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BUOYANCY EFFECTS ON A TURBULENT SHEAR FLOW

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Buoyancy Effects on a Turbulent Shear Flow

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It has long been recognized that the buoyancy force due to density stratification has pronounced effects on the turbulence structure. A number of investigations have utilized stability corrections based on the assumption of the existence of an eddy viscosity or eddy diffusivity. Unfortunately such models are incapable of physically behaving as the measurements in the presence of strong stable or unstable stratifications suggest. Recently Donaldson et al. (1972), Lumley (1972), Daly (1972) and Lee (1974) have proposed closures of the equations of motion in the presence of buoyancy forces which require equations for all Reynold's stresses and heat fluxes. Unfortunately even for a one-dimensional model one must at a minimum then solve simultaneously nine partial differential equations and one algebraic equation. Other theories suggest an even higher total.

Utilizing a simple time dependent one-dimensional example as a test case this report discusses a solution which represents the important characteristics of a buoyancy dominated shear flow by solving four partial differential equations in addition to the mean equations of motion. This suggested model solves equations for total turbulent kinetic energy, k, total turbulent temperature fluctuations, k_t, eddy dissipation, ε, and thermal eddy dissipation, ε_t. Three separate versions of this model are discussed--an algebraic length scale version, a Prandtl-Kolmogorov eddy viscosity version, and an algebraic stress and heat flux model. The final version (requiring six partial differential equations) manages to replicate results for a much more complicated

version (requiring ten partial differential equation). The advantages for two and three dimensional problems are even greater.

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Symbol	Definition
A_1, A_2, A_3 C_D, C_H C_{E1}, C_{E2} C_{Et1}, C_{Et2}, F C_{P1}, C_{P2}, C_{P3}	Constants, See Table 13
g	Gravitational constant
ℓ	Mixing length or length scale
k	Turbulent kinetic energy $(\overline{u_i' u_i'})/2$
k_t	Turbulent temperature fluctuations $(\overline{T'^2})/2$
p	Pressure
Pr	Prandtl number (ν/α)
Re	Reynolds number $\frac{u_{max} L}{\nu}$
Ri	Richardson number $\frac{g}{T} \frac{\Delta T_{max} L}{(u_{max})^2}$
t	Time
T	Temperature
u,v,w	Velocity components
x,y,z	Coordinate directions
X	Body force
α	Bradshaw proportionality constant $\overline{u'w'} = \alpha\, k$
α_T	Eddy diffusivity
δ	Shear layer height
ε	Eddy dissipation of turbulent kinetic energy
ε_t	Eddy dissipation of turbulent temperature fluctuations
ξ	Dimensionless length
λ, Λ	Length scales: Donaldson

Symbol	Definition
ν_T	Eddy viscosity
ρ	Density
σ_k, σ_{k_t}, σ_ε, σ_{ε_t}	Effective Prandtl number for k, k_t, ε, ε_t
ϕ	Gravitational potential

Subscripts

i, j, k	Direction indices

Superscripts

'	Fluctuating guantity
*	Dimensional value

1. Introduction

Neutral stratification is fairly rare in nature; however, since gravity effects are small in aerodynamics there is a tendency to eliminate them from classical expositions on fluid mechanics. If heterogeneity and gravity are both present, the situation is not merely more complicated. Often their interplay produces striking phenomena entirely unexpected! Flow and transport in the atmosphere, in the ocean, and more and more frequently in hydraulics and industrial processes requires consideration of stratification.

The buoyancy force due to density stratification has pronounced effects on the turbulent structure of a shear layer. In fact, the dispersion of atmospheric pollutants is intimately related to the vertical temperature distribution in the atmospheric turbulent boundary layer because of such induced buoyancy effects on turbulent diffusion. Similarly the growth and character of atmospheric clear air turbulence associated with high altitude jet streams or the unusual layered character of certain ocean thermoclines can be attributed to the effect of stratification on turbulence.

Buoyancy acts selectively on the vertical component of turbulence; just as a shear force acts selectively on the longitudinal component of turbulence. Energy is then redistributed to other coordinate directions by pressure and diffusion effects. The potential energy of a density stratified medium in a gravitational field can thus be directly transformed into turbulent energy, and, conversely, turbulent energy can be transformed into potential energy of the medium. There have been numerous laboratory measurements of the effect of buoyancy on the concepts of eddy viscosity, eddy conductivity, and eddy diffusivity (Webster, 1964; Merrit & Rudinger, 1973; Ellison and Turner, 1960 & 1959; Jacobsen, 1913; Rider, 1954; Young, 1973; Chaudhry and Meroney, 1970). Webster (1964) and Arya (1971) have also measured other mean square fluctuating quantities at various values of the Richardson number. Monin and Yaglom (1971) and Lumley and Panofsky (1964) discuss in detail early geophysical evidence concerning turbulence in a thermally stratified medium. Detailed atmospheric data recently recorded by Wyngaard and other meteorologists is carefully reviewed in Workshop in Micrometeorlogy - AMS (1973).

These experimental data invariably show that for positive values of the Richardson number all turbulent fluctuating properties are suppressed by the action of the buoyancy force, while for negative values of the Richardson number turbulent fluctuating properties are accentuated. However there is a vast difference between the behavior of $\overline{w'T'}$, $\overline{u'T'}$, and $\overline{u'w'}$ when $\partial T/\partial z < 0$ and when $\partial T/\partial z > 0$. In the stable case the heat flux is often very small or negligible despite finite gradients of temperature, yet momentum transport may still be finite. In the unstable case a large heat flux is established quite rapidly, momentum transport may be significantly smaller in proportion, yet temperature gradients may be near zero. It appears that buoyancy-generated eddies cause relatively little momentum transport, but they are quite effective at carrying thermal energy. In other words, the rates of the associated turbulent diffusivities for heat and momentum is much larger than one, Reynold's analogy does not apply, and the idea of a simple eddy diffusivity in a stratified medium is completely wrong. Use of the diffusivity concept in calculations thus would tend to develop too rapid dissipation of inversions, and too slow a growth of turbulence in unstable situations.

These physical considerations suggest that an adequate theory for the treatment of the interaction of stratification, gravity, and a turbulent field must include transport equations for the second order correlations or their equivalent. Work by Donaldson and Rosenbaum (1972), Donaldson (1973), Lewellen and Teske (1973), and Mellor (1973) do consider the second order correlation equations including stratification effects. Lumley (1972) has also proposed sets of equations closed at the third order correlations, while Lee (1974) has developed a set of expressions based on analogies between turbulence and Brownian motion utilizing the Fokker-Planck equations. The ability of such formulations to follow the effects of stratification on turbulence are impressive. Unfortunately one must simultaneously solve a set of at least nine to as many as twelve partial differential equations for even a one-dimensional incompressible flow situation. For the equivalent two- or three-dimensional cases the ranges required are from ten to thirteen and from fourteen to seventeen partial differential equations respectively.

