FLOW CHARACTERISTICS IN A DOUBLY PERIODIC PATTERN OF

INJECTION AND PRODUCTION WELL LINES FOR A MOBILITY RATIO OF 1

by

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TECHNICAL MEMORANDUM
FLOW CHARACTERISTICS IN A DOUBLY PERIODIC PATTERN OF INJECTION AND PRODUCTION WELL LINES FOR A MOBILITY RATIO OF 1 May 20, 1963

SUMMARY

A method is indicated to solve flow problems in arrays of wells when the pattern is doubly periodic. What is meant by doubly periodic pattern is illustrated in Figure 1. The whole flow field is constituted of a basic cell (ABCD) that is reproduced indefinitely and periodically in two directions with period vectors $\vec{\omega}_1$ and $\vec{\omega}_2$.

The fluid being assumed incompressible, all flow characteristics can be derived from the knowledge of the complex potential $W(z) = \varphi(x,y) + i\psi(x,y)$ or of the complex velocity $\Omega(z) = u - iv$. Because of the doubly periodic pattern of the wells, $\Omega(z)$ is a doubly periodic function. Due to the existence of sources and sinks in each basic cell, $\Omega(z)$ possesses singularities of the simple pole type.

Thus $\Omega(z)$ is an analytic, doubly-periodic function with simple pole type singularities. It is, therefore, an elliptic function.[1]

Solutions for $\Omega(z)$ and $W(z)$ have been obtained for several particular patterns. Thus equations of the potential and streamlines, values of breakthrough sweep efficiency and pressure distributions are given for the normal (square) repeated five-spot, the general (rectangular) repeated 5-spot, the repeated 2-spot, 3-spot, 7-spot and 9-spot. Many other patterns could be solved in the same manner.

B CONSTRUCTION OF $\Omega(z)$

The construction of $\Omega(z)$ is considered, first, for the case of a basic cell that contains 1 source and 1 sink (cell ABCD of Figure 1). $\Omega(z)$, in this case, is an elliptic function of order 2.

The complex potential has logarithmic singularities at the sources and the sinks. If $f(z)$ is an analytic function, $W(z) = \log\{f(z)\}$ can be a solution provided $f(z)$ possesses both poles and zeros. At the zeros of $f(z)$, $W(z)$ will show the logarithmic singularity of a source and at the poles of $f(z)$, $W(z)$ will show the logarithmic singularity of a sink. Now, $\Omega(z) = \frac{dW}{dz} = \frac{f'(z)}{f(z)}$ must be a doubly periodic function. If we choose as $f(z)$ a doubly periodic function, so is $f'(z)$ and consequently $\Omega(z)$. The periods of will be those of $f(z)$ or a fraction of them. Thus a basic cell for $f(z)$ (assuming $f(z)$ and $\Omega(z)$ have the same periods) should include a zero and a pole, where the basic well cell shows a source and a sink. But there are no elliptic functions of order 1. This difficulty can be resolved by selecting as basic cell for $f(z)$ a cell that is twice the size of the basic cell for $\Omega(z)$ (say the periods of $f(z)$ are $\vec{\omega}_1$ and $2\vec{\omega}_2$). Such a cell (AEFB) is illustrated in Figure 1. The location of the poles and zeros of $f(z)$ also are indicated in Figure 1. Then $f(z)$ is often an elliptic function of order 2 of basic cell AEFB and $\Omega(z)$ is (formally) an elliptic function of order 4 in that same cell. Since the poles and the zeros of an elliptic function determine it uniquely to a multiplicative constant, $\Omega(z) = \frac{d}{dz}\{\log[f(z)]\}$ is determined uniquely. It is easy to verify that $\Omega(z)$ degenerates into an elliptic function of order 2 for each half-cell of AEFB, i.e. is an elliptic function of order 2 in the basic cell ABCD.

C ELEMENTARY (TABULATED) JACOBIAN ELLIPTIC FUNCTIONS

The basic cells and the locations of poles and zeros for $sn(z,k)$, $cn(z,k)$ and $dn(z,k)$ are illustrated in Figure 2.

D THE REPEATED FIVE-SPOT

1 The Complex Potential

A look at Figure 3 makes it clear that the function $f(z)$ for this particular case is $cn(z+K, k.)$

The complex potential is then given by the relation

$$W(z) = \frac{1}{2\pi} \log\left\{cn(z+K,k)\right\} \qquad (1)$$

(the coefficient $\frac{1}{2\pi}$ corresponds to a choice of a source strength of 1).

The potential $\varphi(x,y) = Re\{W(z)\}$ can be explicitly obtained from formula (1). Calculations are indicated in Appendix 1 with the following result:

$$\varphi(x,y) = p(x,y) = \frac{1}{4\pi} \log\left(\frac{[1-cn^2x\ cn^2y]k'^2}{k^2cn^2x + k'^2cn^2y}\right) \qquad (2)$$

where, $cn\,x$ and $cn\,y$ are shorthand for $cn(x,k)$ and $cn(y,k')$ with k and k' being the usual complementary moduli. This notation will be followed throughout this paper.

Similarly $\psi(x,y)$ is obtained

$$\psi(x,y) = \frac{1}{2\pi} \text{Arc tang}\left(\frac{sn\,y\ dn\,y\ cn\,x}{sn\,x\ dn\,x\ cn\,y}\right) \qquad (3)$$

2 Breakthrough Areal Sweep-Efficiency

a) Square repeated five-spot

In this case (see Figure 3) $k = k' = 1/\sqrt{2}$ and $K = K' = 1.85407$ The breakthrough streamline is the diagonal line $y = x$ (for a 1/8 of the repeated five-spot). The breakthrough time t_b is

$$t_b = \int_0^K \frac{dx}{V_b}$$

where V_b is x (or y) component of the velocity along the breakthrough streamline. Now

$$V = -\frac{\partial \psi}{\partial x} = -\frac{1}{2\pi} \cdot \frac{\frac{sny\, dny}{cny}}{1 + \frac{sn^2y\, dn^2y\, cn^2x}{sn^2x\, dn^2x\, cn^2y}} \cdot \frac{d}{dx}\left(\frac{cnx}{snx\, dnx}\right)$$

Knowing that

$$\frac{d}{dx}(snx) = cnx\, dnx$$

$$\frac{d}{dx}(cnx) = -snx\, dnx$$

$$\frac{d}{dx}(dnx) = -k^2\, snx\, cnx$$

$$V = \frac{1}{2\pi} \cdot \frac{sny\, cny\, dny\, (k'^2 + k^2 cn^4x)}{sn^2x\, dn^2x\, cn^2y + sn^2y\, dn^2y\, cn^2x} \qquad (4)$$

Along the breakthrough streamline $y = x$ and since $k^2 = k'^2 = 1/2$

$$V_b = \frac{1}{8\pi} \cdot \frac{1 + cn^4x}{snx\, cnx\, dnx}$$

Thus

$$t_b = \int_0^K 8\pi \frac{snx\, dnx\, cnx\, dx}{1 + cn^4x} = -4\pi \int_0^K \frac{d(cn^2x)}{1 + cn^4x} = 4\pi \int_0^1 \frac{du}{1+u^2}$$

Finally

$$t_b = 4\pi \text{ Arc tang } 1 = \pi^2$$

The sweep efficiency (since the source strength or flow rate is 1) is

$$S.E. = \pi^2/\text{area} = \frac{\pi^2}{4KK'} = \frac{\pi^2}{4 \times (1.85407)^2} = .7178$$

The sweep efficiency for the square repeated five-spot is

$$\boxed{S.E. = 71.78\ \%} \qquad (5)$$

b) Rectangular repeated five-spot (staggered line drive)

The stream function $\psi(x,y)$ is still given by equation (3) but the breakthrough streamline is no longer a straight line. However for symmetry reason the value of ψ along this streamline is $\psi_b = \psi\left(\frac{K}{2}, \frac{K}{2}\right) = \frac{1}{8}$

Along the breakthrough streamline $\dfrac{\operatorname{sn}x \operatorname{dn}x}{\operatorname{cn}x} = \dfrac{\operatorname{sn}y \operatorname{dn}y}{\operatorname{cn}y}$

Let $\dfrac{\operatorname{sn}x \operatorname{dn}x}{\operatorname{cn}x} = \lambda(x)$. Thus $\psi(x,y) = \dfrac{1}{2\pi} \operatorname{Arc\,tang}\left[\lambda(y)/\lambda(x)\right]$

Then $$t_b = \int_0^K \frac{dx}{u} = 2\pi \int_0^K \frac{\lambda(x)^2 + \lambda(y)^2}{\lambda'(y)\,\lambda(x)}\,dx$$

The function $\lambda'(y)$ being elliptic can be expressed algebraically in terms of $\lambda(y)$. Results of calculations (Appendix 1) show that

$$\lambda'(y) = \sqrt{k^2\left(1+\lambda^2(y)\right)^2 + k'^2\left[1-\lambda^2(y)\right]^2} \tag{6}$$

Since along the breakthrough streamline $\lambda(x) = \lambda(y)$

$$t_b = 4\pi \int_0^K \frac{\lambda(x)\,dx}{\sqrt{k^2[1+\lambda^2(x)]^2 + k'^2[1-\lambda^2(x)]^2}}$$

A similar relation to (6) holds for $\lambda(x)$, so that letting $\lambda^2(x) = \omega$

$$t_b = 2\pi \int_0^\infty \frac{d\omega}{\sqrt{\{k^2(1+\omega)^2 + k'^2(1-\omega)^2\}\{k'^2(1+\omega)^2 + k^2(1-\omega)^2\}}}$$

Thus t_b is given by an elliptic integral which can be brought into standard form (Appendix 1) with the result that $t_b = 2\pi K\left[(k^2-k'^2)^2\right]$

and the sweep efficiency is

$$S.E. = \frac{\pi K\{(k^2-k'^2)^2\}}{2KK'} \tag{7}$$

A plot of sweep efficiency versus d/a is given in Figure 9.

E THE REPEATED TWO-SPOT

1 The Complex Potential

From Figure 4 it is clear that the function $f(z)$ is $\operatorname{sn}(z,k)$.

$$\varphi(x,y) = \frac{1}{4\pi} \log \left(\frac{\quad}{cn^2 y + k^2 sn^2 x \, sn^2 y} \right)$$

Therefore,

$$W(z) = \frac{1}{2\pi} \log (sn\, z) \tag{8}$$

from which (Appendix 2) we derive

$$\varphi(x,y) = \frac{1}{4\pi} \log \left(\frac{sn^2 x + sn^2 y - sn^2 x \, sn^2 y}{1 - sn^2 y + k^2 sn^2 x \, sn^2 y} \right) \quad \text{or} \tag{9}$$

and

$$\psi(x,y) = \frac{1}{2\pi} \operatorname{Arctang} \left(\frac{cnx \, dnx \, sny \, cny}{snx \, dny} \right) \tag{10}$$

2 Breakthrough Sweep Efficiency

The breakthrough time (Appendix 2) is obtained easily

$$t_b = -\frac{\pi}{1-k^2} \log (k^2)$$

and

$$S.E. = \frac{-\pi}{1-k^2} \cdot \frac{\log (k^2)}{4KK'} \tag{11}$$

A plot of sweep efficiency versus d/a is given in Figure 10.

F THE REPEATED 3-SPOT (FIGURE 5)

1 The Complex Potential

The complex velocity $\Omega(z)$ in a basic cell possesses a source of strength 2 and 2 sinks of strength 1. The function $f(z)$ must have a double zero at A, a pole at P and a pole at Q. In the cell ABCD (basic cell for $sn\, z$) $f(z)$ is an elliptic function of order 4. The simplest possible algebraic relation between $f(z)$ and $sn\, z$ is a ratio of quadratic expressions in $sn\, z$.

The function $\frac{1}{sn^2 z}$ has a double pole at A (its inverse has a double zero at A). The zeros of $\frac{1}{sn^2 z}$ are double zeros where $sn\, z$ has poles. However, the zeros of $\frac{1}{sn^2 z} + \lambda^2$ are simple zeros (Appendix 3) and the function $\frac{1}{sn^2 z} + \lambda^2$ still has a double pole at A. Thus the function $f(z) = \frac{sn^2 z}{1+\lambda^2 sn^2 z}$ has double zeros at A and A', simple poles at P, P', Q and Q'. To fit the particular repeated 3-spot of Figure 5 it suffices now to give η the value $2K'/3$.

Then $sn^2\left(\frac{2K'}{3}, k'\right) = \frac{1}{1+\lambda^2}$ $\quad \lambda^2 = cs^2\left(\frac{2K'}{3}, k'\right)$

The complex potential is:

$$W(z) = \frac{1}{2\pi}\log\left(\frac{sn^2 z}{1+\lambda^2 sn^2 z}\right) = \frac{1}{\pi}\log(sn\, z) - \frac{1}{2\pi}\log(1+\lambda^2 sn^2 z) \tag{12}$$

Consequently (Appendix 3)

$$\varphi(x,y) = \frac{1}{2\pi}\log(1-cn^2x\, cn^2y) - \frac{1}{4\pi}\log\left\{cn^2y + k^2 sn^2x\, sn^2y)^2 + 2\lambda^2(sn^2x\, dn^2y - cn^2x\, dn^2x\, sn^2y\, cn^2y) + \lambda^4(1-cn^2x\, cn^2y)^2\right\} \tag{13}$$

In the case $\lambda = 0$ this formula reduces to (9).

The stream function is obtained as usual

$$\psi(x,y) = \frac{1}{\pi}\,\mathrm{Arctang}\left(\frac{cnx\, dnx\, sny\, cny}{snx\, dny}\right) - \frac{1}{2\pi}\,\mathrm{Arctang}\left\{\frac{2\lambda^2 snx\, cnx\, dnx\, sny\, cny\, dny}{(cn^2y + k^2 sn^2x\, sn^2y)^2 + \lambda^2(sn^2x\, dn^2y - cn^2x\, dn^2x\, sn^2y\, cn^2y)}\right\} \tag{14}$$

2 Breakthrough Sweep Efficiency for a Repeated 3-Spot

Calculations (Appendix 3) yield the following:

$$S.E. = \frac{-\pi}{4KK'}\left\{\frac{k^2+\lambda^2}{k^2 k'^2}\log(dn^2\eta) - \frac{\lambda^2}{k^2}\log(cn^2\eta)\right\} \tag{15}$$

In the case $\eta = K'$ $\lambda = 0$ and formula (15) reduces to formula (11).

For the case corresponding to Figure 5, $\eta = \frac{2K'}{3}$ $\lambda^2 = cs^2\left(\frac{2K'}{3}\right)$

Formula (15) becomes

$$S.E. = \frac{-\pi}{4KK'}\left\{\frac{ds^2(2K'/3)}{k^2k'^2}\log\left(dn^2\frac{2K'}{3}\right) - \frac{cs^2(2K'/3)}{k^2}\log cn^2\left(\frac{2K'}{3}\right)\right\}$$

A plot of sweep-efficiency versus $\frac{K'}{3K} = \frac{d}{a}$ is given in Figure 11.

G THE SQUARE REPEATED 9-SPOT (FIGURE 6)

The function $f(z)$ is obtained easily by mere inspection. Assuming strengths of sources to be 3 for the central injector A, p for B then at C it is $q = 3-2p$ and

$$f(z) = (sn\, z)^3\left\{\frac{1}{dn(z+iK')}\right\}^p\left[cn(z+K)\right]^q = \frac{sn^3 z}{(cn\, z)^p(dn\, z)^{3-2p}} \qquad (16)$$

Therefrom

$$W(z) = \frac{3}{2\pi}\log(sn\, z) - \frac{p}{2\pi}\log(cn\, z) - \frac{(3-2p)}{2\pi}\log(dn\, z) \qquad (17)$$

and (Appendix 4)

$$\begin{aligned}\varphi(x,y) = {} & \frac{3}{4\pi}\log(1-cn^2x\, cn^2y) - \frac{(3-2p)}{4\pi}\log(cn^2x + cn^2y) \\ & -\frac{p}{4\pi}\left\{\log(2cn^2x + sn^2x\, sn^2y) + \log(2cn^2y + sn^2x\, sn^2y)\right\}\end{aligned} \qquad (18)$$

Evaluation of sweep efficiency (Appendix 4) along the x-axis yields

$$S.E. = \frac{\pi}{4KK'}\sqrt{\frac{3}{3-2p}}\left\{2\log\left(1+\sqrt{\frac{3-2p}{3}}\right) + \log\left(\frac{3}{2p}\right)\right\} \qquad (19)$$

along the diagonal line AC

$$S.E. = \frac{\pi^2}{4KK'}\sqrt{\frac{3}{3-2p}} \tag{20}$$

The 2 sweep efficiencies are the same when

$$\pi = 2\log\left(1+\sqrt{\frac{3-2p}{3}}\right) + \log\left(\frac{3}{2p}\right)$$

$$= \log\left\{\frac{1}{p}\left(3-p+3\sqrt{\frac{3-2p}{3}}\right)\right\} = \log v$$

$$\log_{10} v = \frac{\pi}{M} = \frac{3.1416}{2.3026} = 1.3644 \qquad v = 23.14$$

Thus: $$3-p+3\sqrt{\frac{3-2p}{3}} = 23.14p$$

for which the solution is $$p = \frac{6\times 23.14}{(24.14)^2} = .238$$

For this value of p the sweep efficiency for a repeated 9-spot is

$$\frac{.7178}{\sqrt{1-\frac{2\times .238}{3}}} = \frac{.7178}{\sqrt{.8413}} = \frac{.7178}{.9172} = .7826$$

The optimum sweep-efficiency for a repeated 9-spot is

$$S.E. = 78.26\ \%$$

which is attained for a ratio of sink's strength $\frac{q}{p} = \frac{2.524}{.238} = 10.6$

Sweep efficiency versus p is given in Figure 12.

