DISSERTATION

OPEN AND CLOSED GROMOV-WITTEN THEORY OF THREE-DIMENSIONAL TORIC CALABI-YAU ORBIFOLDS

Submitted by

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ABSTRACT

OPEN AND CLOSED GROMOV-WITTEN THEORY OF THREE-DIMENSIONAL TORIC CALABI-YAU ORBIFOLDS

We develop the *orbifold topological vertex*, an algorithm for computing the all-genus, open and closed Gromov-Witten theory of three-dimensional toric Calabi-Yau orbifolds. We use this algorithm to study Ruan's crepant resolutions conjecture and the orbifold Gromov-Witten/Donaldson-Thomas correspondence.

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CHAPTER 1

Introduction

The primary investigations of this dissertation lie in the study of various conjectural correspondences in Gromov-Witten theory. In particular, we study these correspondences for toric Calabi-Yau 3-folds. In this introductory chapter, we provide the reader with some of the motivations, historical developments, and breakthroughs in this active field, while also giving a sense of where the work carried out herein lies within the larger context.

1.1. Gromov-Witten Theory

Gromov-Witten (GW) theory has its roots both in enumerative geometry and theoretical physics. The origins of the theory lie in papers of Gromov on pseudoholomorphic curves in symplectic manifolds [32] and Witten on topological strings [64], with other major developments in the early stages of the theory provided by Kontsevich, Manin, Ruan, and Tian [38, 39, 59, 61].

A central theme in virtual curve counting theories in general (and GW theory in particular) is to fix a smooth, projective variety X and to consider appopriate compact parameter spaces of algebraic curves in X with fixed genus g and degree $\beta \in H_2(X, \mathbb{Z})$. The necessity of compactness forces us to allow our curves to degenerate to singular objects. In recent years, mathematicians have developed a plethora of ways in which to allow degenerations depending on whether the curves in X are viewed as subschemes, ideal sheaves, maps, quasi-maps, etc. (see [54] for an introduction to these various points of view). As the first successful approach in building a compact parameter space with desirable properties, GW theory views the curves in X as *parametrized* curves, ie. we keep track of maps $f : C \to X$.

More specifically, GW theory studies moduli spaces of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$. These moduli spaces parametrize (up to isomorphism) pairs (C,f) where

(1) C is a connected genus g (at worst) nodal curve with n marked, distinct points,

(2) $f : C \to X$ is a holomorphic map of degree β: $f_*([C]) = \beta \in H_2(X,\mathbb{Z})$, and

(3) there are finitely many automorphisms of C which commute with f.

Because of the finite automorphism condition, spaces of stable maps naturally inherit the structure of Deligne-Mumford stacks. Moreover, they are compact, as can be seen by using properties of stable reduction for curves ([29]). However, the moduli stacks $\overline{\mathcal{M}}_{g,n}(X,\beta)$ can be terrible to work with: possibly reducible, non-reduced, and not of pure dimension. Notwithstanding, the stacks admit a perfect obstruction theory and therefore (by [4]) they support a *virtual fundamental class* $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ of the expected dimension:

$$\int_C c_1(X) + (\dim X - 3)(1-g). \tag{1–1}$$

The importance of Calabi-Yau 3-folds (CY3s) in the subject becomes obvious in light of (1–1): it is exactly the condition for which we expect a zero-dimensional space of curves (and hence a virtual "curve count") for all g and β. When the expected dimension is positive, curve counts can still be computed by imposing certain incidence conditions as we will see below. Physical principles in string theory tell us that the curve counts thus obtained should witness a significant amount of structure inherent in the corresponding physical system.

Moduli stacks of stable maps have natural evaluation maps $ev_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X$ for each marked point. For $\alpha_i \in H^*(X)$, (primary) GW invariants are defined as the intersection

numbers

$$\langle \alpha_1 \cdots \alpha_n \rangle_{g,n}^{X,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_i ev_i^*(\alpha_i) \in \mathbb{Q}.$$

Intuitively, we think of each $ev_i^*(\alpha_i)$ as imposing an incidence condition on the image of the ith point.

It is difficult to get a sense of the structure inherent to GW theory simply by considering the invariants individually. As we will see below, the structures we are after can only be witnessed after packaging the invariants into natural generating functions. Choosing a basis $\phi_1, ..., \phi_N$ of $H^*(X)$ with dual coordinates $x_1, ..., x_N$, the *GW Potential* is defined by:

$$GW(X; x_1, ..., x_N, q, u) := \sum_{g,n_i,\beta} \langle \phi_1^{n_1} \cdots \phi_N^{n_N} \rangle_{g,\sum n_i}^{X,\beta} \prod_i \frac{x_i^{n_i}}{n_1!} q^\beta u^{2g-2}. \tag{1–2}$$

Physics not only makes predictions about curve counts in smooth varieties, but it also naturally extends to *orbifold* targets, ie. varieties with finite quotient singularities. Motivated by the guiding physical principles of string theory on orbifolds, Chen and Ruan developed orbifold stable maps in [18] in the symplectic category (the analogous development in the algebraic category followed in [2]). One of the guiding aspects of the theory is to allow the source curves to obtain orbifold points to probe the orbifold structure of the targets. With some modification, all of the foundational theory goes through in the orbifold case: there is a virtual fundamental class of the expected dimension, natural evaluation maps (with image in the inertia orbifold), and orbifold GW invariants can be encoded exactly as in (1–2).

1.2. Toric Calabi-Yau Threefolds

In the years since the original developments of GW theory, direct computations have proven to be extremely elusive. Indeed, even for the quintic 3-fold (the prototypical example

of a projective CY3), GW invariants have only been computed mathematically in genus ≤ 1. Due to virtual localization techniques ([31]), one arena where direct computations have actually been successful is for toric targets. A variety (or orbifold) X is *toric* if it contains a dense open subset isomorphic to an algebraic torus and the action of the torus naturally extends to an action on X. For any class α, virtual localization tell us that

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \alpha = \sum_F \int_F \frac{i_F^*(\alpha)}{e^{eq}(N_F)}$$

where the sum is over the fixed loci of the torus action, $i_F : F \hookrightarrow X$ is the inclusion, and we formally invert the equivariant Euler class of the normal bundle in the localized equivariant cohomology ring. The contributions at the fixed loci can be evaluated in terms of graph sums of Hodge integrals on moduli spaces of curves. Therefore, localization reduces the computation to that of graph combinatorics and Hodge integrals.

If we restrict further to (smooth) toric CY3s, the computation of GW invariants is completely solved with the *topological vertex* algorithm. In particular, motivated by large N duality, Aganagic, Klemm, Mariño, and Vafa ([3]) suggested the existence of *open* GW invariants counting maps from Riemann surfaces with boundary and they introduced the topological vertex as a certain generating function of open invariants of $\mathbb{C}^3$. They proposed a specific *gluing* algorithm for obtaining the GW potential of any toric CY3 X in terms of a topological vertex contribution defined near each torus fixed point of X. Moreover, via large N-duality, they gave a concrete prediction for the topological vertex in terms of Schur functions evaluated on the formal variables. A key aspect of the theory is that the topological vertex encodes invariants of all genera at once making it significantly more efficient than the traditional localization techniques where invariants must be computed genus by genus.

In the smooth case, the predictions of [3] have been completely verified mathematically in [47] (see also [22, 37, 41, 42, 43, 50]).

Given the effectiveness of the topological vertex in the smooth case, we are naturally confronted with the question of whether such an algorithm exists for toric CY3 *orbifolds*. The first contribution of this dissertation (Chapter 2) is to develop such an algorithm. In particular, we define open orbifold GW invariants of $[\mathbb{C}^3/G]$ where G is any finite abelian group acting trivially on the volume form and we use these invariants to define the *orbifold vertex*. The key result that we prove is an explicit gluing algorithm analogous to that in [3] for reducing the GW theory of any toric orbifold CY3 to the orbifold vertex.

With the orbifold vertex in hand, we obtain a local-to-global approach for investigating conjectural correspondences related to GW theory of toric CY3 orbifolds: first prove the correspondence for the vertex, then show compatibility with the gluing algorithm. A major motivation of the work carried out in this dissertation is to gather support for this approach in the study of two particular problems: Ruan's crepant resolution conjecture and the orbifold Gromov-Witten/Donaldson-Thomas correspondence (described below).

The orbifold vertex is naturally a generating function of G-Hodge integrals on moduli spaces of stable orbifold curves. In the smooth case, these generating functions were proven to be expressible in terms of Schur functions (arising from certain knot invariants in Chern-Simons theory). In the orbifold case, we do not have physical predictions for these generating functions, but the orbifold GW/DT correspondence suggests that in special cases we can express the orbifold vertex in terms of *loop* Schur functions (cf. Chapter 4). The ultimate usefulness of the orbifold vertex will be determined by a comprehensive study of

the corresponding G-Hodge integrals which arise. One of the overlying themes of Chapters 3 - 5 is to lay down the first steps in this direction.

1.3. Ruan's Conjecture

If $\mathfrak{X}$ is a Gorenstein orbifold and Y is a crepant resolution of singularities, then physical principles suggest that the string theory of the two spaces should be equivalent, in some sense. With this guiding physical principle, Ruan conjectured in ([60]) what is commonly referred to as the *crepant resolution conjecture* (CRC). Roughly, he asserted that the GW theory of a Gorenstein orbifold should be equivalent to that of a crepant resolution. The formulation of this conjecture was one of the motivating factors in the development of orbifold GW theory in [18]. In the years since its original formulation, the CRC has attracted a significant amount of attention in the field (eg. [9, 10, 11, 20, 21, 30, 63]) and continues to be a very active area of research.

For the purposes of this dissertation, we restrict ourselves to orbifolds satisfying a hard-Lefschetz condition. In that case, Bryan and Graber ([10]) give a refinement of the CRC which we paraphrase:

Conjecture (Ruan-Bryan-Graber). *If $\mathfrak{X}$ is a Gorenstein, hard-Lefschetz orbifold and Y is a crepant resolution, then there exists an affine linear change of formal parameters and analytic continuation so that $GW(\mathfrak{X}) = GW(Y)$.*

We propose that the Ruan-Bryan-Graber conjecture (in the toric CY3 case) should be approached with the orbifold vertex. In particular, we propose a CRC statement for open GW theory and expect that this open CRC is compatible with gluing.

In Chapter 3, we investigate the first nontrivial geometry for which this approach applies. In particular, we prove the first example of the open CRC (Theorem 3.10)[1]. The correspondence is via a change of variables where we have introduced additional *winding* parameters to keep track of how the boundaries map into the target.

Moreover, we show that our open correspondence is compatible with gluing. In particular, we deduce a new example of the Ruan-Bryan-Graber conjecture (Theorem 3.12) by gluing our open correspondence. These verifications provide promising support for our local approach to the toric CY3 CRC.

1.4. Gromov-Witten/Donaldson-Thomas Correspondence

Donaldson-Thomas (DT) theory views curves in X as embedded subschemes, rather than parametrized curves. The relevant compact moduli space is $\mathrm{Hilb}_X(n,\beta)$, the Hilbert scheme of curves in X with fixed Euler characteristic n and one-dimensional support β. In case X is 3-dimensional, the Hilbert scheme has a perfect obstruction theory[2] and therefore supports a virtual fundamental class of the expected dimension ([62]). DT invariants are defined by intersecting against the virtual class. For our purposes, we restrict to the CY3 setting.

If X is a CY3, then the expected dimension of the Hilbert scheme is zero and we define

$$N_{n,\beta} := \int_{[\mathrm{Hilb}_X(n,\beta)]^{vir}} 1 \in \mathbb{Z}.$$

We naturally package these integer counts into the degree β DT *partition function*:

$$DT_\beta(v) := \sum_n N_{n,\beta} v^n$$

[1]In recent work with A. Brini and R. Cavalieri, we generalize the open CRC to all type A 3-fold singularities ([7]).

[2]Rather, the natural perfect obstruction theory comes from the isomorphic moduli scheme of ideal sheaves.

and we define the reduced partition function by formally removing the degree 0 contributions:

$$DT^\bullet_\beta(v) := \frac{DT_\beta(v)}{DT_0(v)}.$$

Similarly we obtain a degree β GW partition function $GW^\bullet_\beta(u)$ defined by the following formula:

$$\sum_\beta GW^\bullet_\beta(u) q^\beta = \exp\left(\sum_{\substack{\beta\neq 0 \\ g\geq 0}} \left(\int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{vir}} 1 \right) u^{2g-2} q^\beta \right).$$

Notice that $GW^\bullet_\beta(u)$ encodes virtual counts of maps from possibly disconnected curves which are not allowed to contract entire connected components. Maulik, Nekrasov, Okounkov, and Pandharipande conjectured the Gromov-Witten/Donaldson-Thomas correspondence relating these series ([45, 46]).

Conjecture (Maulik-Nekrasov-Okounkov-Pandharipande).

$$GW^\bullet_\beta(u) = DT^\bullet_\beta(-\mathrm{e}^{\mathrm{i}u}).$$

In particular, the correspondence does not allow the conjecture to be checked term-by-term, rather we must know every invariant on one side to compute a single invariant on the other. A significant amount of work has culminated in a proof of the GW/DT correspondence for toric 3-folds (equivariant, with descendents) in [47] and a recent proof for CY complete intersections in products of projective spaces in [53].

DT theory is generalized to orbifold targets by Bryan, Cadman, and Young in [8] where they develop a DT analog of the orbifold vertex. They show that for certain geometries, the DT orbifold vertex can be computed combinatorially generalizing the Schur function evaluations in the smooth case. Naturally, we ask whether an extension of the GW/DT

correspondence holds for orbifolds and whether this correspondence can be witnessed on the level of the orbifold vertex. This question is the motivation behind Chapter 5.

As a natural starting point, we investigate the correspondence for the one-leg A_{n-1} vertex $[\mathbb{C}^3/\mathbb{Z}_n]$. The special case in the smooth setting ($n = 1$) is the Gopakumar-Mariño-Vafa (GMV) formula (named after the physicists who first made the prediction) and was proven independently in [42] and [50]. In Chapter 5, we prove the A_{n-1} generalization of this formula which we playfully call the gerby GMV formula. The proof of the gerby GMV formula requires a significant number of new tools, including a combinatorial generalization of the Murnaghan-Nakayama rule for Schur functions (Theorem 4.1). The tools utilized in the proof of the gerby GMV formula, in particular the use of the wreath Fock space and the combinatorics of loop Schur functions, appear to forge a promising path forward in further investigations of the orbifold vertex.

Moreover, at the end of Chapter 5 we compile more evidence for the local-to-global approach of the orbifold vertex by showing that the gerby GMV formula is compatible with gluing. In particular, we state and prove the orbifold GW/DT correspondence for all local $\mathbb{Z}_n$-gerbes over $\mathbb{P}^1$ – this serves as the first example of the orbifold GW/DT correspondence where curve classes lie entirely in the singular locus.[3]

1.5. Organization of the Dissertation

Chapter 2 develops the GW orbifold vertex which serves as the basic building block for the GW theory of toric CY3 orbifolds. This chapter serves as the backbone of the local-to-global approach pursued in later chapters. The main content from this chapter previously appeared in [56].

[3]A related class of examples of the orbifold GW/DT vertex correspondence appeared in [65].

Chapter 3 investigates an open version of the Ruan-Bryan-Graber crepant resolution conjecture and its compatibility with gluing. The results of this chapter were obtained in collaboration with R. Cavalieri and appeared previously in [17].

Chapter 4 contains a study of loop Schur functions and a proof of the loop Murnaghan-Nakayama rule. The results in this chapter are important in the final chapter, but this chapter can also be read independently of the rest of the dissertation. This content appeared previously in [57].

Chapter 5 proves the gerby GMV formula and deduces from it the GW/DT correspondence for a large class of local orbifold lines. The results in this chapter were obtained in collaboration with Z. Zong and previously appeared in [58].

CHAPTER 2

The Gromov-Witten Orbifold Vertex

In this chapter we generalize the Gromov-Witten topological vertex developed in [22, 37, 41] to the orbifold setting. In particular, we show that the orbifold Gromov-Witten theory of any 3-dimensional toric Calabi-Yau orbifold can be reduced to appropriate generating functions of open orbifold Gromov-Witten invariants, local to each torus fixed orbifold point of the target. The formalism which is developed in this chapter is the starting point of our local-to-global approach and will be utilized in subsequent chapters to investigate the crepant resolution conjecture and the Gromov-Witten/Donaldson-Thomas correspondence for orbifolds.

2.0.1. Statement of Results. The first step in our program is to generalize the definition of open GW invariants to the affine orbifold $[\mathbb{C}^3/G]$ where G is any finite abelian group. After suitably generalizing the work of [37] and [6], we give a definition (Section 2.2) of the *GW Orbifold Vertex* as a generating function of open GW invariants of $[\mathbb{C}^3/G]$ where G is any finite abelian group. The orbifold vertex is naturally a generating function of G-Hodge integrals on moduli spaces of orbifold curves.

The technical heart of this chapter then lies in proving that the GW orbifold vertex of Definition 2.12 glues, ie. that it is a building block for the GW theory of 3-dimensional toric CY orbifolds. We present the main gluing result in Section 2.3.

Theorem (Theorem 2.17). *The GW theory of 3-dimensional toric CY orbifolds is determined by the GW orbifold vertex and a suitable gluing algorithm.*

Given the *toric diagram* $\Gamma = \{\text{vertices}, \text{edges}\}$ associated to the target orbifold $\mathcal{X}$, the gluing algorithm has the following form:

$$GW^\bullet(\mathcal{X}) = \sum_{\Lambda} \prod_{\text{edges}} E(e,\Lambda) \prod_{\text{vertices}} V(v,\Lambda)$$

where $GW^\bullet(\mathcal{X})$ is the GW partition function of the target, $V(v,\Lambda)$ is the GW orbifold vertex and $E(e,\Lambda)$ consists simply of an automorphism correcting combinatorial factor and a sign which depends on the geometry of the target near e. The sum is over all possible ways of assigning decorated partitions to the edges (Section 2.3.2).

In Section 2.5, we make an explicit connection between our formalism and earlier work in the smooth case and we suggest a relationship between the orbifold vertex defined herein and the DT orbifold vertex defined in [8].

2.1. Preliminaries

2.1.1. Toric Calabi-Yau Orbifolds. By a *Calabi-Yau orbifold* we mean a smooth, quasi-projective Deligne-Mumford stack over $\mathbb{C}$ with trivial canonical class. We do not require the isotropy of the generic point to be trivial, but we will require that our orbifolds are Gorenstein (see below). A toric Calabi-Yau orbifold is defined to be such a stack with the action of a Deligne-Mumford torus $\mathcal{T} = T \times \mathcal{B}G$ having an open dense orbit isomorphic to $\mathcal{T}$ (cf. [27]). To a toric CY orbifold $\mathcal{X}$ of dimension three we can associate a planar trivalent graph $\Gamma_{\mathcal{X}} = \{\text{Edges, Vertices}\}$ where the vertices correspond to the torus fixed points and the edges correspond to the torus invariant lines. Following [8], we make the following definition.

DEFINITION 2.1. Let Γ be a trivalent planar graph with a chosen planar representation. An *orientation* of Γ is a choice of direction for each edge and an ordering of the edges incident to each vertex which is compatible with the counterclockwise cyclic ordering.

2.1.2. THE TARGET SPACE $[\mathbb{C}^3/G]$. We set up notation here that will be used throughout. Locally near a torus invariant point of a toric CY 3-fold, the space can be modelled as a global quotient $[\mathbb{C}^3/G]$ where G preserves the coordinates and acts trivially on the volume form (this is the Gorenstein condition mentioned above). We allow G to be any finite abelian group. More specifically, if $G = \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_l}$, then the action of G on $\mathbb{C}^3$ can be described with *weights* $(\vec{\alpha}^1, \vec{\alpha}^2, \vec{\alpha}^3) \in G^3$ summing to 0 where the generator ϵ_i of $\mathbb{Z}_{n_i}$ acts on the coordinates of $\mathbb{C}^3$ as

$$\epsilon_i \cdot (z_1, z_2, z_3) = (e^{\frac{2\pi\sqrt{-1}\alpha_i^1}{n_i}} z_1, e^{\frac{2\pi\sqrt{-1}\alpha_i^2}{n_i}} z_2, e^{\frac{2\pi\sqrt{-1}\alpha_i^3}{n_i}} z_3).$$

Define

$$g_i = \operatorname{lcm}\left\{ \frac{n_j}{\gcd\left(\alpha_j^i, n_j\right)} : j = 1, ..., l \right\}$$

Then $\mathbb{Z}_{g_i}$ is the effective part of the G action along the ith coordinate axis.

We define three Lagrangian suborbifolds $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ inside $[\mathbb{C}^3/G]$ as follows. We can view $[\mathbb{C}^3/G]$ as the neigborhood of the (image of) zero in the global quotient

$$[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/G] \tag{2–1}$$

where z_1 is the coordinate in the base direction. Define an anti-holomorphic involution on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

$$\sigma(z_1, z_2, z_3) = (1/\overline{z_1}, \overline{z_1 z_3}, \overline{z_1 z_2}).$$

One checks that σ descends to an involution σ_G on the quotient (2–1). $\mathcal{L}_1$ is defined to be the fixed locus of σ_G. $\mathcal{L}_2$ and $\mathcal{L}_3$ are defined analogously.

The GW orbifold vertex is defined in Definition 2.12 to be the oriented open GW potential of a formal neighborhood of the coordinate axes in $[\mathbb{C}^3/G]$, relative to the Lagrangian $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.

2.1.3. Toric Orbifold Lines. In order to prove the gluing formula in Section 2.3.4, we recall some basic facts about toric orbifold lines, ie. orbifolds with coarse space $\mathbb{P}^1$.

It follows from the classification theorem of [27] that any toric orbifold line (with finite abelian stabilizers) is an abelian gerbe over a football $\mathbb{P}^1_{n_0,n_\infty}$, and any such orbifold can be constructed via successive root constructions over the football. Recall that the nth root construction of a line bundle L on a space X is defined as the fibered product:

$$\begin{array}{ccc} X^{(L,n)} & \longrightarrow & \mathcal{B}\mathbb{C}^* \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\lambda\to\lambda^n} \\ X & \xrightarrow{L} & \mathcal{B}\mathbb{C}^* \end{array}$$

where the bottom map classifies the line bundle L. The top map classifies a line bundle M on $X^{(L,n)}$ with $M^{\otimes n} = \psi^* L$. We denote M by $L^{1/n}$ and refer to it as the nth root of L.

Generalizing this notion, if $L_1, ..., L_l$ are line bundles on X and $n_1, ..., n_l \in \mathbb{N}$, then $X^{(\mathbf{L_i},\mathbf{n_i})}$ can be defined as the fiber product:

$$\begin{array}{ccc} X^{(\mathbf{L_i},\mathbf{n_i})} & \longrightarrow & \mathcal{B}\mathbb{C}^* \times ... \times \mathcal{B}\mathbb{C}^* \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\lambda_i\to\lambda_i^{n_i}} \\ X & \longrightarrow & \mathcal{B}\mathbb{C}^* \times ... \times \mathcal{B}\mathbb{C}^* \end{array}$$

The orbifold Picard group of $X^{(\mathbf{L_i},\mathbf{n_i})}$ is generated by bundles pulled back from X via ψ along with the n_ith root of L_i.

We will be concerned with the case when X is the football $\mathbb{P}^1_{n_0,n_\infty}$. The line bundles on $\mathbb{P}^1_{n_0,n_\infty}$ are of the form

$$L := \mathcal{O}\left(\frac{a}{n_0}[0] + \frac{b}{n_\infty}[\infty] + c\right). \tag{2–2}$$

The *numerical degree* of L is defined to be the Chern-Weil class of the bundle (cf. [19]). In this case, we compute $\deg(L) = \frac{a}{n_0} + \frac{b}{n_\infty} + c$. We also make the following definition which will be useful in the computations of Section 2.3.

Definition 2.2. Any line bundle on $\mathbb{P}^1_{n_0,n_\infty}$ can be written uniquely in the form (2–2) with $0 \leq a \leq n_0 - 1$ and $0 \leq b \leq n_\infty - 1$. Given such a representation, we define $\mathrm{Int}(L) := c$.

Suppose that $L_i = \mathcal{O}\left(\frac{a_i}{n_0}[0] + \frac{b_i}{n_\infty}[\infty] + c_i\right)$. The line bundles on $\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{n_0,n_\infty}$ are of the form

$$\psi^*\left(\mathcal{O}\left(\frac{a}{n_0}[0] + \frac{b}{n_\infty}[\infty] + c\right)\right) \otimes \left(L_1^{1/n_1}\right)^{m_1} \otimes ... \otimes \left(L_l^{1/n_l}\right)^{m_l}. \tag{2–3}$$

We denote such a line bundle by $\mathcal{O}(a, b, c; m_1, ..., m_l)$. The degree of $\mathcal{O}(a, b, c; m_1, ..., m_l)$ is

$$\frac{a}{n_0} + \frac{b}{n_\infty} + c + \sum_{i=1}^{l} \frac{m_i}{n_i}\left(\frac{a_i}{n_0} + \frac{b_i}{n_\infty} + c_i\right).$$

An orbifold line bundle contains the information of a representation of the isotropy on the fibers. For example, in the case of the bundle (2–2) on $\mathbb{P}^1_{n_0,n_\infty}$, the generator of $\mathbb{Z}_{n_0}$ acts on the fiber over 0 as $\frac{a}{n_0}$ and the generator of $\mathbb{Z}_{n_\infty}$ acts on the fiber over ∞ as $\frac{b}{n_\infty}$ (we have identified S^1 with $\mathbb{R}/\mathbb{Z}$).

Following section 2.1.5 of [35], we now give a concrete description of the isotropy of $\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{n_0,n_\infty}$ as well as how the groups act on the fibers of the orbifold line bundle $\mathcal{O}(a, b, c; m_1, ..., m_l)$. Over any point other than 0 or ∞, the isotropy is simply $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_l}$ and the generator

of $\mathbb{Z}_{n_i}$ acts on the fiber of $\mathcal{O}(a,b,c;m_1,...,m_l)$ by $\frac{m_i}{n_i}$. Over 0, the isotropy group G_0 is an extension of $\mathbb{Z}_{n_0}$ by $\mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_l}$:

$$0 \to \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_l} \to G_0 \to \mathbb{Z}_{n_0} \to 0.$$

Similarly for the isotropy G_∞. The extension G_0 can be determined by the 2-cocycle $\gamma_0 : \mathbb{Z}_{n_0} \times \mathbb{Z}_{n_0} \to \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_l}$ defined by

$$\gamma_0(r,r') = \begin{cases} (a_1,...,a_l) & r+r' \geq n_0 \\ 0 & r+r' < n_0. \end{cases}$$

Explicitly, as a set G_0 consists of $(l+1)$-tuples (r,s) with $r \in \mathbb{Z}_{n_0}$, $s \in \mathbb{Z}_{n_1} \times ... \times \mathbb{Z}_{n_l}$. The operation is defined by

$$(r,s) + (r',s') := (r+r', s+s'+\gamma_0(r,r')).$$

On the fiber of $\mathcal{O}(a,b,c;m_1,...,m_l)$ over 0, we compute that $(1,(0,...,0)) \in G_0$ acts as

$$\frac{a}{n_0} + \sum_{i=1}^{l} \frac{m_i a_i}{n_i n_0}$$

while $(0,(0...,1_i,...,0)) \in G_0$ acts as

$$\frac{m_i}{n_i}.$$

When a rank 2 bundle N over $\mathbb{P}^{(L_i,n_i)}_{n_0,n_\infty}$ splits as the sum of two line bundles, the Calabi-Yau condition reduces to:

$$N = \mathcal{O}(a,b,c;m_1,...,m_k) \oplus \mathcal{O}(-a-1,-b-1,-c;-m_1,...,-m_l).$$

Finally, the following lemma gives us a characterization of when certain torus fixed maps from a football to $\mathbb{P}^{(L_i,n_i)}_{n_0,n_\infty}$ exist.

LEMMA 2.3. *Suppose $\mathcal{C}$ is an orbifold with coarse space $\mathbb{P}^1$ and orbifold structure only over 0 and ∞. Suppose $f : \mathcal{C} \to \mathbb{P}^{(L_i,n_i)}_{n_0,n_\infty}$ is a $\mathbb{C}^*$ fixed map of degree d twisted at 0 by $(k_0^0, (k_1^0, ..., k_l^0)) \in G_0$ and at ∞ by $(k_0^\infty, (k_1^\infty, ..., k_l^\infty)) \in G_\infty$. Then*

$$k_0^0 = k_0^\infty = d$$

and

$$\frac{dc_i}{n_i} - \frac{k_i^0}{n_i} - \frac{k_i^\infty}{n_i} \in \mathbb{Z} \qquad \forall i \geq 1.$$

PROOF. This is Lemmas II.12 and II.13 in [35]. □

In particular, once the degree and twisting at either 0 or ∞ are fixed, then the other twisting is determined.

2.1.4. OPEN GROMOV-WITTEN INVARIANTS. In [37], Katz and Liu propose a tangent obstruction theory for the moduli space $\overline{\mathcal{M}}_{g;h}(X, L|d; \gamma_1, ..., \gamma_h)$ of stable maps from h-holed Riemann surfaces into X with degree given by $d \in H_2(X, L; \mathbb{Z})$ and boundary conditions given by $\gamma_i \in H_1(L; \mathbb{Z})$ provided the following two conditions are met:

- L is the fixed locus of an anti-holomorphic involution $\sigma : X \to X$ and
- (X, L) can be equipped with a well-behaved $\mathbb{C}^*$ action with the real subtorus S^1 fixing L.

An important aspect of their theory is that varying the choice of torus action leads to a family of localized virtual fundamental classes.

In particular, if $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$, L is the fixed locus of the involution $\sigma : (z, u, v) \to (1/\bar{z}, \bar{z}\bar{v}, \bar{z}\bar{u})$, and we consider degree d maps from a disk (D^2, S^1), then the proposed virtual fundamental class is

$$e(R^1\pi_* f^* N(d)) \cap \overline{\mathcal{M}}_{0,1}(D^2, S^1|d)$$

where $N(d)$ is a Riemann-Hilbert bundle defined in Example 3.4.4 of [37]. Assuming that the virtual localization formula of [31] naturally generalizes to this setting, Katz and Liu suggested that the contribution of such a disk invariant to the open GW potential of (X, L) should be given by

$$\frac{t}{d^2} e^{eq}(-R^\bullet \pi_* f^*(T_X, T_L)) = \frac{t}{d^2} \frac{e^{eq} H^1(N(d))}{e^{eq} H^0(L(2d))}$$

where $L(2d)$ is defined in Example 3.4.3 of [37], the Euler classes are S^1 equivariant, and t is generator of $H^*_{S^1}(\{pt\})$. The motivation for $N(d)$ and $L(2d)$ is that their holomorphic doubles are precisely the bundles $\mathcal{O}_{\mathbb{P}^1}(-d) \oplus \mathcal{O}_{\mathbb{P}^1}(-d)$ and $\mathcal{O}_{\mathbb{P}^1}(2d)$, respectively. In other words, the Riemann-Hilbert bundles are 'half' of the bundles obtained by pulling back T_X along a degree d map to the base $\mathbb{P}^1$.

In order to generalize this setup to orbifolds, we define certain orbifold Riemann-Hilbert bundles in the next section which naturally generalize $L(2d)$ and $N(d)$.

2.1.5. Orbifold Riemann-Hilbert Bundles. We define two classes of Riemann-Hilbert bundles on the orbifold disks $(D_r, S^1) := [(D^2, S^1)/\mathbb{Z}_r]$ which play a crucial role in the computation of the open GW potential of $([\mathbb{C}^3/G], \mathcal{L})$.

2.1.5.1. *$L(m)$:* For $m > 0$, consider the orbifold line bundle $\mathcal{O}(m, m, 0)$ on the football $\mathbb{P}^1_{r,r}$. Let z, u be local coordinates near 0 on the coarse space. We define an anti-holomorphic

involution by:

$$\sigma : (z, u) \to (1/\overline{z}, -\overline{z}^{-2m/r}\overline{u})$$

We then define the Riemann-Hilbert bundle $L(m)$ on (D_r, S^1) by

$$L(m) := (\mathcal{O}(m, m, 0)_{|D_r}, \mathcal{O}(m, m, 0)^{\sigma}_{|S^1}).$$

The global sections of $L(m)$ are by definition the σ invariant global sections of $\mathcal{O}(m, m, 0)$. The global sections of $\mathcal{O}(m, m, 0)$ are generated by

$$\left\{ z^{\langle \frac{m}{r} \rangle + i} \right\}_{i=0}^{2\lfloor \frac{m}{r} \rfloor}.$$

Since

$$\sigma : \left(z, \alpha_i z^{\langle \frac{m}{r} \rangle + i}\right) \to \left(z, -\overline{\alpha}_i z^{2\lfloor \frac{m}{r} \rfloor + \langle \frac{m}{r} \rangle - i}\right),$$

then the global sections of $L(m)$ are:

$$\sum_{i=0}^{\lfloor \frac{m}{r} \rfloor - 1} \left(\alpha_i z^{\langle \frac{m}{r} \rangle + i} - \overline{\alpha}_i z^{2\lfloor \frac{m}{r} \rfloor + \langle \frac{m}{r} \rangle - i}\right) + \sqrt{-1}\beta z^{\frac{m}{r}} \tag{2–4}$$

with $\alpha_i \in \mathbb{C}$ and $\beta \in \mathbb{R}$. We can embed the sections (2–4) torus equivariantly into the global sections of $\mathcal{O}(m, 0, 0)$ by mapping them to

$$\sum_{i=0}^{\lfloor \frac{m}{r} \rfloor - 1} \alpha_i z^{\langle \frac{m}{r} \rangle + i} + \beta z^{\frac{m}{r}}.$$

Remark 2.4. This natural choice of identification determines an orientation for the sections.

2.1.5.2. *$N(m,n,l)$:* Given $m,n \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Q}_{\geq 0}$ with $-l+\frac{n}{r} \in \mathbb{Z}$, consider the rank 2 bundle

$$N_1 \oplus N_2 = \mathcal{O}(m,-m-n,-l+\frac{n}{r}) \oplus \mathcal{O}(-m-n,m,-l+\frac{n}{r})$$

on $\mathbb{P}^1_{r,r}$. There is an anti-holomorphic involution

$$\sigma : (z,u,v) \to (1/\overline{z}, \overline{z}^l \overline{v}, \overline{z}^l \overline{u}).$$

We define the Riemann-Hilbert bundle $N(m,n,l)$ on (D_r, S^1) by:

$$N(m,n,l) := \left((N_1 \oplus N_2)_{|_{D_r}}, (N_1 \oplus N_2)^{\sigma}_{|_{S^1}} \right).$$

The H^1 sections of $N(m,n,l)$ are by definition the σ invariant H^1 sections of $N_1 \oplus N_2$. The H^1 sections of $N_1 \oplus N_2$ are generated by

$$\left\{ \left(z^{\langle \frac{m}{r} \rangle - i}, z^{\langle \frac{-m-n}{r} \rangle - j} \right) \right\}_{i,j=1}^{l+\langle \frac{m}{r} \rangle + \langle \frac{-m-n}{r} \rangle - 1}.$$

Since

$$\sigma : \left(z, \alpha_i z^{\langle \frac{m}{r} \rangle - i}, \beta_j z^{\langle \frac{-m-n}{r} \rangle - j} \right) \to \left(z, \overline{\beta}_j z^{-l-\langle \frac{-m-n}{r} \rangle + j}, \overline{\alpha}_i z^{-l-\langle \frac{m}{r} \rangle + i} \right),$$

the σ invariant sections are:

$$\left(\sum_{i=1}^{l+\langle \frac{m}{r} \rangle + \langle \frac{-m-n}{r} \rangle - 1} \frac{\alpha_i}{z^{i-\langle \frac{m}{r} \rangle}}, \sum_{i=1}^{l+\langle \frac{m}{r} \rangle + \langle \frac{-m-n}{r} \rangle - 1} \frac{\overline{\alpha}_i}{z^{l+\langle \frac{m}{r} \rangle - i}} \right)$$

Remark 2.5. In order to compute the equivariant Euler class of these bundles, we face the choice of embedding the sections into either N_1 or N_2. We will denote the corresponding Euler classes by $e(H^1(N(m,n,l)))^+$ or $e(H^1(N(m,n,l)))^-$, respectively.