Such methods must thus be limited to research areas for the great majority of cases. Those situations requiring planning or engineering information generally must consider many case permutations; thus they require a method which retains the essential physical characteristics but with a lower order of solution complexity. This report discusses the efficacy of three such solution techniques. These will be discussed under the titles of

a) An algebraic lengths scale model (ALM),

b) A differential length scale model (DLM), and

c) An algebraic stress model (ASM).

The number of partial differential equations required are of the order of six, seven, and eight for one-, two-, and three-dimensional motions.

2.0 Governing Equations and Turbulent Models

The turbulent model utilized herein and the governing equations from which they are derived are discussed in the following sections. It is recognized that alternative approaches to the turbulence problem have each met with their own degree of success and satisfaction, it is not intended to infer here that the relations developed are final or exclusive--just that this path may be adequate and economical.

2.1 Governing Equations for Stratified Fluids

Complete governing equations for compressible and incompressible, variable property, chemically reacting and nonreactive, neutral and stratified flows have been summarized by a number of authors. (Monin and Yaglom, 1972)(Lumley and Panofsky, 1964)(Daly and Harlow, 1970) (Rotta, 1968)(Hinze, 1959). Donaldson (1973a) has acknowledged the influence of altitude on atmospheric motion and derived a non-Boussinesq set of equations for an atmospheric shear layer. Donaldson (1973b) has included the effects of diffusion and chemical reaction. Rodi (1970) reviews the equations of change and dissipation for both Cartesian and cylindrical coordinates.

For the purposes of this discussion a complete set of equations for an incompressible, constant property fluid assuming no chemical reaction, small departures from equilibrium, and the adequacy of the Boussinesq approximation are displayed in Cartesian coordinates in Table 6.

2.2 Turbulent Models

In any model developed for turbulent closure one would like to have the method possess width of applicability, accuracy, economy of computational time, and simplicity. In the search for these elusive features many closures for the turbulent equations of change have been proposed (Spalding and Launder, 1972). These efforts may be categorized in terms of complexity in the order of additional partial differential equations required beyond the equations of change for the mean quantities--hence zero, one, two, and multi-equation models. Table 1, 2, 3, and 4 summarize some prominant efforts at each of these levels for neutral turbulent flow fields. Parallel efforts for stratified fluids are grouped together in Table 5.

Reynolds (1968) has proposed in the 1968 Stanford "Olympics" on calculational techniques a morphology for classifying methods of closure. He suggests methods which make use of eddy viscosity or mixing length concepts will be called "mean field methods," (MF) whereas methods which relate the Reynolds stress to the turbulence and hence require calculation of some aspects of the turbulence fields will be called "turbulent-field methods." (MTF) Subsequent reviewers of turbulent models have accepted this decision as a critical distinction (Bradshaw, 1972, 1973). Mellor and Herring (1973) suggest two subsets of the MTF group. Those which include a turbulent kinetic energy transport equation and some accommodation for length scales will be "mean turbulent energy" closures (MTE); whereas a "mean Reynolds stress" closure (MRS) implies a closed set of equations which include equations for all nonzero components of the Reynolds stress. Chou (1945) seems to be the first to have studied the full set of equations with an eye to closure. It was Rotta (1951), however, who laid the foundation for almost all the current MRS models.

As is always the case a difficult problem soon becomes muddled again even with respect to categories such as the above. The recent work by Hanjalic and Launder (1972), Rodi (1972) and the present suggestions may lie somewhat between the MTE and MRS classifications.

2.2.1 "Zero Order" or Mean Field Methods: (MFR)

Boussinesq's 1877 paper relating Reynolds stresses to local mean velocity gradient through an eddy viscosity provided an impetus for contribution such as Prandtl's mixing length hypothesis. Despite clearly unphysical implications eddy viscosity and mixing length correlations have had good success in a large number of practical cases. Bradshaw (1972) suggests the explanation for such reliability lies in their application primarily to flows in a state of self preservation or of local equilibrium. Eddy viscosity formulae can be no better than a first approximation in non-self preserving flows where the behavior of the transport term is complicated. An alternate explanation may be that such success means our standards are not high.

Despite such reservations the method continued to have popularity, and authors such as Pandolfo (1968) have prepared eddy viscosity models including effects of stratification (see Table 7). For time dependent stratified flows the method appears unsatisfactory. Nappo (1972) tested a variety of such formulations for flow above a heated, moist earth surface and discovered unsatisfactory results for non-neutral cases. Donaldson, et al. (1972) remarked as a result of their own investigation "the idea of a simple eddy diffusivity in the atmosphere is completely wrong...."

2.2.2 Mean Turbulent Energy Methods (MTE)

A basis for both the MTF and the MTE calculations began with semi-heuristic models of Kolmogarov (1942) and Prandtl (1945). They suggest the use of a turbulent kinetic energy transport equation, a turbulent-energy related eddy viscosity, and a prescribed length scale function or a differential equation for length scale. A rational for the expression relating eddy viscosity to turbulent energy and length scale is displayed in Table 8.

Bradshaw (1973, 1972) has been very critical of methods which retain an explicit algebraic relation between stresses and the mean flow. His criticisms are related to the ad hoc nature of any eddy-viscosity transport relation, the failure to provide correct results in those cases where there is finite transport and zero velocity gradient, and the basically regressive concept of going to the trouble to solve additional transport equations and then reapplying a local-equilibrium assumption to relate stress and gradient. Mellor and Herring (1973) appear more optimistic, they try to show how MTE models derive logically from the MRS models and how both involve essentially the same empirical information. Launder and Spalding (1972) have reviewed the results of most of the effort in this area.

In an acid but illuminating article by Scorer (1972) further critiscm is heaped upon K-theory (and mathematical turbulence theories in general). He notes that "mathematics has a respectability among engineers which is quite unmerited" and goes on to examine the physics of transport in stratified flow in detail. He concludes that in unstable flow for example, the separate bodies of buoyant fluid surround themselves with horizontal vortex rings which transfer the fluid upward. Since the vortices are horizontal they are not affected by the shearing motion in a horizontally moving fluid. Thus the vortices are not stretched, i.e., differences in velocity between the layers are not transferred to momentum transporting eddy motion. Thus, as Scorer concludes, a K-theory approach is at best a "swindle," since it is often used just because it gives formulas, and it is dependent on the exact experimental circumstances in which it was measured. Thus, the excercise is circular and meaningless.

