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ABSTRACT

A simple two-dimensional model for the heating effects of the earth's surface on the distribution of the atmospheric flow and pressure patterns is considered. Attention is focused especially on the stationary flow patterns caused by stationary thermal forcing functions and located in the circumpolar regions. The governing relation is the linearized equation for the vertical vorticity component in which the solenoidal term is retained. The first part of this term is due to the effect of the advection of background temperature, assumed to have zonal symmetry, by the perturbation motion. The other term represents the forcing due to the temperature perturbation. In order to provide for damping of transient disturbances Rayleigh friction is included in the model.

For simplicity, conditions in a rectangular Cartesian Beta plane are considered first. Let the forcing function be in the form of a standing sine wave, and neglect friction. Then the resulting pressure wave is in phase (180° out of phase) with this forcing function if the wave length of the perturbation is larger (smaller) than the wave length of the stationary Rossby wave L_s, which is determined by the velocity of the basic zonal current, the meridional width of the perturbation, and the Beta parameter (see Table I). Correspondingly, the amplitudes of the pressure (and stream line) perturbations, produced by a forcing function of given amplitude, are largest for the waves closest to L_s and decrease for forcing functions with shorter and longer wave lengths. With Rayleigh friction, the phase differences between forcing function and resulting perturbation are no longer

exactly 0° and 180°, respectively, but the temperature and pressure perturbations are still nearly in phase for waves longer than L_s, and nearly 180° out of phase for waves shorter than L_s. The amplitudes of the forced pressure perturbations are reduced by friction, and the reductions are largest for waves whose lengths are closest to L_s.

For the study of circumpolar waves a plane polar coordinate system centered at the pole and here tangential to the earth's surface is used. The results are similar to those for a rectangular Cartesian Beta plane, although there are quantitative as well as qualitative differences because of the singularity at the pole. In particular the meridional structure is now described not by trigonometric, but by Bessel functions. Nevertheless, for stationary waves the phase relations between forcing functions and forced pressure and flow patterns are the same as for the rectangular Cartesian case, namely in phase (180° out of phase) if the perturbation wave length is longer (shorter) than the stationary Rossby wave length for the polar-coordinate geometry (Table III). Further, the response of the forced pressure patterns to a forcing function of given amplitude is larger, the closer the wave length of the forcing function is to that of the stationary Rossby wave (Table III).

Examples are given to show how fast pressure and flow patterns adjust to thermal forcing (Fig. 1), and how wave patterns, whose centers of symmetry are eccentric to the pole, may be represented by series of wave modes of the type discussed in this paper (Figs. 4 and 5, Tables V-VII).

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A_{kd}, $A_{k\ell}$	=	amplitude of perturbation-temperature mode
C_1, C_2	=	factor of cosine and sine in the forced-perturbation stream function
$C_1^{(p)}$, $C_2^{(p)}$	=	same factors for the perturbation-pressure distribution
C	=	$(C_1^2 + C_2^2)^{1/2}$
D	=	see Eq. (2.8)
E	=	6.37×10^6 m, earth's radius
$F(x,y)$	=	steady, forced oscillation
J_k	=	Bessel function of the first kind of order k
$K_{k\ell}$	=	amplitude of free wave mode
L_x	=	$2\pi k^{-1}$
P	=	basic pressure
Q	=	basic density
R	=	287 $m^2 s^{-2} K^{-1}$, gas constant for air
S	=	solenoidal term
T, $\overline{T}$	=	basic temperature, mean basic temperature
U, V_λ	=	basic eastward velocity component
$a(t)$	=	see Eq. (2.11)
b	=	see Eq. (3.5)
c	=	j_{n1}/d
d	=	radial extent of perturbation
f	=	$2\omega \sin \phi$, Coriolis parameter
j_{ks}	=	s'th zero of J_k
k	=	wave number in the x direction, or in the λ direction
ℓ	=	wave number in the y direction
m	=	$(k^2 + \ell^2)^{1/2}$

n	=	number of waves around a latitude circle or around the center of symmetry
p	=	perturbation pressure
r	=	linear pole distance
r'	=	distance from center of symmetry
r_o	=	distance between geographic pole and center of symmetry
t	=	time
x,y	=	eastward and northward coordinates in the beta plane
α	=	basic angular zonal velocity
$\alpha_{k\ell}$	=	phase constant of forced oscillation F
β	=	latitudinal derivative of f in the beta plane
β^*	=	see Eq. (2.3)
$\gamma(t)$	=	see Eq. (2.11)
Δ_{kd}	=	see Eq. (3.9)
ε	=	angular coordinate with reference to the center of symmetry
Z,ζ	=	vertical vorticity component, basic and perturbation motion
η	=	coefficient of Rayleigh friction
θ	=	north pole distance (angular measure)
λ	=	geographic longitude
ν	=	kinematic coefficient of eddy viscosity
σ,σ^*	=	real, complex wave frequency
τ	=	perturbation temperature
ϕ	=	geographic latitude
ψ	=	perturbation stream function
ω	=	$7.292 \times 10^{-5}\ s^{-1}$, frequency of the earth's rotation

1. INTRODUCTION

It is well known that the unequal distribution of the continents and oceans with their different thermal effects exerts a pronounced influence on the atmosphere and its flow patterns. At high latitudes this influence is further enhanced by the variable snow cover over land and the variable ice distribution over the oceans. One particular aspect of this problem has been discussed in a paper by Haurwitz and Bridger (1977), namely the appearance of stationary flow patterns which are not centered at the geographic pole, but at another point. It has been shown that this eccentricity can be attributed to a similar eccentricity of the forcing function which in turn is due to the asymmetric distribution of land and water with respect to the pole.

We shall here discuss these thermally induced large-scale atmospheric motions at greater detail, although still by means of a highly simplified mathematical model which leads to a simple relation between forced pressure patterns and the thermal forcing. Specifically, it will again be assumed that the motion is strictly horizontal and has no divergence so that it can be represented by a horizontal stream function. The effects of changes in the vertical will be included in a later paper. However, the model to be considered here includes, in contrast to the one used before, a dissipative term in the form of Rayleigh friction, and allows also for the effect of the mean meridional temperature gradient.

2. CONDITIONS ON A BETAPLANE

For a first orientation, it is useful to discuss conditions on a beta plane. Accordingly, a rectangular Cartesian coordinate system is used with the x-axis towards the east and the y-axis towards the north. The basic motion, U, may be zonal, constant and geostrophic. Hence

$$fU = - Q^{-1} \, \partial P/\partial y. \tag{1}$$

Superimposed on this basic motion is a perturbation assumed to be purely horizontal and non-divergent and subjected to Rayleigh friction, that is friction proportional to the perturbation wind vector, with η being the constant coefficient of friction. Then the perturbation vorticity equation is

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} + \eta\right) \nabla^2\psi + \beta \frac{\partial \psi}{\partial x} = S .$$

Here ψ is the perturbation stream function which exists since the motion is non-divergent, β is the latitudinal change of the Coriolis parameter, and $S(x,y)$ is the solenoidal term which will be regarded as the forcing function (Haurwitz, 1940). From the derivation of the vorticity equation it follows that the solenoidal term

$$S = - \frac{R}{P} \left(\frac{\partial \tau}{\partial x} \frac{\partial P}{\partial y} - \frac{\partial T}{\partial y} \frac{\partial p}{\partial x}\right) ,$$

where R is the gas constant for air, $T(y)$ and P are the basic temperature and pressure, τ and p their perturbations. The second term in S represents the effect of a meridional temperature gradient, which could also be expressed by the vertical shear of the zonal current. We may regard the perturbation motion as nearly geostrophic since we are considering here long waves, and since the coefficient of friction η will be assumed much smaller than the Coriolis parameter. Then, approximately,

$$\partial p/\partial x = fQ\ \partial\psi/\partial x\ , \tag{2}$$

so that the second term of S becomes

$$T^{-1}\ (dT/dy)\ f\ \partial\psi/\partial x\ .$$

This term may be combined with the last term on the left side of the vorticity equation by introducing

$$\beta^* = \beta - T^{-1}\ (dT/dy)\ f\ . \tag{3}$$

If the temperature difference between the north pole and the equator is -60 K and the mean temperature T = 280 K, then, at 60° N, $\beta = 1.14 \times 10^{-11}$ $m^{-1}\ s^{-1}$ while $T^{-1}(\partial T/\partial y)f = 0.252 \times 10^{-11}\ m^{-1}\ s^{-1}$ so that β^* is about 20 percent larger than β. The first term of S becomes with (1)

$$(fU/T)\ \partial\tau/\partial x\ ,$$

where a representative constant mean value may be used for T, and where f may also be regarded as constant, in accordance with the usual beta plane approximation. Then the vorticity equation takes the form

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} + \eta\right)\nabla^2\psi + \beta^*\frac{\partial\psi}{\partial x} = fUT^{-1}\frac{\partial\tau}{\partial x}\ . \tag{4}$$

Here the right-hand side is the thermal forcing function which will be regarded as known and as unaffected by the motion. The effect of the coupling between the fields of temperature and of motion will be studied in a later investigation.