H THE REPEATED 7-SPOT (FIGURE 7)

The function $f(z)$ can be obtained as a combination of the function $f(z)$ for the particular repeated 3-spot when $\eta = \frac{2K'}{3}$ and of $f(z-K-iK')$

Thus

$$f(z) = \frac{sn^2 z}{(1+\lambda^2 sn^2 z)}\cdot\frac{sn^2(z-K-iK')}{(1+\lambda^2 sn^2(z-K-iK'))} \tag{21}$$

where $\lambda^2 = cs^2\left(\frac{2K'}{3}, k'\right) = \frac{2-\sqrt{3}}{2\sqrt{3}}$, $\frac{K'}{K} = \sqrt{3}$ and

$$k^2 = \sin^2(\pi/12) = \frac{2-\sqrt{3}}{4}$$

From (21) the complex potential is obtained (Appendix 5)

$$W(z) = \frac{1}{\pi}\log(\operatorname{sn} z) + \frac{1}{\pi}\log(\operatorname{dn} z) - \frac{1}{2\pi}\log(1+\lambda^2 \operatorname{sn}^2 z) - \frac{1}{2\pi}\log(1+\mu^2 \operatorname{sn}^2 z) \qquad (22)$$

with $\mu^2 = -\frac{k^2(1+\lambda^2)}{k^2+\lambda^2} = -k^2 \operatorname{nd}^2\left(\frac{2K'}{3}\right) = -\frac{1}{2}$

The potential is obtained easily (Appendix 5) with the result, expliciting some constants:

$$\varphi(x,y) = \frac{1}{2\pi}\log(1-\operatorname{cn}^2 x \operatorname{cn}^2 y) + \frac{1}{2\pi}\log\left\{(2-\sqrt{3})\operatorname{cn}^2 x + (2+\sqrt{3})\operatorname{cn}^2 y\right\}$$
$$-\frac{1}{4\pi}\log\left\{\left(\operatorname{cn}^2 y + \frac{2-\sqrt{3}}{4}\operatorname{sn}^2 x \operatorname{sn}^2 y\right)^2 + \frac{2-\sqrt{3}}{\sqrt{3}}\left(\operatorname{sn}^2 x \operatorname{dn}^2 y - \operatorname{cn}^2 x \operatorname{dn}^2 x \operatorname{sn}^2 y \operatorname{cn}^2 y\right) + \frac{(2-\sqrt{3})^2}{12}\left(1-\operatorname{cn}^2 x \operatorname{cn}^2 y\right)^2\right\}$$
$$-\frac{1}{4\pi}\log\left\{4\left(\operatorname{cn}^2 y + \frac{(2-\sqrt{3})}{4}\operatorname{sn}^2 x \operatorname{sn}^2 y\right)^2 - 4\left(\operatorname{sn}^2 x \operatorname{dn}^2 y - \operatorname{cn}^2 x \operatorname{dn}^2 x \operatorname{sn}^2 y \operatorname{cn}^2 y\right) + \left(1-\operatorname{cn}^2 x \operatorname{cn}^2 y\right)^2\right\}$$
$$+ C \qquad C = -\frac{1}{2\pi}\log 2$$

The sweep efficiency is obtained in the usual manner (Appendix 5) with the result

$$S.E. = 74.37\,\%$$

I SINGLY PERIODIC ARRAYS (SEE FIGURE 8)

The singly periodic function with zeros at ABC... is $\sin z$. Thus for the elementary pattern of one source and one sink in a strip

$$f(z) = \frac{\sin z}{1 + i\mu \sin z}$$

The complex potential is

$$W(z) = \frac{1}{2\pi}\log\left(\frac{\sin z}{1+i\mu\sin z}\right) \qquad (24)$$

The constant μ is such that $i\mu \sin z + 1 = 0$ for $x = 0, \pi ..$ $y = \eta$

Thus

$$i\mu \sin(i\eta) = -\mu \operatorname{Sh}\eta = -1 \qquad \mu = 1/\operatorname{sh}\eta$$

No

and $$W(z) = \frac{1}{2\pi} \log\left(\frac{\sin z \, Sh\eta}{Sh\eta + i \sin z}\right)$$

from which we derive

$$\varphi(x,y) = \frac{1}{4\pi} \log\left(\frac{\sin^2 x + Sh^2 y}{\sin^2 x + Sh^2 y - 2 Sh\eta \, Shy \cos x + Sh^2\eta}\right) \tag{25}$$

$$\psi(x,y) = \frac{1}{2\pi} \text{Arc tang}\left(\frac{Sh\eta \, Shy \cos x - \sin^2 x - Sh^2 y}{Sh\eta \sin x \, Chy}\right) \tag{26}$$

Solutions to complex problems are obtained by superposition of the elementary solution for $f(z)$.

J ISOLATED (N+1) SPOT

When there is no periodicity at all, solutions are obtained by superposition. The function $f(z)$ is a rational fraction of z .

For N sinks on a circle of radius 1 surrounding a source $f(z) = \frac{z^N}{z^N - 1}$ and therefore

$$W(z) = \frac{1}{2\pi} \log z - \frac{1}{2\pi N} \log(z^N - 1) \tag{27}$$

From $\frac{dW}{dz} = u - iv = \frac{1}{2\pi z} - \frac{z^{N-1}}{2\pi(z^N - 1)}$

we find u on the breakthrough streamline.

$$u = \frac{1}{2\pi x} - \frac{x^{N-1}}{2\pi(x^N - 1)} = \frac{-1}{2\pi x(x^N - 1)}$$

The breakthrough time is

$$t_b = 2\pi \int_0^1 (x - x^{N+1}) \, dx = \pi \frac{N}{N+2}$$

The sweep efficiency (relative to the circle) is

$$S.E. = \frac{N}{N+2} \tag{28}$$

K ISOLATED (N+1) SPOT WITH LEAK AT INFINITY

This is an immediate extension of § J . In this case

$$f(z) = \frac{z^{N(1+\alpha)}}{z^N - 1} \tag{29}$$

and therefore

$$W(z) = \frac{1}{2\pi}\log z - \frac{1}{2\pi N(1+\alpha)}\log(z^N - 1) + C$$

The potential is

$$\varphi(x,y) = \frac{1}{4\pi}\log|z|^2 - \frac{1}{4\pi N(1+\alpha)}\log|z^N - 1|^2$$

In the case N = 4 (isolated 5-spot)

$$\varphi(x,y) = \frac{1}{4\pi}\log(x^2+y^2) - \frac{1}{16\pi(1+\alpha)}\log\left\{\left[(x^2+y^2)^2 - 1\right]^2 + 16x^2y^2\right\}$$

In polar coordinates

$$\varphi^*(r,\theta) = \frac{1}{16\pi}\log\left\{\frac{r^8}{(r^8 - 2r^4\cos 4\theta + 1)^{\beta}}\right\} \tag{30}$$

where $\beta = \frac{1}{1+\alpha}$ = production efficiency

For a given value of φ^* and of r , θ can be easily obtained

$$\theta = \frac{1}{4}\arc\cos\left\{\frac{1 + r^8 - \left(r^8 e^{-16\pi\varphi^*}\right)^{1+\alpha}}{2r^4}\right\} \tag{31}$$

The equation of a streamline is in polar coordinates

$$\psi^*(r,\theta) = \frac{\theta}{2\pi} + \frac{\beta}{8\pi}\,\mathrm{Arc\,tang}\left(\frac{r^4\sin 4\theta}{1 - r^4\cos 4\theta}\right) \tag{32}$$

For a given value of ψ^* and of θ , r is easily obtained

$$r = \left\{\frac{\mathrm{tang}\left[4(2\pi\psi^* - \theta)(1+\alpha)\right]}{\sin 4\theta + \cos 4\theta\,\mathrm{tang}\left[4(2\pi\psi^* - \theta)(1+\alpha)\right]}\right\}^{1/4} \tag{33}$$

Breakthrough time is $t_b = \int_0^1 \frac{dr}{\left(\frac{\partial\varphi}{\partial r}\right)_{\theta=0}}$

For $\theta = 0$ $\varphi^*(r,0) = \frac{1}{2\pi}\log r - \frac{1}{8\pi(1+\alpha)}\log(r^4-1)$

$$\left(\frac{\partial\varphi^*}{\partial r}\right)_{\theta=0} = \frac{1}{2\pi}\left\{\frac{1}{r} + \frac{r^3}{(1+\alpha)(1-r^4)}\right\} = \frac{1+\alpha-\alpha r^4}{2\pi(1+\alpha)r(1-r^4)}$$

$$t_b = 2\pi(1+\alpha)\int_0^1 \frac{r(1-r^4)}{1+\alpha-\alpha r^4}\,dr$$

Let $r^2 = u$ $\qquad t_b = \pi(1+\alpha)\int_0^1 \frac{(1-u^2)\,du}{1+\alpha-\alpha u^2}$

For $\alpha > 0$ we can integrate with the result

$$t_b = \frac{\pi(1+\alpha)}{\alpha}\left\{1 - \frac{1}{\sqrt{\alpha(1+\alpha)}}\log\left(\sqrt{1+\alpha}+\sqrt{\alpha}\right)\right\} \qquad (34)$$

Therefrom

$$S.E. = \frac{1+\alpha}{\alpha}\left\{1 - \frac{1}{\sqrt{\alpha(1+\alpha)}}\log\left(\sqrt{\alpha}+\sqrt{1+\alpha}\right)\right\}$$

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APPENDIX 1

1 Potential for the ~~repeated five-spot~~ staggered line drive

$$\varphi(x,y) = \frac{1}{2\pi}\log|cn(x+K+iy)|$$

Let us calculate $cn(x+K+iy)$

$$cn(x+K+iy) = \frac{cn(x+K)cn(iy) - sn(x+K)sn(iy)\,dn(x+K)\,dn\,iy}{1-k^2 sn^2(x+K)\,sn^2(iy)}$$

Knowing that $sn(iy,k) = i\,sn(y,k')/cn(y,k') = i\,sny/cny$

$$cn(iy) = 1/cny$$

$$dn(iy) = dny/cny$$

$$cn(x+K+iy) = \frac{cn(x+K)cny - i\,sn(x+K)dn(x+K)sny\,dny}{cn^2y + k^2 sn^2(x+K)\,sn^2y}$$

Now $sn(x+K) = cnx/dnx$

$$cn(x+K) = -k'snx/dnx$$

$$dn(x+K) = k'/dnx$$

$$cn(x+K+iy) = \frac{-k'snx\,dnx\,cny - ik'cnx\,sny\,dny}{cn^2y\,dn^2x + k^2cn^2x\,sn^2y}$$

$$|cn(x+K+iy)|^2 = \frac{k'^2(sn^2x\,dn^2x\,cn^2y + cn^2x\,sn^2y\,dn^2y)}{(dn^2x\,cn^2y + k^2cn^2x\,sn^2y)^2}$$

Since $sn^2x = 1-cn^2x \qquad sn^2y = 1-cn^2y$

$$dn^2x = k'^2+k^2cn^2x \qquad dn^2y = 1-k'^2sn^2y$$

$$= 1-k^2sn^2x$$

$$\frac{|cn(x+K+iy)|^2}{k'^2} = \frac{(1-cn^2x)(k'^2+k^2cn^2x)cn^2y + cn^2x(1-cn^2y)(k^2+k'^2cn^2y)}{[cn^2y(k'^2+k^2cn^2x) + k^2cn^2x(1-cn^2y)]^2}$$

$$\frac{|cn(x+K+iy)|^2}{k'^2} = \frac{k'^2 cn^2 y\left[(1-cn^2 x)+cn^2 x\,(1-cn^2 y)\right]+k^2 cn^2 x\left[(1-cn^2 x)cn^2 y+(1-cn^2 y)\right]}{(k'^2 cn^2 y+k^2 cn^2 x)^2}$$

$$= \frac{(k'^2 cn^2 y + k^2 cn^2 x)(1-cn^2 x\, cn^2 y)}{(k'^2 cn^2 y + k^2 cn^2 x)^2} = \frac{1-cn^2 x\, cn^2 y}{k'^2 cn^2 y + k^2 cn^2 x}$$

Finally $$\varphi(x,y) = \frac{1}{4\pi} \log\left\{\frac{(1-cn^2 x\, cn^2 y)k'^2}{k^2 cn^2 x + k'^2 cn^2 y}\right\} \qquad (2)$$

An equipotential (or equipressure) line has the following equation

$$cn^2 x\, cn^2 y + (k^2 cn^2 x + k'^2 cn^2 y)\, e^{4\pi\varphi_0} - 1 = 0$$

2 Stream function for the ~~repeated five-spot~~ staggered line drive

$$\psi(x,y) = Im\left\{\log cn\,(z+K)\right\}$$

From the expression of $cn(x+K+iy)$ it follows that

$$\psi(x,y) = \frac{1}{2\pi}\, Arc\ tang\left(\frac{sn\,y\, dn\,y\, cn\,x}{sn\,x\, dn\,x\, cn\,y}\right) \qquad (3)$$

The equation of a streamline is

$$tang^2(2\pi\psi_0) = (1-cn^2 y)\,cn^2 x\,(k^2+k'^2 cn^2 y)\Big/(1-cn^2 x)\,cn^2 y\,(k'^2+k^2 cn^2 x)$$

Let $tang^2(2\pi\psi_0)\,(1-cn^2 x)(k'^2+k^2 cn^2 x)/cn^2 x = A(x,\psi_0) = A > 0$

$$A\, cn^2 y - (1-cn^2 y)(k^2+k'^2 cn^2 y) = 0 \qquad k^2 = m \quad k'^2 = m'$$

$$m' cn^4 y + (A + m - m')\,cn^2 y - m = 0$$

The only acceptable root of this equation is

$$cn^2 y = \frac{-(A+m-m') + \sqrt{(A+m-m')^2 + 4mm'}}{2m'}$$

$$cn\,y = \sqrt{\frac{-(A+1-2m') + \sqrt{(A+1)^2 - 4m'A}}{2m'}}$$

3 Sweep efficiency for the general repeated five-spot

Defining as $\lambda(x)$ the function $\frac{\operatorname{sn}x \operatorname{dn}x}{\operatorname{cn}x}$, the integral for t_b is

$$\int_0^K \frac{dx}{\frac{\partial\psi}{\partial y}} = 4\pi \int_0^K \frac{\lambda(x)}{\lambda'(y)}\, dx$$

$$\lambda'(y) = \frac{(\operatorname{sn}y \operatorname{dn}y)'\operatorname{cn}y - (\operatorname{cn}y)'\operatorname{sn}y \operatorname{dn}y}{(\operatorname{cn}y)^2}$$

$$\lambda'(y) = \frac{\operatorname{cn}^2y \operatorname{dn}^2y - k'^2\operatorname{sn}^2y \operatorname{cn}^2y + \operatorname{sn}^2y \operatorname{dn}^2y}{\operatorname{cn}^2y} = \frac{k'^2\operatorname{cn}^4y + k^2}{\operatorname{cn}^2y}$$

$$\lambda'(y) = k'^2\operatorname{cn}^2y + \frac{k^2}{\operatorname{cn}^2y} = k'^2\alpha + k^2\beta$$

Now $\lambda^2(y)$ can be expressed in terms of cn^2y

$$\lambda^2(y) = \frac{\operatorname{sn}^2y \operatorname{dn}^2y}{\operatorname{cn}^2y} = \frac{(1-\operatorname{cn}^2y)(k^2 + k'^2\operatorname{cn}^2y)}{\operatorname{cn}^2y}$$

$$\lambda^2(y) = k^2/\operatorname{cn}^2y + (k'^2 - k^2) - k'^2\operatorname{cn}^2y = -k'^2\alpha + k^2\beta + (k'^2 - k^2)$$