Bradshaw et al. (1967) avoided the eddy-viscosity concept entirely by assuming that profiles of all turbulence quantities at a given x in a thin shear layer could be uniquely related, empirically, to the shear stress profile. Thus the assumption $\overline{u'w'} = \alpha k$, together with a transport equation for k, turbulent kinetic energy, and an algebraic length scale equation reduced the equations of motion to a unique set of hyperbolic boundary layer equations. Unfortunately the basic assumption breaks down in rapidly changing flows.

In a similar spirit Hanjalic and Launder (1972) proposed to close the turbulence problem with transport equations for k, $\overline{u'w'}$, and ε (eddy dissipation). Their effort may well represent an optimum method for two-dimensional flow fields. As Bradshaw notes "it seems to be the best compromise between flexibility and tractability." Unfortunately its extension to three-dimensional flow fields in stratified fluids is not much less complex than complete MRS methods.

Spalding (1971) proposed the use of a scaler transport equation which might include $k_t = \overline{T'^2}/2$; however his relation included no coupling between the stratification and the turbulence levels. Launder (1973) has proposed an algebraic expression relating $\overline{w'T'}$ to $\partial T/\partial z$ as adjusted for $\overline{T'^2}$; however levels of k and ε would also be affected by $\overline{T'^2}$. Finally Bradshaw and Ferris (1968) developed a transport equation for $\overline{T'^2}$ and hence $\overline{w'T'}$ but again uncoupled from gravitational affects.

2.2.3 Mean Reynolds Stress Methods (MRS)

As noted before MRS closure implies a closed set of equations which include equations for all nonzero components of the Reynolds stress tensor. Rotta (1951) laid the foundation for future efforts when he proposed the pressure-velocity correlation terms in the Reynolds stress equations be proportional to a deviation from isotropy. This assumption was of course an approximation and was subject to modification by subsequent investigators. Other terms in the Reynolds stress equations such as the dissipation and diffusion terms have also been modeled differently by various investigators.

Only a few MRS calculations have been made for comparison with experiment. To date Donaldson et al. (1972), (1968), (1973) have compared results with flat plate boundary layer flows, free turbulent shear flows, transport in the atmospheric shear layer, and in vortex motions. Daly and Harlow (1970), using a dissipation length scale transport equation, have made MRS calculations for a channel flow. Mellor and Herring (1973) compared MRS results utilizing an algebraic length scale expression with zero and adverse pressure gradient boundary layer experiments. Comparison of numerical and experimental results suggest the models selected were of the right order. Since there were in each case a number of disposable empirical constants a certain adjustment occurred to optimize comparisons.

Among the more interesting aspects of the results was the radically different behavior predicted for the heat flux correlation $\overline{w'T'}$ depending on whether $\partial T/\partial z$ was greater or less than zero. Donaldson examined such effects both in a clear air turbulence model (1972) and in an atmospheric surface layer model (1973). Lewellen and Teske (1973) utilized Donaldson's second order invariant modeling technique to examine the Monin-Obukhov similarity functions for mean velocity, temperature, Richardson number, rms vertical velocity and temperature fluctuations and horizontal heat flux. The results agree very favorably with experimental results over the complete range of stability conditions.

In most respects the authors of MRS models agree on general points. Primary differences center around the use of algebraic or transport equations for dissipation rates, and the presence or absence of such terms as the mean strain rate in pressure strain. There are, however, numerous details over which they disagree with one another, especially philosophically in approach to model selection. Donaldson and his co-workers put much faith in the principle of invariant modeling to limit choices for pressure correlations, third order correlations, etc. Other investigators stress ad hoc empiricism and dimensional analysis.

Critics such as Bradshaw suggest the proliferation of new models has outrun the existence of quality experimental data upon which to make tractable judgements. As he states in his 1972 turbulent flow review there is the "unhealthy situation of too many computers chasing too few facts." All but the very simplest second-order closures he suggests, "require empirical information about turbulence quantities which have not been measured to the accuracy needed in calculation methods--if they have ever been measured at all."

It is generally acknowledged that transport equations for length scales (or dissipation) are less well understood than their Reynolds stress counterparts. There is no universal agreement on which length scale one should use, indeed more than one may be needed, and how can one justify developing a transport equation for what is often an integral scale (i.e., dependence on conditions of volume ℓ^3) on purely local point values and gradients?* Nevertheless it is generally acknowledged that MRS closure has already shown great promise, and, when based more firmly on measurements, the new methods may produce results for situations heretofore considered intractable. In particular terms relating to flow curvature, Coriolis forces, and buoyancy may enter automatically in a system based on the second order correlations. Therefore the hope exists that the influence of these agencies will be accounted for with models of this kind without recourse to ad hoc adjustments.

2.2.4 Algebraic Stress Models: (ASM)

A very novel compromise between the simplicity of the MTE approach and the universality and greater range of predictability of the MRS method, has been proposed by Launder and Ying (1971), Rodi (1972), Launder et al. (1972), and Date (1973). Transport equations for turbulent fluctuational energy and eddy dissipation (or length scale) are combined with algebraic equations for each Reynolds stress. The additional algebraic stress equations are derived directly from their exact transport equation counterparts.

Following Rodi (1972), one notes that it is the convection and diffusion terms in the $\overline{u_i u_j}$ equations which make them differential relationships. If such terms are eliminated from the transport equations for $\overline{u_i u_j}$ one produces a set of algebraic relations of the form

$$\overline{u_i u_j} = f(\overline{u_p u_q}, \frac{\partial u_\ell}{\partial x_m}, k, \varepsilon)$$

Of course the simplest way to simulate such terms is to neglect them out of hand. This however produces inconsistances in other than

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*Bradshaw (1968) ... "we are always in danger of ruining a high grade model by making a low grade assumption."

equilibrium situations where production exactly balances dissipation. Rodi postulated that

$$(\text{Convection - Diffusion})_{\text{of } \overline{u_i u_i}} = \frac{\overline{u_i u_j}}{k} (\text{Convection - Diffusion})_{\text{of } k}$$

$$= \frac{\overline{u_i u_j}}{k} (\text{Production - Dissipation})_{\text{of } k}$$

or that $\overline{u_i u_j}/k$ varies but slowly across the flow. (An assumption closely linked to the successful suggestions of Bradshaw for thin shear layers.) Hence theoretically,

$$\overline{u_i u_j} = k \frac{(\text{Production - Pressure Strain})_{\text{of } \overline{u_i u_j}}}{(\text{Production - Dissipation})_{\text{of } k}}$$

however, as a matter of practice to avoid singularities one may usually rearrange the relationship to give

$$\overline{u_i u_j} = \frac{(\text{Production})_{\text{of } \overline{u_i u_j}}}{\dfrac{(\text{Pressure Strain})}{\overline{u_i u_j}}_{\text{of } \overline{u_i u_j}} + \dfrac{1}{k} (\text{Production - Dissipation})_{\text{of } k}}$$

For neutral stratification when $(P-\varepsilon)_{\text{of } k} \equiv 0$ the result reduces to a Prandtl-Kolmogarov type formulation for an eddy viscosity.