The term $\eta\nabla^2\psi$ in (4) arises because the friction is assumed proportional to the perturbation velocity. A similar expression could have been obtained by introducing on the right hand sides of the momentum equations the frictional term $\nu\nabla^2 u$ and $\nu\nabla^2 u$ where ν represents a kinematic coefficient of eddy viscosity. Then the frictional term in the vorticity equation would be

$$\nu\nabla^2 \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \nu\nabla^2\nabla^2\psi \ .$$

Since it will be assumed that

$$\psi \propto e^{ikx} \, e^{i\ell y} \ ,$$

a frictional term of the form $\nu(k^2+\ell^2)\nabla^2\psi$ would thus appear on the left-hand side of (4) instead of $\eta\nabla^2\psi$. Consequently, the dissipation would be larger for shorter waves, as is plausible since the effect of friction increases with increasing curvature of the wind profile.

Since the forcing term in (4) is independent of time let

$$\psi = K_{k\ell} \exp\left[i\,(\sigma^* t + kx + \ell y)\right] + F\,(x,y)\,, \tag{5}$$

where the first term represents the free oscillation and $F(x,y)$ represents the stationary forced part of the disturbance. By substitution in (4) it follows that

$$\sigma^* = k\,\left(\frac{\beta^*}{k^2+\ell^2} - U\right) + i\eta \ . \tag{5a}$$

Thus, the amplitude of the time-dependent part of ψ decreases as $e^{-\eta t}$ while the real frequency has the familiar form of the Rossby wave, modified by the effect of meridional temperature advection. For the time-independent part $F(x,y)$ of ψ it is found that

$$\left(\frac{\partial}{\partial x} + \frac{\eta}{U}\right) \nabla^2 F + \frac{\beta^*}{U}\frac{\partial F}{\partial x} = fT^{-1}\frac{\partial \tau}{\partial x} . \tag{6}$$

The temperature perturbation in the beta plane may be assumed in the form

$$\tau = A_{k\ell} \sin kx \cos \ell y . \tag{7}$$

More general distributions of τ can be expressed by double Fourier series. The forcing term in (6) becomes with (7)

$$T^{-1} fk A_{k\ell} \cos kx \cos \ell y .$$

Putting

$$F = (C_1 \cos kx + C_2 \sin kx) \cos \ell y ,$$

we find by substitution in (6) that

$$C_1 = - T^{-1} fk A_{k\ell} D^{-1} \eta m^2/U,$$

and

$$C_2 = - T^{-1} fk A_{k\ell} D^{-1} k (m^2 - \beta^*/U),$$

where

$$m^2 = k^2 + \ell^2$$

and

$$D = k^2 (m^2 - \beta^*/U)^2 + (\eta m^2 U)^2 . \tag{8}$$

Thus, F may also be written in the form

$$F = T^{-1} fk A_{k\ell} D^{-\frac{1}{2}} \sin (kx - \alpha_{k\ell}) \cos \ell y , \tag{8a}$$

where

$$\tan \alpha_{k\ell} = - C_1/C_2 = \frac{\eta m^2/U}{- k (m^2 - \beta^*/U)} . \tag{9}$$

Thus if $\eta = 0$, the phase constant $\alpha_{k\ell}$ is either zero or 180^o, and F is in phase with τ if $m^2 < \beta^*/U$, and 180^o out of phase if $m^2 > \beta^*/U$. In the case of resonance, $m^2 = \beta^*/U$, friction prevents the amplitude of F from becoming infinite. In the foregoing equations η is always multiplied by m^2 indicating a strong inverse dependence of frictional effects on wave length. This dependence would be further enhanced by assuming that $\eta = m^2\nu$, as discussed above in connection with the hydrodynamic expression for viscous dissipation.

If the forcing term in (6) vanishes, $A_{k\ell} = 0$ and C_1 and C_2 are zero, unless D = 0. If friction vanishes this condition is satisfied when $m^2 = \beta^*/U$, but if $\eta \neq 0$ we have D > 0, since k and ℓ must be real to satisfy the periodicity condition in the x and y directions.

For a numerical example of such a thermally induced stationary wave the following numerical values were chosen:

Disturbance extends from 85° to 45° in latitude,

hence $\pi/\ell = 4.444 \times 10^6$ m and

$f = f(65°) = 1.322 \times 10^{-4}\ s^{-1}$, $\beta = 9.676 \times 10^{-12}\ m^{-1}\ s^{-1}$,

Mean temperature T = 280 K, $dT/dy = - 6 \times 10^{-6}\ K\ m^{-1}$,

hence $\beta^* = 1.251 \times 10^{-11}\ m^{-1}\ s^{-1}$,

$U = 10\ m\ s^{-1}$,

$\eta = 1.653 \times 10^{-6}\ s^{-1}$, thus e-folding time 7 days, and

n = number of waves around latitude circle, thus for n = 1

at 65° lat., $L_x = 1.6915 \times 10^7$ m.

The amplitude $A_{k\ell}$ of the temperature perturbation has been assumed equal to 1 K. The results of the calculations are shown in Table I for n from 1 to 6 and in the absence and presence of friction. Instead of the harmonic constants C_1, C_2 and $C = (C_1{}^2 + C_2{}^2)^{\frac{1}{2}}$ for the perturbation stream function F(x,y) those for the pressure perturbation, denoted by $C^{(p)}$, are shown. They are, according to (2), obtained by multiplying C etc. by fQ, where it has been assumed that the density Q = 1.29 kg m^{-3}. Because of the geostrophic assumption, eq. (2), the streamlines are, of course, also isobars, but values of the pressure perturbation may indicate more clearly than those of the stream function the magnitude of the forced perturbation. With the numerical values assumed, we see that, in the case of no friction, for wave numbers one and two, the pressure perturbation is in phase with the temperature perturbation, but 180° out of phase for higher wave numbers (Table I). Resonance would occur for a zonal wave length L_x = 7250 km, but at 65° latitude no integral wave number exists for this wave length. With friction, the pressure amplitude is smaller than without friction, as would be expected. The reduction is quite pronounced for small n, but very small for the larger n. The reason for this can be seen from (9) which shows the ratio, of C_1, the factor of the term representing mainly the frictional influence, to C_2. The constant C_2 is proportional to k which increases with increasing n. Therefore, with our assumption of a constant η,

TABLE I

Harmonic Constants for Pressure Perturbation
in a Cartesian Beta Plane
(1 Pa = 10^{-2} mb)

	$\eta = 0$			$\eta = 1.653 \times 10^{-6}\ s^{-1} = (7\ day)^{-1}$				
n	$L_x = 2\pi k^{-1}$ km	$C_2^{(p)}$ Pa	$\alpha_{k\ell}$ degrees	$C_1^{(p)}$ Pa	$C_2^{(p)}$ Pa	$C^{(p)}$ Pa	$\alpha_{k\ell}$ degrees	$\alpha_{k\ell}$ radians
1	16915	131.2	0	-50.0	108.1	119.2	24.8	.433
2	8458	403.8	0	-199.4	169.9	262.0	49.6	.865
3	5638	164.1	180°	-67.7	-128.5	145.2	152.2	2.656
4	4229	55.2	180°	-11.0	-53.0	54.1	168.3	2.938
5	3383	29.8	180°	-3.8	-29.3	29.6	172.6	3.012
6	2819	19.1	180°	-1.8	-18.9	19.0	174.5	3.045

shorter waves must appear less and less influenced by friction. A more realistic picture of the effect of friction on the shorter waves is obtained by assuming η proportional to m^2, as discussed earlier in this Section. With this assumption the amplitudes of the waves 4-6 would be reduced to about 3/4 of the values given in Table I for $\eta = 0$.