Solution of the system of equations

$$m'\alpha + m\beta = \lambda'(y)$$
$$-m'\alpha + m\beta = \lambda^2(y) + m - m'$$

gives

$$\beta = \frac{1}{2m}\left[\lambda'(y) + \lambda^2(y) + m - m'\right]$$

$$\alpha = \frac{1}{2m'}\left[\lambda'(y) - \lambda^2(y) - (m - m')\right]$$

and since $\alpha\beta = 1$

$$\lambda'(y)^2 - \left[\lambda^2(y) + (m - m')\right]^2 = 4mm'$$

$$\lambda'(y) = \lambda^4(y) + 2(m-m')\lambda^2(y) + (m-m')^2 + 4mm'$$

Since $m + m' = 1$ and $(m-m')^2 + 4mm' = (m+m')^2 = 1$

$$\lambda'(y) = (m+m')\lambda^4(y) + 2(m-m')\lambda^2(y) + (m+m')$$

$$\lambda'(y) = m[\lambda^2(y)+1]^2 + m'[\lambda^2(y)-1]^2$$

Since we know that $\lambda'(y)$ is > 0

$$\lambda'(y) = \sqrt{k^2[1+\lambda^2(y)]^2 + k'^2[1-\lambda^2(y)]^2}$$

Similarly $\quad \lambda'(x) = \sqrt{k'^2[1+\lambda^2(x)]^2 + k^2[1-\lambda^2(x)]^2}$

The integral for t_b becomes $\quad 2\pi \int_0^K \frac{d[\lambda^2(x)]}{\lambda'(x)\,\lambda'(y)}$

Letting $\lambda^2(x) = \omega = \lambda^2(y)$ on the breakthrough streamline

$$t_b = 2\pi \int_0^\infty \frac{d\omega}{\sqrt{\left\{k^2(1+\omega)^2 + k'^2(1-\omega)^2\right\}\left\{k'^2(1+\omega)^2 + k^2(1-\omega)^2\right\}}}$$

Rewriting the expression under the radical, and observing that the change of variable of ω into $\frac{1}{\omega}$ leaves the integral unchanged.

$$t_b = 4\pi \int_0^1 \frac{d\omega}{\sqrt{(\omega^2 + 2(m-m')\omega + 1)(\omega^2 + 2\omega(m'-m) + 1)}}$$

Using Legendre's procedure to bring this integral to standard form, we let

$\omega = \frac{1-Y}{1+Y}$, and $m - m' = \Delta m$

$$t_b = \frac{4\pi}{\sqrt{1-\Delta m^2}} \int_0^1 \frac{dy}{\sqrt{\left(y^2 + \frac{1+\Delta m}{1-\Delta m}\right)\left(y^2 + \frac{1-\Delta m}{1+\Delta m}\right)}}$$

$$= \frac{2\pi}{kk'} \int_0^1 \frac{dy}{\sqrt{\left(y^2 + \frac{k^2}{k'^2}\right)\left(y^2 + \frac{k'^2}{k^2}\right)}} = \frac{2\pi}{kk'} I$$

Tables of standard elliptic integrals[2] indicate that the integral I is

$$\frac{k'}{k}\, sn^{-1}\left\{k,\ \frac{\sqrt{\Delta m}}{m}\right\}$$

The sweep efficiency is

$$S.E. = \frac{\pi}{2KK'} \cdot \frac{sn^{-1}}{k^2}\left(k,\ \frac{\sqrt{k^2 - k'^2}}{k^2}\right)$$

Prats[3] gives the more elegant formula

$$S.E. = \frac{\pi}{2KK'}\, K\left\{\left|k^2 - k'^2\right|^{\cancel{2}}\right\} \qquad (7)$$

$$\text{i.e. } K(k^*)$$

$$k^* = |k^2 - k'^2| = |2k^2 - 1|$$

APPENDIX 2

1 Potential for the repeated two-spot

Since $W(z) = \frac{1}{2\pi} \log(\mathrm{sn}\, z)$

$$\varphi(x,y) = \frac{1}{2\pi} \log|\mathrm{sn}[x+iy]|$$ we calculate $\mathrm{sn}(x+iy)$

$$\mathrm{sn}(x+iy) = \frac{\mathrm{sn}x\, \mathrm{cn}(iy)\, \mathrm{dn}(iy) + \mathrm{sn}(iy)\, \mathrm{cn}x\, \mathrm{dn}x}{1 - k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2(iy)}$$

Using Jacobi's formulae of transformation for imaginary arguments leads to

$$\mathrm{sn}(x+iy) = \frac{\mathrm{sn}x\,\mathrm{dn}y + i\,\mathrm{cn}x\,\mathrm{dn}x\,\mathrm{sn}y\,\mathrm{cn}y}{\mathrm{cn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y}$$

$$|\mathrm{sn}(x+iy)|^2 = \frac{\mathrm{sn}^2 x\, \mathrm{dn}^2 y + \mathrm{cn}^2 x\, \mathrm{dn}^2 x\, \mathrm{sn}^2 y\, \mathrm{cn}^2 y}{(\mathrm{cn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y)^2}$$

$$= \frac{\mathrm{sn}^2 x(1 - k'^2 \mathrm{sn}^2 y) + \mathrm{sn}^2 y\,(1 - k^2 \mathrm{sn}^2 x)(1 - \mathrm{sn}^2 x)(1 - \mathrm{sn}^2 y)}{(1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y)^2}$$

$$= \frac{\mathrm{sn}^2 x + \mathrm{sn}^2 y - (k^2 + k'^2)\mathrm{sn}^2 x\, \mathrm{sn}^2 y - \mathrm{sn}^2 y(1 - k^2 \mathrm{sn}^2 x)[\mathrm{sn}^2 x + \mathrm{sn}^2 y - \mathrm{sn}^2 x\, \mathrm{sn}^2 y]}{(1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y)^2}$$

$$= \frac{(\mathrm{sn}^2 x + \mathrm{sn}^2 y - \mathrm{sn}^2 x\, \mathrm{sn}^2 y)(1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y)}{(1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y)^2}$$

$$= \frac{\mathrm{sn}^2 x + \mathrm{sn}^2 y - \mathrm{sn}^2 x\, \mathrm{sn}^2 y}{1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y}$$

$$\varphi(x,y) = \frac{1}{4\pi} \log\left(\frac{\mathrm{sn}^2 x + \mathrm{sn}^2 y - \mathrm{sn}^2 x\, \mathrm{sn}^2 y}{1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\, \mathrm{sn}^2 y}\right)$$

$$\psi(x,y) = \mathrm{Im}\{W(z)\} = \frac{1}{2\pi} \mathrm{Arctang}\left(\frac{\mathrm{cn}x\,\mathrm{dn}x\,\mathrm{sn}y\,\mathrm{cn}y}{\mathrm{sn}x \quad \mathrm{dn}y}\right)$$

The calculations of the velocity components u and v are straightforward with the results:

$$u = \frac{1}{2\pi} \frac{\mathrm{sn}x\,\mathrm{cn}x\,\mathrm{dn}x\,(\mathrm{cn}^4 y - k^2\mathrm{sn}^4 y)}{(\mathrm{sn}^2 x + \mathrm{sn}^2 y - \mathrm{sn}^2 x\,\mathrm{sn}^2 y)(1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\,\mathrm{sn}^2 y)}$$

$$v = \frac{1}{2\pi} \frac{\mathrm{sn}y\,\mathrm{cn}y\,\mathrm{dn}y\,(1 - k^2\mathrm{sn}^4 x)}{(\mathrm{sn}^2 x + \mathrm{sn}^2 y - \mathrm{sn}^2 x\,\mathrm{sn}^2 y)(1 - \mathrm{sn}^2 y + k^2 \mathrm{sn}^2 x\,\mathrm{sn}^2 y)}$$

2 Breakthrough sweep efficiency

Along the breakthrough streamline (x = 0) which corresponds to

$$\psi = \frac{1}{4}\ , \quad v = \frac{1}{2\pi}\frac{\mathrm{dn}y}{\mathrm{sn}y\,\mathrm{cn}y}$$

Thus

$$t_b = \int_0^{K'} 2\pi \frac{\mathrm{sn}y\,\mathrm{cn}y}{\mathrm{dn}y}\,dy = 2\pi \int_0^{K'} \frac{\mathrm{sn}y\,\mathrm{cn}y\,\mathrm{dn}y}{\mathrm{dn}^2 y}\,dy$$

Since

$$\frac{d}{dy}(\mathrm{dn}y) = -k'^2\,\mathrm{sn}y\,\mathrm{cn}y$$

$$t_b = -\frac{\pi}{k'^2}\int_0^{K'} \frac{d(\mathrm{dn}^2 y)}{\mathrm{dn}^2 y} = -\frac{\pi}{k'^2}\log\left(\mathrm{dn}^2(K', k')\right)$$

and using the relations

$$\mathrm{dn}^2(K', k') = 1 - k'^2\mathrm{sn}^2(K', k') = 1 - k'^2 = k^2$$ one obtains finally

$$t_b = -\pi/k'^2 \log(k^2) \tag{11}$$

and

$$S.E. = -\frac{\pi}{k'^2}\frac{\log(k^2)}{4KK'}$$

APPENDIX 3

1 Zeros of $\frac{1}{sn^2 z} + \lambda^2$ (λ^2 is a real parameter)

Let us write $sn\, z$ explicitly. Equation $1+\lambda^2 sn^2 z = 0$ becomes

$$1+\lambda^2\left(\frac{snx\, dny + i\, cnx\, dnx\, sny\, cny}{cn^2 y + k^2 sn^2 x\, sn^2 y}\right)^2 = 0$$

For $x=0$, $x=2K$ $sn(x)=0$ and the equation in y is

$$1+\lambda^2\left(\frac{i\, sny\, cny}{cn^2 y}\right)^2 = 1-\frac{\lambda^2 sn^2 y}{cn^2 y} = \frac{1-(1+\lambda^2) sn^2 y}{cn^2 y} = 0$$

Letting $y = sn^{-1}\left(\frac{1}{\sqrt{1+\lambda^2}}, k'\right) = \eta$ roots of $1+\lambda^2 sn^2 z = 0$

are $(0,\eta),(2K,\eta)$ and points congruent to $(0,-\eta),(2K,-\eta)$

Since $1+\lambda^2 sn^2 z$ is an elliptic function of order 4 there are not other roots.

The zeros of $\frac{1+\lambda^2 sn^2 z}{sn^2 z}$ are located at points of coordinates $(0,\eta)$; $(2K,\eta)$; $(0, 2K'-\eta)$; $(2K, 2K'-\eta)$

The complex potential is made of 2 terms:

$$W(z) = \frac{1}{\pi}\log(sn\, z) - \frac{1}{2\pi}\log(1+\lambda^2 sn^2 z) \tag{12}$$

Consequently,

$$\varphi(x,y) = \frac{1}{2\pi}\log\left(\frac{1-cn^2 x\, cn^2 y}{cn^2 y + k^2 sn^2 x\, sn^2 y}\right) - \frac{1}{2\pi} Re\left\{\log\left(1+\lambda^2 sn^2 z\right)\right\}$$

Let us calculate $sn^2 z$ explicitly

From Appendix 2 $sn(x+iy) = \dfrac{snx\, dny + i\, cnx\, dnx\, sny\, cny}{cn^2 y + k^2 sn^2 x\, sn^2 y}$

$$sn^2(x+iy) = \frac{sn^2 x\, dn^2 y - cn^2 x\, dn^2 x\, sn^2 y\, cn^2 y + 2i\, snx\, cnx\, dnx\, sny\, cny\, dny}{(cn^2 y + k^2 sn^2 x\, sn^2 y)^2}$$

$$1+\lambda^2 sn^2 z = \frac{(cn^2 y + k^2 sn^2 x\, sn^2 y)^2 + \lambda^2(sn^2 x\, dn^2 y - cn^2 x\, dn^2 x\, sn^2 y\, cn^2 y) + 2i\lambda^2 snx\, cnx\, dnx\, sny\, cny\, dny}{(cn^2 y + k^2 sn^2 x\, sn^2 y)^2}$$

$$|1+\lambda^2 sn^2 z|^2 = \frac{\left[\left(cn^2y + k^2 sn^2x\, sn^2y\right)^2 + \lambda^2\left(sn^2x\, dn^2y - cn^2x\, dn^2x\, sn^2y\, cn^2y\right)\right]^2 + 4\lambda^4 sn^2x\, cn^2x\, dn^2x\, sn^2y\, cn^2y\, dn^2y}{\left[cn^2y + k^2 sn^2x\, sn^2y\right]^4}$$

$$|1+\lambda^2 sn^2 z|^2 = \frac{\left(cn^2y + k^2 sn^2x\, sn^2y\right)^4 + 2\lambda^2\left(cn^2y + k^2 sn^2x\, sn^2y\right)^2\left(sn^2x\, dn^2y - cn^2x\, dn^2x\, sn^2y\, cn^2y\right) + \lambda^4\left[sn^2x\, dn^2y + cn^2x\, dn^2x\, sn^2y\, cn^2y\right]^2}{\left(cn^2y + k^2 sn^2x\, sn^2y\right)^4}$$

Now $sn^2x\, dn^2y + cn^2x\, dn^2x\, sn^2y\, cn^2y$ can be written

$$sn^2x\left(1-k'^2 sn^2y\right) + \left(1-sn^2x\right)\left(1-k^2 sn^2x\right) sn^2y\left(1-sn^2y\right) =$$

$$sn^2x\left(1-k'^2 sn^2y\right) + \left(1-k^2 sn^2x\right)\left(sn^2y - sn^2x\, sn^2y - sn^4y + sn^2x\, sn^4y\right) =$$

$$sn^2x\left[1 - sn^2y\left(1-k^2 sn^2x\right)\right] + sn^2y\left[-k'^2 sn^2x + 1 - k^2 sn^2x - sn^2y\left(1-sn^2x\right)\left(1-k^2 sn^2x\right)\right] =$$

$$sn^2x\left[1-sn^2y\left(1-k^2 sn^2x\right)\right] + sn^2y\left(1-sn^2x\right)\left[1-sn^2y\left(1-k^2 sn^2x\right)\right] =$$

$$\left[sn^2x + sn^2y\left(1-sn^2x\right)\right]\left(cn^2y + k^2 sn^2x\, sn^2y\right) =$$

$$\left(1-cn^2x\, cn^2y\right)\left(cn^2y + k^2 sn^2x\, sn^2y\right)$$

The expression $|1+\lambda^2 sn^2 z|^2$ simplifies to:

$$\frac{\left(cn^2y + k^2 sn^2x\, sn^2y\right)^2 + 2\lambda^2\left(sn^2x\, dn^2y - cn^2x\, dn^2x\, sn^2y\, cn^2y\right) + \lambda^4\left(1-cn^2x\, cn^2y\right)^2}{\left(cn^2y + k^2 sn^2x\, sn^2y\right)^2}$$

and $\varphi(x,y)$ reduces to:

$$\frac{1}{2\pi}\log(1-cn^2x\,cn^2y)-\frac{1}{4\pi}\log\left\{(cn^2y+k^2sn^2x\,sn^2y)^2+2\lambda^2(sn^2x\,dn^2y-cn^2x\,dn^2x\,sn^2y\,cn^2y)+\lambda^4(1-cn^2x\,cn^2y)^2\right\}$$

The stream function is

$$\psi(x,y)=\frac{1}{\pi}\operatorname{Arctang}\left(\frac{cnx\,dnx\,sny\,cny}{snx\,dnx}\right)-\frac{1}{2\pi}\operatorname{Arctang}\left\{\frac{2\lambda^2 snx\,cnx\,dnx\,sny\,cny\,dny}{(cn^2y+k^2sn^2x\,sn^2y)^2+\lambda^2(sn^2x\,dn^2y-cn^2x\,dn^2x\,sn^2y\,cn^2y)}\right\}$$

2 Sweep efficiency for a repeated 3-spot

Breakthrough time along the breakthrough streamline $\left(x=0,\ \psi=\frac{1}{2}\right)$ is

$$t_b=\int_0^{?}\frac{dy}{V}$$

Now $$V=\frac{\partial\varphi}{\partial y}=-\frac{2cn^2x\,cny\,(cny)'}{2\pi(1-cn^2x\,cn^2y)}-\frac{1}{4\pi}\frac{D'}{D}$$

$$D=(cn^2y+k^2sn^2x\,sn^2y)^2+2\lambda^2(sn^2x\,dn^2y-cn^2x\,dn^2x\,sn^2y\,cn^2y)+\lambda^4(1-cn^2x\,cn^2y)^2$$

$$D_{x=0}=cn^4y-2\lambda^2sn^2y\,cn^2y+\lambda^4sn^4y=(cn^2y-\lambda^2sn^2y)^2$$

$$\begin{aligned}D'=\ &2(cn^2y+k^2sn^2x\,sn^2y)\left\{2cny(cny)'+2k^2sn^2x\,sny\,(sny)'\right\}\\&+2\lambda^2\left[2dny\,sn^2x\,(dny)'-cn^2x\,dn^2x\left(2sny\,cn^2y\,(sny)'+2sn^2y\,cny\,(cny)'\right)\right]\\&-4\lambda^4(1-cn^2x\,cn^2y)\,cn^2x\,cny\,(cny)'\end{aligned}$$

Along the breakthrough streamline

$$D'=4cn^3y(cny)'-4\lambda^2sny\,cny\left\{cny\,(sny)'+sny\,(cny)'\right\}-4\lambda^4sn^2y\,cny\,(cny)'$$

$$D'=-4cn^3y\,sny\,dny-4\lambda^2sny\,cny\left\{cn^2y\,dny-sn^2y\,dny\right\}+4\lambda^4sn^3y\,cny\,dny$$

$$D'=4cny\,sny\,dny\left[-cn^2y-\lambda^2cn^2y+\lambda^2sn^2y+\lambda^4sn^2y\right]$$

$$D' = 4\,\mathrm{cn}y\,\mathrm{sn}y\,\mathrm{dn}y\,(1+\lambda^2)(\lambda^2\mathrm{sn}^2y-\mathrm{cn}^2y)$$

$$V_b = \frac{1}{\pi}\frac{\mathrm{cn}y\,\mathrm{dn}y}{\mathrm{sn}y} + \frac{\mathrm{cn}y\,\mathrm{sn}y\,\mathrm{dn}y}{\pi}\frac{(1+\lambda^2)}{(\mathrm{cn}^2y-\lambda^2\mathrm{sn}^2y)}$$

$$V_b = \frac{1}{\pi}\frac{\mathrm{cn}y\,\mathrm{dn}y}{\mathrm{sn}y} + \frac{\mathrm{cn}y\,\mathrm{sn}y\,\mathrm{dn}y\,(1+\lambda^2)}{\pi\,[1-(1+\lambda^2)\mathrm{sn}^2y]}$$

$$V_b = \frac{1}{\pi}\,\mathrm{sn}y\,\mathrm{cn}y\,\mathrm{dn}y\,\frac{1}{\mathrm{sn}^2y\,[1-(\lambda^2+1)\mathrm{sn}^2y]}$$

$$t_b = \int_0^{\eta}\pi\,\frac{\mathrm{sn}y\,[1-(\lambda^2+1)\mathrm{sn}^2y]\,dy}{\mathrm{cn}y\,\mathrm{dn}y}$$

$$t_b = \int_0^{\mathrm{sn}^2\eta}\frac{\pi}{2}\,\frac{[1-(\lambda^2+1)\mathrm{sn}^2y]}{(1-\mathrm{sn}^2y)(1-k'^2\mathrm{sn}^2y)}\,d(\mathrm{sn}^2y)$$

$$t_b = \int_0^{\mathrm{sn}^2\eta}\frac{\pi}{2}\,\frac{1-(\lambda^2+1)u}{(1-u)(1-k'^2u)}\,du$$

$$\frac{1-(\lambda^2+1)u}{(1-u)(1-k'^2u)} = \frac{k^2+\lambda^2}{k^2}\cdot\frac{1}{1-k'^2u} - \frac{\lambda^2}{k^2(1-u)}$$

$$t_b = \frac{\pi}{2}\,\frac{k^2+\lambda^2}{k^2}\int_0^{\mathrm{sn}^2\eta}\frac{du}{1-k'^2u} - \frac{\pi}{2}\,\frac{\lambda^2}{k^2}\int_0^{\mathrm{sn}^2\eta}\frac{du}{1-u}$$

which integrates to:

$$t_b = \frac{-\pi}{2k^2k'^2}(k^2+\lambda^2)\log(dn^2\eta) + \frac{\pi}{2k^2}\lambda^2\log(cn^2\eta)$$

The sweep efficiency is

$$S.E. = \frac{2t_b}{4KK'} = \frac{-\pi}{4KK'}\left\{\frac{k^2+\lambda^2}{k^2k'^2}\log(dn^2\eta) - \frac{\lambda^2}{k^2}\log(cn^2\eta)\right\} \qquad (15)$$

APPENDIX 4

1 Potential for a square repeated 9-spot

$$\varphi(x,y) = \frac{3}{4\pi}\log|\mathrm{sn}\,z|^2 - \frac{k}{4\pi}\log|\mathrm{cn}\,z|^2 - \frac{(3-2k)}{4\pi}\log|\mathrm{dn}\,z|^2$$

From Appendix 2 $$|\mathrm{sn}\,z|^2 = \frac{2(1-\mathrm{cn}^2x\,\mathrm{cn}^2y)}{2\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{sn}^2y}$$

and from Appendix 1

$$\mathrm{cn}\,z = 2\left(\frac{\mathrm{cn}x\,\mathrm{cn}y - i\,\mathrm{sn}x\,\mathrm{dn}x\,\mathrm{sn}y\,\mathrm{dn}y}{2\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{sn}^2y}\right)$$

$$|\mathrm{cn}\,z|^2 = \frac{4(\mathrm{cn}^2x\,\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{dn}^2x\,\mathrm{sn}^2y\,\mathrm{dn}^2y)}{(2\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{sn}^2y)^2}$$

The numerator can be rearranged in terms of sn^2x and cn^2y, for

$$4(\mathrm{cn}^2x\,\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{dn}^2x\,\mathrm{sn}^2y\,\mathrm{dn}^2y) = 4(1-\mathrm{sn}^2x)\mathrm{cn}^2y + \mathrm{sn}^2x(2-\mathrm{sn}^2x)(1-\mathrm{cn}^2y)(1+\mathrm{cn}^2y)$$

$$= -\mathrm{cn}^4y\,\mathrm{sn}^2x(2-\mathrm{sn}^2x) + \mathrm{cn}^2y\,4(1-\mathrm{sn}^2x) + \mathrm{sn}^2x(2-\mathrm{sn}^2x)$$

$$= -\mathrm{cn}^4y\,\mathrm{sn}^2x(2-\mathrm{sn}^2x) + \mathrm{cn}^2y\left[(2-\mathrm{sn}^2x)^2 - \mathrm{sn}^4x\right] + \mathrm{sn}^2x(2-\mathrm{sn}^2x)$$

$$= \left[-\mathrm{sn}^2x\,\mathrm{cn}^2y + 2 - \mathrm{sn}^2x\right]\left[(2-\mathrm{sn}^2x)\mathrm{cn}^2y + \mathrm{sn}^2x\right]$$

$$= \left(-\mathrm{sn}^2x\,\mathrm{cn}^2y + 2 - \mathrm{sn}^2x\right)\left(2\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{sn}^2y\right)$$

The expression for $|\mathrm{cn}\,z|^2$ reduces to:

$$\frac{2\mathrm{cn}^2x + \mathrm{sn}^2x\,\mathrm{sn}^2y}{2\mathrm{cn}^2y + \mathrm{sn}^2x\,\mathrm{sn}^2y}$$

Now

$$\mathrm{dn}\,z = \frac{\mathrm{dn}x\,\mathrm{dn}(iy) - k^2\,\mathrm{sn}x\,\mathrm{sn}(iy)\,\mathrm{cn}x\,\mathrm{cn}(iy)}{1-k^2\mathrm{sn}^2x\,\mathrm{sn}^2(iy)}$$

$$\mathrm{dn}\,z = \frac{\mathrm{dn}\,x \cdot \dfrac{\mathrm{dn}\,y}{\mathrm{cn}\,y} - k^2\,\mathrm{sn}\,x\; i\,\dfrac{\mathrm{sn}\,y}{\mathrm{cn}\,y}\cdot\dfrac{\mathrm{cn}\,x}{\mathrm{cn}\,y}}{1 - k^2\,\mathrm{sn}^2 x\left(i\dfrac{\mathrm{sn}\,y}{\mathrm{cn}\,y}\right)^2}$$

$$\mathrm{dn}\,z = \frac{\mathrm{dn}\,x\,\mathrm{dn}\,y\,\mathrm{cn}\,y - i k^2\,\mathrm{sn}\,x\,\mathrm{sn}\,y\,\mathrm{cn}\,x}{\mathrm{cn}^2 y + k^2\,\mathrm{sn}^2 x\,\mathrm{sn}^2 y}$$

Since $k^2 = 1/2$

$$\therefore \mathrm{dn}\,z = \frac{2\,\mathrm{dn}\,x\,\mathrm{dn}\,y\,\mathrm{cn}\,y - i\,\mathrm{sn}\,x\,\mathrm{sn}\,y\,\mathrm{cn}\,x}{2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y}$$

$$|\mathrm{dn}\,z|^2 = \frac{4\,\mathrm{dn}^2 x\,\mathrm{dn}^2 y\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y\,\mathrm{cn}^2 x}{(2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y)^2}$$

$$= \frac{(2-\mathrm{sn}^2 x)(1+\mathrm{cn}^2 y)\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,(1-\mathrm{sn}^2 x)(1-\mathrm{cn}^2 y)}{(2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y)^2}$$

$$= \frac{\mathrm{cn}^4 y\,(2-\mathrm{sn}^2 x) + \mathrm{cn}^2 y\,(2 - 2\,\mathrm{sn}^2 x - \mathrm{sn}^4 x) + \mathrm{sn}^2 x\,(1-\mathrm{sn}^2 x)}{(2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y)^2}$$

$$= \frac{[\mathrm{cn}^2 y\,(2-\mathrm{sn}^2 x) + \mathrm{sn}^2 x][\mathrm{cn}^2 y + (1-\mathrm{sn}^2 x)]}{(2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y)^2} = \frac{\mathrm{cn}^2 y + \mathrm{cn}^2 x}{2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y}$$

Finally,

$$\varphi(x,y) = \frac{3}{4\pi}\log\left(\frac{1-\mathrm{cn}^2 x\,\mathrm{cn}^2 y}{2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y}\right) - \frac{p}{4\pi}\log\left(\frac{2\,\mathrm{cn}^2 x + \mathrm{sn}^2 x\,\mathrm{sn}^2 y}{2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y}\right) - \frac{(3-2p)}{4\pi}\log\left(\frac{\mathrm{cn}^2 x + \mathrm{cn}^2 y}{2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y}\right)$$

$$\varphi(x,y) = \frac{3}{4\pi}\log(1-\mathrm{cn}^2 x\,\mathrm{cn}^2 y) - \frac{p}{4\pi}\log(2\,\mathrm{cn}^2 x + \mathrm{sn}^2 x\,\mathrm{sn}^2 y) - \frac{(3-2p)}{4\pi}\log(\mathrm{cn}^2 x + \mathrm{cn}^2 y)$$
$$- \frac{p}{4\pi}\log(2\,\mathrm{cn}^2 y + \mathrm{sn}^2 x\,\mathrm{sn}^2 y)$$

2 Breakthrough sweep efficiency

We evaluate the velocity component u from (18)

$$\mu = \frac{3}{4\pi}\,\frac{2cn^2y\,cnx\,snx\,dnx}{(1-cn^2x\,cn^2y)} + \frac{(3-2p)}{4\pi}\,\frac{2cnx\,snx\,dnx}{cn^2x+cn^2y}$$

$$-\frac{p}{4\pi}\,\frac{\left[-4cnx\,snx\,dnx+2sn^2y\,snx\,cnx\,dnx\right]}{(2cn^2x+sn^2x\,sn^2y)} - \frac{p}{4\pi}\,\frac{2snx\,sn^2y\,cnx\,dnx}{2cn^2y+sn^2x\,sn^2y}$$

a) along the x-axis ($y = 0$)

$$\mu = \frac{3}{2\pi}\,\frac{cnx\,dnx}{sn\,x} + \frac{(3-2p)}{4\pi}\,\frac{cnx\,snx}{dnx} + \frac{p}{2\pi}\,\frac{snx\,dnx}{cnx}$$

$$u = \frac{1}{4\pi}\,snx\,cnx\,dnx\left\{\frac{6}{sn^2x} + \frac{(3-2p)}{dn^2x} + \frac{2p}{cn^2x}\right\}$$

$$u = \frac{1}{4\pi}\,snx\,cnx\,dnx\left(\frac{p\,cn^4x + 2(3-p)cn^2x + p}{sn^2x\,cn^2x\,dn^2x}\right)$$

$$t_b = 4\pi\int_0^K \frac{snx\,cnx\,dnx}{p\,cn^4x + 2(3-p)cn^2x + p}\,dx$$

Let $cn^2x = u$ then $t_b = 2\pi\displaystyle\int_0^1 \frac{du}{pu^2+2(3-p)u+p}$

Let again $p = \frac{3}{2} - \frac{q}{2}$

$$pu^2+2(3-p)u+p = \left(\frac{3}{2}-\frac{q}{2}\right)u^2 + 2\left(\frac{3}{2}+\frac{q}{2}\right)u + \frac{3}{2} - \frac{q}{2}$$

$$= \frac{3}{2}\left\{(1+u)^2 - \frac{q}{3}(1-u)^2\right\}$$

$$= \frac{3}{2}\left[1+u-\sqrt{\frac{q}{3}}(1-u)\right]\left[1+u+\sqrt{\frac{q}{3}}(1-u)\right]$$

$$= \frac{3}{2}\left\{u\left(1+\sqrt{\frac{q}{3}}\right)+1-\sqrt{\frac{q}{3}}\right\}\left\{u\left(1-\sqrt{\frac{q}{3}}\right)+1+\sqrt{\frac{q}{3}}\right\}$$

$$pu^2 + 2(3-p)u + p = p\left\{u + \frac{1-\sqrt{q/3}}{1+\sqrt{\frac{q}{3}}}\right\}\left\{u + \frac{1+\sqrt{\frac{q}{3}}}{1-\sqrt{\frac{q}{3}}}\right\}$$

$$\frac{1}{pu^2 + 2(3-p)u + p} = \frac{1}{6\sqrt{\frac{q}{3}}}\left\{\frac{1}{u + \frac{1-\sqrt{q/3}}{1+\sqrt{\frac{q}{3}}}} - \frac{1}{u + \frac{1+\sqrt{\frac{q}{3}}}{1-\sqrt{\frac{q}{3}}}}\right\}$$

$$t_b = \frac{\pi}{3\sqrt{\frac{q}{3}}}\left\{\log\left[\frac{\left(2/1+\sqrt{q/3}\right)}{\left(1-\sqrt{\frac{q}{3}}\right)\left(1+\sqrt{\frac{q}{3}}\right)} \; \frac{\left(1+\sqrt{\frac{q}{3}}\right)/\left(1-\sqrt{\frac{q}{3}}\right)}{2/\left(1-\sqrt{\frac{q}{3}}\right)}\right]\right.$$

$$t_b = \frac{\pi}{3\sqrt{\frac{q}{3}}}\log\left(\frac{1+\sqrt{\frac{q}{3}}}{1-\sqrt{\frac{q}{3}}}\right)$$

or

$$t_b = \frac{\pi}{3}\sqrt{\frac{3}{q}}\left\{2\log\left(1+\sqrt{\frac{q}{3}}\right) - \log\left(\frac{2p}{3}\right)\right\}$$

When $p = 0$ $\quad t_b = \infty$ (stagnation corner of repeated 5-spot)

$p = \frac{3}{2}$ $\quad t_b = \frac{2\pi}{3}$ (minimum for t_b ; S.E. = .4569)

$$S.E. = \frac{\pi}{4KK'}\sqrt{\frac{3}{q}}\left\{2\log\left(1+\sqrt{\frac{q}{3}}\right) - \log\left(\frac{2p}{3}\right)\right\} \tag{19}$$

b) along a diagonal line (y = x)

Formula for the velocity u reduces to

$$u = \frac{3}{4\pi}\,\frac{cn^3 x}{sn x\, dn x} + \frac{(3-2p)}{4\pi}\,\frac{sn x\, dn x}{cn x} + \frac{p}{\pi}\,\frac{cn^3 x\, sn x\, dn x}{1+cn^4 x}$$

$$u = \frac{3cn^4 x\,(1+cn^4 x) + (3-2p)sn^2 x\, dn^2 x\,(1+cn^4 x) + 4p\, cn^4 x\, sn^2 x\, dn^2 x}{4\pi\,(1+cn^4 x)\, cn x\, sn x\, dn x}$$

$$u = \frac{(3-2p)cn^8 x + 2(3+2p)cn^4 x + (3-2p)}{8\pi\,(1+cn^4 x)\, cn x\, sn x\, dn x}$$

Breakthrough time is $$t_b = 8\pi \int_0^K \frac{(1+cn^4 x)\, cn x\, sn x\, dn x}{(3-2p)cn^8 x + 2(3+2p)cn^4 x + (3-2p)} dx$$

Let $cn^2 x = u$

$$t_b = 4\pi \int_0^1 \frac{(1+u^2)\, du}{(3-2p)u^4 + 2(3+2p)u^2 + 3-2p}$$

$$\frac{1+u^2}{(3-2p)u^4 + 2(3+2p) + 3-2p} = \frac{1+u^2}{\{u^2(\sqrt{3}+\sqrt{2p}) + \sqrt{3}-\sqrt{2p}\}\{u^2(\sqrt{3}-\sqrt{2p}) + \sqrt{3}+\sqrt{2p}\}}$$

$$= \frac{1+u^2}{(3-2p)\left(u^2 + \frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3}+\sqrt{2p}}\right)\left(u^2 + \frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}}\right)}$$

$$= \frac{1}{2\sqrt{3}(\sqrt{3}+\sqrt{2p})\left(u^2 + \frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3}+\sqrt{2p}}\right)} + \frac{1}{2\sqrt{3}(\sqrt{3}-\sqrt{2p})\left(u^2 + \frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}}\right)}$$

$$= \frac{1}{2\sqrt{3}(\sqrt{3}-\sqrt{2p})\left\{1 + \frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}} u^2\right\}} + \frac{1}{2\sqrt{3}(\sqrt{3}+\sqrt{2p})\left\{1 + \frac{(\sqrt{3}-\sqrt{2p})}{\sqrt{3}+\sqrt{2p}} u^2\right\}}$$

Let $\left(\frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}}\right)^{1/2} u = v \qquad \left(\frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3}+\sqrt{2p}}\right)^{1/2} u = w$

$$t_b = \int_0^1 \frac{2\pi \left(\frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}}\right)^{-1/2} dv}{\sqrt{3}(\sqrt{3}-\sqrt{2p})(1+v^2)} + \frac{2\pi}{\sqrt{3}(\sqrt{3}+\sqrt{2p})} \int_0^1 \frac{\left(\frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3}+\sqrt{2p}}\right)^{-1/2} dw}{1+w^2}$$

$$t_b = \frac{2\pi}{\sqrt{3}(\sqrt{3}-\sqrt{2p})}\left(\frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3}+\sqrt{2p}}\right)^{1/2} \text{Arc tang}\left(\frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}}\right)^{1/2}$$

$$+ \frac{2\pi}{\sqrt{3}(\sqrt{3}+\sqrt{2p})}\left(\frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3}-\sqrt{2p}}\right)^{1/2} \text{Arc tang}\left(\frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3}+\sqrt{2p}}\right)^{1/2}$$

$$t_b = \frac{2\pi}{\sqrt{3}\sqrt{3-2p}}\left\{\text{Arc tang}\left(\frac{\sqrt{3}+\sqrt{2p}}{\sqrt{3-2p}}\right) + \text{Arc tang}\left(\frac{\sqrt{3}-\sqrt{2p}}{\sqrt{3-2p}}\right)\right\}$$

$$t_b = \frac{2\pi}{\sqrt{3}\sqrt{3-2p}} \cdot \frac{\pi}{2} = \frac{\pi^2}{3}\sqrt{\frac{3}{3-2p}}$$

Therefrom,

$$S.E. = \frac{\pi^2}{4KK'}\sqrt{\frac{3}{3-2p}}$$

APPENDIX 5

1 Potential for the repeated 7-spot

Let us write $sn(z-K-iK')$ explicitly

$$sn(z-K-iK') = \frac{-dn z}{k\,cn z} = -\frac{1}{k}\,\frac{\dfrac{dnx\,dny\,cny - ik^2 snx\,sny\,cnx}{cn^2y + k^2 sn^2x\, sn^2y}}{\dfrac{cnx\,cny - i\,snx\,dnx\,sny\,dny}{cn^2y + k^2 sn^2x\, sn^2y}}$$

$$sn(z-K-iK') = -\frac{1}{k}\,\frac{(dnx\,dny\,cny - ik^2 snx\,sny\,cnx)}{(cnx\,cny - i\,snx\,dnx\,sny\,dny)}$$

Now

$$\frac{sn^2(z-K-iK')}{1+\lambda^2 sn^2(z-K-iK')} = \frac{dn^2 z}{k^2cn^2z + \lambda^2 dn^2 z} = \frac{dn^2 z}{(k^2+\lambda^2)\left\{1 - \dfrac{k^2(1+\lambda^2)sn^2z}{k^2+\lambda^2}\right\}}$$

$$\text{or}\quad \frac{1}{\lambda^2+k^2}\cdot\frac{dn^2 z}{1+\mu^2 sn^2 z} \qquad \text{with } \mu^2 = -\frac{k^2(1+\lambda^2)}{k^2+\lambda^2} = -k^2 nd^2\eta$$

$$W(z) = \frac{1}{2\pi}\log\left\{\frac{sn^2 z}{(1+\lambda^2 sn^2 z)}\cdot\frac{dn^2 z}{(1+\mu^2 sn^2 z)}\right\}$$ flowrate is 2

From (Appendix 3) we can write immediately

$$\varphi(x,y) = \frac{1}{2\pi}\log(1-cn^2x\,cn^2y) + \frac{1}{2\pi}\log(k^2cn^2x + k'^2cn^2y)$$
$$-\frac{1}{4\pi}\log\left\{(cn^2y + k^2sn^2x\,sn^2y)^2 + 2\lambda^2(sn^2x\,dn^2y - cn^2x\,dn^2x\,sn^2y\,cn^2y) + \lambda^4(1-cn^2x\,cn^2y)^2\right\}$$
$$-\frac{1}{4\pi}\log\left\{(cn^2y + k^2sn^2x\,sn^2y)^2 + 2\mu^2(sn^2x\,dn^2y - cn^2x\,dn^2x\,sn^2y\,cn^2y) + \mu^4(1-cn^2x\,cn^2y)^2\right\}$$

2 Sweep efficiency for the repeated 7-shot ?

It is obtained from the integral

$$t_b = \int_0^{\frac{2K'}{3}} \frac{dy}{(V)_{x=0}}$$

For x = 0 the formula for the potential simplifies to

$$\varphi(0,y) = \frac{1}{\pi}\log(\operatorname{sn}y) + \frac{1}{\pi}\log(\operatorname{dn}y) - \frac{1}{2\pi}\log\left\{1-(1+\lambda^2)\operatorname{sn}^2y\right\}$$
$$-\frac{1}{2\pi}\log\left\{1-(1+\mu^2)\operatorname{sn}^2y\right\}$$

Thus

$$V_{x=0} = \frac{1}{\pi}\frac{\operatorname{cn}y\operatorname{dn}y\operatorname{sn}y}{\operatorname{sn}^2y} - \frac{k'^2}{\pi}\frac{\operatorname{sn}y\operatorname{cn}y\operatorname{dn}y}{\operatorname{dn}^2y}$$
$$+\frac{1+\lambda^2}{\pi}\frac{\operatorname{sn}y\operatorname{cn}y\operatorname{dn}y}{1-(1+\lambda^2)\operatorname{sn}^2y} + \frac{1+\mu^2}{\pi}\frac{\operatorname{sn}y\operatorname{cn}y\operatorname{dn}y}{1-(1+\mu^2)\operatorname{sn}^2y}$$

$$V = \frac{\operatorname{cn}y\operatorname{dn}y\operatorname{sn}y}{\pi}\left\{\frac{1}{\operatorname{sn}^2y\left[1-(1+\lambda^2)\operatorname{sn}^2y\right]} + \frac{(1+\mu^2)\operatorname{dn}^2y - k'^2\left[1-(1+\mu^2)\operatorname{sn}^2y\right]}{\operatorname{dn}^2y\left[1-(1+\mu^2)\operatorname{sn}^2y\right]}\right\}$$

$$V = \frac{\operatorname{cn}y}{\pi\operatorname{sn}y\operatorname{dn}y}\left\{\frac{1-2k'^2\operatorname{sn}^2y+\operatorname{sn}^4y\left\{k'^2(2+\lambda^2+\mu^2)-(1+\lambda^2)(1+\mu^2)\right\}}{\left[1-(1+\lambda^2)\operatorname{sn}^2y\right]\left[1-(1+\mu^2)\operatorname{sn}^2y\right]}\right\}$$

$$t_b = \int_0^{\frac{2K'}{3}} \frac{\pi\operatorname{cn}y\operatorname{sn}y\operatorname{dn}y}{\operatorname{cn}^2y}\frac{\left[1-(1+\lambda^2)\operatorname{sn}^2y\right]\left[1-(1+\mu^2)\operatorname{sn}^2y\right]}{\left[1-2k'^2\operatorname{sn}^2y+\operatorname{sn}^4y\left\{k'^2(2+\lambda^2+\mu^2)-(1+\lambda^2)(1+\mu^2)\right\}\right]}dy$$

Let $\operatorname{sn}^2y = u \qquad 1+\lambda^2=\alpha \qquad 1+\mu^2=\beta$

$$t_b = \frac{\pi}{2}\int_0^{\operatorname{sn}^2\left(\frac{2K'}{3}\right)} \frac{(1-\alpha u)(1-\beta u)}{(1-u)\left[1-2k'^2u+u^2\left\{k'^2(\alpha+\beta)-\alpha\beta\right\}\right]}du$$

Now $\alpha+\beta = 2+\lambda^2+\mu^2 = 2+\lambda^2-\dfrac{k^2(1+\lambda^2)}{k^2+\lambda^2} = 1+\dfrac{\lambda^2(1+\lambda^2)}{k^2+\lambda^2}$

$$\alpha\beta = (1+\lambda^2)\left(1-k^2\frac{1+\lambda^2}{k^2+\lambda^2}\right) = k'^2\,\frac{\lambda^2(1+\lambda^2)}{k^2+\lambda^2}$$

The expression $k'^2(\alpha+\beta)-\alpha\beta$ reduces to k'^2

The integrand can be expanded in a sum of rational fractions

$$\frac{(1-\alpha u)(1-\beta u)}{(1-u)(k'^2u^2-2k'^2u+1)} = \frac{A}{1-u}+\frac{Bu+C}{k'^2u^2-2k'^2u+1}$$

One obtains easily

$$A = -\lambda^2\frac{(1+\lambda^2)}{\lambda^2+k^2} \qquad B = -2k'^2\lambda^2\frac{(1+\lambda^2)}{\lambda^2+k^2} \qquad C = 1+\lambda^2\frac{(1+\lambda^2)}{\lambda^2+k^2}$$

The integral for t_b can be rewritten

$$\frac{\pi}{2}\int_0^{sn^2\left(\frac{2K'}{3}\right)}\left\{\frac{\lambda^2(1+\lambda^2)}{k^2+\lambda^2}\left[\frac{1}{u-1}-\frac{2k'^2u-2k'^2+2k'^2-1}{k'^2u^2-2k'^2u+1}\right]+\frac{1}{k'^2u^2-2k'^2u+1}\right\}du$$

$$t_b = \frac{\pi}{2}\lambda^2\left(\frac{1+\lambda^2}{k^2+\lambda^2}\right)\left\{\log\left(cn^2\frac{2K'}{3}\right)-\log\left(k'^2sn^4\frac{2K'}{3}-2k'^2sn^2\frac{2K'}{3}+1\right)\right\}+D\int^{sn^2\frac{2K'}{3}}\frac{du}{k'^2u^2-2k'^2u+1}$$

with $D = \dfrac{\pi}{2}\left[1-\dfrac{\lambda^2(2k'^2-1)(1+\lambda^2)}{k^2+\lambda^2}\right]$

Now $\displaystyle\int_0^{sn^2\frac{2K'}{3}}\frac{du}{k'^2u^2-2k'^2u+1} = \int_0^{sn^2\frac{2K'}{3}}\frac{\frac{k}{k'}\left(\frac{k'}{k}du\right)}{k^2\left[1+\frac{k'^2}{k^2}(u-1)^2\right]} = \frac{1}{kk'}\,\mathrm{Arc\,tang}\,\frac{k'}{k}(u-1)\Big|_0^{sn^2\frac{2K'}{3}}$

$$= \frac{1}{kk'}\left\{\mathrm{Arc\,tang}\,\frac{k'}{k}-\mathrm{Arc\,tang}\left(\frac{k'}{k}cn^2\frac{2K'}{3}\right)\right\} = \frac{1}{kk'}\,\mathrm{Arc\,tang}\left(kk'sd^2\frac{2K'}{3}\right)$$

Expliciting λ and observing that $kk' = \sin\frac{\pi}{12}\cdot\cos\frac{\pi}{12} = \frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{4}$

$$t_b = \frac{\pi}{2}\,\frac{cs^2\left(\frac{2K'}{3}\right)}{dn^2\,\frac{2K'}{3}}\,\log\left(\frac{cn^2\,\frac{2K'}{3}}{k^2+k'^2 cn^4\,\frac{2K'}{3}}\right)$$

$$+\,2\pi\left[1-(2k'^2-1)\frac{cs^2(2K'/3)}{dn^2(2K'/3)}\right]\text{Arc tang}\left(\frac{sd^2(2K'/3)}{4}\right)$$

For the particular value of $k = \sin\frac{\pi}{12}$ $\quad k^2 = \frac{2-\sqrt{3}}{4}$

$$cn^2\,\frac{2K'}{3} = \frac{2-\sqrt{3}}{2+\sqrt{3}} \qquad 2k'^2-1 = \cos^2\frac{\pi}{12} - \sin^2\frac{\pi}{12} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

from which we derive $\quad sn^2\,\frac{2K'}{3} = \frac{2\sqrt{3}}{2+\sqrt{3}} \quad dn^2\,\frac{2K'}{3} = \frac{2-\sqrt{3}}{2}$

$$sd^2\left(\frac{2K'}{3}\right) = 4\sqrt{3} \qquad \frac{cs^2(2K'/3)}{dn^2(2K'/3)} = \frac{\sqrt{3}}{3} \qquad \frac{cn^2(2K'/3)}{k^2+k'^2cn^4\,\frac{2K'}{3}} = 1$$

The formula for t_b reduces considerably to

$$t_b = 2\pi\left(1-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{3}\right)\text{Arc tang}\left(\sqrt{3}\right) = \frac{\pi^2}{3}$$

Thus $$S.E. = \frac{\pi^2}{3KK'} = \frac{\pi^2}{3\times 1.5981\times 2.7681} = .7437 \qquad (23)$$

since area of hexagon is 2KK' and flow rate = 2

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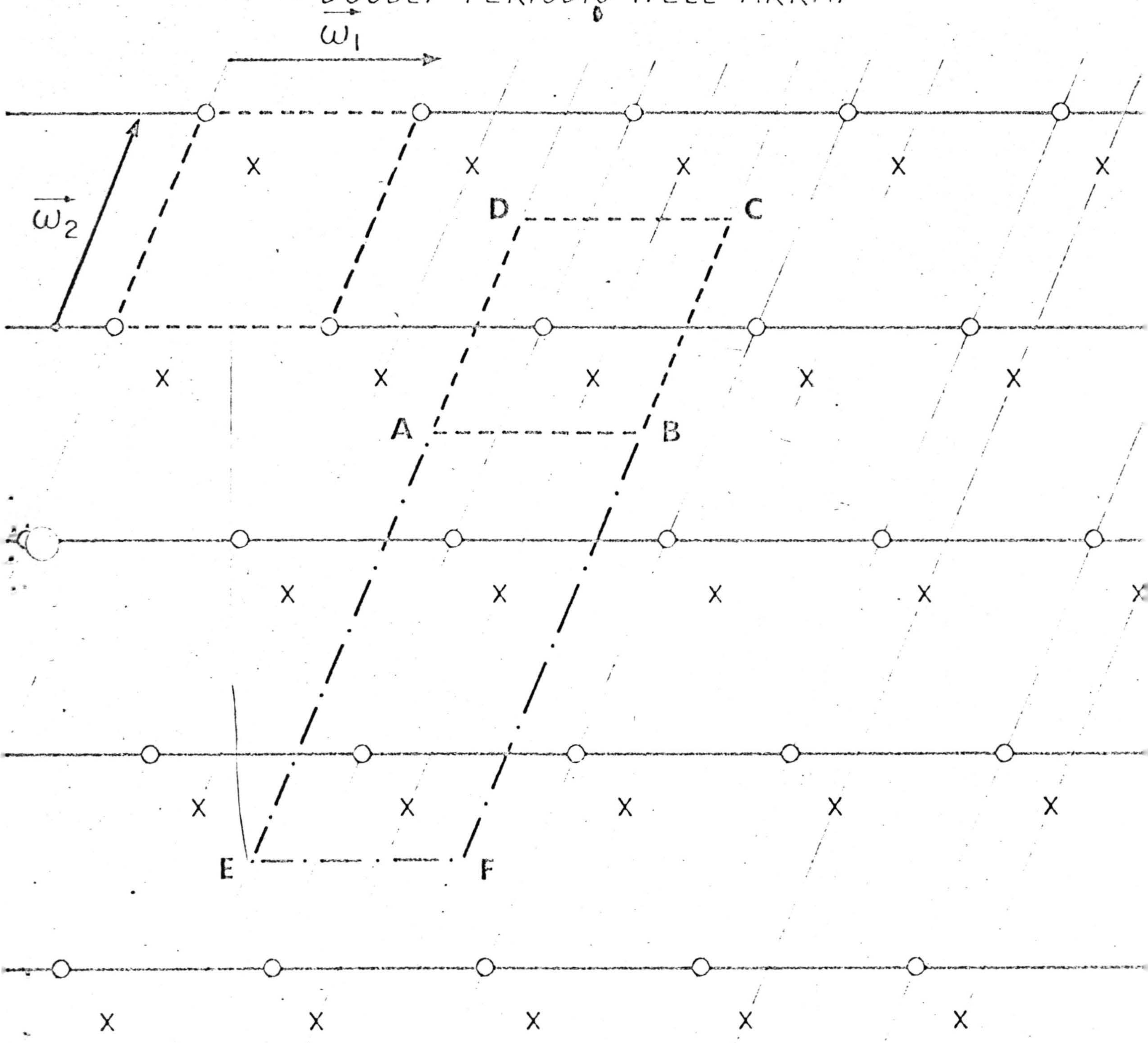
DOUBLY PERIODIC WELL ARRAY



X : INJECTION WELL   O : PRODUCTION WELL

FIGURE 1

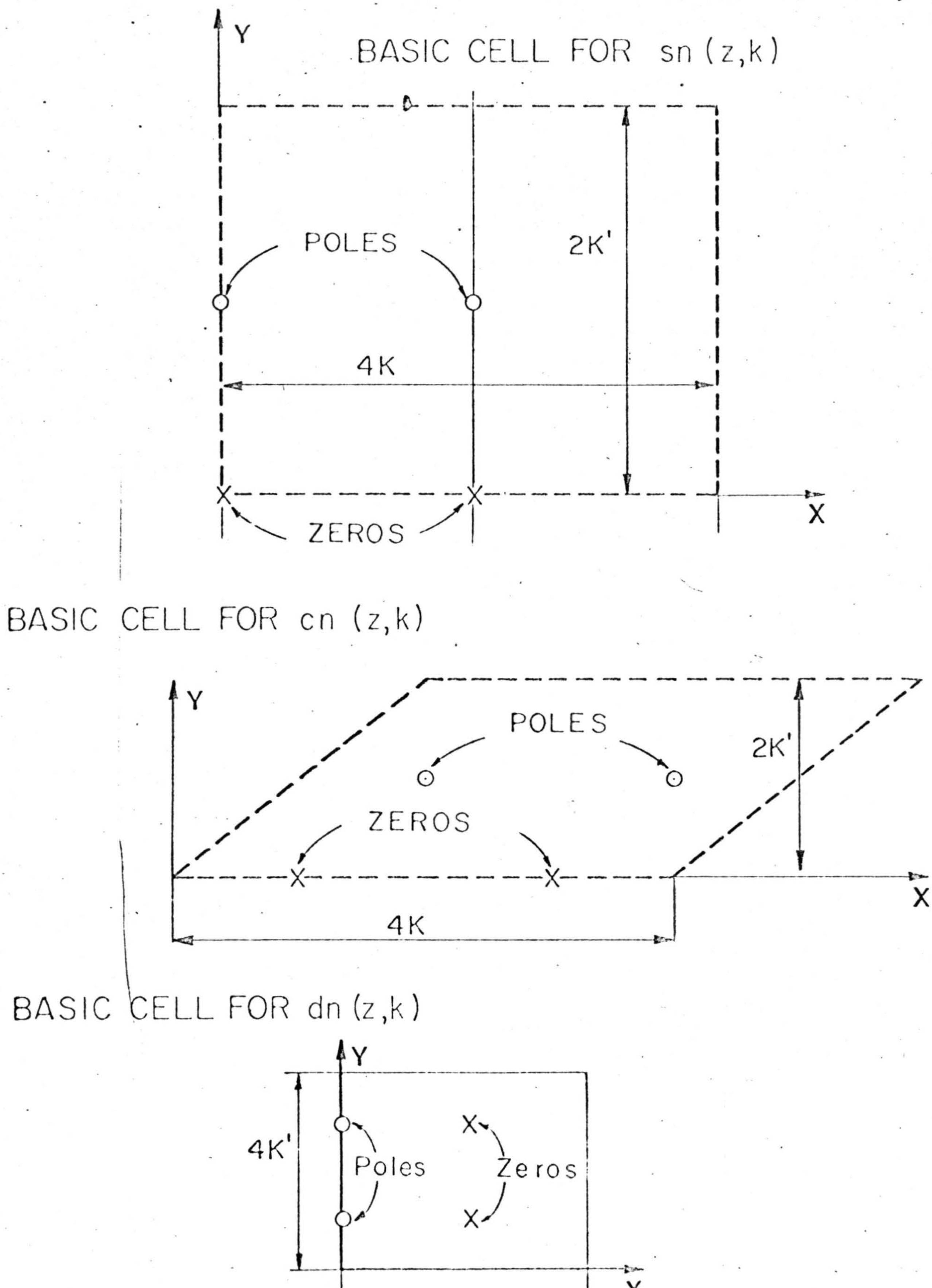
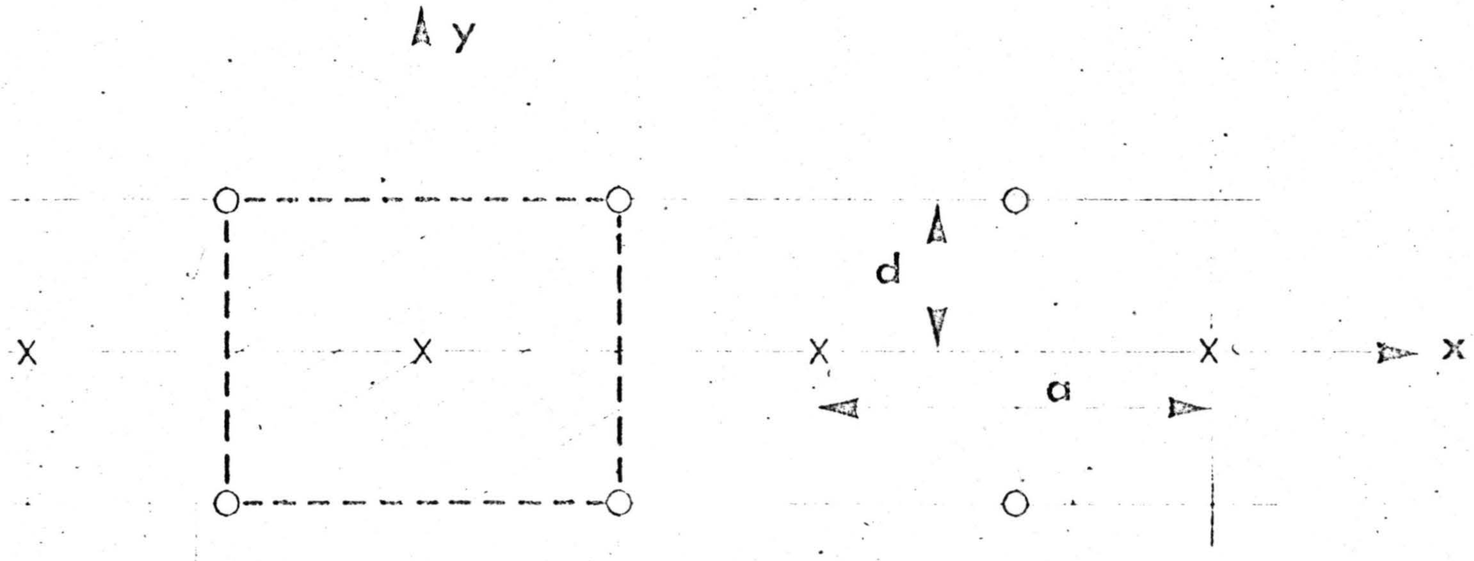

BASIC CELL FOR sn (z,k)
Y
POLES
2K'
4K
ZEROS
X
BASIC CELL FOR cn (z,k)
Y
POLES
2K'
ZEROS
X
4K
BASIC CELL FOR dn (z,k)
Y
4K'
Poles
Zeros
X
2K

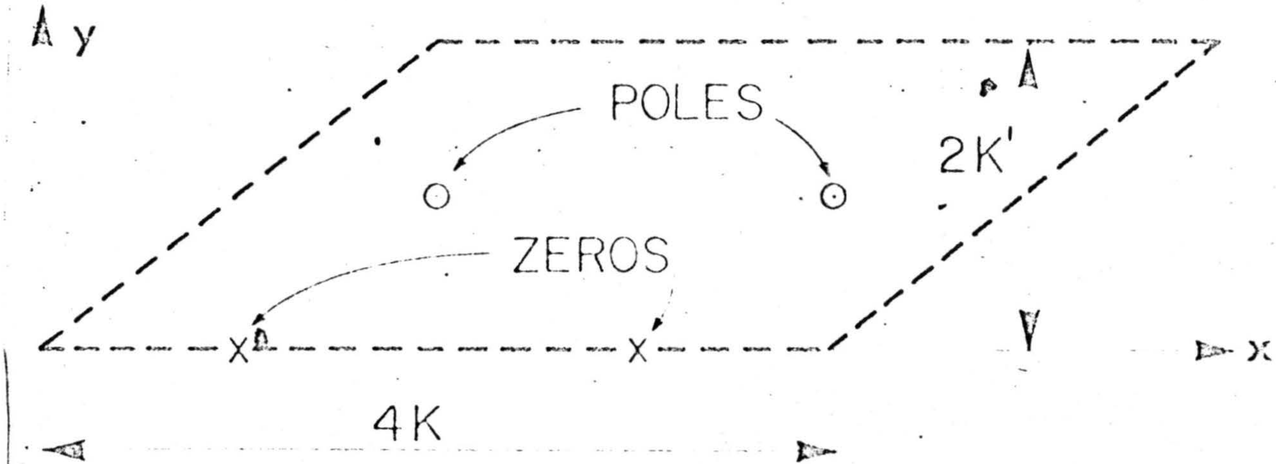
FIGURE 2

staggered line drive

~~REPEATED FIVE SPOT~~



BASIC CELL FOR cn (z,k)



COMPLEX POTENTIAL

$$w(z) = \frac{1}{2\pi} \log \left[cn\,(z + K, k) \right]$$

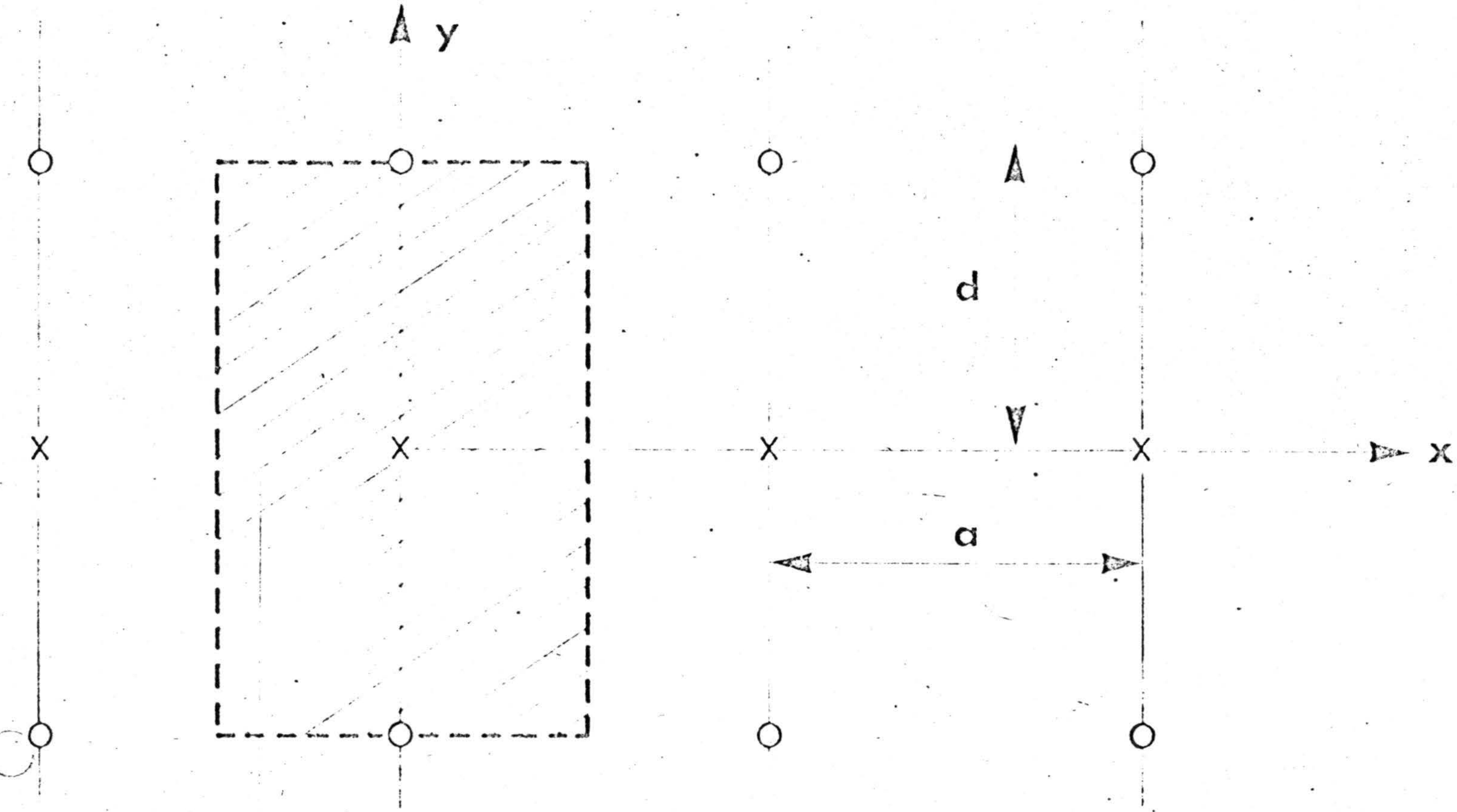
POTENTIAL FOR A SQUARE REPEATED 5-SPOT

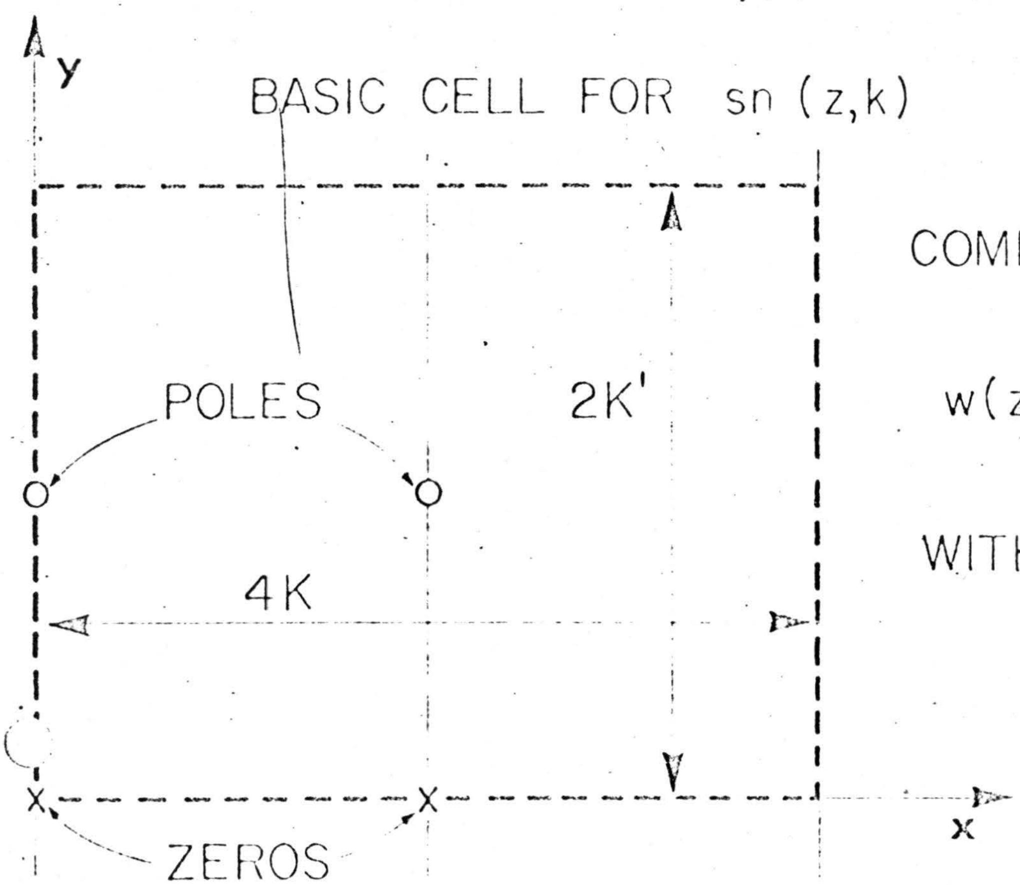
$$\phi\,(x,y) = \frac{1}{4\pi} \log \left[\frac{1 - cn^2 x \; cn^2 y}{cn^2 x + cn^2 y} \right] , \quad k^2 = k'^2 = \frac{1}{2}$$

FIGURE 3

DIRECT LINE-DRIVE

~~REPEATED TWO-SPOT~~





COMPLEX POTENTIAL

$$w(z) = \log\left[\mathrm{sn}(z,k)\right]$$

WITH k SUCH THAT:

$$\frac{d}{a} = \frac{K'}{2K}$$

FIGURE 4

DIRECT REPEATED 3 - SPOT

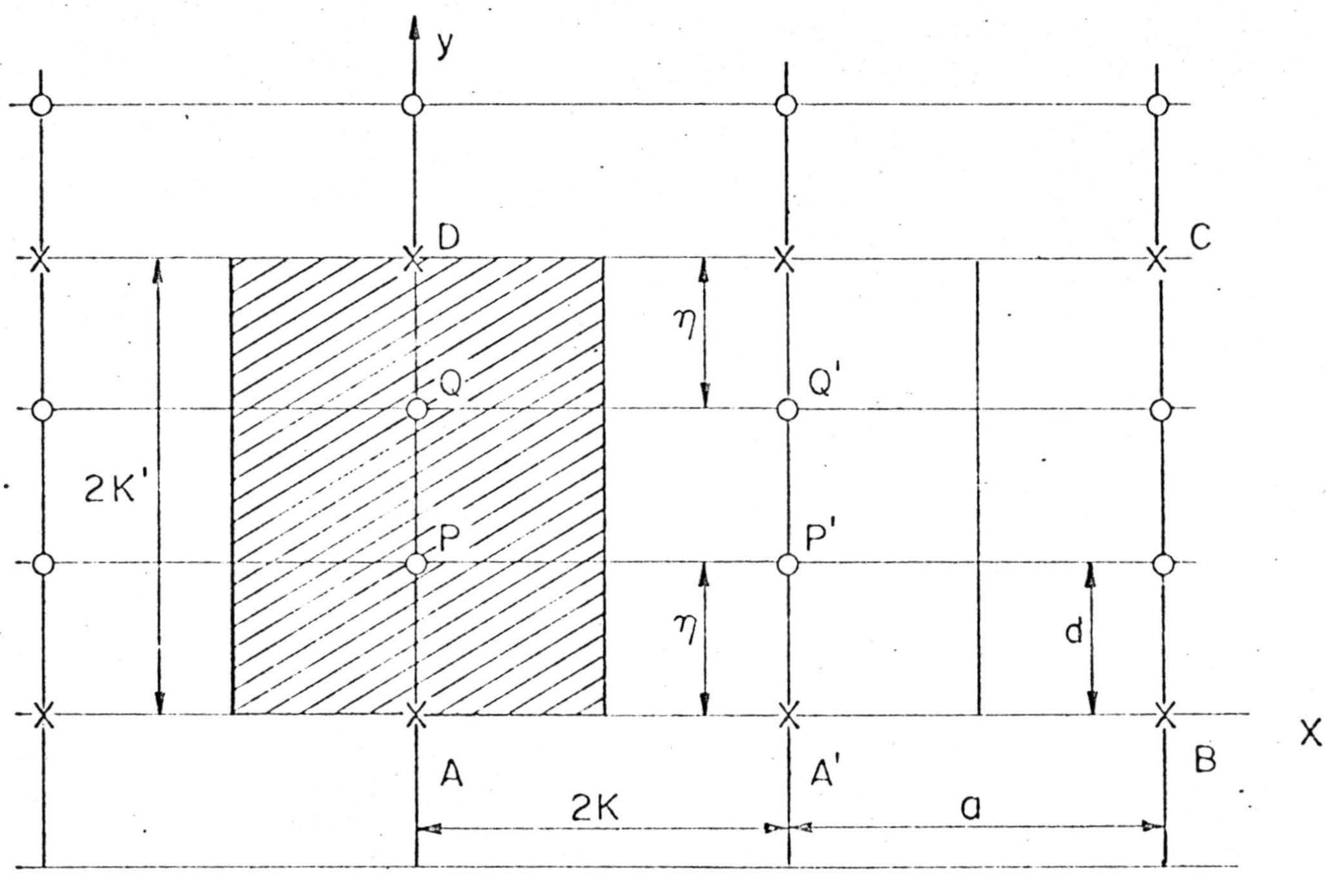


FIGURE 5

REPEATED 9-SPOT

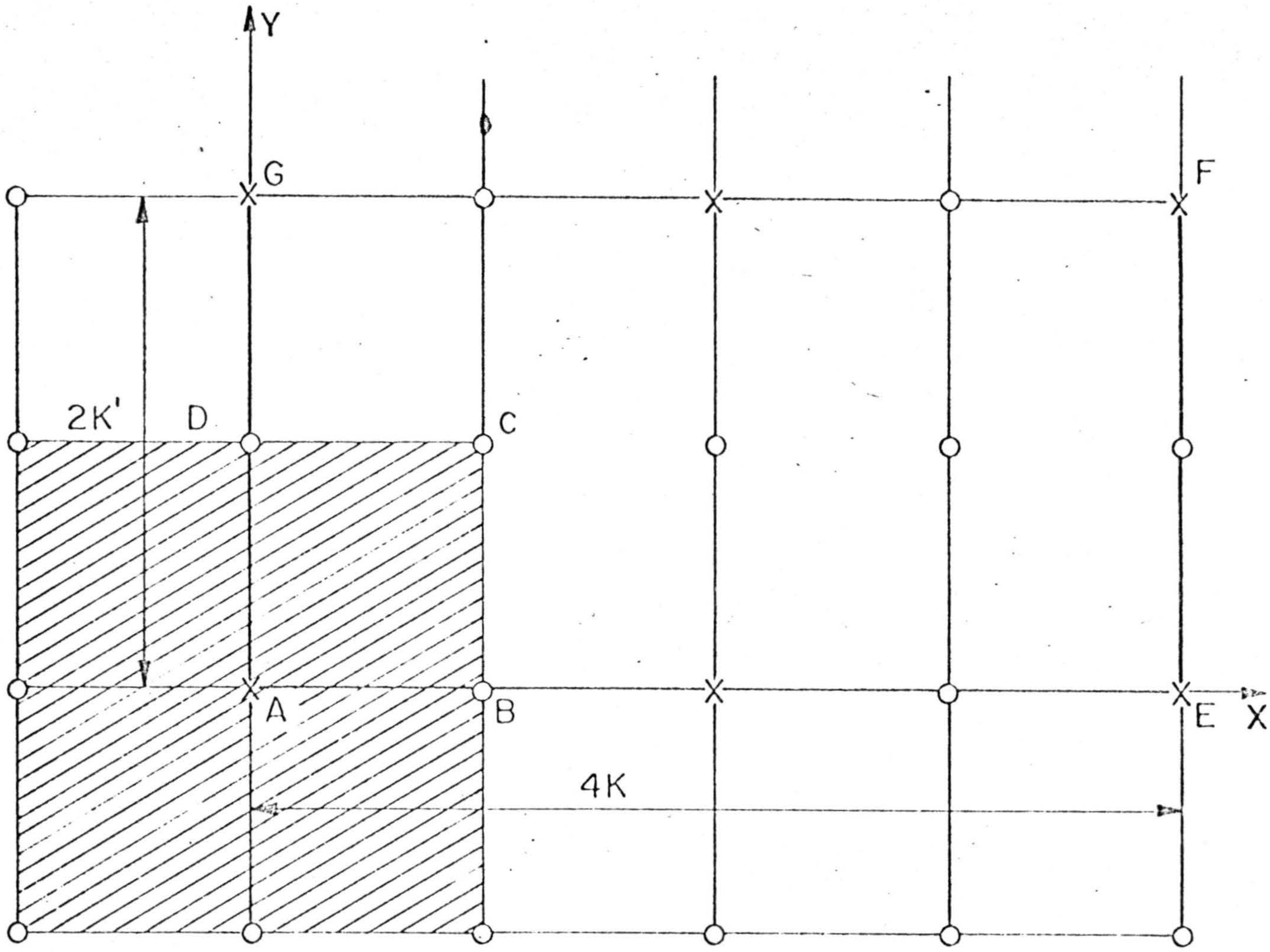


FIGURE 6

REPEATED 7 - SPOT

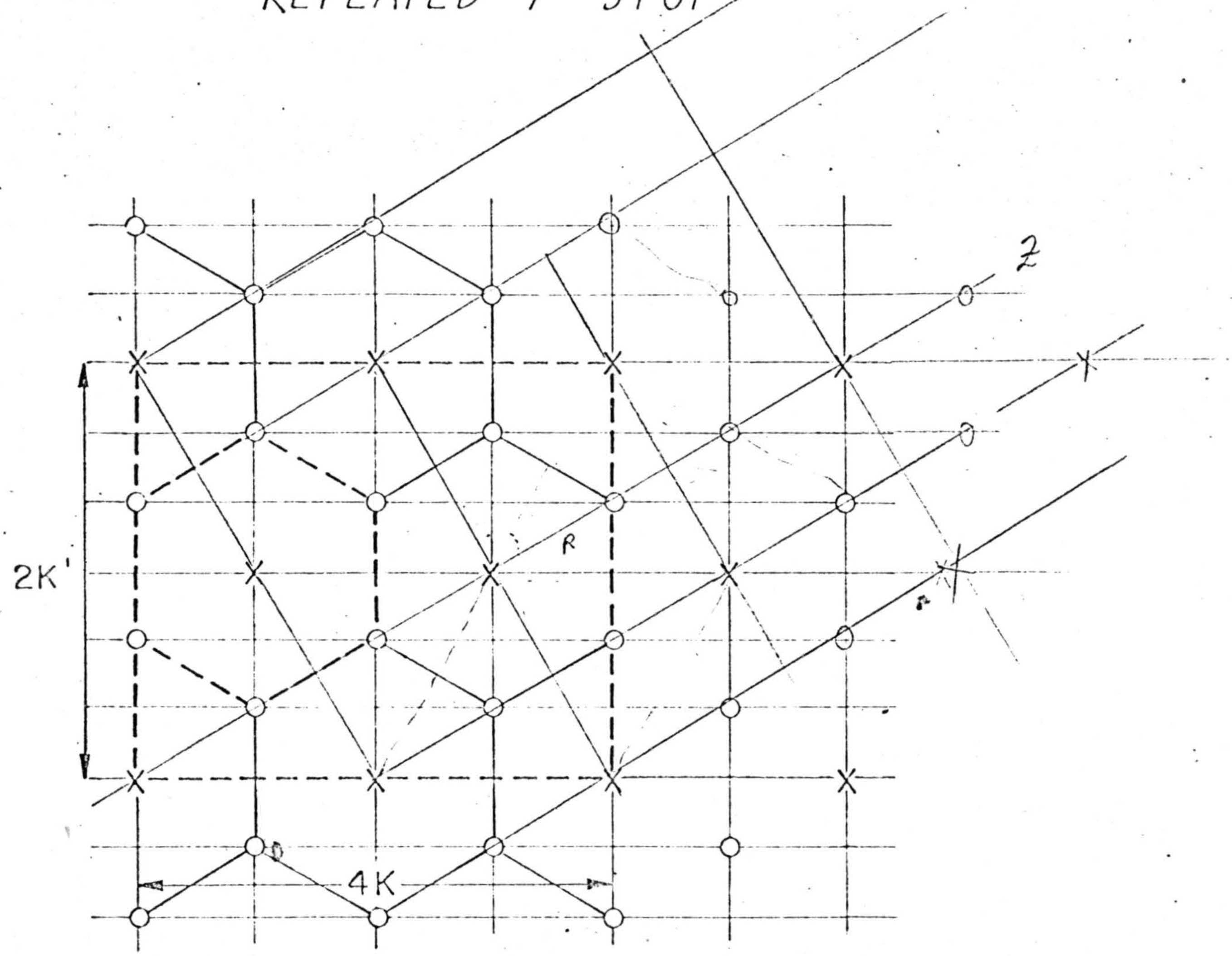
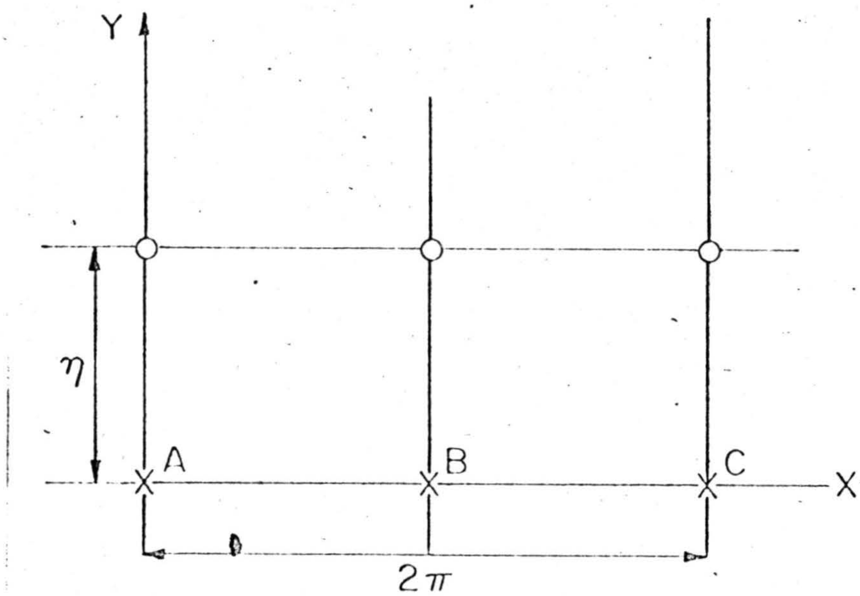
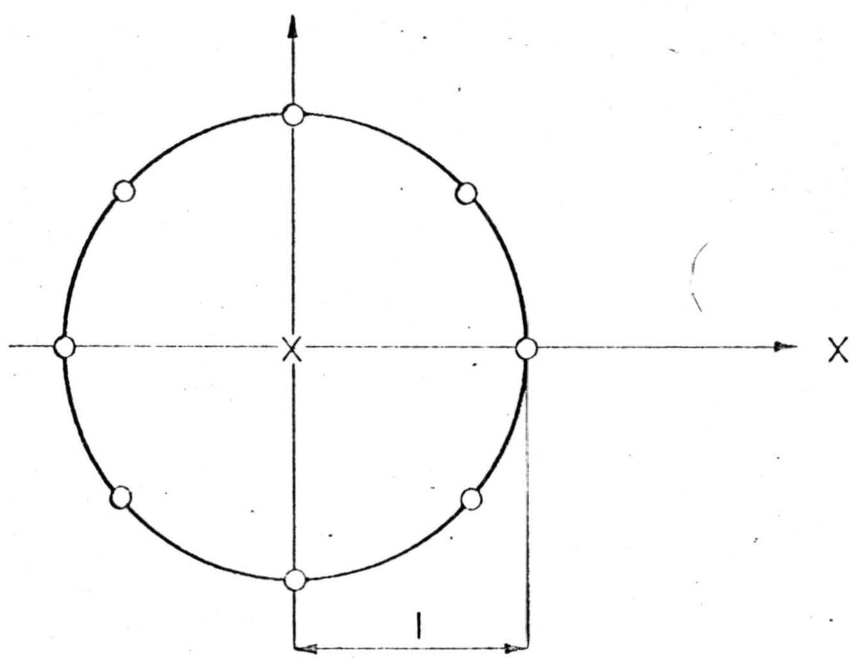