Launder and Ying (1971) applied an ASM formulation to turbulent flow in a rectangular channel. The method predicted the order of secondary notions found in channels with sharp corners and the distribution of lateral Reynolds stresses. Rodi (1972) produced profiles of $\overline{u_k' u_j'}/k$ in plane jets and wakes where conventional two equation models undergo both strong and weak strain. Launder et al. (1972) in the NASA "free-shear flow computational olympics" compared six turbulence models and concludes MRS and ASM models produced results of comparable quality. Finally Date (1973) has produced shear and heat flux results for flow in a tube containing a twisted tape by means of ASM type approximations.

One concludes therefore that the algebraic stress models may combine the most important features of the MRS type (the influence of complex strain fields on the stresses) with (almost) the numerical simplicity of a MTE model.

3.0 A Turbulent Model for Stratified Flow

The models developed in the following sections required the solution of partial differential equations for total turbulent kinetic energy, k, total turbulent temperature fluctuations, k_t, eddy dissipation, ε, and thermal eddy dissipation, ε_t. Three separate versions of this model are discussed--an algebraic length scale version, a Prandtl-Kolmogarov eddy viscocity version, and an algebraic stress and heat flux model. For purposes of demonstration simple time dependent one-dimensional versions of the governing equations will be applied to a set of free shear flow test cases for which a complete MRS solution is available.

3.1 One-Dimension Equations of Transport

It is instructive to isolate the effect of integral problem characteristics by specifying dimensional scales in length, velocity, temperature, and time. Table 9 lists the scales utilized herein. When these relations are applied to the general equation of change (Table 6) and the result contracted for a one-dimensional case the resulting relations appear as in Table 10. These are for the constant property, Boussinesq-assumption, high Reynolds number situation. Even for a one-dimensional case these equations are intractable because they contain higher order correlations in u_i', p', and T'.

3.2 Modeling of the Equations

In order to close the equation system, some of the correlations must be approximated in terms of quantities that can be calculated. Model assumptions about turbulence are thereby introduced which may not be entirely realistic. These assumptions relate the chosen higher order correlations in u_i', p', and T' to other time-averaged quantities; they are expressed in differential and/or algebraic equations which help produce a mathematically closed set.

Turbulence models have been proposed which differ greatly in physical justification, complexity, and universiality of application. A short review of these efforts has been given in Sections 2.2.1 to 2.2.4. At the level of MTE and MRS approximations, however, most models display a common thread of accepted assumptions with difference often due to philosophical taste. Since it is the intent here to primarily examine the additional effects of stratification the decision was made to follow the practice for all other terms of the team of researchers in the Department of Mechanical Engineering, Imperial College, London (i.e., Spalding, Launder, Patankar, Rodi). These investigators have tested their assumptions over a very wide set of boundary layer, free shear, and pipe or duct flow case studies; thus there is some confidence the resulting expressions have a desirable universiality. All constants will be identified and evaluated in a separate section 3.2.7.

3.2.1 Prandtl-Kolmogarov Eddy Diffusivity Models

Kolmogarovf (1942) and Prandtl (1945) introduced an assumption relating shear stress to local velocity gradients through an eddy viscosity based on local effects of turbulence, $\nu_T \propto \sqrt{k}\, \ell$. This assumption has been found satisfactory for cases where stress and velocity gradients have the same sign in flow fields near local equilibrium (i.e., production of kinetic turbulent energy $\simeq$ dissipation). Table 8 suggests a typical heuristic justification for this formulation. A number of alternative length scale relations have been studied; that chosen here $\ell = k^{3/2}/\varepsilon$

was originally proposed by Harlow and Nakayama in 1967. The final expression is then

$$\nu_T \cong C_D k^2/\varepsilon, \tag{1}$$

A series of parallel arguments applied to the vertical heat flux equation will yield an expression for eddy conductivity

$$\alpha_T \equiv C_H\ k\ k_T/\varepsilon_T. \tag{2}$$

The use of these types of expressions require knowledge of the quantities k, ε, k_T, and ε_T over the whole flow field. One acquires this knowledge by solving differential or algebraic transport equations for each quantity. Rodi (1972) has also suggested that the constant C_D (or C_H) will be a function of weak strain. He suggests $C_D = f(P/\varepsilon)$, where P/ε (Production/Dissipation) can be considered a measure of the importance of convection and diffusion.

3.2.2 The k-Equation

The exact equation for k as derived from the Navier-Stokes equation is found in Table 10, Equation 3. We must now make assumptions about the correlations in the diffusion and production terms in order that the equation contains only knowable quantities.

Diffusion of k: We assume that k diffuses down its gradient; thus we write

$$-(\overline{kw'} + \overline{p'w'}) = \frac{\nu_T}{\sigma_k}\frac{\partial k}{\partial z} \tag{3}$$

Production of k: For the algebraic length model (ALM) and the differential length model (DLM) the new expression is

$$-\overline{u'w'}\ \frac{\partial u}{\partial z} = \nu_T\left(\frac{\partial u}{\partial z}\right)^2 \tag{4}$$

Stratification Effects on k: For the ALM and DLM cases the new expression is

$$Ri\ \overline{w'T'} = Ri\ \alpha_T(\frac{\partial T}{\partial z}) \tag{5}$$

The final expressions are summarized in Table 11 and 12.

3.2.3 The k_t-Equation

Again the exact equation for k_T is found in Table 10, Equation 4. Note there is no explicit additional term due to the effect of stratification.

Diffusion of k_t: We assume that k_T diffuses down its gradient; thus we write

$$-(\overline{k_t'w'}) = \frac{\alpha_T}{\sigma_{kt}} \frac{\partial k_T}{\partial z} \tag{6}$$

Production of k_t: For the ALM and DLM models the relation becomes

$$-\overline{w'T'}\ \frac{\partial T}{\partial z} = \alpha_T(\frac{\partial T}{\partial z})^2 \tag{7}$$

Final expressions are found in Table 11 and Table 12, Equations (4).

