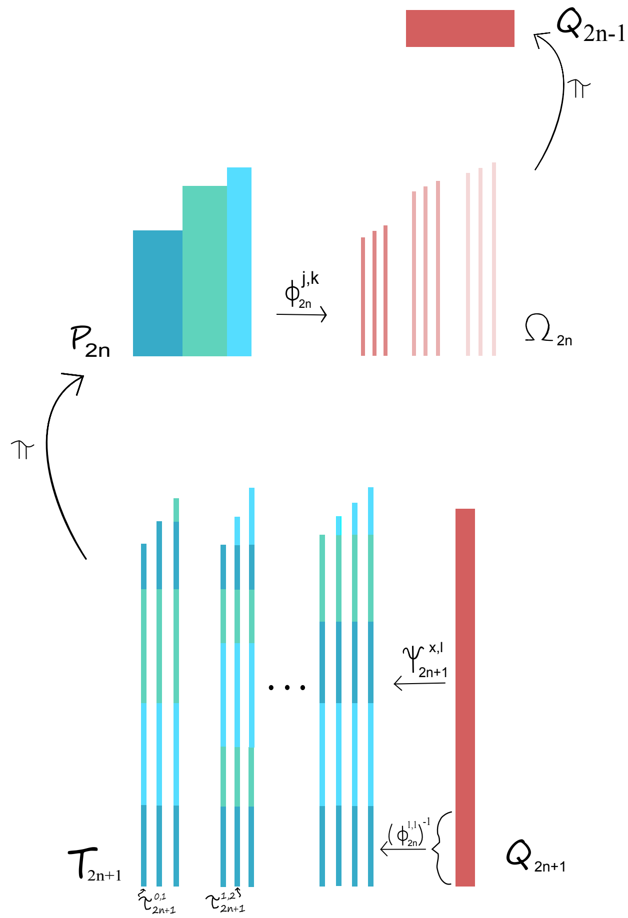

FIGURE 5.3. Stage $2n+1$ of the tower and template machinery.

We will now give an informal account of the stage $2n+1$ construction which allows us to define the partial interval bijections from Q_{2n+1} to the the $\mathcal{T}_{2n+1}$ templates. Each $\tau^{x,l}_{2n+1} \in \mathcal{T}_{2n+1}$ for $x \in \mathcal{R}$ and $l = 1, \ldots, m(x)$ has an α_{2n} name, which canonically breaks $\tau^{x,l}_{2n+1}$ into a certain sequence of blocks of length t^*_{2n}, $* \in \{1, \ldots, w\}$ and gives each block a label from α_{2n}.

DEFINITION 5.4.4. *A tiling of* $\mathcal{Q}_{2n+1}$ *by* Ω_{2n} *is an ordered list of templates* $\omega^{j(i),k(i)}_{2n}$ *from* Ω_{2n} *such that* $h_{2n+1} = \sum_{i=1}^{\tilde{r}} |\omega^{j(i),k(i)}_{2n}|$. *We also say that we tile* $\mathcal{Q}_{2n+1}$ *with* Ω_{2n}.

Given a template $\tau_{2n+1}^{x,l} \in \mathcal{T}_{2n+1}$, the tiling of $\mathcal{Q}_{2n+1}$ by Ω_{2n} *corresponding to* $\tau_{2n+1}^{x,l}$ is said to be *good* if based on the sequence of blocks in $\tau_{2n+1}^{x,l}$, we select a sequence $\omega_{2n}^{j(i),k(i)}$ of templates from Ω_{2n} with which to tile $\mathcal{Q}_{2n+1}$ such that if the r^{th} block of $\tau^{x,l}$ is of length t_{2n}^{J} and α_{2n}-name J, then the r^{th} tile in the tiling of $\mathcal{Q}_{2n+1}$ is one of the variant $\omega_{2n}^{J,k}$ templates, and

$$h_{2n+1} = \sum_{i=1}^{R} |\omega^{j(i),k(i)}|$$

where R is the number of blocks in $\tau^{x,l}$.

The partial interval bijection $\hat{\psi}_{2n+1}^{x,l}$ shall be a concatenation of the inverses of the $\phi_{2n}^{j(i),k(i)}$, where the i^{th} piece of the concatenation is the inverse of $\phi_{2n}^{j(i),k(i)}$ if the i^{th} tile of the tiling for $\mathcal{Q}_{2n+1}$ is $\omega_{2n}^{j(i),k(i)}$. We then extend up to all but a small fraction of the tower. We are careful in this process to select a tiling for $\mathcal{Q}_{2n+1}$ for each of the $\tau_{2n+1}^{x,l}$ templates in such a way that the resultant partial interval bijections $\psi_{2n+1}^{x,k}$ agree for most of the levels in $\mathcal{Q}_{2n+1}$. We accomplish this first by defining the pibs from $\mathcal{Q}_{2n+1}$ to the master templates, and then modeling the pibs from $\mathcal{Q}_{2n+1}$ to the representative templates after the pibs for the master templates.

So, the goal is to build $\psi_{2n+1}^{x,l}$ using concatenations of inverses of the $\phi_{2n}^{j,k}$, and then extensions, in such a way that the $\psi_{2n+1}^{x,l}$ agree on at least a fraction $1 - \tilde{\epsilon}_{2n+1}$ of the levels of $\mathcal{Q}_{2n+1}$. We will do this in essentially two steps for each template, beginning with the master templates for the orbit of zero. First, we will pick a height approximately $1 - \tilde{\epsilon}_{2n+1}$ up $\mathcal{Q}_{2n+1}$. Then, we will pick a height for the master templates which is the same or just slightly taller. As this height is still shorter than the shortest master template, the orbits of each master template are identical up to this height. We will create a tiling of $\mathcal{Q}_{2n+1}$ up to this height using templates from Ω_{2n}. Then, for each of the master templates, $\tau_{2n+1}^{0,1}$

through $\tau_{2n+1}^{0,m(0)}$, we will create a distinct tiling of the remaining upper portion of $\mathcal{Q}_{2n+1}$ corresponding to the remaining upper portion of each template, and combine the tiling for the bottom portion with the tiling of the top portion. These tilings lead to the choice of the $\phi_{2n}^{*,k}$ whose inverses are concatenated to define $\psi_{2n+1}^{x,l}$, ensuring that the pibs from $\mathcal{Q}_{2n+1}$ to the master templates agree for at least $1-\tilde{\epsilon}_{2n+1}$ of the tower. After the process is completed for the master templates, we then repeat the process for the representative templates with a few small tweaks. Note that the representative templates are initial segments of orbits of points within ϵ_{2n} of 0. Whenever $T^h 0 \notin S_{2n}^{\epsilon_{2n}}$ and $T^h 0 \in \text{base}(\mathcal{P}_{2n})$, $T^h 0$ and $T^h x$ are in the same base of a tower $\mathcal{P}_{2n}^{*}$ for some $*$, and this occurs most of the time. It is this fact which ensures that the tiling created for each of the representative templates matches for the most part with the tiling created for the master templates, and hence the pibs $\psi_{2n+1}^{f,l}$ agree on most of $\mathcal{Q}_{2n+1}$. Let us now give a formal account of defining the pibs for stage $2n+1$.

Let

$$q = \min\{mh_{2n-1} : mh_{2n-1} > (1-\tilde{\epsilon}_{2n+1})h_{2n+1} \text{ and } m \in \mathbb{N}\}.$$

Starting with the master templates $\tau_{2n+1}^{0,k}$, select

$$H = \min\{h \geq q : T^h(0) \in B_{\epsilon_{2n}}(0)\}.$$

Let $A_R(x)$ denote the R^{th} return time to $\text{base}(\mathcal{P}_{2n}) = \mathcal{B}_{\eta_{2n}}(0)$, and let $\hat{H}$ be such that $A_{\hat{H}}(0) = H$. Note that $A_R(x) - A_{R-1}(x) \in \{t_{2n}^1, \ldots, t_{2n}^w\}$. Define $B_R(0)$ for $R = 0, \ldots, \hat{H}$ to be the closest integer multiple of h_{2n-1} to $\frac{q}{H}A_R(0)$. If $\frac{q}{H}A_R(0)$ is equidistant from two multiples of h_{2n-1}, choose the smaller multiple. So

$$\left|\frac{q}{H}A_R(0) - B_R(0)\right| \leq \frac{1}{2}h_{2n-1} \tag{5.4.3}$$

for $R = 0, \dots, \hat{H}$.

LEMMA 5.4.5. *For $R \in \{1, \dots, \hat{H}\}$ if $A_R(0) - A_{R-1}(0) = t^J_{2n}$ and $T^{A_R(0)}0 \in base(\mathcal{P}^J_{2n})$ for some $J \in \{1, \dots, w\}$, then there exists an $\omega^{J,k}_{2n} \in \Omega_{2n}$ for some $k \in \{1, \dots, 5\}$ such that*

$$|\omega^{J,k}| = B_R(0) - B_{R-1}(0)$$

.

PROOF. Suppose $A_R(0) - A_{R-1}(0) = t^J_{2n}$ and $T^{A_R(0)}(0) \in base(\mathcal{P}^J_{2n})$ for some $J \in \{1, \dots, w\}$. Then by 5.4.3

$$\left|\frac{q}{H}t^J_{2n} - (B_R(0) - B_{R-1}(0))\right| \leq h_{2n-1}.$$

$$\Rightarrow \frac{q}{H}t^J_{2n} - h_{2n-1} < B_R(0) - B_{R-1}(0) < \frac{q}{H}t^J_{2n} + h_{2n-1}. \tag{5.4.4}$$

Basically, we need to know that one of our five $\omega^{J,k}_{2n}$ templates corresponding to the tower of height t^J_{2n} has length $B_R(0) - B_{R-1}(0)$, which occurs if

$$|\omega^{J,1}_{2n}| \leq B_R(0) - B_{R-1}(0) \leq |\omega^{J,5}_{2n}|.$$

Note that

$$|\omega^{J,1}_{2n}| \leq t^J_{2n} - (h_{2n-1} + 1) \text{ and } t^J_{2n} + 2h_{2n-1} \leq |\omega^{J,5}_{2n}|. \tag{5.4.5}$$

By 5.4.4 and 5.4.5, it suffices to show that

$$t^J_{2n} - (h_{2n-1} + 1) \leq \frac{q}{H}t^J_{2n} - h_{2n-1} \text{ and} \tag{5.4.6}$$

$$\frac{q}{H}t^J_{2n} + h_{2n-1} \leq t^J_{2n} + 2h_{2n-1}. \tag{5.4.7}$$

For 5.4.7, as $H > q$, $\frac{q}{H}t^J_{2n} < t^J_{2n}$, so there is a template which is long enough. For 5.4.6, we need

$$t^J_{2n} - 1 \leq \frac{q}{H}t^J_{2n}$$

.

$$\Rightarrow \frac{H-q}{H} \leq \frac{1}{t^J_{2n}}$$

and using $h_{2n} \geq t^J_{2n}$ we then want

$$\frac{H-q}{H} \leq \frac{1}{h_{2n}}.$$

Recall that $q \leq H \leq q + \kappa$ and $q \geq (1 - \tilde{\epsilon}_{2n+1})h_{2n+1} \Rightarrow$

$$\frac{\kappa}{(1-\tilde{\epsilon}_{2n+1})h_{2n+1}} \leq \frac{1}{h_{2n}}$$

$$\Rightarrow \frac{\kappa h^{2n}}{1-\tilde{\epsilon}_{2n+1}} \leq h_{2n+1}$$

which is met by Requirement 5.4.3(2). □

A quick and hopefully helpful aside: as one can see, if $\mathcal{Q}_{2n+1}$ is selected so that is "tall enough", an amount specified by Requirement 5.4.3, then $\frac{q}{H}$ is "close enough" to 1 so that $\frac{q}{H}(A_R(0) - A_{R-1}(0))$ is approximately equal to $A_R(0) - A_{R-1}(0)$, which is the height of one of the towers in $\mathcal{P}_{2n}$. Then, $B_R(0) - B_{R-1}(0)$ is approximately equal to $A_R(0) - A_{R-1}(0) \pm h_{2n}$. Suppose the tower $\mathcal{P}^J_{2n}$ is of height $A_R(0) - A_{R-1}(0)$ and $T^{A_{R-1}(0)}0 \in \text{base}(\mathcal{P}^J_{2n})$. Notice that the five templates $\omega^{J,k}_{2n}$ for $k = 1, \ldots 5$ from the template set Ω^J_{2n} cover at least the range of heights from $A_R(0) - A_{R-1}(0) - h_{2n} - 1$ to $A_R(0) - A_{R-1}(0) + 2h_{2n}$, so one of those templates will have length $B_R(0) - B_{R-1}(0)$, and that template will be the $R-1$ tile in the tiling of $\mathcal{Q}_{2n+1}$ corresponding to the bottom portion of the master template. It is truly this argument that we are making repeatedly, in detail, and in slightly different guises in order to create tilings for the towers in correspondence to the different templates which we shall use to define the pibs.