FIGURE 7



ELEMENTARY SINGLY-PERIODIC ARRAY



ISOLATED (N+1) SPOT

FIGURE 8

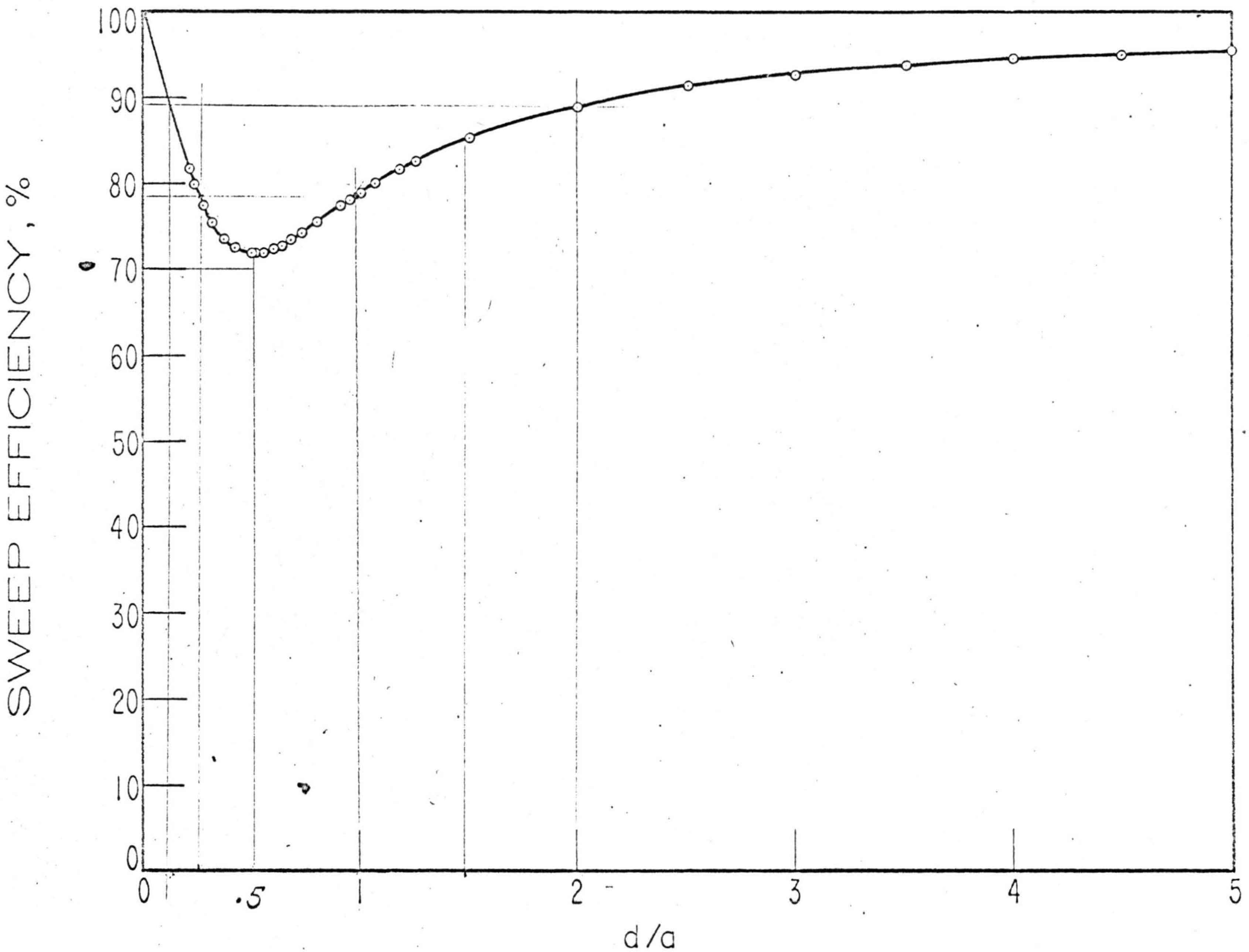


FIGURE 9

SWEEP EFFICIENCY OF STAGGERED LINE DRIVE

6-26-63 FEH

LE 42-906

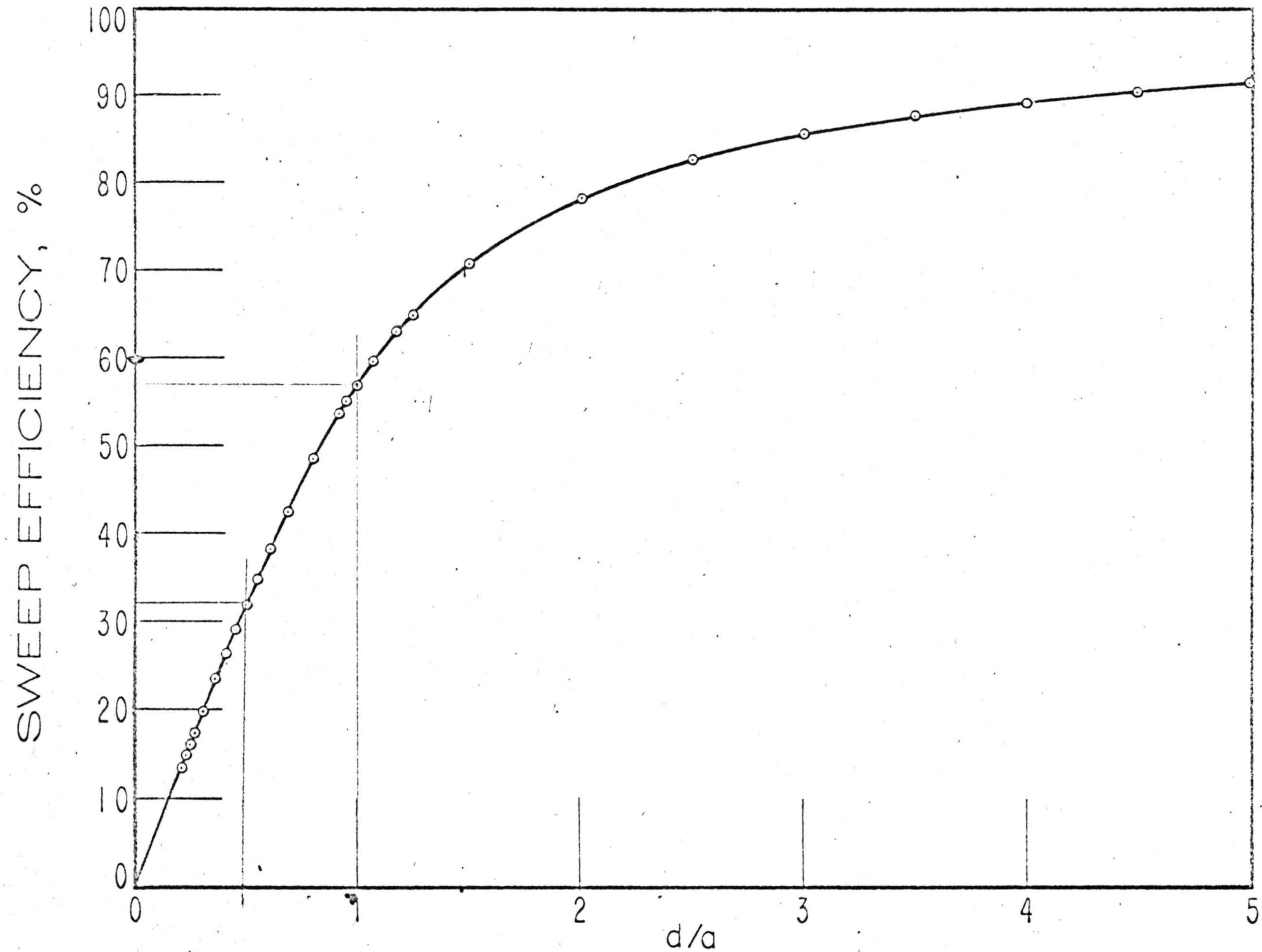


FIGURE 10

SWEEP EFFICIENCY OF REPEATED 2 - SPOT

6-26-63 FEH

F 42-902

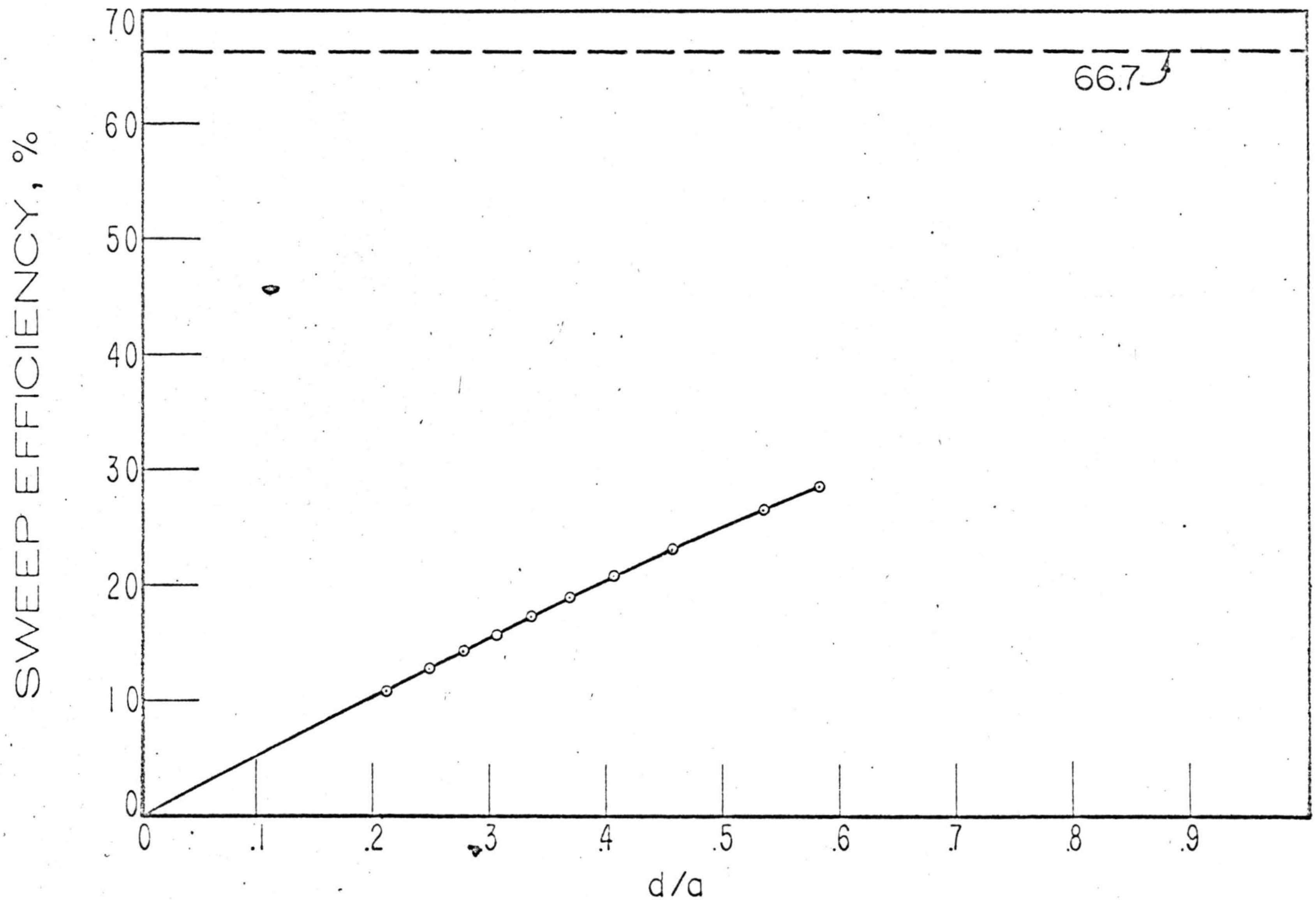


FIGURE 11 (Fig 20 in report)

SWEEP EFFICIENCY OF REPEATED 3-SPOT

5-22-63 FEH

LE 42 908

REPEATED 9-SPOT

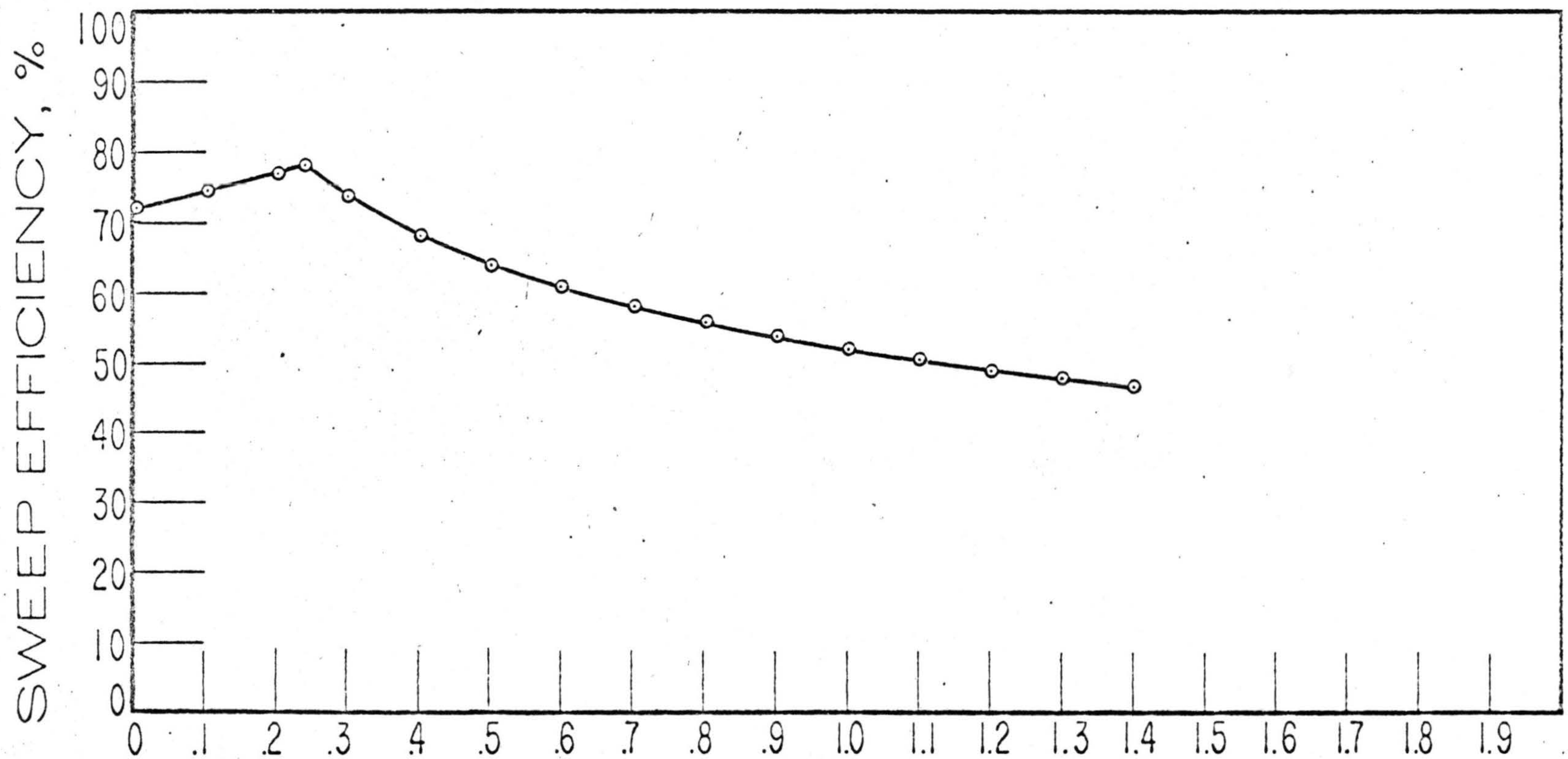


FIGURE 12

SWEEP EFFICIENCY VERSUS PARAMETER p

5-22-63 FEH

LC 12-000

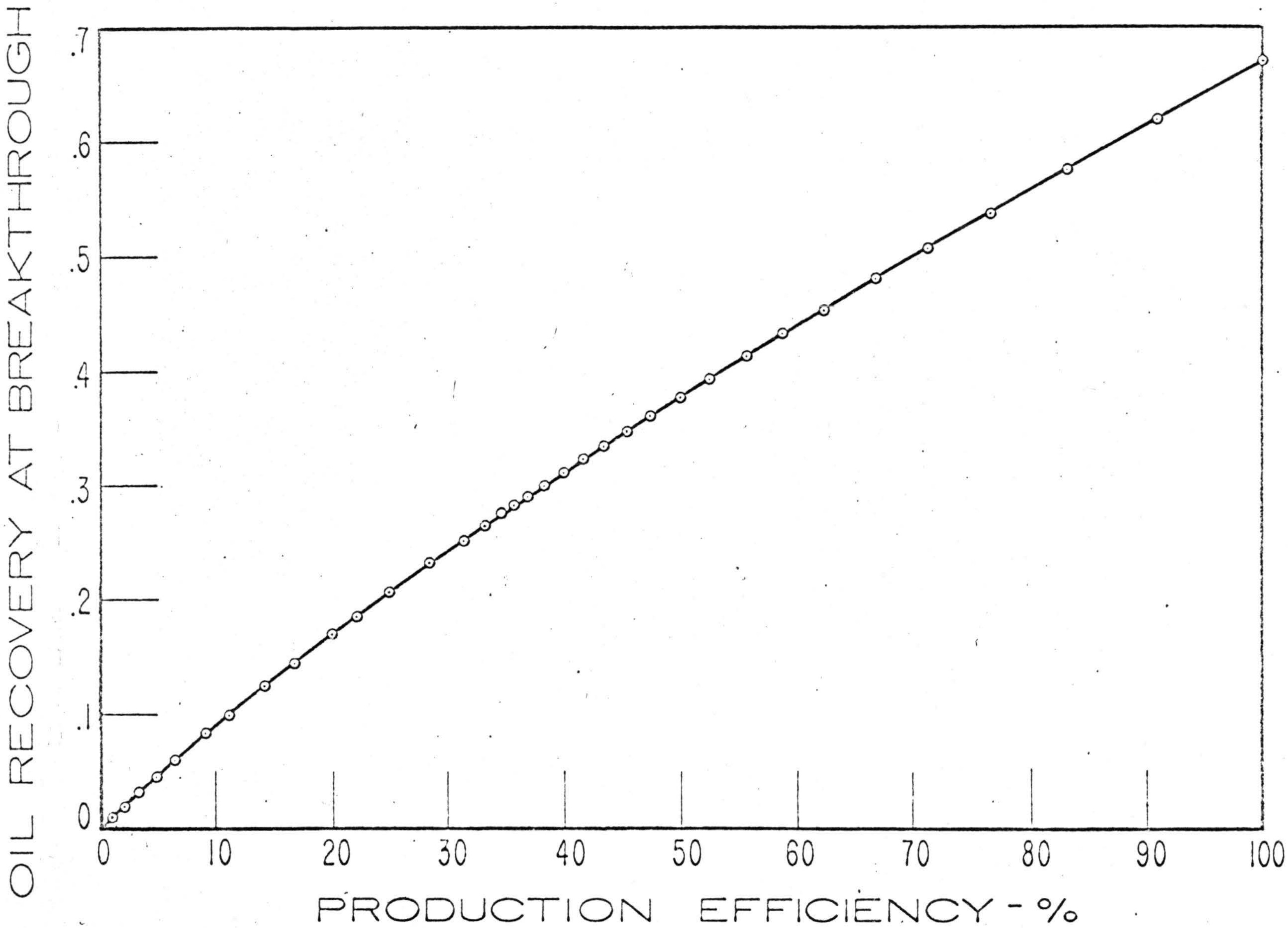


FIGURE 13

BREAKTHROUGH RECOVERY VERSUS PRODUCTION EFFICIENCY

ISOLATED 5-SPOT WITH LEAK AT ∞

LE 42-910