3.2.4 The ε-Equation

The dissipation of total turbulent kinetic energy, ε, is defined here as

$$\varepsilon = \frac{1}{Re}\ [\overline{(\frac{\partial u'}{\partial z})^2} + \overline{(\frac{\partial u'}{\partial z})^2} + \overline{(\frac{\partial w'}{\partial z})^2}] \tag{8}$$

The mathematical complexity of the exact ε-transport equation is so great that even in a one-dimensional form the relation is quite long (Table 10). Dissipation of other $\overline{u_iu_j}$ terms may be related to ε by a suggestion of Harlow

$$\frac{2}{Re}\ \overline{\frac{\partial u_i'}{\partial x_k}\ \frac{\partial u_j'}{\partial x_k}} = \varepsilon\ \frac{\overline{u_iu_j}}{k} \tag{9}$$

When one assumes the turbulence to be locally isotropic at high Reynolds number, then it is probable that

$$\frac{2}{Re} \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}} = \frac{2}{3} \varepsilon \text{ for } i = j$$

$$= 0 \quad \text{for } i \neq j \tag{10}$$

The truth, for high Reynolds numbers, probably lies somewhere between expressions (9) and (10).

Diffusion of ε:

$$- [\overline{\varepsilon' w'} + \overline{\frac{\partial p'}{\partial z} \frac{\partial w'}{\partial z}}] = \frac{\nu_T}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial z} \tag{11}$$

Production of ε: Following the dimensional arguments of Rodi (1972) or Hanjalic and Launder (1972)

$$- \frac{2}{Re} [\frac{\partial u}{\partial z} (\overline{\frac{\partial u'}{\partial z} \frac{\partial w'}{\partial z}}) + \frac{\partial^2 u}{\partial z^2} (\overline{w' \frac{\partial u'}{\partial z}})] = - C_{E1} \frac{\varepsilon}{k} \overline{u'w'} (\frac{\partial u}{\partial z}) \tag{12a}$$

which for the DLM case becomes

$$= C_{E1} C_D k (\frac{\partial u}{\partial z})^2 \tag{12b}$$

Destruction of ε: The fourth and fifth bracketed terms from Eq. (5) Table 10 are generally considered together. The fourth term represents the generation rate of velocity fluctuations through the self-stretching action of turbulence, while the fifth represents the decay of dissipation rate through the action of viscosity. For high Reynolds number flow-fields the sum of these terms are controlled by the dynamics of the energy cascade process and is thus independent of viscosity. Again by dimensional homogeneity one concludes

$$\frac{2}{Re}\left[\overline{\left(\frac{\partial u'}{\partial z}\right)^2 \frac{\partial w'}{\partial z}} + \overline{\left(\frac{\partial v'}{\partial z}\right)^2 \frac{\partial w'}{\partial z}} + \overline{\left(\frac{\partial w'}{\partial z}\right)^3}\right] + \frac{2}{Re}\left[\overline{\left(\frac{\partial^2 u'}{\partial z^2}\right)^2} + \overline{\left(\frac{\partial^2 w'}{\partial z^2}\right)^2} + \overline{\left(\frac{\partial^2 v'}{\partial z^2}\right)^2}\right]$$

$$= C_{E2}\ \varepsilon^2/k \qquad (13)$$

Stratification Effects on ε:

Lumley (1972) fails to retain any effect of stratification upon ε; however if one pursues the exact relationships as suggested by Daly and Harlow (1970) the term as shown in Table 10 appears. Order of magnitude arguments would suggest its inclusion is critical in order to track the effects of stratification on k. Daly and Harlow (1970) suggest that the stratification term might be modeled as

$$2Ri\ \overline{\left(\frac{T'}{z}\ \frac{w'}{z}\right)} \propto Ri\ \overline{w'^2}\ f(\xi)\frac{\partial T}{\partial z}$$

where $\xi = S(2k)^{1/2}/\nu$ is a turbulence Reynolds number and $f(\xi) \sim 0(1)$. To remain consistent with the earlier choices made for the transport of k it is proposed to use

$$2Ri\ \overline{\left(\frac{T'}{z}\ \frac{w'}{z}\right)} = +F\ Ri\ \overline{w'T'}\ \left(\frac{\varepsilon}{k}\right). \qquad (14a)$$

When this is evaluated for the DLM case one finds

$$= -\ F\ Ri\ C_H k_t \left(\frac{\varepsilon}{\varepsilon_t}\right)\left(\frac{\partial T}{\partial z}\right) \qquad (14b)$$

which resembles the Daly-Harlow suggestion. Final relations are summarized in Tables 11 and 12, Eq. (5).

3.2.5 The ε_t-Equation:

Lumley (1970) recommends the use of a temperature fluctuation equation also. Indeed Temmekes and Lumley (1972, p. 102) comment that the problem of buoyant convection is one with two time (or length) scales which may differ by an order of magnitude. Hence it was considered critical that an additional length scale relation or its counterpart be used for this stratified flow discussion.

Diffusion of ε_t:

$$- (\overline{\varepsilon_t' w'}) = \frac{\alpha_T}{\sigma_{\varepsilon t}} (\frac{\partial \varepsilon_t}{\partial z}) \tag{15}$$

Production of ε_t:

$$- \frac{2}{RePr} [\frac{\partial u}{\partial z} \overline{(\frac{\partial T'}{\partial z})^2} + \frac{\partial T}{\partial z} \overline{(\frac{\partial T'}{\partial z})(\frac{\partial u'}{\partial z})} + \frac{\partial^2 T}{\partial z^2} \overline{(w' \frac{\partial T'}{\partial z})}]$$

$$= -C_{Et1} \overline{w'T'} (\frac{\varepsilon}{k}) (\frac{\partial T}{\partial z}) \tag{16a}$$

or for DLM

$$= C_{Et1} k_t (\frac{\varepsilon}{\varepsilon_t}) (\frac{\partial T}{\partial z})^2 \tag{16b}$$

Destruction of ε_t:

$$\frac{2}{RePr} [\overline{\frac{\partial w'}{\partial z} (\frac{\partial T'}{\partial z})^2}] + \frac{2}{RePr} \overline{(\frac{\partial^2 T'}{\partial z^2})}^2 = C_{Et2} \frac{\varepsilon \varepsilon_t}{k} \tag{17}$$

Final relationships are found in Tables 11 and 12, Eq. (6).

3.2.6 The Algebraic Stress and Heat Flux Relations

As indicated in Section 2.2.4 closure for the ASM model requires algebraic expressions for $\overline{u'w'}$, $\overline{u'T'}$, and $\overline{w'T'}$. These new expressions are dependent upon an accurate representation of the production-dissipation terms in their respective exact relationships. Stress-equation models have been recommended by Rotta (1951), Donaldson (1968), Daly and Harlow (1970), Rodi (1972), and Launder et al. (1973).