To determine how fast the atmosphere adjusts to thermal forcing, one may introduce the stationary solution (8a) into (5) and determine the arbitrary constant $K_{k\ell}$ by an initial condition. For an estimate of the time required for an adjustment, a very simple, although somewhat unrealistic, condition may be assumed, namely that the stream function ψ representing the free plus forced oscillation is zero at $t = 0$ while the forcing temperature field is fully established. Then

$$\psi = K_{k\ell}\, e^{-\eta t} \sin(\sigma t + kx - \alpha_{k\ell}) \cos \ell y + \frac{fk\, A_{k\ell}}{TD^{\frac{1}{2}}} \sin(kx - \alpha_{k\ell}) \cos \ell y\,.$$

Here a suitable real form of the free oscillation is taken, and

$$\sigma = k\left(\frac{\beta^*}{k^2 + \ell^2} - U\right).$$

Since $\psi(t=0) = 0$,

$$K_{k\ell} = -\frac{fk\, A_{k\ell}}{TD^{\frac{1}{2}}}\,,$$

and, therefore,

$$\psi = \frac{fkA_{k\ell}}{TD^{\frac{1}{2}}} \cos \ell y \left\{ \sin(kx - \alpha_{k\ell}) - e^{-\eta t} \sin(\sigma t + kx - \alpha_{k\ell}) \right\}. \quad (10)$$

The total oscillation is thus represented by a stationary and a traveling part, the latter decreasing exponentially with time. For a given value of η, the amplitude reduction after one period $2\pi/\sigma$ is evidently larger, the longer the period. To illustrate the superposition of the forced and damped free oscillations, (10) may be written in the form

$$\psi = \frac{fkA_{k\ell}}{D^{\frac{1}{2}}T} \cos \ell y\; a(t) \sin\left[kx - \alpha_{k\ell} - \gamma(t)\right], \quad (10a)$$

where

$$a(t) = \left[1 - 2e^{-\eta t} \cos \sigma t + e^{-2\eta t}\right]^{\frac{1}{2}} \quad (11)$$

and

$$\tan \gamma(t) = e^{-\eta t} \sin \sigma t \left[1 - e^{-\eta t} \cos \sigma t\right]^{-1}.$$

Selecting the same numerical values as for Table I, we obtain for wave numbers n = 2 and 4 the time variations of the amplitude factor

$a(t)$ and of the time variable part of the phase angle $\gamma(t)$ shown in Fig. 1. For $n = 2$, the free period is about 52 days (see Table II), more than seven times larger than the e-folding time of seven days. Consequently, γ_2 decreases almost monotonically from its value of $\tan^{-1} \sigma/\eta$ at $t = 0$ to its asymptotic value of zero as $t \to \infty$. A first zero is reached after 25 days, and thereafter γ_2 oscillates only slightly around zero. Similarly, a_2 grows during the first half period to slightly more than its asymtotic value of one, but has almost reached this value after 25 days. Consequently, both $a(t)$ and $\gamma(t)$ become essentially independent of time halfway through the first period of the oscillation.

For wave number 4, the period, 9.1 days, is only a little larger than the e-folding time of 7 days. Also, the doppler-shifted frequency, σ, is negative, indicating that the traveling wave moves eastward. Consequently γ_4 is negative ($-\gamma_4$ has been plotted in Fig. 1). Both a_4 and γ_4 show damped variations during the first few periods, and the time intervals between subsequent maxima or minima approximate the free period. The maximum of ψ for $y = 0$ occurs at $x_{max} = [\pi/2 + \alpha_{k\ell} + \gamma(t)]/k$. For $n = 4$, after 7 days, $\gamma_4 \simeq 21°$, after 11 days $\gamma_4 \simeq -11°$, after 16 days $\gamma_4 \approx 6°$, while $a_4(t)$ has closely approached the value one after 16 days. Thus, during the first stage of its development, the perturbation does move around its equilibrium position, but the deviations decrease rapidly with time.

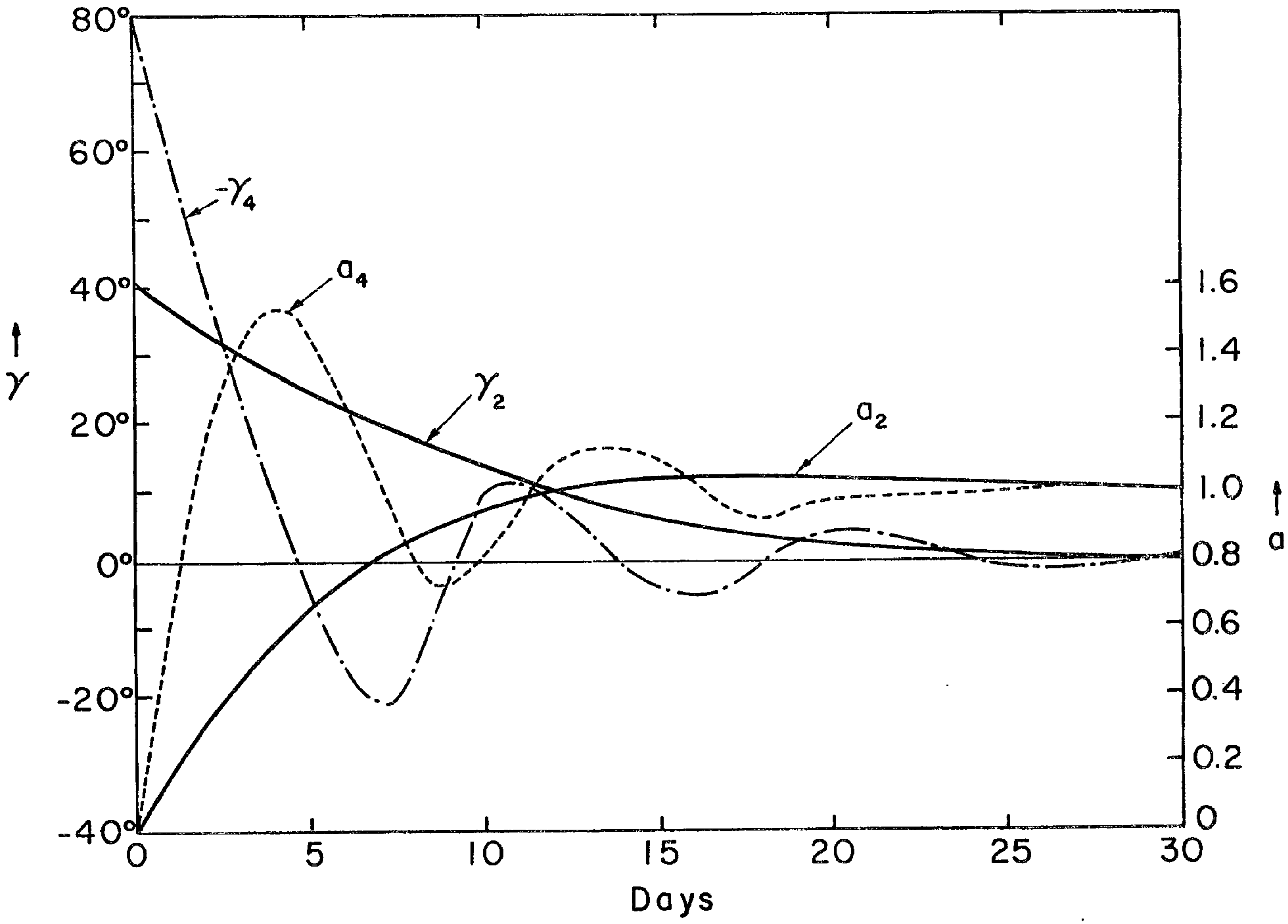

Fig. 1. Development of forced oscillations for wave numbers 2 (full curves) and 4 (dashed and dotted curves). The scale for the time change a(t) of the amplitude, from zero to one, is the right-hand ordinate. The scale for the time change of the phase difference γ(t) between the forcing function and the stream function is the left-hand ordinate. (For more details see text).