2.1.6. Tautological Classes. We recall the definitions of some natural classes on $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$, the moduli stack of (orbifold) stable maps to the classifying stack $\mathcal{B}G$. For $\gamma = (\gamma_1, ..., \gamma_n)$ an n-tuple of elements in G, we denote by $M_{g,\gamma}(\mathcal{B}G)$ the open and closed substack $M_{g,n}(\mathcal{B}G) \cap \bigcap_{i=1}^{n} ev_i^*(\mathbb{1}_{\gamma_i})$ where $\mathbb{1}_{\gamma_i}$ is the fundamental class of the component in the inertia stack indexed by γ_i.

By the definition of $\mathcal{BG}$, $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ parametrizes degree $|G|$ covers of the source curve, ramified over the twisted points, with an action of G which exhibits the source curve as a quotient of the cover. Let

$$p : \mathcal{U}_h \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$$

be the universal covering curve of genus h where h is computed via the Riemann-Hurwitz formula. The Hodge bundle on $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ is the rank h bundle defined by

$$\mathbb{E} := p_* \omega_h$$

where ω_h is the relative dualizing sheaf of p. G naturally acts on $\mathbb{E}$ and its dual $\mathbb{E}^\vee$. For any $\zeta \in G$, we define $\mathbb{E}_\zeta$ and $\mathbb{E}_\zeta^\vee$ to be the ζ-eigenbundles of $\mathbb{E}$ and $\mathbb{E}^\vee$, respectively. They are related by the formula $(\mathbb{E}_\zeta)^\vee = \mathbb{E}^\vee_{\zeta^{-1}}$. We also have the formula

$$\mathbb{E}^\vee_{\zeta^{-1}} = R^1 \pi_* f^* \mathcal{O}_\zeta$$

where π is the map from the universal curve, f is the universal map, and $\mathcal{O}_\zeta$ is the line bundle with isotropy acting by multiplication by ζ. The lambda classes are defined as the chern classes of these bundles:

$$\lambda_j^\zeta := c_j(\mathbb{E}_\zeta)$$

For later convenience we introduce the notation

$$\Lambda^{\xi}(t) := (-1)^{\mathrm{rk}} \sum_{j=0}^{\mathrm{rk}} (-t)^{\mathrm{rk}-j} \lambda_j^{\xi}$$

with $\mathrm{rk} := \mathrm{rk}(\mathbb{E}_\zeta)$.

By forgetting the orbifold structure of the curve, there is a universal coarse curve

$$q : \mathcal{U}_{g,|\gamma|+|\mu|} \to \overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$$

along with a section s_p for each marked point p. We define the cotangent line bundles by

$$\mathbb{L}_p := s_p^* \omega_g$$

where ω_g is the relative dualizing sheaf of q. The psi classes on $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{B}G)$ are defined by

$$\psi_p := c_1(\mathbb{L}_p)$$

2.2. The Orbifold Vertex

In this section, the orbifold vertex is defined via localization. The result is an expression in terms of G-Hodge integrals on $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G)$ and a combinatorial disk function.

2.2.1. Torus Action and Fixed Loci.

Refer to Figure 2.1 throughout this section.

Set $\mathcal{X} := [\mathbb{C}^3/G]$ and choose an orientation for $\Gamma_{\mathcal{X}}$ consistent with the labelling (x_1, x_2, x_3) of the coarse coordinates for $\mathcal{X}$. To equip $\mathcal{X}$ with a $\mathbb{C}^*$ action, begin with an action on $T\mathbb{C}^3$ having CY weights

$$\vec{w} := (w_1, w_2, w_3) \text{ with } w_1 + w_2 + w_3 = 0.$$

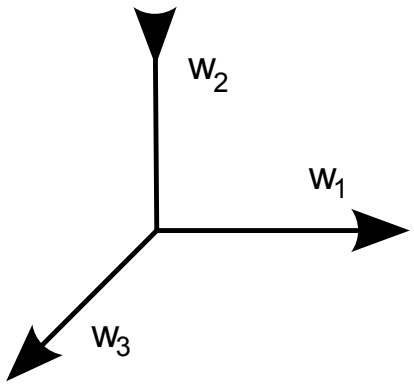


FIGURE 2.1. The toric diagram $\Gamma_{\mathcal{X}}$. The labelling of the weights coincides with the counterclockwise orientation of the coordinates. The orientations of the edges show that disk invariants are computed with D^+ on the 1st and 3rd coordinates and D^- on the 2nd.

This action descends to the quotient and the corresponding weights on the coordinates of the coarse space are g_1w_1, g_2w_2, g_3w_3. One checks that $S^1 \subset \mathbb{C}^*$ fixes each $\mathcal{L}_i$.

Pulling the torus action back to the moduli space of stable bordered maps, the S^1 fixed loci consist of maps $f : \Sigma \to \mathcal{X}$ such that

- Σ is a compact curve attached to some number of disks at possibly twisted nodes,
- f contracts the compact curve to the origin,
- f maps a disk onto the ith axis with local expression $z \to z^d$ (d is the winding number) and with boundary mapping to

$$\mathcal{L}_i \cap \{x_j = 0 | j \neq i\} \cong S^1.$$

Such a fixed locus can be encoded with the datum of:

- The genus of the contracted curve,
- Winding profiles: Three vectors of positive integers, $\vec{d^i} = (d^i_1, ..., d^i_{l_i})$ $i = 1, 2, 3$, corresponding to l_i disks mapping to $\mathcal{L}_i$ with windings determined by d^i_j, and

- Twisting profiles: Three vectors of elements of G, $\vec{k^i} = (k_1^i, ..., k_{l_i}^i)$ $i = 1, 2, 3$, corresponding to the twisting of the nodes attaching the corresponding disk to the contracted curve.

In order to more easily track such a locus, set $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$ where

$$\mu^i = \left\{(d_1^i, k_1^i), ..., (d_{l_i}^i, k_{l_i}^i)\right\}. \tag{2–5}$$

REMARK 2.6. μ^i can naturally be identified with a conjugacy classes in the wreath product $G \wr S_d$ with $d = \sum_j d_j^i$ (cf. [44]).

We denote such a fixed locus by $F(g, \vec{\mu})$. $F(g, \vec{\mu})$ can be identified with a product of a zero dimensional stack and $\overline{\mathcal{M}}_{g,-\vec{k}}(\mathcal{B}G) := \overline{\mathcal{M}}_{g,n}(\mathcal{B}G) \cap ev^*(-\vec{k})$ where $n = l_1 + l_2 + l_3$ and $ev^*(-\vec{k})$ is shorthand for pulling back all of the $-k_j^i$-twisted points via the appropriate evaluation maps. The zero-dimensional stack encodes automorphisms of the fixed loci which will be accounted for in Section 2.2.2

The next lemma gives a condition on each disk which is necessary for the above locus to be nonempty (cf. [6], section 2.2.2).

LEMMA 2.7. *Suppose $h : (D_r, S^1) \to (\mathcal{X}, \mathcal{L}_i)$ is a $\mathbb{C}^*$ fixed winding d map, twisted by $k = (k_1, ..., k_l) \in G$. Then*

$$\frac{-d}{g_i} + \sum_j \frac{k_j \alpha_j^i}{n_j} \in \mathbb{Z}.$$

PROOF. Doubling the map h, we get a degree d map $\hat{h} : \mathbb{P}^1_{r,r} \to [\mathcal{O}(-1) \oplus \mathcal{O}(-1)/G]$ which is twisted by k at 0 and $-k$ at ∞. The target is a rank 2 bundle $N_1 \oplus N_2$ over $[\mathbb{P}^1/G]$.

Consider the bundle $\hat{h}^* N_1$ on $\mathbb{P}^1_{r,r}$. Define

$$r_i := r \sum_j \frac{k_j \alpha_j^i}{n_j}$$

Then the generator of the isotropy on $\mathbb{P}^1_{r,r}$ at 0 acts on the fiber of the pullback by $\frac{r_{i+1}}{r}$ and the generator of the isotropy at ∞ acts by $\frac{-r_i - r_{i+1}}{r}$. Furthermore, the degree of $\hat{h}^* N_1$ is $\frac{-d}{g_i}$. Since the isotropy at 0 contributes a fractional part of $\langle \frac{r_{i+1}}{r} \rangle$ to the degree and the isotropy at ∞ contributes a fractional part of $\langle \frac{-r_i - r_{i+1}}{r} \rangle$ to the degree, we see that

$$\frac{-d}{g_i} - \left\langle \frac{r_{i+1}}{r} \right\rangle - \left\langle \frac{-r_i - r_{i+1}}{r} \right\rangle \in \mathbb{Z}. \tag{2–6}$$

The result follows from (2–6). □

2.2.2. The Obstruction Theory. In this section, we give a precise formula for the restriction of the obstruction theory to a fixed locus in terms of the combinatorial data of that locus. We parse the contributions into three components.

- **Compact Curve**: A contracting compact curve contributes a Hodge part:

$$\Lambda^{\vec{\alpha}^1}(w_1)\Lambda^{\vec{\alpha}^2}(w_2)\Lambda^{\vec{\alpha}^3}(w_3) := e^{eq}\left(\mathbb{E}^\vee_{\vec{\alpha}^1}(w_1) \oplus \mathbb{E}^\vee_{\vec{\alpha}^2}(w_2) \oplus \mathbb{E}^\vee_{\vec{\alpha}^3}(w_3)\right)$$

  where $\mathbb{E}^\vee_{\vec{\alpha}^i}(w_i)$ denotes the dual of the α^i-character sub-bundle twisted by the torus weight w_i. The contracting curve also contributes a factor of w_i for each direction in which the curve can be perturbed, ie. if the contraction to $\mathcal{B}G$ factors through $\mathcal{B}G_i$ where G_i is the isotropy along the ith axis. We denote this contribution by δ.

- **Nodes**: Each node contributes a gluing factor of $|G|$ explained in Section 2.1 of [13]. There is a contribution of w_i for each direction which the node can be perturbed. There

appears a node smoothing contribution of

$$\left(\frac{g_i w_i}{d_j^i} - \psi_j^i\right)^{-1}.$$

There is a term

$$\left(\frac{w_i g_i}{d_j^i}\right)^{-\delta_{0,k_j^i}}$$

to cancel the infinitesimal automorphism at the origin of the disk. There is also a contribution of w_l for each direction which the node can be perturbed. This contributions is

$$\delta(k_j^i) := \prod\{w_l : k_j^i \in G_l\}.$$

- **Disks**: A disk mapping to the ith leg with winding d and twisting $k \in G$ contributes a combinatorial factor of $D_k^{\sigma_i}(d; \vec{w})$ which is described in the next section.

REMARK 2.8. We have abused notation at this point. Having previously defined the w_i to be numbers we are now treating them as equivariant cohomology classes. We have done this to ease notation and one can simply interpret the w_i in this context as $w_i t$ where t is the generator of $H^*_{S^1}(\{pt\})$.

2.2.3. THE DISK FUNCTION. Suppose $h : (D_r, S^1) \to (\mathcal{X}, \mathcal{L}_i)$ is a $k = (k_1, ..., k_l)$ twisted, winding d map from an orbidisk. As in Lemma (2.3), define integers

$$r_i := r \sum_j \frac{k_j \alpha_j^i}{n_j} \text{ for } i = 1, 2, 3.$$

Then the generator of the isotropy of D_r acts on the fiber of the pullback of the three normal directions to the origin in $\mathcal{X}$ as $\frac{r_i}{r}$.

The disk has $d \cdot \frac{|G|}{g_i}$ global automorphisms and an infinitesimal automorphism factor $\frac{w_i g_i}{d}$. We define

$$D_k^{\sigma_i}(d;\vec{w}) := \left(\frac{g_i w_i}{d}\right)^{\delta_{0,k}} \frac{g_i}{d|G|} \frac{e^{eq}\left(H^1 N\left(r_{i+1}, r_i, \frac{d}{g_i}\right)\right)^{\sigma_i}}{e^{eq}\left(H^0 L\left(\frac{rd}{g_i}\right)\right)}$$

where

$$\sigma_i := \begin{cases} + & \text{if the } i\text{th leg is oriented outward,} \\ - & \text{if the } i\text{th leg is oriented inward.} \end{cases}$$

REMARK 2.9. As a notational convenience, we define $r_4 := r_1$ and $r_5 := r_2$ to reflect the cyclic labeling of the coordinates.

REMARK 2.10. We can define $N\left(r_{i+1}, r_i, \frac{d}{g_i}\right)$ (ie. it satisfies the hypothesis of Example 2.1.5.2) due to Lemma 2.7.

Making the identifications of Example 2.1.5.1, we see that

$$\begin{aligned} e^{eq}\left(H^0 L\left(\frac{rd}{g_i}\right)\right) &= \prod_{j=0}^{\left\lfloor \frac{d}{g_i} \right\rfloor - 1} \left(w_i - \frac{g_i w_i}{d}\left(\left\langle \frac{d}{g_i} \right\rangle + j\right)\right) \\ &= \prod_{j=1}^{\left\lfloor \frac{d}{g_i} \right\rfloor} \frac{g_i w_i}{d} j \\ &= \left(\frac{g_i w_i}{d}\right)^{\left\lfloor \frac{d}{g_i} \right\rfloor} \left\lfloor \frac{d}{g_i} \right\rfloor ! \end{aligned}$$

where we have left out the weight 0 contribution from $h^*(\frac{\partial}{\partial z_i})$ as usual (cf. section 27.4 of [33]). Choosing the 'positive' orientation of sections discussed in Example 2.1.5.2, we

compute

$$e^{eq}\left(H^1N\left(r_{i+1},r_i,\frac{d}{g_i}\right)\right)^+ = \prod_{j=1}^{\frac{d}{g_i}+\left\langle\frac{r_{i+1}}{r}\right\rangle+\left\langle\frac{r_{i+2}}{r}\right\rangle-1}\left(w_{i+1}+\frac{g_iw_i}{d}\left(j-\left\langle\frac{r_{i+1}}{r}\right\rangle\right)\right)$$

$$= \left(\frac{g_iw_i}{d}\right)^{\frac{d}{g_i}+\left\langle\frac{r_{i+1}}{r}\right\rangle+\left\langle\frac{r_{i+2}}{r}\right\rangle-1}\frac{\Gamma\left(\frac{dw_{i+1}}{g_iw_i}+\left\langle\frac{r_{i+2}}{r}\right\rangle+\frac{d}{g_i}\right)}{\Gamma\left(\frac{dw_{i+1}}{g_iw_i}-\left\langle\frac{r_{i+1}}{r}\right\rangle+1\right)}.$$

One can check (using $w_1+w_2+w_3=0$) that the 'negative' orientation of the sections merely introduces a factor of

$$(-1)^{\frac{d}{g_i}+\left\langle\frac{r_{i+1}}{r}\right\rangle+\left\langle\frac{r_{i+2}}{r}\right\rangle-1}.$$

Putting this all together, we compute

$$D_k^+(d;\vec{w}) = \left(\frac{g_iw_i}{d}\right)^{\mathrm{age}(k)+\delta_{0,k}-1}\frac{g_i}{d|G|\left\lfloor\frac{d}{g_i}\right\rfloor!}\frac{\Gamma\left(\frac{dw_{i+1}}{g_iw_i}+\left\langle\frac{r_{i+2}}{r}\right\rangle+\frac{d}{g_i}\right)}{\Gamma\left(\frac{dw_{i+1}}{g_iw_i}-\left\langle\frac{r_{i+1}}{r}\right\rangle+1\right)}.$$

REMARK 2.11. This generalizes the disk function defined in Section 2.2.3 of [6].

2.2.4. THE ORBIFOLD VERTEX. We now put together the contributions coming from the compact curve, the nodes, and the disks. We assign formal variables to track the discrete invariants. We let λ track the Euler characteristic $2g-2+n$. For $\vec{\mu}$ as in (2–5), define the formal variables

$$\mathbf{p}_{\vec{\mu}} := p^1_{\mu^1}\cdot p^2_{\mu^2}\cdot p^3_{\mu^3}$$

with formal multiplication given by

$$\left(p^1_{\mu^1}\cdot p^2_{\mu^2}\cdot p^3_{\mu^2}\right)\cdot\left(p^1_{\nu^1}\cdot p^2_{\nu^2}\cdot p^3_{\nu^2}\right) := \left(p^1_{\mu^1\cup\nu^1}\cdot p^2_{\mu^2\cup\nu^2}\cdot p^3_{\mu^2\cup\nu^3}\right).$$

DEFINITION 2.12. Define

$$V_{\mathcal{X},g,\vec{\mu}}(\vec{w}) := \frac{\prod\left(|G|\delta(k_j^i)D_{k_j^i}^{\sigma_i}(d_j^i;\vec{w})\right)}{\delta\cdot|\mathrm{Aut}(\vec{\mu})|}\int_{\overline{\mathcal{M}}_{g,-\vec{k}}(\mathcal{B}G)}\frac{\Lambda^{\vec{\alpha}^1}(w_1)\Lambda^{\vec{\alpha}^2}(w_2)\Lambda^{\vec{\alpha}^3}(w_3)}{\prod\left(\frac{g_iw_i}{d_j^i}-\psi_j^i\right)}$$

where

$$D_k^+(d;\vec{w}) = \left(\frac{g_iw_i}{d}\right)^{\mathrm{age}(k)-1}\frac{g_i}{d|G|\left\lfloor\frac{d}{g_i}\right\rfloor!}\frac{\Gamma\left(\frac{dw_{i+1}}{g_iw_i}+\left\langle\frac{r_{i+2}}{r}\right\rangle+\frac{d}{g_i}\right)}{\Gamma\left(\frac{dw_{i+1}}{g_iw_i}-\left\langle\frac{r_{i+1}}{r}\right\rangle+1\right)}$$

and

$$D_k^-(d;\vec{w}) = (-1)^{\frac{d}{g_i}+\left\langle\frac{r_{i+1}}{r}\right\rangle+\left\langle\frac{r_{i+2}}{r}\right\rangle-1}D_k^+(d;\vec{w}).$$

We define the *connected open GW potential* of $\mathcal{X}$ by

$$V_{\mathcal{X}}(\lambda,\mathbf{p};\vec{w}) := \sum_{g,\vec{\mu}}V_{\mathcal{X},g,\vec{\mu}}(\vec{w})\lambda^{2g-2+n}\mathbf{p}_{\vec{\mu}}.$$

We define the *disconnected open GW potential* of $\mathcal{X}$ by

$$V_{\mathcal{X}}^\bullet(\lambda,\mathbf{p};\vec{w}) := \exp\left(V_{\mathcal{X}}(\lambda,\mathbf{p};\vec{w})\right).$$

Finally, we define the *oriented GW orbifold topological vertex* $V_{\mathcal{X},\vec{\mu}}^\bullet(\lambda;\vec{w})$ to be the coefficient of $\mathbf{p}_{\vec{\mu}}$ in $V_{\mathcal{X}}^\bullet(\lambda,\mathbf{p};\vec{w})$.

REMARK 2.13. If the moduli space in the above definition does not exist, we set the contribution equal to zero except in two exceptional cases where we make the following conventions for the unstable integrals. We set

$$\int_{\overline{\mathcal{M}}_{0,1}(\mathcal{B}G)\cap ev^*(0)}\frac{1}{a-\psi} := \frac{a}{|G|}$$

and

$$\int_{\overline{\mathcal{M}}_{0,2}(\mathcal{B}G)\cap ev_i^*(k_i)} \frac{1}{(a_1-\psi_1)(a_2-\psi_2)} := \begin{cases} \frac{1}{|G|(a_1+a_2)} & \text{if } k_1=-k_2\in G \\ 0 & \text{else.} \end{cases}$$

2.3. Gluing Algorithm

In this section, we describe an algorithm for gluing open amplitudes. We prove that the GW orbifold vertex and the gluing algorithm determine the GW invariants of any toric CY orbifold of dimension 3.

2.3.1. The Geometric Setup. Let $\mathcal{Y}$ be a toric Calabi-Yau orbifold of dimension 3 and $\Gamma_{\mathcal{Y}}$ the corresponding toric diagram with a chosen orientation as in Definition 2.1. Let $V := \{v_i\}$ be the set of vertices in Γ and let G_i be the isotropy group at the corresponding point $y_i \in \mathcal{Y}$. We set $E^c := \{e_{ij}\}$ to be the compact edges in $\Gamma_{\mathcal{Y}}$ directed from v_i to v_j. Let $\mathcal{C}_{ij}$ denote the corresponding line connecting y_i to y_j. If $e \in E^c$, let C_e be the corresponding curve in $\mathcal{Y}$, let $D_e \in H^2(\mathcal{Y},\mathbb{Q})$ be the dual of $[C_e] \in H_2(\mathcal{Y},\mathbb{Q})$ under the natural pairing, and let G_e be the isotropy over C_e. Define $N_{e,r}$ ($N_{e,l}$) to be the orbifold line bundle on C_e corresponding to the toric divisor to the right (left) of e.

Choose a Calabi-Yau $\mathbb{C}^*$ action on $\mathcal{Y}$. Let $\mathcal{Y}_i$ be a neighborhood of y_i so that $\mathcal{Y}_i \cong [\mathbb{C}^3/G_i]$. Let (x_1^i, x_2^i, x_3^i) be the coarse coordinates at $\mathcal{Y}_i$ so that the cyclic ordering of the coordinates coincides with the orientation of $\Gamma_{\mathcal{Y}}$ at y_i. These coordinates inherit an orientation and a $\mathbb{C}^*$ action from $\mathcal{Y}$, we label the weights of the action $\vec{w}(i) := (w_1^i, w_2^i, w_3^i)$. We define Lagrangians in $\mathcal{Y}_i$ as in Section 2.2. This gives us everything we need to compute amplitudes as in Definition 2.12.

2.3.2. Edge Assignments. We now define the edge assignments which are permitted by the geometry of the orbifold at each torus invariant line.

DEFINITION 2.14. We say that the triple $(d, k, k') \in \mathbb{N} \times G_i \times G_{i'}$ is *admissible with respect to* $e_{ii'}$ if the map

$$f : \mathbb{P}^1_{r,s} \to \mathcal{C}_{ii'}$$

given by $z \to z^d$ and twisted by k at y_i and k' at $y_{i'}$ exists. In this case, define

$$n(e, d, k, k') := \mathrm{Int}(f^* N_{e,r}) + 1$$

(cf. Definition 2.2).

REMARK 2.15. Lemma 2.3 gives a characterization of admissible triples.

DEFINITION 2.16. An *edge assignment* for $\Gamma_{\mathcal{Y}}$ is a finite set of admissible triples for each $e \in E^c$.

Let $A_{\mathcal{Y}}$ be the set of edge assignments. For $\Lambda \in A_{\mathcal{Y}}$, $e \in E^c$, define Λ_e to be the set of admissible triples over e. Define

$$n(\Lambda_e) := \sum_{\Lambda_e} n(e, d, k, k').$$

Define $\rho^k_d(\Lambda_e)$ to be the number of times the triple (d, k, k') appears in Λ_e. Also define

$$d(\Lambda_e) := \sum_{\Lambda_e} d.$$

Finally, define Λ_i to be the induced triple of twisting/winding profiles at y_i.

2.3.3. GLUING ALGORITHM. We are now ready to state the main algorithm for gluing the orbifold vertex.

THEOREM 2.17. *The Gromov-Witten potential of $\mathcal{Y}$ is*

$$GW^\bullet(\mathcal{Y}) := \sum_{\Lambda \in A_\mathcal{Y}} \prod_{v_i \in V} V^\bullet_{\mathcal{Y}_i,\Lambda_i}(\lambda; \vec{w}(i)) \prod_{e \in E^c} \prod_{d,k} (d|G_e|)^{\rho^d_k(\Lambda_e)} \rho^d_k(\Lambda_e)! (-1)^{n(\Lambda_e)} Q_e^{d(\Lambda_e)}$$

where Q_e are formal variables tracking the degree and we impose $Q_e = Q'_e$ if $[C_e] = [C'_e] \in H_2(\mathcal{Y})$.

REMARK 2.18. It is a consequence of the theorem that $GW^\bullet(\mathcal{Y})$ is independent of the choices of orientation and torus action.

REMARK 2.19. The obvious extension of this algorithm can be used to define/compute the open GW potential of $\mathcal{Y}$ with inner and/or outer branes. These potentials will depend on the choices of orientation and torus action.

2.3.4. PROOF OF GLUING FORMULA. By localization arguments, it need only be checked that disk contributions glue to multiple cover contributions on a given edge. Using the notation of Section 2.1.3, a given edge is isomorphic to $\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}$ with isotropy group G_0 at 0 and G_∞ at ∞. To prove the gluing formula, we compute the contribution to the potential from a representable $\mathbb{C}^*$ fixed map from a football $f : \mathcal{F} \to \mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s} \subset \mathcal{Y}$ twisted by $\vec{k}^0 = (d, (k^0_1, ..., k^0_l)) \in G_0$ at 0 and $\vec{k}^\infty = (d, (k^\infty_1, ..., k^\infty_l)) \in G_\infty$ at ∞. We show that this contribution decomposes as the corresponding disk contributions along with the prescribed gluing factor.

Since $\mathcal{Y}$ is a toric Calabi-Yau 3-fold, the normal bundle splits (cf. Section 2.1.3) as

$$\begin{aligned} N_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}/\mathcal{Y}} &= N_l \oplus N_r \\ &= \mathcal{O}(a, b, c; m_1, ..., m_l) \oplus \mathcal{O}(-a-1, -b-1, -c; -m_1, ..., -m_l). \end{aligned}$$

The total space of the normal bundle inherits an orientation and a Calabi-Yau $\mathbb{C}^*$ action from $\mathcal{Y}$. We label the oriented weights of this action (on $\mathbb{C}^3$) by w_1, w_2, w_3 at 0 and

$$w_1' := \frac{-rw_1}{s},$$

$$w_2' := w_3 - rw_1\left(\frac{-a-1}{r} + \frac{-b-1}{s} - c - \sum m_i\left(\frac{a_i}{rn_i} + \frac{b_i}{sn_i} + \frac{c_i}{n_i}\right)\right),$$

and

$$w_3' := w_2 - rw_1\left(\frac{a}{r} + \frac{b}{s} + c + \sum m_i\left(\frac{a_i}{rn_i} + \frac{b_i}{sn_i} + \frac{c_i}{n_i}\right)\right)$$

at ∞. Near $\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s} \subset \mathcal{Y}$, $\Gamma_{\mathcal{Y}}$ can be decorated as in Figure 2.2 to account for the orientation and the $\mathbb{C}^*$ action.

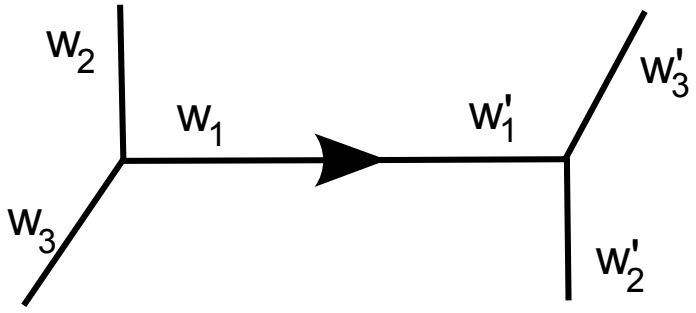


FIGURE 2.2. The neighborhood of $\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s} \subset \mathcal{Y}$

Using the usual obstruction theory for local invariants and the virtual localization formula, the contribution of the map f is

$$\frac{w_1 w_1'}{\tau_f \tau_f' d^3 n} \frac{e^{eq}\left(-R^\bullet \pi_* f^* N_l\right) \cdot e^{eq}\left(-R^\bullet \pi_* f^* N_r\right)}{e^{eq}\left(R^0 \pi_* f^* T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}}\right)} \tag{2–7}$$

where $n = \sum n_i$ and as before τ_f and τ_f' cancel the infinitesimal automorphism if 0 and/or ∞ is either marked or a node.

REMARK 2.20. In the computations that follow, we make the following index and product convention for *decreasing* indices: Suppose $m < n$, set

$$\{A_i\}_{i=n}^{m} = \{A_{m+1}, A_{m+2}, ..., A_{n-1}\},$$

ie. we forget the first and last terms if the index decreases. Similarly, we set

$$\prod_{i=n}^{m} A_i = (A_{m+1} \cdot A_{m+2} \cdot ... \cdot A_{n-1})^{-1}.$$

2.3.4.1. *Numerator of (2–7).* We begin by computing $e^{eq}(R^\bullet \pi_* f^* N_l)$. Suppose s_0 is a minimally vanishing section of $f^* N_1$ on the complement of ∞ and s_∞ is a minimally vanishing section on the complement of 0. If z is a local coordinate of $\mathcal{F}$ at 0, then $s_\infty = s_0 z^M$ where

$$M := \mathrm{Int}(f^* N_l) = \deg(f^* N_l) - \mathrm{ord}_0(s_0) - \mathrm{ord}_\infty(s_\infty).$$

Therefore the vector space $H^*(\mathcal{F}, f^* N_l)$ is generated by $\{s_0 z^p\}_{p=0}^{M}$ with equivariant weights

$$\left\{ W_1(p) := \left(w_2 - \frac{r w_1}{d} \left(\mathrm{ord}_0(s_0) + p \right) \right) \right\}_{p=0}^{M}.$$

Similarly, if we let t_0 and t_∞ be minimally vanishing sections of $f^* N_r$, then $t_\infty = t_0 z^M$ where

$$N := \mathrm{Int}(f^* N_r) = \deg(f^* N_r) - \mathrm{ord}_0(t_0) - \mathrm{ord}_\infty(t_\infty).$$

Therefore the vector space $H^*(\mathcal{F}, f^* N_r)$ is generated by $\{t_0 z^q\}_{q=0}^{N}$ with equivariant weights

$$\left\{ W_2(q) := \left(w_3 - \frac{r w_1}{d} \left(\mathrm{ord}_0(t_0) + q \right) \right) \right\}_{q=0}^{N}.$$

Multiplying the classes, we get

$$e^{eq}\left(R^{\bullet}\pi_* f^* N_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}/\mathcal{Y}}\right) = \prod_{p=0}^{M} W_1(p) \prod_{q=0}^{N} W_2(q).$$

Define positive integers

$$\hat{r} := \frac{d}{r} + \mathrm{ord}_0(s_0) + \mathrm{ord}_0(t_0)$$

and

$$\hat{s} := \frac{d}{s} + \mathrm{ord}_\infty(s_\infty) + \mathrm{ord}_\infty(t_\infty).$$

Using the Calabi-Yau condition on the weights, ie. $w_1 + w_2 + w_3 = 0$, one easily computes:

$$-W_2(-p-\hat{r}) = W_1(p). \tag{2–8}$$

Therefore we can write

$$e^{eq}\left(R^{\bullet}\pi_* f^* N_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}/\mathcal{Y}}\right) = (-1)^{N+1} \prod W_1(p) \tag{2–9}$$

where the product is indexed by

$$0 \longrightarrow M \qquad \text{and} \qquad -\hat{r} - N \longrightarrow -\hat{r}.$$

Recalling the index convention of Remark 2.20 and using the fact that $M + N + \hat{r} + \hat{s} = 0$ (this depends on the CY condition), we see that many of the terms in (2–9) cancel and the remaining terms are indexed by

$$0 \longrightarrow -\hat{r} \qquad \text{and} \qquad M + \hat{s} \longrightarrow M.$$

Making the appropriate cancellations and inverting, we compute

$$e^{eq}\left(-R^\bullet\pi_* f^* N_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}/\mathcal{Y}}\right) = (-1)^{N+1}\prod_{p=-\hat{r}+1}^{-1} W_1(p)\prod_{p=M+1}^{M+\hat{s}-1} W_1(p)$$
$$= (-1)^{N+1}\prod_{p=1}^{\hat{r}-1} W_1(-p)\prod_{p=1}^{\hat{s}-1} W_1(p+M)$$

Using the definition of W_1, we conclude that the numerator of (2–7) is

$$e^{eq}\left(-R^\bullet\pi_* f^* N_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}/\mathcal{Y}}\right) = (-1)^{N+1}\left(\frac{rw_1}{d}\right)^{\hat{r}-1}\frac{\Gamma\left(\frac{dw_2}{rw_1}+\mathrm{ord}_0(t_0)+\frac{d}{r}\right)}{\Gamma\left(\frac{dw_2}{rw_1}-\mathrm{ord}_0(s_0)+1\right)}$$
$$\cdot\left(\frac{sw_1'}{d}\right)^{\hat{s}-1}\frac{\Gamma\left(\frac{dw_3'}{sw_1'}+\mathrm{ord}_\infty(t_\infty)+\frac{d}{s}\right)}{\Gamma\left(\frac{dw_3'}{sw_1'}-\mathrm{ord}_\infty(s_\infty)+1\right)}. \tag{2–10}$$

2.3.4.2. *Denominator of (2–7).* Lastly, we must compute $e^{eq}\left(R^0\pi_* f^* T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}}\right)$. Because $T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}} = \mathcal{O}(1,1,0;0,...,0)$, we have $f^*T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}} = \mathcal{O}(d,d,0)$. If v_0 is a minimally vanishing section at 0, then $H^0(\mathcal{F}, f^*T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}})$ is generated by

$$\left\{v_0 z^j\frac{\partial}{\partial z}\right\}_{j=0}^{\lfloor\frac{d}{r}\rfloor+\lfloor\frac{d}{s}\rfloor}$$

where z has weight $-\frac{rw_1}{d}$, v_0 has weight $-\frac{rw_1}{d}\left\langle\frac{d}{r}\right\rangle$ and $T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}}$ is linearized with w_1, w_1'.

Therefore the denominator of (2–7) is

$$e^{eq}\left(R^0\pi_* f^* T_{\mathbb{P}^{(\mathbf{L_i},\mathbf{n_i})}_{r,s}}\right) = \prod_{j=0}^{\lfloor\frac{d}{r}\rfloor-1}\prod_{j=\lfloor\frac{d}{r}\rfloor+1}^{\lfloor\frac{d}{s}\rfloor}\left(w_1-\frac{rw_1}{d}\left(\left\langle\frac{d}{r}\right\rangle+j\right)\right)$$
$$= \left(\frac{rw_1}{d}\right)^{\lfloor\frac{d}{r}\rfloor}\left\lfloor\frac{d}{r}\right\rfloor!\cdot\left(\frac{sw_1'}{d}\right)^{\lfloor\frac{d}{s}\rfloor}\left\lfloor\frac{d}{s}\right\rfloor!. \tag{2–11}$$

Computing (2–7) with equations (2–10) and (2–11) shows that the edge contribution from f towards the closed GW invariants is equal to

$$dn(-1)^{N+1}D^{+}_{\vec{k}0}(d;\vec{w})D^{-}_{\vec{k}\infty}(d;\vec{w}').$$

This concludes the proof. □

2.4. Insertions

In the original formulation of the topological vertex ([3]), insertions of cohomology classes were disregarded. Indeed, for smooth toric CY 3-folds the moduli spaces of stable maps have virtual dimension 0. Due to the fundamental class axiom, only finitely many invariants with fundamental class insertions are nonzero. By dimensional reasons, the rest of the invariants have only divisor insertions, which are easily handled by the divisor equation. In the orbifold case, however, the divisor equation does not hold for the twisted classes in degree 2 and insertions of these classes tend to give interesting invariants. Consequently, we develop our formalism to include insertions.