Following the experience of Rodi (1972) the production (P), dissipation (D), pressure strain (PS), and stratification effects (S) are identified and modeled. Since the diffusion terms are eliminated through the arguments presented previously they are not considered here.

<u>The Reynolds Stress Equations:</u> For high Reynolds number situations Eq. (10) Section 3.2.4 is appropriate

$$D_{ij} = \frac{2}{3} \varepsilon \delta_{ij} \tag{18}$$

Production terms are exact as

$$P_{ij} = - \overline{u_i' u_k'} \frac{\partial u_j}{\partial x_k} - \overline{u_j' u_k'} \frac{\partial u_i}{\partial x_k}$$

$$P = \frac{1}{2} P_{ii} = - \overline{u_k' u_\ell'} \frac{\partial u_k}{\partial x_\ell} \tag{19}$$

Pressure strain terms can as yet not be measured in laboratory flows. Unfortunately they are also of great importance since they are roughly equal and opposite to the production terms in the shear-stress equation; and in the normal-stress equations they redistribute energy to various direction components. Pressure-strain consists of two parts: one due to the interaction of the various fluctuating velocities, and the second originates from the interaction of mean and fluctuating flow. Rotta (1951) proposed to take the first part proportional to the anisotropy of the turbulence. Naot et al. (1970) and Reynolds (1970) proposed the second part should be proportional to the anisotropy of the production of turbulence, thus

$$PS_{ij} = -C_{P1} \frac{\varepsilon}{k} (\overline{u_i u_j} - \delta_{ij} \frac{2}{3} k) - A_1 (P_{ij} - \delta_{ij} \frac{2}{3} P) \tag{20}$$

Stratification effects are directly expressed as

$$S_{ij} = Ri \left(\frac{\partial \phi}{\partial x_i} \overline{u_j' T'} + \frac{\partial \phi}{\partial x_j} \overline{u_i' T'} \right) \tag{21}$$

where ϕ is the gravitational potential.

When these terms are substituted into the exact relationships the following expressions are obtained:

$$(\text{Conv} - \text{Diff})_{\overline{u'w'}} = -(1-A_1)\ \overline{w'^2}\ \frac{\partial u}{\partial z} - C_{P1}\ \frac{\varepsilon}{k}\ \overline{u'w'} + \text{Ri}\ \overline{u'T'} \qquad (22)$$

$$(\text{Conv} - \text{Diff})_{\overline{w'^2}} = -\frac{2}{3}\ A_1\ \overline{u'w'}\ \frac{\partial u}{\partial z} + \frac{2}{3}(C_{P1}-1)\varepsilon - C_{P1}\ \frac{\varepsilon}{k}\ \overline{w'^2}$$

$$+\ 2\text{Ri}\ \overline{w'T'} \qquad (23)$$

As noted previously the above expressions have been prepared to substitute into the expression below

$$(\text{Conv} - \text{Diff})_{\text{of}\ \overline{u_i u_j}} = \frac{\overline{u_i u_j}}{k}\ (\text{Conv} - \text{Diff})_{\text{of}\ k}$$

$$= \frac{\overline{u_i u_j}}{k}(P - \varepsilon + S)_{\text{of}\ k}$$

$$= (P + PS - \varepsilon + S)_{\text{of}\ \overline{u_i u_j}} \qquad (24)$$

The appropriate substitution for the k terms are found on the right side of Eq. (3), Table 12.

When (22) is introduced into (24):

$$\overline{u'w'} = \frac{-(1-A_1)\ \overline{w'^2}\ \frac{\partial u}{\partial z} + \text{Ri}\ \overline{u'T'}}{C_{P1}\ \frac{\varepsilon}{k} + \frac{1}{k}\ (-\ \overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + \text{Ri}\ \overline{w'T'})} \qquad (25)$$

Similarly if (23) is introduced into (24):

$$\overline{w'^2} = \frac{\frac{2}{3}\ (A_1(-\overline{u'w'})\ \frac{\partial u}{\partial z} + \varepsilon(C_{P1}-1) + 3\text{Ri}\ \overline{w'T'})}{C_{P1}\ \frac{\varepsilon}{k} + \frac{1}{k}(-\ \overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + \text{Ri}\ \overline{w'T'})} \qquad (26)$$

Elimination of $\overline{w'^2}$ in Eq. (25) by means of Eq. (26) yields:

$$\overline{u'w'} = \frac{-\frac{2}{3}(1-A_1)k\frac{\partial u}{\partial z}[A_1(-u'w')\frac{\partial u}{\partial z} + \varepsilon(C_{P1}-1) + 3Ri\,\overline{w'T'}]}{(C_{P1}\frac{\varepsilon}{k} + \frac{1}{k}(-\overline{u'w'}\frac{\partial u}{\partial z} - \varepsilon + Ri\,\overline{w'T'})^2}$$

$$+ \frac{R_i\overline{u'T'}}{(C_{P1}\frac{\varepsilon}{k} + \frac{1}{k}(-\overline{u'w'})\frac{\partial u}{\partial z} - \varepsilon + Ri\,\overline{w'T'})} \quad (27)$$

The Heat Flux Equations:

In the $\overline{u'T'}$ and $\overline{w'T'}$ relations at high Reynolds numbers it is again appropriate to suggest that the dissipation terms are

$$D_{\overline{u_i'T'}} \simeq 0. \quad (28)$$

The production and stratification terms may be treated exactly, i.e.,

$$P_{\overline{u_i'T'}} = -\overline{u_i'u_k'}\frac{\partial T}{\partial x_k} - \overline{u_k'T'}\frac{\partial u_i}{\partial x_k} \quad (29)$$

$$S_{\overline{u_i'T'}} = +2Ri\,[\frac{\partial \phi}{\partial x_i}k_T] \quad (30)$$

The pressure scrambling terms may be postulated by analogy to Eq. (20):

$$PS_{\overline{u_i'T'}} = -C_{P2\text{ or }3}\frac{\varepsilon}{k}(\overline{u_i'T'}) - A_{2\text{ or }3}P_{\overline{u_i'T'}} \quad (31)$$

One might argue that the first term should be $-C_{P2\text{ or }3}\frac{\varepsilon_t}{k_t}(\overline{u_iT'})$ in order that the final relation reduce to the Prandtl-Kolmogarov formulation in equilibrium flows. Unfortunately such a choice eliminates the production of $\overline{w'T'}$ in unstable regions where k_T had been set zero as an initial condition. The modeled relations do not contain the physics of thermal instability within themselves.