TABLE II

Wave Speeds and Frequencies for Free Waves

n	σ/k ms^{-1}	σ * s^{-1}	Period days
1	9.62	3.572×10^{-6}	20.36
2	1.90	1.408×10^{-6}	51.63
3	-2.82	-3.140×10^{-6}	-23.16
4	-5.38	-7.992×10^{-6}	-9.10
5	-6.83	-12.690×10^{-6}	-5.73
6	-7.71	-17.189×10^{-6}	-4.23

* $\sigma < 0$ for an eastward propagating wave and vice versa.

3. THERMAL FORCING OF CIRCUMPOLAR WAVES

If the area covered by the perturbation includes the pole, but does not extend too far toward the equator, it is convenient to use plane polar coordinates. Thermal forcing in the circumpolar region has been discussed previously (Haurwitz and Bridger, 1977) in order to explain the existence and stationarity of flow and pressure configurations eccentric to the pole. The following treatment is similar to that given in the paper referred to, but is extended and includes the effect of Rayleigh friction. The model is analogous to that used in the preceding section, with the modifications required by the singularity at the pole.

Let r be the pole distance and λ the geographic longitude. If E is the earth's radius and θ the colatitude, then $r = E\theta$. This value of r is somewhat larger than the radius of the corresponding colatitude circle, $E \sin \theta$, but even at $\theta = 30^{\circ}$ the difference amounts only to 5 percent. Thus we may use a plane tangential to the earth at the pole instead of the actual spherical surface of the earth as an approximation since the distortion is only small.

It is assumed that the undisturbed wind velocity is zonal, $V_{\lambda} = \alpha r$, where α is the angular velocity relative to the earth, assumed to be constant. Then the gradient wind relation is

$$\alpha\ (\alpha + f)\ r = Q^{-1}\ \partial P/\partial r\ . \qquad (1)^*$$

*Equations are numbered beginning with (1) in each Section. If equations from another Section are referred to, they will be cited by giving first the Section number, second the number of the equation, for instance (2.1).

In the following work, the same notation is used as in section 2, unless otherwise stated. The Coriolis parameter $f = 2\,\omega \cos\theta$ since θ is now used instead of ϕ.

Let Z and ζ denote, respectively, the undisturbed and disturbed vertical component of the relative vorticity and let v_r and v_λ denote the meridional and longitudinal perturbation velocities. The perturbation motion is assumed horizontal and non-divergent so that a stream function ψ exists, and

$$\zeta = \nabla^2\psi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2\psi}{\partial\lambda^2}\,. \qquad (2)$$

The meridional variation of the Coriolis parameter is given by

$$\frac{df}{dr} = 2\omega\,\frac{d}{dr}\,\left(1 - \frac{r^2}{E^2}\right) = -\,2\omega r/E^2\,.$$

To make $\partial Z/\partial r$, the meridional variation of the undisturbed relative vorticity, similar to df/dr, we use the expression for a spherical earth (Haurwitz, 1975)

$$\frac{\partial Z}{\partial r} = -\,2\alpha r/E^2\,.$$

The solenoidal term

$$-\,\frac{1}{QT}\left(\frac{dT}{dr}\,\frac{\partial p}{r\partial\lambda} - \frac{\partial\tau}{r\partial\lambda}\;\;\frac{\partial p}{\partial r}\right),$$

may be changed by introducing the gradient wind relation (1) for the undisturbed motion and by assuming, as in Section 2, that the meridional perturbation velocity, v_r, approximates the gradient wind. Then

$$(2\alpha + f)\,\frac{\partial\psi}{r\partial\lambda} = Q^{-1}\,\frac{\partial p}{r\partial\lambda}\,. \qquad (3)$$

To show that (3) represents a satisfactory approximation, consider the complete λ-momentum equation

$$- (2\alpha + f) \frac{\partial\psi}{r\partial\lambda} - \alpha \frac{\partial^2\psi}{r\partial\lambda^2} + \eta \frac{\partial\psi}{\partial r} = - \frac{1}{Q} \frac{\partial p}{r\partial\lambda} .$$

Here, the terms on the left hand side have the following orders of magnitude,

$$0 \left[(2\alpha + f)\ \partial\psi/r\partial\lambda\right] = 10^{-4}\ k\ \psi/r,$$

$$0 \left[\alpha\partial^2\psi/r\partial\lambda^2\right] = 10^{-6}\ k^2\psi/r$$

and

$$0 \left[\eta\partial\psi/\partial r\right] = 10^{-6}\ (j_{ks}/d)\ \partial\ J_k/\partial r.$$

The radial dependence of ψ is expressed by Bessel functions J_k, j_{ks} being the s'th zero of J_k while d is that distance from the pole at which the temperature disturbance vanishes, so that $d \geq r$. With the values of k and j_{ks} in which we are interested, the first term of the λ-momentum equation is more than an order of magnitude larger than the other two on the left-hand side, so that (3) is a satisfactory approximation. With (3) the perturbation vorticity equation becomes

$$\left(\frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial\lambda} + \eta\right) \nabla^2\psi + \left[\frac{2(\alpha + \omega)}{E^2} + \frac{2\alpha + f}{\overline{T}} \frac{dT}{rdr}\right] \frac{\partial\psi}{\partial\lambda} = \frac{\alpha(\alpha + f)}{\overline{T}} \frac{\partial\tau}{\partial\lambda} . \qquad (4)$$

The second term in the bracket on the left side of (4) represents the effect of meridional advection. It will be convenient to treat this term as constant, which implies that dT/dr is proportional to r since the mean temperature $\overline{T}$ may be considered constant. We shall write for brevity, similar to β^* in (2.3),

$$b = \frac{2(\alpha + \omega)}{E^2} + \frac{2\alpha + f}{\overline{T}} \frac{dT}{rdr} . \tag{5}$$

Solutions of the homogeneous equation (4), representing free oscillations, are of the form

$$\psi_h = A_k J_k \left[\left(\frac{kb}{\sigma + \alpha k}\right)^{\frac{1}{2}} r\right] e^{-\eta t} e^{i(\sigma t + k\lambda)} , \tag{6}$$

where A_k is an arbitrary constant, and where it has been assumed that the solution is bounded at zero so that only Bessel function of the first kind give suitable solutions. For a given wave number k, the frequency σ may be fixed by the condition that $\psi = 0$ when $r = d$. Then, if for f in (5) the value at the pole is substituted,

$$\sigma + \alpha k = \frac{2(\alpha + \omega)}{E^2} k \frac{d^2}{j_{ks}^2} \left(1 + \frac{E^2}{\overline{T}} \frac{dT}{rdr}\right) . \tag{7}$$

If in particular ψ is not zero for any $r < d$ the first root, j_{k1}, must be chosen.

Equation (7) for the frequency of free oscillations is similar to an expression derived earlier (Haurwitz, 1975), but without considering the meridional temperature advection. To estimate the magnitude of this effect let, as in Section 2, $dT/dr = 6 \times 10^{-6}$ K m^{-1} while $\overline{T} = 280$ K. With r = 2220 km, corresponding to half the distance between the pole and 50° latitude,

$$\frac{E^2}{\overline{T}} \frac{dT}{rdr} = 0.39 .$$

Although our choice of dT/dr contradicts the requirement that dT/(rdr) be constant, used in deriving (6) and (7), the example shows that the effect of the temperature advection can be appreciable. The periods of free oscillations may thus be substantially smaller than indicated by calculations based on a barotropic atmosphere (Haurwitz, 1975).