Throughout this section, we restrict to the case where $\mathcal{Y}$ is an *effective* orbifold (ie. the isotropy of the generic point is trivial). By the Gorenstein condition, this implies that the nontrivial isotropy is supported in codimension 2. One could presumably carry out the computations for non-effective orbifolds, however we do not pursue that here.

2.4.1. Orbifold Comology of $\mathcal{Y}$. To compute the primary insertion invariants, we must first understand the Chen-Ruan orbifold cohomology of $\mathcal{Y}$. Since we have restricted to the effective case, the only class in degree 0 is the untwisted fundamental class. As in the smooth case, if we disregard the finitely many nonzero invariants with fundamental class

insertions, then by dimensional reasons the rest of the insertions must be in $H^2_{CR}(\mathcal{Y})$. One computes that $H^2_{CR}(\mathcal{Y})$ is generated by the following classes.

- Divisor classes D_e in the untwisted sector.
- Twisted line classes $\mathfrak{c}_{e,h} := (C_e, h) \subset \mathcal{I}\mathcal{Y}$ with $h \in G_e$.
- Twisted point classes $\mathfrak{v}_{i,h} := (y_i, h) \subset \mathcal{I}\mathcal{Y}$ where $\mathcal{Y}_i$ can be identified with $[\mathbb{C}^3/G_i]$, the fixed point set of $h \in G_i$ is $\{y_i\}$, and h acts on $\mathbb{C}^3$ with weights $e^{2\pi i r_j}$ with $\sum r_j = 1$.

Assign formal variables $t_e, c_{e,h}, v_{i,h}$ corresponding to insertions of these cohomology classes.

2.4.2. Orbifold Vertex with Insertions. Suppose $\mathcal{X}$ is as in section 2.2 such that the action of G is effective. Let e_1, e_2, e_3 be the edges and let v be the vertex in $\Gamma_{\mathcal{X}}$. As before, $H^2_{CR}(\mathcal{X})$ is generated by classes $\mathfrak{c}_{e_j,h}$ and $\mathfrak{v}_h$ with corresponding formal variables $c_{e_j,h}$ and v_h. We modify Definition 2.12 to include insertions as follows.

- We denote the invariant with $m_{j,h}$ insertions of the class $\mathfrak{c}_{e_j,h}$ and n_h insertions of the class $\mathfrak{v}_h$ by $V_{\mathcal{X},g,\vec{\mu}}\left((\mathfrak{c}_{e_j,h})^{m_{j,h}} \cdot (\mathfrak{v}_h)^{n_h}; \vec{w}\right)$. The effect of adding these twisted insertions on the localized contribution is simply to prescribe the twisting at the marked points. Therefore, the corresponding vertex is defined by replacing the moduli space in Definition 2.12 with

$$\overline{\mathcal{M}}_{g,n+\sum_h (n_h+\sum_j m_{j,h})}(\mathcal{B}G) \cap ev^*\left(\vec{k}\right) \cap \prod_{h\in G} ev^*(\mathbb{I}_h)^{n_h+\sum_j m_{j,h}}$$

- $V_{\mathcal{X}}(\lambda, \mathbf{p}, \mathbf{c}, \mathbf{v}; \vec{w})$ is defined by taking the sum in (2.12) over all $g, \vec{\mu}$ as well as all possible insertions of $\mathfrak{c}_{e_j,h}$ and $\mathfrak{v}_h$ and including the formal variables

$$\frac{(c_{e_j,h})^{m_{j,h}}(v_h)^{n_h}}{m_{j,h}! n_h!}.$$

- The *oriented GW orbifold vertex with insertions* $V^{\bullet}_{\mathcal{X},\vec{\mu}}(\lambda, \mathbf{c}, \mathbf{v}; \vec{w})$ is defined to be the coefficient of $\mathbf{p}_{\vec{\mu}}$ in

$$V^{\bullet}_{\mathcal{X}}(\lambda, \mathbf{p}, \mathbf{c}, \mathbf{v}; \vec{w}) := \exp\left(V_{\mathcal{X}}(\lambda, \mathbf{p}, \mathbf{c}, \mathbf{v}; \vec{w})\right).$$

2.4.3. Gluing with Insertions. In order to obtain the full open/closed GW potential of $\mathcal{Y}$ (sans fundamental class insertions), we modify the gluing algorithm of Theorem 2.17 as follows.

- At each vertex v_i, we compute $V^{\bullet}_{\mathcal{Y}_i,\Lambda_i}(\lambda, \mathbf{c}(i), \mathbf{v}(i); \vec{w}(i))$.
- We replace Q_e with $\exp(t_e)Q_e$ to account for the divisor equation imposing the relation $t_e = t_{e'}$ if $D_e = D_{e'} \in H^2(\mathcal{Y})$.

2.5. Connection with Earlier Work

In the smooth case, we recover the computations of [22] which can then be related to the topological vertex of [3] and the generating functions $P_{\vec{\mu}}$ of [52]. We summarize these correspondences in this section.

Assume that G is trivial so that $\mathcal{X} = \mathbb{C}^3$. Orient $\Gamma_{\mathcal{X}}$ with all three edges directed outward. $\vec{\mu}$ is simply a triple of partitions (μ^1, μ^2, μ^3) where $\mu^i = (d^i_1, ..., d^i_{l_i})$ determines the winding profile along the ith Lagrangian. From Definition 2.12, we compute

$$V_{g,\vec{\mu}}(\vec{w}) = \frac{\prod_{i,j} w_1 w_2 w_3 D^+(d^i_j; \vec{w})}{|\mathrm{Aut}(\vec{\mu})|} \int_{\overline{\mathcal{M}}_{g,l(\vec{\mu})}} \prod_{i=1}^{3} \frac{\Lambda(w_i) w_i^{l(\vec{\mu})-1}}{w_1 w_2 w_3 \prod_{i,j}\left(\frac{w_i}{d^i_j} - \psi^i_j\right)}$$

where

$$D^+(d^i_j; \vec{w}) = \frac{1}{w_i d!} \frac{\Gamma\left(\frac{d w_{i+1}}{w_i} + d\right)}{\Gamma\left(\frac{d w_{i+1}}{w_i} + 1\right)}.$$

Simplifying, we get

$$V_{g,\vec{\mu}}(\vec{w}) = \frac{1}{|\mathrm{Aut}(\vec{\mu})|}\left[\prod_{i=1}^{3}\prod_{j=1}^{l_i}\frac{\prod_{k=1}^{d_j^i-1}(d_j^i w_{i+1}+kw_i)}{(d_j^i-1)!w_i^{d_j^i-1}}\right]\int_{\overline{\mathcal{M}}_{g,l(\vec{\mu})}}\prod_{i=1}^{3}\frac{\Lambda(w_i)w_i^{l(\vec{\mu})-1}}{\prod_{j=1}^{l_i}(w_i(w_i-d_j^i\psi_j^i))}.$$

This computation was made in Appendix A of [22]. The results of [41] and [47] together imply that

$$(-\sqrt{-1})^{l(\vec{\mu})}V_{\vec{\mu}}^{\bullet}(\lambda;\vec{w}) = C_{\mu^1,\mu^2,\mu^3}^{(\frac{w_2}{w_1},\frac{w_3}{w_2},\frac{w_1}{w_3})}(q) \tag{2–12}$$

where the right side is the framed topological vertex defined in [3] via large N duality where $q = e^{\sqrt{-1}\lambda}$.

2.5.1. Connection with 3d Partitions. In the rest of this section, we extract from the literature the explicit connection between (2–12) and the generating functions of 3d partitions defined in [52].

There are two natural bases for the center of the group ring of S_d, related by the character table. Equation (2–12) is written in terms of the partition basis. The authors of [3] suggest that it is more natural at times to view the vertex in the representation basis. Also in [3], a particular formula is derived for relating different framings of the vertex. Applying this change of basis and framing dependency formula to equation (2–12), we compute that the topological vertex at the canonical framing in the representation basis is given by

$$C_{\vec{\nu}}(q) = \left(e^{\sqrt{-1}\lambda}\right)^{\frac{-1}{2}\left(\sum_{i=1}^{3}\kappa(\nu^i)\frac{w_{i+1}}{w_i}\right)}\sum_{|\mu^i|=|\nu^i|}(-\sqrt{-1})^{l(\vec{\mu})}V_{\vec{\mu}}^{\bullet}(\lambda;\vec{w})\prod_{i=1}^{3}\chi_{\nu^i}(\mu^i). \tag{2–13}$$

where $\kappa(\nu) = 2\sum_{(i,j)\in\nu}(j-i)$, χ_ν is the character of $S_{|\nu|}$ indexed by ν, and $q = e^{\sqrt{-1}\lambda}$ (cf. Proposition 6.6 of [41]).

In [52], generating functions $P_{\vec{\nu}}(q)$ were defined by enumerating 3d partitions with prescribed asymptotics. They proved the following identity relating their generating functions to the topological vertex in the representation basis at canonical framing.

$$\frac{P_{\vec{\nu}}(q)}{M(q)} = q^{-\frac{1}{2}||\vec{\nu}^t||^2} C_{\vec{\nu}}(1/q) \tag{2–14}$$

where $M(q) = \prod_{k\geq 1}(1-q^k)^{-k}$ is the classical MacMahon function and $||\vec{\nu}||^2 := \sum(\nu_i^j)^2$. Putting together equations (2–13) and (2–14), we get the identity

$$\frac{P_{\vec{\nu}}(q)}{M(q)} = \left(e^{\sqrt{-1}\lambda}\right)^{\frac{1}{2}\left(\sum_{i=1}^3 \kappa(\nu^i)\frac{w_{i+1}}{w_i} - ||(\nu^i)^t||^2\right)} \sum_{|\mu^i|=|\nu^i|} (\sqrt{-1})^{l(\vec{\mu})} V_{\vec{\mu}}^{\bullet}(\lambda;\vec{w}) \prod_{i=1}^3 \chi_{\nu^i}(\mu^i). \tag{2–15}$$

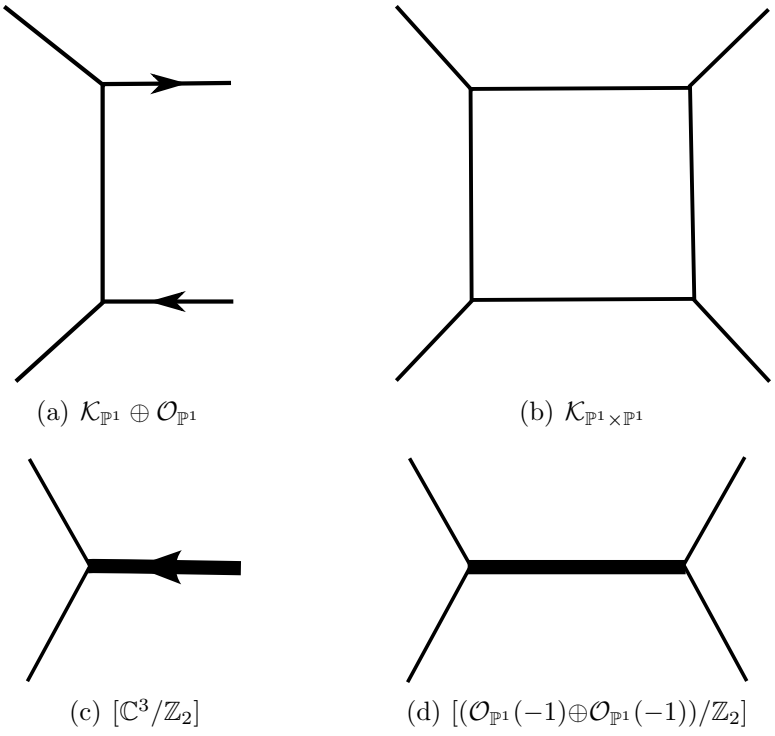
Remark 2.21. Our formalism for the orbifold vertex generalizes $V_{\vec{\mu}}^{\bullet}(\lambda;\vec{w})$ whereas the orbifold vertex formalism of Bryan, Cadman, and Young generalizes $\frac{P_{\vec{\nu}}(q)}{M(q)}$. A relation between the two (the GW/DT correspondence for toric CY orbifolds) should generalize equation (2–15). In Chapter 5, we state and prove such a correspondence for the first class of nontrivial orbifold targets.

CHAPTER 3

The Open Crepant Resolution Conjecture

In this chapter we make explicit, for a specific geometry, our approach of reducing Gromov-Witten correspondences for toric Calabi-Yau 3-folds to local statements at each torus fixed point, utilizing the orbifold vertex formalism of the previous chapter. Our investigation here centers around Ruan's crepant resolution conjecture.

3.0.2. Statement of Results. We give a complete description of our local-to-global approach to the CRC for the geometries in Figure 3.1 at particular framing. The global quotient $\mathfrak{X} = [\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)/\mathbb{Z}_2]$ is a Hard Lefschetz orbifold having $Y = \mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ as its crepant resolution. $\mathfrak{X}$ can be covered by two copies of $[\mathbb{C}^3/\mathbb{Z}_2]$, whose resolutions cover Y.



(a) $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ (b) $\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$

(c) $[\mathbb{C}^3/\mathbb{Z}_2]$ (d) $[(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_2]$

Figure 3.1. $\mathcal{B}\mathbb{Z}_2$ gerbes are denoted in bold and orientations for open invariants have been chosen using the conventions of Chapter 2.

In particular, the main results of this chapter are as follows.

THEOREM (Theorem 3.10). *We make and verify a Crepant Resolution Statement for the open invariants of* $[\mathbb{C}^3/\mathbb{Z}_2]$ *and* $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

This is the first occurence of a crepant resolution statement for open invariants. We compute the genus 0 open potential for $[\mathbb{C}^3/\mathbb{Z}_2]$ (Proposition 3.9) using the methods of Chapter 2. In order to evaluate invariants for more than one boundary component we generalize Theorem 1 of [16] to the case of two-part hyperelliptic Hodge integrals with an arbitrary number of descendant insertions (Theorem 3.5). The open potential for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ is computed (Proposition 3.3) using the techniques of [37]. Some interesting classical combinatorics is required to package the potential in a manageable form.

THEOREM (Theorem 3.12). *We verify the Ruan-Bryan-Graber CRC for* $\mathfrak{X}$ *and* Y.

We prove Theorem 3.12 by showing that our open CRC is compatible with the gluing algorithm of Theorem 2.17. Thus, we have gathered some positive evidence that the CRC, in the toric case, may be addressed locally.

3.0.3. OUTLINE OF THE PROOFS. Sections 3.1 and 3.2 are the computational meat of the paper in which we compute all relevant open invariants. In Section 3.3 we show that the open invariants satisfy the open crepant resolution conjecture by a direct comparison. We show in section 3.4 that the closed CRC for $[(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_2]$ can be deduced from the open CRC.

3.1. OPEN GROMOV-WITTEN INVARIANTS OF $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$

In this section we compute the open GW invariants of $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. We equip the space with a particular $\mathbb{C}^*$ action with (Calabi-Yau) weights on the tangent bundle depicted in Figure 3.2.

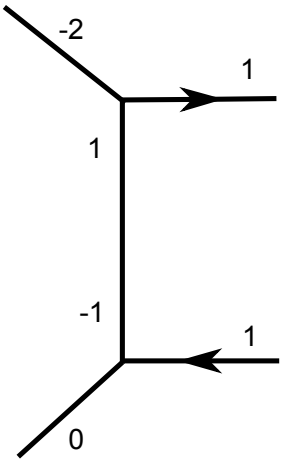


FIGURE 3.2. The web diagram for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, and the specialized toric weights.

The $\mathbb{C}^*$ fixed maps are quite easy to understand:

- The source curve consists of a genus 0 (possibly nodal) closed curve along with attached disks.
- The non-contracted irreducible components of the closed curve must be multiple covers of the torus invariant $\mathbb{P}^1$.
- The disks must map to the fixed fibers of the trivial bundle with prescribed windings at the Lagrangians.

Analyzing the obstruction theory via the normalization sequence of the source curve, one sees that the 0 weight at the bottom vertex limits the possible contributing maps in the following ways:

- Maps with positive dimensional components contracting to the bottom vertex do not contribute.
- Maps with nodes mapping to the bottom vertex contribute only if the node connects a d-fold cover of the invariant $\mathbb{P}^1$ to a disk with winding d.

Fixed loci F_Γ are indexed by localization graphs as in Figure 3.3.

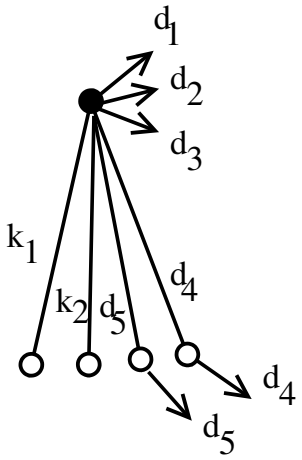


FIGURE 3.3. The open localization graphs have bi-colored vertices to keep track of which vertex components contract to, and decorated arrows to represent disks mapping with given winding.

The combinatorial data is given by three multi-indices:

- $k_1, ..., k_l$ the degrees of the multiple covers of the invariant $\mathbb{P}^1$ which do not attach to a disk at the bottom vertex.
- $d_1, ..., d_m$ the winding profile of the disks with origin mapping to the top vertex.
- $d_{m+1}, ..., d_n$ the winding profile of the disks with origin mapping to the bottom vertex or equivalently if $n > 1$ these are the degrees of the multiple covers of the invariant $\mathbb{P}^1$ which do attach to a disk at the bottom vertex.
- If $n = 1$, we have the possibility of maps from a single disk mapping the origin to the bottom vertex, we label the locus of such maps Γ'.

With the given multi-indices, the fixed locus F_Γ is isomorphic to a finite quotient of $\overline{\mathcal{M}}_{0,n+l}$ where we interpret $\overline{\mathcal{M}}_{0,1}$ and $\overline{\mathcal{M}}_{0,2}$ as points. Define the contribution from a fixed locus Γ to be

$$OGW(\Gamma) := \int_{F_\Gamma} \frac{i^*[\overline{\mathcal{M}}]^{\text{vir}}}{e(N_{\text{vir}})} \tag{3–1}$$

where $i^*[\overline{\mathcal{M}}]^{\text{vir}}$ is the restriction of the virtual fundamental class (proposed in [37]) to the fixed locus and N_{vir} denotes the virtual normal bundle of F_Γ in the moduli space of stable maps.

In order to package the invariants in the Gromov-Witten potential, we assign the following formal variables:

- q tracks the degree of the map on the base $\mathbb{P}^1$
- $y_i^{(t)}$ tracks the number of disks with winding i at the top vertex
- $y_i^{(b)}$ tracks the number of disks with winding i at the bottom vertex
- x tracks insertions of the nontrivial cohomology class (conveniently this class is a divisor).

The open potential is computed by adding the contributions of all fixed loci:

$$\begin{aligned} OGW_{\mathcal{K}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}}(x,q,y_i^{(t)},y_i^{(b)}) &= \sum_{\Gamma'} OGW(\Gamma')y_d^{(b)} \\ &+ \sum_{\Gamma\neq\Gamma'} OGW(\Gamma)(qe^x)^{k+d_{m+1}+\ldots+d_n} y_{d_1}^{(t)}\cdot\ldots\cdot y_{d_m}^{(t)} y_{d_{m+1}}^{(b)}\cdot\ldots\cdot y_{d_n}^{(b)} \end{aligned} \tag{3–2}$$

In (3–2), Γ' denotes graphs consisting of a single white vertex and arrow labelled with winding d.

For non-degenerate graphs $\Gamma \neq \Gamma'$, we denote by $\overline{OGW}(\Gamma)$ the contribution to the potential from the fixed locus indexed by Γ, including invariants with any number of divisor insertions. Following the obstruction theory for open invariants proposed in [37], $OGW(\Gamma)$ are computed using the following ingredients: the euler class of the push-pull of the tangent bundle, the euler class of the normal bundle of F_Γ in the moduli space of stable maps, and

all relevant automorphisms of the map:

$$\frac{1}{|\text{glob. aut.}|}\int_{F_\Gamma}\frac{e(-R^\bullet\pi_* f^* T_{\mathcal{K}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}})\cdot(\text{inf. aut.})}{(\text{smoothing of nodes})} \tag{3–3}$$

For the computational convenience, we organize the computation on each locus Γ into three parts:

- **Closed Curve:** This consists of a closed curve contracting to the upper vertex as well as multiple covers of the torus fixed $\mathbb{P}^1$. We choose not to include the d-covers of the fixed line which are attached to a disk mapping with winding d to the bottom vertex. The contracted component contributes $(-2t^3)^{-1}$ from the push-pull of the tangent bundle and each k-cover contributes

$$\frac{-t}{k^2}\frac{eH^1(\mathcal{O}(-2k))}{eH^0(\mathcal{O})eH^0(\mathcal{O}(2k))}=\frac{(-1)^k}{tk^2}\binom{2k-1}{k}. \tag{3–4}$$

  Here we have included both the global automorphism of the $k:1$ cover and the infinitesimal automorphism at the point ramified over the bottom vertex.

- **Disks:** A disk can either be mapped to the top or the bottom vertex. Following Katz and Liu [37], the contribution of a disk mapping to the top vertex with winding d is given by

$$\frac{1}{d}\frac{eH^1(N(d))}{eH^0(L(2d))}=\frac{(-1)^{d+1}}{td}\binom{2d-1}{d} \tag{3–5}$$

  where $L(2d)$ and $N(d)$ are defined in Examples 3.4.3. and 3.4.4 of [37]. We have divided the contributions in a way that the contribution of a disk mapping to the bottom vertex also includes the contribution of the multiple cover attaching it to the

contracted component. The reason for this is that the combined contribution becomes

$$\begin{aligned}
&\frac{1}{d^2}\frac{eH^1(\mathcal{O}(-2d))}{eH^0(\mathcal{O})eH^0(\mathcal{O}(2d))}\frac{eH^1(N(d))}{eH^0(L(2d))}\frac{eH^0(N_{/X})}{\frac{t}{d}-\frac{t}{d}} \\
&= \frac{1}{d^2}\frac{(-1)^{d+1}}{t^2}\binom{2d-1}{d}\frac{1}{t}\frac{-0t^3}{\frac{t}{d}-\frac{t}{d}} \\
&= \frac{(-1)^{d+1}}{td}\binom{2d-1}{d} \qquad (3\text{–}6)
\end{aligned}$$

which is the same as the contribution of the disk at the top vertex.

REMARK 3.1. In order to interpret the expression $\frac{-0}{1-1}$ in the above equations, first recall that it arises as $\frac{s_1s_2s_3}{s_1+s_2}$ where the s_i sum to 0 by definition. As $s_3 \to 0$, the quotient tends to $-s_1s_2$.

- **Nodes:** Since we have already accounted for the nodes at the bottom vertex (those attaching winding d disks to $d:1$ covers), this piece only contains the contribution from nodes at the top vertex. For each such node connecting either a disk of winding d or a curve of degree d to the contracted component we get a contribution of $-2t^3$ from the push-pull of the tangent sheaf and a contribution of $\frac{1}{(\frac{t}{d}-\psi_i)}$ from node smoothing.

Putting the pieces together:

$$\begin{aligned}
OGW(\Gamma) =& \frac{1}{|\mathrm{Aut}(\Gamma)|}\prod_{i=1}^{l}\frac{(-1)^{k_i}}{tk_i^2}\binom{2k_i-1}{k_i}\prod_{i=1}^{n}\frac{(-1)^{d_i+1}}{td_i}\binom{2d_i-1}{d_i} \\
&\cdot(-2t^3)^{l+n-1}\int_{\overline{M}_{0,n+l}}\frac{1}{\prod(\frac{t}{k_i}-\psi_i)\prod(\frac{t}{d_i}-\psi_{i+l})}. \qquad (3\text{–}7)
\end{aligned}$$

where $\mathrm{Aut}(\Gamma)$ is the product of the automorphisms of the ordered tuples $(k_1,...,k_l)$, $(d_1,...,d_m)$, and $(d_{m+1},...,d_n)$.

Applying the string equation to the integral and simplifying, (3–7) becomes

$$OGW(\Gamma) =$$

$$\frac{-2^{l+n-1}}{|\mathrm{Aut}(\Gamma)|}\left[\prod_{i=1}^{l}\frac{(-1)^{k_i+1}}{k_i}\binom{2k_i-1}{k_i}\right]\left[\prod_{i=1}^{n}(-1)^{d_i}\binom{2d_i-1}{d_i}\right](d+k)^{l+n-3} \tag{3–8}$$

where $d=\sum d_i$ and $k=\sum k_i$.

Recall now that the contribution of a disk is the same regardless of whether it maps to the top or bottom Lagrangian. Therefore, if we let $\Gamma(\bar{d};\bar{k})$ denote all $\Gamma\neq\Gamma'$ with winding profile $\bar{d}=(d_1,...,d_n)$ and fixed $\bar{k}=(k_1,...,k_l)$, we can attach the formal variables to compute the contribution from Γ:

$$\begin{aligned}\sum_{\Gamma\in\Gamma(\bar{d};\bar{k})}\overline{OGW}(\Gamma) &= \frac{-2^{l+n-1}}{|\mathrm{Aut}(\bar{d})|}\prod_{i=1}^{n}\left(y_{d_i}^{(t)}+y_{d_i}^{(b)}(qe^{x})^{d_i}\right)\prod_{i=1}^{n}(-1)^{d_i}\binom{2d_i-1}{d_i}\\ &\cdot\frac{1}{|\mathrm{Aut}(\bar{k})|}\prod_{i=1}^{l}\frac{(-1)^{k_i+1}(qe^{x})^{k_i}}{k_i}\binom{2k_i-1}{k_i}(d+k)^{l+n-3}\end{aligned} \tag{3–9}$$

We now want to sum over all $\bar{k}$ with $\sum k_i=k$. In order to do this, we first define a function $F(X,Y)$ by

$$\begin{aligned}F(X,Y) &:= \exp\left(\sum_{\kappa\geq 1}\frac{(-1)^{\kappa+1}}{\kappa}\binom{2\kappa-1}{\kappa}X^{\kappa}Y\right)\\ &= \sum_{l,k}\sum_{\bar{k}}\frac{1}{|\mathrm{Aut}(\bar{k})|}\left[\prod_{i=1}^{l}\frac{(-1)^{k_i+1}}{k_i}\binom{2k_i-1}{k_i}\right]X^{k}Y^{l}\end{aligned} \tag{3–10}$$

where the second sum is over all l-tuples $\bar{k}=(k_1,...,k_l)$ with $\sum k_i=k$. The sum of all contributions with fixed winding $(d_1,...,d_n)$ and with $(k_1,...,k_l)$ satisfying $\sum k_i=k$ is

obtained by specializing $Y = 2(d+k)$ and multiplying the coefficient of X^k by an appropriate factor:

$$\sum_{|\bar{k}|=k} \sum_{\Gamma \in \Gamma(\bar{d};\bar{k})} \overline{OGW}(\Gamma) = \frac{-2^{n-1}}{|\mathrm{Aut}(\bar{d})|} \prod_{i=1}^{n} \left(y_{d_i}^{(t)} + y_{d_i}^{(b)} (qe^x)^{d_i} \right) \prod_{i=1}^{n} (-1)^{d_i} \binom{2d_i - 1}{d_i}$$

$$\cdot (qe^x)^k (d+k)^{n-3} [F(X, 2(d+k))]_{X^k}. \tag{3–11}$$

To handle (3–11), we find a closed form expression for F. Start with the known generating function

$$\sum_{k \geq 1} \binom{2k-1}{k} (-1)^k X^k = \frac{1}{2} \cdot \frac{1 - \sqrt{1+4X}}{\sqrt{1+4X}} \tag{3–12}$$

If we divide by $-X$ and formally integrate term by term (imposing that the constant term is 0), we get

$$\sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \binom{2k-1}{k} X^k = \ln \left(\frac{1}{2} (1 + \sqrt{1+4X}) \right) \tag{3–13}$$

Finally, we can write

$$\begin{aligned} F &= \exp \left(Y \ln \left(\frac{1}{2} (1 + \sqrt{1+4X}) \right) \right) \\ &= \left[\frac{1}{2} (1 + \sqrt{1+4X}) \right]^Y \end{aligned} \tag{3–14}$$

There are a few interesting comments to make at this point:

- Setting $G := \frac{1}{2}(1 + \sqrt{1+4X})$, we see that $G = 1 + X \cdot C(X)$ where $C(X)$ is the generating function for the Catalan numbers.
- G satisfies the recursive relation $G^n = G^{n-1} + XG^{n-2}$.

- It is easy to see that the recursion and the relation between G and the Catalan numbers are equivalent to the array of coefficients of G^i taking on a slight variation of two classical combinatorial objects, as illustrated in Figure 3.4. Here "slight variation" is probably best described by looking at the first few terms in Table 3.1.

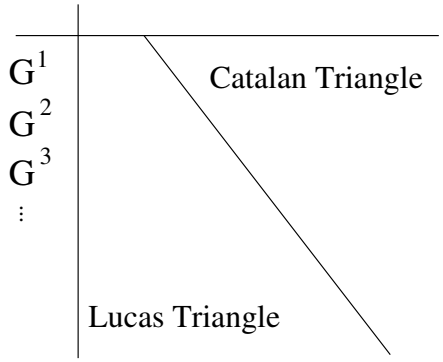


FIGURE 3.4. The coefficients of G^n as classical combinatorial numbers.

TABLE 3.1. The first coefficients of the series of G^n.

	1	x	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
G	1	1	−1	2	−5	14	−42	132	−429	1430
G^2	1	2	−1	2	−5	14	−42	132	−429	1430
G^3	1	3	0	1	−3	9	−28	90	−297	1001
G^4	1	4	2	0	−1	4	−14	48	−165	572
G^5	1	5	5	0	0	1	−5	20	−75	275
G^6	1	6	9	2	0	0	−1	6	−27	110
G^7	1	7	14	7	0	0	0	1	−7	35
G^8	1	8	20	16	2	0	0	0	−1	8
G^9	1	9	27	30	9	0	0	0	0	1
G^{10}	1	10	35	50	25	2	0	0	0	0

Using the recursion and induction, one easily proves the following lemma.

LEMMA 3.2. *If $d > 0$, the X^k coefficient of $G^{2(d+k)}$ is*

$$\binom{k+(2d-1)}{2d-1}\frac{d+k}{d}. \tag{3–15}$$

The X^k coefficient of G^{2k} is 2.

These are precisely the coefficients we need. Therefore, we conclude:

- From equation (3–11), if $(d_1, ..., d_n) \neq \emptyset$, then

$$\sum_{|\bar{k}|=k} \sum_{\Gamma \in \Gamma(\bar{d};\bar{k})} \overline{OGW}(\Gamma) = \frac{-2^{n-1}}{d \cdot |\mathrm{Aut}(\bar{d})|} \prod_{i=1}^{n} \left(y_{d_i}^{(t)} + y_{d_i}^{(b)} (qe^x)^{d_i} \right)$$

$$\cdot \prod_{i=1}^{n} (-1)^{d_i} \binom{2d_i - 1}{d_i} \sum_{k \geq 0} (d+k)^{n-2} \binom{k + (2d-1)}{2d-1} (qe^x)^k. \tag{3–16}$$

- Also from equations (3–11), if $(d_1, ..., d_n) = \emptyset$ and $(k_1, ..., k_l) \neq \emptyset$, then

$$\sum_{|\bar{k}|=k} \sum_{\Gamma \in \Gamma(\emptyset;\bar{k})} \overline{OGW}(\Gamma) = \frac{-1}{k^3} (qe^x)^k. \tag{3–17}$$

Here we have recovered the Aspinwall-Morrison formula for $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$.

Finally recall that:

- If both $\bar{d} = \emptyset$ and $\bar{k} = \emptyset$, then the locus consists of the degree 0 maps with only divisor insertions which can be computed via localization to be

$$\frac{-x^3}{12}. \tag{3–18}$$

- The contribution from a locus Γ' consisting of a single disk mapping to the bottom vertex with winding d is given by

$$\frac{1}{d^2} y_d^{(b)}. \tag{3–19}$$

Adding all contributions we conclude that

$$OGW_{\mathcal{K}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}}(x,q,y_i^{(t)},y_i^{(b)}) = \frac{-1}{2}\frac{x^3}{3!} + \sum_{k\geq 1}\frac{-1}{k^3}(qe^x)^k + \sum_{d\geq 1}\frac{1}{d^2}y_d^{(b)}$$

$$+ \sum_{(d_1,\ldots,d_n)\neq\emptyset}\left[\frac{-2^{n-1}}{d\cdot|\mathrm{Aut}(\vec{d})|}\prod_{i=1}^{n}\left(y_{d_i}^{(t)}+y_{d_i}^{(b)}(qe^x)^{d_i}\right)\prod_{i=1}^{n}(-1)^{d_i}\binom{2d_i-1}{d_i}\right.$$

$$\left.\cdot\sum_{k\geq 0}(d+k)^{n-2}\binom{k+(2d-1)}{2d-1}(qe^x)^k\right] \tag{3–20}$$

In a neighborhood of $x=-\infty$ we have:

$$\sum_{k\geq 0}\binom{k+(2d-1)}{2d-1}(qe^x)^{d+k} = \frac{(qe^x)^d}{(1-qe^x)^{2d}}. \tag{3–21}$$

Using (3–21) we can express (3–20):

$$\sum_{k\geq 0}(d+k)^{n-2}\binom{k+(2d-1)}{2d-1}(qe^x)^k = \frac{1}{(qe^x)^d}\frac{d^{n-2}}{dx^{n-2}}\left(\frac{(qe^x)^d}{(1-qe^x)^{2d}}\right) \tag{3–22}$$

where differentiation/integration is computed formally termwise. When $n\geq 2$, there is no ambiguity as $\frac{d^{n-2}}{dx^{n-2}}$ is a derivative. When $n=1$, we must practice a little bit of caution as the integral is only defined up to translation. Notice that

$$\lim_{x\to-\infty}\sum_{k\geq 0}\frac{1}{k+d}\binom{k+(2d-1)}{2d-1}(qe^x)^{k+d} = 0, \tag{3–23}$$

hence by

$$\int\frac{(qe^x)^d}{(1-qe^x)^{2d}}dx$$

we denote the antiderivative having limit 0 as x approaches $-\infty$.

We conclude this section by putting the open potential in its simplest form:

PROPOSITION 3.3. *The open Gromov-Witten potential (sans fundamental class insertions) for* $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ *is*

$$\begin{aligned} OGW_{\mathcal{K}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}}(x,q,y_i^{(t)},y_i^{(b)}) &= \frac{-1}{12}x^3 + \sum_{k\geq 1}\frac{-1}{k^3}(qe^x)^k \\ &+ \sum_{d\geq 1}\left[\frac{1}{d^2}y_d^{(b)} + \frac{(-1)^{d+1}}{d}\left(y_d^{(t)} + y_d^{(b)}(qe^x)^d\right)\binom{2d-1}{d}\right. \\ &\qquad \left.\cdot\frac{1}{(qe^x)^d}\int\frac{(qe^x)^d}{(1-qe^x)^{2d}}dx\right] \\ &+ \sum_{d_1,\ldots,d_n(n\geq 2)}\left[\frac{-2^{n-1}}{d\cdot|Aut(\overline{d})|}\left[\prod_{i=1}^{n}(-1)^{d_i}\left(y_{d_i}^{(t)} + y_{d_i}^{(b)}(qe^x)^{d_i}\right)\binom{2d_i-1}{d_i}\right]\right. \\ &\qquad \left.\cdot\frac{1}{(qe^x)^d}\frac{d^{n-2}}{dx^{n-2}}\left(\frac{(qe^x)^d}{(1-qe^x)^{2d}}\right)\right]. \end{aligned}$$

The first line is the closed contribution, the next two lines are the contribution from curves with one boundary component, and the final two lines are the contribution from curves with more than one boundary component.