If one now postulates that

$$(\text{Conv - Diff})_{\text{of } \overline{u_i'T'}} = \frac{\overline{u_i'T'}}{k} (\text{Conv - Diff})_{\text{of } k}$$

$$= \frac{\overline{u_i'T'}}{k} (P + PS - D + S)_{\text{of } \overline{u_i'T'}} \tag{32}$$

and substitues Eq. (28), (29), (30), and (31) into Eq. (32) there results new expressions:

$$\overline{w'T'} = \frac{-(1 - A_2)\ \overline{w'^2}\ \frac{\partial T}{\partial z} + 2\ \text{Ri}\ k_t}{C_{P2}\ \frac{\varepsilon}{k} + \frac{1}{k}\ (-\overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + \text{Ri}\ \overline{w'T'})} \tag{33}$$

$$\overline{u'T'} = \frac{(1-A_3)[-\ \overline{u'w'}\ \frac{\partial T}{\partial z} - \overline{w'T'}\ \frac{\partial u}{\partial z}]}{C_{P3}\ \frac{\varepsilon}{k} + \frac{1}{k}\ (-\overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + \text{Ri}\ \overline{w'T'})} \tag{34}$$

Elimination of $\overline{w'^2}$ in Eq. (33) with Eq. (26) yields:

$$\overline{w'T'} = \frac{-\ \frac{2}{3}\ (1-A_2)k\ \frac{\partial T}{\partial z}\ [A_1(-\overline{u'w'})\ \frac{\partial u}{\partial z} + \varepsilon(C_{P1}-1) + 3\text{Ri}\ \overline{w'T'}]}{[C_{P1}\ \frac{\varepsilon}{k} + \frac{1}{k}\ (-\overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + \text{Ri}\ \overline{w'T'})]^2}$$

$$+ \frac{2\text{Ri}k_t}{[C_{P1}\ \frac{\varepsilon}{k} + \frac{1}{k}\ (-\overline{u'w'}\ \frac{u}{z} - \varepsilon + \text{Ri}\ \overline{w'T'})]} \tag{35}$$

Equations (27), (34) and (35) are the final result of the exercise to produce ASM relations. They are repeated as Eqs. (7, (8) and (9) in Table 12.

3.2.7 Turbulence Model Constants

At first glance there appears to be a disagreeably large number of unspecified constants in the modeled relationships. If one assumes the use of an algebraic length scale (ALM) relation four constants exist in Eqs. (1) to (4), Table 11. (C_D, C_H, σ_k, σ_{kt}). For the differential length scale relationships (DLM) there are eleven constant in Eqs. (1) to (6), Table 12. (C_D, C_H, σ_R, σ_{kt}, σ_ε, $\sigma_{\varepsilon T}$, C_{E1}, C_{E2}, C_{Et1}, C_{Et2}, F). Finally in the algebraic stress model approach (ASM) there are six more in Eqs. (7)-(9), Table 12. (C_{P1}, C_{P2}, C_{P3}, A_1, A_2, A_3).

Luckily there is a great deal of experience with these relations from studies of neutral flow field cases. In addition basic studies of decay of turbulence behind grids, near walls, as distorted by plane strain, in homogeneous shear flows, as it passes through asymetric contractions, etc. are available. Table 13 summarizes values utilized by various investigators. Due to the method of formulation most constants are of the order of one. Some researchers have stressed the importance of obtaining a universal limited set of constants. The invariant modeling approach suggested by Donaldson (1968) is one such method which appears to facilitate this goal. It is my opinion, however, that over-zealous search for a universal set may lead to the elimination of physically significant terms, to the discard of viable model relations (many quite satisfactory variant-expressions work quite well for certain cases), and to the attempt to fit experimental data to numerical analysis rather than the opposite. The constants used in this study were chosen to match those used in the MRS comparison study; a recommended set as preferred by this author is also designated in Table 13.

4.0 Test Calculations for Stratified Turbulence Models

Mellor and Herring (1973) recommend that investigators evaluate MTE and MRS models in tandem in order that critical limitations of the MTE approach are identified. Bradshaw (1971) has also proposed such tuning of a "simple" calculation method by a "refined" calculation technique. Such an approach may seem distasteful to the mathematical purist. The concept of "computational fluid dynamics," however, under which this study falls is described by Roache (1973) as basically empirical in nature--"one must rely on rigorous mathematical analysis of simpler, linear, more or less related problems, or heuristic reasoning, physical intuition, wind tunnel experience, and trial and error procedures." "The numerical simulation is then closer to experimental than theoretical fluid dynamics. The performance of each particular calculation on a computer closely resembles the performance of a physical experiment, in that the analyst "turns on" the equations and waits to see what happens...." (accent added).

4.1 Simple Case of Atmospheric Shear

The method chosen for comparison with the ALM, DLM, and ASM methods proposed herein was the MRS technique developed by Donaldson and Rosenbaum (1968). Their "invariant" modeling closure was applied to a hypothetical free-shear clear-air turbulence case in Donaldson, Sullivan, and Rosenbaum (1972). This was a simple, time-dependent, one-dimensional example which characterizes the important effects of buoyancy dominated turbulent shear flows.

The set of one-dimensional, constant property, MRS relations utilized by Donaldson et al. are displayed in Table 14. The concepts of "invariant" modeling as discussed by the authors were used to provide closure for the unknown correlation terms in the Reynolds stress and heat flux equations. Only one length scale equation was utilized to relate the quantities λ and Λ to δ, an appropriately chosen scale of the mean motion. These algebraic relations are

$$\lambda^2 = \Lambda^2/(2.5 + 0.125\ Re_\Lambda)$$

$$\Lambda = 0.064\delta$$

where $Re_\Lambda = \rho\Lambda(2k)^{1/2}/\mu$.

The atmospheric test case is assumed to have an initially 4000 ft band of turbulence that is isotropic with $\overline{u'^2} = \overline{v'^2} = \overline{w'^2} = 1\ (\text{fps})^2$. The band is centered at an altitude of 20,000 ft; however the effects of altitude upon p_o, T_o, and ρ_o are neglected for purposes of the example. The atmosphere is initially at rest, i.e., at $t = 0$, $u = 0$. A body force acts on the atmosphere to create a mean motion, that is

$$X(z,t) = c(1-\xi)^2 \qquad \text{for } t \leq 3000 \text{ sec.}$$

$$= 0 \qquad \text{for } > 3000 \text{ sec.}$$

where $\xi = (z^*-20{,}000)/1000$

for $|\xi| < 1.$

Outside this region the driving force is taken to be zero. The forcing function is such that it produces a shear layer some 2000 ft thick. In the absence of any turbulence or viscosity the constant c is chosen such that the centerline velocity would increase to 30 fps after 3000 sec. After 3000 sec the body force term is removed and the shear layer is left to dissipate.