For $R = 1, \ldots, \hat{H}$, we define $j(R)$ and $k(R)$, two functions such that

$$j(R) = J \text{ if } T^{A_R(0)}(0) \in P_{2n}^J$$

and

$$k(R) = K \text{ if } T^{A_R(0)}(0) \in P_{2n}^J \text{ and } B_{R+1}(0) - B_R(0) = |\omega_{2n}^{J,K}|.$$

This defines a sequence of templates

$$\omega_{2n}^{j(1),k(1)}, \omega_{2n}^{j(2),k(2)}, \ldots, \omega_{2n}^{j(\hat{H}),k(\hat{H})}$$

which is a good tiling of $\mathcal{Q}_{2n+1}$ corresponding to the master templates up to height q.

We now perform similar processes for tiling the top portion of the tower corresponding to the top portions of each template. Let

$$H_l = |\tau_{2n+1}^{0,l}| - H \text{ for } l = 1, \ldots, m(0),$$

and let $\hat{H}_l$ be such that $A_{\hat{H}_l}(0) = |\tau_{2n+1}^{0,l}|$. Let $\tilde{q} = h_{2n+1} - q$. Define $B_R^l(0)$ to be the closest integer multiple of h_{2n-1} to $q + \frac{\tilde{q}}{H_l}(A_R(0) - H)$ for $R = \hat{H} + 1, \ldots, \hat{H}_l$ and $l = 1, \ldots, m(0)$. So

$$\left| \frac{\tilde{q}}{H_l} A_R(0) - B_R^l(0) \right| < \frac{1}{2} h_{2n-1} \tag{5.4.8}$$

for $R = \hat{H} + 1, \ldots, \hat{H}_l$ and $l = 1, \ldots, m(0)$.

Lemma 5.4.6. *For $R \in \{\hat{H}+1, \ldots, \hat{H}_l\}$ and $l \in \{1, \ldots, m(0)\}$, if $A_R(0) - A_{R-1}(0) = t_{2n}^J$ and $T^{A_R(0)}0 \in \mathcal{P}_{2n}^J$, then there exists an $\omega_{2n}^{J,k} \in \Omega_{2n}$ such that $|\omega^{J,k}| = B_R(^l0) - B_{R-1}^l(0)$.*

Proof. We prove the lemma for the two extreme cases, the shortest template $\tau_{2n+1}^{0,1}$ and for the longest template $\tau_{2n+1}^{0,m(0)}$. Start with the shortest template.

Suppose $A_R(0) - A_{R-1}(0) = t_{2n}^J$ and $T^{A_R(0)}(0) \in \text{base}(\mathcal{P}_{2n}^J)$ for some $J \in \{1, \ldots, w\}$. By 5.4.8

$$\left|\frac{\tilde{q}}{H_1} t_{2n}^J - (B_R^1(0) - B_{R-1}^1(0))\right| \leq h_{2n-1}$$

$$\Rightarrow \frac{\tilde{q}}{H_1} t_{2n}^J - h_{2n-1} \leq B_R^1(0) - B_{R-1}^1(0) \leq \frac{\tilde{q}}{H_1} t_{2n}^J + h_{2n-1}. \tag{5.4.9}$$

We need to know that one of our five $\omega_{2n}^{J,k}$ templates corresponding to the tower P_{2n}^J has length $B_R^1(0) - B_{R-1}^1(0)$, which occurs if

$$|\omega_{2n}^{J,1}| \leq B_R^1(0) - B_{R-1}^1(0) \leq |\omega_{2n}^{J,5}|.$$

Recall that

$$|\omega_{2n}^{J,1}| \leq t_{2n}^J - (h_{2n-1} + 1) \text{ and } t_{2n}^J + 2h_{2n-1} \leq |\omega_{2n}^{J,5}|.$$

By 5.4.5 and 5.4.9 it suffices to show that

$$t_{2n}^J - (h_{2n-1} + 1) \quad \leq \quad \frac{\tilde{q}}{H_1} t_{2n}^J - h_{2n-1} \text{ and} \tag{5.4.10}$$

$$\frac{\tilde{q}}{H_1} t_{2n}^J + h_{2n-1} \quad \leq \quad t_{2n}^J + 2h_{2n-1}. \tag{5.4.11}$$

The first inequality comes easily as

$$\tilde{q} > H_1 \Rightarrow t_{2n}^J < \frac{\tilde{q}}{H_1} t_{2n}^J.$$

For 5.4.11, we need that

$$\left(\frac{\tilde{q}}{H_1} - 1\right) t^J_{2n} \leq h_{2n-1}.$$

$$\Rightarrow \frac{\tilde{q} - H_1}{H_1} \leq \frac{h_{2n-1}}{t^J_{2n}}. \tag{5.4.12}$$

Note that

$$\kappa + 1 \leq \tilde{q} - H_1 \leq 3\kappa \text{ and} \tag{5.4.13}$$

$$\tilde{\epsilon}_{2n+1} h_{2n+1} - 3\kappa - h_{2n-1} \leq H_1 \leq \tilde{\epsilon}_{2n+1} h_{2n+1} - \kappa - 1. \tag{5.4.14}$$

Using 5.4.13 and 5.4.14 with 5.4.12, we then need

$$\frac{3\kappa}{\tilde{\epsilon}_{2n+1} h_{2n+1} - 3\kappa - h_{2n-1}} \leq \frac{h_{2n-1}}{t^J_{2n}}$$

$$\Rightarrow \left(\frac{3\kappa t^J_{2n}}{h_{2n-1}} + 3\kappa h_{2n-1}\right) \frac{1}{\tilde{\epsilon}_{2n+1}} \leq h_{2n+1}$$

which is met by Requirement 5.4.3(3).

In the case of the longest master template $\tau^{0,m(0)}_{2n+1}$, let

$$H_{m(0)} = |\tau^{0,m(0)}_{2n+1}| - H.$$

If $A_R(0) - A_{R-1}(0) = t^J_{2n}$ for some J, we need that $|\omega^{J,1}_{2n}| \leq B^{m(0)}_R(0) - B^{m(0)}_{R-1}(0) \leq |\omega^{J,5}_{2n}|$. We know that

$$\Rightarrow \frac{\tilde{q}}{H_{m(0)}} t^J_{2n} - h_{2n-1} \leq B^{m(0)}_R(0) - B^{m(0)}_{R-1}(0) \leq \frac{\tilde{q}}{H_{m(0)}} t^J_{2n} + h_{2n-1} \tag{5.4.15}$$

By 5.4.1, 5.4.2, and 5.4.15, it suffices to show that

$$t_{2n}^{J} - (h_{2n-1} + 1) \leq \frac{\tilde{q}}{H_{m(0)}} t_{2n}^{J} - h_{2n-1} \text{ and} \tag{5.4.16}$$

$$\frac{\tilde{q}}{H_{m(0)}} t_{2n}^{J} + h_{2n-1} \leq t_{2n}^{J} + 2h_{2n-1}. \tag{5.4.17}$$

As $\tilde{q} < H_{m(0)}$, 5.4.17 is true. For 5.4.16,

$$t_{2n}^{J} \left(1 - \frac{\tilde{q}}{H_{m(0)}}\right) \leq 1$$

$$\Rightarrow \frac{H_{m(0)} - \tilde{q}}{H_{m(0)}} \leq \frac{1}{t_{2n}^{J}}. \tag{5.4.18}$$

Using that

$$H_{m(0)} - \tilde{q} \leq 2\kappa \text{ and} \tag{5.4.19}$$

$$2 + \tilde{\epsilon}_{2n+1} h_{2n+1} - h_{2n-1} \leq H_{m(0)} \tag{5.4.20}$$

we require

$$\frac{2\kappa}{2 + \tilde{\epsilon}_{2n+1} h_{2n+1} - h_{2n-1}} \leq \frac{1}{t_{2n}^{J}}$$

$$\Rightarrow \frac{2\kappa t_{2n}^{J} - 2 + h_{2n-1}}{\tilde{\epsilon}_{2n+1}} \leq h_{2n+1}$$

which is met by Requirement 5.4.3(4). □

For $l = 1, \ldots, m(0)$ and $R = \hat{H}+1, \ldots, \hat{H}_l$, we define two functions $j_l(R)$ and $k_l(R)$ such that

$j_l(R) = J$ if $T^{A_R(0)}(0) \in P_{2n}^{J}$ and

$k_l(r) = K$ if $T^{A_R(0)}(0) \in P_{2n}^{J}$ and $B_{R+1}^{l}(0) - B_{R}^{l}(0) = |\omega_{2n}^{J,K}|$.

The functions define a sequence of templates

$$\omega_{2n}^{j_l(\hat{H}+1),k_l(\hat{H}+1)}, \ldots, \omega_{2n}^{j_l(\hat{G}),k_l(\hat{G})}$$

which tile the top portion of the template $\tau_{2n+1}^{0,l}$. Combining the tilings for the bottom and top portions gives a good tiling of $\mathcal{Q}_{2n+1}$ corresponding to a master template. Let

$$\begin{aligned} \Delta(R) &= j(R), k(R) \text{ for } R = 1, \ldots, \hat{H} \text{ and} \\ \Delta_l(R) &= j_l(R), k_l(R) \text{ for } R = \hat{H}+1, \ldots, \hat{H}_l \text{ and } l = 1, \ldots, m(0) \end{aligned}.$$

Define

$$\psi_{2n+1}^{0,l} = (\phi_{2n}^{\Delta(1)})^{-1}|\cdots|(\phi_{2n}^{\Delta(\hat{H})})^{-1}|(\phi_{2n}^{\Delta_l(\hat{H}+1)})^{-1}|\cdots|(\phi_{2n}^{\Delta_l(\hat{H}_l)})^{-1}.$$

Next, we describe the process for defining pibs from $\mathcal{Q}_{2n+1}$ to the representative templates. Let $x \in \mathcal{R}$ be a representative point, $x \neq 0$. Let

$$\begin{aligned} q &= \min\{mh_{2n-1} : mh_{2n-1} > (1 - \tilde{\epsilon}_{2n+1})h_{2n+1}\} \\ H &= \min\{h \geq q : T^h 0 \in \mathcal{B}_{\epsilon_{2n}}(0)\} \\ F(x) &= \min\{h \geq H : T^h x \in \mathcal{B}_{\epsilon_{2n}}(0)\} \end{aligned}$$

and let $\hat{F}(x)$ be such that $A_{\hat{F}(x)} = F(x)$ where $A_R(x)$ is the sequence of return times of $x \in \mathcal{B}_{\epsilon_{2n}}(0)$ to the base of $\mathcal{P}_{2n}$. For $R = 1, \ldots, \hat{F}(x)$, define $B_R(x)$ to be the closest integer multiple of h_{2n-1} to $\frac{q}{H}A_R(x)$.

