DISSERTATION

CONTINUUM LIMITS OF MARKOV CHAINS WITH APPLICATION TO WIRELESS NETWORK MODELING AND CONTROL

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ABSTRACT

CONTINUUM LIMITS OF MARKOV CHAINS WITH APPLICATION TO WIRELESS NETWORK MODELING AND CONTROL

We investigate the continuum limits of a class of Markov chains. The investigation of such limits is motivated by the desire to model networks with a very large number of nodes. We show that a sequence of such Markov chains indexed by N, the number of components in the system that they model, converges in a certain sense to its continuum limit, which is the solution of a partial differential equation (PDE), as N goes to infinity. We provide sufficient conditions for the convergence and characterize the rate of convergence. As an application we approximate Markov chains modeling large wireless networks by PDEs. We first describe PDE models for networks with uniformly located nodes, and then generalize to networks with nonuniformly located, and possibly mobile, nodes. While traditional Monte Carlo simulation for very large networks is practically infeasible, PDEs can be solved with reasonable computation overhead using well-established mathematical tools. Based on the PDE models, we develop a method to control the transmissions in nonuniform networks so that the continuum limit is invariant under perturbations in node locations. This enables the networks to maintain stable global characteristics in the presence of varying node locations.

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CHAPTER 1

INTRODUCTION

1.1 Background and Summary

In this work, we first investigate the convergence of a class of Markov chains to their continuum limits, which are the solutions of certain partial differential equations (PDEs). Second, as an application of the results of such analysis to network modeling, we use solutions of certain PDEs to approximate certain characteristics of large wireless networks modeled by such Markov chains. Finally, based on the PDE models, we develop a method to control the transmissions in the networks so that the continuum limit is invariant under perturbations in network node locations.

The conventional way to study large networks is by computer modeling and simulation [1]. The approach involves representing the network in computer software and then applying a numerical simulation method to study how the network behaves. Typically, each individual component is explicitly represented as a separate entity. As we are confronted with larger and larger networks, the number of its components that have to be represented increases, and this significantly lengthens the time it takes to write, manage, and run computer simulation programs. Simulating large networks typically requires expensive, highly sophisticated supercomputers involving large parallel computing hardware with specialized software. It is not uncommon for a simulation run to take days or weeks, even on a large supercomputer. The larger the network, the longer it takes. The computational overhead associated with direct simulation thus severely limits the size and complexity of networks that can be studied in this fashion.

Our recent papers [2, 3, 4] address this problem by using *continuum modeling* to capture the global characteristics of large networks. In large networks, we are often more interested

in the global characteristics of an entire network than in a particular individual component. Continuum models do away with the need to represent each individual component of a large network as a separate entity, and consider the behavior of the components on the scale of the aggregate rather than of the individual. Similar to treating water as a continuous fluid instead of a large number of individual molecules, continuum modeling treats the large number of communicating components (or nodes) in a network collectively as a continuum. The continuum modeling strategies in [2, 3, 4] use solutions of partial differential equations (PDEs) to approximate large sensor or cellular networks modeled by a certain class of Markov chains. The PDE model represents the global characteristics of the network, while the individual characteristics of the components enter the model through the form and the parameters of the PDE.

PDEs are well-suited to the modeling of continuum behavior. Although uncommon in modeling networks, they are common in modeling many physical phenomena, including heat, sound, electromagnetism, and fluid flow. There are well-established mathematical tools to solve PDEs, such as the finite element method [5] and the finite difference method [6], incorporated into computer software packages such as Matlab and Comsol. (Part of the Matlab code for simulations in this dissertation uses existing code written by Nate Burch in a related research [4].) We can use these tools to greatly reduce computation time, because instead of running many simulations for stochastic networks, we can solve for only *one* deterministic PDE solution. As a result, the effort to run the PDE models in a computer no longer suffers from the curse of sheer size. (In fact, as we will show, the larger the number of network nodes in the given spatial area, the closer the PDE approximates it.) Continuum modeling thus provides a powerful way to deal with the number of components in large networks. This, in turn, would make it possible to carry out—with reasonable computational burden even for extremely large systems—network performance evaluation and prototyping, network design, systematic parameter studies, and optimization of network characteristics.

The work in this dissertation is motivated by the continuum modeling strategies in the

papers [2, 3, 4] mentioned above, and by the need for a rigorous description of the heuristic limiting process underlying the construction of their PDE models. We analyze the convergence of a class of Markov chains to their continuum limits, which are the solutions of certain PDEs. We consider a general Markov chain model in an abstract setting instead of that of any particular network model. We do this for two reasons: first, our network modeling results involve a class of Markov chains modeling a variety of communication networks; second, similar Markov chain models akin to ours arise in several other contexts. For example, a very recent paper [7] on human crowd modeling derives a limiting PDE in a fashion similar to our approach.

In the convergence analysis, we show that a sequence of Markov chains indexed by N, the number of components in the system that they model, converges in a certain sense to its continuum limit, which is the solution of a time-dependent PDE, as N goes to ∞. The PDE solution describes the global spatio-temporal behavior of the model in the limit of large system size. We apply this abstract result to the modeling of a large wireless sensor network by approximating a particular global aspect of the network states (queue length) by a nonlinear convection-diffusion-reaction PDE. This network model includes the network example discussed in [2] as a special case.

1.2 Overview of the Result on Continuum Limits of Markov Chains

In this section we provide a brief description of the abstract result on continuum limits of Markov chains. The application to modeling and control of large networks builds on this result.

1.2.1 Markov Chain Model

Consider N points $V_N = \{v_N(1), \ldots, v_N(N)\}$ in a compact, convex Euclidean domain $\mathcal{D}$ representing a spatial region. We assume that these points form a *uniform* grid. (We later generalizes to nonuniform spacing of points in Chapter ch5.) We refer to these N points in

$\mathcal{D}$ as *grid* points.

We consider a discrete-time Markov chain

$$X_{N,M}(k) = [X_{N,M}(k,1), \ldots, X_{N,M}(k,N)]^\top \in \mathbb{R}^N$$

(the superscript $\top$ represents transpose) whose evolution is described by the stochastic difference equation

$$X_{N,M}(k+1) = X_{N,M}(k) + F_N(X_{N,M}(k)/M, U_N(k)). \tag{1.1}$$

Here, $X_{N,M}(k,n)$ is the real-valued state associated with the grid point $v_N(n)$ at time k, where $n = 1, \ldots, N$ is a *spatial* index and $k = 0, 1, \ldots$ is a *temporal* index; $U_N(k)$ are i.i.d. random vectors that do not depend on the state $X_{N,M}(k)$; M is an "averaging" parameter (explained later); and F_N is a given function.

Treating N and M as indices that grow, the equation (1.1) defines a doubly indexed family $X_{N,M}(k)$ of Markov chains indexed by both N and M. (We will later take M to be a function of N, and treat this family as a sequence $X_N(k)$ of the *single* index N.)

We will give a concrete example of a system described by (1.1) in Chapter 4.

1.2.2 Overview of the Convergence Result

The Markov chain model (1.1) is related to a deterministic difference equation. We set

$$f_N(x) = EF_N(x, U_N(k)), \quad x \in \mathbb{R}^N, \tag{1.2}$$

and define $x_{N,M}(k) = [x_{N,M}(k,1), \ldots, x_{N,M}(k,N)]^\top \in \mathbb{R}^N$ by

$$x_{N,M}(k+1) = x_{N,M}(k) + \frac{1}{M} f_N(x_{N,M}(k)), \quad x_{N,M}(0) = \frac{X_{N,M}(0)}{M} \text{ a.s.} \tag{1.3}$$

("a.s." is short for "almost surely").

We analyze the convergence of the Markov chain to the solution of a PDE using a two-step procedure. The first step depends heavily on the relation between $X_{N,M}(k)$ and $x_{N,M}(k)$. We show that for each N, as $M \to \infty$, the difference between $X_{N,M}(k)/M$ and $x_{N,M}(k)$ vanishes,

by proving that they both converge in a certain sense to the solution of the same ordinary differential equation (ODE). The basic idea of this convergence is that as the "fluctuation size" of the system decreases and the "fluctuation rate" of the system increases, the stochastic system converges to a deterministic "small-fast-fluctuation" limit, which can be characterized as the solution of a particular ODE. In our case, the smallness of the fluctuation size and largeness of the fluctuation rate is quantified by the "averaging" parameter M. We use a weak convergence theorem of Kushner [8] to prove this convergence.

In the second step, we treat M as a function of N, written M_N (therefore treating $X_{N,M_N}(k)$ and $x_{N,M_N}(k)$ as sequences of the *single* index N, written $X_N(k)$ and $x_N(k)$, respectively), and show that for *any sequence* $\{M_N\}$ of N, as $N \to \infty$, $x_N(k)$ converges to the solution of a certain PDE (and we show how to construct the PDE). This is essentially a convergence analysis on the approximating error between $x_N(k)$ and the PDE solution. We stress that this is different from the numerical analysis on classical finite difference schemes (see, e.g., [6, 9, 10]), because our difference equation (1.3), which originates from particular system models, differs from those designed specifically for the purpose of numerically *solving* differential equations. The difficulty in our convergence analysis arises from both the different form of (1.3) and the fact that it is in general nonlinear. We provide not only sufficient conditions for the convergence, but also a practical criterion for verifying such conditions otherwise difficult to check.

Finally, based on these two steps, we show that as N and M_N go to ∞ in a *dependent* way, the continuous-time-space extension (explained later) of the normalized Markov chain $X_N(k)/M_N$ converges to the PDE solution. We also characterize the rate of convergence. We note that special caution is needed for specifying the details of this dependence between the two indices N and M of the doubly indexed family $X_{N,M}(k)$ of Markov chains in the limiting process.

1.3 Related Literature

The modeling and analysis of stochastic systems such as networks is a large field of research, and much of the previous contributions share goals with this work.

In the field of direct numerical simulation approaches, many efforts have been made to accelerate the simulation. For example, parallel simulation techniques have been developed to exploit the computation power of multiprocessor and/or cluster platforms [11, 12, 13, 14]; new mechanisms for executing the simulation have been designed to improve the efficiency of event scheduling in event-driven simulations (see, e.g., [15, 16]); and fluid simulations, in contrast to traditional packet-level ones, have been used to simplify the network model by treating network traffic (*not* nodes) as continuous flows rather than discrete packets [17, 18, 19, 20]. However, as the number of nodes in the network grows extremely large, computer-based simulations involving individual nodes eventually become practically infeasible. For the remainder of this section, we review some existing results on analysis of stochastic networks that do *not* depend on direct numerical simulation.

Kushner's ODE method, which forms the basis for the first step of our analysis, is closely related to the line of research called stochastic approximation. This line of research, started by Robbins and Monro [21] and Kiefer and Wolfowitz [22] in the early 1950s, studies stochastic processes similar to those addressed by Kushner's ODE method, and has been widely used in many areas (see, e.g., [23, 24], for surveys). These convergence results differ from our results in the sense that they essentially study only the single-step "small-fast-fluctuation" limit as the "averaging" parameter (in our case, M) goes to ∞, but do not have the second-step convergence to the "large-system" PDE limit (as $N \to \infty$). In other words, while Kushner's method and related work deal with a fixed state space with fixed N, we treat a sequence of state spaces $\{\mathbb{R}^N\}$ indexed by increasing N. There are systems in which the "averaging" parameter represents some "size" of the system (e.g., population in epidemic models [25, 26]). However, it is still the case that the convergence requires a fixed dimension of the state space of the Markov chain, like the case of Kushner's ODE convergence, and

does not apply to the "large-system" limit in our second step.

Markov chains modeling various systems have been shown to converge to differential equations [27, 28], abstract Cauchy problems [26], or other stochastic processes [8, 29]. These results use methods different from Kushner's, but share with it the principle idea of "averaging out" of the randomness of the Markov chain. Their deeper connection lies in weak convergence theory [8, 29, 30] and methods to prove such convergence that they have in common: the operator semigroup convergence theorem, the martingale characterization method, and identification of the limit as the solution to a stochastic differential equation. The reader is referred to [8, 29] and references therein for additional information on these methods.

There are a variety of other analysis methods for large network systems taking completely different approaches. For example, the well-cited work of Gupta and Kumar [31], followed by many others (e.g., [32, 33]), derives scaling laws of network performance parameters (e.g., throughput); and many efforts based on mean field theory [34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46] or on the theory of large deviations [47, 48, 49] study the limit of the so-called empirical (or occupancy) measure or distribution, which essentially represents the proportion of components in certain states. These approaches differ from our work because they do not study the spatio-temporal characteristics of the system. Note that we can directly compute many such limiting deterministic characteristics of the network once we have computed the solution of the limiting PDE.

Of course, there do exist numerous continuum models in a wide spectrum of areas that formulate spatio-temporal phenomena (e.g., [50, 51, 52, 7]), many of which use PDEs. All these works differ from the work presented here both by the properties of the system being studied and the analytic approaches. In addition, most of them study distributions of limiting processes that are random, while our limiting functions themselves are deterministic. We especially emphasize the difference between our results and those of the mathematical physics of hydrodynamics [53, 54, 55], because the latter have a similar style by deducing macroscopic

behavior from microscopic interactions of individual particles, and in some special cases result in similar PDEs. However, they use an entirely different approach, which usually requires different assumptions on the systems such as translation invariant transition probabilities, conservation of the number of particles, and particular distributions of the initial state; and their limiting PDE is not the direct approximation of system state, but the density of some associated probability measure.

There is a vast literature on the convergence of a large variety of network models different from ours, to essentially two kinds of limits: the fluid limit [56, 57, 58, 59, 60, 61, 62] and the diffusion limit [63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74], with the latter limit mostly studied in networks in heavy traffic. (Some papers study both limits [75, 76, 77].) Unlike our work, this field of research focuses primarily on networks with a fixed number of nodes.

Our work is to be distinguished from approaches where the model is constructed to be a continuum representation from the start. For example, many papers treat nodes as a continuum by considering only the average density of nodes [78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91]; and others model network traffic as a continuum by capturing certain average characteristics of the data packet traffic [92, 93, 94].

1.4 Organization of This Dissertation

The organization of this dissertation is as follows: In Chapter 2, as a motivational example, we approximate the Markov chains modeling multiple i.i.d. random walks by the solutions to certain PDEs. This helps illustrate the specific goal of our modeling method and some basic ideas used in our approach to showing the convergence (e.g., the two-step procedure). In Chapter 3, we investigate the convergence of the Markov chains described by (1.1) to their the continuum limits, which are the solutions to the limiting PDEs. In Chapter 4, we introduce a stochastic networks model, and apply the abstract result from Chapter 3 to derive its continuum limit. In Chapter 5, we generalize to nonuniform networks with more general transmissions, and study their continuum limits. In Chapter 6, we

consider the control of transmission in nonuniform networks so that the continuum limit is invariant under perturbations in node locations.. In Chapter 8, we summarize the major contributions of this dissertation and discuses future work.

CHAPTER 2

CONTINUUM MODELS OF MULTIPLE RANDOM WALKS

In this chapter we present the example of approximating multiple i.i.d. random walks by a PDE. Random walks have been studied with various approaches in numerous contexts (see, e.g., [95, 96, 97, 98, 99]), and their approximation by limits of many sorts are not new results. To illustrate the specific goal of our method and some basic ideas used in our approach to showing the convergence (e.g., the two-step procedure), here we present the example in a setting similar (but not identical) to (1.1), show the convergence in a way similar to that of later chapters, and, in contrast to many previous studies, directly approximate the state of the underlying Markov chain by a PDE (though we do not mean for this chapter to present much novel material). This is only a motivational example, not a special case of the systems defined by (1.1). The procedure to show the convergence here will no longer be valid for systems treated in later chapters.

2.1 Model Setup

First consider the random walk of a single particle on a one-dimensional network consisting of N points uniformly placed over a spatial domain $\mathcal{D}$, as shown in Fig. 2.1. At each time instant, the particle randomly chooses to move to its left or right immediately neighboring point with probability $P_l(n)$ and $P_r(n)$, respectively, if it is at point n, where $n = 1, \ldots, N$. We assume a "sink" boundary condition, i.e., the particle vanishes when it reaches the ends of $\mathcal{D}$ (though "walls" at the boundary are equally treatable).

Now consider the random walks of M particles on the same network, where each particle behaves independently identically as the one in the single random walk described

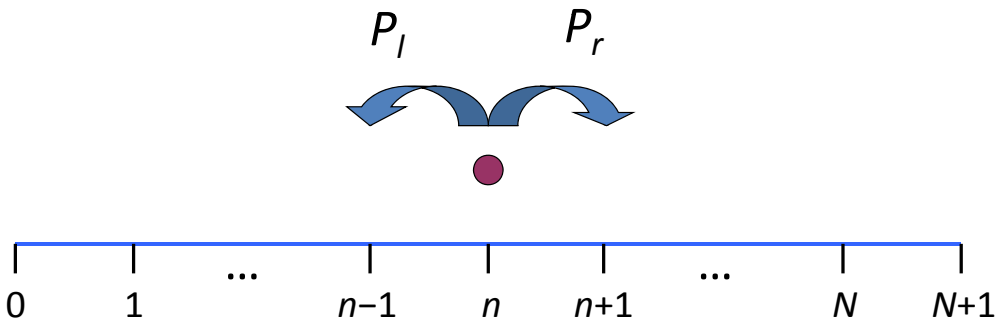


Figure 2.1: An illustration of a one-dimensional single random walk.

above. Let $X_N(k,n)$ be the number of particles at point n at time k. Define $X_N(k) = [X_N(k,1),\ldots,X_N(k,N)]^T$, which forms a discrete-time Markov chain with state space $\mathbb{R}^N$. Suppose that the evolution of $X_N(k)$ is described by

$$X_N(k+1) = X_N(k) + F_N(X_N(k), U(k)), \tag{2.1}$$

where $U(k)$ are i.i.d. random variables and do not depend on the state $X_N(k)$, and F_N is a given function. Notice the difference between this and (1.1).

Define

$$f_N(x) = EF_N(x, U(k)), \quad x \in \mathbb{R}^N, \tag{2.2}$$

where E represents the expectation operator.

To each pair of p_r and p_l in $[0,1]$ satisfying $p_r + p_l \leq 1$, there corresponds a trial with three possible outcomes with probabilities p_r, p_l, and $1-(p_r+p_l)$, respectively. For each pair of such p_r and p_l and each $x = 0, 1, \ldots,$ let $Q_r(x, p_r, p_l, U)$ and $Q_l(x, p_r, p_l, U)$ be the random variables indicating the numbers of the outcomes with probabilities p_r and p_l, respectively, in x independent trials described above, where U is a random variable of a certain distribution (e.g., uniform$[0,1]$) and does not depend on x. Hence the vector

$$[Q_r(x,p_r,p_l,U), Q_l(x,p_r,p_l,U), x - Q_r(x,p_r,p_l,U) - Q_l(x,p_r,p_l,U)]^T$$

of random variables follows a trinomial distribution with parameters x (number of trials) and $p = [p_r, p_l, 1-(p_r+p_l)]^T$ (event probabilities).

According to the behavior of the particles in the random walks, there exist random variables $U(k,n)$, where $k = 0, 1, \ldots$ and $n = 1, \ldots, N$, identically distributed as U above, mutually independent for each k and also independent over k, such that for $k = 0, 1, \ldots$ and $n = 1, \ldots, N$,

$$\begin{aligned}
&X_N(k+1,n) - X_N(k,n) \\
&= -Q_r(X_N(k,n), P_r(n), P_l(n), U(k,n)) - Q_l(X_N(k,n), P_r(n), P_l(n), U(k,n)) \\
&\quad + Q_r(X_N(k,n-1), P_r(n-1), P_l(n-1), U(k,n-1)) \\
&\quad + Q_l(X_N(k,n+1), P_r(n+1), P_l(n+1), U(k,n+1)),
\end{aligned} \tag{2.3}$$

where $X_N(k,n)$ with $n \leq 0$ or $n \geq N+1$ are defined to be zero. Here, the four terms on the right-hand side represent the number of particles moving to the right from, moving to the left from, coming from the left to, and coming from the right to point n at time k, respectively. Then we can write (2.1) with $U(k) = [U(k,1), \ldots, U(k,N)]^T$. Hence by (2.3), for $x = [x_1, \ldots, x_N]^T$, the nth component of $F_N(x, U(k))$, where $n = 1, \ldots, N$, is

$$\begin{aligned}
&- Q_r(x_n, P_r(n), P_l(n), U(k,n)) - Q_l(x_n, P_r(n), P_l(n), U(k,n)) \\
&+ Q_r(x_{n-1}, P_r(n-1), P_l(n-1), U(k,n-1)) \\
&+ Q_l(x_{n+1}, P_r(n-1), P_l(n+1), U(k,n+1)),
\end{aligned} \tag{2.4}$$

where x_n with $n \leq 0$ or $n \geq N+1$ are defined to be zero.

By a property of the trinomial distribution,

$$EQ_r(x, p_r, p_l, U) = xp_r, \text{ and } EQ_r(x, p_r, p_l, U) = xp_l.$$

Therefore, by (2.4) we have that, for $x = [x_1, \ldots, x_N]^T$, the nth component of $f_N(x)$, where $n = 1, \ldots, N$, is

$$-(P_r(n) + P_l(n))x_n + P_r(n-1)x_{n-1} + P_l(n+1)x_{n+1}, \tag{2.5}$$

where x_n with $n \leq 0$ or $n \geq N+1$ are defined to be zero. Hence f_N is linear.

Define a sequence $x_N(k) \in \mathbb{R}^N$ by the deterministic difference equation

$$x_N(k+1) = x_N(k) + f_N(x_N(k)), \tag{2.6}$$

where a.s.,

$$x_N(0) = \frac{X_N(0)}{M}. \tag{2.7}$$

2.2 Sketch of the Limiting PDE

Fix $T > 0$. We seek to approximate the Markov chain $X_N(k)$ by a limiting PDE on $[0, T] \times \mathcal{D}$, where the time and space indices k and n are made continuous as $M \to \infty$ and $N \to \infty$. We first show that $X_N(k)/M$ and $x_N(k)$ are close for large M, and then show that $x_N(k)$ is close to the PDE for large N. Hence $X_N(k)/M$ is close to the PDE for large M and N.

2.2.1 Convergence of $X_N(k)$ to $x_N(k)$

Now we show that for fixed N, as $M \to \infty$, $X_N(k)$ and $x_N(k)$ are asymptotically close. Let $B_i(k) = [B_i(k, 1), \ldots, B_i(k, N)]^T \in \mathbb{R}^N$ represent the ith random walk of a single particle among the M random walks, where $i = 1, \ldots, M$ and $B_i(k, n)$ is the Bernoulli random variable representing the presence of the particle at point n at time k in the ith random walk. Since the random walks are i.i.d., we can write $EB_i(k) = EB(k)$ for each i and k. We have by definition that for each k,

$$X_N(k) = \sum_{i=1}^{M} B_i(k). \tag{2.8}$$

Then it follows that for each k,

$$EX_N(k) = \sum_{i=1}^{M} EB_i(k) = MEB(k). \tag{2.9}$$

By (2.8) and the strong law of large numbers (SLLN), we have that for each k, a.s.,

$$\lim_{M \to \infty} \frac{X_N(k)}{M} = EB(k). \tag{2.10}$$

Hence by (2.9) and (2.10) we have that for each k, a.s.,

$$\lim_{M\to\infty} \|X_N(k) - EX_N(k)\|_\infty^{(N)} = 0, \tag{2.11}$$

where $\|\cdot\|_\infty^{(N)}$ is the ∞-norm on $\mathbb{R}^N$.

It follows from (2.1) that for each k,

$$EX_N(k+1) = EX_N(k) + EF_N(X_N(k), U(k)). \tag{2.12}$$

For each k, since $U(k)$ is independent of $X_N(k)$, by the law of total expectation and (2.2), we have that

$$\begin{aligned} EF_N(X_N(k), U(k)) &= E(E(F_N(X_N(k), U(k))|X_N(k))) = Ef_N(X_N(k)) \\ &= f_N(EX_N(k)), \end{aligned} \tag{2.13}$$

where the last equality follows from the linearity of f_N. Then it follows from (2.12) that for each k,

$$EX_N(k+1) = EX_N(k) + f_N(EX_N(k)). \tag{2.14}$$

By (2.6) and (2.14), $x_N(k)$ and $EX_N(k)$ are governed by the same difference equation; and by (2.7) and (2.11), we have that a.s.,

$$\lim_{M\to\infty} \left\| \frac{EX_N(0)}{M} - x_N(0) \right\|_\infty^{(N)} = 0.$$

Therefore for each k, a.s.,

$$\lim_{M\to\infty} \left\| \frac{EX_N(k)}{M} - x_N(k) \right\|_\infty^{(N)} = 0. \tag{2.15}$$

Then by (2.11), (2.15), and the triangle inequality, we have that for each k, a.s.,

$$\lim_{M\to\infty} \left\| \frac{X_N(k)}{M} - x_N(k) \right\|_\infty^{(N)} = 0. \tag{2.16}$$

This implies that, for each k and n, a.s.,

$$\lim_{M\to\infty} \left| \frac{X_N(k,n)}{M} - x_N(k,n) \right| = 0. \tag{2.17}$$

2.2.2 Convergence of $X_N(k)$ to the PDE

For fixed N, define

$$X_{oN}(t) = X_N(\lfloor Mt \rfloor)/M, \quad t \in [0, T],$$

the continuous-time extension of $X_N(k)$ by piecewise-constant time extensions with interval length $1/M$ and normalized by M.

Define $X_{pN}(t, s)$ to be the continuous-space extension of $X_{oN}(t)$ by piecewise-constant space extensions on $\mathcal{D}$ with interval length ds_N. Thus X_{pN} is the continuous-time-space extension of $X_N(k)$.

Similarly, define $x_{oN}(t) = x_N(\lfloor Mt \rfloor)$, the piecewise-constant continuous-time extension of $x_N(k)$, and $x_{pN}(t, s)$, the piecewise-constant continuous-space extension of $x_{oN}(t)$. Thus x_{pN} is the continuous-time-space extension of $x_N(k)$. Hence we have defined rigorously here the function x_{pN} in Section 2.2. By definition it satisfies (2.21). More details on the construction of these extensions can be found in Section 3.2.

By definition, it follows from (2.17) that X_{oN} and x_{oN} are close for large M in the sense that for each t, a.s.,

$$\lim_{M \to \infty} \|X_{oN}(t) - x_{oN}(t)\|_\infty^{(N)} = 0. \tag{2.18}$$

Note that by definition, for each t,

$$\|X_{pN}(t, \cdot) - x_{pN}(t, \cdot)\|_\infty^{(\mathcal{D})} = \|X_{oN}(t) - x_{oN}(t)\|_\infty^{(N)},$$

where $\|\cdot\|_\infty^{(\mathcal{D})}$ is the ∞-norm on $\mathbb{R}^{\mathcal{D}}$, the space of functions from $\mathcal{D}$ to $\mathbb{R}$. Therefore by (2.18), X_{pN} and x_{pN} are close for large M in the sense that for each t, a.s.,

$$\lim_{M \to \infty} \|X_{pN}(t, \cdot) - x_{pN}(t, \cdot)\|_\infty^{(\mathcal{D})} = 0. \tag{2.19}$$

For $x_N(k) = [x_N(k, 1), \ldots, x_N(k, N)]^T$, by (2.5) we have that for $n = 1, \ldots, N$,

$$\begin{aligned} &x_N(k+1, n) - x_N(k, n) = \\ &\quad -(P_r(n) + P_l(n))x_N(k, n) + P_r(n-1)x_N(k, n-1) + P_l(n+1)x_N(k, n+1), \end{aligned} \tag{2.20}$$

where $x_N(k,n)$ with $n \le 0$ or $n \ge N+1$ are defined to be zero.

Denote the distance between two neighboring points by $ds_N \propto 1/N$. Assume that $P_l(n) = p_l(n\,ds_N)$ and $P_r(n) = p_r(n\,ds_N)$, where $p_l(s)$ and $p_r(s)$ are real-valued functions defined on $\mathcal{D}$. To ensure a finite non-degenerate limit, we assume that for $s \in \mathcal{D}$,

$$p_l(s) = b(s) + c_l(s)ds_N \text{ and } p_r(s) = b(s) + c_r(s)ds_N.$$

Define $c = c_l - c_r$. We call b the diffusion and c the convection, because a greater b means more rapid diffusion and a greater c means a larger directional bias.

Denote by dt the length of time between two consecutive time instants, and set $dt = 1/M$. Without loss of generality, set $\mathcal{D} = [0,1]$. For each N, let the function $x_{pN}(t,s)$ on $[0,T] \times \mathcal{D}$ be such that for each k and n,

$$x_{pN}(k\ dt, n\ ds_N) = x_N(k,n). \tag{2.21}$$

Then by the definition of x_{pN} and by (2.20), we have that for $(t,s) = (k\ dt, n\ ds_N)$ for each k and n,

$$\begin{aligned} x_{pN}(t+dt,s) - x_{pN}(t,s) &= -(p_r(s) + p_l(s))x_{pN}(t,s) \\ &\quad + p_r(s-ds_N)x_{pN}(t,s-ds_N) + p_l(s+ds_N)x_{pN}(t,s+ds_N). \end{aligned} \tag{2.22}$$

Assume that $b \in \mathcal{C}^2$, $c \in \mathcal{C}^1$, and x_{pN} is twice continuously differentiable in s.

Put into the right-hand side of (2.22) the Taylor expansions

$$x_{pN}(t, s \pm ds_N) = x_{pN}(t,s) \pm \frac{\partial x_{pN}}{\partial s}(t,s)ds_N + \frac{\partial^2 x_{pN}}{\partial s^2}(t,s)\frac{ds_N^2}{2} + o(ds_N^2), \tag{2.23}$$

$$b(s \pm ds) = b(s) \pm b_s(s)ds_N + b_{ss}(s)\frac{ds_N^2}{2} + o(ds_N^2), \tag{2.24}$$

and

$$c(s \pm ds_N) = c(s) \pm c_s(s)ds_N + o(ds_N). \tag{2.25}$$

Then we have that

$$x_{pN}(t+dt,s) - x_{pN}(t,s) = b(s)\frac{\partial^2 x_{pN}}{\partial s^2}(t,s)ds_N^2 + (2b_s(s)+c(s))\frac{\partial x_{pN}}{\partial s}(t,s)ds_N^2 + (b_{ss}(s)+c_s(s))x_{pN}(t,s)ds_N^2 + o(ds_N^2), \tag{2.26}$$

where a single subscript s represents first derivative and a double subscript ss represents second derivative.

Now we set the time step $dt := 1/M$ to be ds_N^2, which is a standard time-space scaling approach to ensuring the convergence of the difference equation to a PDE. Since ds_N depends on N, so do dt and M. We hence rewrite them as dt_N and M_N, respectively.

Divide both sides of (2.26) by $dt_N = ds_N^2$ and get

$$\frac{x_{pN}(t+dt_N,s) - x_{pN}(t,s)}{dt_N} = b(s)\frac{\partial^2 x_{pN}}{\partial s^2}(t,s) + (2b_s(s)+c(s))\frac{\partial x_{pN}}{\partial s}(t,s) + (b_{ss}(s)+c_s(s))x_{pN}(t,s) + \frac{o(ds_N^2)}{ds_N^2}.$$

As $N \to \infty$, $ds_N \to 0$, and hence $dt_N = ds_N^2 \to 0$. Assume that x_{pN} is continuously differentiable in t. Then by taking the limit as $N \to \infty$ and rearranging, we get a limiting PDE that x_{pN} satisfies:

$$\dot{x}_{pN}(t,s) = \frac{\partial}{\partial s}\left(b(s)\frac{\partial x_{pN}}{\partial s}(t,s) + (b_s(s)+c(s))x_{pN}(t,s)\right), \tag{2.27}$$

for $(t,s) \in [0,T] \times \mathcal{D}$, with boundary condition $x_{pN}(t,s) = 0$.

Since $ds_N \propto 1/N$, we have that $M_N := 1/dt_N = 1/ds_N^2 \propto N^2$. As $N \to \infty$, $M_N \to \infty$ with the rate $O(N^2)$. Thus N and M_N go to ∞ in a dependent way. Hence for large N, M_N is also large, and by (2.19), for each t, X_{pN}, the continuous-time-space extension of $X_N(k)$, is close to x_{pN}, the continuous-time-space extension of $x_N(k)$. Thus we can approximate $X_N(k)$ by the solution of the PDE (2.27). This PDE is called the one-dimensional diffusion-convection equation, which can be easily solved [97].

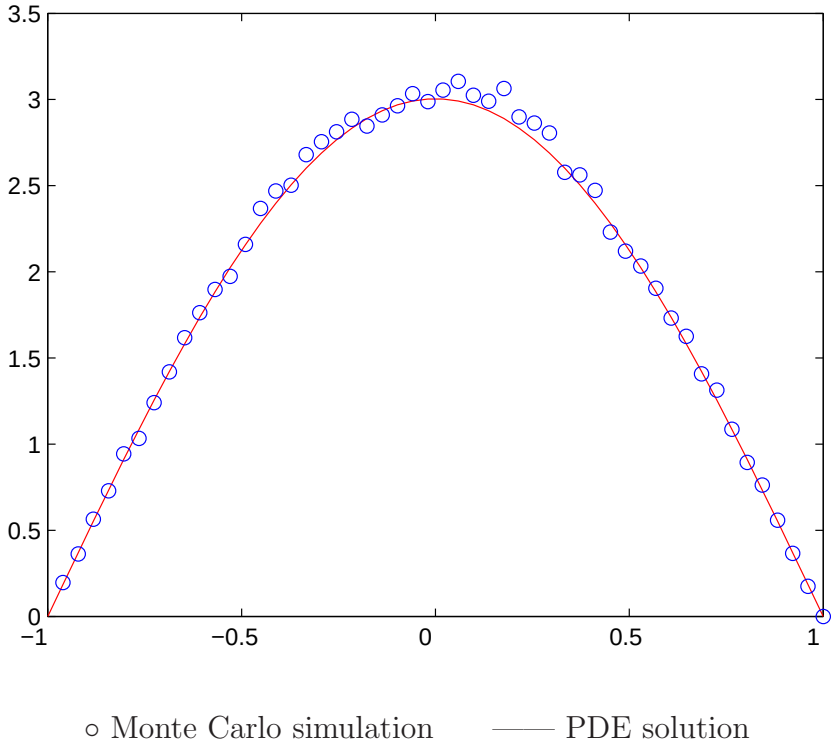


Figure 2.2: The Monte Carlo simulations and the PDE solution of $M = 1000$ one-dimensional random walks on $N = 50$ points, with $b = 1/2$ and $c = 0$, at $t = 1s$.

2.3 Comparisons between the PDE Solution and Monte Carlo Simulations of the Random Walks

We compare the limiting PDE solution with Monte Carlo simulations for $M = 1000$ i.i.d. random walks on $N = 50$ points in the domain $\mathcal{D} = [-1, 1]$. We show the PDE solution and the Monte Carlo simulation results at $t = 1s$. The random walks have diffusion $b = 1/2$ and convection $c = 0$ in Fig. 2.2 and $c = 1/2$ in Fig. 2.3, respectively, where the x-axis denotes the point locations and y-axis denotes the normalized number of particles. There is strong resemblance between the Monte Carlo simulations and the PDE solution. Here the PDEs only took fractions of a second to solve on a computer, while the Monte Carlo simulations took on the order of tens of minutes.

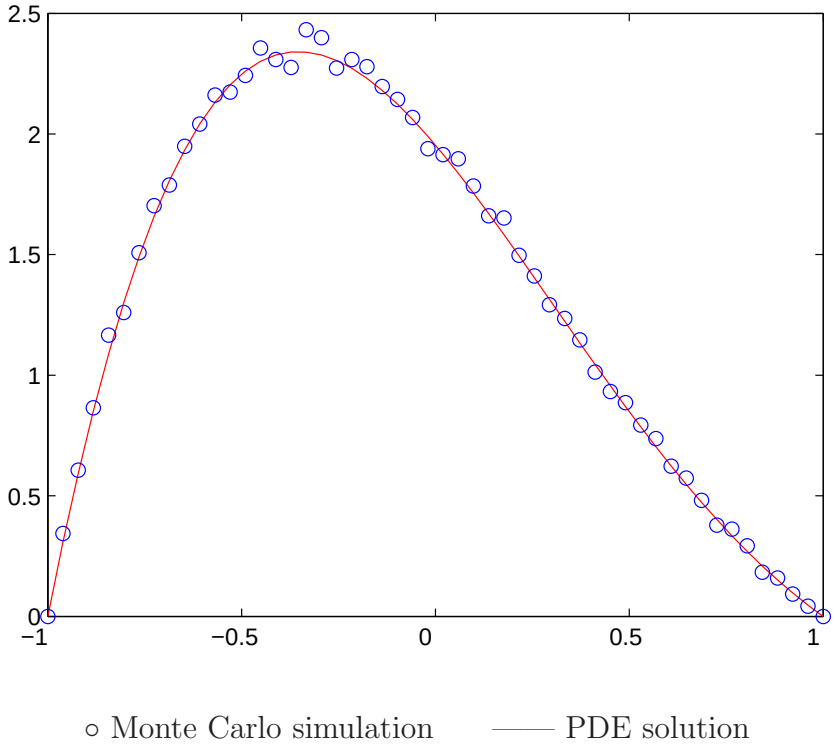


Figure 2.3: The Monte Carlo simulations and the PDE solution of $M = 1000$ one-dimensional random walks on $N = 50$ points, with $b = 1/2$ and $c = 1/2$, at $t = 1s$.

2.4 Discussion

We note again that the result in this chapter is not new, and similar ones may be obtained with different approaches. For example, the well-studied Fokker-Planck equation (also known as the Kolmogorov forward equation) [100] characterizes by a PDE solution the distribution of the limiting random process of a random walk model. On the other hand, we characterize the deterministic limit directly approximating the state of the Markov chain, in a way more consistent with that of our approach in the following chapters.

For the class of Markov chains defined by (1.1), which we study in the following chapter as the main object, this motivational example helps to illustrate the goal and some basic ideas of the approach, but not the detailed means. Specifically, the analysis above does not apply to Markov chains underlying the network introduced later in Chapter 4 of this

dissertation. These Markov chains model systems with more complex behaviors such as interactions between the components, e.g., the interference between the nodes in the network. Consequently the Markov chains are in general no longer the simple superposition of i.i.d. random processes that allows us to use the SLLN; and these systems lack in general the linearity of f_N. Hence the procedure used here to show the convergence of the Markov chain to a deterministic function (as in (2.17)) no longer works for the class of Markov chains in (1.1). Instead, in Section 3.5.1 we use Kushner's ODE convergence theorem in [8] to analyze the convergence similar to (2.17), and provide sufficient conditions for the convergence.

The derivation to show the convergence of x_{pN} to the PDE solution is somewhat heuristic. A rigorous numerical analysis for this involves showing that the truncation error between $x_N(k)$ in the difference equation (2.6) and the PDE solution goes to 0 as $N \to \infty$. For at least some special cases of the random walks example, a rigorous numerical analysis can be found in many textbooks (see e.g., [6, 9]), as the difference equation in those cases coincides with a classical finite difference scheme for solving the PDE. On the other hand, for the difference equation (1.3) of $x_N(k)$ arising from the systems that we treat later, such analysis is no longer readily available, and difficulty rises from both the different form and the nonlinearity of (1.3). We analyze such convergence and provide sufficient conditions for it in Section 3.5.2.

Moreover, the convergence shown in this chapter is uniform with respect to s and pointwise with respect to t. In contrast, in the following chapter we show the convergence of the Markov chain to the PDE solution in a different sense. In addition, based on our analysis, we characterize the rate of convergence.

We stress again that the Markov chain presented in this chapter is not a special case of the one studied in the following chapters, and can not be treated with the same procedure. (For example, F_N in (2.1) above does not satisfy the first assumption of Theorem 1, which we will later assume that F_N in (1.1) satisfies.)

CHAPTER 3

CONTINUUM LIMITS OF MARKOV CHAINS

3.1 Construction of the Limiting PDE

We begin with the construction of the PDE whose solution describes the limiting behavior of the abstract Markov chain model.

For each N and the grid points $V_N = \{v_N(1), \ldots, v_N(N)\} \subset \mathcal{D}$ as introduced in Section 1.2.1, we denote the distance between any two neighboring grid points by ds_N. For any continuous function $w : \mathcal{D} \to \mathbb{R}$, let y_N be the vector in $\mathbb{R}^N$ composed of the values of w at the grid points $v_N(n)$; i.e., $y_N = [w(v_N(1)), \ldots, w(v_N(N))]^\top$. Given a point $s \in \mathcal{D}$, we let $\{s_N\} \subset \mathcal{D}$ be any sequence of *grid points* $s_N \in V_N$ such that as $N \to \infty$, $s_N \to s$. Let $f_N(y_N, s_N)$ be the component of the vector $f_N(y_N)$ corresponding to the location s_N; i.e., if $s_N = v_N(n) \in V_N$, then $f_N(y_N, s_N)$ is the nth component of $f_N(y_N)$.

In order to obtain a limiting PDE, we have to make certain technical assumptions on the asymptotic behavior of the sequence of functions $\{f_N\}$ that insure that $f_N(y_N, s_N)$ is asymptotically close to an expression that looks like the right-hand side of a time-dependent PDE. Such conditions are familiar in the context of PDE limits of Brownian motion. Checking these conditions often amounts to a simple algebraic exercise. We provide a concrete example (the network model) in Chapter 4 where f_N satisfies these assumptions.

We assume that there exist sequences $\{\delta_N\}$, $\{\beta_N\}$, $\{\gamma_N\}$, and $\{\rho_N\}$, functions f and h, and a constant $c < \infty$, such that as $N \to \infty, \delta_N \to 0, \delta_N/\beta_N \to 0, \gamma_N \to 0, \rho_N \to 0$, and:

- Given s in the interior of $\mathcal{D}$, there exists a sequence of functions $\{\phi_N\} : \mathcal{D} \to \mathbb{R}$ such that

$$f_N(y_N, s_N)/\delta_N = f(s_N, w(s_N), \nabla w(s_N), \nabla^2 w(s_N)) + \phi_N(s_N), \tag{3.1}$$

for any sequence of grid points $s_N \to s$, and for N sufficiently large, $|\phi_N(s_N)| \leq c\gamma_N$; and

- Given s on the boundary of $\mathcal{D}$, there exists a sequence of functions $\{\varphi_N\} : \mathcal{D} \to \mathbb{R}$ such that

$$f_N(y_N, s_N)/\beta_N = h(s_N, w(s_N), \nabla w(s_N), \nabla^2 w(s_N)) + \varphi_N(s_N), \tag{3.2}$$

for any sequence of grid points $s_N \to s$, and for N sufficiently large, $|\varphi_N(s_N)| \leq c\rho_N$.

Here, $\nabla^i w$ represents all the ith order derivatives of w, where $i = 1, 2$.

Fix $T > 0$ for the rest of this chapter. Assume that there exists a unique function $z : [0, T] \times \mathcal{D} \to \mathbb{R}$ that solves the limiting PDE

$$\dot{z}(t, s) = f(s, z(t, s), \nabla z(t, s), \nabla^2 z(t, s)), \tag{3.3}$$

with boundary condition

$$h(s, z(t, s), \nabla z(t, s), \nabla^2 z(t, s)) = 0 \tag{3.4}$$

and initial condition $z(0, s) = z_0(s)$.

Recall that $x_{N,M}(k)$ is defined by (1.3). Suppose that we associate the discrete time k with points on the real line spaced apart by a distance proportional to δ_N. Then, the technical assumptions (3.1) and (3.2) imply that $x_{N,M}(k)$ is, in a certain sense, close to the solution of the limiting PDE (3.3) with boundary condition (3.4). Below we develop this argument rigorously.

Establishing existence and uniqueness for the resulting nonlinear models is a difficult problem in theoretical analysis of PDEs in general. The techniques are heavily dependent on the particular form of f. Therefore, as is common with numerical analysis, we assume that this has been established. Later, we apply the general theory to the modeling of networks of particular characteristics. The resulting limiting PDE is a nonlinear reaction-convection-diffusion problem. Existence and uniqueness for such problems for "small" data and short

times can be established under general conditions. Key ingredients are coercivity, which will hold as long as z is bounded away from 1, and diffusion dominance, which will also hold as long as z is bounded above.

3.2 Continuous Time-space Extension of the Markov Chain

Next we define the continuous time-space extension of the Markov chain $X_{N,M}(k)$.

For each N and M, define

$$dt_{N,M} = \frac{\delta_N}{M}, t_{N,M}(k) = k\, dt_{N,M}, K_{N,M} = \left\lfloor \frac{T}{dt_{N,M}} \right\rfloor, \text{ and } \tilde{T}_N = \frac{T}{\delta_N}. \tag{3.5}$$

First, we construct the continuous-time extension $X^{(o)}_{N,M}(\tilde{t})$ of $X_{N,M}(k)$, as the piecewise-constant time interpolant with interval length $1/M$ and normalized by M:

$$X^{(o)}_{N,M}(\tilde{t}) = X_{N,M}(\lfloor M\tilde{t} \rfloor)/M, \quad \tilde{t} \in [0, \tilde{T}_N]. \tag{3.6}$$

Similarly, define the continuous-time extension $x^{(o)}_{N,M}(\tilde{t})$ of $x_{N,M}(k)$ by

$$x^{(o)}_{N,M}(\tilde{t}) = x_{N,M}(\lfloor M\tilde{t} \rfloor), \quad \tilde{t} \in [0, \tilde{T}_N]. \tag{3.7}$$

Let $X^{(p)}_{N,M}(t,s)$, where $(t,s) \in [0,T] \times \mathcal{D}$, be the continuous-space extension of $X^{(o)}_{N,M}(\tilde{t})$ (with $\tilde{t} \in [0, \tilde{T}_N]$) by piecewise-constant space extensions on $\mathcal{D}$ and with time scaled by δ_N so that the time-interval length is $\delta_N/M := dt_{N,M}$. By *piecewise-constant space extension* of $X^{(o)}_{N,M}$, we mean the piecewise-constant function on $\mathcal{D}$ such that the value of this function at each point in $\mathcal{D}$ is the value of the component of the vector $X^{(o)}_{N,M}$ corresponding to the grid point that is "closest to the left" (taken one component at a time). Then $X^{(p)}_{N,M}(t,s)$ is the continuous time-space extension of $X_{N,M}(k)$, and for each t, $X^{(p)}_{N,M}(t,\cdot)$ is a real-valued function defined on $\mathcal{D}$. We illustrate in Fig. 3.1.

The function $X^{(p)}_{N,M}(t,s)$ with $(t,s) \in [0,T] \times \mathcal{D}$ is in the space $D^{\mathcal{D}}[0,T]$ of functions from $[0,T] \times \mathcal{D}$ to $\mathbb{R}$ that are Càdlàg with respect to the time component, i.e., right-continuous

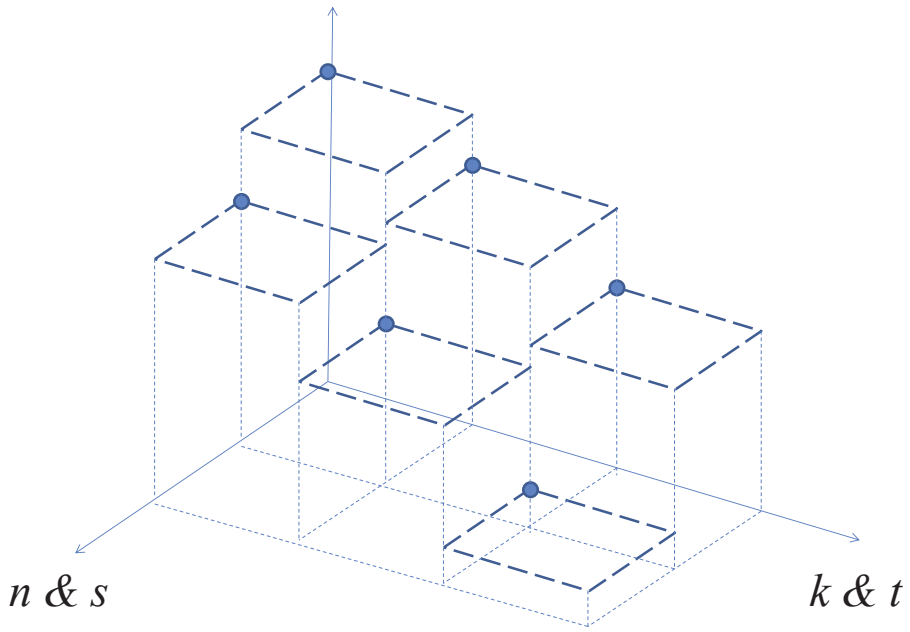


Figure 3.1: An illustration of $X_{N,M}(k)$ and $X^{(p)}_{N,M}(t,s)$ in one dimension, represented by solid dots and dashed-line rectangles, respectively.

at each $t \in [0,T)$, and have left-hand limits at each $t \in (0,T]$. Denote the norm $\|\cdot\|^{(p)}$ on $D^{\mathcal{D}}[0,T]$ such that for $x \in D^{\mathcal{D}}[0,T]$,

$$\|x\|^{(p)} = \sup_{t\in[0,T]} \int_{\mathcal{D}} |x(t,s)|\, ds. \tag{3.8}$$

3.3 Main Results for Continuum Limit of the Abstract Markov Chain Model

In this section, we present the main theorem, Theorem 1, which states that under some conditions, the continuous-time-space extension $X^{(p)}_{N,M}$ of the Markov chain $X_{N,M}(k)$ converges to the solution z of the limiting PDE (3.3) in the norm defined by (3.8), as N and M go to ∞ in a *dependent* way. By this we mean that we set M to be a function of N, written M_N, such that $M_N \to \infty$ as $N \to \infty$. Then we can treat $X_{N,M_N}(k)$, $x_{N,M_N}(k)$, $X^{(p)}_{N,M_N}$, dt_{N,M_N}, t_{N,M_N}, and K_{N,M_N} all as sequences of the *single* index N, written $X_N(k)$, $x_N(k)$, $X^{(p)}_N$, dt_N, t_N, and K_N respectively. We apply such changes of notation throughout the rest

of the paper *whenever* M is treated as a function of N.

Define $z_N(k,n) = z(t_N(k), v_N(n))$ and $z_N(k) = [z_N(k,1), \ldots, z_N(k,N)]^\top \in \mathbb{R}^N$. Define the truncation error

$$u_N(k,n) = \frac{f_N(z_N(k),n)}{\delta_N} - \frac{z_N(k+1,n) - z_N(k,n)}{dt_N}, \tag{3.9}$$

and $u_N(k) = [u_N(k,1), \ldots, u_N(k,N)]^\top \in \mathbb{R}^N$. Define

$$\varepsilon_N(k,n) = x_N(k,n) - z_N(k,n), \tag{3.10}$$

and $\varepsilon_N(k) = [\varepsilon_N(k,1), \ldots, \varepsilon_N(k,N)]^\top \in \mathbb{R}^N$. By (1.3), (3.5), (3.9), and (3.10), we have that

$$\begin{aligned}\varepsilon_N(k+1) &= \varepsilon_N(k) + \frac{1}{M_N}(f_N(x_N(k)) - f_N(z_N(k))) + dt_N u_N(k) \\ &= \varepsilon_N(k) + \frac{1}{M_N}(f_N(z_N(k) + \varepsilon_N(k)) - f_N(z_N(k))) + dt_N u_N(k). \end{aligned} \tag{3.11}$$

Let $\varepsilon_N = [\varepsilon_N(1)^\top, \ldots, \varepsilon_N(K_N)^\top]^\top$ and $u_N = [u_N(0)^\top, \ldots, u_N(K_N - 1)^\top]^\top$ denote vectors in the $(K_N N)$-dimensional vector space $\mathbb{R}^{K_N N}$. Assume that

$$\varepsilon_N(0) = 0. \tag{3.12}$$

Then by (3.11), for fixed z, there exists a function $H_N : \mathbb{R}^{K_N N} \to \mathbb{R}^{K_N N}$ such that

$$\varepsilon_N = H_N(u_N). \tag{3.13}$$

Define the vector norm $\|\cdot\|^{(N)}$ on $\mathbb{R}^{K_N N}$ such that for $x = [x(1)^\top, \ldots, x(K_N)^\top]^\top \in \mathbb{R}^{K_N N}$, where $x(k) = [x(k,1), \ldots, x(k,N)]^\top \in \mathbb{R}^N$,

$$\|x\|^{(N)} = ds_N \max_{k=1,\ldots,K_N} \sum_{n=1}^{N} |x(k,n)|. \tag{3.14}$$

Define

$$\mu_N = \lim_{\alpha \to 0} \sup_{\|u\|^{(N)} \le \alpha} \frac{\|H_N(u)\|^{(N)}}{\|u\|^{(N)}}. \tag{3.15}$$

We now present the main theorem.

Theorem 1: Assume that:

T1.1. there exist a sequence $\{\xi_N\}$ and $c_1 < \infty$ such that as $N \to \infty$, $\xi_N \to 0$, and for N sufficiently large, $\|u_N\|^{(N)} < c_1 \xi_N$;

T1.2. for each N, there exists an identically distributed sequence $\{\lambda_N(k)\}$ of integrable random variables such that for each k and x, $|F_N(x, U_N(k))| \leq \lambda_N(k)$ a.s.;

T1.3. for each N, the function $F_N(x, U_N(k))$ is continuous in x a.s.;

T1.4. for each N, the ODE $\dot{y} = f_N(y)$ has a unique solution on $[0, \tilde{T}_N]$ for any initial condition $y(0)$, where $\tilde{T}_N$ is as defined by (3.5);

T1.5. z is Lipschitz continuous on $[0, T] \times \mathcal{D}$;

T1.6. for each N, (3.12) holds; and

T1.7. the sequence $\{\mu_N\}$ is bounded.

Then a.s., there exist $c_0 < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N \geq N_0$ and each $M_N \geq \hat{M}_N$,

$$\|X_N^{(p)} - z\|^{(p)} < c_0 \max\{\xi_N, ds_N\}.$$

This theorem states that as N and M_N go to ∞ in a dependent way, $X_N^{(p)}$ converges to z in $\|\cdot\|^{(p)}$ a.s. We prove this in Section 3.5.3.

3.4 Sufficient Conditions on f_N for the Boundedness of $\{\mu_N\}$

The key assumption of Theorems 1 is that the sequence $\{\mu_N\}$ is bounded (Assumption T1.7). We present in the following theorem a result that gives specific sufficient conditions on f_N that guarantee that $\{\mu_N\}$ is bounded. This provide a practical criterion to verify this key assumption otherwise difficult to check.

Consider fixed z. We assume that $f_N \in \mathcal{C}^1$ and denote the jacobian matrix of f_N at x by $Df_N(x)$. Define for each N and for $k = 0, \ldots, K_N - 1$,

$$A_N(k) = I_N + \frac{1}{M_N} Df_N(z_N(k)), \tag{3.16}$$

where I_N is the identity matrix in $\mathbb{R}^{N \times N}$.

We denote the 1-norm on $\mathbb{R}^N$ and its induced norm on $\mathbb{R}^{N \times N}$ both by $\| \cdot \|_1^{(N)}$; i.e., for a vector $x = [x_1, \ldots, x_N]^\top \in \mathbb{R}^N$,

$$\|x\|_1^{(N)} = \sum_{n=1}^{N} |x_n|, \tag{3.17}$$

and for a matrix $A \in \mathbb{R}^{N \times N}$ with a_{ij} being its (i,j)th component,

$$\|A\|_1^{(N)} = \max_{j=1,\ldots,N} \sum_{i=1}^{N} |a_{ij}|. \tag{3.18}$$

We then have

Theorem 2: Assume that:

T2.1. for each N, (3.12) holds;

T2.2. for each N, $f_N \in \mathcal{C}^1$; and

T2.3. there exists $c < \infty$ such that for N sufficiently large and for $k = 1, \ldots, K_N - 1$, $\|A_N(k)\|_1^{(N)} \leq 1 + c\, dt_N$, where $\| \cdot \|_1^{(N)}$ is defined by (3.18).

Then $\{\mu_N\}$ is bounded.

We prove this in Section 3.5.4.

In Section 4.1, we will introduce a network model and later show in Section 4.2 that these sufficient conditions hold for that model and use this theorem to prove the convergence of its underlying Markov chain to a PDE.

3.5 Proofs of the Convergence Results

This section is devoted solely to the proofs of the results above. As such, the material here is highly technical and might be tedious to follow in detail, though we have tried our

best to make it as readable as possible. The reader can safely skip this section without doing violence to the main ideas of the paper, though much of our hard work is reflected here.

We first prove Theorems 1 by analyzing the convergence of the Markov chains $X_{N,M}(k)$ to the solution of the limiting PDE in a two-step procedure. In the first step, for each N, we show in Section 3.5.1 that as $M \to \infty$, $X_{N,M}(k)/M$ converges to $x_{N,M}(k)$. In the second step, we treat M as a function of N, written M_N, and for any sequence $\{M_N\}$, we show in Section 3.5.2 that as $N \to \infty$, $x_N(k)$ converges to the PDE solution. Based on the two steps, we show in Section 3.5.3 that as N and M_N go to ∞ in a dependent way, $X_N^{(p)}$ converges to the PDE solution, proving Theorem 1. Finally, we prove Theorem 2 in Section 3.5.4.

3.5.1 Convergence of $X_{N,M}(k)$ and $x_{N,M}(k)$ to the Solution of the Same ODE

In this subsection, we show that for *each* N, $X_{N,M}(k)/M$ and $x_{N,M}(k)$ are close in a certain sense for large M under certain conditions, by proving that both their continuous-time extensions converge to the solution of the same ODE.

For fixed T and N, by (3.5), $\tilde{T}_N$ is fixed. As defined by (3.6) and (3.7) respectively, both $X_{N,M}^{(o)}(\tilde{t})$ and $x_{N,M}^{(o)}(\tilde{t})$ with $\tilde{t} \in [0, \tilde{T}_N]$ are in the space $D^N[0, \tilde{T}_N]$ of $\mathbb{R}^N$-valued Càdlàg functions on $[0, \tilde{T}_N]$. Since they both depend on M, each one of them forms a sequence of functions in $D^N[0, \tilde{T}_N]$ indexed by $M = 1, 2, \ldots$. Define the ∞-norm $\|\cdot\|_\infty^{(o)}$ on $D^N[0, \tilde{T}_N]$; i.e., for $x \in D^N[0, \tilde{T}_N]$,

$$\|x\|_\infty^{(o)} = \max_{n=1,\ldots,N} \sup_{t \in [0, \tilde{T}_N]} |x_n(t)|,$$

where x_n is the nth components of x.

Now we present a lemma stating that under some conditions, for each N, as $M \to \infty$, $X_{N,M}^{(o)}$ converges uniformly to the solution of the ODE $\dot{y} = f_N(y)$, and $x_{N,M}^{(o)}$ converges uniformly to the same solution, both on $[0, \tilde{T}_N]$.

Lemma 1: Assume, for each N, that:

L1.1. there exists an identically distributed sequence $\{\lambda_N(k)\}$ of integrable random variables such that for each k and x, $|F_N(x, U_N(k))| \leq \lambda_N(k)$ a.s.;

L1.2. the function $F_N(x, U_N(k))$ is continuous in x a.s.; and

L1.3. the ODE $\dot{y} = f_N(y)$ has a unique solution on $[0, \tilde{T}_N]$ for any initial condition $y(0)$.

Suppose that as $M \to \infty$, $X^{(o)}_{N,M}(0) \xrightarrow{P} y(0)$ and $x^{(o)}_{N,M}(0) \to y(0)$, where "$\xrightarrow{P}$" represents convergence in probability. Then, for each N, as $M \to \infty$, $\|X^{(o)}_{N,M} - y\|^{(o)}_{\infty} \xrightarrow{P} 0$ and $\|x^{(o)}_{N,M} - y\|^{(o)}_{\infty} \to 0$ on $[0, \tilde{T}_N]$, where y is the unique solution of $\dot{y} = f_N(y)$ with initial condition $y(0)$.

To prove Lemma 1, we first present a lemma due to Kushner [8].

Lemma 2: Assume, for each N, that:

L2.1. the set $\{|F_N(x, U_N(k))| : k \geq 0\}$ is uniformly integrable;

L2.2. for each k and each bounded random variable X,

$$\lim_{\delta \to 0} E \sup_{|Y| \leq \delta} |F_N(X, U_N(k)) - F_N(X + Y, U_N(k))| = 0;$$

and

L2.3. there is a function $\hat{f}_N(\cdot)$ [continuous by $\hat{2}$] such that as $n \to \infty$,

$$\frac{1}{n} \sum_{k=0}^{n} F_N(x, U_N(k)) \xrightarrow{P} \hat{f}_N(x).$$

Suppose that, for each N, $\dot{y} = \hat{f}_N(y)$ has a unique solution on $[0, \tilde{T}_N]$ for any initial condition, and that $X^{(o)}_{N,M}(0) \Rightarrow y(0)$, where "$\Rightarrow$" represents weak convergence. Then for each N, as $M \to \infty$, $\|X^{(o)}_{N,M} - y\|^{(o)}_{\infty} \Rightarrow 0$ on $[0, \tilde{T}_N]$.

We note that in Kushner's original theorem, the convergence of $X^{(o)}_{N,M}$ to y is stated in terms of Skorokhod norm [8], but it is equivalent to the ∞-norm in our case where the time interval $[0, \tilde{T}_N]$ is finite and the limit y is continuous [101].

We now prove Lemma 1 by showing that the Assumptions L2.1, L2.2, and L2.3 of Lemma 2 hold under the Assumptions L1.1, L1.2, and L1.3 of Lemma 1.

Proof of Lemma 1: Since $\lambda_N(k)$ is integrable, as $a \to \infty$, $E|\lambda_N(k)|1_{\{|\lambda_N(k)|>a\}} \to 0$, where 1_A is the indicator function of set A. By Assumption L1.1, for each k, x, and $a > 0$,

$$E|F_N(x, U_N(k))|1_{\{|F_N(x,U_N(k))|>a\}} \leq E|\lambda_N(k)|1_{\{|F_N(x,U_N(k))|>a\}} \leq E|\lambda_N(k)|1_{\{|\lambda_N(k)|>a\}}.$$

Therefore as $a \to \infty$,

$$\sup_{k\geq 0} E|F_N(x, U_N(k))|1_{\{|F_N(x,U_N(k))|>a\}} \to 0;$$

i.e., the family $\{|F_N(x, U_N(k))| : k \geq 0\}$ is uniformly integrable, and hence Assumption L2.1 holds.

By Assumption L1.2, for each k and each bounded X, a.s.,

$$\lim_{\delta\to 0} \sup_{|Y|\leq\delta} |F_N(X, U_N(k)) - F_N(X + Y, U_N(k))| = 0.$$

By Assumption L1.1, for each k and each bounded X and Y, a.s.,

$$\begin{aligned} |F_N(X, U_N(k)) - F_N(X + Y, U_N(k))| &\leq |F_N(X, U_N(k))| + |F_N(X + Y, U_N(k))| \\ &\leq 2\lambda_N(k). \end{aligned}$$

Therefore, for each k, each bounded X, and each δ, a.s.,

$$\left| \sup_{|Y|\leq\delta} |F_N(X, U_N(k)) - F_N(X + Y, U_N(k))| \right| \leq 2\lambda_N(k),$$

an integrable random variable. By the dominant convergence theorem,

$$\begin{aligned} &\lim_{\delta\to 0} E \sup_{|Y|\leq\delta} |F_N(X, U_N(k)) - F_N(X + Y, U_N(k))| \\ &= E \lim_{\delta\to 0} \sup_{|Y|\leq\delta} |F_N(X, U_N(k)) - F_N(X + Y, U_N(k))| = 0. \end{aligned}$$

Hence Assumption L2.2 holds.

Since $U_N(k)$ are i.i.d., by the weak law of large numbers and the definition of f_N in (1.2), as $n \to \infty$,

$$\frac{1}{n}\sum_{k=0}^{n} F_N(x, U_N(k)) \xrightarrow{P} f_N(x).$$

Hence Assumption L2.3 holds.

Therefore, by Lemma 2, for each N, as $M \to \infty$, $\|X^{(o)}_{N,M} - y\|^{(o)}_\infty \Rightarrow 0$ on $[0, \tilde{T}_N]$. For any sequence of random processes $\{X_n\}$, if A is a constant, $X_n \Rightarrow A$ if and only if $X_n \xrightarrow{P} A$. Therefore, as $M \to \infty$, $\|X^{(o)}_{N,M} - y\|^{(o)}_\infty \xrightarrow{P} 0$ on $[0, \tilde{T}_N]$. The same argument implies the deterministic convergence of $x^{(o)}_{N,M}$: as $M \to \infty$, $\|x^{(o)}_{N,M} - y\|^{(o)}_\infty \to 0$ on $[0, \tilde{T}_N]$. □

Based on Lemma 1, we get the following lemma, which states that, for each N, $X^{(o)}_{N,M}$ and $x^{(o)}_{N,M}$ are close with high probability for large M.

Lemma 3: Let the assumptions of Lemma 1 hold. Then for any sequence $\{\zeta_N\}$, for each N and for M sufficiently large,

$$P\{\|X^{(o)}_{N,M} - x^{(o)}_{N,M}\|^{(o)}_\infty > \zeta_N\} \leq 1/N^2 \text{ on } [0, \tilde{T}_N].$$

Proof: By the triangle inequality,

$$\|X^{(o)}_{N,M} - x^{(o)}_{N,M}\|^{(o)}_\infty \leq \|X^{(o)}_{N,M} - y\|^{(o)}_\infty + \|x^{(o)}_{N,M} - y\|^{(o)}_\infty.$$

By Lemma 1, for each N, as $M \to \infty$, $\|X^{(o)}_{N,M} - x^{(o)}_{N,M}\|^{(o)}_\infty \xrightarrow{P} 0$ on $[0, \tilde{T}_N]$. This completes the proof. ■

Since $X^{(o)}_{N,M}$ and $x^{(o)}_{N,M}$ are the continuous-time extensions of $X_{N,M}(k)$ and $x_{N,M}(k)$ by piecewise-constant extensions, respectively, we have the following corollary stating that for each N, as $M \to \infty$, $X_{N,M}(k)/M$ converges uniformly to $x_{N,M}(k)$.

Corollary 1: Let the assumptions of Lemma 1 hold. Then for any sequence $\{\zeta_N\}$, for each N and for M sufficiently large, we have that

$$P\left\{\max_{\substack{k=1,\ldots,K_{N,M}\\ n=1,\ldots,N}} \left|\frac{X_{N,M}(k,n)}{M} - x_{N,M}(k,n)\right| > \zeta_N\right\} \leq \frac{1}{N^2}.$$

3.5.2 Convergence of $x_N(k)$ to the Limiting PDE

For the remainder of this section, we treat M as a function of N, written M_N. We now state conditions under which ε_N converges to 0 for *any sequence* $\{M_N\}$ as $N \to \infty$.

Lemma 4: Assume that:

L4.1 there exist a sequence $\{\xi_N\}$ and $c_1 < \infty$ such that as $N \to \infty$, $\xi_N \to 0$, and for N sufficiently large, $\|u_N\|^{(N)} < c_1\xi_N$;

L4.2 for each N, (3.12) holds; and

L4.3 the sequence $\{\mu_N\}$ is bounded.

Then there exists $c_0 < \infty$ such that for any sequence $\{M_N\}$ and N sufficiently large, $\|\varepsilon_N\|^{(N)} < c_0\xi_N$.

Proof: By the definition of μ_N (3.15), for each N, there exists $\delta > 0$ such that for $\alpha < \delta$,

$$\sup_{\|u\|^{(N)} \leq \alpha} \frac{\|H_N(u)\|^{(N)}}{\|u\|^{(N)}} \leq \mu_N + 1.$$

By Assumption L4.1, $\|u_N\|^{(N)} \to 0$ as $N \to \infty$. Then there exists α_1 such that for N sufficiently large, $\|u_N\|^{(N)} \leq \alpha_1 < \delta$, and hence

$$\frac{\|H_N(u_N)\|^{(N)}}{\|u_N\|^{(N)}} \leq \sup_{\|u\|^{(N)} \leq \alpha_1} \frac{\|H_N(u)\|^{(N)}}{\|u\|^{(N)}} \leq \mu_N + 1.$$

Therefore, for N sufficiently large,

$$\|\varepsilon_N\|^{(N)} = \|H_N(u_N)\|^{(N)} \leq (\mu_N + 1)\|u_N\|^{(N)}.$$

By Assumption L4.3, and because the derivation above does not depend on the choice of the sequence $\{M_N\}$, the proof is completed. ■

3.5.3 Proof of Theorem 1

We now prove the main theorem.

Proof of Theorem 1: By Lemma 4, there exist a sequence $\{\xi_N\}$ and $c_2 < \infty$ such that as $N \to \infty$, $\xi_N \to 0$, and for N sufficiently large, $\|\varepsilon_N\|^{(N)} \leq c_2\xi_N$.

Let $X_N = [X_N(1)^\top, \ldots, X_N(K_N)^\top]^\top / M_N$, $x_N = [x_N(1)^\top, \ldots, x_N(K_N)^\top]^\top$, and $z_N = [z_N(1)^\top, \ldots, z_N(K_N)^\top]^\top$ denote vectors in $\mathbb{R}^{K_N N}$. Hence $\varepsilon_N = x_N - z_N$.

For $x \in \mathbb{R}^{K_N N}$, where $x = [x(1)^\top, \ldots, x(K_N)^\top]^\top$ and $x(k) = [x(k,1), \ldots, x(k,N)]^\top \in \mathbb{R}^N$, we have that

$$\|x\|^{(N)} \leq \max_{\substack{k=1,\ldots,K_N \\ n=1,\ldots,N}} |x(k,n)|.$$

Therefore, by Corollary 1, there exists a sequence $\{\tilde{M}_N\}$ such that if for each N, $M_N \geq \tilde{M}_N$, then

$$\sum_{N=1}^{\infty} P\{\|X_N - x_N\|^{(N)} > \xi_N\} \leq \sum_{N=1}^{\infty} 1/N^2 < \infty.$$

It follows from the first Borel-Cantelli Lemma that a.s., there exists N_1 such that for $N \geq N_1$ and $M_N \geq \tilde{M}_N$, $\|X_N - x_N\|^{(N)} \leq \xi_N$.

By the triangle inequality,

$$\|X_N - z_N\|^{(N)} \leq \|X_N - x_N\|^{(N)} + \|\varepsilon_N\|^{(N)}.$$

Therefore, a.s., there exists N_2 such that for $N \geq N_2$ and $M_N > \tilde{M}_N$,

$$\|X_N - z_N\|^{(N)} < (c_2 + 1)\xi_N. \tag{3.19}$$

Let $z_N^{(p)}(t,s)$, where $(t,s) \in [0,T] \times \mathcal{D}$, be the continuous-time-space extension of $z_N(k)$ defined in the same way as $X_N^{(p)}(t,s)$ is defined from $X_N(k)$. Then by its definition, we have that

$$\|X_N^{(p)} - z_N^{(p)}\|^{(p)} = \|X_N - z_N\|^{(N)}. \tag{3.20}$$

Let $\Omega_N(k,n) = \Omega_N^{(t)}(k) \times \Omega_N^{(s)}(n)$ be the subset of $[0,T] \times \mathcal{D}$ containing $(t_N(k), v_N(n))$ over which $z_N^{(p)}$ is piecewise constant; i.e., $t_N(k) \in \Omega_N^{(t)}(k)$ and $v_N(n) \in \Omega_N^{(s)}(n)$, and for all $(t,s) \in \Omega_N(k,n)$, $z_N^{(p)}(t,s) = z_N^{(p)}(t_N(k), v_N(n)) = z(t_N(k), v_N(n))$.

By (3.5), there exists a sequence $\{\bar{M}_N\}$ such that if for each N, $M_N \geq \bar{M}_N$, then for N sufficiently large, $dt_N \leq ds_N$. By Assumption T1.5, there exists $c_3 < \infty$ such that for N sufficiently large, for $M_N \geq \bar{M}_N$, and for $k = 1, \ldots, K_N$ and $n = 1, \ldots, N$,

$$|z(t_N(k), v_N(n)) - z(t,s)| \leq c_3 ds_N, \quad (t,s) \in \Omega_N(k,n).$$

Then we have that

$$\begin{aligned}
\|z_N^{(p)} - z\|^{(p)} &= \sup_{t\in[0,T]} \int_{\mathcal{D}} |z_N^{(p)}(t,s) - z(t,s)|\, ds \\
&= \sup_{t\in[0,T]} \sum_n \int_{\Omega_N^{(s)}(n)} |z_N^{(p)}(t,s) - z(t,s)|\, ds \\
&= \max_k \sup_{t\in\Omega_N^{(t)}(k)} \sum_n \int_{\Omega_N^{(s)}(n)} |z_N^{(p)}(t,s) - z(t,s)|\, ds \\
&\le \max_k \sum_n \int_{\Omega_N^{(s)}(n)} \sup_{t\in\Omega_N^{(t)}(k)} |z_N^{(p)}(t,s) - z(t,s)|\, ds \\
&= \max_k \sum_n \int_{\Omega_N^{(s)}(n)} \sup_{t\in\Omega_N^{(t)}(k)} |z(t_N(k), v_N(n)) - z(t,s)|\, ds \\
&\le \max_k \sum_n \int_{\Omega_N^{(s)}(n)} c_3 ds_N\, ds = c_3 ds_N |\mathcal{D}|, \qquad (3.21)
\end{aligned}$$

where $|\mathcal{D}|$ is the Lebesgue measure of $\mathcal{D}$.

By the triangle inequality,

$$\|X_N^{(p)} - z\|^{(p)} \le \|X_N^{(p)} - z_N^{(p)}\|^{(p)} + \|z_N^{(p)} - z\|^{(p)}.$$

Set $\hat{M}_N = \max\{\tilde{M}_N, \bar{M}_N\}$. By (3.19), (3.20), and (3.21), a.s., there exist $c_0 < \infty$ and N_0 such that for $N \ge N_0$ and $M_N \ge \hat{M}_N$,

$$\|X_N^{(p)} - z\|^{(p)} < c_0 \max\{\xi_N, ds_N\}.$$

□

3.5.4 Proof of Theorem 2

To prove Theorem 2, we first prove Lemma 5 and 6 below.

First we provide in Lemma 5 a sequence bounding $\{\mu_N\}$ from above. By (3.13), for each N, for $k = 1, \ldots, K_N$ and $n = 1, \ldots, N$, we can write $\varepsilon_N(k,n) = H_N^{(k,n)}(u_N)$, where $H_N^{(k,n)}$ is from $\mathbb{R}^{K_N N}$ to $\mathbb{R}$. Suppose that H_N is differentiable at 0. Define

$$DH_N = \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \max_{j=1,\ldots,N} \sum_{n=1}^{N} \left| \frac{\partial H_N^{(k,n)}}{\partial u(i,j)}(0) \right|. \qquad (3.22)$$

We have that

Lemma 5: Assume that:

L5.1 for each N, (3.12) holds; and

L5.2 for each N, $H_N \in \mathcal{C}^1$ locally at 0.

Then we have that for each N, $\mu_N \leq DH_N$.

Proof: Let J_N be the jacobian matrix of H_N at 0. Note that $J_N \in \mathbb{R}^{K_N N \times K_N N}$. Let $J_N(l,m)$ be its (l,m)th component, where $l,m = 1,\ldots,K_N N$. Then we have that for $k,i = 1,\ldots,K_N$ and $n,j = 1,\ldots,N$,

$$\frac{\partial H_N^{(k,n)}}{\partial u(i,j)}(0) = J_N((k-1)N+n,(i-1)N+j).$$

Let $C_N(k,i)$ be the matrix in $\mathbb{R}^{N\times N}$ such that for $n,j = 1,\ldots,N$, the (n,j)th component of $C_N(k,i)$ is

$$\frac{\partial H_N^{(k,n)}}{\partial u(i,j)}(0);$$

i.e., $C_N(k,i)$ is the (k,i)th block in the partition of J_N into $N\times N$ blocks (there are $K_N \times K_N$ such blocks), where $k,i = 1,\ldots,K_N$. Then by (3.22),

$$DH_N = \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \|C_N(k,i)\|_1^{(N)}. \tag{3.23}$$

($\|\cdot\|_1^{(N)}$ is defined by (3.18).)

By (3.14) and (3.17), for $u = [u(1)^\top,\ldots,u(K_N)^\top]\top \in \mathbb{R}^{K_N N}$, where $u(k) = [u(k,1),\ldots,u(k,N)]^\top \in \mathbb{R}^N$,

$$\begin{aligned}
\|J_N u\|^{(N)} &= ds_N \max_{k=1,\ldots,K_N} \left\| \sum_{i=1}^{K_N} C_N(k,i)u(i) \right\|_1^{(N)} \\
&\leq ds_N \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \|C_N(k,i)u(i)\|_1^{(N)} \\
&\leq ds_N \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \|C_N(k,i)\|_1^{(N)} \|u(i)\|_1^{(N)} \\
&\leq \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \|C_N(k,i)\|_1^{(N)} ds_N \max_{l=1,\ldots,K_N} \|u(l)\|_1^{(N)} \\
&= DH_N \|u\|^{(N)},
\end{aligned}$$

where the last equation follows from (3.23), (3.14), and (3.17). Therefore, for $u \neq 0$,

$$DH_N \geq \frac{\|J_N u\|^{(N)}}{\|u\|^{(N)}}. \tag{3.24}$$

Note that if $u_N = 0$, then by (3.11) and (3.12), $\varepsilon_N = 0$. Therefore

$$H_N(0) = 0. \tag{3.25}$$

By Assumption L5.2 and Taylor's theorem, there exists a function $\tilde{H}_N$ such that

$$H_N(u) = J_N u + \tilde{H}_N(u), \tag{3.26}$$

and

$$\lim_{\alpha \to 0} \sup_{\|u\|^{(N)} \leq \alpha} \frac{\|\tilde{H}_N(u)\|^{(N)}}{\|u\|^{(N)}} = 0. \tag{3.27}$$

By (3.26) and the triangle inequality, we have that

$$\|H_N(u)\|^{(N)} \leq \|J_N u\|^{(N)} + \|\tilde{H}_N(u)\|^{(N)}.$$

Therefore by (3.15),

$$\mu_N \leq \lim_{\alpha \to 0} \sup_{\|u\|^{(N)} \leq \alpha} \left(\frac{\|J_N u\|^{(N)}}{\|u\|^{(N)}} + \frac{\|\tilde{H}_N(u)\|^{(N)}}{\|u\|^{(N)}} \right).$$

Hence by (3.24) and (3.27), we complete the proof. ■

Next we present in Lemma 6 a relationship between f_N and DH_N. Define for each N and for $k, l = 1, \ldots, K_N$,

$$B_N^{(k,l)} = \begin{cases} A_N(k-1)A_N(k-2)\ldots A_N(l), & 1 \leq l < k; \\ I_N, & l = k; \\ 0, & l > k, \end{cases} \tag{3.28}$$

where $A_N(l)$ is as defined by (3.16). We have that

Lemma 6: Assume that:

L6.1 for each N, (3.12) holds; and

L6.2 for each N, $f_N \in \mathcal{C}^1$.

Then we have that for each N, for $k, i = 1, \ldots, K_N$ and $n, j = 1, \ldots, N$,

$$\frac{\partial H_N^{(k,n)}}{\partial u(i,j)}(0) = B_N^{(k,i)}(n,j)dt_N.$$

Proof: By Assumption L6.2 and Taylor's theorem, for fixed z, there exists a function $\tilde{f}_N$ such that

$$f_N(x_N(k)) - f_N(z_N(k)) = Df_N(z_N(k))\varepsilon_N(k) + \tilde{f}_N(z_N(k) + \varepsilon_N(k), z_N(k)),$$

and for each z,

$$\tilde{f}_N(z,z) = 0, \tag{3.29}$$

and

$$\lim_{\|\varepsilon\|^{(N)} \to 0} \frac{\left\| \tilde{f}_N(z+\varepsilon, z) \right\|^{(N)}}{\|\varepsilon\|^{(N)}} = 0. \tag{3.30}$$

Then we have from (3.11) that for $k = 0, \ldots, K_N - 1$,

$$\begin{aligned}\varepsilon_N(k+1) = \varepsilon_N(k) + \frac{1}{M_N} Df_N(z_N(k))\varepsilon_N(k) + \frac{1}{M_N}\tilde{f}_N(z_N(k) + \varepsilon_N(k), z_N(k)) \\ + dt_N u_N(k).\end{aligned}$$

Therefore

$$\varepsilon_N(k+1) = A_N(k)\varepsilon_N(k) + dt_N u_N(k) + \frac{\tilde{f}_N(z_N(k) + \varepsilon_N(k), z_N(k))}{M_N}.$$

For $k = 0, \ldots, K_N - 1$, define

$$\eta_N(k) = dt_N u_N(k) + \frac{\tilde{f}_N(z_N(k) + \varepsilon_N(k), z_N(k))}{M_N}. \tag{3.31}$$

Then $\varepsilon_N(k+1) = A_N(k)\varepsilon_N(k) + \eta_N(k)$. Therefore for $k = 1, \ldots, K_N$,

$$\begin{aligned}\varepsilon_N(k) = A_N(k-1)\ldots A_N(1)\eta_N(0) + A_N(k-1)\ldots A_N(2)\eta_N(1) \\ + \ldots + A_N(k-1)\eta_N(k-2) + \eta_N(k-1).\end{aligned}$$

Then it follows from (3.28) that for $k = 1, \ldots, K_N$,

$$\varepsilon_N(k) = \sum_{l=1}^{k} B_N^{(k,l)} \eta_N(l-1). \tag{3.32}$$

Write $\varepsilon_N(k) = H_N^{(k)}(u_N)$. By (3.31),

$$\eta_N(k) = dt_N u_N(k) + \frac{\tilde{f}_N\left(z_N(k) + H_N^{(k)}(u_N), z_N(k)\right)}{M_N}.$$

Hence by (3.32), for $k = 1, \ldots, K_N$,

$$\varepsilon_N(k) = \sum_{l=1}^{k} B_N^{(k,l)} dt_N u_N(l-1) + \sum_{l=1}^{k} B_N^{(k,l)} \frac{\tilde{f}_N\left(z_N(l-1) + H_N^{(l-1)}(u_N), z_N(l-1)\right)}{M_N}.$$

Denote by $g_N^{(k,l,n)}(\cdot) : \mathbb{R}^{K_N N} \to \mathbb{R}^N$ the nth component of

$$B_N^{(k,l)} \tilde{f}_N\left(z_N(l-1) + H_N^{(l-1)}(\cdot), z_N(l-1)\right).$$

By (3.29) and (3.25), $g_N^{(k,l,n)}(0) = 0$.

Let $\{e(i,j) : i = 1, \ldots, K_N, j = 1, \ldots, N\}$ be the standard basis for $\mathbb{R}^{K_N N}$; i.e., $e(i,j)$ is the element of $\mathbb{R}^{K_N N}$ with the (i,j)th entry being 1 and all other entries being 0. Then

$$\frac{\partial H_N^{(k,n)}}{\partial u(i,j)}(0) = B_N^{(k,i)}(n,j) dt_N + \frac{1}{M_N} \sum_{l=1}^{k} \left(\lim_{h \to 0} \frac{g_N^{(k,l,n)}(h\, e(i,j))}{h} \right).$$

It remains to show that

$$\lim_{h \to 0} \frac{g_N^{(k,l,n)}(h\, e(i,j))}{h} = 0.$$

Denote by $\theta_N^{(l,d)}(\cdot) : \mathbb{R}^{K_N N} \to \mathbb{R}$ the dth component of $\tilde{f}_N(z_N(l) + H_N^{(l)}(\cdot), z_N(l))$. Then

$$g_N^{(k,l,n)}(u) = \sum_{d=1}^{N} B_N^{(k,l)}(n,d) \theta_N^{(l-1,d)}(u).$$

Denote by $\tilde{f}_N^{(l,d)}(\cdot) : \mathbb{R}^N \to \mathbb{R}$ the dth component of $\tilde{f}_N(z_N(l) + (\cdot), z_N(l))$. Then

$$\theta_N^{(l,d)}(u) = \tilde{f}_N^{(l,d)}(H_N^{(l)}(u)). \tag{3.33}$$

Then it remains to show that

$$\lim_{\|u\|^{(N)} \to 0} \frac{\theta_N^{(l,d)}(u)}{\|u\|^{(N)}} = 0. \tag{3.34}$$

By Assumption L6.2 and by induction, it follows from (3.11) that for fixed z, ε_N is a $\mathcal{C}^1$ function of u_N, because the composition of functions in $\mathcal{C}^1$ is still in $\mathcal{C}^1$. Hence Assumption L6.2 here implies Assumption L5.2 of Lemma 5. By Assumption L5.2 and (3.25), there exists c such that $|c| < \infty$, and for each $\varepsilon_1 > 0$, there exists $\delta_1(\varepsilon_1)$ such that for $\|u\|^{(N)} < \delta_1(\varepsilon_1)$, $\left| \frac{\left\|H_N^{(l)}(u)\right\|^{(N)}}{\|u\|^{(N)}} - c \right| < \varepsilon_1$. Hence for $\|u\|^{(N)} < \delta_1(\varepsilon_1)$,

$$\left\|H_N^{(l)}(u)\right\|^{(N)} < (|c| + \varepsilon_1)\|u\|^{(N)}. \tag{3.35}$$

By (3.30), $\lim_{\|x\|^{(N)} \to 0} \frac{\tilde{f}_N^{(l,d)}(x)}{\|x\|^{(N)}} = 0$. Hence for each $\varepsilon_2 > 0$, there exists $\delta_2(\varepsilon_2)$ such that for $\|x\|^{(N)} < \delta_2(\varepsilon_2)$, $\frac{\left|\tilde{f}_N^{(l,d)}(x)\right|}{\|x\|^{(N)}} < \frac{\varepsilon_2}{|c|+1}$. Hence for $0 < \|x\|^{(N)} < \delta_2(\varepsilon_2)$,

$$\left|\tilde{f}_N^{(l,d)}(x)\right| < \frac{\varepsilon_2}{|c|+1}\|x\|^{(N)}. \tag{3.36}$$

For each ε, let $\hat{\varepsilon}(\varepsilon)$ be sufficiently small such that

$$(|c| + \hat{\varepsilon}(\varepsilon))\delta_1(\hat{\varepsilon}(\varepsilon)) < \delta_2(\varepsilon), \tag{3.37}$$

and

$$\hat{\varepsilon}(\varepsilon) < 1. \tag{3.38}$$

Then by (3.35) and (3.37), for $\|u\|^{(N)} < \delta_1(\hat{\varepsilon}(\varepsilon))$, $\left\|H_N^{(l)}(u)\right\|^{(N)} < \delta_2(\varepsilon)$. Therefore, in the case that $\left\|H_N^{(l)}(u)\right\|^{(N)} > 0$, by (3.33) and (3.36),

$$\left|\theta_N^{(l,d)}(u)\right| = \left|\tilde{f}_N^{(l,d)}\left(H_N^{(l)}(u)\right)\right| < \frac{\varepsilon}{|c|+1}\left\|H_N^{(l)}(u)\right\|^{(N)}.$$

By (3.35) and (3.38),

$$\left\|H_N^{(l)}(u)\right\|^{(N)} < (|c| + \hat{\varepsilon}(\varepsilon))\|u\|^{(N)} < (|c|+1)\|u\|^{(N)}.$$

By the above two inequalities,

$$\frac{\left|\theta_N^{(l,d)}(u)\right|}{\|u\|^{(N)}} < \varepsilon. \tag{3.39}$$

By (3.29), $\tilde{f}_N^{(l,d)}(0) = 0$. Therefore, in the case that $\left\|H_N^{(l)}(u)\right\|^{(N)} = 0$, $\theta_N^{(l,d)}(u) = 0$, and thus (3.39) still holds. Therefore, (3.34) holds. ■

Now we prove Theorem 2 using the preceding lemmas.

Proof of Theorem 2: By (3.22), Lemma 5, and Lemma 6, we have that

$$\begin{aligned}
\mu_N &\leq \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \max_{j=1,\ldots,N} \sum_{n=1}^{N} \left| B_N^{(k,i)}(n,j) \right| dt_N \\
&= \max_{k=1,\ldots,K_N} \sum_{i=1}^{K_N} \left\| B_N^{(k,i)} \right\|_1^{(N)} dt_N \\
&\leq \max_{k=1,\ldots,K_N} K_N \max_{i=1,\ldots,K_N} \left\| B_N^{(k,i)} \right\|_1^{(N)} dt_N \\
&\leq T \max_{\substack{k=1,\ldots,K_N \\ i=1,\ldots,K_N}} \left\| B_N^{(k,i)} \right\|_1^{(N)} .
\end{aligned}$$

($\|\cdot\|_1^{(N)}$ is defined by (3.18).) Therefore, by (3.28) and by the sub-multiplicative property of induced norm, we have that

$$\begin{aligned}
\mu_N &\leq T \max_{\substack{k=1,\ldots,K_N \\ i=1,\ldots,k-1}} \|A_N(k-1)A_N(k-2)\ldots A_N(i)\|_1^{(N)} \\
&\leq T \max_{\substack{k=1,\ldots,K_N \\ i=1,\ldots,k-1}} \|A_N(k-1)\|_1^{(N)} \ldots \|A_N(i)\|_1^{(N)} .
\end{aligned}$$

Then by Assumption T2.3, there exists $c < \infty$ such that for N sufficiently large,

$$\mu_N \leq T(1 + c\, dt_N)^{K_N}.$$

As $N \to \infty$, $K_N \to \infty$, and

$$(1 + c\, dt_N)^{K_N} = \left(1 + \frac{c\, T}{K_N}\right)^{K_N} \to e^{cT}.$$

Therefore $\{\mu_N\}$ is bounded. □

CHAPTER 4

CONTINUUM MODELS OF STOCHASTIC NETWORKS

4.1 A Stochastic Network Model

In this section we demonstrate the various objects in the abstract Markov chain model analyzed in this dissertation on a prototypical example. We begin by describing a stochastic model of a wireless sensor network.

Consider a network of N wireless sensor nodes uniformly placed over the domain $\mathcal{D}$. That is, the N nodes are located on the grid points $V_N = \{v_N(1), \ldots, v_N(N)\}$ described above. We label the node at $v_N(n)$ by n, where $n = 1, \ldots, N$. The sensor nodes generate, according to a probability distribution, data messages that need to be communicated to the destination nodes located on the boundary of the domain, which represent specialized devices that collect the sensor data. The sensor nodes also serve as relays for routing messages to the destination nodes. Each sensor node has the capacity to store messages in a *queue*, and is capable of either transmitting or receiving messages to or from its *immediate* neighbors. (Generalization to further ranges of transmission can be found in our paper [102].) At each time instant $k = 0, 1, \ldots$, each sensor node probabilistically decides to be a transmitter or receiver, but not both. This simplified rule of transmission allows for a relatively simple representation. We illustrate such a network over a two-dimensional domain in Fig. 4.1(a).

In this network, communication between nodes is interference-limited because all nodes share the same wireless channel. We assume a simple collision protocol: a transmission from a transmitter to a neighboring receiver is successful if and only if none of the other neighbors of the receiver is a transmitter, as illustrated in Fig. 4.1(b). We assume that in a successful

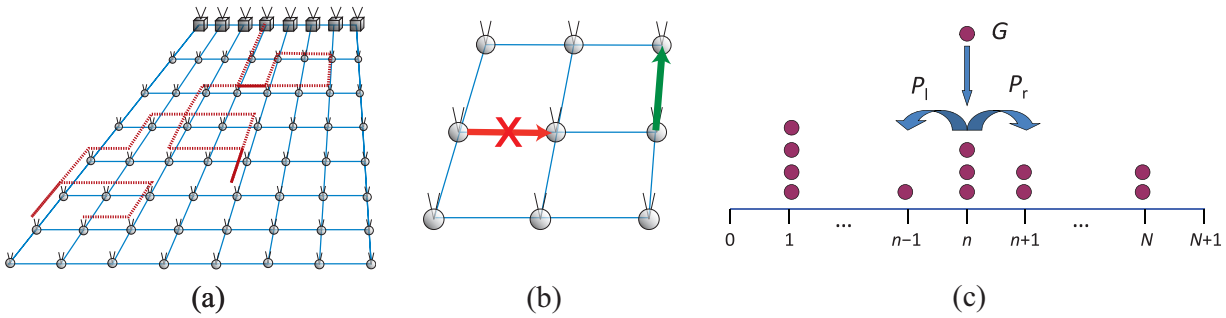


Figure 4.1: (a) An illustration of a wireless sensor network over a two-dimensional domain. Destination nodes are located at the far edge. We show the possible path of a message originating from a node located in the left-front region. (b) An illustration of the collision protocol: reception at a node fails when one of its other neighbors transmits (regardless of the intended receiver). (c) An illustration of the time evolution of the queues in the one-dimensional network model.

transmission, *one* message is transmitted from the transmitter to the receiver.

We assume that the probability that a node decides to be a transmitter is a function of its normalized queue length (normalized by an "averaging" parameter M). That is, at time k, node n decides to be a transmitter with probability $W(n, X_{N,M}(k,n)/M)$, where $X_{N,M}(k,n)$ is the *queue length* of node n at time k, and W is a given function.

In this section, for the sake of explanation, we simplify the problem even further and consider a one-dimensional domain (a two-dimensional example will be given in Section 4.2.3). Here, N sensor nodes are equidistributed in an interval $\mathcal{D} \subset \mathbb{R}$ and labeled by $n = 1, \ldots, N$. The destination nodes are located on the boundary of $\mathcal{D}$, labeled by $n = 0$ and $n = N + 1$.

We assume that if node n is a transmitter at a certain time instant, it randomly chooses to transmit one message to the right or the left immediate neighbor with probability $P_r(n)$ and $P_l(n)$, respectively, where $P_r(n) + P_l(n) \leq 1$. In contrast to strict equality, the inequality here allows for a more general stochastic model of transmission: after a sensor node randomly decides to transmit over the wireless channel, there is still a positive probability that the message is not transferred to its intended receiver (what might be called an "outage").

The special destination nodes at the boundaries of the domain do not have queues; they simply receive any message transmitted to them and never themselves transmit anything.

We illustrate the time evolution of the queues in the network in Fig. 4.1(c).

The queue lengths

$$X_{N,M}(k) = [X_{N,M}(k,1), \ldots, X_{N,M}(k,N)]^\top \in \mathbb{R}^N$$

form a Markov chain network model given by (1.1), where

$$U_N(k) = [Q(k,1), \ldots, Q(k,N), T(k,1), \ldots, T(k,N), G(k,1), \ldots, G(k,N)]^\top$$

is a random vector comprising independent random variables: $Q(k,n)$ are uniform random variables on $[0,1]$ used to determine if the node is a transmitter or not; $T(k,n)$ are ternary random variables used to determine the direction a message is passed, which take values R, L, and S (representing transmitting to the right, the left, and neither, respectively) with probabilities $P_r(n)$, $P_l(n)$, and $1-(P_r(n)+P_l(n))$, respectively; and $G(k,n)$ are the number of messages generated at node n at time k. We model $G(k,n)$ by independent Poisson random variables with mean $g(n)$.

For a generic $x = [x_1, \ldots, x_N]^\top \in \mathbb{R}^N$, the nth component of $F_N(x, U_N(k))$, where $n = 1, \ldots, N$, is

$$\begin{cases} 1+G(k,n) & \text{if} \quad Q(k,x_{n-1}) < W(n-1,x_{n-1}),\ T(k,n-1) = R, \\ & \quad\ Q(k,x_n) > W(n,x_n),\ Q(k,x_{n+1}) > W(n+1,x_{n+1}); \\ & \text{or} \quad Q(k,x_{n+1}) < W(n+1,x_{n+1}),\ T(k,n+1) = L, \\ & \quad\ Q(k,x_n) > W(n,x_n),\ Q(k,x_{n-1}) > W(n-1,x_{n-1}) \\ -1+G(k,n) & \text{if} \quad Q(k,x_n) < W(n,x_n),\ T(k,n) = L, \\ & \quad\ Q(k,x_{n-1}) > W(n-1,x_{n-1}), \\ & \quad\ Q(k,x_{n-2}) > W(n-2,x_{n-2}); \\ & \text{or} \quad Q(k,x_n) < W(n,x_n),\ T(k,n) = R, \\ & \quad\ Q(k,x_{n+1}) > W(n+1,x_{n+1}), \\ & \quad\ Q(k,x_{n+2}) > W(n+2,x_{n+2}) \\ G(k,n) & \text{otherwise}, \end{cases} \tag{4.1}$$

where x_n with $n \leq 0$ or $n \geq N+1$ are defined to be zero; and W is the function that specifies the probability that a node decides to be a transmitter, as defined earlier. Here, the three possible values of F_N correspond to the three events that at time k, node n successfully receives one message, successfully transmits one message, and does neither of the above, respectively. The inequalities and equations on the right describe conditions under which these three events occur: for example, $Q(k, x_{n-1}) < W(n-1, x_{n-1})$ corresponds to the choice of node $n-1$ to be a transmitter at time k, $T(k, n-1) = R$ corresponds to its choice to transmit to the right, $Q(k, x_n) > W(n, x_n)$ corresponds to the choice of node n to be a receiver at time k, and so on.

We simplify the situation further by assuming that $W(n, y) = \min(1, y)$. (We use this assumption throughout the dissertation.) With the collision protocol described earlier, this provides the analog of a network with backpressure routing [103].

For the one-dimensional Markov chain network model introduced in Section 3.1, it follows from (4.1) (with the particular choice of $W(n, y) = \min(1, y)$) that for $x = [x_1, \ldots, x_N]^\top \in [0, 1]^N$, the nth component of $f_N(x)$ in its corresponding deterministic difference equation (1.3), where $n = 1, \ldots, N$, is (after some tedious algebra, as described in [2])

$$\begin{aligned}&(1 - x_n)[P_r(n-1)x_{n-1}(1 - x_{n+1}) + P_l(n+1)x_{n+1}(1 - x_{n-1})] \\ &- x_n[P_r(n)(1 - x_{n+1})(1 - x_{n+2}) + P_l(n)(1 - x_{n-1})(1 - x_{n-2})] + g(n),\end{aligned} \tag{4.2}$$

where x_n with $n \leq 0$ or $n \geq N+1$ are defined to be zero.

4.2 Continuum models of Large Networks

In this section, we apply the main results to show how the Markov chain modeling the network introduced in Section 4.1 can be approximated by the solution of a PDE. This approximation was heuristically developed in [2].

We first deal with the one-dimensional network model. Its corresponding stochastic and deterministic difference equations (1.1) and (1.3) were specified by (4.1) and (4.2), respectively.

For this model we set δ_N (introduced in Section 3.1) to be ds_N^2. Then

$$dt_{N,M} := \delta_N / M = ds_N^2 / M.$$

Assume that

$$P_l(n) = p_l(v_N(n)) \text{ and } P_r(n) = p_r(v_N(n)), \tag{4.3}$$

where $p_l(s)$ and $p_r(s)$ are real-valued functions defined on $\mathcal{D}$ such that

$$p_l(s) = b(s) + c_l(s) ds_N \text{ and } p_r(s) = b(s) + c_r(s) ds_N.$$

Let $c = c_l - c_r$. The values $b(s)$ and $c(s)$ correspond to diffusion and convection quantities in the limiting PDE. Because $p_l(s) + p_r(s) \leq 1$, it is necessary that $b(s) \leq 1/2$. In order to guarantee that the number of messages entering the system from outside over finite time intervals remains finite throughout the limiting process, we set $g(n) = M g_p(v_N(n)) dt_N$, where $g_p : \mathcal{D} \to \mathbb{R}$ is called the message generation rate. Assume that b, c_l, c_r, and g_p are in $\mathcal{C}^1$. Further assume that $x_{N,M}(k) \in [0,1]^N$ for each k. Then f_N is in $\mathcal{C}^1$.

We have assumed above that the probabilities P_l and P_r of the direction of transmission are the values of the continuous functions p_l and p_r at the grid points, respectively. This may correspond to stochastic routing schemes where nodes in close vicinity behave similarly based on some local information that they share; or to those with an underlying network-wide directional configuration that are continuous in space, designed to relay messages to destination nodes at known locations. On the other hand, the results can be extended to situations with certain levels of discontinuity, as discussed in Chapter 8.

By these assumptions and definitions, it follow from (4.2) that the function f in (3.3) for this network model is:

$$\begin{aligned} f(s, z(t,s), \nabla z(t,s), \nabla^2 z(t,s)) = {} & b(s) \frac{\partial}{\partial s} \left((1 - z(t,s))(1 + 3z(t,s)) z_s(t,s) \right) \\ & + 2(1 - z(t,s)) z_s(t,s) b_s(s) \\ & + z(t,s)(1 - z(t,s))^2 b_{ss}(s) \\ & + \frac{\partial}{\partial s} (c(s) z(t,s) (1 - z(t,s))^2) + g_p(s). \end{aligned} \tag{4.4}$$

Here, a single subscript s represents first derivative and a double subscript ss represents second derivative.

Note that the computations needed to obtain (4.4) (and later, (4.5), (4.6), and (4.7)) require tedious but elementary algebraic manipulations. For this purpose, we found it helpful to use the symbolic tools in Matlab.

Based on the behavior of nodes $n = 1$ and $n = N$ next to the destination nodes, we derive the boundary condition (3.4) of the PDE of this network. For example, the node $n = 1$ receives messages only from the right and encounters no interference when transmitting to the left. Replacing x_n with $n \leq 0$ or $n \geq N + 1$ by 0, it follows that the 1st component of $f_N(x)$ is

$$(1 - x_n)P_l(n + 1)x_{n+1} - x_n[P_l(n) + P_r(n)(1 - x_{n+1})(1 - x_{n+2})] + g(n).$$

Similarly, the Nth component of $f_N(x)$ is

$$(1 - x_n)P_r(n - 1)x_{n-1} - x_n[P_r(n) + P_l(n)(1 - x_{n-1})(1 - x_{n-2})] + g(n).$$

Set β_N, defined in Section 3.1, to be 1. Then from each of the above two functions we get the function h in (3.4) for the one-dimensional network:

$$h(s, z(t, s), \nabla z(t, s), \nabla^2 z(t, s)) = -b(s)z(s)^3 + b(s)z(s)^2 - b(s)z(s). \tag{4.5}$$

Note that the function h is the limit of $f_N(y_N, s_N)/\beta_N$, not $f_N(y_N, s_N)/\delta_N$ (whose limit is f). Solving $h = 0$ for real z, we have the boundary condition $z(t, s) = 0$.

Let z be the solution of the PDE (3.3) with f specified by (4.4) and with boundary condition $z(t, s) = 0$ and initial condition $z(0, s) = z_0(s)$. Assume that (3.12) holds. As in Section 3.3, we treat M as a sequence of N, written M_N. In the following theorem we show the convergence of the Markov chain modeling the one-dimensional network to the PDE solution.

Theorem 3: For the one-dimensional network model, a.s., there exist $c_0 < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N \geq N_0$ and each $M_N \geq \hat{M}_N$, $\|X_N^{(p)} - z\|^{(p)} < c_0 ds_N$.

Proof: We prove this theorem using Theorem 1 and 2. It follows from (4.2) that there exists $c_1, c_2 < \infty$ such that for N sufficient large and $k = 0, \ldots, K_N - 1$,

$$\begin{cases} |u_N(k,n)| < c_1, & n = 1, N; \\ |u_N(k,n)| < c_2 ds_N, & n = 2, \ldots, N-1. \end{cases} \tag{4.6}$$

Therefore, there exists $c_3 < \infty$ such that for N sufficient large,

$$\max_{k=0,\ldots,K_N-1} \sum_{n=1}^{N} |u_N(k,n)| < c_3,$$

and hence by (3.14), we have that for N sufficient large,

$$\|u_N\|^{(N)} < c_3 ds_N.$$

Hence the Assumption T1.1 of Theorem 1 holds.

By (4.2), for each N, for $x = [x_1, \ldots, x_N]^\top \in [0,1]^N$, the (n,m)th component of $Df_N(x)$, where $n, m = 1, \ldots, N$, is

$$\begin{cases} P_l(n)x_n(1-x_{n-1}), & m = n-2; \\ (1-x_n)[P_r(n-1)(1-x_{n+1}) - P_l(n+1)x_{n+1}] + P_l(n)x_n(1-x_{n-2}), & m = n-1; \\ -P_r(n-1)x_{n-1}(1-x_{n+1}) - P_l(n+1)x_{n+1}(1-x_{n-1}) & \\ \quad -P_r(n)(1-x_{n+1})(1-x_{n+2}) - P_l(n)(1-x_{n-1})(1-x_{n-2}), & m = n; \\ (1-x_n)[P_l(n+1)(1-x_{n-1}) - P_r(n-1)x_{n-1}] + P_r(n)x_n(1-x_{n+2}), & m = n+1; \\ P_r(n)x_n(1-x_{n+1}), & m = n+2; \\ 0 & \text{other wise}, \end{cases}$$

where x_n with $n \leq 0$ or $n \geq N+1$ are defined to be zero. It then follows that for each k,

$$\|A_N(k)\|_1^{(N)} = 1. \tag{4.7}$$

Hence Assumption T2.3 of Theorem 2 holds. We note that obtaining (4.7) and (4.6) requires tedious, but elementary, algebraic manipulation. One can also verify that the other assumptions of Theorem 1 and 2 hold. By Theorem 1, this completes the proof. ■

4.2.1 Interpretation of Limiting PDE

Now we make some remarks on how to interpret a given limiting PDE. First, for fixed N and M, the normalized queue length of node n at time k, is approximated by the value of the PDE solution z at the corresponding point in $[0,T] \times \mathcal{D}$; i.e., $\frac{X_{N,M}(k,n)}{M} \approx z(t_{N,M}(k), v_N(n))$.

Second, we discuss how to interpret $C(t_o) := \int_{\mathcal{D}} z(t_o, s)ds$, the area below the curve $z(t_o, s)$ for fixed $t_o \in [0,T]$. Let $k_o = \lfloor t_o/dt_{N,M} \rfloor$. Then we have that $z(t_o, v_N(n))ds_N \approx \frac{X_{N,M}(k_o,n)}{M} ds_N$, the area of the nth rectangle in Fig. 4.2. Therefore

$$C(t_o) \approx \sum_{n=1}^{N} z(t_o, v_N(n))ds_N \approx \sum_{n=1}^{N} \frac{X_{N,M}(k_o, n)}{M} ds_N,$$

the sum of all rectangles. If we assume that all messages in the queue have roughly the same bits, and think of ds_N as the "coverage" of each node, then the area under any segment of the curve measures a kind of "data-coverage product" of the nodes covered by the segment, in the unit of "bit·meter." As $N \to \infty$, the total normalized queue length $\sum_{n=1}^{N} X_{N,M}(k_o, n)/M$ of the network does go to ∞; however, the coverage ds_N of each node goes to 0. Hence the sum of the "data-coverage product" can be approximated by the finite area $C(t_o)$.

4.2.2 Comparisons of the PDE Solutions and Monte Carlo Simulations of the Networks

In the remainder of this section, we compare the limiting PDE solutions with Monte Carlo simulations of the networks.

We first consider a one-dimensional network over the domain $\mathcal{D} = [-1, 1]$. We use the initial condition $z_0(s) = l_1 e^{-s^2}$, where $l_1 > 0$ is a constant, so that initially the nodes in the middle have messages to transmit, while those near the boundaries have very few. We set the message generation rate $g_p(s) = l_2 e^{-s^2}$, where $l_2 > 0$ is a parameter determining the total load of the system.

We use three sets of values of $N = 20, 50, 80$ and $M = N^3$, and show the PDE solution and the Monte Carlo simulation results with different N and M at $t = 1s$. The networks

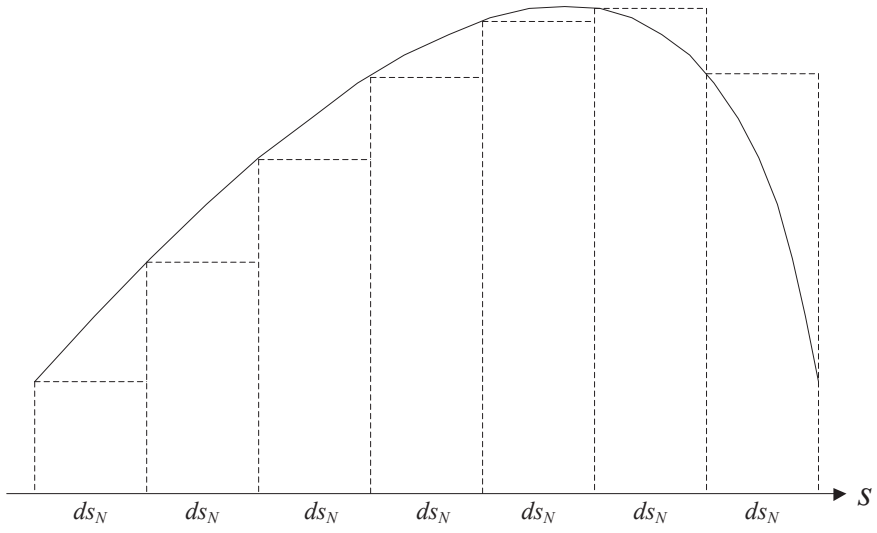


Figure 4.2: The PDE solution at a fixed time that approximates the normalized queue lengths of the network.

have diffusion $b = 1/2$ and convection $c = 0$ in Fig. 4.3 and $c = 1$ in Fig. 4.4, respectively, where the x-axis denotes the node location and y-axis denotes the normalized queue length.

For the three sets of the values of $N = 20, 50, 80$ and $M = N^3$, with $c = 0$, the maximum absolute errors of the PDE approximation are 5.6×10^{-3}, 1.3×10^{-3}, and 1.1×10^{-3}, respectively; and with $c = 1$, the errors are 4.4×10^{-3}, 1.5×10^{-3}, and 1.1×10^{-3}, respectively. As we can see, as N and M increase, the resemblance between the Monte Carlo simulations and the PDE solution becomes stronger. In the case of very large N and M, it is difficult to distinguish the results.

We stress that the PDEs only took fractions of a second to solve on a computer, while the Monte Carlo simulations took on the order of tens of hours.

4.2.3 A Two-dimensional Network

The generalization of the continuum model to higher dimensions is straightforward, except for more arduous algebraic manipulation. Likewise, the convergence analysis is similar

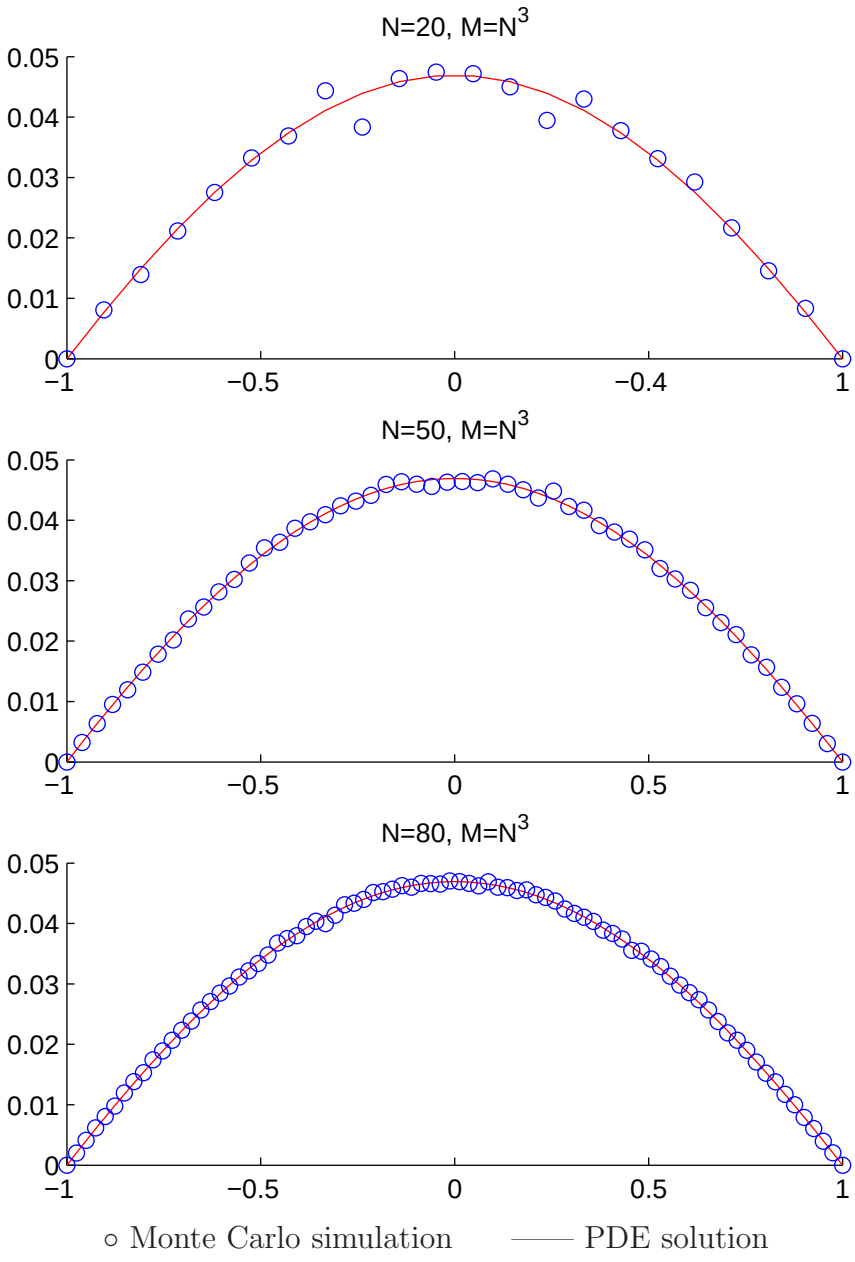


Figure 4.3: The Monte Carlo simulations (with different N and M) and the PDE solution of a one-dimensional network, with $b = 1/2$ and $c = 0$, at $t = 1s$.

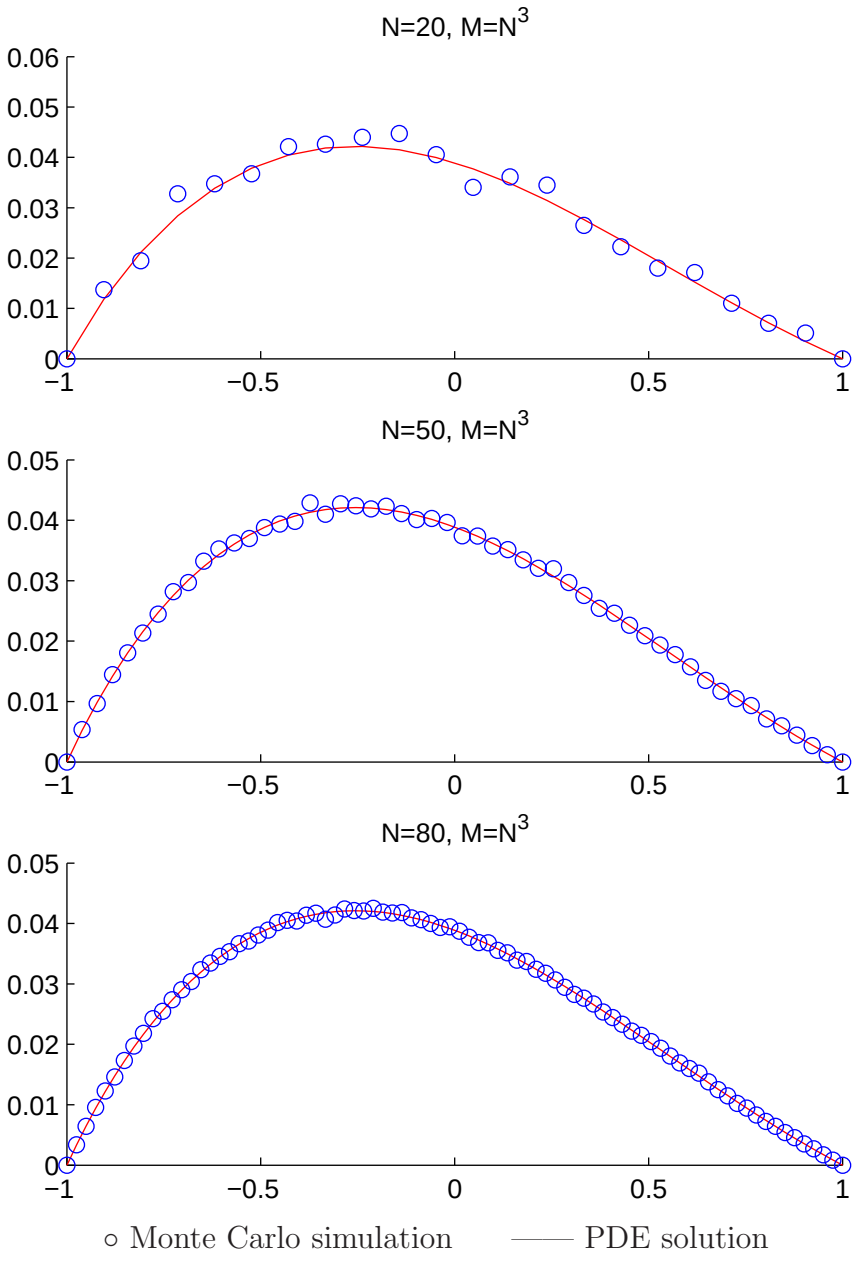


Figure 4.4: The Monte Carlo simulations (with different N and M) and the PDE solution of a one-dimensional network, with $b = 1/2$ and $c = 1$, at $t = 1s$.

to the one dimensional case.

We consider a two-dimensional network of $N = N_1 \times N_2$ sensor nodes uniformly placed over a domain $\mathcal{D} \subset \mathbb{R}^2$. Here we switch to a two-dimensional labeling scheme. We label the nodes by (n_1, n_2), where $n_1 = 1, \ldots, N_1$ and $n_2 = 1, \ldots, N_2$, and denote the grid point in $\mathcal{D}$ corresponding to node (n_1, n_2) by $v_N(n_1, n_2)$. This labeling scheme is more intuitive for this two-dimensional scenario, but is essentially equivalent to the single-label one. (e.g., if we set $n := (n_1 - 1)N_2 + n_2$ and $\hat{v}_N(n) := v_N(n_1, n_2)$, then $\hat{v}_N(n)$ form the same grid.)

Again let the distance between any two neighboring nodes be ds_N. Assume that node (n_1, n_2) randomly chooses to transmit to the east, west, north, or south immediate neighbor with probabilities $P_e(n_1, n_2) = b_1(v_N(n_1, n_2)) + c_e(v_N(n_1, n_2))ds_N$, $P_w(n_1, n_2) = b_1(v_N(n_1, n_2)) + c_w(n_1, n_2))ds_N$, $P_n(n_1, n_2) = b_2(v_N(n_1, n_2)) + c_n(v_N(n_1, n_2))ds_N$, and $P_s(n_1, n_2) = b_2(v_N(n_1, n_2)) + c_s(v_N(n_1, n_2))ds_N$, respectively, where $P_e(n_1, n_2) + P_w(n_1, n_2) + P_n(n_1, n_2) + P_s(n_1, n_2) \leq 1$. Therefore it is necessary that $b_1(s) + b_2(s) \leq 1/2$. Define $c_1 = c_w - c_e$ and $c_2 = c_s - c_n$.

The derivation of the limiting PDE is similar to those of the one-dimensional case, except that we now have to consider transmission to and interference from four directions instead of two. We present the limiting PDE here without the detailed derivation:

$$\dot{z} = \sum_{j=1}^{2} b_j \frac{\partial}{\partial s_j} \left((1 + 5z)(1 - z)^3 \frac{\partial z}{\partial s_j} \right) + 2(1 - z)^3 \frac{\partial z}{\partial s_j} \frac{db_j}{ds_j} + z(1 - z)^4 \frac{d^2 b_j}{ds_j^2} + \frac{\partial}{\partial s_j} \left(c_j z(1 - z)^4 \right) + g_p,$$

with boundary condition $z(t, s) = 0$ and initial condition $z(0, s) = z_0(s)$, where $t \in [0, T]$ and $s = (s_1, s_2) \in \mathcal{D}$.

We now compare the PDE approximation and the Monte Carlo simulations of a network over the domain $\mathcal{D} = [-1, 1] \times [-1, 1]$. We use the initial condition $z_0(s) = l_1 e^{-(s_1^2 + s_2^2)}$, where $l_1 > 0$ is a constant. We set the message generation rate $g_p(s) = l_2 e^{-(s_1^2 + s_2^2)}$, where $l_2 > 0$ is a constant.

We use three different sets of the values of $N_1 \times N_2$ and M, where $N_1 = N_2 = 20, 50, 80$

and $M = N_1^3$. We show the contours of the normalized queue length from the PDE solution and the Monte Carlo simulation results with different sets of values of N_1, N_2, and M, at $t = 0.1s$. The networks have diffusion $b_1 = b_2 = 1/4$ and convection $c_1 = c_2 = 0$ in Fig. 4.5 and $c_1 = -2, c_2 = -4$ in Fig. 5.3, respectively.

For the three sets of values of $N_1 = N_2 = 20, 50, 80$ and $M = N_1^3$, with $c_1 = c_2 = 0$, the maximum absolute errors are 3.2×10^{-3}, 1.1×10^{-3}, and 6.8×10^{-4}, respectively; and with $c_1 = -2, c_2 = -4$, the errors are 4.1×10^{-3}, 1.0×10^{-3}, and 6.6×10^{-4}, respectively. Again the accuracy of the continuum model increases with N_1, N_2, and M.

It took 3 days to do the Monte Carlo simulation of the network at $t = 0.1s$ with 80×80 nodes and the maximum queue length $M = 80^3$, while the PDE solved on the same machine took less than a second. We could not do Monte Carlo simulations of any larger networks or greater values of t because of prohibitively long computation time.

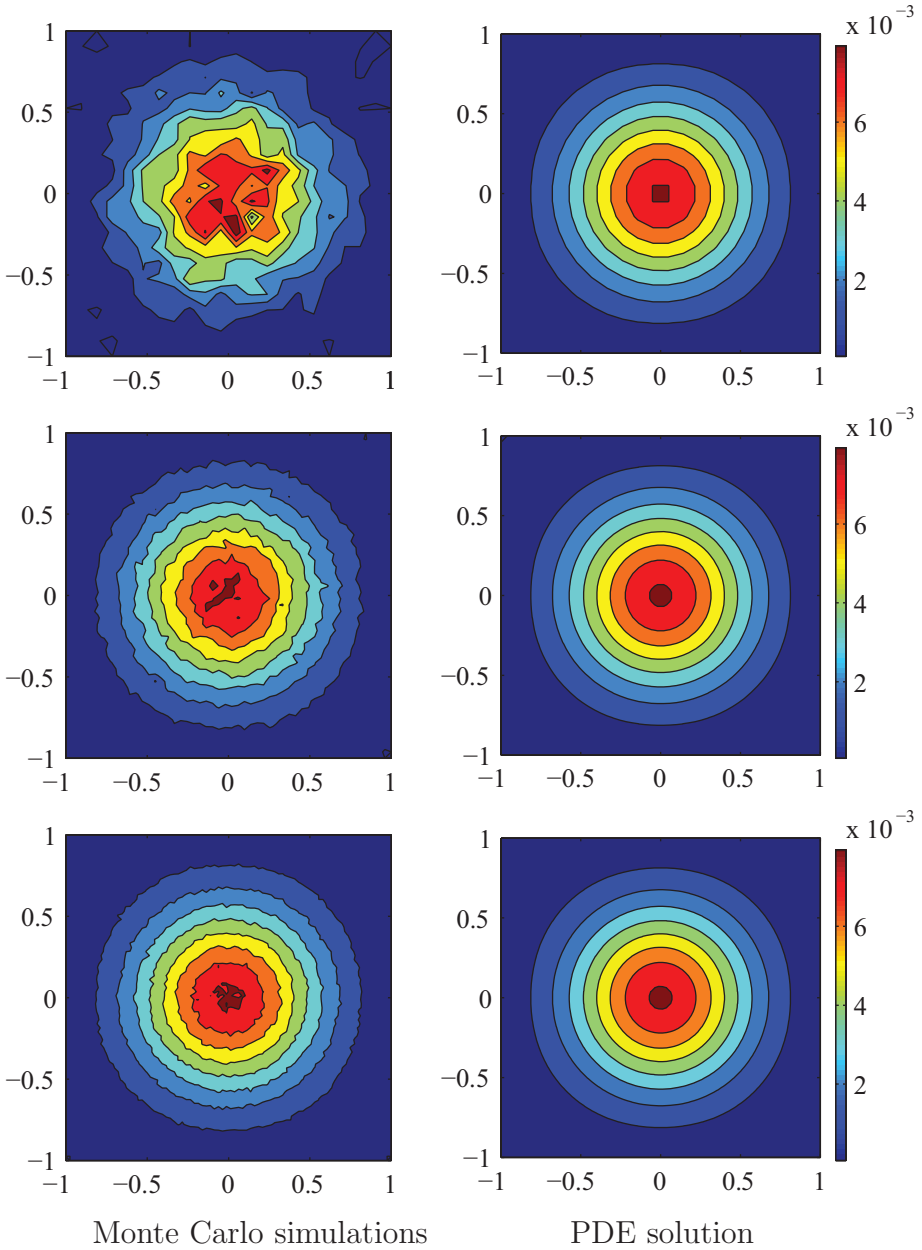


Figure 4.5: The Monte Carlo simulations (from top to bottom, with $N_1 = N_2 = 20, 50, 80$, respectively, and $M = N_1^3$) and the PDE solution of a two-dimensional network, with $b_1 = b_2 = 1/4$ and $c_1 = c_2 = 0$, at $t = 0.1s$.

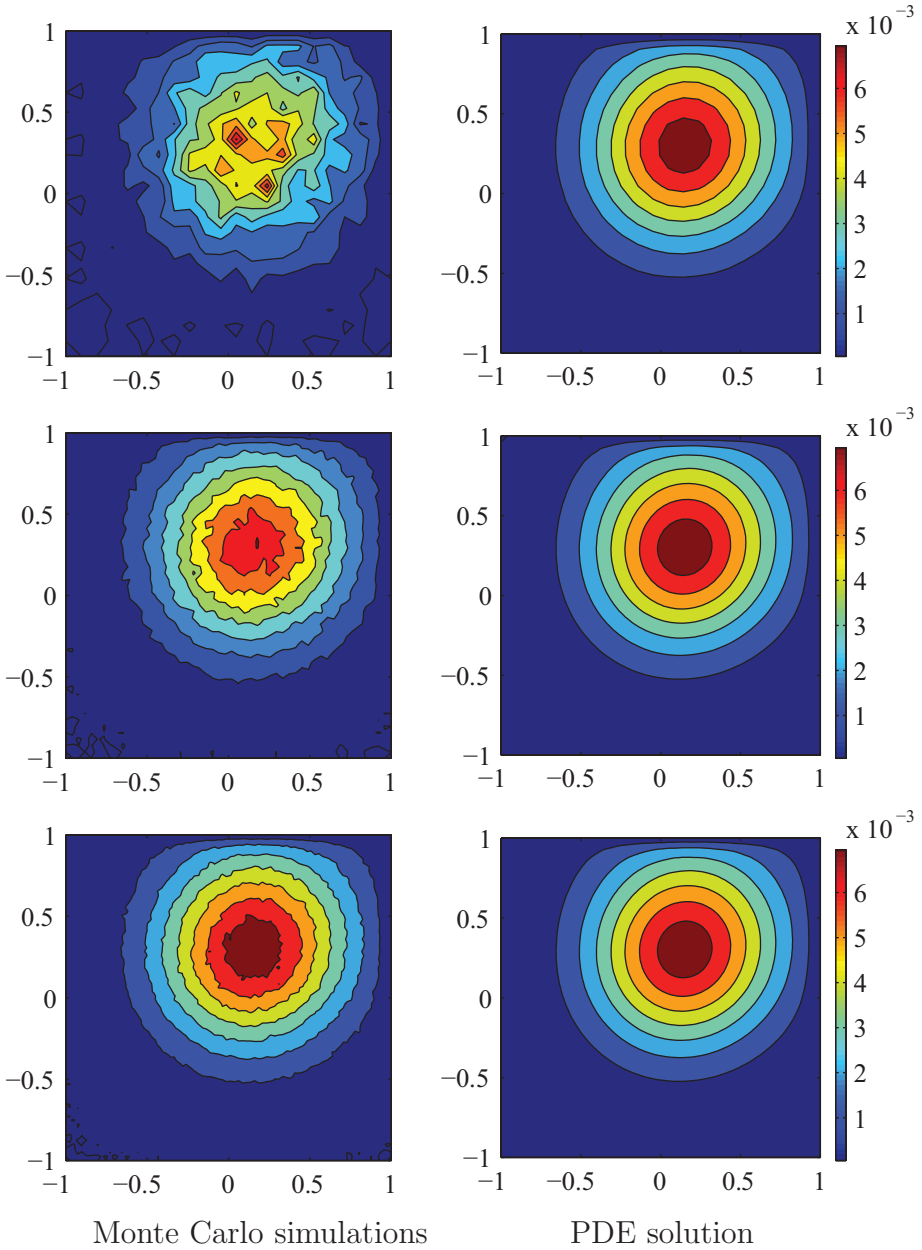


Figure 4.6: The Monte Carlo simulations (from top to bottom, with $N_1 = N_2 = 20, 50, 80$, respectively, and $M = N_1^3$) and the PDE solution of a two-dimensional network, with $b_1 = b_2 = 1/4$ and $c_1 = -2, c_2 = -4$, at $t = 0.1s$.

CHAPTER 5

CONTINUUM MODELS OF NONUNIFORM NETWORKS

The results in previous chapters assumed *uniform* networks, i.e., networks with immobile and uniformly located nodes. Moreover, the model assumes that, the nodes have a *fixed* transmission range in the sense that they communicate (exchange data and interfere) only with their immediate neighbors.

In this chapter, we consider *nonuniform* networks, i.e., networks with nonuniformly located and possibly mobile nodes. We also consider nodes with more general transmission ranges; i.e., they may communicate with neighbors further away than immediate ones. We first present a more general network model than that in the previous chapters, and derive its limiting PDEs in the setting of uniform node locations. This generalization is necessary for the discussion of the control of nonuniform networks later. Then through transformation between uniform and nonuniform node locations, we derive limiting PDEs for nonuniform networks.

5.1 A More General Network Model

We introduced the wireless sensor network model in a simple setting in Chapter 4. In this subsection, we consider uniform networks in a more general setting where the network nodes have more general transmission ranges, and derive their limiting PDEs. Such generalization is necessary for the control scheme of nonuniform networks in Chapter 6 to be possible. We consider nonuniform networks in Section 5.2.

Recall that in Chapter 4, we introduced 1-step networks where the sensor nodes communicate (exchange data and interfere) with their *immediate* neighbors. We now consider

L-step networks where the nodes communicate with their *communicating neighbors*, which can be further away than the immediate ones. To be specific, at each time instant, a transmitter tries to transmit a message to one of its communicating neighbors; and a receiver may receive a message from one of its communicating neighbors. Interference also occurs among communicating neighbors: a transmission from a transmitter to a receiver (one of the communicating neighbors of the transmitter) is successful if and only if none of the other communicating neighbors of the receiver is a transmitter.

For an L-step network, we call the positive integer L its *communication range*, and assume that it determines the communicating neighbors as follows.

In a 1-D L-step network of N nodes, communicating neighbors of the node at $s \in V_N \subset \mathbb{R}$ are the nodes at $s \pm l d s_N$, where $1 \leq l \leq L$.

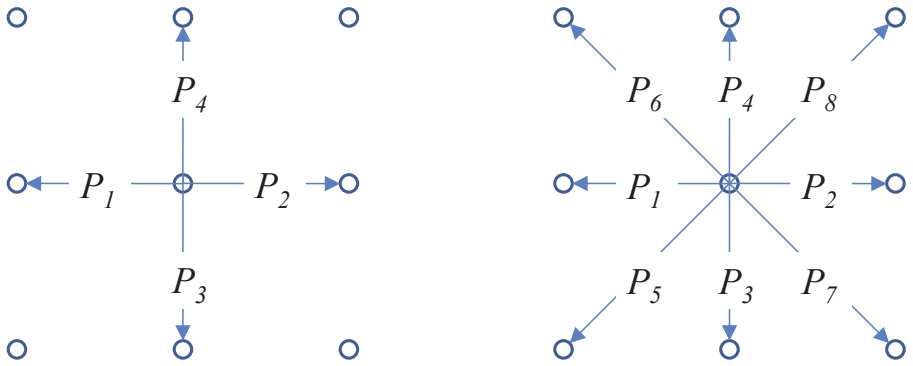
In 2-D networks, we consider two types of communicating neighbors. In a 2-D L-step network of N nodes, for a node at $s = (s_1, s_2) \in V_N \subset \mathbb{R}^2$, its communicating neighbors are the nodes at

$$(s_1 \pm l_1 d s_N, s_2 \pm l_2 d s_N), \text{ where}$$

- for Type-I networks, $0 \leq l_1, l_2 \leq L, l_1 + l_2 > 0$, and $l_1 l_2 = 0$; and
- for Type-II networks, $0 \leq l_1, l_2 \leq L$ and $l_1 + l_2 > 0$.

We illustrate the two types of definition of communicating neighbors for 2-D 1-step networks in Fig. 5.1.

We assume the use of directional antennas and power control to accommodate such routing schemes. Here we consider two types of communicating neighbors because they may correspond to two types of routing schemes, and one may be a better model than the other for networks with different design choices. For example, a Type-II network may offer higher rate in propagating information to the destination nodes at the boundaries, but at the same time may require more complex directional antennas and power control to implement.



Type-I Type-II

Figure 5.1: The two types of communicating neighbors of 2-D 1-step networks. The nodes pointed by the arrows are the communicating neighbors of the node in the center. The labels on the arrows are probabilities of transmitting to the pointed communicating neighbors.

Next we derive the limiting PDEs for this more general network model. The network model above can again be written as (1.1), for which Theorem 3 still holds.

We assume that if at time k, node n is a transmitter, it randomly chooses to transmit a message to its ith communicating neighbor with probability $P_i(k,n)$, where the possible values of i depend on the number of its communicating neighbors. Note that here P_i depends on k, i.e., is time-variant, which generalizes the case in Chapter 4. Correspondingly, we now assume that

$$P_i(k,n) = p_i(kdt_N, v_N(n)); \tag{5.1}$$

and that

$$p_i(t,s) = b_i(t,s) + c_i(t,s)ds_N, \tag{5.2}$$

where b_i and c_i are $\mathcal{C}^1$ functions from $[0,T] \times \mathcal{D}$ to $\mathbb{R}$. We call p_i the *direction function*. We have assumed above that the probabilities P_i of the direction of transmission are the values of the continuous functions p_i at the grid points, respectively. This may correspond to stochastic routing schemes where nodes in close vicinity behave similarly based on some local information that they share; or to those with an underlying network-wide directional configuration that are continuous in space, designed to relay messages to destination nodes at known locations.

For a J-D L-step network, let $\lambda_{(J,L)}$ be the number of the communicating neighbors of its nodes that are away from the boundaries. We have that

$$\lambda_{(J,L)} := \begin{cases} 2LJ, & \text{for Type-I networks;} \\ (1+2L)^J - 1, & \text{for Type-II networks.} \end{cases} \tag{5.3}$$

We assume that the communicating neighbors of each node are indexed according only to their *relative* locations with respect to the node. For example, if we call the left immediate neighbor of *any* node its 1st neighbor, then the left immediate neighbor of *all* nodes must be their 1st neighbor respectively. That is, for a node at $v_N(n)$, if we denote by $v_N(n,i)$ the location of its ith communicating neighbor, then $v_N(n) - v_N(n,i)$ depends on i, but not on n.

We present below the limiting PDE in the sense of Theorem 3 for an arbitrary J-D L-step network with both Type-I and II communicating neighbors. The PDE is derived in a way similar to that of the 1-D 1-step network in Section 4.2, which involves writing down the expression of the corresponding Markov chain (1.1) and then the difference equation (1.3), except that we now have to consider transmission to and interference from more neighbors instead of only the two immediate ones, requiring more arduous, but still elementary, algebraic manipulation. We omit the algebraic details here.

Let $\{e_1, \ldots, e_J\}$ be the standard basis of $\mathbb{R}^J$; i.e., e_j is the element of $\mathbb{R}^J$ with the jth entry being 1 and other entries 0. Define

$$b^{(j)} = \sum_i^{\lambda_{(J,L)}} \frac{((v_N(n,i) - v_N(n))^\top e_j)^2 b_i}{2}, \quad c^{(j)} = \sum_i^{\lambda_{(J,L)}} (v_N(n,i) - v_N(n))^\top e_j c_i. \tag{5.4}$$

Then the limiting PDE for a J-D L-step network is

$$\begin{aligned} \dot{z} = \sum_{j=1}^{J} \Bigg(& b^{(j)} \frac{\partial}{\partial s_j} \left(\left(1 + (\lambda_{(J,L)} + 1)z\right) (1-z)^{(\lambda_{(J,L)}-1)} \frac{\partial z}{\partial s_j} \right) \\ & + 2(1-z)^{(\lambda_{(J,L)}-1)} \frac{\partial z}{\partial s_j} \frac{\partial b^{(j)}}{\partial s_j} + z(1-z)^{\lambda_{(J,L)}} \frac{\partial^2 b^{(j)}}{\partial s_j^2} + \frac{\partial}{\partial s_j} \left(c^{(j)} z (1-z)^{\lambda_{(J,L)}} \right) \Bigg) + g_p, \end{aligned} \tag{5.5}$$

with boundary condition $z(t,s) = 0$. This general PDE works for both Type-I and II

communicating neighbors, provided that $\lambda_{(J,L)}$ is calculated with (5.3) accordingly. We will present some examples of the PDEs and the corresponding network models in Section 5.3.1.

5.2 Continuum Models of Nonuniform Networks

In this section we extend the continuum models to nonuniform and mobile networks. First we introduce the *transformation function*, which is the mapping between the node locations of uniform and nonuniform networks. Then, through the transformation function, we derive the continuum limits of nonuniform and mobile networks with given trajectories and transmissions. We consider the domain $\mathcal{D} \subset \mathbb{R}^J$ and a fixed time interval $[0, T]$.

5.2.1 Location Transformation Function

For networks with the design of uniform node placement, there may be small perturbations to the uniform grid because of imperfect implementation or landscape limitation; and some sensor networks may have nodes with moderate mobility. The study of nonuniform networks here is motivated by the need for modeling these networks. Again we assume the use of directional antennas and power control to preserve the neighborhood structure in the nonuniform or mobile networks.

Consider a nonuniform and possibly mobile network with N nodes indexed by $n = 1, \ldots, N$ over $\mathcal{D}$. The nodes no longer are located at the grid points V_N and possibly change their locations at each time step k.

We denote by $\tilde{v}_N(k, n)$ the location of node n of the nonuniform network at time k. Let $\tilde{v}_N(k) = [\tilde{v}_N(k, 1), \ldots, \tilde{v}_N(k, N)]$ and $\tilde{V}_N = [\tilde{v}_N(0), \ldots, \tilde{v}_N(K_N)]$. Assume that there exists a smooth *transformation function* $\phi(t, s) : [0, T] \times \mathcal{D} \to \mathcal{D}$ such that for each k and n,

$$\tilde{v}_N(k, n) = \phi(kdt_N, v_N(n)), \tag{5.6}$$

and for each t_o, $\phi(t_o, \cdot)$ is bijective. Hence ϕ is the mapping between the nonuniform node locations and uniform grid points.

Note that, for mobile networks, by assuming that $\phi(t_o, \cdot)$ is bijective for each t_o, we focus on a subset of all possible node movements, which simplifies the problem. This restricts the mobility of nodes but is still a reasonable model in many practical scenarios, e.g., in sensor networks where each node collects environmental data from its designated area and moves in a small neighborhood of, instead of arbitrarily far away from, their original locations.

Since $\phi(t_o, \cdot)$ is bijective, its inverse with respect to s exists and we denote it by $\eta : [0, T] \times \mathcal{D} \to \mathcal{D}$; i.e., for each t and s,

$$\eta(t, \phi(t, s)) = s. \tag{5.7}$$

Throughout the chapter we assume fixed nodes on the boundary; i.e., $\phi(t, s) = s$ for s on the boundary of $\mathcal{D}$.

For *given* N and $\tilde{V}_N$, a transformation function ϕ can be constructed using some interpolation scheme. Note that ϕ is not unique because of the freedom we have in choosing different schemes. Let ϕ_j and η_j be the jth components of ϕ and η, respectively, where $j = 1, \ldots, J$. For the rest of the chapter, we assume that for $i \neq j$,

$$\frac{\partial \phi_j}{\partial s_i} = 0. \tag{5.8}$$

Then equivalently for $i \neq j$, $\frac{\partial \eta_j}{\partial s_i} = 0$. This assumption can be achieved by choosing a proper interpolation scheme, and it simplifies the analysis below.

On the other hand, a *given* ϕ, by (5.6), specifies a sequence $\{\tilde{V}_N\}$ of nonuniform node locations indexed by N. We study the continuum limit of a *sequence* of nonuniform networks associated with such $\{\tilde{V}_N\}$; i.e., for each N, the N-node nonuniform network has node locations $\tilde{V}_N$.

5.2.2 Continuum Limits of Mirroring Networks

For an N-node network (uniform or nonuniform), we define its *transmission-interference rule* to be

- the probability that node m sends a message to node n at time k; and

- the fact of whether node m and n interfere at time k,

for $m, n = 1, \ldots, N$ and $k = 0, 1, \ldots, K_N$. The transmission-interference rule specifies how the nodes in a network interact with each other at each time step. At each time step, each node chooses to be a transmitter with a certain probability; and if it chooses to be a transmitter, it then chooses one of its communicating neighbors to send a message to. The first component of this definition is determined by the probabilities of the above choices of *all* the nodes at *all* the time steps. The second component of this definition is determined by the neighborhood structure of the network at each time step; i.e., which nodes are the communicating neighbors of each node (so that they interfere with it) at each time step.

For each N, write $X_N = [X_N(0), \ldots, X_N(K_N)]$. Then we can describe a network during $[0, T]$ entirely by its states X_N. Define the *network behavior* of a network X_N to be the combination of its initial state $X_N(0)$, transmission-interference rule, and incoming traffic $g(n)$. Two sequences $\{X_N\}$ and $\{\tilde{X}_N\}$ of networks indexed by the number N of nodes, with different node locations in general, are said to *mirror* each other if, for each N, X_N and $\tilde{X}_N$ have the same network behavior. We state in the following theorem on the relationship between the continuum limits of mirroring networks.

Theorem 4: Suppose that a sequence $\{\tilde{X}_N\}$ of networks has node locations specified by a given transformation function ϕ with inverse η. If $\{\tilde{X}_N\}$ mirrors a sequence $\{X_N\}$ of uniform networks, then $\{X_N\}$ converges to a function $q(t, s)$ on $[0, T] \times \mathcal{D}$ in the sense of Theorem 3 if and only if $\{\tilde{X}_N\}$ converges to

$$u(t, s) := q(t, \eta(t, s)), \tag{5.9}$$

in the sense that a.s., there exist $c_0 < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N \geq N_0$ and each $M_N \geq \hat{M}_N$,

$$\|\tilde{X}_N^{(p)} - u(t, \phi(t, s))\|^{(p)} < c_0 d s_N, \tag{5.10}$$

where $\tilde{X}_N^{(p)}$ is the continuous time-space extension of $\tilde{X}_N$.

Proof: "⇒": Since $\{X_N\}$ and $\{\tilde{X}_N\}$ mirror each other, they would converge to the same continuum limit on a uniform grid. Therefore, by Theorem 3, a.s., there exist $c_0 < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N \geq N_0$ and each $M_N \geq \hat{M}_N$,

$$\|\tilde{X}_N^{(p)} - q(t,s)\|^{(p)} < c_0 ds_N, \tag{5.11}$$

It follows from (5.7) and (5.9) that

$$q(t,s) = u(t, \phi(t,s)).$$

Then (5.11) is equivalent to (5.10).

"⇐": Done analogously in the opposite direction. ■

5.2.3 Sensitivity of Uniform Continuum Models to Location Perturbation

In networks with nodes not necessarily at, but close to, the uniform grid points, we can use *uniform* continuum models to approximate nonuniform networks, i.e., treat them as uniform while deriving limiting PDEs. Then a certain approximation error arises from ignoring nonuniformity. If we treat such nonuniformities as perturbations to the uniform models, the above theorem enables us to analyze the error *sensitivity* of these models with respect to such perturbation.

Consider a sequence $\{\tilde{X}_N\}$ of nonuniform networks with node locations specified by the transformation function ϕ with inverse η. Suppose that we ignore the nonuniformity and approximate $\{\tilde{X}_N\}$ by the continuum limit q of the sequence $\{X_N\}$ of uniform networks that mirrors $\{\tilde{X}_N\}$. We now characterize the maximum approximation error

$$\varepsilon_N := \|\tilde{X}_N^{(p)} - q(t, \phi(t,s))\|^{(p)} \max_{\substack{k=0,\ldots,K_N \\ n=1,\ldots,N}} \left| \frac{\tilde{X}_N(k,n)}{M_N} - q(kdt_N, \tilde{v}_N(k,n)) \right|$$

by ϕ in the following proposition.

Proposition 1: Almost surely, there exist $c_0, c_1 < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N \geq N_0$ and each $M_N \geq \hat{M}_N$,

$$\varepsilon_N \leq c_0 ds_N + \|q_s(t,s)(s - \phi(t,s))\|^{(p)} + c_1 \|(s - \phi(t,s))^2\|^{(p)}. \tag{5.12}$$

Proof: We have, from the triangle inequality, that

$$\varepsilon_N \leq \|\tilde{X}_N^{(p)} - q(t,s)\|^{(p)} + \|q(t,s) - q(t,\phi(t,s))\|^{(p)}. \tag{5.13}$$

By Theorem 4, a.s., there exist $c_0 < \infty$, N_0, and $\hat{M}_1 < \hat{M}_2 < \hat{M}_3, \ldots$ such that for each $N \geq N_0$ and each $M_N \geq \hat{M}_N$, the first term above is smaller than $c_0 ds_N$.

The second term represents the error caused by location perturbation. By Taylor's theorem, there exists $c_1 < \infty$ such that

$$q(t,s) - q(t,\phi(t,s)) \leq q_s(t,s)(s - \phi(t,s)) + c_1(s - \phi(t,s))^2.$$

Therefore we have that

$$\|q(t,s) - q(t,\phi(t,s))\|^{(p)} \leq \|q_s(t,s)\|^{(p)} \|(s - \phi(t,s))\|^{(p)} + c_1 \|(s - \phi(t,s))^2\|^{(p)}.$$

By (5.13) this completes the proof. ■

This proposition states that, for fixed q and for N and M_N sufficiently large, ε_N is dominated by $\|(s - \phi(t,s))\|^{(p)}$, when it is close to 0. We note that by definition $\|(s - \eta(t,s))\|^{(p)} = \|(s - \phi(t,s))\|^{(p)}$. In the case where $\tilde{X}_N$ are uniform, i.e., $\eta(t,s) = \phi(t,s) = s$, the last two terms on the right hand side of (5.12) vanish.

5.2.4 Limiting PDEs for Nonuniform Networks

Consider a sequence $\{\tilde{X}_N\}$ of networks with *given* network behavior and with node locations specified by a given transformation function ϕ with inverse η. If a sequence $\{X_N\}$ of uniform networks mirrors $\{\tilde{X}_N\}$, from this given network behavior, we can find the continuum limit q of $\{X_N\}$ by constructing its limiting PDE as in Chapter 4. Suppose that this PDE has the form

$$\dot{q}(t,s) = Q\left(s, q(t,s), \frac{\partial q}{\partial s_j}(t,s), \frac{\partial^2 q}{\partial s_j^2}(t,s)\right), \tag{5.14}$$

with initial condition $q(0,s) = q_0(s)$, where $j = 1, \ldots, J$, $t \in [0,T]$, and $s = (s_1, \ldots, s_J) \in \mathcal{D}$. By Theorem 4, we have that the continuum limit $u(t,s)$ of $\{\tilde{X}_N\}$ satisfies (5.9).

However, in general, we can only solve (5.14) numerically instead of analytically. In fact, all the limiting PDEs in this chapter are solved by software using numerical methods. In this case we cannot find the closed-form expression of u from q using (5.9). Instead, we derive a PDE that u satisfies so that we can solve it numerically.

Suppose that $u(t,s)$ solves the PDE

$$\dot{u}(t,s) = \Gamma\left(s, u(t,s), \frac{\partial u}{\partial s_j}(t,s), \frac{\partial^2 u}{\partial s_j^2}(t,s)\right), \tag{5.15}$$

with initial condition $u(0,s) = u_0(s)$, where $j = 1, \ldots, J$ and $(t,s) \in [0,T] \times \mathcal{D}$. We now find Γ from the known PDE (5.14).

By (5.8), (5.9), and the chain rule,

$$\frac{\partial u}{\partial s_j}(t,s) = \frac{\partial \eta_j}{\partial s_j}(t,s)\frac{\partial q}{\partial s_j}(t,\eta(t,s)).$$

By (5.8), the product rule, and the chain rule,

$$\frac{\partial^2 u}{\partial s_j^2}(t,s) = \frac{\partial^2 \eta_j}{\partial s_j^2}(t,s)\frac{\partial q}{\partial s_j}(t,\eta(t,s)) + \left(\frac{\partial \eta_j}{\partial s_j}(t,s)\right)^2 \frac{\partial^2 q}{\partial s_j^2}(t,\eta(t,s)).$$

Note that, without the assumption (5.8), the expression of the derivatives above would be much more complex. Then by (5.9), (5.14) and (5.15) we have

$$\begin{aligned}
&\Gamma\left(s, u(t,s), \frac{\partial u}{\partial s_j}(t,s), \frac{\partial^2 u}{\partial s_j^2}(t,s)\right) \\
&= Q\left(\eta(t,s), u(t,s), \frac{\frac{\partial u}{\partial s_j}(t,s)}{\frac{\partial \eta_j}{\partial s_j}(t,s)}, \frac{\frac{\partial^2 u}{\partial s_j^2}(t,s)}{\left(\frac{\partial \eta_j}{\partial s_j}(t,s)\right)^2} - \frac{\frac{\partial^2 \eta_j}{\partial s_j^2}(t,s)\frac{\partial u}{\partial s_j}(t,s)}{\left(\frac{\partial \eta_j}{\partial s_j}(t,s)\right)^3}\right),
\end{aligned}$$

where $u_0(s) = q_0(\eta(0,s))$. Hence we find the limiting PDE (5.15) of $\{\tilde{X}_N\}$.

We present a concrete numerical example of the nonuniform network and its continuum limit later in Section 5.3.2.

5.3 Numerical Examples

5.3.1 2-D Networks of the Two Types of Communicating Neighbors

We consider 2-D 1-step networks with the two types of communicating neighbors separately (as illustrated in Fig. 5.1).

Type-I communicating neighbors

For 2-D 1-step networks of Type-I communicating neighbors, we define the probabilities P_i of transmitting to the 4 communicating neighbors as in Fig. 5.1. This is the same as the 2-D network studied in [104].

The limiting PDE for this network is:

$$\dot{z} = \sum_{j=1}^{2} \Bigg(b^{(j)} \frac{\partial}{\partial s_j} \left((1+5z)(1-z)^3 \frac{\partial z}{\partial s_j} \right) + 2(1-z)^3 \frac{\partial z}{\partial s_j} \frac{\partial b^{(j)}}{\partial s_j} + z(1-z)^4 \frac{\partial^2 b^{(j)}}{\partial s_j^2} + \frac{\partial}{\partial s_j} \left(c^{(j)} z (1-z)^4 \right) \Bigg) + g_p, \tag{5.16}$$

where $b^{(1)} = (b_1 + b_2)/2$, $b^{(2)} = (b_3 + b_4)/2$, $c^{(1)} = c_1 - c_2$, $c^{(2)} = c_3 - c_4$, and $(s_1, s_2) \in \mathcal{D}$. (We omit the detailed algebraic derivation.)

We consider such a network over the spatial domain $D = [-1, 1] \times [-1, 1]$. We set the number of nodes $N = 80 \times 80$ and the normalizing parameter $M = 80^3$. We set the initial condition

$$\begin{aligned} z_0(s) = {} & r_1 e^{-4\left((s_1+0.65)^2+(s_2+0.75)^2\right)} + r_2 e^{-3\left((s_1-0.75)^2+(s_2-0.85)^2\right)} \\ & + r_3 e^{-2\left((s_1-0.75)^2+(s_2+0.75)^2\right)} + r_4 e^{-3\left((s_1+0.85)^2+(s_2-0.75)^2\right)}, \end{aligned}$$

where the constants $r_1, \ldots, r_4 > 0$, so that initially the nodes near $(-0.65, -0.75)$, $(0.75, 0.85)$, $(0.75, -0.75)$, and $(-0.85, 0.75)$ have more messages to transmit than those far away from these points. We set the incoming incoming traffic function

$$\begin{aligned} z_0(s) = {} & r_5 e^{-4\left((s_1+0.65)^2+(s_2+0.75)^2\right)} + r_6 e^{-3\left((s_1-0.75)^2+(s_2-0.85)^2\right)} \\ & + r_7 e^{-2\left((s_1-0.75)^2+(s_2+0.75)^2\right)} + r_8 e^{-3\left((s_1+0.85)^2+(s_2-0.75)^2\right)}, \end{aligned}$$

where the constants $r_5, \ldots, r_8 > 0$, so that the nodes near $(-0.65, -0.75)$, $(0.75, 0.85)$, $(0.75, -0.75)$, and $(-0.85, 0.75)$ generate more messages to transmit than those far away from these points. This may correspond to four information sources at these four points that generate different rate of data traffic. Set the diffusion functions $b_i = 1/4$, where

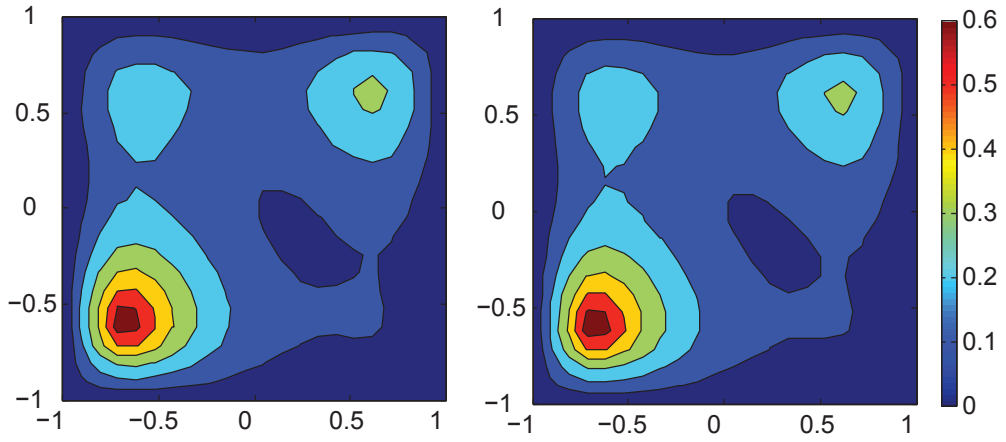


Figure 5.2: The Monte Carlo simulation and the PDE solution of a 2-D 1-step network of Type-I communicating neighbors.

$i = 1, \ldots, 4$, and the convection functions $c_1 = 0$, $c_2 = 1$, $c_3 = 0.1$, and $c_4 = -0.1$. Hence $b^{(1)} = b^{(2)} = 1/4$, $c^{(1)} = -1$, and $c^{(2)} = 0.2$, so that more data traffic in the network is routed to the south and the east. In Fig. 5.2, we show the contour of the PDE solution and the simulation result at $t = 0.1s$. We can again see the resemblance.

Type-II communicating neighbors

For 2-D 1-step networks of Type-II communicating neighbors, we define the probabilities P_i of transmitting to the 8 communicating neighbors as in Fig. 5.1. The limiting PDE is:

$$\dot{z} = \sum_{j=1}^{2} \left(b^{(j)} \frac{\partial}{\partial s_j} \left((1+9z)(1-z)^7 \frac{\partial z}{\partial s_j} \right) + 2(1-z)^7 \frac{\partial z}{\partial s_j} \frac{\partial b^{(j)}}{\partial s_j} + z(1-z)^8 \frac{\partial^2 b^{(j)}}{\partial s_j^2} + \frac{\partial}{\partial s_j} \left(c^{(j)} z (1-z)^8 \right) \right) + g_p, \tag{5.17}$$

where $b^{(1)} = \sum_{l=1,2,5,\ldots,8} \frac{b_l}{2}$, $b^{(2)} = \sum_{l=3,4,5,\ldots,8} \frac{b_l}{2}$, $c^{(1)} = c_1 - c_2 + c_5 - c_7 + c_6 - c_8$, $c^{(2)} = c_3 - c_4 + c_5 - c_6 + c_7 - c_8$.

Again the spatial domain $D = [-1, 1] \times [-1, 1]$. We set the number of nodes $N = 80 \times 80$ and the normalizing parameter $M = 80^3$. We set the initial condition

$$z_0(s) = r_1 e^{-4\left((s_1+0.55)^2+(s_2+0.55)^2\right)} + r_2 e^{(s_1-0.55)^2+(s_2-0.55)^2},$$

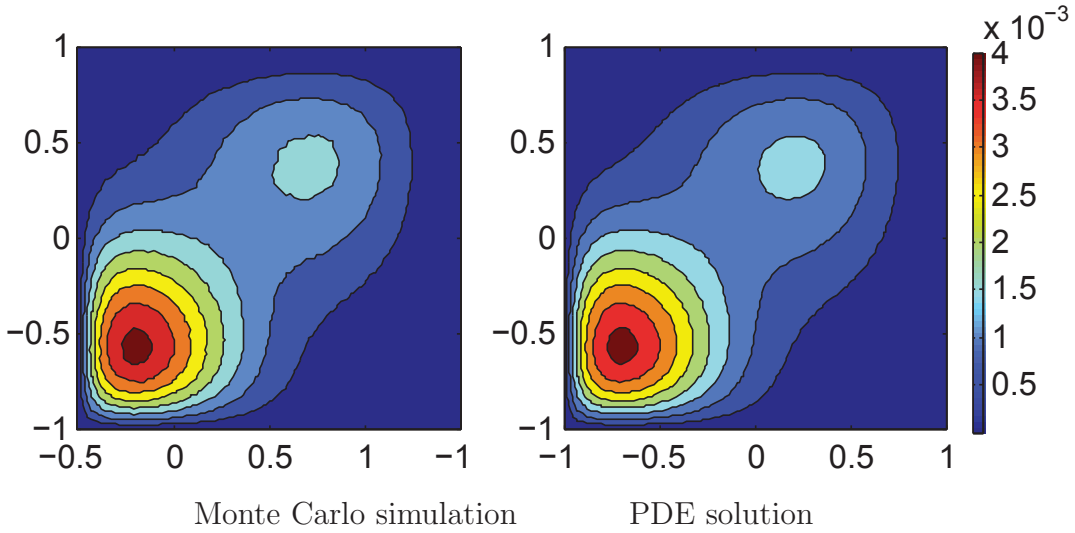


Figure 5.3: The Monte Carlo simulation and the PDE solution of a 2-D 1-step network of Type-II communicating neighbors.

where the constants $r_1, r_2 > 0$, so that initially the nodes near $(-0.55, -0.55)$ and $(0.55, 0.55)$ have more messages to transmit than those far away from these two points. We set the incoming incoming traffic function

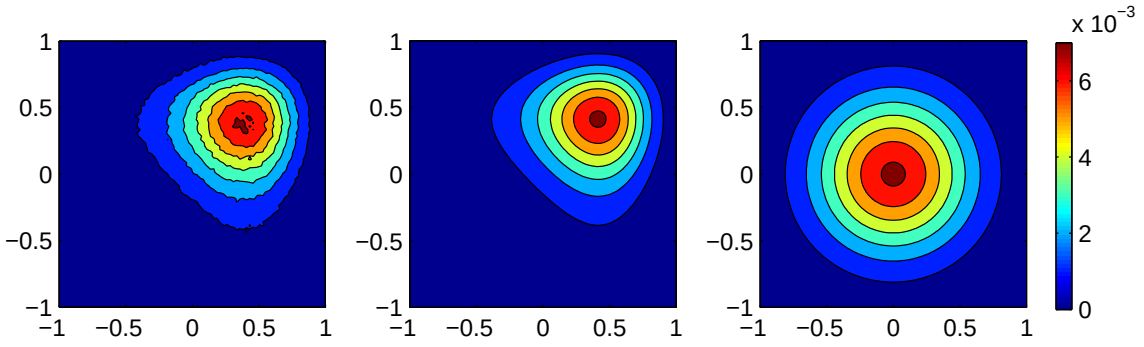
$$g_p(s) = r_3 e^{-4\left((s_1+0.55)^2+(s_2+0.55)^2\right)} + r_4 e^{(s_1-0.55)^2+(s_2-0.55)^2},$$

where the constants $r_3, r_4 > 0$, so that the nodes near $(-0.55, -0.55)$ and $(0.55, 0.55)$ generate more messages to transmit than those far away from these two points. This may correspond to two information sources at these two points that generate different rates of data traffic. In Fig. 5.3, we show the contours of the PDE solution and the simulation results with the diffusion functions $b_i = 1/8$ for $i = 1, \ldots, 8$, and convection functions $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_4 = 4$, $c_5 = -1$, $c_6 = -2$, $c_7 = -3$, and $c_8 = -4$. Hence $b^{(1)} = b^{(1)} = 3/8$, $c^{(1)} = 3$, and $c^{(2)} = 1$, so that more data traffic in the network is routed to the west and the south.

The reader can verify that the two PDEs (5.16) and (5.17) above are special cases of (5.5).

5.3.2 Example of Nonuniform Network

We illustrate a 2-D nonuniform network $\tilde{X}_N$, its continuum limit $u(t, s)$, and the continuum limit $q(t, s)$ of its mirroring uniform network in Fig. 5.4. The spatial domain



Network $\tilde{X}_N$ Continuum limit u Continuum limit q

Figure 5.4: A nonuniform network, its limiting PDE solution, and the limiting PDE solution of its mirroring uniform network.

$D = [-1,1]\times[-1,1]$. We assume that the mirroring uniform network is a 2-D 1-step network of Type-I communicating neighbors. Therefore q satisfies the limiting PDE (5.16). For the mirroring uniform network, we set the initial condition $q_0(s) = l_1 e^{-(s_1^2+s_2^2)}$, and incoming traffic $g_p(s) = l_2 e^{-(s_1^2+s_2^2)}$, where the constants $l_1, l_2 > 0$; and we set the diffusion functions $b_i = 1/4$ and the convection functions $c_i = 0$, for $i = 1,\ldots,4$. The inverse transformation function here is set to be $\eta_j(s) = (s_j+1)^2/2-1$ for $j = 1,2$. (Notice that this satisfies (5.8).) Therefore the continuum limit u of the nonuniform network $\tilde{X}_N$ is $u(t,s) = q(t,\eta(s))$.

CHAPTER 6

CONTROL OF NONUNIFORM NETWORKS

This chapter concerns the control of nonuniform networks. For example, suppose that a uniform network with certain transmissions achieves a steady state that is desirable in terms of global traffic distribution (for example, load is well-balanced over the network). Further suppose that we want the network to maintain such global characteristics if the nodes are no longer at their original uniform locations. Then the problem is to control the transmissions in the network such that its continuum limit remains invariant.

By comparing the limiting PDEs of corresponding uniform and nonuniform networks. we develop a method to control the transmissions of nonuniform networks so that the continuum limit is invariant under node locations. In other words, we can maintain a stable global characteristic for nonuniform networks.

6.1 Control of Nonuniform Networks

The global characteristic of the network is determined by the transmission-interference rule defined in Section 5.2.2 and is described by its limiting PDE. The transmission-interference rule depends entirely on the transmission range L and the probabilities P_i, which in turn by (5.1) depends on the direction function p_i. On the other hand, L and p_i also determine the limiting PDE of a sequence of networks. Therefore we can *control* the transmission-interference rule to obtain the desired limiting PDE, and hence the desired global characteristic of the network, by changing L and p_i.

For uniform networks, this procedure is straightforward because L and p_i relate directly to the form and coefficients of the limiting PDE. For example, for the 1-D 1-step network in Chapter 4, increasing the convection c results in a greater bias of the PDE solution to the

left side of the domain. (A numerical example of this network is provide in Section 4.2.2.)

We now study this kind of control for *nonuniform* and possibly mobile networks. For such networks, we have to take into account the varying node locations in order to still achieve certain global characteristics. The goal is to develop a control method so that the continuum limit is invariant under node locations and mobility, i.e., remains the same as a reference, which is the continuum limit of the sequence of corresponding uniform networks with a certain transmission-interference rule. We then say the sequence has a *location-invariant* continuum limit.

We illustrate this idea in Fig. 6.1. The plus signs in both figures represent the queues of a certain uniform network at a certain time. The solid lines in both figures represent the continuum limit (the limiting PDE solution) of the same uniform network at the same time. Thus they resemble each other. On the left, the diamonds represent the queues of a nonuniform network with the same transmission-interference rule as the uniform network, but no longer resembling the continuum limit because of the changes in node locations. On the right, the circles represent the queues of a second nonuniform network with the same node locations as the first nonuniform network, but under some control over its transmission-interference rule, therefore resembling the continuum limit of the uniform network. In other words, location-invariance in the second nonuniform network has been achieved by network control. Apparently, for this particular network, such a control scheme has to be able to direct more (and the right amount of) data traffic to the right hand side. In what follows, we describe how this can be done by properly increasing the probabilities of the nodes transmitting to the right through the use of the limiting PDEs.

Throughout the chapter we assume no control over node location or motion.

6.1.1 Transmission-interference Rule for Location-invariance

Consider a sequence $\{\tilde{X}_N\}$ of nonuniform networks whose node locations are specified by a given transformation function ϕ with inverse η, and a sequence $\{\hat{X}_N\}$ of uniform networks

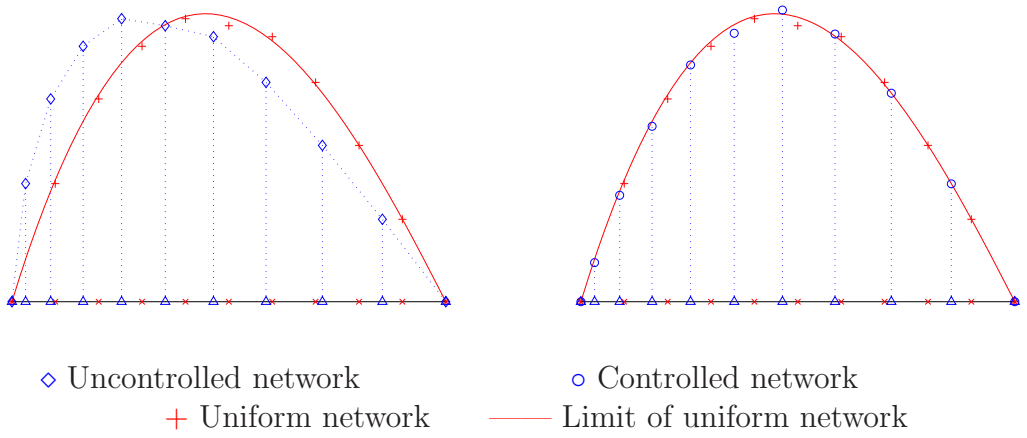


Figure 6.1: An illustration of control of nonuniform networks. On the x-axis, the ×-marks are the uniform grid, and the △-marks are the nonuniform node locations.

with *given* transmission-interference rule and continuum limit u. We want to control the transmission-interference rule of $\{\tilde{X}_N\}$ so that it also converges to u, i.e., obtains the location-invariant continuum limit.