Similar to Section 2, the frictional term $\eta\nabla^2\psi$ in (4) of this section could also be obtained by introducing on the right-hand sides of the momentum equations the frictional terms $\nu\nabla^2 v_r$ and $\nu\nabla^2 v_\lambda$, where ν is the kinematic coefficient of eddy viscosity. Then, with a stream function of the form (6) vanishing at $r = d$, the term $\eta\nabla^2\psi$ is replaced by

$$- \nu \left[\frac{1}{r}\frac{\partial}{\partial r}(r\nabla^2 v_\lambda) - \frac{\partial}{r\partial\lambda}(\nabla^2 v_r)\right] = - \nu\nabla^4\psi = \nu\,\frac{j_{ks}^2}{d^2}\,\nabla^2\psi,$$

indicating that η depends on the scale of the perturbations, as in Section 2.

For the forcing function in (4) it is assumed that

$$\tau = A_{kd} \sin k\lambda\; J_k\,(j_{ks}\; r/d)\,. \tag{8}$$

Here A_{kd} is a constant determined by the temperature perturbation, which is assumed to be zero at $r = d$ and j_{ks} is the s'th zero of the Bessel function of order k, implying that τ vanishes at $s - 1$ colatitudes for which $r < d$ as well as vanishing at $r = d$. More general temperature disturbances can be expressed by superposition of terms of the form (8). The forcing function in (4) becomes thus

$$A_{kd}\,(\alpha + f)\,\overline{T}^{-1}\,k \cos k\lambda\; J_k\,(j_{ks}\; r/d)\,.$$

Since friction will produce a phase difference between ψ and τ, the perturbation stream function will be of the form

$$\psi = (C_1 \cos k\lambda + C_2 \sin k\lambda)\, J_k\, (j_{ks}\, r/d)\,.$$

It is found by substitution in (4) that the following relations hold for C_1 and C_2

$$C_1 \left(\frac{j_{ks}^2}{d^2} - \frac{b}{\alpha}\right) k - C_2 \frac{j_{ks}^2}{d^2} \frac{\eta}{\alpha} = 0$$

$$C_1 \frac{j_{ks}^2}{d^2} \frac{\eta}{\alpha} + C_2 \left(\frac{j_{ks}^2}{d^2} - \frac{b}{\alpha}\right) k = - \frac{\alpha + f}{\overline{T}} k\, A_{kd}\,.$$

With

$$\Delta_{kd} = k^2 \left(\frac{j_{ks}^2}{d^2} - \frac{b}{\alpha}\right)^2 + \left(\frac{j_{ks}^2}{d^2} \frac{\eta}{\alpha}\right)^2, \tag{9}$$

these constants are

$$C_1 = - \frac{\alpha+f}{\overline{T}} k\, A_{kd}\, \Delta_{kd}^{-1} \frac{j_{ks}^2}{d^2} \frac{\eta}{\alpha}$$

$$C_2 = - \frac{\alpha+f}{\overline{T}} k^2\, A_{kd}\, \Delta_{kd}^{-1} \left(\frac{j_{ks}^2}{d^2} - \frac{b}{\alpha}\right).$$

The solution may also be written in the form

$$\psi = \frac{\alpha+f}{\overline{T}} A_{kd}\, k\, \Delta_{kd}^{-\frac{1}{2}}\, J_k\, (j_{ks}\, r/d) \sin (k\lambda - \alpha_{kd}), \tag{10}$$

where

$$\tan \alpha_{kd} = \frac{(j_{ks}^2/d^2)\, \eta/\alpha}{k \left(\frac{b}{\alpha} - j_{ks}^2/d^2\right)}\,. \tag{11}$$

The two equations for C_1 and C_2 show that in the absence of the forcing function a non-zero solution for the stationary case requires that $\Delta_{kd} = 0$. This is impossible in the presence of friction if the perturbation is to have a finite latitudinal extent, as was already found in the case of the Cartesian beta-plane in Section 2.

To give a numerical example, we choose the following values which are as far as possible the same as in the previous Section:

$A_{kd} = 1$ K

$d = 4.444 \times 10^6$ m, corresponding to latitude 50°, no intermediate nodal parallels.

$\alpha = 2.25 \times 10^{-6}\ s^{-1}$, so that $\alpha d = 10\ m\ s^{-1}$

$\overline{T} = 280$ K, $dT/dr = 6 \times 10^{-6}\ K\ m^{-1}$ at $r = 2.222 \times 10^6\ m^{-1}$

thus $T^{-1}\ r^{-1}\ dT/dr = 9.644 \times 10^{-15}\ m^{-2}$

$f = 2\ \omega \sin \phi = 1.322 \times 10^{-4}\ s^{-1}$ at latitude 65°

$\eta = 1.653 \times 10^{-6}\ s^{-1}$, that is e-folding time = 7 days.

As in the case of the Cartesian beta plane of the preceding Section, we compute the constants for the pressure rather than the streamline distribution, making use of the gradient wind approximation, (3), according to which

$$p = (2\alpha + f)\ Q\psi\ . \tag{10a}$$

The density Q will be assumed to be $1.29\ kg\ m^{-3}$.

Values of the amplitudes and phases for the pressure field are shown in Table III. As in the case presented in Table I the temperature and pressure perturbations, in the absence of friction, are in phase for the smaller wave numbers k = 1,2,3, but are 180° out of phase for the larger ones, with the largest pressure amplitude occurring at k = 3.

TABLE III

Harmonic Coefficients for Pressure Perturbations of Circumpolar Waves

	$\eta = 0$		$\eta = 1.653 \times 10^{-6}\ s^{-1}$				
k	$C_2^{(p)}$ Pa	α_{kd} degrees	$C_1^{(p)}$ Pa	$C_2^{(p)}$ Pa	$C^{(p)}$ Pa	α_{kd} degrees	α_{kd} radians
1	43.8	0	-4.8	43.2	43.5	6.4	.112
2	95.0	0	-39.9	72.6	83.0	28.8	.503
3	511.6	0	-151.4	49.6	159.4	71.8	1.253
4	122.9	180	-59.5	-76.6	97.0	142.1	2.480
5	50.7	180	-15.6	-45.4	48.0	161.1	2.812
6	30.5	180	-6.4	-29.1	29.8	167.5	2.923

In the presence of friction, there is a phase difference between the thermal forcing and the isobars (or streamlines), and the amplitudes are reduced, especially in the vicinity of k = 3, for which the forced pressure oscillation is largest with the atmospheric parameters assumed here.

From (10), it follows that the largest response for a given amplitude of the forcing function occurs when $k\,\Delta_{kd}^{-\frac{1}{2}}$ is largest. If friction and the meridional temperature gradient are zero this leads to the resonance condition that

$$d/E \simeq j_{ks} \left[\frac{\alpha}{2(\alpha + \omega)}\right]^{\frac{1}{2}}. \qquad (12)$$

Thus for a zonal wind velocity α the radius d of the perturbation with largest response increases as k, and hence j_{k1}, increases. If d is kept constant the wave number of greatest response becomes smaller as α increases. These relations are illustrated in Fig. 2.