Lemma 5.4.7. *For $R\{1, \ldots, \hat{F}(x)\}$, if $A_R(x) - A_{R-1}(x) = t_{2n}^J$ for some $J \in \{1, \ldots, w\}$, then there exists $\omega_{2n}^{J,k} \in \Omega_{2n}$ such that $|\omega_{2n}^{J,k}| = B_R(x) - B_{R-1}(x)$.*

PROOF. The proof is identical to the proof for the lemma regarding the bottoms of the master templates □

For $R = 1, \ldots, \hat{F}(x)$, we define $j_x(R)$ and $k_x(R)$, two functions such that

$$j_x(R) = J \text{ if } T^{A_R(x)}(x) \in P_{2n}^J$$

and

$$k_x(R) = K \text{ if } T^{A_R(x)}(x) \in P_{2n}^J \text{ and } B_{R+1}(x) - B_R(x) = |\omega_{2n}^{J,K}|.$$

This defines a sequence of templates

$$\omega^{j_x(1),k_x(1)}, \omega^{j_x(2),k_x(2)}, \ldots, \omega^{j_x(\hat{H}),k_x(\hat{H})}$$

which is a good tiling of $\mathcal{Q}_{2n+1}$ corresponding to the representative templates up to height q.

To create a tiling of the top portion of $\mathcal{Q}_{2n+1}$ corresponding to the top portions of the representative templates, we only concern ourselves with showing that a good tiling exists for the extreme cases, the top of the shortest representative templates $\tau_{2n+1}^{x,1}$ and tallest representative templates $\tau_{2n+1}^{x,m(x)}$. Let

$$\begin{aligned} \tilde{q} &= h_{2n+1} - q, \\ C_l &= |\tau_{2n+1}^{x,l}| - F(x), \end{aligned}$$

and let $\hat{C}_l$ be such that $A_{\hat{C}_l} = |\tau_{2n+1}^{x,l}|$ where $A_R(x)$ is the sequence of return times of $x \in \mathcal{B}_{\epsilon_{2n}}(0)$ to the base of $\mathcal{P}_{2n}$ and $l = 1, \ldots, m(x)$. Then, for $R = \hat{F}(x) + 1, \ldots, \hat{C}_1$ (where

$\hat{F}(x)$ is as described for the previous lemma), define $B_R(x)$ to be the closest integer multiple of h_{2n-1} to $F(x) + \frac{\tilde{q}}{C_l}(A_R(x) - F(x))$.

LEMMA 5.4.8. *For $R \in \{\hat{H}+1, \ldots \hat{F}(x)\}$ and $l \in \{1, \ldots, m(x)\}$, if $A_R(x) - A_{R-1}(x) = t^J_{2n}$ for some $J \in \{1, \ldots, w\}$, then there exists $\omega^{J,k}_{2n} \in \Omega_{2n}$ such that $|\omega^{J,k}_{2n}| = B_R(x) - B_{R-1}(x)$.*

PROOF. Considering first the top of the shortest representative template, $\tau^{x,1}_{2n+1}$, for a fixed representative point $x \in \mathcal{R}$, let $A_R(x) - A_{R-1}(x) = t^J_{2n}$ for some $J \in \{1, \ldots, w\}$. Note that

$$\left|\frac{\tilde{q}}{C_1}t^J_{2n} - (B_R(x) - B_{R-1}(x))\right| \leq h_{2n-1}$$

$$\Rightarrow \frac{\tilde{q}}{C_1}t^J_{2n} - h_{2n-1} \leq B_R(x) - B_{R-1}(x) \leq \frac{\tilde{q}}{C_1}t^J_{2n} + h_{2n-1}. \tag{5.4.21}$$

We want

$$|\omega^{J,1}_{2n}| \leq B_R(x) - B_{R-1}(x) \leq |\omega^{J,5}_{2n}|.$$

By 5.4.1, 5.4.2, and 5.4.21, it suffices to show that

$$t^J_{2n} - (h_{2n-1} + 1) \leq \frac{\tilde{q}}{C_1}t^J_{2n} - h_{2n-1} \text{ and} \tag{5.4.22}$$

$$\frac{\tilde{q}}{C_1}t^J_{2n} + h_{2n-1} \leq t^J_{2n} + 2h_{2n-1}. \tag{5.4.23}$$

5.4.22 holds as $\tilde{q} > C_1$. For 5.4.23,

$$t^J_{2n}\left(\frac{\tilde{q}}{C_1} - 1\right) \leq h_{2n-1} \Rightarrow \frac{\tilde{q} - C_1}{C_1} \leq \frac{h_{2n-1}}{t^J_{2n}}. \tag{5.4.24}$$

Using that

$$\tilde{\epsilon}_{2n+1}h_{2n+1} - 4\kappa - h_{2n-1} \leq C_1 \leq h_{2n+1}\tilde{\epsilon}_{2n+1} - \kappa - 1 \tag{5.4.25}$$

and

$$\kappa + 1 \leq \tilde{q} - C_1 \leq 4\kappa, \tag{5.4.26}$$

we need

$$\frac{4\kappa}{\tilde{\epsilon}_{2n+1}h_{2n+1} - 4\kappa - h_{2n-1}} \leq \frac{h_{2n-1}}{h_{2n}}$$
$$\Rightarrow \frac{1}{\tilde{\epsilon}_{2n+1}}\left(\frac{4\kappa h_{2n}}{h_{2n-1}} + 4\kappa + h_{2n-1}\right) \leq h_{2n+1}$$

which is met by Requirement 5.4.3(5).

For the longest template, $\tau_{2n+1}^{x,m(x)}$, similar to 5.4.22 and 5.4.23 it suffices to show that,

$$t_{2n}^J - (h_{2n-1} + 1) \leq \frac{\tilde{q}}{C_{m(x)}} t_{2n}^J - h_{2n-1} \text{ and} \tag{5.4.27}$$

$$\frac{\tilde{q}}{C_{m(x)}} t_{2n}^J + h_{2n-1} \leq t_{2n}^J + 2h_{2n-1}. \tag{5.4.28}$$

We consider three cases:

Case I: $\tilde{q} = C_{m(x)}$

Case II: $C_{m(x)} > \tilde{q}$, wherein we must show that $t_{2n}^J - 1 \leq \frac{\tilde{q}}{C_{m(x)}} t_{2n}^J$

Case III: $\tilde{q} > C_{m(x)}$, wherein we must show that $\frac{\tilde{q}}{C_{m(x)}} t_{2n}^J \leq t_{2n}^J + h_{2n-1}$.

For Case I, 5.4.27 and 5.4.28 are obviously true. For Case II, we need

$$\frac{C_{m(x)} - \tilde{q}}{C_{m(x)}} \leq \frac{1}{t_{2n}^J}. \tag{5.4.29}$$

Now,

$$C_{m(x)} - \tilde{q} \leq 2\kappa \tag{5.4.30}$$

and

$$C_{m(x)} > \tilde{q} = h_{2n+1} - q \geq h_{2n+1} - ((1-\tilde{\epsilon}_{2n+1})h_{2n+1} + h_{2n-1}) = \tilde{\epsilon}_{2n+1}h_{2n+1} - h_{2n-1}. \quad (5.4.31)$$

Using 5.4.30 and 5.4.31 in conjunction with 5.4.29, we require

$$\frac{2\kappa}{\tilde{\epsilon}_{2n+1}h_{2n+1} - h_{2n-1}} \leq \frac{1}{t^J_{2n}}$$

$$\Rightarrow \frac{2\kappa t^J_{2n} + h_{2n-1}}{\tilde{\epsilon}_{2n+1}} \leq h_{2n+1},$$

which is met by Requirement 5.4.3(4).

For Case III, we need

$$\frac{\tilde{q} - C_{m(x)}}{C_{m(x)}} \leq \frac{h_{n-1}}{t^J_{2n}}. \quad (5.4.32)$$

Now,

$$\tilde{q} - C_{m(x)} \leq \kappa - 1 \text{ and} \quad (5.4.33)$$

$$C_{m(x)} \geq \tilde{q} - \kappa + 1 = h_{2n+1} - q + \kappa + 1 \geq \tilde{\epsilon}_{2n+1}h_{2n+1} - h_{2n-1} - \kappa + 1. \quad (5.4.34)$$

By 5.4.33 and 5.4.34 in conjunction with 5.4.32 we require

$$\frac{\kappa - 1}{\tilde{\epsilon}_{2n+1}h_{2n+1} - h_{2n-1} - \kappa + 1} \leq \frac{h_{2n-1}}{t^J_{2n}}$$

$$\Rightarrow \frac{1}{\tilde{\epsilon}_{2n+1}}\left(\frac{(\kappa-1)t^J_{2n}}{h_{2n-1}} + h_{2n-1} - \kappa - 1\right) \leq h_{2n+1}$$

which is met by Requirement 5.4.3(6).

□

We then define the functions $j^l_x(R)$ and $k^l_x(R)$ for $R = \hat{F}(x) + 1 \ldots, \hat{H}_l$ such that

$$\begin{aligned} j^l_x(R) &= J \text{ if } T^{A_R(x)}(x) \in \text{ base} P^J_{2n} \text{ and} \\ k^l_x(R) &= K \text{ if } T^{A_R(x)}(x) \in P^J_{2n} \text{ and } B_{R+1}(x) - B_R(x) = |\omega^{J,K}|. \end{aligned}$$

This gives a sequence of tiles

$\omega^{j^l_x(R), k^l_x(R)}$ for $R = \hat{F}(x) + 1, \ldots, \hat{H}_l$ which is a good tiling of the top portion of $\mathcal{Q}_{2n+1}$ corresponding to the tops of the representative templates. Let

$$\begin{aligned} \Delta_x(R) &= j_x(R), k_x(R) \text{ for } R = 1, \ldots, \hat{F}(x) \text{ and} \\ \Delta^l_x(R) &= j^l_x(R), k^l_x(R) \text{ for } R = \hat{F}(x) + 1, \ldots, \hat{H}_l \text{ and } l = 1, \ldots, m(x) \end{aligned}.$$

Define the partial interval bijection

$$\hat{\psi}^{x,l}_{2n+1} = (\phi^{\Delta_x(1)}_{2n})^{-1} | (\phi^{\Delta_x(2)}_{2n})^{-1} | \cdots | (\phi^{\Delta_x(\hat{F}(x))}_{2n})^{-1} | (\phi^{\Delta^l_x(\hat{F}(x)+1)}_{2n})^{-1} | \cdots | (\phi^{\Delta^l_x(\hat{H}_l)}_{2n})^{-1}.$$

A bad block of points in $\tau^{x,l}_{2n+1}$ for $x \in \mathcal{R}$ is a list of points $T^{A_R(0)}x, \ldots, T^{A_{R+1}(0)-1}x$ such that $T^{A_R(0)}0 \in S^{\epsilon_{2n}}_{2n}$ for some $R \in 1, \ldots \hat{H}_l$. Define $BAD(\tau^{x,l}_{2n+1})$ to be the collection of bad blocks of points in the template $\tau^{x,l}_{2n+1}$. Define $BAD(\mathcal{Q}_{2n+1})$ to be the collection of levels in $\mathcal{Q}_{2n+1}$ which are indexed by $B_R(0), B_R(0) + 1, \ldots, B_{R+1}(0) - 1$ for which $T^{A_R(0)}(0) \in S^{\epsilon_{2n}}_{2n}$ for some R.

Lemma 5.4.9. *If $BADQ_{2n+1}$ is as described above, then*

$$\mu(BAD(\mathcal{Q}_{2n+1})) < \tilde{\epsilon}.$$

Proof. Note that whenever $T^{A_R(0)}(0) \notin S_{2n}^{\epsilon_{2n}}$, the points $T^{A_R(0)}(0)$ and $T^{A_R(0)}(x)$ for $x \in \mathcal{B}_{\epsilon_{2n}}(0)$ lie in the base of the same tower of $\mathcal{P}_{2n}$ and that $T^{A_{R+1}(0)}(0)$ and $T^{A_{R+1}(0)}(x)$ are both in the base of $\mathcal{P}_{2n}$. This implies $A_{R+1}(0) - A_R(0) = A_{\tilde{R}}(x) - A_{\tilde{R}}(x)$ where $\tilde{R}$ is such that $A_R(0) = A_{\tilde{R}}(x)$. Because we scaled $A_{R+1}(0) - A_R(0)$ and $A_{\tilde{R}}(x) - A_{\tilde{R}}(x)$ by the same amount $(\frac{q}{H})$, $B_{R+1}(0) - B_R(0) = B_{\tilde{R}+1}(x) - B_{\tilde{R}}(x)$. Thus, we assigned the same $\omega_{2n}^{j,k}$ template to tile this portion of $\mathcal{Q}_{2n+1}$ for the master templates and for all representative templates. This in turn means that $\hat{\psi}_{2n+1}^{x,l}$ agree for all $x \in \mathcal{R}$ and $l = 1, \ldots, m(x)$. The only time that tilings of $\mathcal{Q}_{2n+1}$ may not agree for different templates occurs when $T^{A_R(0)} \in S_{2n}^{\epsilon_{2n}}$. But, by requirement 6.1.1 and Lemma 5.4.2,

$$\frac{1}{|\tau_{2n+1}^{x,1}|} \sum_{i=0}^{|\tau_{2n+1}^{x,1}|-1} \chi_{S_{2n}^{\epsilon_{2n}}}(T^i x) < \tilde{\epsilon}_{2n}$$

for all $x \in \mathcal{R}$. □

We now extend the pibs: starting at the base of $\mathcal{Q}_{2n+1}$, list the levels up to height $h_{2n+1}(1 - \tilde{\epsilon}_{2n+1})$ of $\mathcal{Q}_{2n+1}$ which are not in $BAD(\mathcal{Q}_{2n+1})$ and for which $\hat{\psi}_{2n+1}^{0,1}$ has not been defined on the indices. Then, starting at the bottom of $\tau_{2n+1}^{0,1}$, list the indices of points of the template $\tau_{2n+1}^{0,1}$ up to height H which are not in $BAD(\tau_{2n+1}^{0,1})$, and to which $\hat{\psi}_{2n+1}^{0,1}$ has not been defined. Note that there are at least as many indices from points in $\tau_{2n+1}^{0,1}$ as there are indices of levels in $\mathcal{Q}_{2n+1}$ by construction. Map as many of these indices of levels to as many these indices of points as possible. Order is not important, but it is important that we extend in the same way for the rest of the $\tau_{2n+1}^{x,l}$, $x \in \mathcal{R}$. This defines $\psi_{2n+1}^{x,l}$. On what we call the good set of $\mathcal{Q}_{2n+1}$, where all of the $\psi_{2n+1}^{x,l}$ agree, define

$$\psi_{2n+1} = \xi \circ \psi_{2n+1}^{0,1}$$

where ξ projects the point indexed by $\psi^{0,1}_{2n+1}$ to a level of $\mathcal{P}_{2n}$. So, for a good level of $\mathcal{Q}_{2n+1}$, the map $\psi_{2n+1} = \xi \circ \psi^{0,1}_{2n+1}$ associates a level in $\mathcal{P}_{2n}$.

5.4.10. Stage 2n+2. The primary differences between stage $2n+2$ and stage $2n+1$ come from the fact that the multi-tower $\mathcal{P}_{2n+2}$ consists of multiple towers, and to each tower, there are only 5 associated templates. As before, all of the templates are initial segments of the orbit of zero.

Requirement 5.4.11. *Select η_{2n+2} such that $0 < \eta_{2n+2} < \min\{\epsilon_{2n}, \tilde{\epsilon}_{2n+2}, \eta_{2n}\}$ and $\mu(\partial\mathcal{B}_{\eta_{2n+2}}(0)) = 0$ and also such that the skyscraper over $\mathcal{B}_{\eta_{2n+2}}(0)$ has minimum height t^1_{2n+2} satisfying:*

(1) $t^1_{2n+2} \geq 3h_{2n+1}$

(2) $t^1_{2n+2} \geq \frac{(h_{2n+1})^2}{1-\tilde{\epsilon}_{2n+1}}$

(3) $t^1_{2n+2} \geq \frac{3h^2_{2n+1}+h_{2n+1}+\kappa}{\tilde{\epsilon}_{2n+2}}$

(4) $t^1_{2n+2} \geq \frac{2h^2_{2n+1}+\kappa}{\tilde{\epsilon}_{2n+2}}$.

where κ is still the largest first return time to $\mathcal{B}_{\epsilon_{2n}}(0)$ for any point in $\mathcal{B}_{\epsilon_{2n}}(0)$, and h_{2n+1} is the height of tower $\mathcal{Q}_{2n+1}$.

Purify the skyscraper over $\mathcal{B}_{\eta_{2n+2}}(0)$ into maximal α_{2n}-pure towers, yielding the multi-tower$\mathcal{P}_{2n+2} = \{\mathcal{P}^*_{2n+2}\}$ where the towers are indexed in non-decreasing order of height. Let $\star$ be the list of indices and $|\star|$ be the number of towers. Define the template set Ω_{2n+2} in the same manner as we defined the Ω_{2n}-template set. For tower $\mathcal{P}^*_{2n+2}$, define the template set, $\hat{\Omega}^*_{2n+2}$, as the set of five templates whose lengths satisfy:

$$|\omega^{*,1}_{2n+2}| < |\omega^{*,2}_{2n+2}| < t^*_{2n+2} \leq |\omega^{*,3}_{2n+2}| < |\omega^{*,4}_{2n+2}| < |\omega^{*,5}_{2n+2}| \text{ and}$$
$$|\omega^{*,k}_{2n+2}| = |\omega^{*,k-1}_{2n+2}| + h_{2n+1} \text{ for } k = 2,3,4,5.$$

Note that

$$|\omega^{*,1}_{2n+2}| \quad < \quad t^*_{2n+2} - (h_{2n+1} + 1) \tag{5.4.35}$$

$$|\omega^{*,5}_{2n+2}| \quad \geq \quad t^*_{2n+2} + 2(h_{2n+1}) \tag{5.4.36}$$

Define

$$\Omega^*_{2n+2} = \bigcup_{k=1}^{5} \omega^{*,k}_{2n+2}$$

the Q_{2n+1}-template set for $\mathcal{P}^*_{2n+2}$ and

$$\Omega_{2n+2} = \bigcup_{*\in\star} \Omega^*_{2n+2}$$

the Q_{2n+1} template set for $\mathcal{P}_{2n+2}$. Define $\xi : \Omega_{2n+2} \to \mathcal{Q}_{2n+1}$ by $\xi(S^k 0) = S^k(\text{base}(\mathcal{Q}_{2n+1}))$ or $\xi(y) = d_{2n+1}(y)$ where $d_{2n+1}(y)$ is the level of $\mathcal{Q}_{2n+1}$ which contains y.

Before we formally describe how to create the stage $2n + 2$ pibs, we give an informal description. This is similar to the stage $2n + 1$ construction with the roles of $\mathcal{Q}_{2n+1}$ and $\mathcal{P}_{2n+2}$ swapped. The multi-tower $\mathcal{P}_{2n+2}$ comes from a cutting and stacking of the multi-tower $\mathcal{P}_{2n}$. So, each tower $\mathcal{P}^*_{2n+2}$ is naturally tiled by the towers of $\mathcal{P}_{2n}$, and each tower $\mathcal{P}^*_{2n+2}$ has an α_{2n}-name in the sense that the initial segments of forward orbits of all points which start in the base of $\mathcal{P}^*_{2n+2}$ and end in the top level of $\mathcal{P}^*_{2n+2}$ have the exact same α_{2n}-name.

Definition 5.4.12. *A tiling of $\mathcal{P}^*_{2n+2}$ by $\mathcal{T}_{2n+1}$ is an ordered list of templates $\tau^{x(R),k(R)}_{2n+1}$ such that $t^*_{2n+2} = \sum_{R=1}^{\hat{G}} |\tau^{x(R),k(R)}|$. We say that we tile $\mathcal{P}^*_{2n+2}$ with $\mathcal{T}_{2n+1}$.*

A tiling of $\mathcal{P}^*_{2n+2}$ by $\mathcal{T}_{2n+1}$ *corresponding to* $\omega^{*,l}_{2n+2}$ is said to be *good* if for some number M of the templates in $\mathcal{T}_{2n+1}$, $t^*_{2n+2} = \sum_{R=1}^{M} |\tau^{x(R),l(R)}_{2n+1}|$ and the α_{2n}-name given by the tiling matches the α_{2n}-name of $\mathcal{P}^*_{2n+2}$. If $\tau^{x(R),l(R)}_{2n+1}$ with $R = 1, \ldots, M$ is the sequence of $\mathcal{T}_{2n+1}$-templates for the tiling, then $\hat{\phi}^{*,l}_{2n+2}$ is the concatenation of inverses of $\psi^{x(R),l(R)}_{2n+1}$ for $R = 1, \ldots, M$, which we then extend to all but a small fraction of the tower. At this stage, we are also careful to select a tiling of each tower $\mathcal{P}^*_{2n+2}$ for each of the $\omega^{*,k}_{2n+2}$ templates, $* \in \star$, $k \in \{1, \ldots, 5\}$, in such a way that the resultant partial interval bijections for differing k agree for a fraction $(1 - \tilde{\epsilon}_{2n+2})$ of the levels in $\mathcal{P}^*_{2n+2}$. We do so by first creating a tiling which is identical for the bottom portion of the tower corresponding to each of the templates, and then creating separate tilings which correspond to the top of each template. We then concatenate the bottom tiling with each of the top portions, separately, to define the pibs.

We now begin the formal process for defining the stage $2n + 2$ pibs. Fix a tower $\mathcal{P}^*_{2n+2}$. Note that base$(\mathcal{P}_{2n+2})$ contains a representative point $\tilde{x} \in \mathcal{R}$. Let

$$p = \min\{h \geq (1 - \tilde{\epsilon}_{2n+2})t^*_{2n+2} : T^h(\tilde{x}) \in \mathcal{B}_{\epsilon_{2n}}(0)\} \text{ and let} \tag{5.4.37}$$

$$G = \min\{h \geq p : h = mh_{2n+1}\} = \hat{R}h_{2n+1} \tag{5.4.38}$$

for some positive integer $\hat{R}$.

Define $B_R(\tilde{x})$ to be the closest integer to $\frac{p}{G}Rh_{2n+1}$ such that $T^{B_R(\tilde{x})}(\tilde{x}) \in \mathcal{B}_{\epsilon_{2n}}(0)$ for $R = 0, 1, \ldots, \hat{R}$. Then

$$\left|\frac{p}{G}Rh_{2n+1} - B_R(\tilde{x})\right| \leq \frac{1}{2}\kappa. \tag{5.4.39}$$

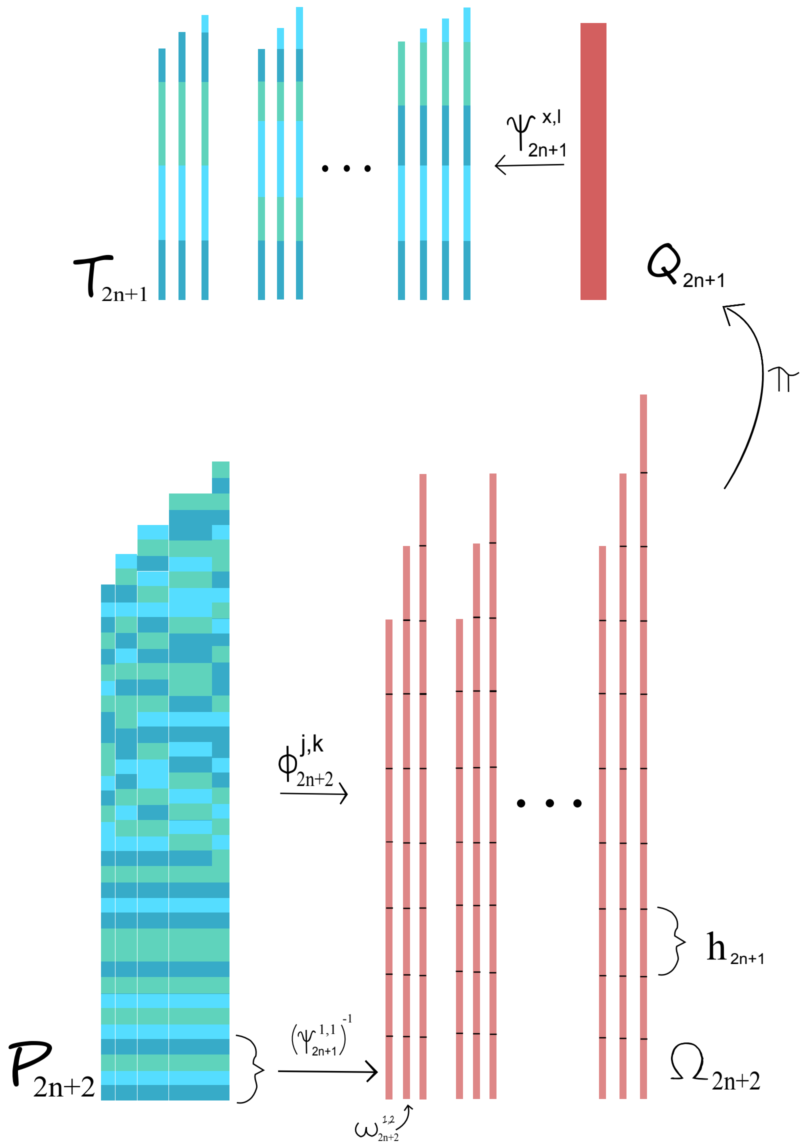

FIGURE 5.4. Stage $2n+2$ of the tower and template machinery.

LEMMA 5.4.13. *For each $R \in \{1, \ldots, \hat{R}\}$, there exists a template $\tau^{x,k}_{2n+1}$ such that $|\tau^{x,k}_{2n+1}| = B_R(\tilde{x}) - B_{R-1}(\tilde{x})$ and the α_{2n}-name of $T^{B_{R-1}(\tilde{x})}\tilde{x}, \ldots, T^{B_R(\tilde{x})-1}\tilde{x}$ equals the α_n-name of $\tau^{x,k}_{2n+1}$.*

PROOF. First of all, by 5.4.39

$$\left| \frac{p}{G}(Rh_{2n+1} - (R-1)h_{2n+1}) - (B_R(\tilde{x}) - B_{R-1}(\tilde{x})) \right| \leq \kappa$$

$$\Rightarrow \left| \frac{p}{G} h_{2n+1} - (B_R(\tilde{x}) - B_{R-1}(\tilde{x})) | \leq \kappa \right|. \tag{5.4.40}$$

$$\Rightarrow \frac{p}{G} h_{2n+1} - \kappa \leq B_R(\tilde{x}) - B_{R-1}(\tilde{x}) \leq \frac{p}{G} h_{2n+1} + \kappa \tag{5.4.41}$$

Note that

$$\kappa < h_{2n+1} - |\tau_{2n+1}^{x,1}| \text{ and } \kappa < |\tau_{2n+1}^{x,m(x)}| - h_{2n+1} \tag{5.4.42}$$

for all $x \in \mathcal{R}$

$$\Rightarrow |\tau_{2n+1}^{x,1}| < h_{2n+1} - \kappa \text{ and } h_{2n+1} + \kappa < |\tau_{2n+1}^{x,m(x)}|. \tag{5.4.43}$$

In order to show that

$$|\tau_{2n+1}^{x,1}| \leq B_R(\tilde{x}) - B_{R-1}(\tilde{x}) \leq |\tau_{2n+1}^{x,m(x)}| \tag{5.4.44}$$

for all $x \in \mathcal{R}$, it suffices by 5.4.41 and 5.4.43 to show that

$$h_{2n+1} - (\kappa + 1) \leq \frac{p}{G} h_{2n+1} - \kappa \text{ and} \tag{5.4.45}$$

$$\frac{p}{G} h_{2n+1} + \kappa \leq h_{2n+1} + (\kappa + 1). \tag{5.4.46}$$

Note that 5.4.46 comes freely as $\frac{p}{G} < 1$. For the first inequality, we need

$$\left(1 - \frac{p}{G}\right) h_{2n+1} \leq 1$$

$$\Rightarrow \frac{G-p}{G} \leq \frac{1}{h_{2n+1}}. \tag{5.4.47}$$

We have that

$$0 \leq G - p < h_{2n+1} \tag{5.4.48}$$

and

$$(1 - \tilde{\epsilon}_{2n+2}) t_{2n+2}^* \leq p \leq G < p + h_{2n+1} \leq (1 - \tilde{\epsilon}_{2n+2}) t_{2n+2}^* + h_{2n+1}. \tag{5.4.49}$$

Using 5.4.48 and 5.4.49 with 5.4.47, we need

$$\frac{h_{2n+1}}{(t-\tilde{\epsilon}_{2n+2})t^*_{2n+2}} \le \frac{1}{h_{2n+1}}$$

$$\Rightarrow \frac{(h_{2n+1})^2}{1-\tilde{\epsilon}_{2n+2}} \le t^*_{2n+2},$$

which is met by Requirement 5.4.11(2). Recall that for each α_{2n}-name of length $h_{2n+1}+2\kappa$ for points in $\mathcal{B}_{\epsilon_{2n}}(0)$, we chose a representative point and a set of templates for that point whose lengths range from at least $\kappa+1$ shorter to $\kappa+1$ longer than h_{2n+1}. Thus, the template set $\mathcal{T}_{2n+1}$ consists of templates whose α_{2n}-names contain the names of all orbits of length $\kappa+1$ shorter to $\kappa+1$ longer than h_{2n+1} which start and end in $\mathcal{B}_{\epsilon_{2n}}(0)$, and there will be a template in the set whose α_n-name matches that of $T^{B_{R-1}(\tilde{x})}\tilde{x},\ldots,T^{B_R(\tilde{x})-1}\tilde{x}$ for each $R=1,\ldots,\hat{R}$. □

Again, let us give a brief aside to help shed light on the above argument and the arguments which follow. If $\mathcal{P}_{2n+2}$ is selected so that all of the towers $\mathcal{P}^*_{2n+2}$ for $* \in \star$ are "tall enough", an amount specified by Requirement 5.4.11, then $\frac{p}{G}$ is "close enough" to 1 so that $\frac{p}{G}(Rh_{2n+1}-(R-1)h_{2n+1})$ is approximately h_{2n+1}, the height of the tower $\mathcal{Q}_{2n+1}$. Let $*$ be fixed. Then, $B_R(\tilde{x})-B_{R-1}(\tilde{x})$ is approximately equal to $h_{2n+1}\pm\kappa$. Notice that the templates in $\mathcal{T}_{2n+1}$ cover at least a range of heights from $h_{2n+1}-\kappa-1$ to $h_{2n+1}+2\kappa$, so one of these templates will have the correct length as well as the correct α_{2n} name, and that template will be the $R-1$ tile in the tiling of $\mathcal{P}^*_{2n+2}$ corresponding to the bottom portion of the $\omega^{*,k}_{2n+2}$ templates. It is roughly this argument that we shall make again for creating tilings of the top portions of the towers $\mathcal{P}^*_{2n+2}$ corresponding to the top portions of the templates $\omega^{*,k}_{2n+2}$, and we then use the tilings to guide the definition of the pibs.

For $R = 1, \ldots, \hat{R}$, we define two functions $x(R) \in \mathcal{R}$ and $k(R) \in \{1, \ldots, m(x(R))\}$ such that

$$|\tau_{2n+1}^{x(R),k(R)}| = B_R(\tilde{x}) - B_{R-1}(\tilde{x})$$

and the α_{2n}-name of the point $x(R)$ of length $|\tau_{2n+1}^{x(R),k(R)}|$ matches the α_{2n}-name of

$$T^{B_{R-1}(\tilde{x})}\tilde{x}, \ldots, T^{B_R(\tilde{x})-1}\tilde{x}.$$

The sequence of templates, $\tau_{2n+1}^{x(1),k(1)}$, $\tau_{2n+1}^{x(2),k(2)}$,...,$\tau_{2n+1}^{x(\hat{R}),k(\hat{R})}$ distinguishes the sequence of pibs whose inverses are concatenated to create the bottom portion of the stage $n+2$ pib. For the indices of levels $1, \ldots, p$ of $\mathcal{P}_{2n+2}^*$, define

$${}^p\hat{\phi}_{2n+2}^* = (\psi_{2n+1}^{x(1),k(1)})^{-1}| \cdots |(\psi_{2n+1}^{x(\hat{R}),k(\hat{R})})^{-1}.$$

We now extend ${}^p\hat{\phi}_{2n+2}^*$ by listing any indices of $\mathcal{P}_{2n+2}^*$ up to level p not in the domain of ${}^p\hat{\phi}_{2n+2}^*$. Then, list indices of points in $\tau_{2n+2}^{*,1}$ up to height G not in the range of ${}^p\hat{\phi}_{2n+2}^*$, and extend the map to as many of the indices of levels to the indices of points as possible. Order does not matter. This defines ${}^p\phi_{2n+2}^*$.

As we continue to define the pibs from indices of the tower $\mathcal{P}_{2n+2}^*$ to indices of the five templates $\omega_{2n+2}^{*,k}$, we perform a similar procedure for the top portions of $\mathcal{P}_{2n+2}^*$, creating a tiling for the top of $\mathcal{P}_{2n+2}^*$ in conjunction with the top portion of each of the $\omega_{2n+2}^{*,k}$ templates. We only worry about showing the existence of a tiling which we use to define the pibs for the shortest and longest templates $\omega_{2n+2}^{*,1}$ and $\omega_{2n+2}^{*,5}$. For the shortest template, first, let

$$\tilde{p} = t_{2n+2}^{*} - p, \tag{5.4.50}$$

$$G_1 = |\omega_{2n+2}^{*,1}| - G, \text{ and} \tag{5.4.51}$$

$$\hat{G}_1 = \frac{|\omega_{2n+2}^{*,1}|}{h_{2n+1}}. \tag{5.4.52}$$

For $R = \hat{R}+1, \ldots, \hat{G}_1$, where $\hat{R}$ is as in the previous lemma, let $B_R^1(\tilde{x})$ be the closest integer to $\frac{\tilde{p}}{G_1} R h_{2n+1}$ such that $T^{B_R^1(\tilde{x})}\tilde{x} \in \mathcal{B}_{\epsilon_{2n}}(0)$, where $\tilde{x}$ is also as in the previous lemma.

Lemma 5.4.14. *For each $R \in \{\hat{R}+1, \ldots, \hat{G}_1\}$, there exists a template $\tau_{2n+1}^{x,l}$ such that $|\tau_{2n+1}^{x,l}| = B_R^1(\tilde{x}) - B_{R-1}^1(\tilde{x})$ and the α_{2n}-name of $T^{B_{R-1}^1(\tilde{x})}\tilde{x}, \ldots, T^{B_R^1(\tilde{x})-1}\tilde{x}$ equals the α_n-name of $\tau_{2n+1}^{x,l}$.*

Proof. First, note that

$$\begin{aligned} \left|B_R^1(\tilde{x}) - B_{R-1}^1(\tilde{x})\right| &\leq \frac{1}{2}\kappa \\ \Rightarrow \left|(B_R^1(\tilde{x}) - B_{R-1}^1(\tilde{x})) - \frac{\tilde{p}}{G_1}h_{2n+1}\right| &\leq \kappa \\ \Rightarrow \frac{\tilde{p}}{G_1}h_{2n+1} - \kappa \leq B_R^1(\tilde{x}) - B_{R-1}^1(\tilde{x}) &\leq \frac{\tilde{p}}{G_1}h_{2n+1} + \kappa. \end{aligned} \tag{5.4.53}$$

For the lemma, it suffices to show that

$$|\tau_{2n+1}^{x,1}| \leq B_R^1(\tilde{x}) - B_{R-1}^1(\tilde{x}) \leq |\tau_{2n+1}^{x,m(x)}|$$

for all $x \in \mathcal{R}$. Recall that

$$|\tau_{2n+1}^{x,1}| < h_{2n+1} - \kappa \text{ and } h_{2n+1} + \kappa + 1 \leq |\tau_{2n+1}^{x,m(x)}|. \tag{5.4.54}$$

One may then show

$$h_{2n+1} - (\kappa + 1) \;\; \le \;\; \frac{\tilde{p}}{G_1} h_{2n+1} - \kappa \text{ and} \tag{5.4.55}$$

$$\frac{\tilde{p}}{G_1} h_{2n+1} + \kappa \;\; \le \;\; h_{2n+1} + \kappa + 1. \tag{5.4.56}$$

Here, $\tilde{p} \ge G_1$ and we automatically have 5.4.55. For 5.4.56, one may show

$$h_{2n+1} \left(\frac{\tilde{p}}{G_1} - 1 \right) \le 1$$

$$\Rightarrow \frac{\tilde{p} - G_1}{G_1} \le \frac{1}{h_{2n+1}}. \tag{5.4.57}$$

Using

$$h_{2n+1} + 1 \le \tilde{p} - G_1 \le 3h_{2n+1} - 1 \text{ and} \tag{5.4.58}$$

$$\tilde{\epsilon}_{2n+2} t^*_{2n+2} - \kappa \le \tilde{p} \le \tilde{\epsilon}_{2n+2} t^*_{2n+2} \tag{5.4.59}$$

in conjunction with 5.4.57, we see that we need

$$\frac{3h_{2n+1} - 1}{\tilde{\epsilon}_{2n+2} t^*_{2n+2} - (\kappa + 3h_{2n+1} - 1)} \le \frac{1}{h_{2n+1}}.$$

Finally, we require

$$\frac{(3h_{2n+1} - 1)h_{2n+1} + \kappa + 3h_{2n+1} - 1}{\tilde{\epsilon}_{2n+2}} < t^*_{2n+2},$$

which is given by Requirement 5.4.11(3). A template of the appropriate length does exist, and, as we have included templates for all α_{2n}-names within a range of lengths $\kappa + 1$ shorter than $\mathcal{Q}_{2n+1}$ and $\kappa + 1$ longer than $\mathcal{Q}_{2n+1}$ for orbits of points starting and ending in $\mathcal{B}_{\epsilon_{2n}}(0)$, there exists a template $\tau_{2n+1}^{x,k}$ such that $|\tau_{2n+1}^{x,k}| = B_R^1(\tilde{x}) - B_{R-1}^1(\tilde{x})$ whose α_n-name matches that of $T^{B_{R-1}^1(\tilde{x})}\tilde{x}, \ldots, T^{B_R^1(\tilde{x})-1}\tilde{x}$. □

And now, for the tallest template, again let p be as in 5.4.37, let $\tilde{p}$ be as in 5.4.50, and let G be as in 5.4.38. Define

$$\begin{aligned} G_5 &= |\omega_{2n+2}^{*,5}| - G \\ \hat{G}_5 &= \tfrac{|\omega_{2n+2}^{J,5}|}{h_{2n+1}}. \end{aligned}$$

For $R = \hat{R}, \ldots, \hat{G}_5$, let $B_R^5(\tilde{x})$ be the closest integer to $\frac{\tilde{p}}{G_5} R h_{2n+1}$ such that $T^{B_R^5(\tilde{x})} \in \mathcal{B}_{\epsilon_{2n}}(0)$.

LEMMA 5.4.15. *For each $R \in \{\hat{R}+1, \ldots, \hat{G}_5\}$, there exists a template $\tau_{2n+1}^{x,k}$ such that $|\tau_{2n+1}^{x,k}| = B_R^5(\tilde{x}) - B_{R-1}^5(\tilde{x})$ and the α_{2n}-name of $T^{B_{R-1}^5(\tilde{x})}\tilde{x}, \ldots, T^{B_R^5(\tilde{x})-1}\tilde{x}$ equals the α_n-name of $\tau_{2n+1}^{x,k}$.*

PROOF. Following similar steps as in proof of Lemma 5.4.14, we need to know that

$$h_{2n+1} - (\kappa + 1) \leq \frac{\tilde{p}}{G_5 h_{2n+1} - \kappa} \text{ and} \tag{5.4.60}$$

$$\frac{\tilde{p}}{G_5} h_{2n+1} + \kappa \leq h_{2n+1} + (\kappa + 1). \tag{5.4.61}$$

As $G_5 \geq \tilde{p}$, 5.4.61 holds automatically. For 5.4.60, we need

$$\frac{G_5 - \tilde{p}}{G_5} \leq \frac{1}{h_{2n+1}}. \tag{5.4.62}$$

Using

$$2 \leq G_5 - \tilde{p} \leq 2h_{2n+1} \text{ and} \tag{5.4.63}$$

$$2 + \tilde{\epsilon}_{2n+1} t_{2n+2}^{*} - \kappa \leq G_5 \leq \tilde{\epsilon}_{2n+2} t_{2n+2}^{*} + 2h_{2n+1}, \tag{5.4.64}$$

in conjunction with 5.4.62, we require

$$\frac{2h_{2n+1}}{2+\tilde{\epsilon}_{2n+2}t^*_{2n+2}-\kappa} \leq \frac{1}{h_{2n+1}}$$

$$\Rightarrow \frac{(2h_{2n+1})^2+\kappa-2}{\tilde{\epsilon}_{2n+2}} \leq t^*_{2n+2},$$

which is given by Requirement 5.4.11(4) A template of the appropriate length does exist. And, as we have included templates for all words within a range of lengths $\kappa+1$ shorter than $\mathcal{Q}_{2n+1}$ and $\kappa+1$ longer than $\mathcal{Q}_{2n+1}$ for orbits of points starting and ending in $\mathcal{B}_{\epsilon_{2n}}(0)$, there exists a template $\tau^{x,k}_{2n+1}$ such that $|\tau^{x,k}_{2n+1}| = B^5_R(\tilde{x}) - B^5_{R-1}(\tilde{x})$ whose α_n-name matches that of $T^{B^5_{R-1}(\tilde{x})}\tilde{x}, \ldots, T^{B^5_R(\tilde{x})-1}\tilde{x}$. □

We now describe how to build the pib from the indices of the top portion of $\mathcal{P}^*_{2n+2}$ (above height p) to the indices of the top portions of each of the five Ω_{2n+2} templates for $l = 1, \ldots, 5$. For $R = \hat{R}+1, \ldots, \hat{G}_l$, define two functions $x_l : [\hat{R}+1, \ldots, \hat{G}_l] \rightarrow \mathcal{R}$ and $k_l : [\hat{R}+1, \ldots, \hat{G}_l] \rightarrow [1, \ldots, 5]$ such that $B^l_R(\tilde{x}) - B^l_{R-1}(\tilde{x}) = |\tau^{x_l(R),k_l(R)}_{2n+1}|$, and the α_{2n}-name of $x_l(R), \ldots, T^{|\tau^{x_l(R),k_l(R)}|-1}x_l(R)$ matches the α_{2n}-name of $T^{B^l_{R-1}(\tilde{x})}\tilde{x}, \ldots, T^{B^l_R(\tilde{x})-1}\tilde{x}$. This gives the sequence of $\mathcal{T}_{2n+1}$ template-tiles for the upper portion of $\mathcal{P}^*_{2n+2}$, and hence mandates the sequence of stage $2n+1$ pibs whose inverses are concatenated to build the pib on the upper portion of the $\mathcal{P}^*_{2n+2}$. Define

$$\phi^{*,l}_{2n+2} = {}^p\phi^*_{2n+2}|(\psi^{x_l(\hat{R}+1),k_l(\hat{R}+1)}_{2n+1})^{-1}|\cdots|(\psi^{x_l(\hat{G}_1),k_l(\hat{G}_l)}_{2n+1})^{-1}.$$

On the lower portion of the tower, up to height p, where defined, the templates agree. For stage $2n+2$, we shall call all levels up to height p good. On this good set of levels, let

$$\varphi^*_{2n+2} = \xi \circ \phi^{*,1}_{2n+2},$$

and let

$$\varphi_{2n+2} = \bigcup_{*\in\star} \varphi^*_{2n+2}.$$

Note that φ_{2n+2} associates levels of $\mathcal{Q}_{2n+1}$ with levels of $\mathcal{P}_{2n+2}$ in the following manner: $\phi^{*,1}_{2n+2}$ maps indices of levels in $\mathcal{P}_{2n+2}$ to indices of points in $\omega^{*,1}_{2n+2}$. ξ then projects the point in $\omega^{*,1}_{2n+2}$ given by the index $\phi^{*,1}_{2n+2}$ to the level of $\mathcal{Q}_{2n+1}$ containing this point.

5.5. Good Sets and the Convergence of PIBs

For $n \geq 0$, let

$$\mathcal{G}^*_{2n+1} = \{ \text{ points in levels up to height } (1-\tilde{\epsilon}_{2n+1})t^*_{2n+1} \text{ of tower } \mathcal{P}^*_{2n+1}\} \setminus S_{2n+1}.$$

Let

$$\mathcal{G}_{2n+1} = \bigcup_{*\in\star} \mathcal{G}^*_{2n+1}.$$

For $n \geq 0$, let

$$\begin{aligned} \mathcal{H}_{2n} \quad = \quad & \{ \text{ points in levels up to height } (1-\tilde{\epsilon}_{2n})h_{2n} \text{ of the tower } \mathcal{Q}_{2n}\} \setminus \\ & \{\{0, S0, \ldots, S^{h_{2n}-1}0\} \cup BAD\mathcal{Q}_{2n}\}. \end{aligned}$$

For $n \geq 0$ and a point $x \in X$, let $a_{2n+1}(x)$ denote the level of $\mathcal{P}_{2n+1}$ which contains x as well as the index (by height) of the level containing x. For $n \geq 0$ and a point $y \in Y$, let

$d_{2n}(y)$ denote the level of $\mathcal{Q}_{2n}$ which contains y as well as the index (by height) of the level containing y.

LEMMA 5.5.1. *If for n-odd, $x \in \mathcal{G}_n$ and for $m > 0$, $x \in \mathcal{G}_{n+2m}$, then*

$$\varphi_{n+2m}(a_{n+2m}(x)) \subset \varphi_n(a_n(x)).$$

PROOF. As $x \in \mathcal{G}_n$, the $\phi_n^{*,k}$ map the $a_n(x)$ to the same point from $\mathcal{Q}_{n-1}$ in each of the $\omega_n^{*,k}$, and so $\phi_n(a_n(x))$ is a single level of $\mathcal{Q}_{n-1}$. Note that we create a tiling for the $\mathcal{P}_{n+2}$ towers using the $\mathcal{T}_{n+1}$-templates, and the $\mathcal{T}_{n+1}$-templates are segments of forward orbits of points from the base of $\mathcal{P}_n$. We may think of the $\mathcal{P}_{n+2}$ towers as being "tiled" by the $\mathcal{P}_n$-columns, and we see that the $\phi_{n+2}^{*,k}$ are concatenations of the $\phi_n^{*,k}$. So, by construction, the $\phi_{n+2m}^{*,k}$ are concatenations of extensions of the $\phi_n^{*,k}$. We took care to create tilings of $\mathcal{P}_{n+2}$ so that the pibs defined on $\mathcal{P}_{n+2}$ respect and preserve the pibs defined in $\mathcal{P}_n$. As $x \in \mathcal{G}_{n+2m}$, the $\phi_{n+2m}^{*,k}$ all map the $a_{n+2m}(x)$ to the same point from $\mathcal{Q}_{n+2m-1}$ in each of the $\omega_{n+2m}^{*,k}$. So, $\phi_{n+2m}(a_{n+2m}(x))$ is a single level of $\mathcal{Q}_{n+2m-1}$, which also projects to the level of $\mathcal{Q}_{2n-1}$ containing the projection of the image of $a_n(x)$ via $\phi_n^{*,k}$.

□

LEMMA 5.5.2. *If, for n-even, $y \in \mathcal{H}_n$ and for $m > 0$ $y \in \mathcal{H}_{n+2m}$, then*

$$\psi_{n+2m}(d_{n+2m}(y)) \subset \psi_n(d_n(y)).$$

PROOF. The argument is identical to the above argument for the odd steps. □

Suppose $x \in X$ is in level $a_n(x)$ of the tower $\mathcal{P}_n^*$. $\phi_n^{*,k}(a_n(x))$ maps the index (by height) of $a_n(x)$ to the index of a point in the Ω_n-templates $\omega_n^{*,k}$. As φ_n is only defined where all of the pibs $\phi_n^{*,k}(a_n(x))$ agree, i.e. on the good part of $\mathcal{P}_n$, $\phi_n^{*,k}(a_n(x))$ indicates the same point in the forward orbit of $0 \in Y$ for each of the five templates. ξ projects the point in the Ω_n-template indicated by the index $\phi_n^{*,k}(a_n(x))$ to a level in the $\mathcal{Q}_{2n-1}$ tower. So, $\varphi_n(a_n(x))$ gives a level of $\mathcal{Q}_{2n-1}$. Note that for points of X which lie in the good sets infinitely often, the sets $\varphi_n(a_n(x))$ for $x \in \mathcal{G}_n$ form a nested sequence of clopen intervals by 5.5.1 which intersect to a single point as the heights of all of the towers go to infinity.

Let $n_m(x)$ be the odd subsequence such that $x \in \mathcal{G}_{n_m(x)}$ for all $n_m(x)$. Then, define

$$\varphi(x) = \bigcap_{m=1}^{\infty} \varphi_{n_m(x)}(a_{n_m(x)}(x)).$$

Suppose $y \in Y$ is in level $d_n(y)$ of $\mathcal{Q}_n$. $\psi_n^{x,l}(d_n(y))$ maps the index (by height) of $d_n(y)$ to the index (by height) of a point in the $\mathcal{T}_n$-tempaltes, $\tau_n^{x,l}$, for each representative point $x \in \mathcal{R}$ and $l = 1, \ldots, m(x)$. As ψ_n is only defined where all of the pibs $\tau_n^{x,l}$ agree, i.e. on the good part of $\mathcal{Q}_n$, the $\tau_n^{x,l}(d_n(y))$ give the indices of points which project via ξ to a single level of the $\mathcal{P}_{2n-1}$ tower. So, $\psi_n(d_n(y))$ gives a level of $\mathcal{P}_{2n-1}$. Then, for points of Y which lie in the good set infinitely often, the sets $\psi_n(d_n(y))$ for $y \in \mathcal{H}_n$ form a nested sequence of clopen intervals by 5.5.2 which intersect to a single point as the height of $\mathcal{Q}_n$, $h_n \to \infty$.

Let $n_p(y)$ be the even subsequence such that $y \in \mathcal{H}_{n_p(y)}$. Define

$$\psi(y) = \bigcap_{p=1}^{\infty} \psi_{n_p(y)}(d_{n_p(y)}(y)).$$

LEMMA 5.5.3. *φ and ψ are defined on G_δ sets of full measure.*

PROOF. First, note that for n-odd, $\mathcal{G}_n$ is an open set. Next, let

$$\mathcal{G} = \bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{G}_{2n+1}.$$

$\mathcal{G}$ is a G_δ set. Points in $\mathcal{G}$ appear infinitely often in the good parts of the $\mathcal{P}_n$ towers. So, for $x \in \mathcal{G}$, we are able to find an infinite subsequence $n_m(x)$ such that $x \in \mathcal{G}_{n_m(x)}$ for all m, meaning that φ is defined on all of $\mathcal{G}$. Note that $\mu(\mathcal{G}_n) > 1 - \tilde{\epsilon}_n \Rightarrow \mu(\mathcal{G}) = 1$.

Next, note that for n-even, $\mathcal{H}_n$ is open. Let

$$\mathcal{H} = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \mathcal{H}_{2n}.$$

$\mathcal{H}$ is a G_δ set. Points in $\mathcal{H}$ appear infinitely often in the good parts of the $\mathcal{Q}_n$ towers. So, for $y \in \mathcal{H}$, we are able to find an infinite subsequence $n_p(y)$ such that $y \in \mathcal{H}_{n_p(y)}$, meaning that ψ is defined on all of $\mathcal{H}$. As $\nu(\mathcal{H}_n) > 1 - 2\tilde{\epsilon}_n$, $\nu(\mathcal{H}) = 1$. □

5.6. Properties of the Maps

LEMMA 5.6.1. *φ and ψ are nearly continuous.*

PROOF. Let $x \in \mathcal{G}$, and let $\epsilon > 0$ be given. Select a tower $\mathcal{Q}_{2n+1}$ such that $h_{2n+1} > \frac{1}{\epsilon}$. Then, $\nu(\text{base}(\mathcal{Q}_{2n+1})) < \epsilon$. Let $2m > 2n+1$ be such that $x \in \mathcal{G}_{2m}$. Then, x is in a level of $\mathcal{P}^*_{2m}$ for some $*$. Let $\delta = \frac{1}{2}\rho(x, d_{2m}(x)^C)$. Every x' such that $\rho(x, x') < \delta$ is also in $\mathcal{G}_{2m}$ and $\varphi_{2m}(d_{2m}(x)) = \varphi_{2m}(d_{2m}(x')) \Rightarrow \rho(\varphi(x), \varphi(x')) < \epsilon$.

Let $y \in \mathcal{H}$, and let $\epsilon > 0$ be given. Select a tower $\mathcal{P}_{2n}$ such that

$$\min_{* \in \star}\{\text{diam}(\text{base}(\mathcal{P}^*_{2n}))\} < \epsilon.$$

Let $2m+1 > 2n$ be such that $y \in \mathcal{H}_{2m+1}$. Let $\delta = \frac{1}{h_{2m+1}}$. Then, every y' such that $\rho(y, y') < \delta$ is also in $\mathcal{H}_{2m+1}$ and $\psi_{2m+1}(d_{2m+1}(y)) = \psi_{2m+1}(d_{2m+1}(y')) \Rightarrow \rho(\psi(y), \psi(y')) < \epsilon$. □

LEMMA 5.6.2. *φ and ψ are measure preserving.*

PROOF. For φ, let d be a level in $\mathcal{Q}_n$. $\nu(d) = \frac{1}{h_n}$. Let $m > n$. Ω_m contains $\frac{1}{h_n}$ points in level d. So, there are at most $\frac{1}{h_n}$ levels in $\mathcal{P}_m$ whose indices map to one of these points, and, for each $*$, there are at least $(1 - \tilde{\epsilon}_n)\frac{t^*_m}{h_n}$ levels in each tower $\mathcal{P}^*_m$ whose indices map to one of the points in level d via $\phi^{*,l}_m$. Summing up over the towers,

$$\begin{aligned}
\sum_{*\in\star} \frac{1-\tilde{\epsilon}_m}{h_n} t^*_m \mu(\text{base}(\mathcal{P}^*_m)) &= \frac{1-\tilde{\epsilon}_m}{h_n} \sum_{*\in\star} t^*_m \mu(\text{base}(\mathcal{P}^*_m)) && (5.6.1)\\
&= \frac{1-\tilde{\epsilon}_m}{h_n} && (5.6.2)\\
&\le \mu(\{ \text{ levels of } \mathcal{P}_m \text{ which map into } d.\}) && (5.6.3)
\end{aligned}$$

Now, as $m \to \infty$,

$$\frac{1}{h_n} \le \mu(\varphi^{-1}(d)) \le \frac{1}{h_n}.$$

For ψ, let a be a level in $\mathcal{P}_n$. If a is in $\mathcal{P}^*_n$, then $\mu(a) = \mu(\text{base}(\mathcal{P}^*_n))$. Select $m > n$ large enough so that, by nearly unique ergodicity of T, the points in the templates $\mathcal{T}_m$ lie in level a the fraction $\mu(a) \pm \delta_m$ of the time and $\lim_{m\to\infty} \delta_m = 0$. Then, there are at most $\mu(a) + \delta_m$ levels in Q_m whose indices map to points in level a, and at least $(1 - \tilde{\epsilon}_m)(\mu(a) + \delta_m)$. Then

$$(1 - \tilde{\epsilon}_m)(\mu(a) + \delta_m) \le \nu(\{ \text{ levels of } \mathcal{Q}_m \text{ mapping into a } \}) \le \mu(a) + \delta_m.$$

As $m \to \infty$, $\mu(a) \le \nu(\psi^{-1}(a)) \le \mu(a)$. □

LEMMA 5.6.3. *φ and ψ are inverses.*

PROOF. Let $x \in \mathcal{G}$ and let $\varphi(x) \in \mathcal{H}$. Now, let n be such that $\varphi(x) \in \mathcal{H}_n$, and let $m > n$ be such that $x \in \mathcal{G}_m$. Then $\varphi_m(x) \subset a_n(\varphi(x))$. By construction, $\phi_m^{*,l}$ is a concatenation of the $(\psi_n^{x,l})^{-1}$ so that $\psi_n(a_n(\varphi_m(x))) = d_{2n-1}(x)$.

$$\psi_n(a_n(\varphi(x))) = \psi_n(a_n(\varphi_m(x))) = d_{2n-1}(x)$$

and as $n \to \infty$, $\psi_n(a_n(\varphi(x))) \to \psi \circ \varphi(x)$ and $d_{2n-1}(x) \to x$ meaning $\psi \circ \varphi(x) = x$.

The argument is identical for the opposite direction. □

5.7. Orbit Equivalence and Conjugacy on Nearly Clopen Sets

LEMMA 5.7.1. *ϕ is an orbit equivalence.*

PROOF. Suppose that x_1 and $x_2 \in \mathcal{G}$ and $x_2 = T^r x_1$, $r > 0$. After some point in the construction, because the heights of towers go to infinity, x_2 is in a level in a tower of a tower for T, directly above x_1. Let N be the first time that $a_N(x_2) = T^r(a_N(x_1))$, which occurs from this time onward. Suppose that $x_2 \in \mathcal{G}_N$, meaning $x_1 \in \mathcal{G}_N$. Then, $\phi_N^{*,k}(a_N(x_2)) = S^t \phi_N^{*,k}(a_N(x_1))$ for some $t \in \mathbb{Z}$. Note that $\phi_N^{*,k}(a_N(x_2))$ and $\phi_N^{*,k}(a_N(x_1))$ could, at this stage, project via ξ to levels of $\mathcal{Q}_{N-1}$ which are not, as of yet, t apart. We thus look forward in the construction to a stage $N+2p$ for $p > 0$ with $x_2 \in \mathcal{G}_{N+2p}$ and with $h_{N+2p-1} > t$ where h_{N+2p-1} is the height of $\mathcal{Q}_{N+2p-1}$. $\mathcal{P}^*_{N+2p}$ is naturally tiled by the $\mathcal{P}^*_N$, and x_1 and x_2 are located within a single $\mathcal{P}^*_N$-tile. This single $\mathcal{P}^*_N$-tile is located within a single $\mathcal{T}_{N+2p-1}$-tile of $\mathcal{P}^*_{N+2p}$ and thus maps to a segment of $\omega^{*,1}_{N+2p}$ between heights jh_{N+2p-1} and $(j+m)h_{N+2p-1}$ for some j and m. Since $h_{2N+p-1} > t$, $\phi^{*,k}_{N+2p} a_{N+2p}(x_2)$ and $\phi^{*,k}_{N+2p}(a_{N+2p}(x_1))$, will project to levels in $\mathcal{Q}_{N+2p-1}$ which are t apart so that $\varphi_{N+2p}(x_2) = S^t \varphi_{N+2p}(x_1)$. This occurs infinitely often. □

LEMMA 5.7.2. *ϕ is a conjugacy when restricted to $\mathcal{G}_1$.*

PROOF. Suppose that $x_1,\ x_2 \in \mathcal{G}$ and $x_2 = T^r x_1,\ r > 0$. After some point in the construction, as the heights of towers go to infinity, x_2 is in a level in a tower of a tower for T directly above x_1. Let N be the first time that $a(x_2) = T^r(a(x_1))$. Suppose that $x_2 \in \mathcal{G}_N$. Now, as $x_1, x_2 \in \mathcal{G}_1$, pibs are defined on the levels containing these points in the very first stage. At stage N, the pibs $\phi_N^{*,k}$ are concatenations of the $\phi_1^{*,k}$, and the $\phi_1^{*,k}$ pib acting on the $\mathcal{T}_{N-1}$-tile containing x_1 comes before the $\phi_1^{\hat{*},\hat{k}}$ pib acting on the $\mathcal{T}_{N-1}$-tile containing x_2. Hence, $\phi_N^{*,k}(a_N(x_2)) = S^t \phi_N^{*,k}(a_N(x_1))$ for some $t > 0$. During all future odd stages, $\phi_{N+2p}^{*,k}$ is a concatenation of the $\phi_N^{*,k}$, and thus acts on the segment containing x_1 and x_2 as ONE $\phi_N^{*,k}$ so that $\varphi_{N+2p}(x_2) = S^t \varphi_{N+2p}(x_1)$, and order is preserved on the orbits of points from $\mathcal{G}_1$. □

5.8. THE FINAL SETS

Let

$$X_1 = \bigcap_{i=-\infty}^{\infty} T^{-i}\mathcal{G}$$

and

$$Y_1 = \bigcap_{i=-\infty}^{\infty} S^{-i}\mathcal{H}.$$

Let

$$X^* = X_1 \bigcap \varphi^{-1}(Y_1) \text{ and } Y^* = \varphi(X^*).$$

Let

$$A = \mathcal{G}_1 \bigcap X^* \text{ and } B = \varphi(A).$$

Theorem 5.8.1. *All minimal isometries of compact metric spaces are nearly continuously evenly Kakutani equivalent to the binary odometer*

Proof. X^* and Y^* are nearly full, invariant subsets of X and Y on which φ is an orbit equivalence, and A and B are nearly clopen sets of positive, equal measure for which φ is a conjugacy between T_A and S_B. $\square$

CHAPTER 6

Conclusion and Open Problems

This concludes our exploration of nearly continuous dynamical systems and nearly continuous Kakutani equivalence. We end with a discussion of intriguing encounters along the way, differences between our use and previous uses of towers and templates, and possible avenues for continued exploration.

One item of interest in this dissertation relates to the differences between the definition for nearly continuous Kakutani equivalence given by [5] and the definition given within this thesis. Note that an orbit equivalence is not necessary in the definition, as shown in Theorem 4.1.1 of Chapter Four, and as experienced in our proof of Theorem 4.3.3.2. However, in Chapter Five, with the tower and template machinery, we found that establishing an orbit equivalence between nearly full subsets was essential to our ability to create a nearly clopen set $B \subset Y$ of the same measure as A such that T_A was nearly continuously conjugate to S_B. In the measure-theoretic category, if A is a measurable subset of X, then T_A on A and T on X are evenly Kakutani equivalent. We discovered that, in the nearly continuous category, this holds for the n.c. even Kakutani equivalence class containing irrational rotations of the circle. Does this hold in general?

The tower and template machinery provides a clever way of overcoming one of the principle problems within the theory of nearly continuous dynamics; that one can modify a nearly clopen set only finitely many times in order to ensure that it remains nearly clopen. Thus, our method for proving Alpern's lemma in Chapter Three differs significantly from the original. Consider for a moment rank one transformations–transformations with a refining sequence of towers whose levels generate the topology. Within the measure-theoretic definition of rank one, levels are only required to be measurable. For an irrational rotation of the circle, by taking a sequence of towers, and using the bounded return times to any measurable subset, one can modify the towers countably many times in order to produce a refining sequence of towers whose levels generate the topology, i.e., one may build a cutting and stacking consisting of towers whose levels are measurable, and such that there is only one stack at each stage. This proof that irrational rotations are measure-theoretic rank one does not extend into the nearly continuous category as one cannot refine a nearly clopen tower countably many times and be assured that the tower remains nearly clopen. This leads to an interesting question about the nature of irrational rotations of the unit circle: are irrational rotations strongly rank one? The answer is yes in the measure-theoretic sense, and no if one requires the levels to be intervals, but can one create a sequence of refining towers using nearly clopen sets?

In the measure-theoretic category, if A is a measurable subset, then T_A is evenly Kakutani equivalent to T. In the nearly continuous category, for the n.c. even Kakutani equivalence class of transformations containing irrational rotations, we showed that if A is a nearly clopen set of positive measure, then T_A is n.c evenly Kakutani equivalent to T. The proof relied upon the nearly unique ergodicity of irrational rotations of the unit circle and adding machines. For any n.c. dynamical system, (X, τ, μ, T), and nearly clopen subset $A \subset X$, is it possible that T_A is n.c. evenly Kakutani equivalent to T?

Returning to our discussion of towers and templates, one wonders whether it is possible to implement them in a simpler fashion. One might wish to eliminate the templates in the process and only create maps from levels of towers for one system to levels of towers of the other systems. One quickly encounters a problem in that one produces a non-invertible map. Consider first mapping from levels of towers of the first system to levels of towers of the second system. To build an inverse-map, one might then cut and stack each system to produce much taller towers and attempt to map in the backwards-direction, by necessity keeping the same assignments of levels to levels. However, one finds that the cutting-and-stacking produced different numbers of copies of the previous towers, and a measure-preserving map cannot be produced. This, then, leads to the introduction of templates. Now, templates may be thought of in several ways–they are a bit like very tall towers through which we project to shorter towers. They are also representatives for all of the partition names of a certain length which appear in a tower. By using templates of a variety of heights, we are able to define maps in such a way as to keep the number of copies of towers the same, hence producing a measure-preserving map in the end. Recall the discussion from above on the inability to refine a nearly clopen set a countable number of times. After introducing the templates, in order to ensure that the maps produced by composing a template with a projection is well-defined, we may only refine a decision to map a certain level of one tower to a certain level of another tower at specific stages, stages where all of the partial interval bijections agree for that level. Once it is decided that level a of a tower for one system maps to level b of a tower for another system, subsets of a must map to subsets of b, which may not be possible at every stage of construction. Templates provide a book-keeping device, tracking the destination of levels so that, whenever a refinement is possible, we did not forget to-where a level was previously assigned, keeping the new map consistent with those from previous stages. The templates also produce a way to ensure that the domains of the maps produced are nearly full sets since the domains consist of points which lie in the good sets infinitely often.

Our use of towers and templates differs from the templates used in [18] and [10] in a few key ways. First of all, in [18] and [10], only three templates and three partial interval bijections are used at each stage in the construction. For [18], this comes from the fact that for odometers, the orbits of all points starting in the bottom level share the same name. In [10], while they only use three templates and three pibs at each stage, they must modify the partial interval bijections at each stage by conjugating with functions which correct the errors produced by the fact that the forward orbits of points starting in the base of the same tower could have different names. Due to less predictable return times, we needed to allow for the use of more templates to cover a range of heights. Also, in order to account for differences in names of orbits, we elected to use more templates, the representative templates, instead of attempting to modify partial interval bijections for a set of templates which start from a single point, i.e. the master templates. Our technique for tiling the towers using previous-stage templates also differed. The previous papers use arguments from number theory to first select the appropriate number of templates with certain heights, and then modify the templates. Instead, we make sure to choose towers so tall for each stage that the difference in heights between towers and templates is minuscule by comparison, and then we rescale a template and choose a tiling based on comparing the tower to the rescaled template. In the end, our proof relies heavily on the fact that minimal isometries of compact metric spaces are uniquely ergodic and that they are isometries.

From here, one wishes to continue the quest to understand nearly continuous even Kakutani equivalence and the different classes of transformations. While a classical cutting and stacking does not exist for generic minimal isometries of compact metric spaces, we still relied upon the ability to build a sequence of refining towers/skyscrapers. Perhaps the next task lies in considering systems without an easy tower/skyscraper construction, or which fail to be an isometry. Dykstra and Şahin [11] show that the morse minimal system, which is not an isometry, in nearly continuously Kakutani equivalent to the binary odometer. The class of zero entropy loosely Bernoulli transformations, which are known to be Kakutani equivalent in the measure-theoretic category, provide a nice test-bed in the hunt for a more natural dynamical system which fails to be n.c. evenly Kakutani equivalent. Marina Ratner [17] proved that horocycle flows are loosely Bernoulli. Thus, horocycle flows provide another example of a system which is not an isometry and which lack an easy tower construction and provide a logical next step to test the adaptability of the tower and template machinery. It is also our hope to continue to develop the theory of nearly continuous dynamics and to further understand the interplay between ergodic theory and topological dynamics.

Bibliography

[1] P. Arnoux, D. Ornstein, and B., Weiss, *Cutting and stacking, interval exchanges, and geometric models*, Israel J. Math. (1985), vol. 50, no. 1-2, 160-168.

[2] Vladimir Bogachev, 2007. *Measure theory* Vol I, II. Berlin: Springer-Verlag.

[3] R. V. Chacon, *Weakly mixing transformations which are not strongly mixing*, Proc. Amer. Math. Soc. 22 1969, 559-562.

[4] R. V. Chacon, and D. Ornstein, *A general ergodic theorem* , Illinois J. Math.4 (1960), 153160.

[5] A. del Junco, D. Rudolph and B. Weiss, *Measured topological orbit and Kakutani equivalence*, Discrete Contin. Dyn. Syst. Ser. 2 (2009), no. 2, 221-238.

[6] Andres del Junco and Ayşe Şahin *Dye's Theorem in the Nearly Continuous Category*, Israel J. Math. 173 (2009), 235-251.

[7] Andres del Junco and Daniel Rudolph. *Kakutani equivalence of ergodic* $\mathbb{Z}^n$ *actions*, Ergodic Theory Dynam. Systems 4 (1984), no. 1, 89-104.

[8] Manfred Denker and Michael Keane, *Almost topological dynamical systems*, Israel J. Math. 34 (1979), no. 1-2, 139-160 .

[9] H. A. Dye, *On groups of measure preserving transformations I*, American Journal of Mathematics, 81., No. 1 (1959), 119-159.

[10] Andrew Dykstra and Daniel Rudolph, *Any two irrational rotations are nearly continuously Kakutani equivalent*, J. Anal. Math 110 (2010), 339-384.

[11] Andrew Dykstra and Ayşe Şahin, *The morse minimal system is nearly continuously evenly Kakutani equivalent to the binary odometer*, preprint.

[12] Ryszard Engelking, *General topology*, Second edn. Sigma Series in Pure Mathematics, vol. 6. Berlin: Heldermann Verlag.

[13] Paul R. Halmos, *In general a measure preserving transformations is mixing*, Ann. of Math. (2) 45, (1944), 786-792.

[14] Shizou Kakutani *Induced measure preserving transformations*, Proc. Emp. Acad. Tokyo 19 (1943), 635-641.

[15] D. Ornstein, D. Rudolph, and B. Weiss, *Equivalence of measure preserving transformations*, Mem. Amer. Math. Soc. 37 (1982), no. 262.

[16] Karl Petersen, *Ergodic Theory*, 2000. Cambridge University Press. First edition 1983.

[17] Marina Ratner, *Horocycle flows are loosely Bernoulli*, Israel J. Math., 31 (1978), no. 2, p. 122-132.

[18] M.K. Roychowdhury and D. J. Rudolph, *Nearly continuous Kakutani equivalence of adding machines,* J. Mod. Dyn. 3 (2009), no. 1, 103-119.

[19] Karl Stromberg, *Probability for Analysts*, Chapman and Hall, New York: 1994.

[20] V. Zhuravlev, (2009) *Two Theorems of Dye in the Almost Continuous Category* Unplublished Thesis. University of Toronto.