Again we do not assume a known closed-form expression of u. Instead, assume that $u(t,s)$ solves (5.15), except that Γ is now *given*.

Define

$$q(t,s) = u(t,\phi(t,s)). \tag{6.1}$$

Suppose that a sequence $\{X_N\}$ of uniform networks has continuum limit $q(t,s)$. By Theorem 4, for $\{\tilde{X}_N\}$ to converge to this desired $u(t,s)$, it suffices that $\{\tilde{X}_N\}$ mirrors $\{X_N\}$. Therefore all we have to do is to specify the transmission-interference rule of $\{X_N\}$ to $\{\tilde{X}_N\}$. Next we find this transmission-interference rule.

Suppose that $q(t,s)$ solves (5.14), except that Q is now *unknown*. Again using the product rule and the chain rule as we did in Section 5.2.4, by (5.14), (5.15), and (6.1), we have that

$$\begin{aligned} &Q\left(s, q(t,s), \frac{\partial q}{\partial s_j}(t,s), \frac{\partial^2 q}{\partial s_j^2}(t,s)\right) \\ &= \Gamma\left(\phi(t,s), q(t,s), \frac{\frac{\partial q}{\partial s_j}(t,s)}{\frac{\partial \phi_j}{\partial s_j}(t,s)}, \frac{\frac{\partial^2 q}{\partial s_j^2}(t,s)}{\left(\frac{\partial \phi_j}{\partial s_j}(t,s)\right)^2} - \frac{\frac{\partial^2 \phi_j}{\partial s_j^2}(t,s)\frac{\partial q}{\partial s_j}(t,s)}{\left(\frac{\partial \phi_j}{\partial s_j}(t,s)\right)^3}\right), \end{aligned} \tag{6.2}$$

and $q_0(s) = u_0(\phi(0,s))$, where $j = 1, \ldots, J$.

Since $q(t,s)$ is the continuum limit of a sequence of uniform networks, (5.14) must be a case of (5.5), the general limiting PDE. Therefore we can replace the left-hand side of (6.2) by the right-hand side of (5.5) and get

$$\begin{aligned}
&\sum_{j=1}^{J}\Bigg(b^{(j)}(t,s)\frac{\partial}{\partial s_j}\left(\left(1+(\lambda_{(J,L)}+1)z(t,s)\right)(1-z(t,s))^{(\lambda_{(J,L)}-1)}\frac{\partial z}{\partial s_j}(t,s)\right) \\
&\quad + 2(1-z(t,s))^{(\lambda_{(J,L)}-1)}\frac{\partial z}{\partial s_j}(t,s)\frac{\partial b^{(j)}}{\partial s_j}(t,s) + z(t,s)(1-z(t,s))^{\lambda_{(J,L)}}\frac{\partial^2 b^{(j)}}{\partial s_j^2}(t,s) \\
&\quad + \frac{\partial}{\partial s_j}\left(c^{(j)}(t,s)z(t,s)(1-z(t,s))^{\lambda_{(J,L)}}\right)\Bigg) + g_p(t,s) \\
&= \Gamma\left(\phi(t,s), q(t,s), \frac{\frac{\partial q}{\partial s_j}(t,s)}{\frac{\partial \phi_j}{\partial s_j}(t,s)}, \frac{\frac{\partial^2 q}{\partial s_j^2}(t,s)}{\left(\frac{\partial \phi_j}{\partial s_j}(t,s)\right)^2} - \frac{\frac{\partial^2 \phi_j}{\partial s_j^2}(t,s)\frac{\partial q}{\partial s_j}(t,s)}{\left(\frac{\partial \phi_j}{\partial s_j}(t,s)\right)^3}\right).
\end{aligned} \tag{6.3}$$

We call this the *comparison equation*. If we can solve it for L, p_l, and g_p, our goal is accomplished because they determine the network behavior, which includes the transmission-interference rule, for each N-node uniform network in the mirroring sequence $\{X_N\}$. If we assign these transmission-interference rule to $\{\tilde{X}_N\}$, then it has the location-invariant continuum limit $u(t,s)$.

We note a constraint for (6.3): by (5.1), for each i, p_i has to be sufficiently small such that for each k and n,

$$P_i(k,n) \in [0,1], \text{ and } \sum_i P_i(k,n) \in [0,1]. \tag{6.4}$$

In turn by (5.2), b_i and c_i have to be sufficiently small for (6.4) to hold. By further observing (5.3) and (5.4), it follows that the transmission range L has to be sufficiently large. For this reason, it is necessary to generalize from 1-step to L-step transmission range, as we did in Section 5.1. Note that with this constraint, (6.3) is still underdetermined. Such freedom gives us a class of transmission-interference rules to assign to $\{\tilde{X}_N\}$ instead of just one.

One way to solve (6.3) is this. Suppose that we have chosen L sufficiently large. Since (5.15) is now given, we know the numerical form of u and in turn that of q by (6.1). For

fixed t_o, we put $q(t_o, s)$ in (6.3). For each j, if we fix $b^{(j)}(t_o, s)$, then we can solve (6.3), which is now an ordinary differential equation (ODE), for $c^{(j)}(t_o, s)$. Similarly, fixing $c^{(j)}(t_o, s)$ makes (6.3) an ODE that we can solve for $b^{(j)}(t_o, s)$. Then by (5.4) we can further choose b_i and c_i, and further determine p_i by (5.2). Thus we have found P_i by (5.1), which together with L determines the transmission-interference rule.

6.1.2 Distributed Control Using Local Information

The control method presented above is centralized in the sense that it requires knowledge of the transformation function ϕ over $\mathcal{D}$. This assumes that each node knows the location of all other nodes. However, this is generally not the case in practice, especially for networks without a central control unit. In this subsection we present a distributed version of our control method, where only the locations of nearby nodes are needed for each node to determine its transmission-interference rule. We can do this because all the information needed to solve the comparison equation (6.3) can be approximated locally at each node.

The derivatives of ϕ in (6.3) can be approximated from the locations of neighboring nodes using a certain finite difference method. For example, in the 1-D case, we can use the approximation:

$$\frac{\partial \phi}{\partial s}(t, s) \approx \frac{\phi(kdt_N, v_N(n+1)) - \phi(kdt_N, v_N(n-1))}{2ds_N} = \frac{\tilde{v}_N(k, n+1) - \tilde{v}_N(k, n-1)}{2ds_N},$$

where $t = kdt_N$ and $s \in [v_N(n-1), v_N(n+1))$. Note that we can also use the location information of further neighbors to get a more accurate approximation of $\partial\phi/\partial s$. The trade-off between locality and accuracy can be flexibly adjusted.

The ODE for $b^{(j)}$ or $c^{(j)}$ can also be solved based on local information using numerical procedures such as Euler's method [105].

We present two concrete examples of network control in 1-D and 2-D case in Section 6.2.1 and Section 6.2.2, respectively.

6.2 Examples of Control of Nonuniform Networks

6.2.1 1-D Example

Let the domain $\mathcal{D} = [-1, 1]$. Let $u(t, s)$ be the continuum limit of a sequence $\{\hat{X}_N\}$ of 1-D 1-step uniform networks with transmission range $\hat{L} = 1$, the diffusion function $\hat{b} = 1/2$, the convection function $\hat{c} = 0$, and a given incoming traffic function $\hat{g}_p$ for all $(t, s) \in [0, T] \times \mathcal{D}$. A given transformation function ϕ specifies the node locations of a sequence $\{\tilde{X}_N\}$ of nonuniform networks. We show how to find the transmission-interference rule for $\{\tilde{X}_N\}$ to converge to $u(t, s)$. As the continuum limit of this particular 1-D 1-step network, $u(t, s)$ solves the PDE

$$\dot{u} = \frac{\partial}{\partial s}\left(\frac{1}{2}(1-u)(1+3u)\frac{\partial u}{\partial s}\right) + g_p, \tag{6.5}$$

with boundary condition $u(t, s) = 0$ and initial condition $u(0, s) = u_0(s)$.

In this case $\lambda_{(J,L)} = 2L$. Let $\theta = 1/\left(2\left(\frac{\partial \phi}{\partial s}\right)^2\right)$. Then the comparison equation (6.3) becomes

$$\begin{aligned} & b^{(1)}\frac{\partial}{\partial s}\left((1+(2L+1)q)(1-q)^{(2L-1)}\frac{\partial q}{\partial s}\right) + 2(1-q)^{(2L-1)}\frac{\partial q}{\partial s}\frac{\partial b^{(1)}}{\partial s} \\ & + q(1-q)^{2L}\frac{\partial^2}{\partial s^2}b^{(1)} + \frac{\partial}{\partial s}\left(c^{(1)}q(1-q)^{2L}\right) + \hat{g}_p \\ & = \theta(1-q)(1+3q)\frac{\partial^2 q}{\partial s^2} + 2(1-3q)\theta\left(\frac{\partial q}{\partial s}\right)^2 + \frac{1}{2}(1-q)(1+3q)\frac{\partial \theta}{\partial s}\frac{\partial q}{\partial s} + g_p(\phi), \end{aligned} \tag{6.6}$$

where q is the continuum limit of the mirroring sequence $\{X_N\}$ of $\{\tilde{X}_N\}$.

We assume that $\hat{g}_p(s) = g_p(\phi(t, s))$, which corresponds to the assumption that the continuum limit of the incoming traffic is invariant under node locations and mobility. This assumption is feasible in a large class of networks where traffic load depends directly on actual physical location. For example, in a wireless sensor network that detects environmental events such as a forrest fire, the event-triggered data traffic depends on the distribution of heat rather than the node locations.

Suppose that we set

$$b^{(1)} = \theta. \tag{6.7}$$

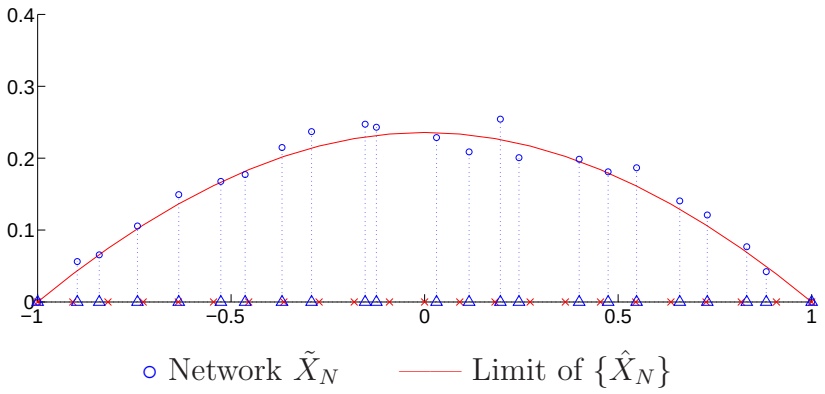


Figure 6.2: The comparison of the 1-D controlled network and the location-invariant continuum limit at $t = 1s$. On the x-axis, the $\times$ marks are the uniform grid, and the $\triangle$ marks are the nonuniform node locations.

Since q is known to be the solution of (6.5), (6.6) has now become a first-order linear ODE for $c^{(1)}$.

We can use Euler's method to solve this ODE based on local information. For fixed t_o, suppose the ODE is written in the form $\Phi(t_o, s, c^{(1)}) = \frac{dc^{(1)}}{ds}$. We first choose $c^{(1)}(t_o, s(1))$ such that $P_i(k_o, 1)$ satisfies (6.4), where $t_o = k_o dt_N$. Then we can approximate $c^{(j)}(t_o, s(n))$ by $\hat{c}(t_o, n)$, where $\hat{c}(t_o, 1) = c^{(j)}(t_o, s(1))$, and $\hat{c}(t_o, n+1) = \hat{c}(t_o, n) + \Phi(t_o, s(n), \hat{c}(t_o, n))ds_N$, for $n = 1, \ldots, N$.

With this given ϕ, the transmission range L of the mobile network has to be greater or equal to 2 for (6.4) to hold. We choose $L = 2$. Then any b_i, c_i, where $i = 1, 2$, that satisfy (6.6) and (6.7) will give us the desired transmission-interference rule of networks in $\{X_N\}$, and hence that of $\{\tilde{X}_N\}$.

We simulate a 51-node controlled mobile network $\tilde{X}_N$ in the sequence $\{\tilde{X}_N\}$ that mirrors $\{X_N\}$, whose node locations are specified by this given ϕ. In Fig. 6.2, we compare the simulation result with the continuum limit of $\{\hat{X}_N\}$, at $t = 1s$. We set the initial condition $z_0(s) = r_1 e^{-s^2}$ and the incoming traffic function $g_p(s) = r_2 e^{-s^2}$, where the constants $r_1, r_2 > 0$. As we can see, the global characteristic of $\tilde{X}_N$ resembles $u(t, s)$, the continuum limit of $\{\hat{X}_N\}$.

6.2.2 2-D Example

Let the domain $\mathcal{D} = [-1,1] \times [-1,1]$. Let $u(t,s)$ be the continuum limit of a sequence $\{\hat{X}_N\}$ of 2-D 1-step uniform networks of Type-II communicating neighbors with transmission range $\hat{L} = 1$, the diffusion functions $\hat{b}_i(t,s) = 1/8$, for $i = 1,\ldots,8$, the convection functions $\hat{c}^{(j)} = 0$, for $j = 1,2$, and given incoming traffic function $\hat{g}_p$ for all $(t,s) \in [0,T] \times \mathcal{D}$. Again denote the given transformation function that specifies the node locations of $\{\tilde{X}_N\}$ by $\phi(t,s)$.

As the continuum limit of this particular 1-D 1-step network, $u(t,s)$ solves the PDE

$$\dot{u} = \frac{3}{8}\sum_{j=1}^{2}\frac{\partial}{\partial s_j}\left((1+9u)(1-u)^7\frac{\partial u}{\partial s_j}\right) + \hat{g}_p, \tag{6.8}$$

with boundary condition $u(t,s) = 0$ and initial condition $u(0,s) = u_0(s)$.

Let $\theta_j = 1/\left(2\left(\frac{\partial \phi_j}{\partial s_j}\right)^2\right)$. Then the comparison equation (6.3) becomes

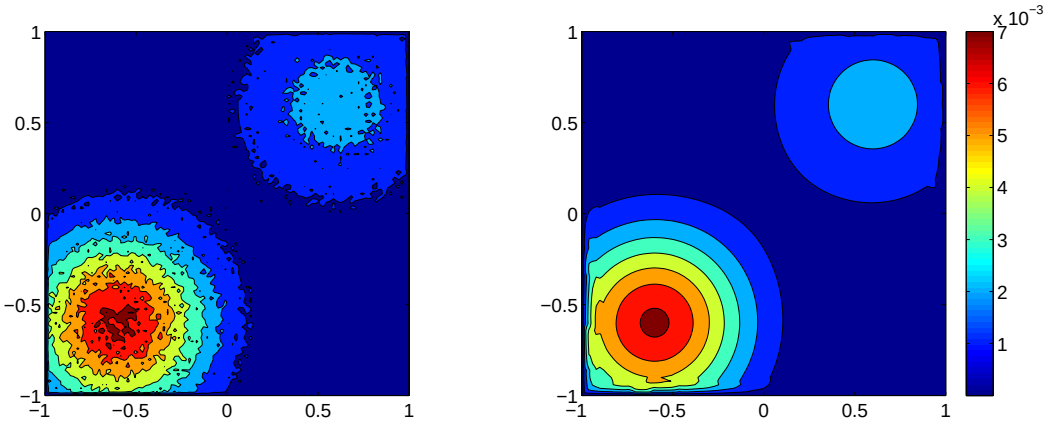
$$\begin{aligned}
&\sum_{j=1}^{2}\Bigg(b^{(j)}\frac{\partial}{\partial s}\left((1+(\lambda_{(2,L)}+1)q)(1-q)^{(\lambda_{(2,L)}-1)}\frac{\partial q}{\partial s}\right) + 2(1-q)^{(\lambda_{(2,L)}-1)}\frac{\partial q}{\partial s}\frac{\partial \hat{b}_j}{\partial s_j} \\
&+q(1-q)^{\lambda_{(2,L)}}\frac{\partial^2 b^{(j)}}{\partial s^2} + \frac{\partial}{\partial s}\left(c^{(j)}q(1-q)^{\lambda_{(2,L)}}\right)\Bigg) + \hat{g}_p \\
&= \sum_{j=1}^{2}\Bigg(\frac{3}{4}(1-q)^7(1+9q)\theta_j\frac{\partial^2 q}{\partial x_j^2} + \frac{3}{8}(1-q)^7(1+9q)\frac{\partial \theta_j}{\partial x_j}\frac{\partial q}{\partial x_j} \\
&+\frac{3}{2}(1-36q)(1-q)^6\theta_j\left(\frac{\partial q}{\partial x_j}\right)^2\Bigg) + g_p(\phi),
\end{aligned} \tag{6.9}$$

where q is the continuum limit of the mirroring sequence $\{X_N\}$ of $\{\tilde{X}_N\}$. Assume that $\hat{g}_p(t,s) = g_p(\phi(t,s))$ and

$$b^{(j)} = \theta_j. \tag{6.10}$$

Since q is known to be the solution of (6.8), we have two first-order linear ODEs of $c^{(j)}$, where $j = 1,2$.

For this given ϕ, $L = 2$ is sufficient for (6.4) to hold. Then any $b_i, c_i, l = 1,2$ that satisfy (6.9) and (6.10) will give us the desired transmission-interference rule for $\{X_N\}$, and hence $\{\tilde{X}_N\}$.



Network $\tilde{X}_N$ Limit of $\{X_N\}$

Figure 6.3: The comparison of the 2-D controlled network and the location-invariant continuum limit at $t = 1s$.

We simulate a (100×100)-node controlled mobile network $\tilde{X}_N$ in the sequence $\{\tilde{X}_N\}$ that mirrors $\{X_N\}$, whose node locations are specified by ϕ. In Fig. 6.3, we compare the simulation result with the continuum limit of $\{\hat{X}_N\}$, at $t = 1s$. We set the initial condition

$$z_0(s) = r_1 e^{-4\left((s_1+0.6)^2+(s_2+0.6)^2\right)} + r_2 e^{-3\left((s_1-0.6)^2+(s_2-0.6)^2\right)},$$

and the incoming traffic function

$$g_p(s) = r_3 e^{-4\left((s_1+0.6)^2+(s_2+0.6)^2\right)} + r_4 e^{-3\left((s_1-0.6)^2+(s_2-0.6)^2\right)},$$

where the constants $r_1, \ldots, r_4 > 0$. Again, the global characteristic of $\tilde{X}_N$ resembles $u(t, s)$, the continuum limit of $\{\hat{X}_N\}$.

CHAPTER 7

STABILITY

In this chapter we discuss the stability of the systems studied in this dissertation.

We first present a lemma on the stability of the system described by the deterministic difference equation (1.3).

Lemma 7: Assume that for each N, there exists $x_{eN} \in \mathbb{R}^N$ of (1.3) such that

$$f_N(x_{eN}) = 0, , \tag{7.1}$$

Further assume that the jacobian matrix $Df_N(0)$ of f_N at 0 has only eigenvalues with negative real part. Then for each N, for M sufficiently large, $x_{eN}$is asymptotically stable.

Proof: Define

$$\delta_N(k) = x_N(k) - x_{eN}. \tag{7.2}$$

Then we have from (1.3) that

$$\delta_N(k+1) = \delta_N(k) + \frac{1}{M} f_N(\delta_N(k) + x_{eN}). \tag{7.3}$$

By (7.1) and (7.2) we have that (7.3) has $0 \in \mathbb{R}^N$ as its equilibrium.

By the second assumption, for each N, for M sufficiently large, all eigenvalues of $I_N + \frac{1}{M} Df_N(0)$ lie in the open unit disc. Therefore 0 is asymptotically stable [106, 107, 108]. Then by (7.2), x_{eN} is equivalently asymptotically stable. ■

For the one-dimensional network model, we now provide a sufficient condition for the assumption on Df_N in the above Lemma.

Lemma 8: In the one-dimensional network case, if $P_l(n) + P_r(n) > 0$ for all n, then $Df_N(0)$ has only negative eigenvalues.

Proof: We have that

$$Df_N(0) =$$

$$\begin{bmatrix} -2(P_r(1)+P_l(1)) & P_l(2) & & & & \\ P_r(1) & -2(P_r(2)+P_l(2)) & P_l(3) & & & \\ & \ddots & & & & \\ & P_r(n-1) & -2(P_r(n)+P_l(n)) & P_l(n+1) & & \\ & & \ddots & & & \\ & & P_r(N-1) & -2(P_r(N)+P_l(N)) \end{bmatrix}$$

Let A be a complex $N \times N$ matrix, with entries a_{ij}. For $i \in \{1, \ldots, N\}$. let $R_j = \sum_{i \neq j} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the jth column. A corollary of the Gershgorin circle Theorem [109] states that every eigenvalue of A lies within at least one of the Gershgorin discs $D(a_{jj}, R_j)$, the closed disc centered at a_{jj} with radius R_j.

For $Df_N(0)$, we have that its Gershgorin disc $D(a_{jj}, R_j)$ is the closed disc centered at $a_{jj} = -(P_l(j) + P_r(j))$ with radius $R_j = P_l(j) + P_r(j)$ for $j = 2, \ldots, N-1$, $R_1 = P_r(1)$, and $R_N = P_l(N)$.

Since $Df_N(0)$ is a real tridiagonal matrix, and the signs of its entries are symmetric, its eigenvalues are real [110]. It can be shown that the determinant of $Df_N(0)$ is the continuant [111]

$$(-1)^N (N+1) \prod_{n=1}^{N} (P_l(n) + P_r(n)).$$

Therefore, since $P_l(n) + P_r(n) > 0$ for all n, $Df_N(0)$ is invertible, and consequently, 0 is not one of its eigenvalues. Therefore, by the corollary of Gershgorin circle Theorem, $Df_N(0)$ has all negative eigenvalues. ■

CHAPTER 8

CONCLUSION AND FUTURE WORK

8.1 Summary of Contributions

In this dissertation we analyze the convergence of a sequence of Markov chains to its continuum limit, the solution of a PDE, in a two-step procedure. To the best of our knowledge, for the class of Markov chains studied in this dissertation, our work is the first to directly approximate their time-space characteristics by continuum limits. Specifically, our approach directly approximates the *state* of the Markov chain, not its distribution; and it approximates the state at specific temporal and spatial points, not the number or proportion of components in a certain state. We provide precise sufficient conditions for the convergence and the explicit rate of convergence. Based on such convergence we approximate the Markov chain modeling a large wireless sensor network by a nonlinear diffusion-convection PDE.

With the well-developed mathematical tools available for PDEs, this approach provides a framework to model and simulate networks with a very large number of components, which is practically infeasible for Monte Carlo simulation. Such a tool enables us to tackle problems such as performance analysis and prototyping, resource provisioning, network design, network parametric optimization, network control, network tomography, and inverse problems, for very large networks. For example, we can now use the PDE model to optimize certain performance metrics (e.g., throughput) of a large network by adjusting the placement of destination nodes or the routing parameters (e.g., coefficients in convection terms), with relatively negligible computation overhead compared with that of the same task done by Monte Carlo simulations.

We then study the modeling of nonuniform and possibly mobile networks via nonlinear PDEs, and develop a distributed method to control their transmission-interference rules to

maintain certain global characteristics.

8.2 Discussion and Future Work

The assumption made in (4.3) that the probabilities of transmission behave continuously insures that there is a limiting behavior in the limit of large numbers of nodes and relates the behavior of networks with different numbers of nodes. The convergence results can be extended to the situation in which the probabilities change discontinuously at a finite number of lower dimensional linear manifolds (e.g., points in one dimension, lines in two dimensions, planes in three dimensions) in space provided that all of the discrete networks under consideration have nodes on the manifolds of discontinuity.

There are other considerations regarding the network that can significantly affect the derivation of the continuum model. For example, transmissions could happen beyond immediate nodes, and the interference between nodes could behave differently in the presence of power control; we can consider more boundary conditions other than sinks, including walls, semi-permeating walls, and their composition; and we can seek to establish continuum models for other domains such as the Internet, cellular networks, traffic networks, and human crowds. Our network modeling and control method can be extended to other network models. The freedom in the control method mentioned in Chapter 6 can also be further exploited to improve the network performance.

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