3.2. Open Orbifold Gromov-Witten Invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$

In this section we compute the open orbifold GW invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$ following [6]. We will need an evaluation of certain hyperelliptic Hodge integrals in order to package the open invariants.

3.2.1. Hyperelliptic Hodge Integrals. In this section we prove a closed formula for a generating function which packages the hyperelliptic Hodge integrals of the form

$$L(g,i,\overline{m}) := \int_{\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_g\lambda_{g-i}(\overline{\psi})^{\overline{m}} \tag{3–24}$$

where $\overline{m}$ is a multi-index $(m_1, ..., m_l)$, $|\overline{m}| := m_1 + ... + m_l = i - 1$, and

$$\overline{\psi}^{\overline{m}} := \psi_1^{m_1} \cdot ... \cdot \psi_l^{m_l}.$$

REMARK 3.4. Recall that $\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)$ is the moduli space of maps from genus zero curves into $\mathcal{B}\mathbb{Z}_2$ with $(2g + 2)$ twisted marked points. Each such map corresponds to a (possibly nodal) genus g double cover of the source curve ramified over the marked points. We have two natural forgetful maps:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2) & \xrightarrow{F} & \overline{\mathcal{M}}_g \\ \downarrow{\scriptstyle \pi} & & \\ \overline{\mathcal{M}}_{0;2g+2} & & \end{array}$$

by sending a map to the corresponding double cover of its source curve. The lambda classes on $\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)$ are defined to be

$$\lambda_i := c_i(F^*\mathbb{E})$$

where $\mathbb{E}$ is the Hodge bundle on $\overline{\mathcal{M}}_g$. Recall that the psi classes are defined via pull-back from $\overline{\mathcal{M}}_{0;2g+2}$.

For a fixed i and $\overline{m}$ with $|\overline{m}| = i - 1$, define the generating function

$$\mathcal{L}_{i,\overline{m}}(x) := \sum_g L(g, i, \overline{m}) \frac{x^{2g}}{(2g)!}. \tag{3–25}$$

We know from the $\lambda_g\lambda_{g-1}$ computation [5, 12, 26] that

$$\mathcal{L}_{1,\emptyset} = \log \sec \left(\frac{x}{2}\right) \tag{3–26}$$

and we also know from [16] that

$$\mathcal{L}_{i,(i-1)} = \frac{2^{i-1}}{i!}\mathcal{L}_{1,\emptyset}^{i} \tag{3–27}$$

The following theorem generalizes (3–27).

THEOREM 3.5.

$$\mathcal{L}_{i,\overline{m}} = \binom{m_1 + ... + m_l}{m_1, ..., m_l} \frac{2^{i-1}}{i!}\mathcal{L}_{1,\emptyset}^{i}. \tag{3–28}$$

REMARK 3.6. This formula appeared independently in Danny Gillam's PhD dissertation. He computationally verified the result for $l \leq 4$.

PROOF. We use induction on the multi-index $\overline{m}$. Given a multi-index $\overline{m} = (m_1, ..., m_k)$ with $|\overline{m}| = j - 1$, we know the result is true if either $j = 1$ or $k = 1$. Suppose the lemma holds in the following cases:

(1) $j < i$ and

(2) $j = i, k \leq l$.

Under these assumptions, we show (3–28) holds when $j = i$ and $k = l + 1$.

Notation. Write $\overline{m} = (m_1, ..., m_l, m_{l+1})$ and set $\overline{m}' = (m_1, ..., m_{l-1}, m_l')$ where $m_l' := m_l + m_{l+1}$. For a subset $A \subseteq \{1, ..., l+1\}$, we write $\overline{m}(A)$ for the multi-index which is equal to $\overline{m}$ in the entries indexed by numbers in A and equal to 0 in the other entries. A^c denotes the complement of A. $\overline{m}[k]$ denotes the multi-index $\overline{m}$ with the first entry replaced by k.

We prove the recursion by evaluating via localization auxiliary integrals on $\overline{\mathcal{M}}_{0;2g+2,0}(\mathbb{P}^1 \times \mathcal{B}\mathbb{Z}_2, 1)$. This moduli space parametrizes double covers of the source curve with a special

rational component picked out. By postcomposing the usual evaluation maps with projection onto the first factor, we have evaluation maps to $\mathbb{P}^1$ which we denote by e_i. The auxiliary integrals are:

$A1$:

$$\int \lambda_g \lambda_{g-i} \overline{\psi}^{\overline{m}(\{1\}^c)} e_l^*(0) e_{l+1}^*(0) e_{2g+2}^*(\infty)$$

$A2$:

$$\int \lambda_g \lambda_{g-i} \overline{\psi}^{\overline{m'}(\{1\}^c)} e_l^*(0) e_{l+1}^*(0) e_{2g+2}^*(\infty)$$

Let us breifly explain the notation in the integrals.

(1) In each integrand, we do not include the ψ_1 part of the Hodge integral. The ψ_1 classes in the result make an appearance through node smoothing. The other ψ classes correspond to the marked points with the matching index.

(2) We have abused notation in order to make the expression legible. By λ_{g-i} we intend $c_{g-i}^{eq.}(R^1\pi_* f^*\mathcal{O})$ where the trivial bundle is linearized with 0 weights: the lambda classes are how these classes restrict to the fixed loci. By $e_i^*(0)$ (resp. $e_i^*(\infty)$) we denote $c_1^{eq.}(e_l^*\mathcal{O}(1))$ linearized with weight 1 over 0 and weight 0 over ∞ (resp. 0 over 0 and -1 over ∞). These classes essentially localize to require the corresponding mark point to map over 0 (resp. ∞).

(3) The difference in the two auxiliary integrals is that we have "spread" the ψ classes on the two points fixed over 0 in two different ways.

(4) Both integrals vanish by dimensional reasons. In both integrals the degree of the class we integrate is $m_2 + ... + m_{l+1} + 3 + 2g - i$ and this is strictly less than $2g + 2$ (because $m_1 + ... + m_{l+1} = i - 1$ and $m_1 > 0$).

(5) Localizing $A1$ yields relation (3–30) among Hodge integrals where all terms are already known by induction. Localizing $A2$ we get a relation (3–31) computing one unknown Hodge integral in terms of inductively known ones. Noticing that (3–30) and (3–31) are proportional to each other allows one to determine the desired integral.

Analyzing the obstruction theory via the normalization sequence of the source curve, one sees that the maps in the contributing fixed loci satisfy the following properties ([15] for more details):

- The preimages of 0 and ∞ in the corresponding double cover must be connected.
- One distinguished projective line in the source curve maps to the main component of the target with degree 1. The corresponding double cover has a rational component over the distinquished projective line.
- The lth and $(l+1)$th marked points must map to 0 while the $(2g+2)$th marked point must map to ∞.

The contributing fixed loci are:

F_g: All marked points except for the $(2g+2)$th map to 0. The corresponding double cover contracts a genus g component over 0 and does not have a positive dimensional irreducible component over ∞. This locus is isomorphic to $\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)$.

F_{g_1,g_2}: $2g_1+1$ marked points map to 0 and $2g_2+1$ marked points map to ∞ (this includes the points that are already forced to map to 0 and ∞). The corresponding double cover contracts a genus g_1 component over 0 and a genus g_2 component over ∞. This locus is isomorphic to $\overline{\mathcal{M}}_{0;2g_1+2,0}(\mathcal{B}\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0;2g_2+2,0}(\mathcal{B}\mathbb{Z}_2)$.

The mirror analog of F_g is not in the fixed locus because we are requiring that at least 2 marked points map to 0.

The first integral evaluates on the two types of fixed loci to:

F_g:

$$\frac{(-1)^i}{t^{m_1}}\int_{\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)}\lambda_g\lambda_{g-i}\overline{\psi}^{\overline{m}}=\frac{(-1)^i}{t^{m_1}}L(g,i,\overline{m})$$

F_{g_1,g_2}:

$$\frac{2(-1)^i}{t^{m_1}}\sum_{k=1}^{i-1}\sum_{A\subseteq\{2,\ldots,l-1\}}\binom{2g+1-l}{2g_1+1-|A|}(-1)^{k-|\overline{m}(A^c)|-1}$$
$$\cdot\int_{\overline{\mathcal{M}}_{0;2g_1+2,0}(\mathcal{B}\mathbb{Z}_2)}\lambda_{g_1}\lambda_{g_1-i+k}\psi_1^{i-k-|\overline{m}(A)|-1}\overline{\psi}^{\overline{m}(A)}\psi_l^{m_l}\psi_{l+1}^{m_{l+1}}$$
$$\cdot\int_{\overline{\mathcal{M}}_{0;2g_2+2,0}(\mathcal{B}\mathbb{Z}_2)}\lambda_{g_2}\lambda_{g_2-k}\psi_1^{k-|\overline{m}(A^c)|-1}\overline{\psi}^{\overline{m}(A^c)}$$

where we only sum over subsets A which keep the powers of ψ classes nonnegative. The subset A determines which ψ classes map to 0 and the binomial coefficient corresponds to the number of ways to distribute the marked points without a ψ class.

Now write $\overline{n}_{A,k}$ for the multi-index $\overline{m}(A^c)[k-|\overline{m}(A^c)|-1]$. The vanishing of the integral and the above computations lead to the following recursive relations which are satisfied by L:

$$L(g,i,\overline{m})=2\sum_{g_1}\sum_{k=1}^{i-1}\sum_{A\subseteq\{2,\ldots,l-1\}}\binom{2g+1-l}{2g_1+1-|A|}(-1)^{k-|\overline{m}(A^c)|}$$

$$\cdot L(g_1,i-k,\overline{m}-\overline{n}_{A,k})\cdot L(g_2,k,\overline{n}_{A,k}) \tag{3–29}$$

Evaluating the auxiliary integral for all genera and packaging (3–29) in generating function form:

$$\frac{d^{l-1}}{dx^{l-1}}\mathcal{L}_{i,\overline{m}} =$$

$$2\sum_{k=1}^{i-1}\sum_{A\subseteq\{2,\ldots,l-1\}}(-1)^{k-|\overline{m}(A^c)|}\left(\frac{d^{l-1-|A|}}{dx^{l-1-|A|}}\mathcal{L}_{i-k,\overline{m}-\overline{n}_{A,k}}\right)\left(\frac{d^{|A|}}{dx^{|A|}}\mathcal{L}_{k,\overline{n}_{A,k}}\right) \tag{3–30}$$

The second integral leads to a very similar relation:

$$\frac{d^{l-1}}{dx^{l-1}}\mathcal{L}(i,\overline{m}') =$$

$$2\sum_{k=1}^{i-1}\sum_{A\subseteq\{2,\ldots,l-1\}}(-1)^{k-|\overline{m}'(A^c)|}\left(\frac{d^{l-1-|A|}}{dx^{l-1-|A|}}\mathcal{L}_{i-k,\overline{m}'-\overline{n}'_{A,k}}\right)\left(\frac{d^{|A|}}{dx^{|A|}}\mathcal{L}_{k,\overline{n}'_{A,k}}\right) \tag{3–31}$$

By definition, $\overline{n}_{A,k} = \overline{n}'_{A,k}$, so

$$\frac{d^{|A|}}{dx^{|A|}}\mathcal{L}_{k,\overline{n}_{A,k}} = \frac{d^{|A|}}{dx^{|A|}}\mathcal{L}_{k,\overline{n}'_{A,k}} \tag{3–32}$$

Also, the induction hypothesis implies (because $k \geq 1$) that

$$\frac{d^{l-1-|A|}}{dx^{l-1-|A|}}\mathcal{L}_{i-k,\overline{m}-\overline{n}_{A,k}} = \frac{(m_l+m_{l+1})!}{m_l!m_{l+1}!}\frac{d^{l-1-|A|}}{dx^{l-1-|A|}}\mathcal{L}_{i-k,\overline{m}'-\overline{n}'_{A,k}} \tag{3–33}$$

Therefore the left hand sides of (3–30) and (3–31) are term by term proportional and we can conclude,

$$\frac{d^{l-1}}{dx^{l-1}}\mathcal{L}_{i,\overline{m}} = \frac{(m_l+m_{l+1})!}{m_l!m_{l+1}!}\frac{d^{l-1}}{dx^{l-1}}\mathcal{L}_{i,\overline{m}'}. \tag{3–34}$$

Now recall that $l(\overline{m}) = l+1$, so in order for $\int \lambda_g\lambda_{g-i}\overline{\psi}^{\overline{m}}$ to be defined, we must have at least $l+1$ marked points in our moduli space. Thus, in order to get a nontrivial integral, we

must have $2g+2 \geq l+1$. All coefficients of monomials of smaller degree than x^{l-1} in both generating functions vanish and we can conclude that

$$\begin{aligned}\mathcal{L}_{i,\overline{m}} &= \frac{(m_l+m_{l+1})!}{m_l!m_{l+1}!}\mathcal{L}_{i,\overline{m}'} \\ &= \frac{(m_l+m_{l+1})!}{m_l!m_{l+1}!}\binom{m_1+...+m_l'}{m_1,...,m_l'}\frac{2^{i-1}}{i!}\mathcal{L}_{1,\emptyset}^i \\ &= \binom{m_1+...+m_{l+1}}{m_1,...,m_{l+1}}\frac{2^{i-1}}{i!}\mathcal{L}_{1,\emptyset}^i \end{aligned} \tag{3–35}$$

where we use the induction hypothesis again on the second equality. □

All $\mathcal{L}_{i,\overline{m}}$ can be further packaged in one jumbo generating function (with infinitely many symmetric variables q_i keeping track of all possible descendant insertions):

$$\mathcal{L}(x,\overline{q}) := \sum_{i,\overline{m}} \mathcal{L}_{i,\overline{m}}\overline{q}^{\overline{m}} \tag{3–36}$$

Corollary 3.7.

$$\mathcal{L} = \frac{1}{(2\sum q_i)}\exp\left(\left(2\sum q_i\right)\mathcal{L}_{1,\emptyset}\right) = \frac{1}{2\sum q_i}\sec^{2\sum q_i}\left(\frac{x}{2}\right) \tag{3–37}$$

Proof. The first equality follows immediately from theorem (3.5). The second is obtained by plugging (3–26) for $\mathcal{L}_{1,\emptyset}$. □

3.2.2. Open Invariants of $[\mathbb{C}^3/\mathbb{Z}_2]$. We now define a $\mathbb{C}^*$ action on the orbifold with weights on the tangent bundle depicted in Figure 3.5.

We characterize the $\mathbb{C}^*$ fixed maps:

- The source curve consists of a genus 0 closed curve along with attached disks. The closed component can carry (possibly twisted) marks whereas a disk can only carry a

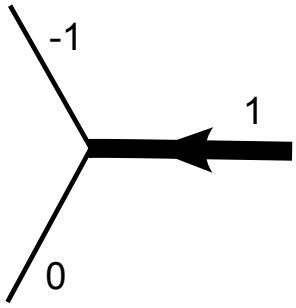


FIGURE 3.5. Toric diagram for $[\mathbb{C}^3/\mathbb{Z}_2]$ and $\mathbb{C}^*$ weights.

mark at the origin (and then only if it is not attached to a closed component). The attaching points of the nodes must carry inverse twisting.

- The closed curve must contract to the vertex.
- The disks must map to the twisted $\mathbb{C}$ with prescribed windings at the Lagrangian.

Since we are working with a $\mathbb{Z}_2$ quotient, we simply refer to points as twisted or untwisted as there is no ambiguity. A careful analysis of the obstruction theory via the normalization sequence of the source curve shows that the 0 weight conveniently kills all contributions where a disk attaches to a contracted component at an untwisted node. By dimensional reasons, all other marks must be twisted.

Combinatorially, the fixed loci Λ are indexed by

- m the number of insertions of the twisted sector and
- $d_1, ..., d_n$ the winding profile of the disks.

REMARK 3.8. Since all nodes and marked points are twisted, the maps restricted to the contracted component (maps into $\mathcal{B}\mathbb{Z}_2$) classify double covers of the contracted component with simple ramification over $m+n$ points. Since such a cover only exists if $m+n$ is even, the loci are non-empty only when $m+n$ is even.

If we let z and w_d be formal variables tracking the twisted sector insertions and the winding d disks, then the open orbifold potential can be computed as

$$OGW_{[\mathbb{C}^3/\mathbb{Z}_2]}(z, w_i) = \sum_{\Lambda} OGW(\Lambda) \frac{z^m}{m!} w_{d_1} \cdot \ldots \cdot w_{d_n} \tag{3–38}$$

We now group the computation of $OGW(\Lambda)$ into three components:

- **Closed Curve:** The closed curve contracted to the vertex essentially carries the information of a map into $\mathcal{B}\mathbb{Z}_2$ along with the weights of the $\mathbb{C}^*$ action on the three normal directions. This classifies a double cover of the source curve. Analogous to [13, section 2.1], the contribution from the closed component is the equivariant euler class of two copies of the dual of the Hodge bundle on the cover twisted by the weights of the action on the untwisted fixed fibers:

$$e(\mathbb{E}^{\vee}_{-1}(-1) \oplus \mathbb{E}^{\vee}_{-1}(0)) \tag{3–39}$$

  We also get a contribution of t^{-1} from the weight of the action on the twisted sector.

- **Disks:** The disk contribution is laid out in [6, section 2.2.3]. This contribution is a combinatorial function depending on the winding at the Lagrangian and the twisting at the origin of the disk. The localization step simplifies the disk contribution to two cases, either the origin of the disk is marked and twisted (possibly a node) or the origin is unmarked. For the particular case at hand, a disk with winding d and with twisting at the origin contributes

$$\frac{1}{2d} \frac{(2d-1)!!}{(2d)!!} \tag{3–40}$$

whereas a disk with no mark and no twisting at the origin contributes

$$\frac{1}{2d^2}. \tag{3–41}$$

- **Nodes:** We consider the nodes attaching a winding d disk to the closed component. Each one gets a t from the weight of the action on the twisted sector. Smoothing the node contributes $\frac{1}{\frac{t}{2d}-\frac{\psi_i}{2}}$.

Putting together the three parts described above, we find that $OGW(\Lambda)$ is:

$$\frac{1}{|\mathrm{Aut}(\bar{d})|}\left[\prod_{i=1}^{n}\frac{1}{2d}\frac{(2d_i-1)!!}{(2d_i)!!}\right]\int (2)^n t^{n-1}\frac{e^{eq}(\mathbb{E}_{-1}^{\vee}(-1)\oplus\mathbb{E}_{-1}^{\vee}(0))}{\prod_{i=1}^{n}(\frac{t}{d_i}-\psi_i)}$$
$$=\frac{1}{|\mathrm{Aut}(\bar{d})|}\left[\prod_{i=1}^{n}\frac{(2d_i-1)!!}{(2d_i)!!}\right]\sum_{i=1}^{g-1}\sum_{|\bar{j}|=i-1}\int \lambda_g\lambda_{g-i}(\overline{d\psi})^{\bar{j}}$$

where the integral is taken over $\overline{\mathcal{M}}_{0;m+n,0}(\mathcal{B}\mathbb{Z}_2)$, $g=\frac{m+n-2}{2}$ (the genus of the cover of the closed curve) and $(\overline{d\psi})^{\bar{j}}$ and $|\bar{j}|$ are defined in section 3.2.1.

Summing over all m (equivalently g) and specializing $q_i=d_i$ in Thereom 3.7, we see that the contribution to the open potential from all maps with a fixed winding profile $d_1,...,d_n$ is given by

$$\frac{1}{|\mathrm{Aut}(\bar{d})|}\left[\prod_{i=1}^{n}\frac{(2d_i-1)!!}{(2d_i)!!}\right]\frac{d^{n-2}}{dz^{n-2}}\frac{\sec^{2d}(z/2)}{2d} \tag{3–42}$$

There is no ambiguity for $n\geq 2$, but we must again be careful when $n<2$.

When $n=1$ the above formula still holds, but since integrals are only defined up to translation, we must make sure and get the correct constant term. The constant term corresponds to the contribution from maps with one boundary component and no marked

points. The only type of map in the fixed locus that satisfies this criteria is a disk with no marked points mapping with winding d. We've seen that the contribution from such a map is $\frac{1}{2d^2}$.

When $n = 0$, we must compute the closed contribution. The maps must have at least 3 marked points to be stable, but any map into $\mathcal{B}\mathbb{Z}_2$ must have an even number of twisted points (see Remark (3.8)). Since there are no disk or node smoothing factors, the contribution is

$$H(z) = \sum_{g\geq 1} \int_{\overline{\mathcal{M}}_{0;2g+2,0}(\mathcal{B}\mathbb{Z}_2)} \lambda_g \lambda_{g-1} \frac{z^{2g+2}}{(2g+2)!}. \tag{3–43}$$

By the $\lambda_g\lambda_{g-1}$ result of Faber and Pandharipande [26], $\frac{d^2}{dz^2}H(z) = \log(\sec(z/2))$.

Pulling together everything from the above discussion, we prove the following result:

Proposition 3.9. *The open orbifold Gromov-Witten potential (sans fundamental class insertions) of* $[\mathbb{C}^3/\mathbb{Z}_2]$ *is*

$$\begin{aligned} OGW_{[\mathbb{C}^3/\mathbb{Z}_2]}(z, w_i) = \quad & H(z) \\ + \quad & \sum_{d\geq 1} \left(\frac{1}{2d^2} + \frac{(2d-1)!!}{(2d)!!} \int \frac{\sec^{2d}(z/2)}{2d} dz \right) w_d \\ + \quad & \sum_{d_1,\ldots,d_n (n\geq 2)} \frac{1}{|Aut(\overline{d})|} \left(\prod_{i=1}^{n} \frac{(2d_i-1)!!}{(2d_i)!!} \right) \left(\frac{d^{n-2}}{dz^{n-2}} \frac{\sec^{2d}(z/2)}{2d} \right) w_{d_1} \cdot \ldots \cdot w_{d_n}, \end{aligned} \tag{3–44}$$

where the antiderivative is chosen to vanish at $z = 0$.

3.3. The Open Crepant Resolution Conjecture

Now that we have computed the open potentials for $[\mathbb{C}^3/\mathbb{Z}_2]$ and its crepant resolution $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$, we show that there is a change of variables which equates the stable terms. We

start with the contribution from a given winding profile on the orbifold, we consider all contributions on the resolution with that same winding profile, and we show that the change of variables equates these contributions. More specifically, we show the following.

THEOREM 3.10. *Under the change of variables*

$$\begin{array}{ll} q \to -1 & x \to iz \\ y_d^{(b)} \to \frac{i}{2} w_d & y_d^{(t)} \to \frac{i}{2} w_d (-e^{iz})^d \end{array} \tag{3–45}$$

the open GW potential of $\mathcal{K}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ *analytically continues to the open GW potential of* $[\mathbb{C}^3/\mathbb{Z}_2]$ *up to unstable terms.*

PROOF. The closed portion of the Theorem was proven in [10, Section 3.2]. Consider the winding d disk contribution on the resolution:

$$\frac{1}{d^2} y_d^{(b)} + \frac{(-1)^{d+1}}{d} \left(y_d^{(t)} + y_d^{(b)} (qe^x)^d \right) \binom{2d-1}{d} \frac{1}{(qe^x)^d} \left[\int \frac{(qe^x)^d}{(1-qe^x)^{2d}} dx \right].$$

Making the change of variables, it becomes

$$\begin{aligned} &\frac{i}{2d^2} w_d + \frac{(-1)^{d+1}}{d} \left(\frac{i}{2} w_d (-e^{iz})^d + \frac{i}{2} w_d (-e^{iz})^d \right) \binom{2d-1}{d} \frac{1}{(-e^{iz})^d} \left[i \int \frac{(-e^{iz})^d}{(1+e^{iz})^{2d}} dz \right] \\ &= \frac{i}{2d^2} w_d + \frac{i(-1)^{d+1}}{d} w_d \binom{2d-1}{d} \left[i \int \frac{(-e^{iz})^d}{(1+e^{iz})^{2d}} dz \right] \\ &= \frac{i}{2d^2} w_d + \frac{-i}{d} w_d \binom{2d-1}{d} \cdot \left[i \int \frac{\sec^{2d}(z/2)}{(2^{2d})} dz \right] \end{aligned}$$

Here we do not pay attention to the constant terms in the anti-derivatives since they correspond to unstable terms about which we make no claims. Hence we obtain:

$$\frac{1}{d}\binom{2d-1}{d} w_d \int \frac{\sec^{2d}(z/2)}{(2^{2d})} dz = \frac{(2d-1)!!}{(2d)!!} w_d \int \frac{\sec^{2d}(z/2)}{2d} dz, \tag{3–46}$$

the disk potential computed on the orbifold.

Finally, consider a general term with winding profile $d_1, ..., d_n$:

$$\frac{-2^{n-1}}{d \cdot |\mathrm{Aut}(\bar{d})|} \left[\prod_{i=1}^{n} (-1)^{d_i} \left(y_{d_i}^{(t)} + y_{d_i}^{(b)} (qe^x)^{d_i} \right) \binom{2d_i - 1}{d_i} \right] \frac{1}{(qe^x)^d} \frac{d^{n-2}}{dx^{n-2}} \left(\frac{(qe^x)^d}{(1-qe^x)^{2d}} \right) \tag{3–47}$$

Making the change of variables, this becomes:

$$\begin{aligned}
&\frac{-2^{n-1}}{d \cdot |\mathrm{Aut}(\bar{d})|} (i)^n \prod_{i=1}^{n} w_{d_i} \binom{2d_i - 1}{d_i} \frac{1}{i^{n-2}} \frac{d^{n-2}}{dz^{n-2}} \frac{1}{2^{2d}} \sec^{2d}\left(\frac{z}{2}\right) \\
&= \frac{1}{2d \cdot |\mathrm{Aut}(\bar{d})|} \prod_{i=1}^{n} \frac{w_{d_i}}{2^{2d_i - 1}} \binom{2d_i - 1}{d_i} \left(\frac{d^{n-2}}{dz^{n-2}} \sec^{2d}\left(\frac{z}{2}\right) \right) \\
&= \frac{1}{|\mathrm{Aut}(\bar{d})|} \prod_{i=1}^{n} \frac{(2d-1)!!}{(2d)!!} \left(\frac{d^{n-2}}{dz^{n-2}} \frac{\sec^{2d}\left(\frac{z}{2}\right)}{2d} \right) w_{d_i} \cdot ... \cdot w_{d_n}.
\end{aligned}$$

The final expression coincides with the contribution on the orbifold. □

3.4. The Closed Crepant Resolution Conjecture via Gluing

In this section we deduce the Ruan-Bryan-Graber crepant resolution conjecture for the orbifold $\mathfrak{X} = [\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ and its crepant resolution $Y = \mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ from the results of the previous sections.

The orbifold $\mathfrak{X}$ can be presented in terms of two charts isomorphic to $[\mathbb{C}^3/\mathbb{Z}_2]$. We equip the two charts with the compatible torus action depicted in Figure 3.6.

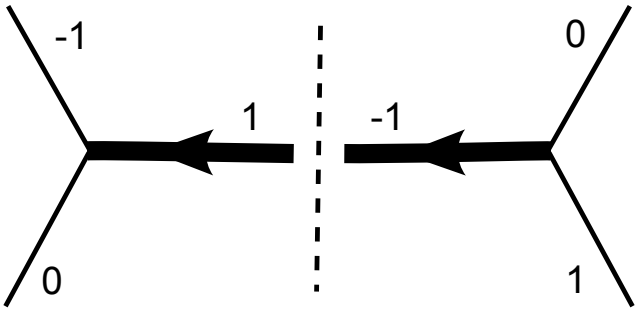


FIGURE 3.6. Symmetry of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$

In section 3.1 we computed disk invariants for the left vertex in figure 3.6. The right vertex with the given weights and orientation gives the same invariants multiplied by a factor of $(-1)^d$. In other words, the open potential for the right vertex in Figure 3.6 can be obtained from the open potential of the left vertex under the change of variables $z \to \tilde{z}$ and $w_d \to -\tilde{w_d}$.

REMARK 3.11. Throughout the rest of this section, variables with a tilde correspond to formal variables on the right sides of the diagrams.

Refer to Figure 3.7 for the resolution.

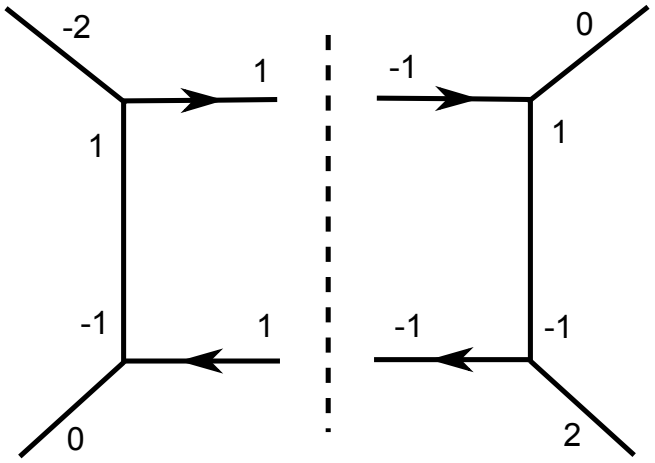


FIGURE 3.7. Symmetry of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$.

It is not hard to check that the disk invariants for the right half of the diagram with the given orientations and weights coincides with disk invariants computed in section 3.1. Therefore, the open potential on the right can be obtained from the open potential on the left by the change of variables:

$$\begin{array}{ll} q \to \tilde{q} & x \to \tilde{x} \\ y_d^{(b)} \to \tilde{y}_d^{(t)} & y_d^{(t)} \to \tilde{y}_d^{(b)} \end{array} \tag{3–48}$$

The setup for the crepant resolution conjecture is as follows. The Chen-Ruan orbifold cohomology of $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ has two generators in degree 2, the fiber over a point of $\mathbb{P}^1$ and the class of the constant function on the twisted $\mathbb{P}^1$. We assign the formal variables W and Z to correspond to insertions of these classes, respectively. Any map into the orbifold is classified by the degree on the twisted $\mathbb{P}^1$, thus we only need one degree variable P. On the resolution, we have two insertion variables, corresponding to the fiber over a point in each $\mathbb{P}^1$, let these be X and Y. We also have two degree variables corresponding to the degree of a map on each $\mathbb{P}^1$; denote them Q and U, where Q corresponds to the $\mathbb{P}^1$ which is dual to the divisor corresponding to X.

THEOREM 3.12. *After the change of variables*

$$\begin{array}{ll} Q \to -1 & U \to -P \\ X \to iZ & Y \to iZ + W \end{array} \tag{3–49}$$

the genus 0 GW potential of $\mathcal{K}_{\mathbb{P}^1 \times \mathbb{P}^1}$ *equals the genus 0 GW potential of* $[\mathcal{O}(-1) \oplus \mathcal{O}(-1)/\mathbb{Z}_2]$ *up to unstable terms.*

First we express the two potentials as a sum over the same set of decorated trees. We then describe how one can extract the contribution to the GW potential from each tree by multiplying vertex and edge contributions. The open crepant resolution statement proved in section 3.3 verifies that the change of variables equates the vertex contributions and edge contributions.

Since the portion of the computation corresponding to degree 0 maps into the orbifold is immediate from the closed computation done in section 3.3, we focus on contributions with nonzero powers of U and P.

3.4.1. Closed Invariants of $[\mathcal{O}(-1)\oplus\mathcal{O}(-1)/\mathbb{Z}_2]$. The closed potential of the orbifold can be expressed as a sum over localization trees:

- black (white) vertices of the tree correspond to components contracting to the left (right) orbifold vertex;
- edges of the tree correspond to multiple covers of the twisted $\mathbb{P}^1$ obtained by gluing disks. Each edge is decorated with a positive integer denoting the degree of the multiple cover.

By the gluing algorithm of Theorem 2.17, closed GW invariants of the orbifold are obtained by gluing open invariants along half edges. For a given localization tree T with more than one edge, the corresponding contribution to the GW potential is given by

$$GW_{\mathfrak{X}}(T) = \prod_{\text{black vertices}} V(v) \prod_{\text{edges } e} E(e) \prod_{\text{white vertices}} \tilde{V}(v) \tag{3–50}$$

In the above formula, $V(v)$ and $\tilde{V}(v)$ are the open invariants with winding profile corresponding to the edges meeting at v (with the formal variables z and $\tilde{z}$ replaced with Z). In

the case that v is univalent, only the contribution from disks with twisted origin is taken. The edge contribution is:

$$E(e) = \frac{(-1)^d 2d (Pe^W)^d}{w_d \tilde{w}_d}. \tag{3–51}$$

where e is an edge marked with d. The Pe^W is from applying the divisor equation to the new divisor class obtained by gluing and the $(-1)^d 2d$ is the gluing factor of Theorem 2.17.

In the case that T' is the tree with a unique edge labeled d, then one must also take into account the contribution from gluing two unmarked disks. The contribution in this case is

$$GW_{\mathfrak{X}}(T') = V(v_1)E(e)\tilde{V}(v_2) + \frac{1}{2d^3}(Pe^W)^d. \tag{3–52}$$

3.4.2. Closed invariants of $\mathcal{K}_{\mathbb{P}^1\times\mathbb{P}^1}$. Again, the Gromov Witten potential is expressed as a sum over localization graphs. For each graph, collapsing all "vertical" edges (ie. edges corresponding to multiple covers of the vertical fixed fibers) produces essentially a tree as in section 3.4.1, with the extra decoration of a subset S of the edges corresponding to edges mapping to the top invariant line. We forget this extra decoration to organize the potential as a sum over the same trees of section 3.4.1.

Again by the gluing algorithm of Theorem 2.17, the contribution to the GW potential from all loci corresponding to a given decorated tree T is:

$$GW_Y(T) = \sum_{S\subset\{\text{edges}\}} \left(\prod_{\text{black vertices}} V^{(S)}(v) \prod_{\text{edges } e} E'(e) \prod_{\text{white vertices}} \tilde{V}^{(S)}(v) \right) \tag{3–53}$$

In the above formula, $V^{(S)}(v)$ and $\tilde{V}^{(S)}(v)$ are the open GW contributions from all fixed loci with winding profile determined by the edges meeting v (we replace the formal variables

$q, \tilde{q}$ with Q and $x, \tilde{x}$ with X). If an adjacent edge is in S, this corresponds to a disk mapping to the upper Lagrangian and vice versa. Also

$$E'(e) = \begin{cases} \frac{-d(Ue^Y)^d}{y_d^{(t)}\tilde{y}_d^{(t)}} & \text{if } e \in S \\ \frac{-d(Ue^Y)^d}{y_d^{(b)}\tilde{y}_d^{(b)}} & \text{if } e \notin S \end{cases} \tag{3–54}$$

where e is an edge labeled with d. The $-d$ is the gluing factor of Theorem 2.17 and the Ue^Y comes from applying the divisor equation to the new divisor class created by gluing.