Attainment of the steady case, if the initial pressure and flow patterns are different, may be treated in the same way as in Section 2. If it is again assumed for simplicity that at $t = 0$ the perturbation flow pattern, and thus also the perturbation pressure pattern, is zero, a suitable special form of ψ_h, eq. (6), is

$$\psi_h = A_k^* J_k (j_{ks} r/d) e^{-\eta t} \sin (\sigma t + k\lambda - \alpha_{kd}) ,$$

where A_k^* is a new arbitrary constant, and σ is given by (7). Since it is now required that the sum of the homogeneous solution ψ_h plus the stationary inhomogeneous solution (10) is zero at $t = 0$,

$$A_k^* = - \frac{\alpha + f}{\overline{T}} A_{kd} \Delta^{-\frac{1}{2}} ,$$

and it follows that

$$\psi = \frac{\alpha+f}{\overline{T}} A_{kd} k\Delta^{-\frac{1}{2}} J_k (j_{ks} r/d) \left[\sin (k\lambda - \alpha_{kd}) - e^{-\eta t} \sin (\sigma t + k\lambda - \alpha_{kd})\right]$$

or

$$\psi = \frac{\alpha+f}{\overline{T}} A_{kd} k\Delta^{-\frac{1}{2}} J_k (j_{ks} r/d) a(t) \sin \left[k\lambda - \alpha_{kd} - \gamma(t)\right] ,$$

where $a(t)$ and $\gamma(t)$ are given by (2.11). Examples of the time dependence of the amplitude and phase are shown in Fig. 1. It should of course, be noted that the frequencies and periods in the case of the circumpolar waves differ from those for a rectangular geometry. These periods and frequencies for the numerical parameters given on p. 19 are shown in Table IV.

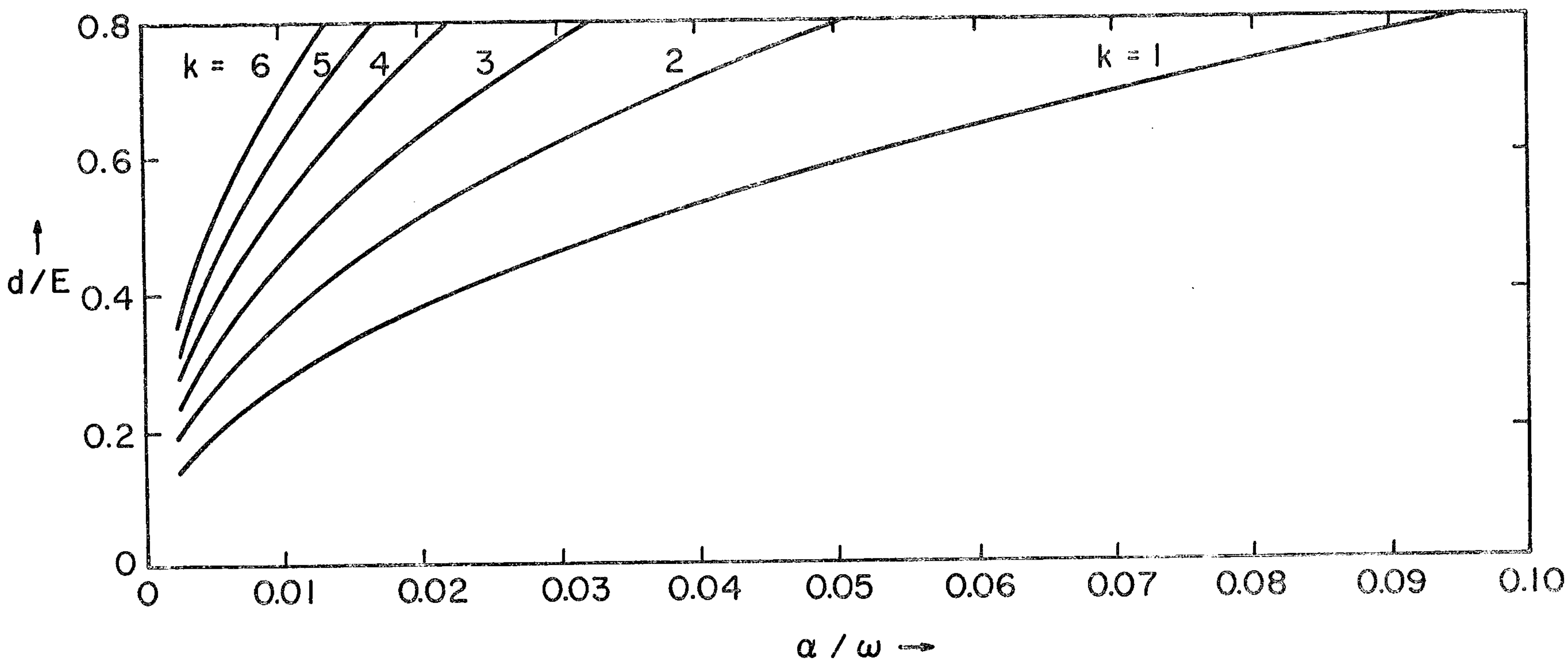

Fig. 2. Resonance conditions for stationary wave modes with different wave numbers k: Pole distance d (unit: earth's radius E) as function of the angular velocity α of the zonal current (as fraction of the frequency ω of the earth's rotation).

TABLE IV

Frequencies and Periods for Circumpolar Waves

k	$\sigma + \alpha k$ s^{-1}	σ s^{-1}	Period days
1	6.94×10^{-6}	4.69×10^{-6}	15.5
2	7.73×10^{-6}	3.23×10^{-6}	22.5
3	7.51×10^{-6}	0.76×10^{-6}	95.6
4	7.08×10^{-6}	-1.92×10^{-6}	-37.9
5	6.62×10^{-6}	-4.63×10^{-6}	-15.7
6	6.12×10^{-6}	-7.38×10^{-6}	-9.9

4. ECCENTRIC WAVE PATTERNS

In the Introduction, reference had already been made to the fact that the atmospheric pressure and flow patterns are frequently not centered at the earth's geographic pole, but some distance away from it. For the special case of circular flow, this eccentricity has been discussed by LaSeur (1954) and Barrett (1958). The observed stationarity of these patterns may be attributed to thermal forcing due to the distribution of water and land (Haurwitz and Bridger, 1977). If the eccentric flow pattern is not strictly circular, but consists of oscillatory streamlines, it can still, like an eccentric circular pattern, be expressed by the geographic longitude λ and the pole distance r.

Let (Fig. 3) C be the center of symmetry with r_o its distance from the geographic pole P, while A is an arbitrary point with pole distance r and longitude λ. Its distance from C is denoted by r', and the angle between r_o and r' is ε. G is the direction toward Greenwich so that λ_o is the longitude of C. It will be convenient in the following to use $\lambda-\lambda_o$ instead of λ as the angular coordinate, but for convenience we shall write λ instead of $\lambda-\lambda_o$.

The stream line distribution around the center of symmetry C may be given by

$$\psi\ (r',\varepsilon) = K\ J_n\ (cr')\ \cos n\varepsilon\ . \tag{1}$$

Here K is an arbitrary constant, n the wave number with respect to C, $c = j_{n1}/d$ where j_{n1} is the first zero of the Bessel function J_n so that the amplitude of ψ vanishes at r' = 0 and r' = d, but not at

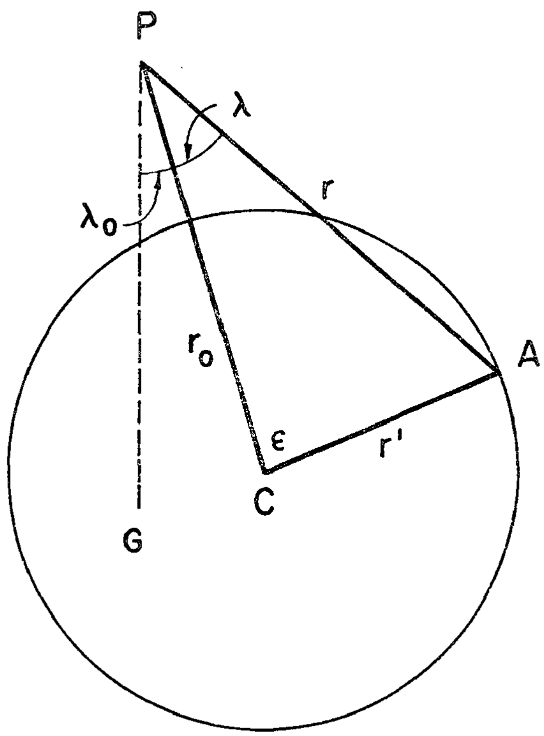

Fig. 3. Showing notations used for eccentric vortices.