Four cases of initial mean temperature distribution were considered as in Figure 1.

Case I: Neutrally stable atmosphere $T(z^*,o) = 0$.

Case II: Unstable-stable-unstable atmosphere where

$T(z^*,o) = (135/132)\xi(2-|\xi|)^2$ for $|\xi| \leq 2$

$T(z^*,o) = 0$ for $|\xi| > 2$

Case III: Stable-unstable-stable atmosphere where

$T(z^*,o) = -(135/132)\xi(2-|\xi|)^2$ for $|\xi| \leq 2$

$T(z^*,o) = 0$ for $|\xi| > 2$

Case IV: Stable-unstable atmosphere where

$T(z^*,o) = 10(1 - (\frac{1}{2}\xi)^2)^2$ for $|\xi| \leq 2$

$T(z^*,o) = 0$ for $|\xi| > 2$.

The maximum temperature difference in each case was 10°R. The other initial conditions utilized by Donaldson et al. were

$u'w'(z^*,o) = 0$, $w'T'(z^*,o) = 0$, $u'T'(z^*,o) = 0$, and

$\overline{T'^2}(z^*,o) = 0$.

The scale δ for all calculations was taken to be the breadth of the mean shear layer as based on the distance between the points where the mean velocity falls to half its maximum value.

4.2 Dimensionless Test Case Format

The equations proposed here in Tables 11 and 12 have been made dimensionless per Table 9. Thus one must identify for comparison purposes the various scales used, i.e., (L^*, $u_{max}{}^*$, $\Delta T_{max}{}^*$). They will be in all cases studied as follows

$L^* = 1000$ ft

$u_{max}{}^* = 30$ fps, and

$\Delta T_{max}{}^* = 10$°R.

Thus the forcing function and initial conditions are:

$X(z,t) = (1.0 - z)^2$ for $t \leq 90$,

$= 0$ for $t > 90$.

Case II

$$T(z,o) = 0.42188\ z\ (2.0 - |z|)^2$$

Case III

$$T(z,o) = -\ 0.42188\ z\ (2.0 - |z|)^2$$

Case IV

$$T(z,o) = (1.0 - (\frac{z}{2.0})^2)^2$$

The initial conditions for the ALM and DLM methods were as follows:

$$k(z,o) = 1.67 \times 10^{-3}\ ,$$

$$\varepsilon(z,o) = 4.0 \times 10^{-4}\ ,$$

$$k_t(z,o) = 1.67 \times 10^{-4}\ , \text{ and}$$

$$\varepsilon_t(z,o) = 4.0 \times 10^{-5}\ .$$

These methods are incapable of initializing thermal fluctuating effects without finite initial values of k_t and ε_t.

The initial conditions for the ASM method were as follows:

$$\left.\begin{aligned} k(z,o) &= 1.67 \times 10^{-3} \\ \varepsilon(z,o) &= 4.0 \times 10^{-4} \end{aligned}\right\} \quad |z| \leq 2.0$$

$$\left.\begin{aligned} k(z,o) &= 1.67 \times 10^{-202} \\ \varepsilon(z,o) &= 4.0 \times 10^{-201} \end{aligned}\right\} \quad |z| > 2.0$$

and

$$k_t(z,o) = 1.67 \times 10^{-200}$$

$$\varepsilon_t(z,o) = 4.0 \times 10^{-201}\ .$$

The other initial conditions that were used for the ASM method are

$$\overline{u'w'}(z,o) = 0,\ \overline{u'T'}(z,o) = 0, \text{ and } \overline{w'T'}(z,o) = 0.$$

The algebraic length method of the MTE type approach suggested here still requires an expression to specify ε and ε_t in the governing partial differential equations. The algebraic relation formulated by

Donaldson et al. (1972) can be re-expressed in terms of equivalent values of dissipation; hence for large Re_Λ one can find

$$\varepsilon = 5.52\ k^{1.5}/z\delta$$

$$\varepsilon_t = 5.52\ k_t k^{0.5}/z\delta\ .$$

where $z_\delta \equiv \delta/L^*$ and δ is defined as before.

Although the MRS method compared to here did not require specification of the series of constants used in the ALM, DLM, and ASM models, consideration of the algebraic length scale equation plus asymptotic forms of the governing equations in those regions where production or dissipation dominate allows one to specify the equivalent constant values. These values as recorded in Table 13 were used as the initial values studied in each model. It should be noted that no inherent value is placed in these constants since comparison with experimental fluid flows would suggest different values. After consideration of the results obtained from the original constants it was only found necessary to adjust C_{E2} and C_{Et2} slightly to arrive at a reasonable comparison to the Donaldson et al. (1972) results.

4.3 Numerical Integration Technique

The partial differential equations to be solved are parabolic differential equations. They are parabolic because the time derivative is first order in time and present in every equation while there is at most a second derivative $\partial^2/\partial z^2$ in a spatial dimension. If one was to ignore all diffusion by microscopic transport the equations could be reduced to a hyperbolic set in the manner of Bradshaw, Ferriss, and Atwell (1967).

The simple boundary conditions involved in a free shear flow situation (all values go to zero for $\pm$ 5L) makes the application of an implicit technique convenient. The main attraction of an implicit technique is its strong stability; convergence can be checked by running the program at one-tenth the forward time step. The continuous derivatives of the exact equations must be replaced by finite difference algorithms. In this case we choose to use uniform step size in z over 101 points.

Finite difference relations for the governing differential equations were obtained by integrating over finite areas. This "tank and tube" analog method enables one to ensure that conservation laws are obeyed over arbitrarily large or small portions of the field. The method also lends itself to straightforward physical interpretation. This method has been discussed in some detail in Patankar and Spalding (1967) (§2.4) and Gosman et al. (1969) (§3.23). In this case the result is equivalent to centered-difference formulae in space evaluated at a forward time step. Time differences were a simple forward difference type. An additional feature of the solution technique was associated with source on sink terms such as dissipation. It is realized that it would be physically meaningless to allow positive definite quantities such as temperature, turbulent energy, or turbulent temperature fluctuations to go negative. To avoid anamalous behavior all sink terms are evaluated as if they were $(\text{sink})^n f^{n+1}/f^n$ (f = dependent variable, n = time step). As long as time stops are small this method will perform satisfactorily.

The finite difference equations were solved by successive use of the TDMA (tridiagonal matrix algorithm) or Thomas algorithm in the z direction. This method requires a double sweep of the z profile and is actually a straightforward Gaussian elimination procedure.

The programs were written in terms of a series of subroutines to facilitate organization. No effort was made to optimize the program for time efficiency. