DISSERTATION

LINEAR SYSTEMS AND RIEMANN-ROCH THEORY ON GRAPHS

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WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY RODNEY JAMES ENTITLED LINEAR SYSTEMS AND RIEMANN-ROCH THEORY ON GRAPHS BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

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Graphs can be viewed as discrete counterparts to algebraic curves, as exemplified by the recent Riemann-Roch formula for integral divisors on multigraphs. We show that for any subring R of the reals, the Riemann-Roch formula can be generalized to R-valued divisors on edge-weighted graphs over R. We also show that a related abelian sandpile model extended to R on edge-weighted graphs leads to a group, which has many interesting properties. The sandpile results are used to prove various properties of linear systems of divisors on graphs, including that the set of divisors with empty linear systems is completely determined by a lattice of nonspecial divisors. We use these properties of linear systems on graphs to study line bundles on binary and ternary algebraic curves that match the dimension of their graph counterparts.

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Chapter 1

Introduction

1.1 Overview

This dissertation concerns the study of algebro-geometric properties of graphs and their connections to algebraic curves, inspired by the recent proof of a Riemann-Roch theorem for graphs by Baker and Norine [3]. In this chapter, an overview of motivating results and related work is presented. New results are contained in the subsequent chapters, which are grouped in two areas.

The first area involves extending the Riemann-Roch theory in [3] from integral divisors to divisors over any subring R of the reals. In Chapter 2, a Riemann-Roch theorem for such R-divisors is shown, based on extending the framework in [3]. Following that, Chapter 3 presents a related result for sandpiles or chip-firing games on graphs which have weighted edges over R. These results are then used to prove results for linear systems in Chapter 4.

The second area of research, contained in Chapter 5, concerns finding compatible line bundles on nodal curves corresponding to their discrete counterparts, divisors on graphs. In this sense, an n-vertex graph and divisor serve as a model for a particular type of curve with n rational components with line bundles corresponding to an effective divisor on the graph. Such a line bundle is called *compatible* if its dimension is equal to the dimension of the linear system of the corresponding graph divisor. Although such curves and graphs both obey Riemann-Roch, it is not obvious or even clear that the discrete graph object and a corresponding algebraic curve have such a compatible divisor-bundle pairing.

1.2 Graph Curves

Graph curves, discussed extensively in [4], [7], [8] and [15], are reducible algebraic curves which are in a sense characterized by graphs.

Let G be a finite, connected graph with vertex set $V(G)$ and edge set $E(G)$, where the degree of each vertex $v \in V(G)$ is at most 3. Such a graph is said to be *sub-trivalent*; if the degree of every vertex is exactly 3, the graph is *trivalent*. Multiple edges and loops (where an edge connects a vertex to itself) are allowed, where we define p_{vw} to be the number of edges joining vertices v and w.

Using G as a model, we construct a connected, reducible algebraic curve X_G over a field k as follows. For each $v \in V(G)$, let $X_v \cong \mathbb{P}^1_k$. The intersection number $X_v \cdot X_w = p_{vw}$, where X_v and X_w meet transversely p_{vw} times at distinct coordinates. The *graph curve* X_G is then defined to be

$$X_G = \bigcup_{v \in V(G)} X_v$$

with the above intersection conditions. The genus of X_G is given by

$$g = |E(G)| - |V(G)| + 1.$$

For graphs that are trivalent, the corresponding graph curves are *stable* curves, in the sense of Deligne and Mumford [9].

We will consider curves of the form X_G where we allow G to be any finite connected graph with multiple edges, with the restriction that loops will not be allowed.

1.3 Baker-Norine Theory

In this section we describe the theory of linear systems on graphs used in [3], much of which was formulated in [1]. Let G be a connected graph with multiple edges allowed, but without loops, with vertices $V(G) = \{v_1, \ldots, v_n\}$ and p_{ij} denoting the number of edges joining vertices v_i and v_j. Note that since loops are not allowed, $p_{ii} = 0$ for all i. Set the number of edges of G to be $m = \sum_{i<j} p_{ij}$, then the *genus* of G is $g = m - n + 1$.

A *divisor* on G is essentially a map $V(G) \to \mathbb{Z}$, which we will write as the formal sum

$$D = \sum_{i=1}^{n} d_i \cdot v_i$$

where each $d_i \in \mathbb{Z}$. The group of divisors is denoted by $\mathrm{Div}(G)$. The *degree* of D is

$$\deg(D) = \sum_{i=1}^{n} d_i.$$

We say that D is *effective* (write $D \geq 0$) if each $d_i \geq 0$. The subgroup of *zero divisors* Div^0 are the divisors with degree zero.

Let $H_j = \deg(v_j) \cdot v_j + \sum_{i \neq j} -p_{ij} \cdot v_i$ for each $j = 1, \ldots, n$, noting that the valence or degree of v_j is $\deg(v_j) = \sum_{i=1}^{n} p_{ij}$. The *principal divisors* $\mathrm{PDiv}(G)$ are generated by the set $\{H_1, \ldots, H_n\}$ over $\mathbb{Z}$. Note that since each $H_j \in \mathrm{Div}^0(G)$, $\mathrm{PDiv}(G) \leq \mathrm{Div}^0(G)$. We say that two divisors $D, D' \in \mathrm{Div}(G)$ are *linearly equivalent* (write $D \sim D'$) if and only if $D - D' \in \mathrm{PDiv}(G)$.

The *complete linear system* associated with a divisor D is defined as

$$|D| = \{D' \in \mathrm{Div}(G) \mid D' \geq 0, D' \sim D\}$$

and the *rank* of D is

$$r(D) = \min\{\deg(E) \mid E \geq 0, |D - E| = \emptyset\} - 1.$$

Let K be the *canonical divisor*

$$K = \sum_{i=1}^{n} (\deg(v_i) - 2) \cdot v_i.$$

The main result of [3] can now be stated, the Riemann-Roch theorem for graphs.

Theorem 1.1 (Baker-Norine)**.** *If* $D \in \mathrm{Div}(G)$ *then*

$$r(D) - r(K - D) = \deg(D) - g + 1.$$

1.4 Tropical Curves

A different way to view a graph in an algebro-geometric sense is as a tropical curve. Following [13] and [12], a (compact) tropical curve is essentially a connected metric graph G. A tropical rational function on G is a real-valued continuous piecewise linear function with integer slopes. Note that tropical functions are defined on the edges of G as well as the vertices. The order of a tropical rational function f at a point $p \in G$, $\mathrm{ord}_p(f)$, is the sum of the slopes of f for all edges emanating from the point p. We will denote the space of all tropical rational functions on G as $M(G)$.

A tropical divisor D on G is a formal sum

$$D = \sum_{p \in G} a_p \cdot p$$

where set of nonzero coefficient a_p is finite, and the degree of D is $\deg(D) = \sum_p a_p$. The divisor D above is called *effective* if each $a_p \geq 0$. A tropical rational function $f \in M(G)$ can be represented as a divisor using its order:

$$(f) = \sum_{p \in G} \mathrm{ord}_p(f) \cdot p.$$

If D is a tropical divisor on G, the space $R(D)$ is the set of all $f \in M(G)$ such that the divisor $D + (f)$ is effective. The dimension $r(D)$ of the space $R(D)$ is

$$r(D) = \max\{n \mid R(D - p_1 - \cdots - p_n) \neq \emptyset \text{ for all choices of } p_1, \ldots, p_n \in G\}.$$

We define the canonical divisor K as before to be

$$K = \sum_{v \in V(G)} (\deg(v) - 2) \cdot v.$$

Independently, Milkhlakin and Zharkov [14], and Gathmann and Kerber [12], recently showed that a Riemann-Roch formula

$$r(D) - r(K - D) = \deg(D) - g + 1$$

holds for tropical curves. The original proof in [14] involved using Jacobians of tropical curves, where the proof in [12] depends on the Baker-Norine result in [3]. A revised version of [14] provides a simpler proof, again based on [3].

1.5 Edge-Weighted Graphs

The finite connected graphs used in Baker-Norine theory can be generalized to edge-weighted graphs in the following way. Let G now be a connected *simple* graph; that is, loops and multiple edges are not allowed. Let $V(G) = \{v_1, \ldots, v_n\}$ be the vertex set of G, and assign each edge a nonnegative *weight*, w_{ij}, corresponding to the edge connecting vertices v_i and v_j. If v_i and v_j are not connected, set $w_{ij} = 0$. If the weights are real-valued, G is then a metric graph, where the lengths are $l_{ij} = w_{ij}^{-1}$ for $w_{ij} = 0$. More generally, we will consider weights in a subring R of the reals.

We define the degree of a vertex $v_j \in V(G)$ to be

$$\deg(v_j) = \sum_{i \neq j} w_{ij}.$$

The weight sum m is

$$m = \sum_{i<j} w_{ij}$$

so that the genus of G is

$$g = m - n + 1.$$

For non-integral weights, it is then possible to have a non-integral, or even negative genus.

For edge weights in $\mathbb{Z}$, an edge-weighted graph G is a multigraph, as described in the sections above, where the number of edges $p_{ij} = w_{ij}$. For edge weights in $\mathbb{R}$, we have a metric graph. Divisors over R are defined as in [3], with one exception: a divisor D is *effective* if its ceiling $\lceil D \rceil \geq 0$, or equivalently $D > -1$. This definition allows for compatibility with linear systems over $\mathbb{Z}$.

In Chapter 2, we develop the theory of linear systems on edge-weighted or R-graphs by extending the results of Baker and Norine. Chapters 3 and 4 use develop new independent results for linear systems on R-graphs.

Chapter 2

Riemann-Roch on R-graphs

2.1 Introduction

Let R be a subring of the real numbers $\mathbb{R}$. An *R-graph* G is a finite connected graph (without loops or multiple edges) where each edge is assigned a weight, which is a positive element of R. If we let the n vertices of G be $\{v_1, \ldots, v_n\}$, we will denote by $p_{ij} = p_{ji}$ the weight of the edge joining v_i and v_j. If there is no edge connecting v_i and v_j, we set $p_{ij} = p_{ji} = 0$.

We define the *degree* of a vertex v_j of G to be the sum of the weights of the edges incident to it:

$$\deg(v_j) = \sum_{i \neq j} p_{ij}.$$

The *edge matrix* P of G is the symmetric $n \times n$ matrix defined by

$$(P)_{ij} = \begin{cases} -p_{ij} & \text{if } i \neq j \\ \deg(v_j) & \text{if } i = j. \end{cases}$$

The *genus* of G is defined as

$$g = \sum_{i<j} p_{ij} - n + 1.$$

An *R-divisor* D on G is a formal sum

$$D = \sum_{i=1}^{n} d_i \cdot v_i$$

where each $d_i \in R$; the divisors form a free R-module $\mathrm{Div}(G)$ of rank n. We write $D_1 \geq D_2$ if the inequality holds at each vertex; for a constant c, we write $D \geq c$ (respectively $D > c$) if $d_i \geq c$ (respectively $d_i > c$) for each i.

The *degree* of a divisor D is

$$\deg(D) = \sum_{i=1}^{n} d_i$$

and the *ceiling* of D is the divisor

$$\lceil D \rceil = \sum_{i=1}^{n} \lceil d_i \rceil \cdot v_i.$$

The degree map is a homomorphism from $\mathrm{Div}(G)$ to R, and the kernel $\mathrm{Div}_0(G)$ of divisors of degree zero is a free R-module of rank $n-1$.

Let $H_j = \deg(v_j) \cdot v_j - \sum_{i \neq j} p_{ij} \cdot v_i$, and set $\mathrm{PDiv}(G) = \{\sum_{i=1}^{n} c_i H_i \mid c_i \in \mathbb{Z}\}$ to be the free $\mathbb{Z}$-module generated by the H_j. (Note that the H_j divisors correspond to the columns of the matrix P.) If G is connected, $\mathrm{PDiv}(G)$ has rank $n-1$. Note that $\mathrm{PDiv}(G) \subset \mathrm{Div}_0(G)$; the quotient group is called the *Jacobian* of G.

For two divisors $D, D' \in \mathrm{Div}(G)$, we say that D is *linearly equivalent* to D', and write $D \sim D'$, if and only if $D - D' \in \mathrm{PDiv}(G)$.

The *linear system* associated with a divisor D is

$$|D| = \{D' \in \mathrm{Div}(G) \mid D \sim D' \text{ with } \lceil D' \rceil \geq 0\} = \{D' \in \mathrm{Div}(G) \mid D \sim D' \text{ with } D' > -1\}.$$

We note that linearly equivalent divisors have the same linear system. The use of the ceiling divisor in the definition above is the critical difference between this theory and the integral theory developed by Baker and Norine [3].

The essence of the Riemann-Roch theorem, for divisors on algebraic curves, is to notice that the linear system corresponds to a vector space of rational functions, and to relate the dimensions of two such vector spaces. In our context we do not have vector spaces; so we measure the size of the linear system in a different way (as does Baker and Norine).

Define the h^0 of an R-divisor D to be

$$h^0(D) = \min\{\deg(E) \mid E \text{ is an } R\text{-divisor, } E \geq 0 \text{ and } |D - E| = \emptyset\}.$$

Note that $h^0(D) = 0$ if and only if $|D| = \emptyset$, and that linearly equivalent divisors have the same h^0.

The *canonical divisor* of G is defined as

$$K = \sum(\deg(v_i) - 2) \cdot v_i.$$

The Riemann-Roch result that we will prove can now be stated.

Theorem 2.1. *Let G be a connected R-graph as above, and let D be an R-divisor on G. Then*

$$h^0(D) - h^0(K - D) = \deg(D) + 1 - g.$$

The results of Baker and Norine (see [3]) are exactly that the above theorem holds in the case of the subring $R = \mathbb{Z}$. Our proof depends on the Baker-Norine Theorem in a critical way; it would be interesting to provide an independent proof.

In [12] and [14], a Riemann-Roch theorem is proved for metric graphs with integral divisors; these results differ from the present result in two fundamental ways. First, our edge weights p_{ij} and the coefficients of the divisors are elements of the ring R. Second, the genus g is in R for the present result, whereas in [12] and [14], g is a nonnegative integer.

As an example, consider the R-graph G with two vertices and edge weight $p > 0$. For convenience, we will write the divisor $a \cdot v_1 + b \cdot v_2$ as the ordered pair (a, b). The principal divisors are $\mathrm{PDiv}(G) = \{(np, -np) \mid n \in \mathbb{Z}\}$, and $K = (p - 2, p - 2)$, with $g = p - 1$. Note that if $p < 1$, we have $g < 0$.

For $(a, b) \in \mathrm{Div}(G)$, the linear system $|(a, b)|$ can be written as

$$\begin{aligned} |(a,b)| &= \{(c,d) \in \mathrm{Div}(G) \mid \lceil (c,d) \rceil \geq 0 \text{ and } (c,d) \sim (a,b)\} \\ &= \{(a + np, b - np) \mid n \in \mathbb{Z}, a + np > -1, b - np > -1\}. \end{aligned}$$

In what follows, we will be brief, and leave most of the details to the reader to verify. One can check that $|(a,b)| \neq \emptyset$ if and only if $\lceil (1+a)/p \rceil + \lceil (1+b)/p \rceil \geq 2$.

Let $\phi_p : R \times R \to \mathbb{Z}$ be defined as

$$\phi_p(x,y) = \lfloor (x+1)/p \rfloor + \lfloor (x+1)/p \rfloor.$$

The value of $h^0((a,b))$ can be computed as follows:

$$h^0((a,b)) = \begin{cases} 0 & \text{if } \phi_p(a,b) < 0 \\ \min\{a+1-p\lfloor (a+1)/p \rfloor, b+1-p\lfloor (b+1)/p \rfloor\} & \text{if } \phi_p(a,b) = 0 \\ a+b-p+2 & \text{if } \phi_p(a,b) > 0 \end{cases}$$

Note that if $D = (a,b) \in \mathrm{Div}(G)$ then $K - D = (p-2-a, p-2-b)$. To check that the Riemann-Roch theorem holds for D, it is easiest to consider the three cases (noted above) for the formula for $h^0((a,b))$. We note that (a,b) is in one of the three cases if and only if $(p-2-a, p-2-b)$ is in the opposite case. It is very straightforward then to check Riemann-Roch in case $\phi_p(a,b) \neq 0$; one of the two h^0 values is zero. It is a slightly more interesting exercise, but still straightforward, to check it in case $\phi_p(a,b) = 0$.

Unfortunately, this method of direct computation becomes intractible for R-graphs with $n > 2$.

2.2 Change of Rings

Note that in the definition of the h^0 of a divisor, the minimum is taken over all non-negative R-divisors. Therefore, a priori, the definition of h^0 depends on the subring R. We note that if $R \subset S \subset \mathbb{R}$ are two subrings of $\mathbb{R}$, then any R-graph G and R-divisor D on G is also an S-graph and an S-divisor. In this section we will see that the h^0 in fact does not depend on the subring.

Any $H \in \mathrm{PDiv}(G)$ can be written as an integer linear combination of any $n-1$ elements of the set $\{H_1, H_2, \ldots H_n\}$. If we exclude H_k, for example, then there are

$n-1$ integers $\{m_j\}_{j\neq k}$ such that $H = \sum_{j\neq k} m_j H_j$, and we can write $H = \sum_{i=1}^n h_i \cdot v_i$ where

$$h_i = \begin{cases} m_i \deg(v_i) - \sum_{j\neq k,i} m_j p_{ij} & \text{if } i \neq k \\ -\sum_{j\neq k} m_j p_{jk} & \text{if } i = k. \end{cases} \tag{2.2}$$

Let P_k be the $(n-1)\times(n-1)$ matrix obtained by deleting the kth row and column from the matrix P. We can write the h_i's other than h_k in matrix form as $\mathbf{h} = P_k\mathbf{m}$ where $\mathbf{h} = (h_i)_{i\neq k}$ and $\mathbf{m} = (m_i)_{i\neq k}$ are the corresponding column vectors.

For any $\mathbf{x} = (x_i) \in \mathbb{R}^{n-1}$ and $c \in \mathbb{R}$, we say $\mathbf{x} \geq c$ if and only if $x_i \geq c$ for each i; similarly for a matrix $A = (a_{ij})$, we write $A \geq c$ if and only if $a_{ij} \geq c$ for each i, j.

Lemma 2.3. *If $\boldsymbol{x} = (x_i)_{i\neq k}$ is a column vector in $\mathbb{R}^{n-1}$ such that $P_k\boldsymbol{x} \geq 0$, then $\boldsymbol{x} \geq 0$. Furthermore, P_k is nonsingular and $P_k^{-1} \geq 0$.*

Proof. Let $V_i = \{i' \mid p_{ii'} > 0, i' \neq k, i' \neq i\}$ be the set of indices of vertices connected to v_i (excluding k). Suppose that it is the case that $x_i < 0$, and that $x_i \leq x_{i'}$ for all $i' \in V_i$. Then

$$\begin{aligned} (P_k\mathbf{x})_i &= x_i \deg(v_i) - \sum_{i'\in V_i} x_{i'} p_{ii'} \\ &= x_i p_{ik} + x_i \sum_{i'\in V_i} p_{ii'} - \sum_{i'\in V_i} x_{i'} p_{ii'} \\ &= x_i p_{ik} + \sum_{i'\in V_i} p_{ii'}(x_i - x_{i'}), \end{aligned}$$

and we note that with our assumptions, no term here is positive. Since the sum is non-negative, we conclude that all terms are zero. We have verified the following therefore, if $P_k\mathbf{x} \geq 0$:

$$x_i < 0 \text{ and } x_i \leq x_{i'} \text{ for all } i' \in V_i \Rightarrow p_{ik} = 0 \text{ and } x_i = x_{i'} \text{ for all } i' \in V_i. \tag{2.4}$$

Now assume that $\mathbf{x} \not\geq 0$; then there is an index j such that $x = x_j < 0$ and $x_j \leq x_i$ for all $i \neq k$. By (2.4), we conclude that $x_i = x$ for all $i \in V_j$, and also that $p_{jk} = 0$. We see, by induction on the distance in G to the vertex v_j, that we must have $x_i = x$ and $p_{ij} = 0$ for all $i \neq k$. This contradicts the connectedness of G: vertex v_k has no edges on it. This proves the first statement.

Now suppose that $\mathbf{x} \in \ker P_k$; then $\mathbf{x} \geq 0$. Also, $-\mathbf{x} \in \ker P_k$, and thus $-\mathbf{x} \geq 0$; we conclude that $\mathbf{x} = \mathbf{0}$. Hence $\ker P_k = \{\mathbf{0}\}$ and P_k is invertible.

Let $\mathbf{y} = P_k\mathbf{x}$. Since $\mathbf{y} \geq 0 \Rightarrow \mathbf{x} \geq 0$ and P_k is invertible, $\mathbf{x} = P_k^{-1}\mathbf{y} \geq 0$ for all $\mathbf{y} \geq 0$. Applying $\mathbf{y} = \mathbf{e}_i$ for each $i \neq k$, where $(\mathbf{e}_i)_j = 1$ for $i = j$ and 0 otherwise, we have $P_k^{-1} \geq 0$. □

We can now prove the main result for this section.

Proposition 2.5. *Suppose that all of the entries of the matrix P are in two subrings R and R', and that all the coordinates of the divisor D are also in both R and R'. Then (using the obvious notation) $h^0 = h^{0'}$.*

Proof. It suffices to prove the statement when one of the subrings is R and the other is $\mathbb{R}$. In this case we'll use the notation Rh^0 and $\mathbb{R}h^0$, respectively, for the two minima in question.

First note that the linear system $|D|$ is clearly independent of the ring; and in particular, whether a linear system is empty or not is also independent.

Therefore, the minimum in question for the $\mathbb{R}h^0$ computation is over a strictly larger set of divisors; and hence there can only be a smaller minimum. This proves that $Rh^0(D) \geq \mathbb{R}h^0(D)$.

Suppose that E is an $\mathbb{R}$-divisor, $E \geq 0$, and $|D - E| = \emptyset$, achieving the minimum, so that $\mathbb{R}h^0(D) = \deg(E)$. If E is an R-divisor, it also achieves the minimum in R and $Rh^0(D) = \mathbb{R}h^0(D)$. We will show that in fact E must be an R-divisor.

Now suppose that E is not an R-divisor, and write $D = \sum_{i=1}^n d_i \cdot v_i$ and $E = \sum_{i=1}^n e_i \cdot v_i$, with k the index of an element such that $e_k \notin R$. Since $\mathbb{R}h^0(D) = \deg(E)$, for any $\epsilon \in \mathbb{R}$ with $0 < \epsilon \leq e_k$, we have that $E - \epsilon \cdot v_k \geq 0$, and therefore $|D - E + \epsilon \cdot v_k| \neq \emptyset$. Hence there are principal divisors H such that $D - E + \epsilon \cdot v_k + H > -1$.

Let $\mathcal{H}_\epsilon$ be the set of all such H; by assumption, this is a nonempty set. Note that if $H \in \mathcal{H}_\epsilon$, and $H = \sum_{i=1}^n h_i \cdot v_i$, then $d_i - e_i + h_i > -1$ for each $i \neq k$, and

$$d_k - e_k + \epsilon + h_k > -1. \tag{2.6}$$

Also, since $|D - E| = \emptyset$, there is a k' such that $d_{k'} - e_{k'} + h_{k'} \leq -1$; combined with the conditions above, the only possibility is $k' = k$. Since $d_k \in R$, $h_k \in R$ and $e_k \notin R$, $d_k - e_k + h_k \neq -1$, and thus $d_k - e_k + h_k < -1$. Hence $-1 - \epsilon < d_k - e_k + h_k < -1$.