intermediate distances. The distribution (1) can be expressed in terms of r and λ (the longitude being reckoned from the meridian through C) by means of the "addition" theorem of the Bessel functions (see for instance Jahnke-Emde, 1943) which may be written in the form

$$J_n(r'c) \cos n\varepsilon = J_n(r_o c)\, J_o(rc) \tag{2}$$

$$+ \sum_{k=1}^{\infty} \{J_k(rc)[J_{n+k}(r_o c) + (-1)^k J_{n-k}(r_o c)] \cos k\lambda\}.$$

Three examples of such eccentric wave patterns have been computed. Their characteristics are listed in Table V. The first two examples, both with wave number 3, are identical except for r_o, which is twice as large in case ii as in case i. The third case, iii, with wave number one, has the same r_o as case ii. The factors of the terms

TABLE V

Examples of Eccentric Wave Patterns: Characteristic Parameters

Example	Distribution around C	K	n	d	$b = j_{n1}/d$	r_o
i	100 $J_3(2 \times 10^{-3} r')$ cos 3ϵ	100	3	3190 km	2×10^{-3} km^{-1}	500 km
ii	100 $J_3(2 \times 10^{-3} r')$ cos 3ϵ	100	3	3190 km	2×10^{-3} km^{-1}	1000 km
iii	100 $J_1(1.59 \times 10^{-3} r')$ cos ϵ	100	1	2405 km	1.59×10^{-3} km^{-1}	1000 km

TABLE VI

Factors of $J_k(rc)$ cos $k\lambda$ in the Series (4,2)

Example	k = 0	1	2	3	4	5	6	7
i	+ 2.0	-11.2	+44.0	-76.5	-44.0	-11.5	- 2.0	-0.2
ii	+12.9	-31.9	+58.4	-22.3	-57.6	-35.5	-12.9	-3.4
iii	+56.9	-20.4	-49.8	-24.0	- 6.9	- 1.4	- 0.2	-

$J_k(rc) \cos k\lambda$ for the three examples at different pole distances are shown in Table VI. Only in example i, with the smallest r_o, does the term with k = n have the largest numerical factor. The streamline distribution in this example is plotted in Fig. 4. Although in this figure the three-wave pattern clearly predominates, and although the term with k = 3 has the largest factor in the series (2), a zonal harmonic analysis would not show maximum amplitudes for k = 3 at all pole distances. The longitudinal ψ-profiles for various pole distances are plotted in Fig. 5, illustrating what is shown by Table VII, namely the predominance of wave number 2 at smaller pole distances, shifting to 3 and finally to 4, as the pole distance increases.

In examples ii, with n = 3, and iii, with n = 1, the pole distance of the center of symmetry is assumed as 1000 km, twice as large as for i. As Table VI shows, the factor of the term $J_k(rc) \cos k\lambda$ is now no longer the largest term. It is seen from Table VII, ii and iii, that a zonal harmonic analysis would nowhere indicate a predominance of wave numbers 3 or 1, respectively, for such strongly eccentric wave patterns.

Equation (2) shows how a forcing function represented by a single mode centered around a point different from the pole may be expressed by a series of terms which are of the form (3.8). The flow and pressure distributions resulting from such a forcing function eccentric to the pole are thus given by series whose terms are of the general form (3.10) or (3.10a) for the pressure distribution. The special case of a forcing function with circular isotherms eccentric to the pole has been considered earlier (Haurwitz and Bridger, 1977). It indicates the procedure to be followed in the general case of eccentric wave patterns.

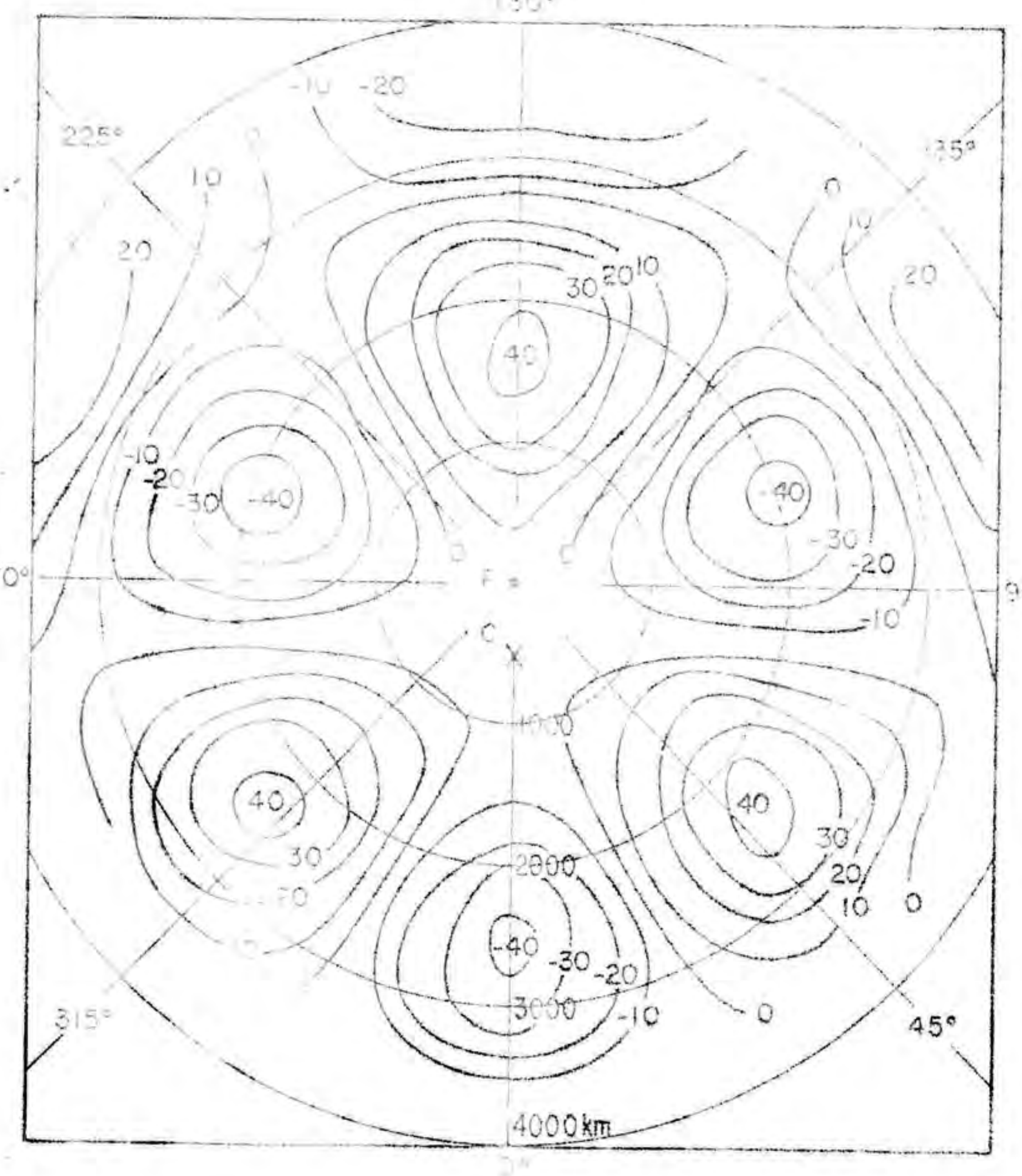

Fig. 4. Eccentric stream line pattern 100 J_3 (2×10^{-3} r') cos 3 ε; r in km, d = 3190 km, r_o = 500 km.