Let $V'(v)$ and $\tilde{V}'(v)$ denote the open contributions corresponding to all fixed loci with winding profile $(d_1, ..., d_n)$ given by the edges $(e_1, ..., e_n)$ meeting v (summing over all possibilitiees for the disks to map to the top edge or the bottom edge). Undoing (3–9), we have:

$$V^{(S)}(v) = \begin{cases} \frac{y_d^{(t)}}{y_d^{(t)}+y_d^{(b)}(Qe^X)^d}\left(V'(v) - \frac{1}{d^2}y_d^{(b)}\right) & v \text{ univalent, } e \in S \\ \frac{y_d^{(b)}(Qe^X)^d}{y_d^{(t)}+y_d^{(b)}(Qe^X)^d}\left(V'(v) - \frac{1}{d^2}y_d^{(b)}\right) + \frac{1}{d^2}y_d^{(b)} & v \text{ univalent, } e \notin S \\ \frac{\prod_{e_i \in S} y_{d_i}^{(t)} \prod_{e_i \notin S} y_{d_i}^{(b)}(Qe^X)^{d_i}}{\prod_{i=1}^n \left(y_{d_i}^{(t)}+y_{d_i}^{(b)}(Qe^X)^{d_i}\right)} V'(v) & \text{else} \end{cases}$$

and

$$\tilde{V}^{(S)}(v) = \begin{cases} \frac{\tilde{y}_d^{(t)}(Qe^X)^d}{\tilde{y}_d^{(t)}+\tilde{y}_d^{(b)}(Qe^X)^d}\left(V'(v) - \frac{1}{d^2}\tilde{y}_d^{(t)}\right) + \frac{1}{d^2}\tilde{y}_d^{(t)} & v \text{ univalent, } e \in S \\ \frac{\tilde{y}_d^{(b)}}{\tilde{y}_d^{(t)}+\tilde{y}_d^{(b)}(Qe^X)^d}\left(V'(v) - \frac{1}{d^2}\tilde{y}_d^{(t)}\right) & v \text{ univalent, } e \notin S \\ \frac{\prod_{e_i \in S} \tilde{y}_{d_i}^{(t)}(Qe^X)^{d_i} \prod_{e_i \notin S} \tilde{y}_{d_i}^{(b)}}{\prod_{i=1}^n \left(\tilde{y}_{d_i}^{(b)}+\tilde{y}_{d_i}^{(t)}(Qe^X)^{d_i}\right)} V'(v) & \text{else} \end{cases}$$

Remark 3.13. In each of the above formulas for the vertex contributions, the third case is the generic case and the other two are adjusted to take into account the Γ' loci of (3–2).

3.4.3. The Crepant Resolution Transformation. In order to verify the Ruan-Bryan-Graber crepant resolution conjecture, we show that after the prescribed change of

variables,

$$GW_Y(T) \to GW_{\mathfrak{X}}(T) \tag{3–55}$$

for every decorated tree T.

Even though our formulas for the vertex and edge contributions of $GW_{\mathfrak{X}}(T)$ and $GW_Y(T)$ involve winding variables, these variables cancel in the product. Hence we can make any substituion for the winding variables and it does not affect the overall product. Motivated by the open crepant resolution transformation, in the above formulas for $GW_Y(T)$ we make the substitutions:

$$\begin{aligned} y_d^{(b)} &\to \frac{i}{2} w_d & \tilde{y}_d^{(b)} &\to \frac{i}{2}(e^{iZ})^d \tilde{w}_d \\ y_d^{(t)} &\to \frac{i}{2}(-e^{iZ})^d w_d & \tilde{y}_d^{(t)} &\to (-1)^d \frac{i}{2} \tilde{w}_d \\ Q &\to -1 & U &\to -P \\ X &\to iZ & Y &\to iZ + W \end{aligned}$$

By Theorem 3.10, under this change of variables $V'(v) \to V(v)$ and $\tilde{V}'(v) \to \tilde{V}(v)$. So for any $S \subset \{\text{edges}\}$, we have:

$$V^{(S)}(v) \to \begin{cases} \frac{1}{2}V(v) - \frac{i}{4d^2} w_d & v \text{ univalent, } e \in S \\ \frac{1}{2}V(v) + \frac{i}{4d^2} w_d & v \text{ univalent, } e \notin S \\ \frac{1}{2^n}V(v) & \text{else} \end{cases} \tag{3–56}$$

and similarly,

$$\tilde{V}^{(S)}(v) \to \begin{cases} \frac{1}{2}\tilde{V}(v) + \frac{i}{4d^2}\tilde{w}_d & v \text{ univalent, } e \in S \\ \frac{1}{2}\tilde{V}(v) - \frac{i}{4d^2}\tilde{w}_d & v \text{ univalent, } e \notin S \\ \frac{1}{2^n}\tilde{V}(v) & \text{else} \end{cases} \tag{3–57}$$

Also, under the change of variables

$$E'(e) \to 2E(e). \tag{3–58}$$

Given any tree T with more than one edge, the extra terms on the univalent vertices cancel by summing over all contributions $e \in S$ and $e \notin S$. Therefore, from (3–56),(3–57) and (3–58);

$$GW_Y(T) =$$

$$\sum_{S \subset \{\text{edges}\}} \prod \frac{1}{2} V(v) \prod 2E(e) \prod \frac{1}{2}\tilde{V}(v) = 2^{\#\{\text{edges}\}} \frac{\prod V(v) \prod E(e) \prod \tilde{V}(v)}{2^{\#\{\text{edges}\}}}$$

$$= GW_{\mathfrak{X}}(T). \tag{3–59}$$

If T' is the tree with a unique edge labeled d:

$$GW_Y(T') = V(v_1)E(e)\tilde{V}(v_2) + \frac{1}{2d^3}(Pe^Y)^d = GW_{\mathfrak{X}}(T'). \tag{3–60}$$

Equations (3–59) and (3–60) establish Theorem 3.12.

CHAPTER 4

The Loop Murnaghan-Nakayama Rule

In this chapter we take an adventurous detour into the land of algebraic combinatorics. As we will see in Chapter 5, the A_{n-1} orbifold vertex is closely linked to combinatorial gadgets called loop Schur functions. The identities which we develop in this chapter will be pivotal in proving the Gromov-Witten/Donaldson-Thomas correspondence of Chapter 5.

4.0.4. Statement of Results. The classical Schur functions $s_\rho(\mathbf{x})$ are a special class of power series defined in infinitely many variables $\mathbf{x} = (x_1, x_2, ...)$ and indexed by partitions ρ (we refer the reader to Section 4.1 for a precise definition). Schur functions are classically known to form an orthonormal, integral basis of the ring of symmetric functions and they have proven ubiquitous in many areas of mathematics.

Another (rational) basis for the ring of symmetric functions is given by products of the power-sum functions $p_k(\mathbf{x})$. The classical Murnaghan-Nakayama rule provides a simple way to write the symmetric function $p_k s_\rho$ in the Schur basis:

$$p_k s_\rho = \sum_\sigma (-1)^{ht(\sigma\backslash\rho)} s_\sigma \tag{4–1}$$

where the sum is over all ways of adding a length k *border strip* to ρ and ht is the height (ie. the number of rows) of the border strip, minus 1.

Loop Schur functions naturally generalize the combinatorial definition of Schur functions and have previously been studied by Lam and Pylyavskyy in the context of loop symmetric functions ([40]). Given a positive integer n, the loop Schur functions $s_\rho[n]$ are power series in infinitely many variables $\{x_{i,j} : i \in \mathbb{Z}_n, j \in \mathbb{N}\}$ and indexed by partitions ρ. There is also

a notion of *loop power-sum functions* $p_k[n]$. In Section 4.2 we provide a combinatorial proof for the natural generalization of the Murnaghan-Nakayama rule.

THEOREM 4.1.

$$p_k[n]s_\rho[n] = \sum_\sigma (-1)^{ht(\sigma\backslash\rho)} s_\sigma[n]$$

where the sum is over all ways of adding length kn border strips to ρ.

By forgetting the index $i \in \mathbb{Z}_n$, Theorem 4.1 specializes to the classical Murnaghan-Nakayama rule and, to the best of our knowledge, our proof provides a *new* combinatorial proof of the classical result.

For any $0 \leq l < n$, we introduce in Section 4.1 the *l-shifted loop Schur functions* $s^l_\rho[n]$, a close variant of the loop Schur functions (in particular, $s^0_\rho[n] = s_\rho[n]$). We prove the following identity in Section 4.3.

THEOREM 4.2. *For* $l \neq 0$,

$$0 = \sum_\sigma (-1)^{ht(\sigma\backslash\rho)} s^l_\sigma[n]$$

where the sum is over all ways of adding length kn border strips to ρ.

4.0.5. OUTLINE OF THE PROOFS. Theorem 4.1 is proven as a corollary of the following identity

THEOREM 4.3. *For any* $N \geq kn + l(\rho)$,

$$p_{k,N}[n]x^\delta s_{\rho,N}[n] = \sum (-1)^{ht(\sigma\backslash\rho)} x^\delta s_{\sigma,N}[n]$$

where

$$x^\delta := (x_{-1,1} \cdot ... \cdot x_{-N,1})(x_{-2,2} \cdot ... \cdot x_{-N,2})...(x_{-N,N}),$$

$p_{k,N}[n]$, $s_{\rho,N}[n]$ are defined by specializing $x_{i,j} = 0$ if $j > N$, and the sum is over all ways of adding length kn border strips to ρ.

To prove Theorem 4.3, we begin in Section 4.2.1 by interpreting the product $x^\delta s_{\rho,N}[n]$ combinatorially in a way which will be convenient for later arguments - the key is a sign-reversing involution previously defined in [14]. In Section 4.2.2, we define a generating function $F_{\rho,N}[n]$ for certain combinatorial gadgets related to those discussed in Section 4.2.1. In Sections 4.2.3 and 4.2.4 we define sign-reversing involutions on the terms in $F_{\rho,N}[n]$ so that the sum of the weights of the fixed terms can be identified with the left and right-hand sides, respectively, of Theorem 4.3. This proves that both sides are equal to $F_{\rho,N}[n]$, thus proving the theorem. Theorem 4.2 is proven in Section 4.3 using similar techniques.

4.1. Definitions and Notation

We now make precise the objects which appeared in the statements of Theorems 4.1 and 4.2. Before defining loop Schur functions, we begin by briefly recalling the classical Schur functions (see eg. [44]). Though originally defined as quotients of antisymmetric functions, Schur functions can be defined combinatorially as generating functions of semi-standard Young tableaux as we now describe.

To a partition ρ we can associate a Young diagram (which we also call ρ), a northwest justified collection of boxes where the rows encode the sizes of the parts of ρ. For example, if ρ is the partition $(4, 3, 3, 2)$, the associated Young diagram is given in Figure 4.1.

A *tableau* of ρ is an assignment of positive integers to the boxes of ρ. A *semi-standard Young tableau* (SSYT) of ρ is a numbering of the boxes so that numbers are weakly increasing left to right and strictly increasing top to bottom – an example is given in Figure 4.2.

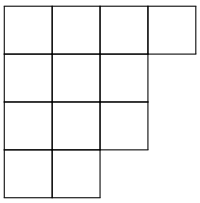
$\rho =$ 

FIGURE 4.1. The Young diagram associated to the partition $(4, 3, 3, 2)$.

$T =$

1	1	2	4
2	3	3	
4	4	6	
7	7		

FIGURE 4.2. A SSYT of the Young diagram.

For each $\square \in \rho$, we define the *weight* $w(\square, T)$ to be the number appearing in that square. To each tableau $T \in SSYT(\rho)$ we can associate a monomial

$$x^T := \prod_{\square \in \rho} x_{w(\square,T)}.$$

For example, to the SSYT in Figure 4.2 we associate the monomial $x^T = x_1^2 x_2^2 x_3^2 x_4^3 x_6 x_7^2$.

The Schur functions can be defined by the rule

$$s_\rho := \sum_{T \in SSYT(\rho)} x^T.$$

It is not obvious, but this definition of Schur functions coincides with the classical definition (cf. [44] or [14] for a combinatorial proof).

The power-sum functions are defined as

$$p_k := \sum_i x_i^k.$$

The sum in the classical Murnaghan-Nakayama rule (4–1) is over all Young diagrams $\sigma \supset \rho$ such that the complement is connected, contains k boxes, and contains no 2×2 square. We say that σ is obtained from ρ by adding a length k *border strip* and $ht(\sigma \setminus \rho)$ is the number of rows the border strip occupies, minus 1.

4.1.1. Loop Schur Functions. In the current paper, we study loop Schur functions which we now define. For a positive integer n and partition ρ, the *colored* Young diagram (ρ, n) is obtained by coloring the boxes of the Young diagram by their *content* modulo n. In other words if $\square$ is in the ith row and the jth column (row and column indexing begins with 1), we color it $c(\square) := j - i \mod n$. For example, if $\rho = (4, 3, 3, 2)$ and $n = 3$, the colored Young diagram is given in Figure 4.3.

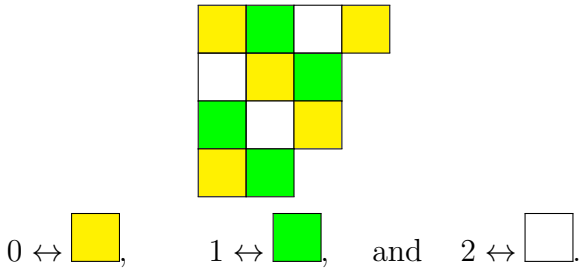


Figure 4.3. The colored Young diagram associated to $(4, 3, 3, 2)$ with $n = 3$. The bottom row describes the correspondence between the chosen colors and elements of $\mathbb{Z}_3$.

We let $\rho[i]$ denote the collection of boxes with color i. To each semi-standard Young tableau $T \in SSYT(\rho, n)$, we associate a monomial in n infinite sets of variables $\{x_{i,j} : i \in \mathbb{Z}_n, j \in \mathbb{N}\}$:

$$x^T := \prod_{i=0}^{n-1} \prod_{\square \in \rho[i]} x_{i,w(\square,T)}. \tag{4–2}$$

For example, we associate the monomial

$$x^T = x_{0,1}x_{0,3}x_{0,4}x_{0,6}x_{0,7}x_{1,1}x_{1,3}x_{1,4}x_{1,7}x_{2,2}^2x_{2,4}$$

to the SSYT given in Figure 4.4.

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 4 & 6 \\ \cline{1-3} 7 & 7 \\ \cline{1-2}\end{array}$$

FIGURE 4.4. A SSYT of the colored Young diagram.

DEFINITION 4.4. The *loop Schur function* associated to (ρ, n) is defined by

$$s_\rho[n] := \sum_{T \in SSYT(\rho,n)} x^T.$$

Power-sum functions also naturally generalize to the colored setting.

DEFINITION 4.5. The *loop power-sum functions* are defined by

$$p_k[n] := \sum_j \left(\prod_{i=0}^{n-1} x_{i,j} \right)^k.$$

REMARK 4.6. By definition we have the following specializations:

$$p_k[n]|_{(x_{i,j}=x_j)} = p_{kn} \text{ and } s_\rho[n]|_{(x_{i,j}=x_j)} = s_\rho.$$

It follows immediately that Theorem 4.1 specializes to the classical identity (4–1) by forgetting the index i.

4.1.2. Shifted Loop Schur Functions. To define the l-shifted loop Schur functions appearing in Theorem 4.2, we define the shifted weight

$$w^l(\square, T) := w(\square, T) + \frac{l \cdot c(\square)}{n}$$

and the corresponding monomial

$$x^{T,l} := \prod_{i=0}^{n-1} \prod_{\square \in \rho[i]} x_{i,w^l(\square,T)} \tag{4–3}$$

where the variables appearing in the monomial now belong to the set $\{x_{i,j} : i \in \mathbb{Z}_n, j \in \frac{1}{n}\mathbb{Z}\}$.

Definition 4.7. The *l-shifted loop Schur function*[1] associated to (ρ, n) is defined by

$$s^l_\rho[n] := \sum_{T \in SSYT(\rho,n)} x^{T,l}.$$

Remark 4.8. By definition, $s^0_\rho[n] = s_\rho[n]$.

4.2. Proof of the Loop M-N Rule

4.2.1. Involutions: Round One. In this section we give a combinatorial description of the product $x^\delta s_{\rho,N}[n]$ which will prove useful in later arguments. For a given Young diagram ρ, and positive integers n and $N > l(\rho)$, define $\hat{\rho}$ to be the diagram obtained by adding adding a staircase of size N to the left of ρ. In other words, we add $N - i + 1$ boxes to the left of the ith row of ρ (if $i > l(\rho)$, the right edge of the new boxes should be justified with the right edge of the new boxes in the rows above it). As before, the diagram is colored by content modulo n. Consider pairs (T, τ) where

[1]The l-shifted Schur functions here should not be confused with the shifted Schur functions defined in [49]. We shift the index of the variables wheras they shift the variables themselves. Moreover, they sum over *reverse* tableaux.

(a) T is a tableau (not necessarily semi-standard) of $\hat{\rho}$, and

(b) $\tau = (\tau_1, ..., \tau_N)$ is a labeling of the N rows of $\hat{\rho}$ with the numbers $1, ..., N$ (considered as a permutation of $\{1, ..., N\}$ given by $i \to \tau_i$).

Let $\mathcal{T}_{\rho,n,N}$ be the set of such pairs (T, τ) which satisfy the following conditions:

(i) T only contains the numbers $1, ..., N$.

(ii) The rows of T are weakly increasing.

(iii) The leftmost entry in the jth row is at least τ_j.

REMARK 4.9. When confusion does not arise, we omit the subscripts and write $\mathcal{T} = \mathcal{T}_{\rho,n,N}$.

EXAMPLE 4.10. For $\rho = (2, 1)$, $n = 3$, and $N = 5$, two examples of elements which belong to $\mathcal{T}$ are given in Figure 4.5.

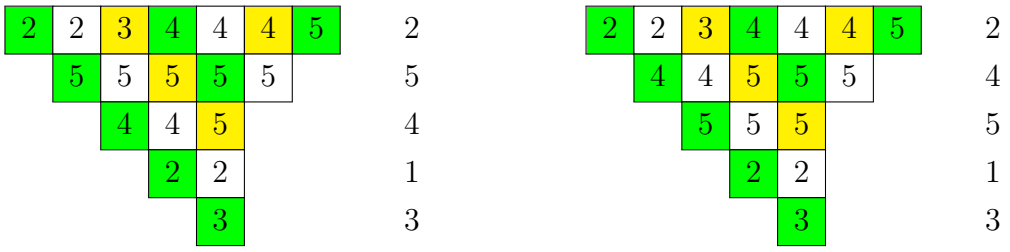


FIGURE 4.5. Two elements of $\mathcal{T}$ which are interchanged by I_1.

As in (4–2), we can associate to each T a monomial x^T. Let $(-1)^\tau$ denote the sign of the permutation τ. We have the following identity.

LEMMA 4.11.

$$x^\delta s_{\rho,N}[n] = \sum_{(T,\tau)\in\mathcal{T}} (-1)^\tau x^T.$$

PROOF. We consider a sign reversing involution which cancels pairs of terms in the sum. We then identify the sum of the fixed terms as $x^{\delta}s_{\rho,N}[n]$. The involution we use is defined in [14], the setting here is only slightly different. We include the details for completeness.

The involution I_1 is defined on a pair (T,τ) as follows:

(I) Look for the rightmost and then highest vertical domino $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ such that the upper entry is at least the lower entry.

(II) Swap every box to the left of the upper box in (I) with the box directly to its southeast.

(III) Swap the elements of τ which index these two rows.

Define $I_1(T,\tau)$ to be the new tableau and permutation obtained through this process. We will often abuse notation and write $I_1(T,\tau) = (I_1(T), I_1(\tau))$ to reference the action on the tableau or the permutation alone. See Figure 4.5 above for two elements of $\mathcal{T}$ which are interchanged by I_1.

First of all, $I_1(T,\tau)$ is an involution because the location of the domino in step (I) is preserved under the action. It is easy to see that $x^T = x^{I_1(T)}$ since the involution moves entries along diagonals on which the colors are constant. It is also easy to see that $(-1)^{\tau} = -(-1)^{I_1(\tau)}$ whenever (T,τ) is not fixed by I_1 because switching two elements of the labeling τ corresponds to multiplying the corresponding permutation by a transposition. Therefore, we conclude that

$$\sum_{\mathcal{T}}(-1)^{\tau}x^T = \sum_{\mathcal{T}^{I_1}}(-1)^{\tau}x^T.$$

where $\mathcal{T}^{I_1}$ is the set of elements in $\mathcal{T}$ which are fixed by I_1.

It is left to analyze $\mathcal{T}^{I_1}$. If (T,τ) is fixed by I_1, then T must be a column-strict tableau. In particular, the column immediately to the left of $\rho \subset \hat{\rho}$ should read $1,...,N$ top to bottom. In particular, this implies that the entries of $\hat{\rho} \setminus \rho$ must be 1 in the first row, 2 in the second

row, etc. and τ is forced to be the identity. The constraint imposed on the entries of ρ are simply that they form a semi-standard tableau. The entries in $\hat{\rho} \setminus \rho$ contribute x^{δ} to each monomial x^T and the sum over all semi-standard tableaux of ρ contributes $s_{\rho,N}[n]$. $\square$

4.2.2. Master Generating Function. In this section we define a master generating function $F_{\rho,N}[n]$ which is shown in subsequent sections to equal both the left and right-hand sides of the identity in Theorem 4.3. To that end, we fix a partition $\rho = (\rho_1, ..., \rho_l)$, positive integers n and k, and a positive integer N satisfying $N \geq kn + l$. For any $i \in \{1, ..., N\}$, let $\hat{\rho}_i$ be the diagram obtained by adding kn boxes to the right of the ith row of $\hat{\rho}$. The combinatorial objects we want to consider are pairs (T, τ) where

(a) T is a tableau of the diagram $\hat{\rho}_i$ for some i, and

(b) $\tau = (\tau_1, ..., \tau_N)$ is a labeling of the N rows of $\hat{\rho}_i$ with the numbers $1, ..., N$ (considered as a permutation in S_N).

Let $\mathcal{S}_{\rho,n,k,N}$ be the set of such tableaux which satisfy the same three conditions (i) - (iii) required of the set $\mathcal{T}$ in Section 4.2.1.

Example 4.12. For $\rho = (2, 1)$, $n = 3$, $k = 1$, and $N = 5$, two examples of elements which belong to $\mathcal{S}$ are given in Figure 4.6.

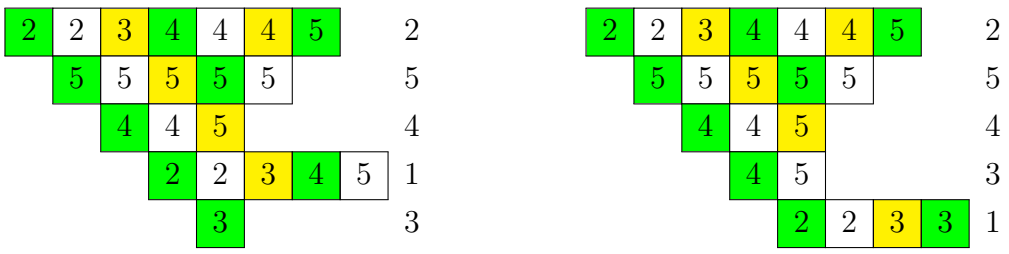


Figure 4.6. Two elements of $\mathcal{S}$ which are interchanged by I_2.

To each (T, τ), we assign a monomial x^T as before. We define the generating function $F_{\rho,N}[n]$ by

$$F_{\rho,N}[n] := \sum_{(T,\tau)\in\mathcal{S}} (-1)^\tau x^T. \tag{4–4}$$

4.2.3. Involutions: Round Two.

Lemma 4.13.

$$F_{\rho,N}[n] = p_{k,N}[n] x^\delta s_{\rho,N}[n]$$

Proof. We define an involution on the terms of $F_{\rho,N}[n]$ which cancels terms in pairs. The remaining terms are seen to coincide with the left-hand side of Theorem 4.3. We define the involution I_2 on sets of pairs (T, τ) as follows.

If T is a tableau of $\hat{\rho}_i$ and the knth entry of row i is τ_i, then define $I_2(T, \tau) = (T, \tau)$. Otherwise, the knth entry of row i is l with $l > \tau_i$ because of conditions (ii) and (iii) in Section 4.2.1. Then $I_2(T, \tau)$ is defined by the following process:

(I) Remove the first kn boxes (along with their labels) of the ith row, and shift the remaining boxes in that row to the left by kn units.

(II) Interchange τ_i and τ_j where j is the row with $\tau_j = l$.

(III) Slide the boxes in row j to the right by kn units and reinsert the kn boxes (along with their labels) in row j.

Notice that when I_2 does not fix an element, it sends a tableau of $\hat{\rho}_i$ to a tableau of $\hat{\rho}_j$ with $j \neq i$. We will again abuse notation and write $I_2(T, \tau) = (I_2(T), I_2(\tau))$. See Figure 4.6 for an illustration of two elements of $\mathcal{S}$ which are interchanged by I_2.

It is easy to see that I_2 is an involution, $x^T = x^{I_2(T)}$,and if (T, τ) is not a fixed point of I_2, then $(-1)^\tau = -(-1)^{I_2(\tau)}$. Therefore, the terms which are not fixed cancel in pairs in the sum (4–4).

By definition, the terms which are fixed correspond to those where the kn leftmost boxes in $\hat{\rho}_i$ all contain the number τ_i. If we set $\underline{N} := \{1, ..., N\}$, then we obtain a bijection between the sets $\mathcal{S}^{I_2}$ and $\mathcal{T} \times \underline{N}$ by mapping (T, τ) to (T', τ, i) where T' is obtained by removing the leftmost kn boxes from row i and sliding the remaining boxes to the left. Moreover, this bijection preserves $(-1)^\tau$ and the weights are related by the equation $x^T = \left(\prod_{j=0}^{n-1} x_{j,i}\right)^k x^{T'}$. We have

$$\begin{aligned}
F_{\rho,N}[n] &= \sum_{(T,\tau)\in\mathcal{S}} (-1)^\tau x^T \\
&= \sum_{(T,\tau)\in\mathcal{S}^{I_2}} (-1)^\tau x^T \\
&= \sum_{(T',\tau,i)\in\mathcal{T}\times\underline{N}} (-1)^\tau \left(\prod_{j=0}^{n-1} x_{j,i}\right)^k x^{T'} \\
&= \left(\sum_{i=1}^{N} \left(\prod_{j=0}^{n-1} x_{j,i}\right)^k\right) \left(\sum_{(T',\tau)\in\mathcal{T}} (-1)^\tau x^{T'}\right) \\
&= p_{k,N}[n] x^\delta s_{\rho,N}[n]
\end{aligned}$$

where the last equality follows from Lemma 4.11 and the definition of the loop power-sum functions. □

4.2.4. Involutions: Round Three.

Lemma 4.14.

$$F_{\rho,N}[n] = \sum (-1)^{ht(\sigma\backslash\rho)} x^\delta s_{\sigma,N}[n]$$

where the sum is over all ways of adding a length kn border strip to ρ.

PROOF. We define a different involution on $\mathcal{S}$ which cancels terms in the sum $F_{\rho,N}[n]$ in pairs. The sum of the weights of the remaining terms is then seen to coincide with $\sum(-1)^{ht(\sigma\setminus\rho)}x^{\delta}s_{\sigma,N}[n]$. The involution I_3 is defined as follows.

First, if (T,τ) is a tableau on $\hat{\rho}_i$ and two rows of $\hat{\rho}_i$ have the same number of boxes, then one of those rows must be i, call the other one j (it is not hard to see that at most two rows can have equal length). Define $I_3(T,\tau) = (T^*,\tau^*)$ where T^* is obtained by swapping the entries of rows i and j and τ^* is obtained by swapping τ_i and τ_j.

EXAMPLE 4.15. See Figure 4.7 for an example of two elements of $\mathcal{S}$ which are interchanged by I_3.

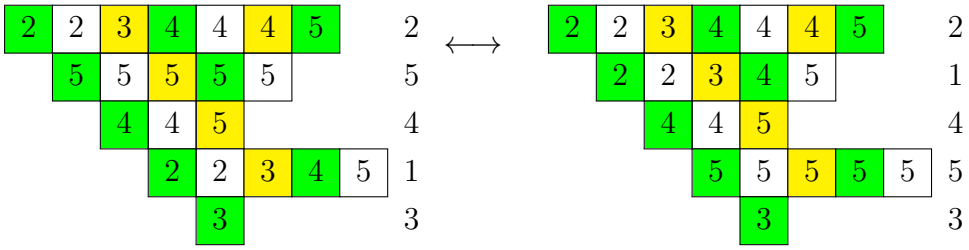


FIGURE 4.7. Two elements of $\mathcal{S}$ which are interchanged by I_3.

If all rows of $\hat{\rho}_i$ have distinct size, then $I_3(T,\tau)$ is obtained as follows:

(I) Slide the ith row of $\hat{\rho}_i$ northwest until the length of the rows are strictly decreasing, slide τ_i upward with the row, call this new tableau (T',τ').

(II) Apply the involution I_1 from the proof of Lemma 4.11 to (T',τ').

(III) Reverse step (I).

REMARK 4.16. The important thing to notice is that the new diagram obtained in Step I can be identified with $\hat{\sigma}$ for some σ which is obtained from ρ by adding a length kn border strip.

EXAMPLE 4.17. Figure 4.8 illustrates the involution I_3.

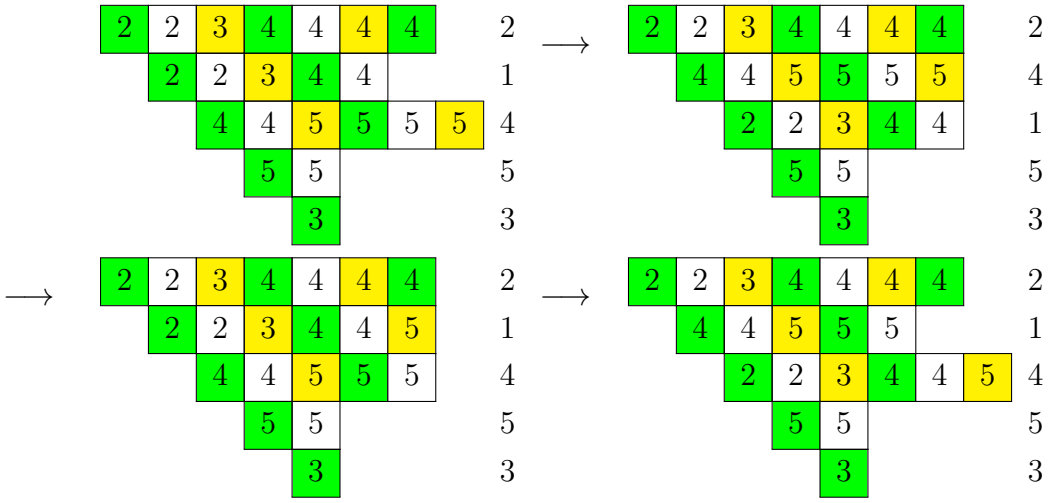


FIGURE 4.8. The involution I_3 in three steps. The first diagram is (T, τ), the second is (T', τ'), the third is $I_1(T', \tau')$, and the fourth is $I_3(T, \tau)$.

As with the other involutions, it is easy to see that I_3 reverses the sign and preserves the weight for all elements $(T, \tau) \in \mathcal{S}$ which are not fixed. The elements of $\mathcal{S}$ which are fixed by I_3 are those which get fixed by I_1 in step (II) above. Therefore, step (I) above defines a map $f : \mathcal{S}_\rho^{I_3} \to \coprod \mathcal{T}_\sigma^{I_1}$ where the union is over all σ which are obtained by adding a length kn border strip to ρ. The map f is clearly invertible, so f is a bijection. The function f preserves the weight but does not quite preserve the sign. In fact, f introduces a factor of -1 for every shift in step (I) (corresponding to multiplying τ by a transposition). This

introduces a factor of $(-1)^{ht(\sigma\backslash\rho)}$. Putting it all together, we have

$$F_{\rho,N}[n] = \sum_{(T,\tau)\in\mathcal{S}_\rho} (-1)^\tau x^T = \sum_{(T,\tau)\in\mathcal{S}_\rho^{I_3}} (-1)^\tau x^T$$
$$= \sum_\sigma (-1)^{ht(\sigma\backslash\rho)} \sum_{(T',\tau')\in\mathcal{T}_\sigma^{I_1}} (-1)^\tau x^T = \sum_\sigma (-1)^{ht(\sigma\backslash\rho)} x^\delta s_{\sigma,n}[N]$$

where the last equality follows from Lemma 4.11. □

Lemmas 4.13 and 4.14 complete the proof of Theorem 4.3. Dividing both sides by x^δ and taking $N \to \infty$ proves Theorem 4.1.

4.3. Proof of Shifted Identity

In order to prove Theorem 4.2, define the degree of the variable $x_{i,j}$ to be j. Then Theorem 4.2 follows from the next result by taking $N \to \infty$.

Theorem 4.18. *The leading term of*

$$\sum (-1)^{ht(\sigma\backslash\rho)} s^l_{\sigma,N}[n]$$

has degree bounded below by $N - kn - \frac{l}{n}N$.

Proof. We define the generating function $F^l_{\rho,N}[n]$ exactly as we defined $F_{\rho,N}[n]$ above, except we use the shifted weight defined in (4–3). Since the involution I_3 preserves the shifted weight (it only moves boxes along diagonals), Lemma 4.14 carries through unchanged and proves that

$$F^l_{\rho,N}[n] = x^{\delta,l} \sum (-1)^{ht(\sigma\backslash\rho)} s^l_{\sigma,N}[n]$$

where $x^{\delta,l}$ is the shifted monomial associated to the *standard* tableau on $\hat{\emptyset}$, illustrated in Figure 4.9

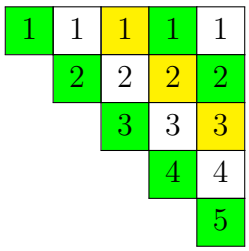


FIGURE 4.9. The standard tableau on $\hat{\emptyset}$.

REMARK 4.19. It is easy to see that $x^{\delta,l}$ has the smallest degree of any tableau on $\hat{\emptyset}$ which weakly increases along rows.

Define $\mathcal{S}' \subset \mathcal{S}$ to be the subset of $\mathcal{S}$ consisting of tableaux of $\hat{\rho}_i$ where the entries in the ith row do not exceed $N - kl$. We define an involution I_4 on the elements of $\mathcal{S}'$ as follows.

(I) Remove the first kn boxes from row i, slide the remaining boxes kn units to the left and add kl to each remaining entry.

(II) If m is the rightmost entry of the boxes which were removed in (I) ($m \leq N - kl$ by definition of $\mathcal{S}'$), subtract kl from from each entry of row j where $\tau_j = m + kl$, and then slide them to the right by kn units and insert the boxes removed in (I).

(III) Switch τ_i and τ_j.

Clearly I_4 is sign reversing and it preserves weight (this is the reason for adding/subtracting kl to the entries when we slide them). Therefore,

$$F^l_{\rho,N}[n] = \sum_{(T,\tau)\in\mathcal{S}\setminus\mathcal{S}'} (-1)^\tau x^{T,l}.$$

But the rightmost entry of the ith row of every tableau in $\mathcal{S} \setminus \mathcal{S}'$ is at least $N - kl$ and this contributes at least $N - kl - \frac{l}{n}N$ to the degree of the associated monomial. This implies that the degree of the associated monomial is at least $deg(x^{\delta,l}) + N - kl - \frac{l}{n}N$. Therefore, the degree of the leading term of $F^l_{\rho,N}[n]$ (and hence $x^{\delta,l}\sum(-1)^{ht(\sigma\setminus\rho)}s^l_{\sigma,N}[n]$) is at least $deg(x^{\delta,l}) + N - kl - \frac{l}{n}N$. Dividing by $x^{\delta,l}$ proves the theorem. $\square$

CHAPTER 5

The Gromov-Witten/Donaldson-Thomas Correspondence

In this chapter we employ *gerby* localization (cf. [13] for the origins of the adjective gerby) as well as various tools from combinatorics, linear algebra, Hurwitz theory, and representation theory of generalized symmetric groups in order to prove the gerby Gopakumar-Mariño-Vafa formula. We describe how this formula should be viewed as a local orbifold Gromov-Witten/Donaldson-Thomas correspondence for certain toric CY 3-folds.