For any $H \in \mathcal{H}_\epsilon$, there are unique integers m_i such that $H = \sum_{i \neq k} m_i H_i$. Let $\mathbf{d} = (d_i)_{i \neq k}$, $\mathbf{e} = (e_i)_{i \neq k}$, and $\mathbf{m} = (m_i)_{i \neq k}$ be the corresponding column vectors, and define $\mathbf{f} = (f_i)_{i \neq k} = \mathbf{d} - \mathbf{e} + P_k \mathbf{m}$. Note that $\mathbf{f} > -1$, and $h_k = -\sum_{i \neq k} m_k p_{ik}$ by (2.2).

We can write $\mathbf{m} = P_k^{-1}(\mathbf{f} - \mathbf{d} + \mathbf{e})$, and by Lemma 2.3, $P_k^{-1} \geq 0$. Therefore, since $\mathbf{e} \geq 0$ and $\mathbf{f} > -1$, the m_i are bounded from below; set $M \leq m_i$ for all $i \neq k$.

We claim that, for $H = \sum_{i \neq k} m_i H_i \in \mathcal{H}_\epsilon$, the possible coordinates $h_k = -\sum_{i \neq k} m_k p_{ik}$ form a discrete set. It will suffice to show that, for any real x, the possible coordinates h_k which are at least $-x$ is a finite set.

To that end, for any $x \in \mathbb{R}$ set $\mathcal{H}_\epsilon(x) = \{H \in \mathcal{H}_\epsilon \mid \sum_{i \neq k} m_i p_{ik} \leq x\}$; for large enough x this set is nonempty.

Fix $x \in \mathbb{R}$ such that $\mathcal{H}_\epsilon(x) \neq \emptyset$ and choose $j \neq k$ such that $p_{jk} > 0$. For $H = \sum_{i \neq k} m_i H_i \in \mathcal{H}_\epsilon(x)$ we then have

$$M \leq m_j \leq \frac{x - \sum_{i \neq j,k} m_i p_{ik}}{p_{jk}} \leq \frac{x - M \sum_{i \in V_k, i \neq j} p_{ik}}{p_{jk}}.$$

Thus the coefficients $m_j \in \mathbb{Z}$ are bounded both below and above, and hence can take on only finitely many values. It follows that the set of possible values of $h_k = -\sum_{i \neq k} m_i p_{ik}$ is also finite, for $H \in \mathcal{H}_\epsilon(x)$. As noted above, this implies that these coordinates h_k, for $H \in \mathcal{H}_\epsilon$, form a discrete set. This in turn implies that there is a maximum value h for the possible h_k, since for all such we have $d_k - e_k + h_k < -1$.

Note that if $\epsilon < \epsilon'$, then $\mathcal{H}_\epsilon \subset \mathcal{H}_{\epsilon'}$.

We may now shrink ϵ (if necessary) to achieve $\epsilon < e_k - d_k - h - 1$. This gives a contradition, since now $d_k - e_k + \epsilon + h_k \leq d_k - e_k + \epsilon + h < -1$ for $H \in \mathcal{H}_\epsilon$, violating (2.6). We conclude that E is in fact an R-divisor as desired, finishing the proof.

$\square$

The result above allows us to simply consider the case of $\mathbb{R}$-graphs.

At the other end of the spectrum, the case of $\mathbb{Z}$-graphs is equivalent to the Baker-Norine theory.

The Baker-Norine dimension of a linear system associated with a divisor D on a graph G defined in [3] is equal to

$$r(D) = \min\{\deg(E) \mid E \in \mathrm{Div}(G), E \geq 0 \text{ and } |D - E|_{BN} = \emptyset\} - 1$$

where here the linear system associated with a divisor D is

$$|D|_{BN} = \{D' \in \mathrm{Div}(G) \mid D' \geq 0 \text{ and } D \sim D'\}.$$

If we are restricted to $\mathbb{Z}$-divisors on $\mathbb{Z}$-graphs, the h^0 dimension is compatible with the Baker-Norine dimension:

Lemma 2.7. *If G is a $\mathbb{Z}$-graph and D a $\mathbb{Z}$-divisor on G, then $h^0(D) = r(D) + 1$.*

Proof. Note that $\lceil D \rceil = D$ since each component of D is in $\mathbb{Z}$. This implies that $|D| = |D|_{BN}$ which gives the result. $\square$

2.3 Reduction to $\mathbb{Q}$-graphs

Note that the definition of $h^0(D)$ depends on the coordinates of D and on the entries of the matrix P which give the edge-weights of the graph G. Indeed, the set $\mathcal{E}$ of divisors with empty linear systems depends continuously on P, as a subset of $\mathbb{R}^n$. (If $\mathcal{F}_0$ is the set of divisors D with $d_i > -1$ for each i, then $\mathcal{E}$ is the complement of the union of all the translates of $\mathcal{F}_0$ by the columns of P.)

The value of $h^0(D)$ is essentially the taxicab distance from D to $\mathcal{E}$. This also depends continuously on the coordinates of D.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, by approximating both P and D by rationals, we see that it will suffice to prove the Riemann-Roch theorem for $\mathbb{Q}$-graphs:

Proposition 2.8. *Suppose that the Riemann-Roch Theorem 2.1 is true for connected $\mathbb{Q}$-graphs. Then the Riemann-Roch Theorem is true for connected $\mathbb{R}$-graphs.*

2.4 Scaling

Suppose that G is an R-graph, with edge weights p_{ij}. For any $a > 0$, $a \in R \subset \mathbb{R}$, define aG to be the R-graph with the same vertices, and edge weights $\{ap_{ij}\}$. In other words, if P defines G, then aG is the R-graph defined by the matrix aP.

We will use subscripts to denote which R-graph we are using to compute with, e.g., $|D|_G$, $h^0_G(D)$, etc. if necessary.

For any divisor D on G and $a > 0$, define

$$T_a(D) = aD + (a-1)I$$

where

$$I = \sum_i 1 \cdot v_i.$$

The transformation T_a is a homothety by a, centered at $-I$.

Lemma 2.9. *Let D be an R-divisor. If $a, b > 0$ with $a, b \in R$, then the following hold:*

1. $T_b \circ T_b = T_{ab}$

2. $T_a(D + H) = T_a(D) + aH$

3. $\lceil D \rceil \geq 0 \Leftrightarrow \lceil T_a(D)) \rceil \geq 0$

4. $|D|_G \neq \emptyset \Leftrightarrow |T_a(D)|_{aG} \neq \emptyset$

5. $|D - E|_G \neq \emptyset \Leftrightarrow |T_a(D) - aE|_{aG} \neq \emptyset$

Proof. 1. Suppose that $D = \sum_i d_i \cdot v_i$. Then:

$$\begin{aligned}
T_a(T_b(D)) &= T_a\left(\sum_i (bd_i + b - 1) \cdot v_i\right) \\
&= \sum_i (a(bd_i + b - 1) + a - 1) \cdot v_i \\
&= \sum_i (abd_i + ab - a + a - 1) \cdot v_i \\
&= \sum_i (abd_i + ab - 1) \cdot v_i \\
&= T_{ab}(D).
\end{aligned}$$

2. Let $a > 0$ and $D, H \in \mathrm{Div}(G)$, then

$$\begin{aligned}
T_a(D + H) &= a(D + H) + (a - 1)I \\
&= aD + aH + (a - 1)I \\
&= T_a(D) + aH.
\end{aligned}$$

3. Let $D = \sum_i d_i \cdot v_i \in \mathrm{Div}(G)$ and $a > 0$. Since $T_a(D) = \sum_i (ad_i + a - 1) \cdot v_i$, we have

$$\begin{aligned}
\lceil T_a(D)) \rceil \geq 0 &\Leftrightarrow ad_i + a - 1 > -1 \text{ for each } i \\
&\Leftrightarrow d_i > -1 \text{ for each } i \\
&\Leftrightarrow \lceil D \rceil \geq 0.
\end{aligned}$$

4. Suppose $|D|_G \neq \emptyset$. Then there is a $H \in \mathrm{PDiv}(G)$ such that $\lceil D + H \rceil \geq 0$. Since $T_a(D + H) = T_a(D) + aH$ and $aH \in \mathrm{PDiv}(aG)$, by part (3) we have $\lceil T_a(D) + aH \rceil \geq 0$ and thus $|T_a(D)|_{aG} \neq \emptyset$.

   The converse is an identical argument.

5. Let $D' = D - E$; then from (4), $|D'|_G \neq \emptyset \Leftrightarrow |T_a(D')|_{aG} \neq \emptyset$ where $T_a(D') = T_a(D - E) = T_a(D) - aE$.

□

Corollary 2.10. $h^0_{aG}(T_a(D)) = ah^0_G(D)$

Proof. Since $a > 0$, from Lemma 2.9 (5) we have

$$\begin{aligned}
h^0_{aG}(T_a(D)) &= \min_{E' \in \mathrm{Div}(aG)} \{\deg(E') \mid E' \geq 0, |T_a(D) - E'|_{aG} = \emptyset\} \\
&= \min_{E \in \mathrm{Div}(G)} \{\deg(aE) \mid aE \geq 0, |T_a(D) - aE|_{aG} = \emptyset\} \\
&= a\left(\min_{E \in \mathrm{Div}(G)} \{\deg(E) \mid E \geq 0, |T_a(D) - aE|_{aG} = \emptyset\}\right) \\
&= a\left(\min_{E \in \mathrm{Div}(G)} \{\deg(E) \mid E \geq 0, |D - E|_{G} = \emptyset\}\right) \\
&= ah^0_G(D).
\end{aligned}$$

□

Lemma 2.11. *Let D be an R-divisor. If $a > 0$ with $a \in R$ then the following hold:*

1. $K_{aG} = T_a(K_G) + (a-1)I$
2. $K_{aG} - T_a(D) = T_a(K_G - D)$
3. $\deg(T_a(D)) = a\deg(D) + (a-1)(n)$
4. $g_{aG} = ag_G + (a-1)(n-1)$.

Proof. 1. Since $K_{aG} = \sum_i (a\deg(v_i) - 2) \cdot v_i$, we have

$$\begin{aligned}
T_a(K_G) &= T_a(\sum_i (\deg(v_i) - 2) \cdot v_i) \\
&= a\sum_i (\deg(v_i) - 2) \cdot v_i + \sum_i (a-1) \cdot v_i \\
&= \sum_i (a\deg(v_i) - 2a + a - 1) \cdot v_i \\
&= \sum_i (a\deg(v_i) - a - 1) \cdot v_i \\
&= K_{aG} - (a-1)I.
\end{aligned}$$

2.

$$
\begin{aligned}
K_{aG} - T_a(D) &= T_a(K_G) + (a-1)I - T_a(D) \\
&= aK_G + (a-1)I + (a-1)I - aD - (a-1)I \\
&= a(K_G - D) + (a-1)I \\
&= T_a(K_G - D).
\end{aligned}
$$

3.

$$
\begin{aligned}
\deg(T_a(D)) &= \deg(aD + (a-1)I) \\
&= a\deg(D) + (a-1)\deg(I) \\
&= a\deg(D) + (a-1)(n).
\end{aligned}
$$

4.

$$
\begin{aligned}
g_{aG} &= \sum_i ap_{ij} - n + 1 \\
&= a\sum_i p_{ij} - an + a + (a-1)n + 1 - a \\
&= ag_G + (a-1)(n-1).
\end{aligned}
$$

□

2.5 Reduction to $\mathbb{Z}$-graphs

Theorem 2.12. *Let $a > 0$; then*

$$h^0_G(D) - h^0_G(K_G - D) = \deg(D) - g_G + 1 \tag{2.13}$$

if and only if

$$h^0_{aG}(T_a(D)) - h^0_{aG}(K_{aG} - T_a(D)) = \deg(T_a(D)) - g_{aG} + 1. \tag{2.14}$$

Proof. Let $a > 0$. Multiplying (2.13) by a, we have

$$ah_G^0(D) - ah_G^0(K_G - D) = a\deg(D) - ag_G + a.$$

The left side of this equation is equal to

$$h_{aG}^0(T_a(D)) - h_{aG}^0(T_a(K_G - D)) = h_{aG}^0(T_a(D)) - h_{aG}^0(K_{aG} - T_a(D))$$

using Corollary 2.10 and Lemma 2.11 (2). The right side of the equation is

$$\deg(T_a(D)) - (a-1)(n) - g_{aG} + (a-1)(n-1) + a = \deg(T_a(D)) - g_{aG} + 1$$

using Lemma 2.11 (3) and (4). This proves that (2.13) implies (2.14); the converse is identical. □

Corollary 2.15. *Suppose that the Riemann-Roch Theorem 2.1 is true for connected $\mathbb{Z}$-graphs. Then the Riemann-Roch Theorem is true for connected $\mathbb{Q}$-graphs.*

Proof. Given a connected $\mathbb{Q}$-graph G and a $\mathbb{Q}$-divisor D on it, there is an integer $a > 0$ such that aG is a connected $\mathbb{Z}$-graph and $T_a(D)$ is a $\mathbb{Z}$-divisor. Therefore by hypothesis, the Riemann-Roch statement (2.14) holds. Hence by Theorem 2.12, (2.13) holds, which is the Riemann-Roch theorem for D on G. □

We now have the ingredients to prove Theorem 2.1.

Proof. First, we note again that the Riemann-Roch Theorem of [3] is equivalent to the Riemann-Roch theorem for connected $\mathbb{Z}$-graphs in our terminology. Therefore, using Corollary 2.15, we conclude that the Riemann-Roch Theorem is true for connected $\mathbb{Q}$-graphs. Then, using Proposition 2.8, we conclude that Riemann-Roch holds for connected $\mathbb{R}$-graphs.

Finally, Proposition 2.5 finishes the proof of the Riemann-Roch theorem for divisors on arbitrary R-graphs, for any subring $R \subset \mathbb{R}$. □

2.6 Conclusions

An independent proof of Theorem 2.1 that does not depend on [3] may be possible using properties of linear systems divisors on R-graphs developed in Chapter 4. This was the original motivation for developing the theory of sandpiles on R-graphs in Chapter 3.

One of the most important observations we made in extending the theory of linear systems to edge-weighted graphs was to note that the divisor $(-1, -1, \ldots, -1)$ plays the role of the *origin* in a sense. This is apparent in the definition of an effective divisor being $\lceil D \rceil \geq 0$, which is equivalent to $D > -1$. As we will see in Chapters 2 and 3, using the -1-*point* is in fact what allows the theory to go through and be consistent with the integral theory for the related sandpiles and linear systems over R on edge-weighted graphs.

Chapter 3

Sandpiles on R-graphs

3.1 Introduction

Biggs [5] showed that the certain configurations of a chip-firing game on a graph led to a group, which he called the *critical group* of a graph. Earlier work by Dhar [10] and later in [11], referred to such configurations as stable sandpile configurations, after the abelian sandpile model introduced in [2]. In the sandpile model, sand particles located on the vertices are *toppled* one vertex at a time until the number of particles is less than a critical number, usually the degree of the vertex. When no more sand particles can be toppled, the configuration is called *stable*. We extend the stable sandpile model to edge-weighted graphs over a subring R of the reals, and consider a different method of toppling where multiple vertices can be toppled concurrently.

Let R be any subring of the reals and G be a connected edge-weighted graph over R with vertex set $V = \{v_0, v_1, \ldots, v_n\}$ and weight set $W = \{w_{ij} \mid i, j = 0, \ldots, n\}$ where each $w_{ij} \in R$. Multiple edges and loops are not allowed, and we set $w_{ij} = 0$ if v_i and v_j are not connected; otherwise, $w_{ij} > 0$. Note that $w_{ii} = 0$. We will define the degree of a vertex v_j to be

$$\deg(v_j) = \sum_{i=0}^{n} w_{ij}$$

and the parameter g (the genus of the graph) to be

$$g = \sum_{i<j} w_{ij} - n.$$

Note that if $R = \mathbb{Z}$, these definitions coincide with those of a multigraph where the number of edges connecting v_i andv_j is w_{ij}.

Define $V_0 = V - \{v_0\}$, where v_0 will be called a *sink*. the choice of v_0 to be the sink is arbitrary, and is used for notational convenience. A *sandpile* on G is a function $s : V_0 \to R$ that satisfies $\lceil s(v) \rceil \geq 0$, or equivalently $s(v) > -1$, for each $v \in V_0$.

A *singleton toppling* T_i for $i \in \{1, \ldots, n\}$ acts on sandpile s by

$$T_i(s)(v_j) = \begin{cases} s(v_j) - \deg(v_j) & \text{if } i = j \\ s(v_j) + w_{ij} & \text{if } j \neq i. \end{cases}$$

If s is such that $s(v) > \deg(v) - 1$ for each $v \in V_0$, we say that s is *full*; if $s(v) \leq \deg(v) - 1$ for each $v \in V_0$, we say that s is *stable*. If s is full, any toppling T_i, $i \in \{1, \ldots, n\}$, can act on s since $T_i(s)(v) > -1$ for each $v \in V_0$. Similarly, if s is stable, no singleton toppling T_i can act on s since $T_i(s)(v_i) \leq -1$ for each i. For $R = \mathbb{Z}$, these definitions are consistent with those for chip-firing and sandpiles on multigraphs such as in [5] and [11].

Let $I \subset \{1, \ldots, n\}$ be nonempty. The *subset toppling* T_I acts on s by

$$T_I(s)(v_j) := \begin{cases} s_j - \deg(v_j) + \sum_{i \in I} w_{ij} & \text{if } j \in I \\ s_j + \sum_{i \in I} w_{ij} & \text{if } j \notin I. \end{cases}$$

Paoletti [16] refers to this as *cluster-toppling*.

We can extend subset toppling to a multiset of $\{1, \ldots, n\}$ by choosing coefficients $c(v) \in \mathbb{Z}^+$ for each $v \in V_0$ (where $\mathbb{Z}^+ = \{n \in \mathbb{Z} \mid n \geq 0\}$); the corresponding *multiset toppling* T is then

$$T := \sum_{i=1}^{n} c(v_i) T_i.$$

which acts on sandpile s by

$$T(s)(v_j) = s(v_j) - c(v_j) \deg(v_j) + \sum_{i=0}^{n} c(v_i) w_{ij}.$$

Since a singleton or subset toppling is a multiset toppling, we will use the term *toppling* to mean a multiset toppling. A toppling $T = \sum_{i=1}^{n} c(v_i)T_i$ is called *allowable* on a sandpile s if $T(s)(v) > -1$ for each $v \in V_0$. If no multiset topplings are allowable on a sandpile s, it is called *superstable*.

3.2 Superstable Sandpiles

It is clear that if s is superstable, no subset topplings are allowable on s. However, it is possible for s to be stable but not superstable. For example, consider a graph with $n = 2$ and $w_{01} = w_{02} = w_{12} = 1$. The sandpile s with $s(v_1) = s(v_2) = 1$ is stable, but not superstable since $T_{\{1,2\}}$ is allowable on s.

Lemma 3.1. *A sandpile s is superstable if and only if no subset topplings are allowable on s.*

Proof. Since s superstable implies that no subset topplings are allowable, we need only show the converse is true.

Suppose s is not superstable. From the definition of an allowable toppling, it follows that there is a $c : V_0 \to \mathbb{Z}^+$ that is not identically zero such that

$$s(v_j) > c(v_j)\deg(v_j) - \sum_{i=1}^{n} c(v_i)w_{ij} - 1$$

for each $j \in \{1, \ldots, n\}$. Let $\alpha = \max\{c_1, \ldots, c_n\}$ and set $(b_1, \ldots, b_n) \in \mathbb{Z}_+^n$ as follows:

$$b_i = \begin{cases} 1 & \text{if } c_i = \alpha \\ 0 & \text{otherwise.} \end{cases}$$

We claim that

$$s(v_j) > b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} - 1$$

for each $j \in \{1, \ldots, n\}$.

If $b_j = 0$, we have

$$b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} - 1 = -\sum_{i=1}^{n} b_i w_{ij} - 1 \leq -1 < s(v_j).$$

Let $A_j = \{i > 0 \mid w_{ij} > 0 \text{ and } c_i < \alpha\}$, $B_j = \{i > 0 \mid w_{ij} > 0 \text{ and } a_i = \alpha\}$. If $b_j = 1$, then $c_j = \alpha$ and

$$\begin{aligned} s(v_j) &> c_j \deg(v_j) - \sum_{i=1}^{n} c_i w_{ij} - 1 \\ &= \alpha \deg(v_j) - \sum_{i \in A_j} c_i w_{ij} - \alpha \sum_{i \in B_j} w_{ij} - 1 \\ &= \alpha(w_{0j} + \sum_{i \in A_j} w_{ij} + \sum_{i \in B_j} w_{ij}) - \sum_{i \in A_j} c_i w_{ij} - \alpha \sum_{i \in B_j} w_{ij} - 1 \\ &= \alpha w_{0j} + \sum_{i \in A_j} (\alpha - c_i) w_{ij} - 1 \\ &\geq w_{0j} + \sum_{i \in A_j} w_{ij} - 1. \end{aligned}$$

Also, we have

$$\begin{aligned} b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} &= \deg(v_j) - \sum_{i \in B_j} w_{ij} \\ &= w_{0j} + \sum_{i \in A_j} w_{ij} + \sum_{i \in B_j} w_{ij} - \sum_{i \in B_j} w_{ij} \\ &= w_{0j} + \sum_{i \in A_j} w_{ij} \end{aligned}$$

thus $s(v_j) > b_j \deg(v_j) - \sum_{i=1}^{n} b_i w_{ij} - 1$ for each $j = 1, \ldots, n$. Since $T = \sum_{i=1}^{n} b_j T_j$ is a subset toppling, s has an allowable subset toppling. $\square$

Lemma 3.2. *If a sandpile s is superstable, then*

$$\sum_{v \in V_0} s(v) \leq g,$$

with equality if and only if there exists a permutation $(j_1, j_2, \ldots, j_n)$ of $(1, 2, \ldots, n)$ such that

$$s(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

for each $k = 1, \ldots, n$, where $j_0 = 0$.

Proof. Suppose that s is superstable, then by Lemma 3.1 all subset topplings T_I, $I \subset \{1, \ldots, n\}$, are not allowable on s. This means that for each I, there exists a

$j \in I$ such that

$$s(v_j) \leq \deg(v_j) - \sum_{i \in I} w_{ij} - 1 = \sum_{i \notin I} w_{ij} - 1. \tag{3.3}$$

Suppose that $I = I_0 = \{1, \ldots, n\}$, and that that (3.3) is satified for $j = j_1 \in I_0$, then

$$s(v_{j_1}) \leq \sum_{i \notin I_0} w_{ij_1} - 1 = w_{0j_1} - 1.$$

Now let $I = I_1 = I_0 - \{j_1\}$, then (3.3) is satified for some $j_2 \in I_1$ so that

$$s(v_{j_2}) \leq \sum_{i \notin I_1} w_{ij_2} - 1 = w_{0j_2} + w_{j_1 j_2} - 1.$$

Similarly, for $I = I_2 = I_1 - \{j_2\}$, (3.3) is satisfied for $j = j_3$ and

$$s(v_{j_3}) \leq \sum_{i \notin I_2} w_{ij_3} - 1 = w_{0j_3} + w_{j_1 j_3} + w_{j_2 j_3} - 1.$$

Continuing this process, set $j_0 = 0$ and let $I_k = I_{k-1} - \{j_k\}$ for $k = 1, \ldots, n-1$ where j_k is the j satisying (3.3) for I_{k-1}, and we have in general

$$s(v_{j_k}) \leq \sum_{i=0}^{k-1} w_{j_i j_k} - 1. \tag{3.4}$$

Note that the resulting n-tuple $(j_1, j_2, \ldots, j_n)$ is a permutation of $(1, 2, \ldots, n)$. If we rewrite (3.4) as

$$s(v_{j_k}) - \sum_{i=0}^{k-1} w_{j_i j_k} + 1 \leq 0$$

and sum over all k, we have

$$\sum_{k=1}^{n} \left(s(v_{j_k}) - \sum_{i=0}^{k-1} w_{j_i j_k} + 1 \right) = \sum_{j=1}^{n} s(v_j) - \sum_{i<j} w_{ij} + n \leq 0$$

or equivalently

$$\sum_{v \in V_0} s(v) \leq \sum_{i<j} w_{ij} - n = g.$$

For the second part, note that if

$$s(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

holds for some $(j_1, \ldots, j_n)$ at each $k \in \{1, \ldots n\}$, then

$$\sum_{v \in V_0} s(v) = g \tag{3.5}$$

follows directly. For the other direction, assume that (3.5) holds. It then follows from above that

$$s(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$. $\square$

3.3 Allowable Topplings

We now show that the maximum of two allowable topplings is allowable.

Lemma 3.6. *Let s be a sandpile. If $T = \sum_{i=1}^{n} c(v_i)T_i$ and $T' = \sum_{i=1}^{n} c'(v_i)T_i$ are allowable topplings on s, then*

$$\max(T, T') = \sum_{i=1}^{n} \max(c(v_i), c'(v_i))T_i$$

is also an allowable toppling on s.

Proof. Since $T(s)(v) > -1$ and $T'(s)(v) > -1$ for each $v \in V_0$, we have

$$s(v_j) > \sum_{i=1}^{n}(c(v_j) - c(v_i))w_{ij} + c(v_j)w_{0j} - 1$$

and

$$s(v_j) > \sum_{i=1}^{n}(c'(v_j) - c'(v_i))w_{ij} + c'(v_j)w_{0j} - 1.$$

If $\max(c(v_j), c'(v_j)) = c(v_j)$, then

$$s(v_j) > \sum_{i=1}^{n}(c(v_j) - \max(c(v_i), c'(v_i)))w_{ij} + c(v_j)w_{0j} - 1$$

and similarly if $\max(c(v_j), c'(v_j)) = c'(v_j)$,

$$s(v_j) > \sum_{i=1}^{n}(c'(v_j) - \max(c(v_i), c'(v_i)))w_{ij} + c'(v_j)w_{0j} - 1.$$

Combining these, we have

$$s(v_j) > \sum_{i=1}^{n} (\max(c(v_j), c'(v_j)) - \max(c(v_i), c'(v_i)))w_{ij} + \max(c(v_j), c'(v_j))w_{0j} - 1$$

for each $j \in \{1, \ldots, n\}$, thus $\max(T, T')$ is allowable on s. □

Let $x : V_0 \to \mathrm{Frac}(R)$ be an arbitrary function, and Δ_0 be the reduced Laplacian of G given by

$$\Delta_0 x(v_j) = x(v_j)\deg(v_j) - \sum_{i=1}^{n} x(v_i)w_{ij}$$

where $j \in \{1, \ldots, n\}$. For $x, y : V_0 \to \mathrm{Frac}(R)$, we will use the following notation:

$$x \geq y \Leftrightarrow x(v) \geq y(v) \text{ for all } v \in V_0$$

and similarly with a scalar $a \in \mathrm{Frac}(R)$

$$x \geq a \Leftrightarrow x(v) \geq a \text{ for all } v \in V_0.$$

Let s be any sandpile. The toppling associated with $c : V_0 \to \mathbb{Z}^+$ is then allowable on s if and only if $s + 1 > \Delta_0 c$.

Lemma 3.7. *Let* $x, y : V_0 \to \mathrm{Frac}(R)$.

1. *If* $\Delta_0 x \geq 0$*, then* $x \geq 0$.

2. *The inverse* Δ_0^{-1} *exists, and if* $y \geq 0$ *then* $\Delta_0^{-1} y \geq 0$.

3. *If* $x \geq 0$*,* $y \geq 0$ *and* $y \geq \Delta_0 x$*, then* $\Delta_0^{-1} y \geq x$.

Proof. (1): Suppose that $\Delta_0 x \geq 0$ and and $x(v_i) < 0$ for some i. Let $\beta = \min\{x(v) \mid v \in V_0\}$ and set

$$K_j = \{i \mid w_{ij} > 0 \text{ and } 0 < i \leq n\}.$$

We can then write

$$\deg(v_j) = w_{0j} + \sum_{i \in K_j} w_{ij}$$

and

$$\Delta_0 x(v_j) = x(v_j)\deg(v_j) - \sum_{i \in K_j} x(v_i) w_{ij} \geq 0.$$

If $x(v_j) = \beta$, then $w_{0j} = 0$ and

$$x(v_j)\deg(v_j) = \sum_{i \in K_j} x(v_i) w_{ij}$$

thus $x(v_i) = \beta$ for each $i \in K_j$. Since G is finite and connected, we have $w_{0k} = 0$ for each $1 \leq k \leq n$, which implies that the sink v_0 is not connected to any other vertex; thus by contradiction, $x(v) \geq 0$ for all $v \in V_0$.

(2): Assume (1) holds, and $\Delta_0 x(v) = 0$ for each $v \in V_0$ for some $x : V_0 \to \mathrm{Frac}(R)$. Since $\Delta_0 x \geq 0 \Rightarrow x \geq 0$ and $\Delta_0 x = 0 \Rightarrow \Delta(-x) = 0$, we have $x = 0$. Since $\Delta_0 0 = 0$, $\ker(\Delta_0) = \{0\}$ and thus Δ_0 has an inverse. Suppose $y = \Delta_0 x \geq 0$. By (1), $\Delta_0^{-1} y = x \geq 0$.

(3): Since (2) $\Rightarrow \Delta_0^{-1} y \geq 0$ whenever $y \geq 0$, and $y \geq \Delta_0 x$, $y - \Delta_0 x \geq 0$ thus we have $\Delta_0^{-1}(y - \Delta_0 x) = \Delta_0^{-1} y - x \geq 0$, and $\Delta_0^{-1} y \geq x$. □

Let $\mathcal{A}(s) = \{c : V_0 \to \mathbb{Z}^+ \mid \sum_{i=1}^{n} c(v_i) T_i \text{ allowable on } s\}$. Note that if s is superstable, then $0 \in \mathcal{A}(s) \neq \emptyset$.

Lemma 3.8. *For any sandpile s, there exists a function $b : V_0 \to \mathrm{Frac}(R)$ such that $b \geq c$ for all $c \in \mathcal{A}(s)$.*

Proof. Let $c \in \mathcal{A}(s)$, then $s + 1 > \Delta_0 c$, where $s + 1 > 0$. By Lemma 3.7, we have $\Delta_0^{-1} s \geq c$, thus set $b = \Delta_0^{-1} s$. □

Lemma 3.9. *For each sandpile s, there is a unique $c \in \mathcal{A}(s)$ such that $s - \Delta_0 c$ is superstable.*

Proof. By Lemmas 3.6 and 3.8, there is a unique maximal $c \in \mathcal{A}(s)$. If $s - \Delta_0 c$ is not superstable, then there is a $d \in \mathcal{A}(s - \Delta_0 c)$ and $s - \Delta_0 c + 1 > \Delta_0 d$. Then, we would have $s + 1 > \Delta_0(c + d)$, which means that $c + d \in \mathcal{A}(s)$. Since $d \neq 0$, c cannot be maximal in $\mathcal{A}(s)$, which leads to a contradiction. Hence $s - \Delta_0 c$ is superstable. □

3.4 Superstable Group

Let $c \in \mathcal{A}(s)$ be the unique maximal element in Lemma 3.9 and define $[s] := s - \Delta_0 c$. If r and s are any two sandpiles, we write $r \sim s$, if and only if $[r] = [s]$. Let $\mathcal{S}$ be the set of all superstable sandpiles (on G). For any two superstable sandpiles $r, s \in \mathcal{S}$, define $r \oplus s$ to be $[r + s]$.

Define the set of *zero* sandpiles to be

$$\mathcal{Z} = \{s \mid [s] = 0\} = \{\Delta_0 c \mid c : V_0 \to \mathbb{Z}^+\}.$$

Lemma 3.10. *Given any sandpile s, there is a $\hat{s} \in \mathcal{Z}$ such that $\hat{s} \geq s$.*

Proof. Let $C_0 = \{x : V_0 \to \mathbb{R}) \mid \Delta_0 x \geq 0\}$, $C_s = \{x : V_0 \to \mathbb{R} \mid \Delta_0 x \geq s\}$, and $K = \{x : V_0 \to \mathbb{R} \mid x \geq 0\}$. By Lemma 3.7, $C_s \subset C_0 \subset K$. If $x, y \in C_0$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta \geq 0$, then since Δ_0 is linear

$$\Delta_0(\alpha x + \beta y) = \alpha \Delta_0 x + \beta \Delta_0 y \geq 0$$

and thus $\alpha x + \beta y \in C_0$ (which is a cone), and since Δ_0 is injective C_0 has an interior. Hence C_s is a nondegenerate affine cone in $\mathbb{R}^n$ and $C_s \cap (\mathbb{Z}^+)^n$ contains an infinite number of integer lattice points. Let $c \in C_s \cap (\mathbb{Z}^+)^n$ such that $\Delta_0 c \geq s$ and set $\hat{s} = \Delta_0 c \in \mathcal{Z}$. $\square$

Theorem 3.11. *The set of superstable sandpiles $\mathcal{S}$ form a group under $\oplus$.*

Proof. Addition on $\mathcal{S}$ is clearly commutative and closed. The zero sandpile 0 is the unit for $\mathcal{S}$ since $s \oplus 0{=}s$ for all $s \in \mathcal{S}$.

Let $q, r, s \in \mathcal{S}$. By Lemma 3.9 there are $a, b : V_0 \to \mathbb{Z}^+$ such that $q \oplus r = q + r - \Delta_0 a$ and $(q \oplus r) \oplus s = (q \oplus r) - \Delta_0 b$. Similarly, there are $c, d : V_0 \to \mathbb{Z}^+$ such that $r \oplus s = r + s - \Delta_0 c$ and $q \oplus (r \oplus s) = q + (r \oplus s) - \Delta_0 d$. Since we have

$$(q \oplus r) \oplus s = (q + r - \Delta_0 a) + s - \Delta_0 b = q + r + s - \Delta_0(a + b)$$

and also

$$q \oplus (r \oplus s) = q + (r + s - \Delta_0(c)) - \Delta_0(d) = q + r + s - \Delta_0(c + d),$$

the fact that $[q+r+s]$ is unique implies that $a+b=c+d$, hence $\oplus$ is associative.

Let $s \in \mathcal{S}$. From Lemma 3.10, let $\hat{s} \in \mathcal{Z}$ be such that $\hat{s} \geq s$, and set $s' = [\hat{s} - s]$. Since $s' \oplus s = [\hat{s} - s + s] = [\hat{s}] = 0$, $s' \in \mathcal{S}$ is the inverse of s. □

3.5 Conclusions

In Chapter 4 we show that the superstable group is isomorphic to the graph Jacobian $\mathrm{Jac}(G)$. Biggs [5] showed that the *critical group* of the graph, which is equivalent to the abelian sandpile group of Dhar [10], is also isomorphic to $\mathrm{Jac}(G)$. Thus for $R = \mathbb{Z}$, the superstable group and the critical group are isomorphic. These two groups do not generally have the same indentity, however, since the superstable identity is the zero sandpile, while the critical group identity is almost always not the zero sandpile (see for example [6]).

For a general subring R, however, it appears that we need to show that the set of stable sandpiles obtained by a sequence of singleton topplings from a *full* sandpile (where each $s(v) > \deg(v) - 1$) form a group that is isomorphic to the superstable group. This would then generalize the critical group to R-valued sandpiles on R-graphs.

Chapter 4

Applications to Linear Systems

4.1 Introduction

In this chapter we apply results from Chapter 3 to linear systems of divisors on graphs. We will use the notation of Chapter 3, where R is a subring of the reals, G is a R-graph with vertices $V(G) = \{v_0, \ldots, v_n\}$ and edge weights $\{w_{ij}\}$.

Recall that a divisor $D \in \mathrm{Div}(G)$ on the R-graph G is a function $D : V \to R$, which we can write as the formal sum

$$D = \sum_{i=0}^{n} D(v_i) \cdot v_i.$$

The degree of D is defined as $\deg(D) = \sum_{v \in V} D(v)$, and the divisors of degree zero are denoted $\mathrm{Div}^0(G)$. The principal divisors $\mathrm{PDiv}(G)$ are generated by $\{H_j \mid j = 0, \ldots, n\}$ over $\mathbb{Z}$, where

$$H_j = \deg(v_j) \cdot v_j - \sum_{i \neq j} w_{ij} \cdot v_i.$$

We can extend a sandpile s from V_0 to V by defining

$$s(v_0) = -\sum_{i=1}^{n} s(v_i),$$

creating a natural map from sandpiles to divisors of degree zero on G. Similarly, a multiset toppling $T = \sum_{i=1}^{n} c(v_i)T_i$ applied to s corresponds to the divisor

$$(s(v_0) + \sum_{j=1}^{n} c(v_j)w_{0j})) \cdot v_0 + \sum_{j=1}^{n}(s(v_j) - c(v_j)\deg(v_j) + \sum_{i=1}^{n} c(v_i)w_{ij}) \cdot v_j$$

which can also be written as

$$(\sum_{i=0}^{n} s(v_i) \cdot v_i) - (\sum_{j=1}^{n} c(v_j)H_j).$$

Note that the second term $\sum_{j=1}^{n} c(v_j)H_j \in \mathrm{PDiv}(G)$, thus multiset toppling corresponds to translation of a divisor by a principal divisor.

4.2 Graph Jacobian

The graph Jacobian $\mathrm{Jac}(G)$ (or degree zero Picard group $\mathrm{Pic}^0(G)$) is the quotient $\mathrm{Div}^0(G)/\mathrm{PDiv}(G)$. The correspondence between multiset toppling and principal divisors can be used to show that the superstable sandpile group $\mathcal{S}$ is isomorphic to $\mathrm{Jac}(G)$.

Theorem 4.1. $\mathcal{S} \cong \mathrm{Div}^0(G)/\mathrm{PDiv}(G)$.

Proof. Define $\phi : \mathcal{S} \to \mathrm{Div}^0(G)/\mathrm{PDiv}(G)$ by $s \mapsto \sum_{i=0}^{n} s(v_i) \cdot v_i + \mathrm{PDiv}(G)$ as defined above. If $s, s' \in \mathcal{S}$, we clearly have $\phi(s + s') = \phi(s) + \phi(s')$ so ϕ is a homomorphism.

Choose $D \in \mathrm{Div}^0(G)$. By Lemma 3.10, choose $\hat{s} \in \mathcal{Z}$ such that

$$\hat{s} \geq |\min_i\{D(v) \mid v \in V_0\}|$$

and set $s(v) = \hat{s}(v) + D(v)$ for each $v \in V_0$. Then $s(v) \geq 0$ for each $v \in V_0$ and there is a unique $c : V_0 \to \mathbb{Z}^+$ such that $[s] = s - \Delta_0 c \in \mathcal{S}$. Since $\phi(\hat{s}) = 0 + \mathrm{PDiv}(G)$, we have $\phi([s]) = D + \mathrm{PDiv}(G)$, and thus ϕ is surjective.

Let $s \in \ker\phi$, and suppose that $s \neq 0$. Since $\phi(s) = \mathrm{PDiv}(G)$, we must have $\sum_{i=0}^{n} s(v_i) \cdot v_i \in \mathrm{PDiv}(G)$, so $s = \Delta_0 a$ for some $a : V_0 \to \mathbb{Z}^+$. By Lemma 3.7, $s \geq 0$ implies that $a \geq 0$, and thus either s is not superstable or $s = 0$, which leaves us with $\ker\phi = 0$ and ϕ injective. $\square$

It follows directly from Theorem 4.1 that the superstable sandpile group $\mathcal{S}$ is independent (up to isomorphism) of choice of sink v_0.

Corollary 4.2. *Up to isomorphism, $\mathcal{S}$ is independent of choice of sink v_0 in $V(G)$.*

Up to this point, we have assume v_0 to be the sink, and our notation has reflected this assumption by using the subscript 0 for $V_0 = V(G) - \{v_0\}$ and Δ_0 for the reduced laplacian. We will generally continue using v_0 as the sink for notational convenience, but Corollary 4.2 implies that the sandpile results hold for any sink in $V(G)$. Thus, in general any v_k could the sink, with $V_k = V(G) - \{v_k\}$, etc. We will exploit this symmetry in the next section.

4.3 Empty Linear Systems

Recall that two divisors D and D' are linearly equivalent, written $D \sim D'$, if $D - D' \in \mathrm{PDiv}(G)$, and the linear system associated with D is

$$|D| = \{D' \mid D' \sim D, D' > -1\}.$$

We aim to describe in this section the divisors whose linear system is empty.

Define the set of divisors with empty linear systems as

$$\mathcal{E}(G) = \{D \in \mathrm{Div}(G) \mid |D| = \emptyset\},$$

and let

$$\mathcal{N}(G) = \{D \in \mathrm{Div}(G) \mid \deg(D) = g - 1, |D| = \emptyset\} \subset \mathcal{E}(G).$$

Lemma 4.3. *Let $D \in \mathrm{Div}(G)$. There is a unique $D_0 \in \mathrm{Div}(G)$ such that $D_0 \sim D$ and the sandpile defined by $D_0(v)$ for $v \in V_0$ is superstable.*

Proof. Define the sandpile s on G as follows: set

$$\alpha = \min_{v \in V_0}\{D(v)\}$$

and $z \in \mathcal{Z}$ such that $z(v) \geq \max\{0, -\alpha\}$, then let

$$s'(v) = D(v) + z(v)$$

for $v \in V_0$, thus $s'(v) \geq 0$ for each $v \in V_0$. Now, let $s = [s'] = s' - z'$ where $z' \in \mathcal{Z}$ and set

$$\beta = D(v_0) + \sum_{i=1}^{n} z'(v_i) - z(v_i)$$

so that s is the unique superstable sandpile (by Lemma 3.9). Define the divisor D_0 by

$$D_0(v) = \begin{cases} \beta & v = v_0 \\ s(v) & v \in V_0 \end{cases}$$

where we have $D \sim D_0$. $\square$

We will call such a divisor D_0 in Lemma 4.3 the *superstable* divisor corresponding to D. Such divisors are referred to as *reduced* in [3], and are an example of a so-called G-parking function (see [17]).

An immediate application of Lemma 4.3 gives a sufficient condition for a linear system of a divisor to be nonempty.

Lemma 4.4. *Let* $D \in \mathrm{Div}(G)$*. If* $\deg(D) > g - 1$ *then* $|D| \neq \emptyset$*.*

Proof. Let D be a divisor with $\deg(D) > g-1$, and let D_0 be the unique superstable divisor such that $D_0 \sim D$. Since D_0 restricted to V_0 is a superstable sandpile by Lemma 3.2

$$\sum_{v \in V_0} D_0(v) \leq g.$$

By assumption we have

$$\deg(D) = \deg(D_0) = D_0(v_0) + \sum_{v \in V_0} D_0(v) > g - 1,$$

or equivalently

$$D_0(v_0) > -\sum_{v \in V_0} D_0(v) + g - 1,$$

thus $D_0(v_0) > -1$. Since $D_0(v) > -1$ for each $v \in V_0$, $|D| \neq \emptyset$. $\square$

Lemma 4.5. *If D_0 be a superstable divisor, then $|D_0| \neq \emptyset$ if and only if $D_0(v_0) > -1$.*

Proof. If $D_0(v_0) > -1$, then $D_0(v) > -1$ for all $v \in V$ and $|D_0| \neq \emptyset$. Now assume that $|D_0| \neq \emptyset$, thus there is a $P \in \mathrm{PDiv}(G)$ such that $D_0 + P > -1$. Since D_0 restricted to V_0 a superstable sandpile, there are no allowable topplings on V_0, hence the only $P \in \mathrm{PDiv}(G)$ which would satisfy $D_0 + P > -1$ must have $P(v) \geq 0$ for all $v \in V_0$. Since $\deg(P) = 0$, $P(v_0) \leq 0$, thus we must have $D_0 > -1$ in order for $|D_0|$ to be nonempty. $\square$

Finally, we turn our attention to the subset of empty divisors $\mathcal{N}(G)$.

Lemma 4.6. *If D_0 is a superstable divisor with $\deg(D_0) = g - 1$ and $|D_0| = \emptyset$, then*

$$D_0(v_{j_k}) = \begin{cases} -1 & k = 0 \\ \sum_{i=0}^{k-1} w_{j_i j_k} - 1 & k > 0 \end{cases}$$

where $j_0 = 0$ and $(j_1, \ldots, j_n)$ is a permutation of $(1, \ldots, n)$.

Proof. Since D_0 is superstable with $|D_0| = \emptyset$, by Lemma 4.5 $D_0(v_0) \leq -1$, thus by Lemma 3.2 we then have $\sum_{i=1}^{n} D_0(v_i) = g$, $D_0(v_0) = -1$ and

$$D_0(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$. $\square$

We will denote the set of such superstable divisors by

$$\mathcal{N}_0(G) = \{D \in \mathcal{N}(G) \mid D \text{ is superstable } \}.$$

Note that $|\mathcal{N}_0(G)| \leq n!$. A direct consequence of Lemma 4.6 then gives us the composition of $\mathcal{N}(G)$, which is a lattice generated by $\mathcal{N}_0(G)$.

Lemma 4.7. $\mathcal{N}(G) = \{D \in \mathrm{Div}(G) \mid D \sim D_0$ *where* $D_0 \in \mathcal{N}_0(G)\}$.

Proof. If $D \in \mathcal{N}(G)$, then by Lemma 4.3 there is a $D_0 \in \mathcal{N}_0(G)$ such that $D \sim D_0$. $\square$

The canonical divisor on G is given by $K = \sum_v (\deg(v) - 2) \cdot v$. We shall see that $\mathcal{N}(G)$ is invariant under the map $D \mapsto K - D$.

Lemma 4.8. *$D \in \mathcal{N}(G)$ if and only if $K - D \in \mathcal{N}(G)$.*

Proof. Since any $D \in \mathcal{N}(G)$ can be written as $D = N_0 + P$ for some $P \in \mathrm{PDiv}(G)$, it is sufficient to assume $D \in \mathcal{N}_0(G)$.

Assume that D is superstable and that $D = N_0$ for some $N_0 \in \mathcal{N}_0(G)$. By Lemma 4.6 $N_0(v_{j_0}) = -1$ and $N_0(v_{j_k}) = \sum_{i=0}^{k-1} w_{j_i j_k} - 1$ for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$ with $j_0 = 0$. Since $K(v_i) = \sum_{j=0}^{n} w_{ij} - 2$, for $k > 0$ we have

$$(K - D)(v_{j_k}) = \sum_{i=0}^{n} w_{j_i j_k} - 2 - \sum_{i=0}^{k-1} w_{j_i j_k} + 1 = \sum_{i=k}^{n} w_{j_i j_k} - 1$$

and for $k = 0$

$$(K - D)(v_{j_0}) = \sum_{i=0}^{n} w_{j_i j_0} - 1.$$

Note that $(K - D)(v_{j_n}) = -1$. Let $l_k = j_{n-k}$ for $k = 0, \ldots, n$. We then have

$$(K - D)(v_{l_0}) = -1$$

and

$$(K - D)(v_{l_k}) = \sum_{i=0}^{k} w_{l_i l_k} - 1.$$

Thus, since $(l_1, \ldots, l_n) = (j_{n-1}, \ldots, j_0)$ is a permutation of n-tuple derived from the vertex index space V_{j_n}, and by Corollary 4.2 $K - D \in \mathcal{N}_{j_n}(G) \subset \mathcal{N}(G)$.

Now assume that $K - D \in \mathcal{N}_0(G)$. Let $D' = K - D$, and from above we have $K - D' = D \in \mathcal{N}(G)$. $\square$

We now give a description of the empty set $\mathcal{E}(G)$.

Theorem 4.9. *If $D \in \mathcal{E}(G)$, then $D \leq N$ for some $N \in \mathcal{N}(G)$.*

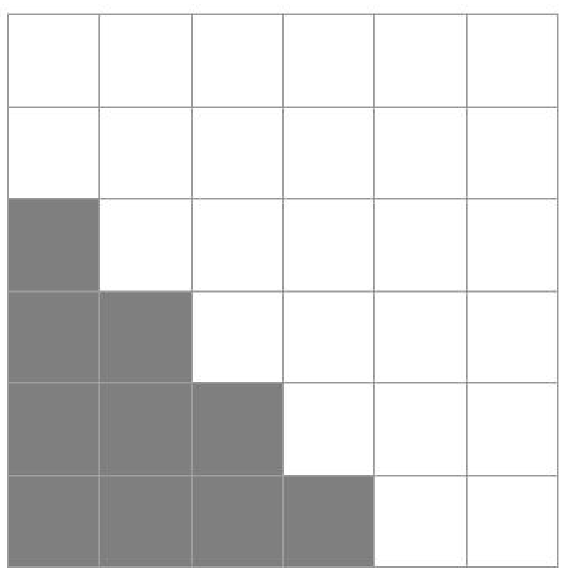
Proof. Let $D \in \mathrm{Div}(G)$ with $|D| = \emptyset$. By Lemma 4.3, there is a unique superstable divisor $D_0 \sim D$. Since $|D_0| = \emptyset$, Lemma 4.5 implies that $D_0(v_0) \leq -1$. By the proof of Lemma 3.2, we have that (3.4) holds for each $D_0(v)$ where $v \in V_0$, so for some permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$,

$$D_0(v_{j_k}) \leq \sum_{i=0}^{k-1} w_{j_i j_k} - 1$$

and thus $D_0 \leq N_0$ for one of the $N_0 \in \mathcal{N}_0(G)$. Let $P \in \mathrm{PDiv}(G)$ such that $D = D_0 + P$, and let $N = N_0 + P$. Then we have $D \leq N$ where $N \in \mathcal{N}(G)$. $\square$

The set of divisor with empty linear systems is thus generated by the set of superstable divisors of degree $g-1$ with empty linear systems. Over $\mathbb{R}$, the boundary of this set is a polyhedral surface in $(n+1)$-space.

As an example, consider a graph G with $n = 1$ (two vertices) and edge weight p. Let $D = (a, b) \in \mathrm{Div}(G)$, then following the example in §2.1, we have $|D| = \emptyset$ if and only if $\lceil (1+a)/p \rceil + \lceil (1+b)/p \rceil < 2$. We can use this condition to plot the empty set (the gray region) in $\mathbb{R}^2$ as shown below for the two-vertex graph with $p = 1$. The center point of the plot is $(0, 0)$ with unit grid spacing in both directions. The empty set $\mathcal{E}(G)$ is then the set of all points (a, b) such that $a \leq np - 1$ and $b \leq -np + p - 1$ for all $n \in \mathbb{Z}$.



For a three vertex graph with $n = 2$, suppose the edge-weights are $p_{01} = p$, $p_{12} = q$, and $p_{02} = r$. The genus is $g = p+q+r-2$, and the two superstable divisors of degree

$g-1$ with empty linear systems are $(-1, p-1, q+r-1)$ and $(-1, p+q-1, r-1)$. These two points generate a lattice in the plane of divisors of degree $g-1$ via linear equivalence.

By defining the empty set geometrically, we can define the dimension $h^0(D)$ as the minimum *distance* in R^{n+1} from D to $\mathcal{E}(D)$. Define the *distance* function $d : \mathrm{Div}(G) \times \mathrm{Div}(G) \to R$ to be

$$d(D, D') = \sum_{v \in V(G)} |D(v) - D(v')|$$

for $D, D' \in \mathrm{Div}(G) \cong R^{n+1}$.

Lemma 4.10. *For any* $D \in \mathrm{Div}(G)$*,*

$$h^0(D) = \min\{d(D, E) \mid E \in \mathcal{E}(G), D - E \geq 0\}.$$

Proof. The result follows directly from the definition of $h^0(D)$:

$$\begin{aligned} h^0(D) &= \min\{\deg(D-E) \mid D - E \geq 0, |E| = \emptyset\} \\ &= \min\{d(D, E) \mid E \in \mathcal{E}(G), D - E \geq 0\}. \end{aligned}$$

□

4.4 Conclusions

Perhaps the most interesting result in this chapter is that the set of divisors of degree $g-1$ with empty linear systems is generated by a known finite set of size $\leq n!$ by Lemma 4.6. This enables the set of divisors with empty linear systems to be defined by cones from the points in $\mathcal{N}(G)$. Exploiting the symmetry of the $\mathcal{N}(G)$ set may lead to an independent proof of Theorem 2.1. Also, knowing what the points in $\mathcal{N}(G)$ are allows the computation of $h^0(D)$ for a fixed G, which we will use for $\mathbb{Z}$-graphs in Chapter 5.

Chapter 5

Compatible Line Bundles

5.1 Introduction

In this chapter we wish to address the following question: Given a n-vertex $\mathbb{Z}$-graph G with an effective divisor $D = (d_1, \ldots, d_n)$, can we find a nodal curve X_G and a line bundle L_D with multidegree $(d_1, \ldots, d_n)$ on X_G such that the dimension of L_D matches $h^0(D)$?

Consider a $\mathbb{Z}$-graph G with two vertices v_1 and v_2 joined by p edges. G corresponds to the curve $X_G = X_1 \cup X_2$ where $X_1, X_2 \cong \mathbb{P}^1$, with X_1 and X_2 intersecting transversely p times. If $D = (d_1, d_2) \in \mathrm{Div}(G)$ is an effective divisor, let L_1,L_2 be line bundles on X_1,X_2 (respectively) with degree d_1, d_2. The kth intersection condition for the bundles L_1 and L_2 on X_1 and X_2 is

$$f_1(q_{12k}) = \lambda f_2(q_{21k})$$

where q_{12k} and q_{21k} are the respective coordinates of the kth intersection point with $1 \leq k \leq p$, f_1 and f_2 are polynomials of degree d_1 and d_2, and λ is a nonzero parameter. Let $f_i(x) = \sum_{j=0}^{d_1} a_{ij} x^j$ for $i = 1, 2$, and the intersection condition for the kth point is

$$a_{10} + a_{11} q_{12k} + \cdots + a_{1d_1} q_{12k}^{d_1} = \lambda \left(a_{20} + a_{21} q_{21k} + \cdots + a_{2d_2} q_{21k}^{d_2} \right). \qquad (5.1)$$

Now assume G is a $\mathbb{Z}$-graph with n vertices $V(G) = \{v_1, \ldots, v_n)$, and $p_{ij} \geq 0$ edges joining vertices v_i and v_j. Set $m = \sum_{i<j} p_{ij}$ to be the total number of edges. The corresponding nodal curve is $X_G = \cup_{i=1}^n X_i$.

Suppose $D = \sum_{i=1}^n d_i \cdot v_i$ is an effective divisor on G, with corresponding line bundle L_D on X_G with multidegree $(d_1, \ldots, d_n)$. Let A_{ij}^d be the Vandermonde matrix

$$A_{ij}^d = \begin{pmatrix} 1 & q_{i,j,1} & q_{i,j,1}^2 & \cdots & q_{i,j,1}^d \\ 1 & q_{i,j,2} & q_{i,j,2}^2 & \cdots & q_{i,j,2}^d \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & q_{i,j,p_{ij}} & q_{i,j,p_{ij}}^2 & \cdots & q_{i,j,p_{ij}}^d \end{pmatrix}$$

corresponding to the polynomial $f_i(x) = a_{i,0} + a_{i,1}x + \cdots + a_{i,d_i}x^{d_i}$ evaluated at the points q_{ijk} for $1 \leq k \leq p_{ij}$. Note that if we assume the q_{ijk} are distinct, this matrix has full rank.

Let $\mathbf{f}_i$ be the coefficient vector

$$\mathbf{f}_i = \begin{pmatrix} a_{i,0} \\ a_{i,1} \\ \vdots \\ a_{i,d_i} \end{pmatrix}$$

and set $\underline{\lambda}_{ij} = (\lambda_{ij1}, \ldots, \lambda_{ijp_{ij}})$, the row vector of *gluing data*, with each $\lambda_{ijk} \in \mathbb{C}^*$. The intersection condition (5.1) for $X_i \cdot X_j$ can then be written as

$$A_{ij}^{d_i}\mathbf{f}_i - \lambda_{ij} A_{ji}^{d_j}\mathbf{f}_j.$$

For the entire graph G, we construct by concatenation the coefficient vector

$$\mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \end{pmatrix}$$

and we can represent the intersection conditions by the following block matrix

$$M = \begin{pmatrix} A_{12}^{d_1} & -\underline{\lambda}_{12}A_{21}^{d_2} & 0 & \cdots & 0 & 0 \\ A_{13}^{d_1} & 0 & -\underline{\lambda}_{13}A_{31}^{d_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ A_{1n}^{d_1} & 0 & 0 & \cdots & 0 & -\underline{\lambda}_{1n}A_{n1}^{d_n} \\ 0 & A_{23}^{d_2} & -\underline{\lambda}_{23}A_{32}^{d_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & A_{2n}^{d_2} & 0 & \cdots & 0 & -\underline{\lambda}_{2n}A_{n2}^{d_n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{n-1,n}^{d_{n-1}} & -\underline{\lambda}_{n-1,n}A_{n,n-1}^{d_n} \end{pmatrix}$$

as $M\mathbf{f}$, where 0 above represents the appropriate zero block matrix. Note that the size of M is $m \times (\deg(D) + n)$, and that each row has two nonzero blocks and each column has $n-1$ nonzero blocks. The dimension of the linear system can then be computed using the rank-nullity theorem by

$$\dim(H^0(X_G, L_D)) = \deg(D) + n - \operatorname{rank}(M). \tag{5.2}$$

Our aim in this chapter is to describe the conditions on the parameters λ and intersection coordinates q such that dimension of the bundle on X_G matches that of the dimension of the corresponding divisor D on G, $h^0(D)$, which is given by

$$h^0(D) = \min\{\deg(E) \mid E \in \operatorname{Div}(G), E \geq 0 \text{ and } |D - E| = \emptyset\} \tag{5.3}$$

as in §2, which is equivalent to the $r(D) + 1$ in [3].

5.2 Binary Curves

We will begin with a binary curve, which is described by a graph with two vertices v_1 and v_2 connected by p edges. The block matrix for the intersection conditions is

$$M = \begin{pmatrix} A_{12}^{d_1} & -\underline{\lambda}A_{21}^{d_2} \end{pmatrix}$$

where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_p)$.

Theorem 5.4. *Let G be a two vertex graph with p edges, and $D = (d_1, d_2)$ an effective divisor on G. If the corresponding intersection points on each component of X_G are equal, that is $q_{12k} = q_{21k}$ for each $k = 1, \ldots, p$, then*

$$\max_{\underline{\lambda}} \left\{ \dim(H^0(X_G, L_D)) \right\} = h^0(D).$$

Proof. Since $\dim(H^0(X_G, L_D)) = \deg(L_D) + n - \operatorname{rank}(M)$, the maximum dimension occurs when M has minimum rank.

If $p \leq \max\{d_1, d_2\}$, each of the Vandermonde matrices $A_{12}^{d_1}$ and $A_{21}^{d_2}$ have rank p. Then M also has rank p and $\dim(H^0(X_G, L_D)) = d_1 + d_2 + 2 - p$. Since $h^0(D) = \deg(D) - p$ (see the example at the end of §2.1), we have $\dim(H^0(X_G, L_D)) = h^0(D)$.

If $p > \max\{d_1, d_2\}$, we need to determine λ_k's such that

$$M = \begin{pmatrix} 1 & q_{121} & \cdots & q_{121}^{d_1} & -\lambda_1 & -\lambda_1 q_{211} & \cdots & -\lambda_1 q_{211}^{d_2} \\ 1 & q_{122} & \cdots & q_{122}^{d_1} & -\lambda_2 & -\lambda_2 q_{212} & \cdots & -\lambda_2 q_{212}^{d_2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & q_{12p} & \cdots & q_{12p}^{d_1} & -\lambda_p & -\lambda_p q_{21p} & \cdots & -\lambda_p q_{21p}^{d_2} \end{pmatrix}$$

has minimal rank. Since $q_{12k} = q_{21k}$ for each k, if we set $\lambda_k = \lambda$ for each k, M will have a minimal set of linearly independent columns and have rank $\max\{d_1, d_2\} + 1$, thus

$$\dim(H^0(X_G, L_D)) = d_1 + d_2 - \max\{d_1, d_2\} + 1 = \min\{d_1, d_2\} + 1.$$

Since $h^0(D) = \min\{d_1, d_2\} + 1$, we have again $\dim(H^0(X_G, L_D)) = h^0(D)$. $\square$

Note that if $q_{12k} \neq q_{21k}$ for each k, in the case $p > \max\{d_1, d_2\}$ above, $\operatorname{rank}(M) = \min\{p, d_1 + d_2\}$ when $\lambda_k = \lambda$, thus the requirement that the intersection points have the same coordinates on both components is in general a necessary condition.

5.3 Ternary Curves

For a three-vertex graph, the intersection condition matrix with the condition $q_{ijk} = q_{jik}$ at each intersection point is

$$M = \begin{pmatrix} A_{12}^{d_1} & -\lambda_{12} A_{12}^{d_2} & 0 \\ A_{13}^{d_1} & 0 & -\lambda_{13} A_{13}^{d_3} \\ 0 & A_{23}^{d_2} & -\lambda_{23} A_{23}^{d_3} \end{pmatrix}$$

where the λ_{ij} are again vectors of length p_{ij}. We shall see that for ternary curves, we do not have the nice result as above with binary curves. Consider the following examples.

Example 5.5. *Let $p_{12} = p_{13} = 1$, $p_{23} = 2$, with divisor $D = 3 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$. Note that this curve is not stable. The corresponding intersection condition matrix is*

$$M = \left(\begin{array}{cccc|c|c} 1 & q_1 & q_1^2 & q_1^3 & -\lambda_1 & 0 \\ \hline 1 & q_2 & q_2^2 & q_2^3 & 0 & -\lambda_2 \\ \hline 0 & 0 & 0 & 0 & 1 & -\lambda_3 \\ 0 & 0 & 0 & 0 & 1 & -\lambda_4 \end{array}\right)$$

which has rank 4 for general λ's and thus

$$\dim(H^0(X_G, L_D)) = 3 + 3 - 4 = 2$$

The graph dimension is $h^0(D) = 2$ for G, so the general line bundle has the correct dimension. However, with $\lambda_1 = \lambda_2$ and $\lambda_3 = \lambda_4 = 1$, M has rank 3 and thus

$$\dim(H^0(X_G, L_D)) = 3 + 3 - 3 = 3$$

and thus

$$\max_{\underline{\lambda}} \left\{ \dim(H^0(X_G, L_D)) \right\} > h^0(D).$$

Example 5.6. *Let G be a three-vertex graph with edges $p_{12} = 1, p_{13} = p_{23} = 2$ and again set $D = 3 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$. This graph corresponds to a stable curve.*

The graph dimension for this graph is $h^0(D) = 2$. *For general* $\underline{\lambda}$, $\operatorname{rank}(M) = 5$, *thus* $\dim(H^0(X_G, L_D)) = 3 + 3 - 5 = 1$. *If we set* $\underline{\lambda}_{13} = \underline{\lambda}_{23}$, $\operatorname{rank}(M) = 4$ *and* $\dim(H^0(X_G, L_D)) = 2$.

Lemma 5.7. *If* $D_0 \in \operatorname{Div}(G)$ *is superstable and effective, then* $h^0(D_0) \leq d_1 + 1$.

Proof. Assume that v_1 is the sink. For D_0 to be superstable and effective, we have

$$d_1 \geq 0$$

with either

$$0 \leq d_2 \leq p_{12} - 1$$
$$0 \leq d_3 \leq p_{13} + p_{23} - 1$$

or

$$0 \leq d_3 \leq p_{13} - 1$$
$$0 \leq d_2 \leq p_{12} + p_{23} - 1.$$

From [3], we know that we can compute $h^0(D)$ by

$$h^0(D) = \min\{\deg^+(D' - \nu) \mid \nu \in \mathcal{N}(G), D' \sim D\}$$

where

$$\deg^+(D) = \sum_{d_i \geq 0} d_i$$

and $\mathcal{N}(G) = \{D \in \operatorname{Div}(G) \mid \deg(D) = g - 1, |D| = \emptyset\}$. We know from the results of Chapter 4 that $\mathcal{N}(G)$ is generated from $\mathcal{N}_0(G) = \{N_1, N_2\}$ where

$$\begin{aligned} N_1 &= (-1, p_{12} - 1, p_{13} + p_{23} - 1) \\ N_2 &= (-1, p_{12} + p_{23} - 1, p_{13} - 1). \end{aligned}$$

Thus, we have

$$h^0(D_0) \leq \min\{\deg^+(D_0 - N_1), \deg^+(D_0 - N_2)\} \leq d_1 + 1.$$

$\square$

Theorem 5.8. *Let G be a three vertex graph with $g = 0$. If D is an effective divisor on G, then there is a L_D on X_G such that $h^0(D) = \dim(H^0(X_G, L_D))$.*

Proof. A genus zero graph has two edges, and there are three such configurations. It suffices to show the statement holds for one of the three, so we will choose $p_{12} = p_{13} = 1$ and $p_{23} = 0$. Let $D = (d_1, d_2, d_3)$. The intersection matrix is

$$M = \begin{pmatrix} A_{12}^{d_1} & -\lambda_{12}A_{12}^{d_2} & 0 \\ A_{13}^{d_1} & 0 & -\lambda_{13}A_{13}^{d_3} \end{pmatrix}$$

where $\mathrm{rank}(A_{12}) = \mathrm{rank}(A_{13}) = 1$. It follows that $\mathrm{rank}(M) = 2$, thus

$$\dim(H^0(X_G, L_D)) = \deg(D) + 1.$$

Since $g = 0$, Riemann-Roch implies that

$$h^0(D) \geq \deg(D) + 1.$$

Let $D_0 = (\bar{d}_1, \bar{d}_2, \bar{d}_3)$ be a superstable divisor such that $D \sim D_0$ by Lemma 4.3. From Lemma 5.7, we have

$$h^0(D_0) \leq \bar{d}_1 + 1$$

and since $g = 0$, this forces $\bar{d}_2 = \bar{d}_3 = 0$. We then have

$$h^0(D_0) \leq \deg(D_0) + 1$$

and thus

$$\dim(H^0(X_G, L_D)) = h^0(D).$$

□

There are two types of three-vertex graphs of genus one: the $p_{12} = p_{13} = p_{23} = 1$ graph, and six variants of $p_{12} = 2, p_{13} = 1, p_{23} = 0$. We show below that in the first case, there is a compatible L_D for any effective D on G.

Theorem 5.9. *Let G be a three vertex graph with one edge connecting each vertex to the other two vertices. If D is an effective divisor on G, then there is a line bundle L_D on X_G such that $h^0(D) = \dim(H^0(X_G, L_D))$.*

Proof. The intersection matrix for L_D is

$$M = \begin{pmatrix} A_{12}^{d_1} & -\lambda_{12}A_{12}^{d_2} & 0 \\ A_{13}^{d_1} & 0 & -\lambda_{13}A_{13}^{d_3} \\ 0 & A_{23}^{d_2} & -\lambda_{23}A_{23}^{d_3} \end{pmatrix}$$

where $\mathrm{rank}(A_{ij}^{d_k}) = 1$ for each i, j, k since each $p_{ij} = 1$, and thus $\mathrm{rank}(M) \leq 3$. Also, M could be of the form

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

which has a rank of 2, thus $\dim(H^0(X_G, L_D))$ is either $\deg(D)$, or $\deg(D) + 1$ if $\mathrm{rank}(M) = 2$.

Let $D_0 \sim D$ be superstable, in which case either $d_2 = 0$ and $d_3 \leq 1$ or $d_3 = 0$ and $d_2 \leq 1$. Riemann-Roch implies that $h^0(D) \geq \deg(D)$. For this configuration, we have $N_1 = (-1, 0, 1)$ and $N_2 = (-1, 1, 0)$, and principal divisors $P_1 = (2, -1, -1)$ and $P_2 = (-1, 2, -1)$. Let $N_3 = N_1 + P_1$, $N_4 = N_2 + P_1$, $N_5 = N_2 - P_2$ and $N_6 = N_1 - P_1 - P_2$; collectively, the N_i represent the six permutations of $(-1, 1, 0)$. Using the argument in Lemma 5.7, we have

$$h^0(D_0) \leq \min\{\deg^+(D_0 - N_j) \mid j = 1, \ldots, 6\}.$$

Note that if $\deg(D_0) \geq 1$, we have $h^0(D_0) \leq \deg(D_0)$. If $\deg(D_0) = 0$, we have $h^0(D_0) \leq 1 = \deg(D_0) + 1$, and since $|D_0| \neq \emptyset$, it is in fact an equality. In this case, a compatible L_D corresponds to the special rank 2 matrix above. For $\deg(D_0) \geq 1$, the general L_D gives the correct dimension, and thus we have $h^0(D) = \dim(H^0(X_G, L_D))$. $\square$

It seems probable that this is always the case for any three-vertex graph G, and we have yet to find a counterexample, but proving the general case would seem to require a different technique than used above.

5.4 Conclusions

The original motivation of the work in this chapter was to find a way of describing the (X_G, L_D) pairs that correspond to a given graph-divisor pair (G, D), where the dimensions of L_D and D match. Ideally, one would prefer to find a family of such (X_G, L_D) that can be easily described as with the two-vertex case, but such a description has been elusive thus far. In fact, although it seems probable, we do not know that for any (G, D), such a (X_G, L_D) exists. We conjecture that this is indeed the case:

Conjecture 5.10. *Let G be a connected multigraph. If D is an effective divisor on G, then there is a line bundle L_D on X_G such that*

$$h^0(D) = \dim(H^0(X_G, L_D)).$$

Immediate future work involves proving the conjecture for ternary curves. Ultimately, beyond proving the conjecture for any (G, D), we would like to understand much more about the deeper connections between Riemann-Roch theory for graphs and that of algebraic curves.

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