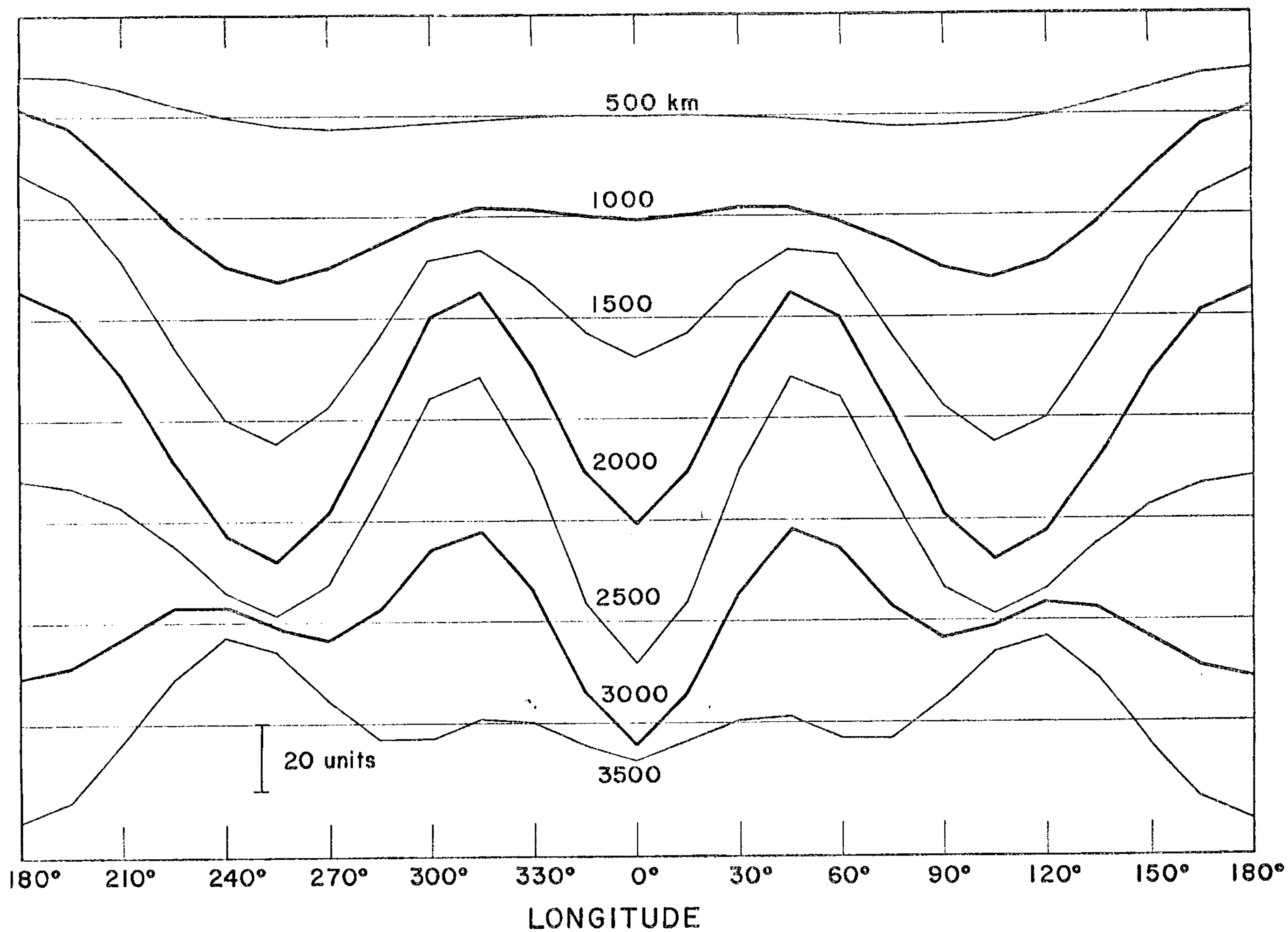

Fig. 5. Zonal streamline profile at different pole distances for the eccentric stream line pattern of Fig. 4.

TABLE VII

Factors of cos kλ for Eccentric Wave Patterns

r/k	0	1	2	3	4	5
			For example i of Table V			
0 km	1.96	0	0	0	0	0
500	1.50	- 4.95	5.06	- 1.5	- 0.11	0
1000	0.44	- 6.48	15.53	- 9.86	- 1.50	- 0.08
1500	- 0.51	- 3.81	21.40	-23.65	- 5.81	- 0.49
2000	- 0.78	0.74	16.03	-32.92	-12.37	- 1.52
2500	- 0.35	3.68	2.04	-27.91	-17.22	- 3.00
3000	0.30	3.11	-10.69	- 8.79	-15.73	- 4.16
3500	0.59	0.05	-13.27	12.82	- 6.94	- 4.00
			For example ii of Table V			
0 km	12.89	0	0	0	0	0
500	9.86	-14.03	6.71	- 0.44	- 0.14	0
1000	2.89	-18.38	20.59	- 2.87	- 1.96	- 0.25
1500	- 3.35	-10.81	28.37	- 6.88	- 7.61	- 1.52
2000	- 5.12	2.10	21.25	- 9.58	-16.20	- 4.66
2500	- 2.29	10.44	2.72	- 8.12	-22.55	- 9.21
3000	1.94	8.82	-14.18	- 2.56	-20.62	-12.77
3500	3.87	0.15	-17.59	3.73	- 9.10	-12.27
			For example iii of Table V			
0 km	56.9	0	0	0	0	
500	48.2	- 7.5	- 3.7	- 0.2	0	
1000	26.1	-11.6	-12.7	- 1.7	- 0.1	
1500	0.4	-10.6	-21.4	- 4.7	- 0.4	
2000	-18.0	- 5.4	-24.1	- 8.2	- 1.1	
2500	-22.7	1.2	-18.3	-10.3	- 1.9	
3000	-14.0	6.0	- 6.1	- 9.6	- 2.6	
3500	+ 1.1	6.9	- 7.0	- 5.7	- 2.7	

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15. Supplementary Notes

16. Abstracts

A simple two-dimensional model for the heating effects of the earth's surface on the distribution of the atmospheric flow and pressure patterns is considered. Attention is focused especially on the stationary flow patterns caused by stationary thermal forcing functions and located in the circumpolar regions. The governing relation is the linearized equation for the vertical vorticity component in which the solenoidal term is retained. The first part of this term is due to the effect of the advection of background temperature, assumed to have zonal symmetry, by the perturbation motion. The other term represents the forcing due to the temperature perturbation. In order to provide for damping of transient disturbances Rayleigh friction is included in the model.

For simplicity, conditions in a rectangular Cartesian Beta plane are considered first. Let the forcing function be in the form of a standing sine wave, and neglect friction. Then the resulting pressure wave is in phase (180° out of phase) with

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this forcing function if the wave length of the perturbation is larger (smaller) than the wave length of the stationary Rossby wave L_s, which is determined by the velocity of the basic zonal current, the meridional width of the perturbation, and the Beta parameter (see Table I). Correspondingly, the amplitudes of the pressure (and stream line) perturbations, produced by a forcing function of given amplitude, are largest for the waves closest to L_s and decrease for forcing functions with shorter and longer wave lengths. With Rayleigh friction, the phase differences between forcing function and resulting perturbation are no longer exactly 0° and 180°, respectively, but the temperature and pressure perturbations are still nearly in phase for waves longer than L_s, and nearly 180° out of phase for waves shorter than L_s. The amplitudes of the forced pressure perturbations are reduced by friction, and the reductions are largest for waves whose lengths are closest to L_s.

For the study of circumpolar waves a plane polar coordinate system centered at the pole and here tangential to the earth's surface is used. The results are similar to those for a rectangular Cartesian Beta plane, although there are quantitative as well as qualitative differences because of the singularity at the pole. In particular the meridional structure is now described not by trigonometric, but by Bessel functions. Nevertheless, for stationary waves the phase relations between forcing functions and forced pressure and flow patterns are the same as for the rectangular Cartesian case, namely in phase (180° out of phase) if the perturbation wave length is longer (shorter) than the stationary Rossby wave length for the polar-coordinate geometry (Table III). Further, the response of the forced pressure patterns to a forcing function of given amplitude is larger, the closer the wave length of the forcing function is to that of the stationary Rossby wave (Table III).

Examples are given to show how fast pressure and flow patterns adjust to thermal forcing (Fig. 1), and how wave patterns, whose centers of symmetry are eccentric to the pole, may be represented by series of wave modes of the type discussed in this paper (Figs. 4 and 5, Tables V-VII).