5.0.1. Statement of Results. The GMV formula, proven independently in [42] and [50], evaluates certain generating functions of cubic Hodge integrals on moduli spaces of curves in terms of Schur functions, a special basis of the ring of symmetric functions. The formula can be interpreted as one instance of the GW/DT correspondence for CY3s. In this chapter, we generalize the GMV formula to $\mathbb{Z}_n$-Hodge integrals and we show that this formula can be viewed as one instance of the orbifold GW/DT correspondence.

In particular, we consider generating functions $\tilde{V}_\mu^\bullet(a)$ of cubic $\mathbb{Z}_n$-Hodge integrals on moduli spaces of stable maps to the classifying space $\mathcal{B}\mathbb{Z}_n$. These generating functions are indexed by conjugacy classes μ of the generalized symmetric group $\mathbb{Z}_n \wr S_d$ and are closely related to the GW orbifold vertex developed in Chapter 2. In place of the Schur functions in the usual GMV formula, we introduce generating functions $\tilde{P}_\lambda(a)$ which are specializations of *loop* Schur functions, discussed in Chapter 4. These generating functions are indexed by irreducible representations λ of $\mathbb{Z}_n \wr S_d$ and are closely related to the DT orbifold vertex developed in [8]. The main result is a relation via the character values $\chi_\lambda(\mu)$ of $\mathbb{Z}_n \wr S_d$.

THEOREM (Theorem 5.7). *After an explicit change of variables,*

$$\tilde{V}_\mu^\bullet(a) = \sum_\lambda \tilde{P}_\lambda(a) \frac{\chi_\lambda(\mu)}{z_\mu}$$

There are n distinct $\mathbb{Z}_n$-gerbes $\mathcal{G}_k$ $(0 \leq k < n)$ over $\mathbb{P}^1$ classified by $H^2(\mathbb{P}^1, \mathbb{Z}_n)$. We define $\mathcal{X}$ to be a *local $\mathbb{Z}_n$-gerbe over $\mathbb{P}^1$* if $\mathcal{X}$ is isomorphic to the total space of a rank two Calabi-Yau orbifold bundle over some $\mathcal{G}_k$. Applying the gluing algorithm of Theorem 2.17 and [8], Theorem 5.7 leads to a proof of the orbifold GW/DT correspondence for local $\mathbb{Z}_n$-gerbes over $\mathbb{P}^1$.

THEOREM (Theorem 5.8). *After an explicit change of variables, the GW potential of any local $\mathbb{Z}_n$-gerbe over $\mathbb{P}^1$ is equal to the reduced, multi-regular DT potential.*

This is the first example of the GW/DT correspondence for orbifold targets with non-trivial curve classes contained in the singular locus.

5.0.2. OUTLINE OF PROOF. After setting up notation and giving a precise statement of Theorems 5.7 and 5.8 in Section 5.1, we study the geometry of the *framed* GW vertex $\tilde{V}_\mu^\bullet(a)$ in Section 5.2. In particular, we develop a set of bilinear equations relating the GW vertex to generating functions of certain rubber integrals. In Section 5.2.5, we interpret these rubber integrals in terms of wreath Hurwitz numbers and apply the Burnside formula to write the bilinear relations in terms of the characters of the generalized symmetric group $\mathbb{Z}_n \wr S_d$. We then show in Sections 5.2.6 and 5.3 that these relations uniquely determine the GW vertex. Sections 5.4, 5.5, and 5.6 are devoted to proving that the DT vertex also satisfies these bilinear relations. In Section 5.4, we reinterpret the main results from Chapter 4 in the current context and we recall a hook-length formula from [24] and [48] which relates the loop

Schur functions to the framed DT vertex $\tilde{P}_\lambda(a)$. In Section 5.5, we study the representation theory of $\mathbb{Z}_n \wr S_d$ where the main tool is the wreath Fock space. Finally, in Section 5.6 we put everything together to prove Theorem 5.7. In Section 5.7 we use the gluing rules developed Chapter 2 and [8] to show how the GW/DT correspondence for local $\mathbb{Z}_n$-gerbes over $\mathbb{P}^1$ follows from Theorem 5.7.

5.1. Definitions and Notation

In this section we set up notation which will be used throughout the chapter and we give a precise statement of the main results.

5.1.1. Partitions. For each positive integer n we fix a generator of the cyclic group

$$\mathbb{Z}_n = \left\langle \xi_n := e^{\frac{2\pi\sqrt{-1}}{n}} \right\rangle.$$

When no confusion arises, we write the generator simply as ξ. It is well known that n-tuples of partitions naturally correspond to conjugacy classes and irreducible representations of $\mathbb{Z}_n \wr S_d$, see eg. [44]. We will use μ and ν to denote n-tuples of partitions corresponding to conjugacy classes and reserve λ and σ to refer to irreducible representations. We let $\chi_\lambda(\mu)$ denote the value of the character of the irreducible representation λ on the conjugacy class μ.

Consider the n-tuple of partitions

$$\mu = \left((d_1^0, ..., d_{l_0}^0), ..., (d_1^{n-1}, ..., d_{l_{n-1}}^{n-1}) \right)$$

with $d_j^i \in \mathbb{N}$ (we assume when using this notation that $d_1^i \geq d_2^i \geq ...$). Let $\mu^i = (d_1^i, ..., d_{l_i}^i)$ denote the partition indexed by i and let μ^{tw} correspond to the n-tuple of *twisted* partitions

$(\emptyset, \mu^1, ..., \mu^{n-1})$. At times it will be convenient to write μ as a multiset $\{\xi^i d_j^i\}$ where the power of ξ keeps track of which μ^i the d_j^i came from. Let $l(\mu) := \sum l_i$ denote the length of μ. Set $|\mu^i| := \sum_j d_j^i$ and $|\mu| := \sum |\mu^i|$. Let $\underline{\mu}$ denote the underlying partition of μ that forgets the $\mathbb{Z}_n$ decorations. We define $-\mu := \{\xi^{-i} d_j^i\}$, ie. it is the n-tuple of partitions with opposite twistings. We also define

$$z_\mu := |\mathrm{Aut}(\mu)| \prod n d_j^i$$

to be the order of the centralizer of any element in the conjugacy class of μ.

Suppose $\lambda = (\lambda_0, ..., \lambda_{n-1})$. Via n-quotients (described explicitly in Section 5.5.2) λ can be identified with a partition of nd where $d = |\lambda|$. We denote this corresponding partition by $\bar{\lambda}$. We write $\bar{\lambda} = \{(i, j)\}$ where i indexes the rows and j indexes the columns of the Young diagram corresponding to $\bar{\lambda}$. We will often think of $\bar{\lambda}$ as a colored Young diagram where the box (i, j) has color $j - i \mod n$. We denote the boxes with color k by $\bar{\lambda}[k]$. For $\square \in \bar{\lambda}$, we let $h_k(\square)$ denote the number of color k boxes in the hook defined by $\square$ and we define

$$n_k(\bar{\lambda}) := \sum_i (i-1)(\#\text{ of color k boxes in the } i\text{th row}).$$

We let γ denote a tuple of *nontrivial* elements in $\mathbb{Z}_n$. We define $m_i(\gamma)$ to be the number of occurrences of $\xi^i \in \mathbb{Z}_n$ in γ.

5.1.2. Gromov-Witten Theory. Given μ and γ as above, let $\overline{\mathcal{M}}_{g,\gamma+\mu}(\mathcal{B}\mathbb{Z}_n)$ denote the moduli stack of stable maps to the classifying space with $m_i(\gamma) + l_i(\mu)$ marked points twisted by ξ^i. The marked points in μ are indexed by $\{(i, j) : 0 \leq i < n, 1 \leq j \leq l_i\}$ and we denote the corresponding psi classes by $\psi_{i,j}$.

For any $a \in \frac{1}{n}\mathbb{Z}$, the special cubic Hodge integrals we are interested in are:

$$V_{g,\gamma}(\mu;a) := \frac{(a+1)^{l_0}}{|\mathrm{Aut}(\mu)|} \prod_{i=0}^{n-1} \prod_{j=1}^{l_i} \frac{\prod_{k=0}^{d_j^i-1}(ad_j^i + \frac{i}{n} + k)}{(-1)^{d_j^i} d_j^i \cdot d_j^i!}$$

$$\int_{\overline{\mathcal{M}}_{g,\gamma+\mu}(\mathcal{B}\mathbb{Z}_n)} \frac{\Lambda^0(1)\Lambda^1(a)\Lambda^{-1}(-a-1)}{\delta(a)\prod_{i=0}^{n-1}\prod_{j=1}^{l_i}\left(\frac{1}{d_j^i} - \psi_{i,j}\right)} \tag{5–1}$$

where

$$\Lambda^i(t) := (-1)^{\mathrm{rk}} \sum_{j=0}^{\mathrm{rk}} (-t)^{\mathrm{rk}-j} \lambda_j^{\xi^i}$$

with $\mathrm{rk} := \mathrm{rk}\left(\mathbb{E}_{\xi^i}\right)$ and $\delta(a)$ is the function which takes value $-a^2 - a$ on the connected component of the moduli space which parametrizes trivial covers of the source and takes value 1 on all other components.

Remark 5.1. The parameter a is often referred to as the *framing*. In the notation of Chapter 2, we have $\vec{w} = (1, -a-1, a)$.

Introduce formal variables, u and x_i to track genus and marks. Also introduce the variables p_μ with formal multiplication defined by concatenating the indexing partitions. Then we define

$$V_\mu^\bullet(x,u;a) := \exp\left(\sum_{g,\gamma,\nu} V_{g,\gamma}(\nu;a) u^{2g-2+l(\nu)} \frac{x^\gamma}{\gamma!} p_\nu\right)[p_\mu]$$

where

$$\frac{x^\gamma}{\gamma!} := \prod_{i=1}^{n-1} \frac{x_i^{m_i(\gamma)}}{m_i(\gamma)!}$$

and $[p_\mu]$ denotes "the coefficient of p_μ". By definition, $V_\mu^\bullet(x,u;a)$ is the one-leg A_{n-1} orbifold GW vertex defined in Chapter 2.

DEFINITION 5.2. The framed GW vertex is defined by

$$\tilde{V}_\mu^\bullet(a) := \prod_{i=1}^{n} (\sqrt{-1}\xi_{2n}^i)^{l_i} V_\mu^\bullet(x,u;a). \tag{5–2}$$

where $l_n := l_0$.

5.1.3. DONALDSON-THOMAS THEORY. Let $q_0, ..., q_{n-1}$ be formal variables (always assume that the index of q_k is computed modulo n) and define $q := q_0 \cdots q_{n-1}$. For $\bar{\lambda}$ as above, define

$$P_\lambda(q_0, ..., q_{n-1}) := \frac{1}{\prod_{\square \in \bar{\lambda}} \left(1 - \prod_i q_i^{h_i(\square)}\right)}. \tag{5–3}$$

By Theorem 12 in [8], $P_\lambda(-q_0, ..., q_{n-1})$ is the reduced, multi-regular one-leg A_{n-1} orbifold DT vertex.

REMARK 5.3. Notice the sign discrepancy between (5–3) and the DT vertex.

DEFINITION 5.4. The framed DT vertex is defined by

$$\tilde{P}_\lambda(a) := \left(\left((-\xi_{2n})^{|\lambda|} \prod \xi_n^{l|\lambda_l|} \right)^n \prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^{-a} \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)} q^{\frac{d}{2}} (-1)^d \prod_i q_i^{n_i(\bar{\lambda})} P_\lambda(q_0, ..., q_{n-1}) \tag{5–4}$$

REMARK 5.5. $\chi_{\bar{\lambda}}$ is a character of S_{dn} whereas $\dim(\lambda)$ is the dimension of an irreducible representation of $\mathbb{Z}_n \wr S_d$. As we will see in Section 5.5.4, the quotient $\frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)}$ is simply a compact way of keeping track of a sign.

REMARK 5.6. In Corollary 5.35, we relate $\tilde{P}_\lambda(0)$ to loop Schur functions.

5.1.4. THE CORRESPONDENCE. We will prove the following formula.

THEOREM 5.7. *After the change of variables*

$$q \to \mathrm{e}^{\sqrt{-1}u}, \quad q_k \to \xi_n^{-1}\mathrm{e}^{-\sum_i \frac{\xi_n^{-ik}}{n}\left(\xi_{2n}^i - \xi_{2n}^{-i}\right)x_i} \ (k>0),$$

$$\tilde{V}_\mu^\bullet(a) = \sum_\lambda \tilde{P}_\lambda(a)\frac{\chi_\lambda(\mu)}{z_\mu}$$

In Section 5.7, we use Theorem 5.7 to deduce the Gromov-Witten/Donaldson-Thomas correspondence for local $\mathbb{Z}_n$-gerbes over $\mathbb{P}^1$.

THEOREM 5.8. *Let $\mathcal{X}$ be a local $\mathbb{Z}_n$-gerbe over $\mathbb{P}^1$ and let $GW(\mathcal{X})$ and $DT'_{mr}(\mathcal{X})$ denote the GW potential and the reduced, multi-regular DT potential of $\mathcal{X}$, respectively. After the change of variables*

$$q \to -\mathrm{e}^{\sqrt{-1}u}, \quad q_k \to \xi_n^{-1}\mathrm{e}^{-\sum_i \frac{\xi_n^{-ik}}{n}\left(\xi_{2n}^i - \xi_{2n}^{-i}\right)x_i} \ (k>0),$$

$$GW(\mathcal{X}) = DT'_{mr}(\mathcal{X}).$$

REMARK 5.9. Notice the sign difference in the change of variables of Theorems 5.7 and 5.8 – this difference is an artifact of Remark 5.3.

REMARK 5.10. The change of variables in Theorems 5.7 and 5.8 is predicted by Iritani's stacky Mukai vector [34] and previously appeared in [65]. We thank Jim Bryan for explaining this change of variables to us.

5.2. GEOMETRY

In this section we set up auxilary integrals on moduli spaces of relative maps into $\mathbb{P}^1$-gerbes in order to obtain bilinear relations between the vertex $\tilde{V}_\mu^\bullet(a)$ and certain rubber

integrals $\tilde{H}^\bullet_{\nu,\mu}(a)$. The rubber integrals in $\tilde{H}^\bullet_{\nu,\mu}(a)$ can be interpreted as wreath Hurwitz numbers and can be computed via Burnside's formula in terms of the represenation theory of the wreath product $\mathbb{Z}_n \wr S_d$. We use this interpretation to show that the localization relations uniquely determine $\tilde{V}^\bullet_\mu(a)$ from $\tilde{H}^\bullet_{\nu,\mu}(a)$. The method of localizing maps into gerbes in order to obtain useful relations of Hodge integrals first appeared in [13] where it was used to compute the GW invariants of $[\mathbb{C}^3/\mathbb{Z}_3]$.

5.2.1. Cyclic Gerbes over $\mathbb{P}^1$. Cyclic $\mathbb{P}^1$ gerbes will be important both for the localization computations in Section 5.2.4 and in the GW/DT comparisons in Section 5.7. We briefly collect the necessary details here. For each line bundle $\mathcal{O}(-k)$ with $0 \leq k < n$, we can define a $\mathbb{P}^1$-gerbe $\mathcal{G}_k$ with isotropy group $\mathbb{Z}_n$ and an orbifold line bundle L_k as follows.

Definition 5.11. The gerbe $\mathcal{G}_k$ is defined by pullback

$$\begin{array}{ccc} \mathcal{G}_k & \longrightarrow & \mathcal{B}\mathbb{C}^* \\ \downarrow & & \downarrow{\scriptstyle \lambda \to \lambda^n} \\ \mathbb{P}^1 & \xrightarrow{\mathcal{O}(-k)} & \mathcal{B}\mathbb{C}^* \end{array}$$

and L_k is defined to be the line bundle parametrized by the top map.

Note that the numerical degree of L_k is $-k/n$ and the action of $\mathbb{Z}_n$ on the fibers is given by multiplication by ξ_n (cf. Section 2.1.1).

The $\mathcal{G}_k$ are only distinct if we choose an isomorphism of each isotropy group with $\mathbb{Z}_n$. In other words, for each $\phi \in \mathrm{Aut}(\mathbb{Z}_n)$, we obtain an equivalence $\tilde{\phi}_k : \mathcal{G}_k \xrightarrow{\cong} \mathcal{G}_{\phi(k)}$ for each k. However, it is not true in general that $\tilde{\phi}_k^* \left(L_{\phi(k)}\right) = L_k$. This fact will be important in our discussion of 3-fold targets in Section 5.7.

One of the most useful aspects of localizing maps of curves into $\mathbb{P}^1$ gerbes is that it allows us to control the orbifold structure over 0 and ∞. To make this precise, let $\mathcal{C}$ be an orbifold

with coarse space $\mathbb{P}^1$ and orbifold structure only at 0 and ∞. Let $f : \mathcal{C} \to \mathcal{G}_k$ be a $\mathbb{C}^*$ fixed degree d map with twisting k_0 at 0 and k_∞ at ∞. Then

$$k_\infty = -dk - k_0 \mod n.$$

A more general characterization of this property was given in Section 2.1.1. To keep track of this twisting compatibility, we make the following definition.

Definition 5.12. For a decorated partition $\mu = \{\xi^i d_j^i\}$, we define the involution $g_k(\mu)$ by

$$g_k(\mu) := \{\xi^{d_j^i k - i} d_j^i\}$$

If f is a $\mathbb{C}^*$ fixed map from a disjoint union of orbifold $\mathbb{P}^1$s with degree and twisting over 0 given by μ, then the degree and twisting over ∞ is determined by $-g_k(-\mu)$ (the convention with signs seems cumbersome at the moment but it will become natural in later formulas).

5.2.2. Auxilary Integrals. In this section we set up integrals on the moduli spaces $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_k, \mu[\infty])$ which parametrize maps with fixed ramification and isotropy profile over ∞. These moduli spaces were developed in [1]. The integrals we will investigate are the following.

$$\frac{1}{|\mathrm{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0,\mu[\infty])} \mathrm{e}(R^1\pi_*((\hat{f}^* L_0)(-D) \oplus \hat{f}^* L_0^\vee(-1))) \tag{A–1}$$

where D is the locus of relative points on the universal curve with trivial isotropy and $\hat{f}$ contracts the degenerated target and maps all the way to $\mathcal{G}_0$, and for $1 \le k \le n-1$

$$\frac{1}{|\mathrm{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_k,\mu[\infty])} \mathrm{e}(R^1\pi_*(\hat{f}^* L_k \oplus \hat{f}^* L_k^\vee(-1))). \tag{A–2}$$

5.2.3. Partial Evaluations. In certain cases, we can evaluate the integrals (A–1) and (A–2) explicitly. We collect these computations here.

We begin with the first integral. As we will see in Section 5.2.4, (A–1) is equal to $V_{g,\gamma}(\mu;0)$. Therefore, we consider special choices of μ for which we can evaluate $V_{g,\gamma}(\mu;0)$. Recall that $\{d\}$ denotes the n-tuple of partitions with one untwisted part. The following evaluation will be extremely useful.

Lemma 5.13.

$$V_{g,\gamma}(\{d\};0) = \delta_{|\gamma|,0}\frac{(-1)^{d-1}}{n}\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g (d\psi)^{2g-2}.$$

Proof. By (5–1), $V_{g,\gamma}(\{d\};0)$ vanishes away from the locus of maps which parametrize trivial covers. In particular, since γ consists of nontrivial elements in $\mathbb{Z}_n$, the cover can only be trivial if $\gamma = \emptyset$. On the locus of maps which parametrize trivial covers, $\mathbb{E}_\xi \cong \mathbb{E}_{\xi^{-1}} \cong \mathbb{E}_1$. Therefore we can apply the Mumford relation to the integrand in the definition of $V_{g,\emptyset}((d);0)$. The lemma follows by pushing forward to $\overline{\mathcal{M}}_{g,1}$ which is a degree $\frac{1}{n}$ map. □

Corollary 5.14.

$$V^\bullet_\mu(0) = \left(\frac{1}{z_{\mu^0}}\prod_{j=1}^{l_0}\frac{(-1)^{d_j^0-1}}{2}\csc\left(\frac{d_j^0 u}{2}\right)\right) V^\bullet_{\mu^{tw}}(0)$$

Proof. By (5–1), the only nonzero vertex terms $V_{g,\gamma}(\mu,0)$ with $\mu^0 \neq \emptyset$ are those with $l_0 = 1$ – these invariants were computed in Lemma 5.13. The evaluations of Lemma 5.13 can be packaged using the Faber-Pandharipande identity ([25]):

$$\sum_g \left(\int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi^{2g-2}\right) t^{2g} = \frac{t}{2}\csc\left(\frac{t}{2}\right).$$

The result then follows by passing from the connected invariants to the disconnected ones by exponentiating. □

From these evaluations, we see that the $a = 0$ vertex is completely determined once we know the contributions coming from partitions μ with $\mu^0 = \emptyset$.

For the integral (A–2), we obtain the following vanishing result.

LEMMA 5.15. *The integral* (A–2) *vanishes if any of the parts of* μ *are untwisted.*

PROOF. The integral vanishes by dimensional reasons. The dimension of the moduli space is $|\mu| + 2g - 2 + |\gamma| + l(\mu)$. The degree of the integrand is $|\mu| + 2g - 2 + |\gamma| + l(\mu^{tw})$ which can be computed by the orbifold Riemann-Roch formula ([2] - Theorem 7.2.1). □

5.2.4. BILINEAR RELATIONS. We now compute the integrals (A–1) and (A–2) via localization. Beginning with (A–1), we give the target the standard $\mathbb{C}^*$ action with weight 1 (-1) on the fibers of the tangent bundle over 0 (∞). This defines a $\mathbb{C}^*$ action on the moduli space by postcomposing the map with the action. In order to choose an equivariant lift of the integrand, we lift the action from the target to the bundles $T(-\infty)$, $L_0^\vee$, and $L_0(-1)$ so that $\mathbb{C}^*$ acts on the fibers over 0 and ∞ with weights summarized in Table 5.1.

TABLE 5.1. Weights of the torus action.

	$T(-\infty)$	L_0	$L_0^\vee(-1)$
0	1	a	$-a-1$
∞	0	a	$-a$

Each fixed locus of the torus action on the moduli space can be encoded by a bipartite graph Γ with white (black) vertices corresponding to the connected components of $\hat{f}^{-1}(0)$ $(\hat{f}^{-1}(\infty))$. The vertices and edges are decorated with the following data:

- Each vertex v is labeled with a tuple γ_v of nontrivial elements in $\mathbb{Z}_n$ corresponding to the twisted marks on that component and an integer g_v corresponding to the genus.
- Each edge e is labeled with a complex number $(\xi^{k_e} d_e)$ which induces a n-tuple of partitions $\nu_v \in \mathrm{Conj}(\mathbb{Z}_n \wr S_{d_v})$ at each white vertex and $-\nu_v \in \mathrm{Conj}(\mathbb{Z}_n \wr S_{d_v})$ at each black vertex.
- In addition, each black vertex is labeled with a n-tuple of partitions μ_v such that $|\mu_v| = |\nu_v|$ and the union of all μ_v is μ.

To a white vertex, we associate the contribution

$$\mathrm{Cont}(v) = V_{g_v,\gamma_v}(\nu_v; a)$$

and to a black vertex we associate the contribution

$$\mathrm{Cont}(v) = \frac{(-1)^{l_0(\nu_v)+g-1+\sum_{i\neq 0}\frac{n-i}{n}(m_i(\gamma_v)+l_i(\mu_v)+l_{n-i}(\nu_v))}(a)^{2g_v-2+|\gamma_v|+l(\mu_v)+l(\nu_v)}}{|\mathrm{Aut}(\nu_v)|} \cdot \left(\prod_{i=1}^{l(\nu_v)} n d_i\right) \int_{\overline{\mathcal{M}}_{g_v,\gamma_v}(\mathcal{G}_0;-\nu_v[0],\mu_v[\infty])//\mathbb{C}^*} -(-\psi_0)^{2g_v-3+|\gamma_v|+l(\nu_v)+l(\mu_v)},$$

where ψ_0 is the target psi class. By the localization formula for orbifold stable maps we compute the integral

$$(\text{A–1}) = \frac{1}{|\mathrm{Aut}(\mu)|} \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v} \mathrm{Cont}(v).$$

Remark 5.16. In the simplification of the black vertex contribution, we used the $\mathbb{Z}_n$-Mumford relation ([11]), namely:

$$\Lambda^1(a)\Lambda^{-1}(-a) = (a)^{rk(\mathbb{E}_\xi)}(-a)^{rk(\mathbb{E}_{\xi^{-1}})}$$

where the ranks can be computed by the orbifold Riemann-Roch formula.

Setting $a = 0$, we observe that the contributions from black vertices vanish and the integral is equal to $V_{g,\gamma}(\mu; 0)$.

Define the rubber integral generating function

$$H_{\nu,\mu}(x,u) := \frac{1}{|\mathrm{Aut}(\nu)||\mathrm{Aut}(\mu)|} \sum_{g,\gamma} \int_{\overline{\mathcal{M}}} \psi_0^{r+|\gamma|-1} u^r \frac{x^\gamma}{\gamma!}$$

where $r := 2g - 2 + l(\mu) + l(\nu)$, $\overline{\mathcal{M}}$ is the space of relative maps into the rubber target: $\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0; \nu[0], \mu[\infty])//\mathbb{C}^*$.

For notational convenience, we define

$$\tilde{H}^\bullet_{\nu,\mu}(a) := \exp\left(H_{\nu,\mu}(a\xi_{2n}^{-1}x_1, ..., a\xi_{2n}^{1-n}x_{n-1}, \sqrt{-1}au)\right)$$

The above localization computations amount to the following bilinear relations between V and H:

$$\tilde{V}^\bullet_\mu(0) = \sum_{|\nu|=|\mu|} \tilde{V}^\bullet_\nu(a) z_\nu \tilde{H}^\bullet_{-\nu,\mu}(a). \tag{R–1}$$

Remark 5.17. Notice that the $-\nu$ appearing in the rubber integrals is equal to $g_0(\nu)$ defined in Definition 5.12.

We also compute (A–2) via localization. Again we equip the moduli space with a $\mathbb{C}^*$ action via the standard $\mathbb{C}^*$ action on the target. We lift the integrand with the choice of linearizations summarized in Table 5.2.

TABLE 5.2. Weights of the torus action.

	$T(-\infty)$	L_k	$L_k^\vee(-1)$
0	1	0	-1
∞	0	k/n	$-k/n$

The localization computation of (A–2) is nearly identical to that of (A–1) and leads to the relations

$$0 = \sum_{|\nu|=|\mu|} \tilde{V}_\nu^\bullet(0) z_\nu \tilde{H}^\bullet_{g_k(\nu),\mu}\left(\frac{k}{n}\right) \tag{R–2}$$

where μ is any partition with at least one untwisted part.

5.2.5. WREATH HURWITZ NUMBERS. In the non-orbifold case, it was shown in [42, 43] that certain rubber integrals can be interpreted in terms of double Hurwitz numbers. In this section, we generalize their result to the orbifold case.

Hurwitz numbers classically count degree d ramified covers of Riemann surfaces with monodromy around the branch points prescribed by conjugacy classes in S_d. Cyclic wreath Hurwitz numbers are defined to be analogous counts of degree dn ramified covers where the monodromy is prescribed by conjugacy classes μ in $\mathbb{Z}_n \wr S_d$. Since $\mathbb{Z}_n$ is in the center of $\mathbb{Z}_n \wr S_d$, such covers have a natural $\mathbb{Z}_n$ action and the quotient is a classical Hurwitz cover with monodromy given by the underlying partitions $\underline{\mu}$.

We define now the particular wreath Hurwitz numbers which arise in our context.

DEFINITION 5.18. Let $H^{g,\gamma}_{\nu,\mu}$ be the automorphism-weighted count of wreath Hurwitz covers $f : C \to \mathbb{P}^1$ where the branch locus consists of a set of $|\gamma| + r + 2$ fixed points (we fix the last two points at 0 and ∞) and the maps satisfy the following conditions:

- The quotient $C/\mathbb{Z}_n$ is a connected genus g curve,

- The monodromy around 0 and ∞ is given by ν and μ,
- The monodromy around the branch point corresponding to $\gamma_i \in \gamma$ is given by the conjugacy class $\{\gamma_i, 1, ..., 1\}$,
- The monodromy around the r additional branch points is given by the conjugacy class $\{2, 1, ..., 1\}$.

REMARK 5.19. Here we use the multiset notation for n-tuples of partitions introduced in Section 5.1.1.

The next theorem relates the rubber integrals which arose in the localization computations to the wreath Hurwitz numbers $H^{g,\gamma}_{\nu,\mu}$.

THEOREM 5.20.

$$H^{g,\gamma}_{\nu,\mu} = \frac{r!}{|\mathrm{Aut}(\nu)||\mathrm{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])//\mathbb{C}^*} \psi_0^{r-1+|\gamma|}.$$

PROOF. Via the forgetful map $F : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty]) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1;\underline{\nu}[0],\underline{\mu}[\infty])$, we obtain a branch morphism $Br : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty]) \to \mathrm{Sym}^r\mathbb{P}^1 \cong \mathbb{P}^r$ by postcomposing F with the usual branch morphism. For each of the n (twisted) marked points, we also obtain maps $e\tilde{v}_i : \overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty]) \to \mathbb{P}^1$ by postcomposing the usual evaluation map with the natural map to $\mathbb{P}^1$. Then the wreath Hurwitz numbers can be expressed as

$$H^{g,\gamma}_{\nu,\mu} = \frac{1}{|\mathrm{Aut}(\nu)||\mathrm{Aut}(\mu)|} \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])} Br^*(pt) \cdot \prod e\tilde{v}_i^*(pt). \tag{5–5}$$

It is left to show that

$$\int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])} Br^*(pt) \cdot \prod e\tilde{v}_i^*(pt) = r! \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])//\mathbb{C}^*} \psi_0^{r-1+|\gamma|}$$

and we accomplish this via localization.

We equip the moduli space with a torus action by fixing the $\mathbb{C}^*$ action on the target $t \cdot [z_0 : z_1] = [z_0 : tz_1]$ so that the tangent bundle is linearized with weights 1 at $0 = [0 : 1]$ and -1 and $\infty = [1 : 0]$. The isomorphism $\mathbb{P}^r = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(r))) \to \mathrm{Sym}^r\mathbb{P}^1$ is given by $s \to Div(s)$ where the basis $\langle z_0^r, z_0^{r-1}z_1, ..., z_1^r\rangle$ for $H^0(\mathbb{P}^1, \mathcal{O}(r))$ corresponds to the homogeneous coordinates $(y_0 : y_1 : ... : y_r)$. We equip $\mathbb{P}^r$ with the torus action $t \cdot (y_0 : y_1 : ... : y_r) = (y_0 : ty_1 : ... : t^r y_r)$ which makes Br an equivariant map. We lift $[pt] \in H^{2r}(\mathbb{P}^r)$ to $\prod_{i=0}^{r-1}(H + i\hbar) \in H^{2r}_{\mathbb{C}^*}(\mathbb{P}^r)$ where $\hbar$ is the equivariant parameter. The preimage of this lift is the locus of maps where the simple ramification points map to ∞. Likewise we lift

$$\tilde{ev}_i^*(pt) = c_1(\tilde{ev}_i^*\mathcal{O}(1))$$

by linearizing $\mathcal{O}(1)$ with weights 0 at 0 and -1 at ∞.

With these choices of linearizations, we see that the integrand vanishes on all fixed loci where any of the $n + r$ points with nontrivial monodromy map to 0. This leaves exactly one fixed locus where the target expands over ∞ and everything interesting happens over the expansion. On this locus, the integrand specializes to $(-\hbar)^{r+n}r!$ and the inverse of the equivariant Euler class of the normal bundle is $(-\hbar - \psi_0)^{-1}$.

Therefore the contribution, and hence the integral in (5–5), is equal to

$$r! \int_{\overline{\mathcal{M}}_{g,\gamma}(\mathcal{G}_0;\nu[0],\mu[\infty])//\mathbb{C}^*} \psi_0^{r+n-1}.$$

□

COROLLARY 5.21.

$$H^{\bullet}_{\nu,\mu}(x,u) = \exp\left(\sum_{g,\gamma} H^{g,\gamma}_{\nu,\mu}\frac{u^r}{r!}\frac{x^\gamma}{\gamma!}\right)$$
$$= \sum_{g,\gamma} H^{\chi,\gamma\bullet}_{\nu,\mu}\frac{u^r}{r!}\frac{x^\gamma}{\gamma!}$$

where $H^{\chi,\gamma\bullet}_{\nu,\mu}$ is the wreath Hurwitz number with possibly disconnected covers.

By the Burnside formula ([23]), we compute

$$H^{\chi,\gamma\bullet}_{\nu,\mu} = \sum_{|\lambda|=d} (f_T(\lambda))^r \prod (f_i(\lambda))^{m_i(\gamma)} \frac{\chi_\lambda(\mu)}{z_\mu}\frac{\chi_\lambda(\nu)}{z_\nu}$$

where $f_T(\lambda)$ and $f_i(\lambda)$ are the *central characters* defined by

$$f_T(\lambda) := \frac{nd(d-1)\chi_\lambda(\{2,1,...,1\})}{2\cdot \dim\lambda}$$

and

$$f_i(\lambda) := \frac{d\chi_\lambda(\{\xi^i,1,...,1\})}{\dim\lambda}.$$

We thus obtain the following form for the generating function of wreath Hurwitz numbers:

$$H^{\bullet}_{\nu,\mu}(x,u) = \sum_{|\lambda|=d} \frac{\chi_\lambda(\mu)}{z_\mu}\frac{\chi_\lambda(\nu)}{z_\nu} \mathrm{e}^{f_T(\lambda)u+\sum f_i(\lambda)x_i}. \tag{5–6}$$

Using the fact that $\chi_\lambda(-\nu) = \overline{\chi_\lambda(\nu)}$, orthogonality of characters gives us the following relations:

$$H^{\bullet}_{\nu,\mu}(x+y,u+v) = \sum_{\sigma} H^{\bullet}_{\nu,\sigma}(x,u) z_\sigma H^{\bullet}_{-\sigma,\mu}(y,v) \tag{5–7}$$

and

$$H^{\bullet}_{\nu,-\mu}(0,0) = \frac{1}{z_\mu}\delta_{\nu,\mu}. \tag{5–8}$$

The relations (5–7) and (5–8) also have a geometric meaning – (5–7) is the degeneration formula for the target $\mathbb{P}^1$ where x and y keep track of whether the corresponding point of ramification maps to one side of the node or the other, and (5–8) counts covers with ramification only over 0 and ∞.

5.2.6. Invertibility. In this section we show that the relations (R–1) can be inverted explicitly. We also state the main result concerning the relations (R–2) but we defer the proof to the next section.

The next lemma follows immediately from Equations (5–7) and (5–8).

Lemma 5.22. *Framing dependence in the conjugacy basis:*

$$\tilde{V}^{\bullet}_{\mu}(a) = \sum_{|\nu|=|\mu|} \tilde{V}^{\bullet}_{\nu}(0) z_\nu \tilde{H}^{\bullet}_{-\nu,\mu}(-a)$$

In particular, Lemma 5.22 determines the general framed vertex from the $a = 0$ vertex and characters of $\mathbb{Z}_n \wr S_d$.

Define

$$\hat{P}_\lambda(a) := \sum_{\mu} \tilde{V}^{\bullet}_{\mu}(a)\chi_\lambda(-\mu)$$

or equivalently

$$\tilde{V}^{\bullet}_{\mu}(a) = \sum_{\lambda} \hat{P}_\lambda(a)\frac{\chi_\lambda(\mu)}{z_\mu}.$$

Then Lemma 5.22 is equivalent to the following.

LEMMA 5.23. *Framing dependence in the representation basis:*

$$\hat{P}_\lambda(a) = \mathrm{e}^{-\sqrt{-1}af_T(\lambda)u - a\sum \xi_{2n}^{-i} f_i(\lambda) x_i} \hat{P}_\lambda(0)$$

Therefore, once we know that $\tilde{P}_\lambda(a)$ and $\tilde{P}_\lambda(0)$ are related by the exponential factor of Lemma 5.23, we only need to prove Theorem 5.7 for the case $a = 0$.

The relations (R–2) are significantly more difficult to work with and do not admit a convenient inverse as far as we know. Nonetheless, we prove that they are invertible.

THEOREM 5.24. *Relations* (R–2) *uniquely determine* $V_\mu(0)$ *from the partial evaluations of Corollary* (5.14) *and characters of* $\mathbb{Z}_n \wr S_d$.

The proof of Theorem 5.24 is rather involved and we defer it to the next section. In the meantime, we gather formally the reductions which we have made while the formulas are fresh in our minds.

REDUCTION 5.25. *To prove Theorem 5.7, it suffices to check that the following properties hold after the prescribed change of variables.*

(I) The framing factors are consistent:

$$\left(\left((-\xi_{2n})^{|\lambda|} \prod \xi_n^{l|\lambda_l|} \right)^n \prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^a = \mathrm{e}^{\sqrt{-1}af_T(\lambda)u + a\sum \xi_{2n}^{-i} f_i(\lambda) x_i}.$$

(II) $\tilde{P}_\lambda(0)$ *satisfy the partial evaluations of Corollary 5.14:*

$$\sum_{|\lambda|=|\mu|} \tilde{P}_\lambda(0) \frac{\chi_\lambda(\mu)}{z_\mu} = \left(\frac{1}{z_{\mu^0}} \prod_{j=1}^{l_0} \frac{\sqrt{-1}(-1)^{d_j^0}}{2} \csc\left(\frac{d_j^0 u}{2} \right) \right) \left(\sum_{|\sigma|=|\mu^{tw}|} \tilde{P}_\sigma(0) \frac{\chi_\sigma(\mu^{tw})}{z_{\mu^{tw}}} \right).$$

(III) $\tilde{P}_\lambda(0)$ *satisfy the relations* (R–2) *for all* μ *with at least one untwisted part:*

$$\sum_\nu \left(\sum_\lambda \tilde{P}_\lambda(0) \frac{\chi_\lambda(\nu)}{z_\nu} \right) z_\nu \left(\sum_\sigma \frac{\chi_\sigma(g_k(\nu))}{z_{g_k(\nu)}} \frac{\chi_\sigma(\mu)}{z_\mu} \mathrm{e}^{\frac{k}{n}(\sqrt{-1} f_T(\sigma) u + \sum \xi_{2n}^{-i} f_i(\sigma) x_i)} \right) = 0.$$

Proof. If $\tilde{P}_\lambda(0)$ satisfies (II) and (III), then Corollary 5.14 and Theorem 5.24 imply that Theorem 5.7 is true in the case $a = 0$. The general framed correspondence then follows from the definition of the framed DT vertex and Lemma 5.23. □

The proofs of identities (I) - (III) are given in Section 5.6 after developing the necessary combinatorial and representation theoretic identities in Sections 5.4 and 5.5.

5.3. Linear Algebra

This section is devoted to the proof of Theorem 5.24. By Corollary 5.14, the only vertices left to be determined are those $V_\nu^\bullet(0)$ with $\nu^{tw} = \nu$. So let us rewrite (R–2) as

(R–2') $$0 = \sum_{|\nu|=|\mu|} \tilde{V}_{\nu^{tw}}^\bullet(0) \tilde{V}_{\nu^0}^\bullet(0) z_\nu \tilde{H}_{g_k(\nu),\mu}^\bullet \left(\frac{k}{n} \right).$$

Let us begin by reinterpreting relations (R–2') in terms of matrix equations. Define the vector $\alpha_d = (\tilde{V}_\nu^\bullet(0))$ with indexing set $\{\nu : |\nu| \leq d, \nu = \nu^{tw}\}$ and the vector

$$\beta_d = \left(- \sum_{|\tau|=|\mu|, \tau=\tau^0} \tilde{V}_\tau^\bullet(0) z_\tau \tilde{H}_{g_k(\tau),\mu}^\bullet \left(\frac{k}{n} \right) \right)$$

with indexing set $\{(\mu, k) : |\mu| \leq d, \mu \neq \mu^{tw}, k \neq 0\}$. We introduce a matrix

$$\Phi_d(u; x) = (\Phi_d^{(\mu,k),\nu}(u; x))_{(\mu,k),\nu}$$

with the same indexing sets defined by

$$\Phi_d^{(\mu,k),\nu}(u;x) = \begin{cases} 0, & \text{if} |\nu| > |\mu| \\ z_\nu \tilde{H}^\bullet_{g_k(\nu),\mu}\left(\frac{k}{n}\right), & \text{if} |\nu| = |\mu| \\ \sum_{|\tau|=|\mu|-|\nu|,\tau^0=\tau} \tilde{V}^\bullet_\tau(0) z_{(\tau\sqcup\nu)} \tilde{H}^\bullet_{g_k(\tau\sqcup\nu),\mu}\left(\frac{k}{n}\right), & \text{if} |\nu| < |\mu| \end{cases}$$

Then the collection of relations (R–2') is equivalent to the collection of matrix equations

$$\Phi_d(u;x)\alpha_d = \beta_d$$

Our task is to show that $\Phi_d(u;x)$ has full (column) rank for all d.

5.3.1. Matrix Reductions. We begin by making a sequence of reductions. First note that Φ_d is block upper triangular and decomposes as in Figure 5.1 where Φ'_d is defined by restricting the indexing sets to partitions of size d..

$$\Phi_d = \begin{pmatrix} \Phi'_d & * & \cdots & * \\ 0 & \Phi'_{d-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \Phi'_1 \end{pmatrix}$$

Figure 5.1. Decomposition of Φ_d.

Therefore, it suffices to prove that Φ'_d has full rank and to do this we need only prove that the specialization $\tilde{\Phi}_d := \Phi'_d|_{x_2=\ldots=x_{n-1}=u=0}$ has full rank.

Remark 5.26. By setting $u = 0$, notice that $\tilde{\Phi}_d$ is a generating function of wreath Hurwitz numbers counting covers for which the $\mathbb{Z}_n$ quotient is a disjoint union of $\mathbb{P}^1$s, each one fully ramified over 0 and ∞. Moreover, by setting $x_2 = \cdots = x_{n-1} = 0$, the only nontrivial monodromy away from 0 and ∞ is given by conjugacy classes $\{\xi, 1, \ldots, 1\}$.

By the first part of Remark 5.26, if $\underline{\mu} \neq \underline{\nu}$, then the entry $\tilde{\Phi}_d^{(\mu,k),\nu} = 0$. This implies that $\tilde{\Phi}_d$ is block diagonal and decomposes as in Figure 5.2 where $\tilde{\Phi}_\tau$ is defined by restricting the indexing sets to a single underlying partition $\underline{\nu} = \underline{\mu} = \tau$ of size d.

$$\tilde{\Phi}_d = \begin{pmatrix} \tilde{\Phi}_{\tau^1} & 0 & \cdots & 0 \\ 0 & \tilde{\Phi}_{\tau^2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots \end{pmatrix}$$

FIGURE 5.2. Decomposition of $\tilde{\Phi}_d$.

Therefore, we have reduced our task to showing that $\tilde{\Phi}_\tau$ has full rank for a fixed partition τ.

To this end, fix τ once and for all and write $\tau = (\tau_1, ..., \tau_l)$ with nonincreasing parts. We henceforth suppress τ from the notation and write $\tilde{\Phi}$ for $\tilde{\Phi}_\tau$. We also write x for x_1 when no confusion arises.

In order to prove that $\tilde{\Phi}$ has full rank, we will restrict the row index to a suitable subset and show that the resulting submatrix is invertible. In order to do this, we must first introduce some subtle notation.

5.3.2. ORDERING CONVENTION. We introduce an order on the parts of each μ with $\underline{\mu} = \tau$. Begin by defining $c_i := \gcd(\tau_i, n)$. If $\underline{\mu} = \tau$, we can write μ as the multiset $\mu = \{\xi^{t_i}\tau_i\}$ where $t_i \in \{0, ..., n-1\}$. We define $\bar{t_i} := t_i (\text{mod } c_i)$ and we set $(\tau_i, t_i) > (\tau_j, t_j)$ if one of the following is true

(1) $\tau_i > \tau_j$, or

(2) $\tau_i = \tau_j$ and $\bar{t_i} < \bar{t_j}$, or

(3) $\tau_i = \tau_j$ and $\bar{t_i} = \bar{t_j}$ and $t_i < t_j$.

Then μ can be written uniquely as

$$\mu = ((\tau_1, m_1), \ldots, (\tau_l, m_l))$$

where the pairs are nonincreasing. This ordering convention will be important in defining the square submatrix $\hat{\Phi}$ in Section 5.3.3 and in proving its invertibility in Section 5.3.4. At present, we use the ordering convention to define

$$\tilde{\mu} := \mu \setminus (\tau_1, m_1)$$

and we define the *twisting partition* of μ to be

$$t(\mu) := (m_1, ..., m_l).$$

5.3.3. A Square Submatrix. We now explain a particular way to reduce the row index of $\tilde{\Phi}$ to a suitable subset so that the resulting submatrix $\hat{\Phi}$ is square. We will show in Section 5.3.4 that $\hat{\Phi}$ is invertible which proves that $\tilde{\Phi}$ has full rank.

For $d \geq 1$ and $h \in \{1, ..., n-1\}$, let $c := \gcd(n, d)$ and $\bar{h} = h (\text{mod } c) \in \{0, ..., n-1\}$. We define

$$\Sigma_{d,h} := \{k \in \{1, \cdots, n-1\} | -h + dk = -\bar{h} (\text{mod } n)\}$$

Remark 5.27. $\bar{h}$ has the following interpretation: For each $k \in \{1, ..., n-1\}$ consider the unique $\mathbb{C}^*$ fixed map from an effective orbifold $\mathbb{P}^1$ with orbifold ramification (d, h) at 0. Then the twisting over ∞ is fixed (c.f. Section 5.2.1) and $\bar{h}$ is the smallest possible twisting at ∞ as we vary k. Moreover, $\Sigma_{d,h}$ is exactly the set of k for which the minimal twisting is obtained.

Notice that

$$|\Sigma_{d,h}| = \begin{cases} c-1 & \text{if } h \in \{1,\dots,c-1\} \\ c & \text{else.} \end{cases}$$

The set $\Sigma_{d,h}$ has a natural order as a subset of $\{1,...,n-1\}$, so we can write $\Sigma_{d,h} = \{k_1,...,k_{|\Sigma_{d,h}|}\}$. We define

$$k_d(h) := \begin{cases} k_{\bar{h}} \in \Sigma_{d,h} & \text{if } h \in \{1,...,c-1\} \\ k_{\bar{h}+1} \in \Sigma_{d,h} & \text{else.} \end{cases}$$

LEMMA 5.28. *$k_d(-)$ defines a bijection on the set $\{1,...,n-1\}$.*

PROOF. We show that the map is injective. Suppose $k_d(h) = k_d(h')$. This implies that there is some $k \in \Sigma_{d,h} \cap \Sigma_{d,h'}$. Chasing the definitions, this implies that $h - \bar{h} = h' - \bar{h}' (\text{mod } n)$. In particular, if we define the sets

$$D_d^i := \{j \in \{1,\dots,n-1\} : (i-1)c \leq j < ic\}, \tag{5–9}$$

then h and h' belong to the same D_d^i and it follows that $\Sigma_{d,h} = \Sigma_{d,h'}$. But for a fixed i, each element in D_d^i has different reduction mod c. Since $k_{\bar{h}} = k_{\bar{h}'} \in \Sigma_{d,h} = \Sigma_{d,h'}$, then we must have $\bar{h} = \bar{h}'$ implying that $h = h'$. □

We saw in the proof of Lemma 5.28 that $h, h' \in D_d^i$ if and only if $\Sigma_{d,h} = \Sigma_{d,h'}$. For this reason, we adopt the notation Σ_d^i.

We are now ready to cut down the rows in the matrix $\tilde{\Phi}$.

Using the above ordering convention, for any ν with $\nu^{tw} = \nu$ and $\underline{\nu} = \tau$, we can write

$$\nu = ((\tau_1, h_1), \ldots, (\tau_l, h_l).$$

We define $\hat{\Phi}$ to be the matrix obtained from $\tilde{\Phi}$ by restricting the row index to the set

$$\{(\mu, k) : m_1 = 0, k = k_{\tau_1}(h_1), \tilde{\mu} = -g_k(\tilde{\nu}) \text{ for some } \nu = \nu^{tw}\}.$$

The fact that $\hat{\Phi}$ is square follows from Lemma 5.28.

5.3.4. The Invertibility of $\hat{\Phi}$. To prove that $\hat{\Phi}$ is invertible over $\mathbb{C}((x))$, we proceed in two steps. We first define certain blocks $\hat{\Phi}^i_h$ in $\hat{\Phi}$ with the following properties:

(1) Each $\hat{\Phi}^i_h$ is invertible over $\mathbb{C}((x))$.

(2) Each row and column of $\hat{\Phi}$ intersects exactly one $\hat{\Phi}^i_h$.

(3) If $f(x)$ is an entry in some $\hat{\Phi}^i_h$ and $g(x)$ is an entry of $\hat{\Phi}$ in the same column, then $\mathrm{ord}_x f(x) \leq \mathrm{ord}_x g(x)$.

If the inequality in (3) were strict, we would be done because the least degree term of the determinant of $\hat{\Phi}$ would be a signed product of the least degree terms in the determinants of the $\hat{\Phi}^i_h$ (by (2)) which are nonzero (by (1)). However, the inequality is not always strict as we will see below. The second step is to use elementary matrix operations to take care of terms where the inequality is not strict.

We now define the blocks $\hat{\Phi}^i_h$. For $h \in \{1, ..., n-1\}^{l-1}$ and for $1 \leq i \leq \frac{n}{c_1}$ define

$$\begin{aligned} B^i_h &= \{h_1 \in D^i_{\tau_1}, t(\tilde{\nu}) = h\} \\ C^i_h &= \{k \in \Sigma^i_{\tau_1}, t(g_k(-\tilde{\mu})) = h\}. \end{aligned}$$

Then we define the sub-matrix $\hat{\Phi}_h^i$ by intersecting the indexing sets of $\hat{\Phi}$ with B_h^i and C_h^i.

REMARK 5.29. The above definitions might seem a bit obscure, a priori, but the motivation is simple. From Remark 5.26, we know that the wreath Hurwitz numbers encoded by $\hat{\Phi}$ are rather simple. In particular, the $\mathbb{Z}_n$ quotient of the cover is a disjoint union of $\mathbb{P}^1$s and the only allowable monodromy over $\mathbb{C}^* \subset \mathbb{P}^1$ are x_1 points. For a fixed $\nu \in B_h^i$, the pairs $(\mu, k) \in C_h^i$ were chosen to be exactly those pairs such that there exists a wreath cover with the following three properties:

(1) The $\mathbb{Z}_n$ monodromy over 0 and ∞ for the ith $\mathbb{P}^1$ is identified with $-h_i + \tau_i k$ and m_i, respectively,

(2) The $\mathbb{Z}_n$ monodromy over the first $\mathbb{C}^* \subset \mathbb{P}^1$ has the minimal possible number of x_1 points as we vary over all choices (μ, k) (this minimal number is $\bar{h}_1$), and

(3) The $\mathbb{Z}_n$ monodromy over the other $\mathbb{C}^*$s is trivial.

If we vary $\nu \in B_h^i$, the set of (μ, k) with these properties remains constant and they define the matrix $\hat{\Phi}_h^i$.

REMARK 5.30. That each column and each row of $\hat{\Phi}$ intersects exactly one $\hat{\Phi}_h^i$ follows from the fact that $\mathcal{D}_{\tau_1}^i$ and $\Sigma_{\tau_1}^i$ both partition the set $\{1, \ldots, n-1\}$.

LEMMA 5.31. *Let $\underline{\hat{\Phi}_h^i}$ denote the matrix of leading terms in $\hat{\Phi}_h^i$. Then $\underline{\hat{\Phi}_h^i}$ is invertible. In particular, $\hat{\Phi}_h^i$ is invertible over $\mathbb{C}((x))$.*

PROOF. By Remark 5.29, the lowest degree term of the $((\mu, k), \nu)$ entry of $\hat{\Phi}_h^i$ has coefficient

$$z_\nu \frac{(\xi_{2n}^{-1} \frac{k}{n})^{\bar{h}_1}}{\bar{h}_1!} H_{\mu, g_k(\nu)}^{2l, \gamma(h_1)\bullet} \tag{5–10}$$

where $\bar{h}_1$ is *independent* of $(\mu, k) \in C_h^i$ and $\gamma(h_1)$ is a $\bar{h}_1$-tuple of ξ's. The wreath Hurwitz numbers appearing in (5–10) are easy to compute, explicitly we have

$$\begin{aligned} & z_\nu \frac{(\xi_{2n}^{-1}\frac{k}{n})^{\bar{h}_1}}{\bar{h}_1!} H^{2l,\gamma(h_1)\bullet}_{\mu,g_k(\nu)} \\ &= z_\nu \frac{(\xi_{2n}^{-1}\frac{k}{n})^{\bar{h}_1}}{\bar{h}_1!} \frac{1}{|\mathrm{Aut}(\mu)|} \nu_1^{\bar{h}_1} \prod_{i=1}^{l} \frac{1}{n\tau_i} \\ &= \frac{|\mathrm{Aut}(\nu)|}{|\mathrm{Aut}(\mu)|} \frac{(\xi_{2n}^{-1}\frac{k}{n}\tau_1)^{\bar{h}_1}}{\bar{h}_1!} \end{aligned}$$

Therefore $\det\left(\underline{\hat{\Phi}_h^i}\right)$ is equal to

$$\left(\prod_{(\mu,k)\in C_h^i} \frac{1}{|\mathrm{Aut}(\mu)|}\right) \left(\prod_{\nu\in B_h^i} \frac{|\mathrm{Aut}(\nu)|(\xi_{2n}^{-1}\nu_1 x)^{\bar{h}_1}}{\bar{h}_1!}\right) \det\left(\left(\frac{k}{n}\right)^{\bar{h}_1}\right)_{(\mu,k)\in C_h^i, \nu\in B_h^i}$$

This is nonzero because $\det\left(\left(\frac{k}{n}\right)^{\bar{h}_1}\right)$ is the determinant of a Vandermonde matrix with different k in different rows. □

Theorem 5.24 now follows from the next result.

LEMMA 5.32. *$\hat{\Phi}$ is invertible over $\mathbb{C}((x))$.*

PROOF. For any fixed column α_ν of $\hat{\Phi}$, there is a unique sub-matrix $\hat{\Phi}_h^i$ that intersects with this column. The degrees of the entries that lie in the intersection of α_ν and $\hat{\Phi}_h^i$ are $\bar{h}_1$. By the ordering convention introduced above, the degrees of the other entries of α_ν are greater or equal to $\bar{h}_1$ (note that m_1 is always trivial). The equality holds for an entry in the row indexed by $(\mu, k) \notin C_h^i$ only if the following conditions are satisfied:

(1) There exists a $j > 1$ such that $\tau_j = \tau_1$, $\bar{h}_1 = \bar{h}_j$, and $h_1 < h_j$,

(2) $-h_j + \tau_j k = -\bar{h}_j (\mathrm{mod}\ n)$, and

(3) $g_k(-\tilde{\mu}) = \hat{\nu}$ where $\hat{\nu} = \nu \setminus \{(\tau_j, h_j)\}$.

If these conditions are met for some $(\mu, k) \notin C_h^i$, then there is a unique sub-matrix $\hat{\Phi}_{h'}^{i'}$ that intersects this row. By definition, $h' = t(\hat{\nu})$ and i' is determined by the property $k \in \Sigma_{\tau_1}^{i'}$. It is not hard to see that every other entry that lies in the intersection of α_ν and a row of $\hat{\Phi}_{h'}^{i'}$ also has minimal degree $\bar{h}_1$. See figure 5.3

$$\begin{pmatrix} & & & * & \cdots & \cdots & \cdots \\ & \hat{\Phi}_h^i & & * & \cdots & \cdots & \cdots \\ & & & * & \cdots & \cdots & \cdots \\ * & * & * & \ddots & * & * & * \\ * & * & x^{\bar{h}_1} & * & & & \\ \vdots & \vdots & \vdots & * & & \hat{\Phi}_{h'}^{i'} & \\ * & * & x^{\bar{h}_1} & * & & & \end{pmatrix}$$

FIGURE 5.3. General layout of $\hat{\Phi}_h^i$ and $\hat{\Phi}_{h'}^{i'}$.

For every column $\alpha_{\nu'}$ that intersects $\hat{\Phi}_{h'}^{i'}$, we know $\widetilde{\nu'} = \hat{\nu}$. In particular, $\bar{h}_2' = \bar{h}_1$ implying that $\bar{h}_1' \leq \bar{h}_1$ by the ordering convention. If $\bar{h}_1' = \bar{h}_1$, then $\bar{h}_1' = \bar{h}_j$ (by (1)) and $h_1', h_j \in D_{\tau_1}^{i'}$ (the latter inclusion follows from (2)). This would imply that $h_1' = h_j'$, i.e. $\nu = \nu'$ – a contradiction. Therefore we conclude that $\bar{h}_1' < \bar{h}_1 = \bar{h}_2'$. In other words, condition (1) can never be satisfied by ν'. In particular, the degrees of the entries in $\alpha_{\nu'}$ which are not contained in $\hat{\Phi}_{h'}^{i'}$ are strictly greater than $\bar{h}_1'$.

By Lemma 5.31, we can transform the matrix $\hat{\Phi}_{h'}^{i'}$ to a matrix $\Psi_{h'}^{i'}$ such that $\Psi_{h'}^{i'}|_{x=0}$ is the identity matrix. More specifically, we first multiply each column by $x^{-\bar{h}_1'}$ where ν' is the index of the column, then we apply elementary column operations (over $\mathbb{C}$) to reduce the matrix of (constant) leading terms to the identity. Extending these column operations to the columns of $\hat{\Phi}$, we can replace the sub-matrix $\hat{\Phi}_{h'}^{i'}$ by $\Psi_{h'}^{i'}$ in such a way that the following two properties are satisfied (see Figure 5.4):

(a) For each column intersecting $\Psi^{i'}_{h'}$, the entries which do not lie in $\Psi^{i'}_{h'}$ have vanishing constant terms, and

(b) The transformed matrix is invertible over $\mathbb{C}((x))$ if and only if the original matrix is invertible over $\mathbb{C}((x))$.

$$\begin{pmatrix} & & & * & & & \\ & \hat{\Phi}^i_h & & * & & O(x) & \\ & & & * & & & \\ * & * & * & \ddots & * & * & * \\ * & * & x^{\bar{h}_1} & * & & & \\ \vdots & \vdots & \vdots & * & & \Psi^{i'}_{h'} = I + O(x) & \\ * & * & x^{\bar{h}_1} & * & & & \end{pmatrix}$$

FIGURE 5.4. After transforming the matrix $\hat{\Phi}^{i'}_{h'}$.

We can now use the columns intersecting $\Psi^{i'}_{h'}$ to cancel the degree $\bar{h}_1$ terms of the entries that lie in the intersection of α_ν and rows of $\Psi^{i'}_{h'}$. By property (a), this does not affect the degree $\bar{h}_1$ terms in the entries of α_ν which do not lie in rows which intersect $\Psi^{i'}_{h'}$. In particular, the smallest degree term in $\det\left(\hat{\Phi}^i_h\right)$ is not affected. We can repeat this process until the least degree terms in each column are contained in the sub-matrix $\hat{\Phi}^i_h$ (or Ψ^i_h if it has been transformed). Call the resulting matrix Ψ. Then the least degree term of $\det(\Psi)$ is the product of least degree terms of determinants of matrices of the form $\hat{\Phi}^i_h$ or Ψ^i_h, all of which are nonzero. Therefore Ψ is invertible over $\mathbb{C}((x))$. By property (b), $\hat{\Phi}$ is invertible over $\mathbb{C}((x))$. □

5.4. COMBINATORICS

In this section, we investigate the framed Donaldson-Thomas vertex $\tilde{P}_\lambda(a)$ and relate it to loop Schur functions.

5.4.1. Loop Schur Functions. We refer the reader to Chapter 4 for the relevant definitions of the functions $s_\rho[n]$ and $s^k_\rho[n]$. In the current setting, we are only concerned with the case where $\rho = \bar{\lambda}$ arises from an n-tuple of partitions λ via n-quotients (cf. Section 5.5.2). This is equivalent to the following condition.

Definition 5.33. We call the colored Young diagram ρ *balanced* if $|\rho[i]| = |\rho[j]|$ for all i, j.

Denote by S_λ and S^k_λ the functions in n variables $(q_0, ..., q_{n-1})$ obtained by making the substitution $x_{i,j} = q_i^j$ in $s_{\bar{\lambda}}[n]$ and $s^k_{\bar{\lambda}}[n]$, respectively. The following result appears in both [24] and [48].

Lemma 5.34 ([24, 48]).

$$S_\lambda = \frac{\prod_i q_i^{n_i(\bar{\lambda})}}{\prod_{\square\in\bar{\lambda}} \left(1 - \prod_i q_i^{h_i(\square)}\right)}.$$

As a consequence, we have the following identity:

Corollary 5.35.

$$\tilde{P}_\lambda(0) = \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)} q^{\frac{d}{2}} (-1)^d S_\lambda.$$

Remark 5.36. Notice the specialization $s^0_\rho[n] = s_\rho[n]$, and hence similarly with S.

Since S^k_λ differs from S_λ only by a monomial factor, we have the following natural generalization of Corollary 5.35.

Lemma 5.37.

$$\tilde{P}_\lambda(0) = \frac{\chi_{\bar{\lambda}}(n^d)}{\dim(\lambda)} q^{\frac{d}{2}} (-1)^d S^k_\lambda \left(\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^{\frac{-k}{n}}.$$

5.4.2. Combinatorial Identities. We now rephrase the results from Chapter 4.

Theorem 5.38.

$$\frac{1}{1-(q_0 \dots q_{n-1})^l} S_\lambda = \sum (-1)^{ht(\bar{\sigma} \backslash \bar{\lambda})} S_\sigma$$

where the sum is over all ways of adding a length ln border strip to $\bar{\lambda}$.

Theorem 5.39. *For a fixed* $\bar{\lambda}$ *and* $k \neq 0$,

$$\sum (-1)^{ht(\bar{\sigma} \backslash \bar{\lambda})} S_\sigma^k = 0$$

where the sum is over all ways of adding a length ln border strip to $\bar{\lambda}$.

5.5. Representation Theory

In this section we investigate certain characters of the generalized symmetric group which arose in Section 5.2.5. Our main tool is the wreath Fock space. We begin by recalling the basic definitions and results concerning the usual Fock space.

5.5.1. The Infinite Wedge. The infinite wedge provides a convenient setting for studying the representation theory of the symmetric group in terms of combinatorial manipulations of partitions and Maya diagrams. For a more thorough treatment of the infinite wedge and some of its applications in Gromov-Witten theory, see for example [50, 51] or for an application in double Hurwitz numbers, see [36].

Let V be the infinite vector space with spanning set indexed by half integers:

$$V := \bigoplus_{i \in \mathbb{Z}} \left\langle \underline{i + \frac{1}{2}} \right\rangle_{\mathbb{C}}.$$

DEFINITION 5.40. The *infinite wedge* $\bigwedge^{\frac{\infty}{2}} V$ is the vector space

$$\bigwedge^{\frac{\infty}{2}} V := \bigoplus_{(i_k)} \langle \underline{i_1} \wedge \underline{i_2} \wedge ... \rangle$$

where (i_k) is a decreasing sequence of half integers such that

$$i_k + k - \frac{1}{2} = c$$

for some constant c and $k \gg 0$. We call c the *charge* of the vector.

We will only be concerned with the subvector space spanned by vectors of charge 0. We denote this space by $\bigwedge_0^{\frac{\infty}{2}} V$.

5.5.1.1. *Maya Diagrams.* The primary combinatorial tool for us will be Maya diagrams. A Maya diagram is a collection of stones placed at the half integers such that the half integers without stones are bounded below and the half integers with stones are bounded above. A Maya diagram has *charge zero* if the number of stones at positive half integers is equal to the number of negative half integers without stones.

The basis vectors of $\bigwedge_0^{\frac{\infty}{2}} V$ can be identified with charge zero Maya diagrams canonically as follows. Let $S = \{i_k\}$ where (i_k) corresponds to a charge 0 vector. Then we canonically obtain a charge zero Maya diagram from S by placing a stone in the ith place if and only if $i \in S$.

5.5.1.2. *Partitions.* The charge zero basis vectors can also be canonically identified with partitions. If we let α be the increasing sequence of half integers in $S \cap \mathbb{Q}_{>0}$ and β the increasing sequence of half integers in $-(S^c \cap \mathbb{Q}_{<0})$, then $(\alpha|\beta)$ is the modified Frobenius coordinate of a partition ρ. In other words, representing ρ as a Young diagram, α_i is the

number of boxes (half-boxes included) in the ith row to the right of the main diagonal and β_i is the number of boxes in the ith column below the main diagonal.

Equivalently, the partition $\rho = (\rho_1, \rho_2, ...)$ is determined by writing the vector v_S in the following form.

$$v_S = \underline{\rho_1 - 1/2} \wedge \underline{\rho_2 - 3/2} \wedge .....$$

To relate partitions to Maya diagrams, rotate the corresponding Young diagram counter-clockwise by 135 and place 0 directly below the vertex. The stones in the Maya diagram lie directly below outer edges of the Young diagram which have slope 1. This correspondence is illustrated in Figure 5.5.

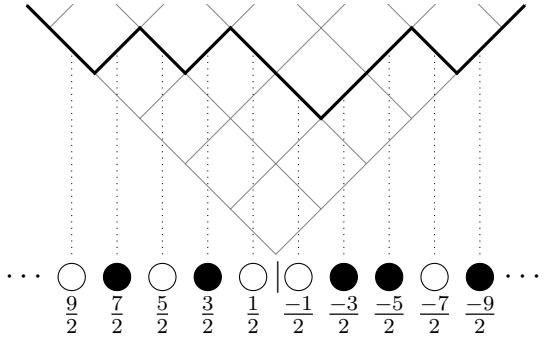


FIGURE 5.5. Correspondence between the different combinatorial bases of $\bigwedge_0^{\frac{\infty}{2}} V$.

5.5.1.3. *One Basis.* With the above correspondences, we will think of $\bigwedge_0^{\frac{\infty}{2}} V$ simultaneously as the vector space spanned by

- Sequences S of the half integers with charge 0,
- Maya diagrams with charge 0, or
- Partitions.

For simplicity, we will denote the basis elements by v_ρ keeping in mind that the partition ρ corresponds canonically to a Maya diagram m_ρ and a set of half integers S_ρ. We denote by

$v_\emptyset$ the *vacuum vector* which is the vector corresponding to the trivial partition which has no nonzero parts.

5.5.1.4. *Operators.* In order to relate the infinite wedge to the representation theory of S_d, we define several operators on $\bigwedge_0^{\frac{\infty}{2}} V$ via their action on basis elements v_ρ.

For any half integer k and basis element v_ρ, the operator $E_{k,k}$ acts on v_ρ as follows:

$$E_{k,k}v_\rho = \begin{cases} v_\rho & k > 0, k \in S_\rho \\ -v_\rho & k < 0, k \notin S_\rho \\ 0 & \text{else.} \end{cases}$$

For k a positive integer, the creation operator α_{-k} acts on v_ρ as follows:

$$\alpha_{-k}v_\rho = \sum_\tau (-1)^{ht(\tau\backslash\rho)} v_\sigma$$

where the sum is over all ways of adding length k border strips to ρ. In terms of Maya diagrams, the sum is over all ways of moving a stone k places to the left and the sign corresponds to the number of stones jumped during such a move.

Recall that each partition ρ corresponds to an irreducible representation of S_d with character χ_ρ. Given a partition $\tau = (d_1, ..., d_l)$ corresponding to a conjugacy class in S_d, we define the operator

$$\alpha_{-\tau} := \prod_{i=1}^{l} \alpha_{-d_i}$$

The following identity follows from the Murnaghan-Nakayama formula.

$$\alpha_{-\tau}v_\emptyset = \sum_\rho \chi_\rho(\tau) v_\rho. \tag{5–11}$$

We also define the operator

$$\mathcal{F}_T := \sum_k \frac{k^2}{2} E_{k,k}.$$

If T is the conjugacy class of transpositions and $f_T(\lambda) := \frac{|T|\chi_\lambda(T)}{\dim(\lambda)}$, then each v_λ is an eigenvector of $\mathcal{F}_T$ with eigenvalue $f_T(\lambda)$:

$$\mathcal{F}_T \cdot v_\lambda = f_T(\lambda) v_\lambda. \tag{5–12}$$

5.5.2. Wreath Fock Space. The wreath product generalization of the Fock space gives a combinatorial tool for manipulating the representation theory of the groups $G \wr S_d$. These spaces and their corresponding operators have been developed in eg. [28, 35, 55]. We merely focus on the cyclic case which is all we require. To that end, the wreath Fock space can be defined as

$$\mathcal{Z}_n := \bigotimes_{\{0,\ldots,n-1\}} \bigwedge_0^{\frac{\infty}{2}} V.$$

Basis vectors correspond to n-tuples of partitions $\lambda = (\lambda_0, ..., \lambda_{n-1})$ or, equivalently, n-tuples of Maya diagrams.

In the wreath Fock space, there is an additional way by which we will distinguish a basis element. Given an n-tuple of Maya diagrams, we can interlace them to get a single Maya diagram by sending a stone in the kth place of the ith Maya diagram to position $n\left(k - \frac{1}{2}\right) + (i + \frac{1}{2})$ in the new Maya diagram. An example of this identification is shown in Figure 5.6.

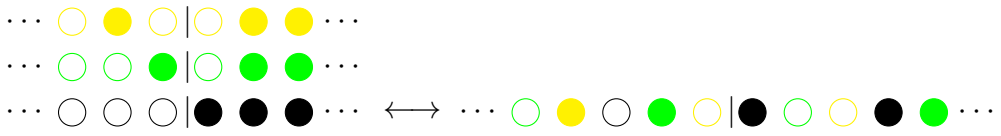


Figure 5.6. A example of a 3-quotient.

This new Maya diagram corresponds to a partition of nd which we denote $\bar{\lambda}$. Reversing this process is usually referred to as an *n-quotient*. It is well known that taking n-quotients gives a bijection between balanced Young diagrams $\bar{\lambda}$ (cf. Definition 5.33) and n-tuples of partitions λ.

For any operator M on $\bigwedge_0^{\frac{\infty}{2}} V$ and any integer $0 \leq k \leq n-1$, we define the operator M^k to act on the wreath fock space $\mathcal{Z}_n$ by acting as M on the kth factor and acting trivially on all of the other factors.

Given λ, we can canonically identify it with an irreducible representation of $\mathbb{Z}_n \wr S_d$ with character χ_λ. Similarly, given an n-tuple of partitions $\mu = (\mu^0, ..., \mu^{n-1})$ with $\mu^k = (d_1^k, ..., d_{l_k}^k)$, we can be canonically identify it with a conjugacy class. We have the following important generalizations of (5–11) and (5–12):

$$\prod_{k=0}^{n-1} \prod_{i=0}^{l_k} \left(\sum_{j=0}^{n-1} \xi^{-kj} \alpha^j_{-d_i^k} \right) v_\emptyset = \sum_\lambda \chi_\lambda(\mu) v_\lambda. \tag{5–13}$$

and

$$\left(n \sum_{i=0}^{n-1} \mathcal{F}_T^i \right) \cdot v_\lambda = f_T(\lambda) v_\lambda. \tag{5–14}$$

5.5.3. Central Characters. We now use the combinatorics of colored partitions and Maya diagrams to study the central characters $f_i(\lambda)$ and $f_T(\lambda)$ which arose in the computations of Section 5.2.5.

Lemma 5.41. *Let $\lambda = (\lambda_0, ..., \lambda_{n-1})$ with $|\lambda_i| = d_i$. Then*

(i) $f_i(\lambda) = \sum_j \xi_n^{-ij} d_j$

(ii) $f_T(\lambda) = \sum_{(i,j) \in \bar{\lambda}[0]} j - i$

PROOF. To prove identity (i), recall that

$$f_i(\lambda) = \frac{d\chi_\lambda(\{\xi^i, 1^{d-1}\})}{\dim(\lambda)} = \frac{d\chi_\lambda(\{\xi^i, 1^{d-1}\})}{\chi_\lambda(\{1^d\})}$$

where the exponent of 1 in the multiset denotes repetition. For $\mu = \{1^d\}$, the coefficient of v_λ in (5–13) can be interpreted as the number of ways to build the n-tuple of Young diagrams $\lambda = (\lambda_0, ..., \lambda_{n-1})$ one box at a time. Equivalently, this can be interpreted as the number of standard Young tableaux of λ, ie. the number of ways to fill the boxes of the λ_i with the numbers $1, ..., d$ with the property that numbers always increase along rows and down columns. This is easily computed:

$$\chi_\lambda(\{1^d\}) = \binom{d}{d_0, ..., d_{n-1}} \prod \dim(\lambda_i) \tag{5–15}$$

where we use the fact that $\dim(\lambda_i)$ is the number of standard tableaux of λ_i.

On the other hand, for $\mu = \{\xi^i, 1^{d-1}\}$, the coefficient of v_λ in (5–13) can be interpreted as a weighted count of ways to build λ one box at a time, where the weight is ξ^{-ij} if the first box is a part of λ_j. This is also easily computed:

$$\chi_\lambda(\{\xi^i, 1^{d-1}\}) = \sum_{j=0}^{n-1} \xi_n^{-ij} \binom{d-1}{d_0, ..., d_j - 1, ..., d_{n-1}} \prod \dim(\lambda_i). \tag{5–16}$$

Identity (i) follows by dividing (5–16) by (5–15) and multiplying by d.

To prove identity (ii), begin by writing $\bar{\lambda} = (\alpha|\beta)$ in modified Frobenius notation (cf. Section 5.5.1). Then the number of boxes in $\bar{\lambda}[0]$ to the right (below) the ith diagonal element is given by $\left\lfloor \frac{\alpha_i}{n} \right\rfloor$ $\left(\left\lfloor \frac{\beta_i}{n} \right\rfloor\right)$. If we compute the sum in (ii) over these $\left\lfloor \frac{\alpha_i}{n} \right\rfloor$ $\left(\left\lfloor \frac{\beta_i}{n} \right\rfloor\right)$ terms, we get

a contribution of

$$n + 2n + ... + n\left\lfloor \frac{\alpha_i}{n} \right\rfloor \qquad \left(-n - 2n - ... - n\left\lfloor \frac{\beta_i}{n} \right\rfloor\right).$$

Therefore, the right side of the (ii) can be written as

$$\sum_{(i,j)\in\bar{\lambda}[0]} j - i = n\sum_{i=1}^{\infty}\left(\frac{\left\lfloor \frac{\alpha_i}{n} \right\rfloor^2 + \left\lfloor \frac{\alpha_i}{n} \right\rfloor}{2} - \frac{\left\lfloor \frac{\beta_i}{n} \right\rfloor^2 + \left\lfloor \frac{\beta_i}{n} \right\rfloor}{2}\right). \tag{5–17}$$

To compute the left side of (ii), we consider equation (5–14). Via the n-quotient correspondence described above, we can interpret v_λ as a vector $v_{\bar{\lambda}} \in \bigwedge_0^{\frac{\infty}{2}}$. Under this correspondence, the operator $n\sum_{i=0}^{n-1}\mathcal{F}_T^i$ becomes

$$n\sum_k \frac{1}{2}\left(\left\lfloor \frac{k}{n} \right\rfloor + \frac{1}{2}\right)^2 E_{kk}.$$

Each summand acts simply by multiplying $v_{\bar{\lambda}}$ by an appropriate scalar. This scalar is zero unless $k = \alpha_i > 0$ or $k = -\beta_i < 0$ for some i. In these cases, the scalar is

$$n\frac{1}{2}\left(\left\lfloor \frac{\alpha_i}{n} \right\rfloor + \frac{1}{2}\right)^2$$

and

$$-n\frac{1}{2}\left(\left\lfloor \frac{\beta_i}{n} \right\rfloor + \frac{1}{2}\right)^2.$$

We obtain (5–17) by summing over all such i. □

LEMMA 5.42. *After the change of variables prescribed by Theorem 5.7,*

$$\left(\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i}\right)^{1/n} = (-\xi_{2n})^{-d}\left(\xi_n^{-\sum kd_k}\right)\mathrm{e}^{\frac{1}{n}\left(\sqrt{-1}f_T(\lambda)u + \sum \xi_{2n}^{-k} f_k(\lambda)x_k\right)} \tag{5–18}$$

PROOF. If $\lambda = (\lambda_0, ..., \lambda_{n-1})$ with $|\lambda_k| = d_k$, then in terms of Maya diagrams we can interpret the d_i as follows: d_k is the number of moves it takes to build the Maya diagram of λ_k from the empty Maya diagram by only moving stones one place at a time. Moreover, each such move has the effect of adding a length n border strip to $\bar{\lambda}$, the northeast-most box in the strip having color k. The quantity $j - i$ decreases uniformly by 1 as we move south and west along the strip so each such move contributes to $\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i}$ a factor of

$$q_k^l q_{k-1}^{l-1} \cdots q_1^{l-k+1} q_0^{l-k} q_{n-1}^{l-k-1} \cdots q_{k+1}^{l-n+1} \tag{5–19}$$

for some l. In order to apply the change of variables, we need to collect the q_0's into q's. Borrowing the necessary q_i's from the other squares in the border strip, (5–19) becomes

$$q^{l-k} \left(q_k^k q_{k-1}^{k-1} \cdots q_1^1 q_{n-1}^{-1} \cdots q_{k+1}^{k-n+1} \right).$$

Combining these factors for all k, we find

$$\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} = q^M \prod_{k=0}^{n-1} \left(q_k^k q_{k-1}^{k-1} \cdots q_1^1 q_{n-1}^{-1} \cdots q_{k+1}^{k-n+1} \right)^{d_k} \tag{5–20}$$

where $M = \sum_{(i,j)\in\bar{\lambda}[0]} (j - i)$ is the total power of q_0 which we know is equal to $f_T(\lambda)$ from Lemma 5.41.

It is left to investigate what happens to the factors in (5–20) after the change of variables. Since $q \to \mathrm{e}^{\sqrt{-1}u}$ and $M = f_T(\lambda)$, then we see immediately that the u factors on either side of (5–18) agree.

We now compute the coefficient of $d_i x_j$ in the exponent of (5–20) after the change of variables. To do this, we must compute the coefficient of x_j in the factor $q_i^i q_{i-1}^{i-1} \cdots q_1^1 q_{n-1}^{-1} \cdots q_{i+1}^{i-n+1}$.

Applying the change of variables, this coefficient is

$$-\sum_{r=1}^{i} \frac{r\xi_n^{-jr}}{n}\left(\xi_{2n}^{j}-\xi_{2n}^{-j}\right)-\sum_{s=i+1}^{n-1} \frac{(s-n)\xi_n^{-js}}{n}\left(\xi_{2n}^{j}-\xi_{2n}^{-j}\right). \tag{5–21}$$

Setting $y := \xi_n^{-j}$, (5–21) can be written as

$$\begin{aligned}&\frac{-y^{-\frac{1}{2}}}{n}\left(\sum_{r=1}^{i}\left(ry^r-ry^{r+1}\right)+\sum_{s=i+1}^{n-1}\left((s-n)y^s-(s-n)y^{s+1}\right)\right)\\ &\qquad=\frac{-y^{-\frac{1}{2}}}{n}\left(-ny^{i+1}+\sum_{r=1}^{n} y^r\right).\end{aligned} \tag{5–22}$$

Using the fact that $\sum_{r=1}^{n} y^r = 0$, (5–22) is equal to $\xi_{2n}^{j(-2i-1)}$. Therefore, the coefficient of x_j is

$$\xi_{2n}^{-j}\sum \xi_n^{-ij} d_i = \xi_{2n}^{-j} f_j(\lambda)$$

where the equality follows from the first identity of Lemma 5.41.

Finally, notice that the root of unity which factors out of the term

$$\left(q_i^i q_{i-1}^{i-1}\cdots q_1^1 q_{n-1}^{-1}\cdots q_{i+1}^{i-n+1}\right)^{1/n}$$

after the change of variables is $-\xi_{2n}^{-1}\xi_n^{-i}$. Putting all of this together proves the result. □

5.5.4. Signs. If $\bar{\sigma}$ is obtained from $\bar{\lambda}$ by adding a length kn border strip, then the Maya diagrams corresponding to σ are obtained from those corresponding to λ by moving a stone k places in the ith Maya diagram. Notice that k and i are both determined by $\bar{\sigma}$ and $\bar{\lambda}$. For notational convenience, we make the following definition.

Definition 5.43. If $\bar{\sigma}$ is obtained from $\bar{\lambda}$ by adding a length kn border strip, let $\beta(\sigma \setminus \lambda)$ denote the number of stones in the ith Maya diagram which are skipped over.

Notice that $(-1)^{\beta(\sigma\setminus\lambda)}$ is the coefficient of v_σ in $\alpha^i_{-k}(v_\lambda)$.

The next lemma allows us to deal with the sign $\frac{\chi_{\bar\lambda}(n^d)}{\dim(\lambda)}$ appearing in Theorem 5.7.

LEMMA 5.44. *If $\bar\sigma$ is obtained from $\bar\lambda$ by adding a length kn border strip, then*

$$\frac{\chi_{\bar\sigma}(n^{d+k})}{\dim(\sigma)} = (-1)^{\beta(\sigma\setminus\lambda)+ht(\bar\sigma\setminus\bar\lambda)-1}\frac{\chi_{\bar\lambda}(n^d)}{\dim(\lambda)}.$$

PROOF. By (5–11), $\chi_{\bar\lambda}(n^d)$ is the weighted sum of ways to create the Maya diagram of $\bar\lambda$ from the vacuum diagram by moving stones n places at a time; the weight is ±1 depending on whether the total number of stones jumped over is even or odd. It is not hard to see that the weight of any such sequence is equal to the weight of any other. Since $\dim(\lambda)$ is the total number of such sequences, we see that $\frac{\chi_{\bar\lambda}(n^d)}{\dim(\lambda)}$ is equal to the weight of any one of them.

Now suppose $\bar\sigma$ is obtained from $\bar\lambda$ by adding a length kn border strip. We can think of $\bar\sigma$ as being obtained from $\bar\lambda$ by moving a single stone kn places to the left in the Maya diagram of $\bar\lambda$, $ht(\bar\sigma\setminus\bar\lambda)-1$ is the total number of stones jumped while $\beta(\sigma\setminus\lambda)$ counts the number of jumped stones which are $n, 2n, 3n, ...$ positions to the left of where the stone sat originally.

On the other hand, the Maya diagram of $\bar\sigma$ can be obtained from that of $\bar\lambda$ by choosing a sequence of length n jumps. As above, $\frac{\chi_{\bar\sigma}(n^{d+k})}{\dim(\sigma)} = (-1)^*\frac{\chi_{\bar\lambda}(n^d)}{\dim(\lambda)}$ where $*$ is equal to the total number of stones jumped during the sequence of moves. With the above interpretations for $ht(\bar\sigma\setminus\bar\lambda)-1$ and $\beta(\sigma\setminus\lambda)$, we see that the number of stones jumped in this process is $(ht(\bar\sigma\setminus\bar\lambda)-1)-\beta(\sigma\setminus\lambda)$. □

The final lemma of this section allows us to compare $\chi_\lambda(\mu)$ with $\chi_\lambda(g_k(\mu))$.

LEMMA 5.45. *If* $\lambda = (\lambda_0, ..., \lambda_{n-1})$ *with* $|\lambda_j| = d_j$, *then*

$$\chi_\lambda(g_k(\mu)) = \xi_n^{-k\sum jd_j}\chi_\lambda(-\mu).$$

PROOF. Write $\mu = (\mu^0, ..., \mu^{n-1})$ with $\mu^s = (d_1^s, ..., d_{l_s}^s)$ and define $(,)$ to be the inner product for which $\{v_\lambda\}$ is an orthonormal basis. By (5–13), we have

$$\begin{aligned}
\chi_\lambda(g_k(\mu)) &= \left(\prod_{s=0}^{n-1}\prod_{i=0}^{l_s}\left(\sum_{j=0}^{n-1}\xi_n^{-d_i^s kj+sj}\alpha_{-d_i^s}^j\right)v_\emptyset, v_\lambda\right) \\
&= \xi_n^{-k\sum jd_j}\left(\prod_{s=0}^{n-1}\prod_{i=0}^{l_s}\left(\sum_{j=0}^{n-1}\xi_n^{sj}\alpha_{-d_i^s}^j\right)v_\emptyset, v_\lambda\right) \\
&= \xi_n^{-k\sum jd_j}\chi_\lambda(-\mu).
\end{aligned}$$

□

5.6. PROOF OF THE GERBY GMV FORMULA

We now check identities (I) - (III) of Reduction 5.25.

IDENTITY (I). This follows immediately from Lemma 5.42.

IDENTITY (II). Since $z_\mu = z_{\mu^0}z_{\mu^{tw}}$, we must show that

$$\sum_{|\lambda|=|\mu|}\tilde{P}_\lambda(0)\chi_\lambda(\mu) = \left(\prod_{j=1}^{l_0}\frac{\sqrt{-1}(-1)^{d_j^0}}{2}\csc\left(\frac{d_j^0 u}{2}\right)\right)\left(\sum_{|\sigma|=|\mu^{tw}|}\tilde{P}_\sigma(0)\chi_\sigma(\mu^{tw})\right).$$

after the change of variables. To do this, it is equivalent to show

$$\sum_{|\lambda|=|\mu|+k}\tilde{P}_\lambda(0)\chi_\lambda(\mu\cup\{k\})) = \frac{\sqrt{-1}(-1)^k}{2}\csc\left(\frac{ku}{2}\right)\left(\sum_{|\sigma|=|\mu|}\tilde{P}_\sigma(0)\chi_\sigma(\mu)\right)$$

which is equivalent (before the change of variables) to

$$\sum_{|\lambda|=|\mu|+k} \tilde{P}_\lambda(0)\chi_\lambda(\mu \cup \{k\}) = \frac{(-1)^k q^{\frac{k}{2}}}{1-q^k} \sum_{|\sigma|=|\mu|} \tilde{P}_\sigma(0)\chi_\sigma(\mu). \tag{5–23}$$

Fix σ. Then

$$\begin{aligned}
\frac{(-1)^k q^{\frac{k}{2}}}{1-q^k}\tilde{P}_\sigma(0)\chi_\sigma(\mu) &= \frac{(-1)^{k+|\mu|} q^{\frac{k}{2}}}{1-q^k}\chi_\sigma(\mu)\frac{\chi_{\bar{\sigma}}(n^{|\mu|})}{\dim(\sigma)} q^{\frac{|\mu|}{2}} S_\sigma \\
&= (-1)^{k+|\mu|} q^{\frac{|\mu|+k}{2}}\chi_\sigma(\mu)\frac{\chi_{\bar{\sigma}}(n^{|\mu|})}{\dim(\sigma)} \sum_{\bar{\lambda}\supset\bar{\sigma}} (-1)^{ht(\bar{\lambda}\setminus\bar{\sigma})-1} S_\lambda \\
&= \chi_\sigma(\mu) \sum_{\bar{\lambda}\supset\bar{\sigma}} (-1)^{\beta(\lambda\setminus\sigma)} \frac{\chi_{\bar{\lambda}}(n^{|\mu|+k})}{\dim(\lambda)} q^{\frac{|\lambda|}{2}} (-1)^{|\lambda|} S_\lambda \\
&= \chi_\sigma(\mu) \sum_{\bar{\lambda}\supset\bar{\sigma}} (-1)^{\beta(\lambda\setminus\sigma)} \tilde{P}_\lambda(0). \qquad \text{(5–24)}
\end{aligned}$$

where the sum is over $\bar{\lambda}$ obtained from $\bar{\sigma}$ by adding a kn strip. The first and fourth equalities follows from Corollary 5.35, the second equality follows from Theorem 5.38, and the third equality follows from Lemma 5.44.

From (5–13), we know

$$\chi_\lambda(\mu \cup \{k\}) = \sum_\sigma \chi_\sigma(\mu)(-1)^{\beta(\lambda\setminus\sigma)}, \tag{5–25}$$

where the sum is over all σ such that $\bar{\sigma}$ is obtained from $\bar{\lambda}$ by removing a kn strip. Summing (5–24) over all σ proves identity (5–23) and thus (II).

Identity (III). Applying Lemma 5.45, (III) is equivalent to

$$\sum_\nu \left(\sum_\lambda \tilde{P}_\lambda(0) \frac{\chi_\lambda(\nu)}{z_\nu} \right) z_\nu \left(\sum_\sigma \xi_n^{-k\sum j|\sigma_j|} \frac{\chi_\sigma(-\nu)}{z_\nu} \frac{\chi_\sigma(\mu)}{z_\mu} \mathrm{e}^{\frac{k}{n}(\sqrt{-1} f_T(\sigma)u + \sum \xi_{2n}^{-i} f_i(\sigma) x_i)} \right) = 0.$$

Summing over all ν and using orthogonality of characters, the left side becomes

$$\sum_{\lambda} \tilde{P}_\lambda(0) \frac{\chi_\lambda(\mu)}{z_\mu} \xi_n^{-k\sum j|\lambda_j|} \mathrm{e}^{\frac{k}{n}(\sqrt{-1}f_T(\lambda)u + \sum \xi_{2n}^{-i} f_i(\lambda) x_i)}.$$

Applying Lemma 5.42, we then see that (III) is equivalent to

$$\sum_{\lambda} \tilde{P}_\lambda(0)\chi_\lambda(\mu) \left(\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n} = 0$$

for any μ with at least one untwisted part. This is equivalent to

$$\sum_{\lambda} \tilde{P}_\lambda(0)\chi_\lambda(\nu \cup \{k\}) \left(\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n} = 0 \tag{5–26}$$

for any ν. Fix σ with $|\sigma| = |\nu|$. Then

$$\begin{aligned}
0 &= \sum_{\bar{\lambda}\supset\bar{\sigma}} (-1)^{ht(\bar{\lambda}\setminus\bar{\sigma})-1} S_\lambda^k \\
&= \sum_{\bar{\lambda}\supset\bar{\sigma}} \frac{\chi_{\bar{\sigma}}(n^{|\sigma|})}{\dim(\sigma)} \chi_\sigma(\nu)(-1)^{ht(\bar{\lambda}\setminus\bar{\sigma})-1} S_\lambda^k \\
&= \sum_{\bar{\lambda}\supset\bar{\sigma}} (-1)^{\beta(\lambda\setminus\sigma)} \chi_\sigma(\nu) q^{-\frac{|\lambda|}{2}} (-1)^{|\lambda|} \tilde{P}_\lambda(0) \left(\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n}.
\end{aligned}$$

The first equality is Theorem 5.39, the second holds because σ is fixed, and the third follows from Lemmas 5.37 and 5.44.

Since $|\lambda|$ is constant over the sum, it follows that

$$0 = \chi_\sigma(\nu) \sum_{\bar{\lambda}\supset\bar{\sigma}} (-1)^{\beta(\lambda\setminus\sigma)} \tilde{P}_\lambda(0) \left(\prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{j-i} \right)^{k/n}.$$

Summing over all σ (using equation (5–25)) proves (5–26) and thus finishes the proof of Theorem 5.7.

5.7. GW/DT for local $\mathbb{Z}_n$-gerbes over $\mathbb{P}^1$

We conclude by giving an application of the gerby Gopakumar-Mariño-Vafa formula. In particular, we prove that the Gromov-Witten potential of any local $\mathbb{Z}_n$-gerbe over $\mathbb{P}^1$ equals the reduced, multi-regular Donaldson-Thomas potential after an explicit change of variables.

Definition 5.46. A *local $\mathbb{Z}_n$-gerbe over $\mathbb{P}^1$* is the total space of a rank two Calabi-Yau bundle $L_1 \oplus L_2$ over some $\mathcal{G}_k$ with trivial generic isotropy.

The CY condition implies that $\deg(L_1)+\deg(L_2) = -2$. Because of the generically trivial isotropy, we know that the $\mathbb{Z}_n$ isotropy acts on the fibers of L_1 by a generator $\zeta \in \mathbb{Z}_n$ and on the fibers of L_2 by its inverse ζ^{-1}. The automorphism of $\mathbb{Z}_n$ which maps $\zeta \to \xi$ induces an isomorphism of the total space which allows us to assume that the isotropy always acts on the fibers of L_1 and L_2 with weights ξ and ξ^{-1}, respectively (cf. remarks after Definition 5.11).

Fix $k \in \{0, ..., n-1\}$ and set $e := \gcd(k, n)$. Then $\mathrm{Pic}(\mathcal{G}_k) = \frac{e}{n}\mathbb{Z} \oplus \mathbb{Z}_e$. For each $b \in \frac{e}{n}\mathbb{Z} \oplus \mathbb{Z}_e$ we let $\mathcal{L}_b$ denote the corresponding orbifold line bundle. The subset of $\mathrm{Pic}(\mathcal{G}_k)$ where $\mathbb{Z}_n$ acts on fibers as multiplication by ξ is given by $(\mathbb{Z} - \frac{k}{n}) \oplus \{1\}$. Every local $\mathbb{Z}_n$-gerbe over $\mathbb{P}^1$ is isomorphic to $\mathcal{X}_{k,b} := \mathrm{Tot}\,(\mathcal{L}_b \oplus \mathcal{L}_{-b-2})$ for some $k \in \{0, ..., n-1\}$ and $b \in \mathbb{Z} - \frac{k}{n}$.

By the gluing algorithm of Theorem 2.17, the degree d Gromov-Witten potential of $\mathcal{X}_{k,b}$ is given by

$$GW_d(\mathcal{X}_{k,b}) = \sum_{\mu} V_{\mu}^{\bullet}(b)\, z_{\mu} V_{g_k(\mu)}^{\bullet}(0) \prod_{i,j} (-1)^{d_j^i b + 1 + \delta_{0,i} + \delta_{0,(-d_j^i k - i) \bmod n} + \frac{i}{n} + \frac{(d_j^i k - i) \bmod n}{n}}. \tag{5–27}$$

where the sign is the gluing term in Theorem 2.17.

Analyzing the modification in (5–2), we see that (5–27) is equivalent to

$$GW_d(\mathcal{X}_{k,b}) = (-1)^{db} \sum_{\mu} \tilde{V}_{\mu}^{\bullet}(b)\, z_{\mu} \tilde{V}_{g_k(\mu)}^{\bullet}(0)\,. \tag{5–28}$$

Applying the change of variables in Theorem 5.7, then using Lemma 5.45 and orthogonality of characters, we find that

$$\begin{aligned} GW_d(\mathcal{X}_{k,b}) &= (-1)^{db} \sum_{\mu} \left(\sum_{\lambda} \tilde{P}_{\lambda}(b)\, \frac{\chi_{\lambda}(\mu)}{z_{\mu}} \right) z_{\mu} \left(\sum_{\sigma} \tilde{P}_{\sigma}(0)\, \frac{\chi_{\sigma}(g_k(\mu))}{z_{g_k(\mu)}} \right) \\ &= (-1)^{db} \sum_{\lambda} \xi_n^{-k \sum i|\lambda_i|} \tilde{P}_{\lambda}(b)\, \tilde{P}_{\lambda}(0) \end{aligned}$$

From equation (5–4), we see that this last expression is

$$\sum_{\lambda} P_{\lambda}(q_0, q_1, ..., q_{n-1}) E_{\lambda} P_{\lambda'}(q_0, q_{n-1}, ..., q_1) \tag{5–29}$$

where

$$E_{\lambda} := \prod_{(i,j)\in\bar{\lambda}} q_{j-i}^{(b+2)i-bj-1} (-1)^{dnb}.$$

By the main result of [8], (5–29) is equal to the reduced, multi-regular, degree d Donaldson-Thomas potential $DT'_{mr,d}(\mathcal{X}_{k,b})$ after the substitution $q_0 \to -q_0$. This proves Theorem 5.8.

BIBLIOGRAPHY

[1] D. Abramovich and B. Fantechi. Orbifold techniques in degeneration formulas. Preprint: math/1103.5132v1, 2011.

[2] D. Abramovich, T. Graber, and A. Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.

[3] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa. The topological vertex. *Comm. Math. Phys.*, 254(2):425–478, 2005.

[4] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[5] A. Bertram, R. Cavalieri, and G. Todorov. Evaluating tautological classes using only Hurwitz numbers. *Trans. Amer. Math. Soc.*, 360(11):6103–6111, 2008.

[6] A. Brini and R. Cavalieri. Open orbifold Gromov-Witten invariants of $[\mathbb{C}^3/\mathbb{Z}_n]$: localization and mirror symmetry. *Selecta Math. (N.S.)*, 17(4):879–933, 2011.

[7] A. Brini, R. Cavalieri, and D. Ross. The open string McKay correspondence for type A singularities. Preprint: math/1303.0723, 2013.

[8] J. Bryan, C. Cadman, and B. Young. The orbifold topological vertex. *Adv. Math.*, 229(1):531–595, 2012.

[9] J. Bryan and A. Gholampour. The quantum McKay correspondence for polyhedral singularities. *Invent. Math.*, 178(3):655–681, 2009.

[10] J. Bryan and T. Graber. The crepant resolution conjecture. In *Algebraic geometry—Seattle 2005. Part 1*, volume 80 of *Proc. Sympos. Pure Math.*, pages 23–42. Amer. Math. Soc., Providence, RI, 2009.

[11] J. Bryan, T. Graber, and R. Pandharipande. The orbifold quantum cohomology of $\mathbb{C}^2/Z_3$ and Hurwitz-Hodge integrals. *J. Algebraic Geom.*, 17(1):1–28, 2008.

[12] J. Bryan and R. Pandharipande. The local Gromov-Witten theory of curves. *J. Amer. Math. Soc.*, 21(1):101–136, 2008. With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande.

[13] C. Cadman and R. Cavalieri. Gerby localization, Z_3-Hodge integrals and the GW theory of $[\mathbb{C}^3/Z_3]$. *Amer. J. Math.*, 131(4):1009–1046, 2009.

[14] J.O Carbonara, A. Kulikauskas, and J.B Remmel. A combinatorial proof of the equivalence of the classical and combinatorial definitions of Schur function. *J. Combin. Theory Ser. A*, 72(2):293–301, 1995.

[15] R. Cavalieri. Hodge-type integrals on moduli spaces of admissible covers. In *The interaction of finite-type and Gromov-Witten invariants (BIRS 2003)*, volume 8 of *Geom. Topol. Monogr.*, pages 167–194. Geom. Topol. Publ., Coventry, 2006.

[16] R. Cavalieri. Generating functions for Hurwitz-Hodge integrals. *Adv. Math.*, 218(5):1419–1429, 2008. arXiv:mathAG/0608590.

[17] R. Cavalieri and D. Ross. Open Gromov-Witten theory and the crepant resolution conjecture. *Michigan Math. J.*, 61(4):807–837, 2012.

[18] W. Chen and Y. Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002.

[19] W. Chen and Y. Ruan. A new cohomology theory of orbifold. *Comm. Math. Phys.*, 248(1):1–31, 2004.

[20] T. Coates, A. Corti, H. Iritani, and H.-H. Tseng. Computing genus-zero twisted Gromov-Witten invariants. *Duke Math. J.*, 147(3):377–438, 2009.

[21] Tom Coates. On the crepant resolution conjecture in the local case. *Comm. Math. Phys.*, 287(3):1071–1108, 2009.

[22] D.-E. Diaconescu and B. Florea. Localization and gluing of topological amplitudes. *Comm. Math. Phys.*, 257(1):119–149, 2005.

[23] R. Dijkgraaf. Mirror symmetry and elliptic curves. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 149–163. Birkhäuser Boston, Boston, MA, 1995.

[24] Ö. N. Eğecioğlu and J. B. Remmel. A combinatorial proof of the Giambelli identity for Schur functions. *Adv. in Math.*, 70(1):59–86, 1988.

[25] C. Faber and R. Pandharipande. Hodge integrals and Gromov-Witten theory. *Invent. Math.*, 139(1):173–199, 2000.

[26] C. Faber and R. Pandharipande. Logarithmic series and Hodge integrals in the tautological ring. *Michigan Math. J.*, 48:215–252, 2000. With an appendix by Don Zagier, Dedicated to William Fulton on the occasion of his 60th birthday.

[27] B. Fantechi, E. Mann, and F. Nironi. Smooth toric Deligne-Mumford stacks. *J. Reine Angew. Math.*, 648:201–244, 2010.

[28] I. Frenkel and W. Wang. Virasoro algebra and wreath product convolution. *J. Algebra*, 242(2):656–671, 2001.

[29] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[30] William D. Gillam. *Hyperelliptic Gromov-Witten theory.* ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Columbia University.

[31] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.

[32] M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

[33] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2003. With a preface by Vafa.

[34] H. Iritani. Ruan's conjecture and integral structures in quantum cohomology. In *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, volume 59 of *Adv. Stud. Pure Math.*, pages 111–166. Math. Soc. Japan, Tokyo, 2010.

[35] P. Johnson. *Equivariant Gromov-Witten theory of one dimensional stacks.* ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Michigan.

[36] P. Johnson. Double Hurwitz number via the infinite wedge. Preprint: arXiv:1008.3266v1, 2010.

[37] S. Katz and C.-C. Liu. Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc. In *The interaction of finite-type and Gromov-Witten invariants (BIRS 2003)*, volume 8 of *Geom. Topol. Monogr.*, pages 1–47. Geom. Topol. Publ., Coventry, 2006.

[38] M. Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser

Boston, Boston, MA, 1995.

[39] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry [MR1291244 (95i:14049)]. In *Mirror symmetry, II*, volume 1 of *AMS/IP Stud. Adv. Math.*, pages 607–653. Amer. Math. Soc., Providence, RI, 1997.

[40] T. Lam and P. Pylyavskyy. Total positivity in loop groups, I: Whirls and curls. *Adv. Math.*, 230(3):1222–1271, 2012.

[41] J. Li, C.-C. Liu, K. Liu, and J. Zhou. A mathematical theory of the topological vertex. *Geom. Topol.*, 13(1):527–621, 2009.

[42] C-.C. Liu, K. Liu, and J. Zhou. A proof of a conjecture of Mariño-Vafa on Hodge integrals. *J. Differential Geom.*, 65(2):289–340, 2003.

[43] C.-C. Liu, K. Liu, and J. Zhou. A formula of two-partition Hodge integrals. *J. Amer. Math. Soc.*, 20(1):149–184, 2007.

[44] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[45] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. I. *Compos. Math.*, 142(5):1263–1285, 2006.

[46] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory. II. *Compos. Math.*, 142(5):1286–1304, 2006.

[47] D. Maulik, A. Oblomkov, A. Okounkov, and R. Pandharipande. Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds. *Invent. Math.*, 186(2):435–479, 2011.

[48] K. Nakada. q-hook formula of Gansner type for a generalized Young diagram. In *21st International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2009)*, Discrete Math. Theor. Comput. Sci. Proc., AK, pages 685–696. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2009.

[49] A. Okounkov and G. Olshanski. Shifted Schur functions. II. The binomial formula for characters of classical groups and its applications. In *Kirillov's seminar on representation theory*, volume 181 of *Amer. Math. Soc. Transl. Ser. 2*, pages 245–271. Amer. Math. Soc., Providence, RI, 1998.

[50] A. Okounkov and R. Pandharipande. Hodge integrals and invariants of the unknot. *Geom. Topol.*, 8:675–699, 2004.

[51] A. Okounkov and R. Pandharipande. The equivariant Gromov-Witten theory of $\mathbf{P}^1$. *Ann. of Math. (2)*, 163(2):561–605, 2006.

[52] A. Okounkov, N. Reshetikhin, and C. Vafa. Quantum Calabi-Yau and classical crystals. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 597–618. Birkhäuser Boston, Boston, MA, 2006.

[53] R. Pandharipande and A. Pixton. Gromov-Witten/Pairs correspondence for the quintic 3-fold. Preprint: math/1206.5490, 2012.

[54] R. Pandharipande and R. Thomas. 13/2 ways of counting curves. To appear in proceedings of "School on Moduli Spaces", Isaac Newton Institute, Cambridge 2011, Preprint: math:1111.1552, 2011.

[55] Z. Qin and W. Wang. Hilbert schemes of points on the minimal resolution and soliton equations. In *Lie algebras, vertex operator algebras and their applications*, volume 442 of *Contemp. Math.*, pages 435–462. Amer. Math. Soc., Providence, RI, 2007.

[56] D. Ross. Localization and gluing of orbifold amplitudes: The Gromov-Witten orbifold vertex. *Trans. Amer. Math. Soc.* in press, 2011. Preprint: math/1109.5995.

[57] D. Ross. The loop Murnaghan-Nakayama rule. *J. Algebraic Combin.* in press, 2012. Preprint: math:1208.4369.

[58] D. Ross and Z. Zong. The gerby Gopakumar-Mariño-Vafa formula. Preprint: math/1208.4342, 2013.

[59] Y. Ruan. Topological sigma model and Donaldson-type invariants in Gromov theory. *Duke Math. J.*, 83(2):461–500, 1996.

[60] Y. Ruan. The cohomology ring of crepant resolutions of orbifolds. In *Gromov-Witten theory of spin curves and orbifolds*, volume 403 of *Contemp. Math.*, pages 117–126. Amer. Math. Soc., Providence, RI, 2006.

[61] Y. Ruan and G. Tian. A mathematical theory of quantum cohomology. *J. Differential Geom.*, 42(2):259–367, 1995.

[62] R. P. Thomas. A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations. *J. Differential Geom.*, 54(2):367–438, 2000.

[63] J. Wise. The genus zero Gromov-Witten invariants of the symmetric square of the plane. *Comm. Anal. Geom.*, 19(5):923–974, 2011.

[64] E. Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

[65] Z. Zong. Generalized Mariño-Vafa formula and local Gromov-Witten theory of orbicurves. Preprint: math/1109.4992, 2011.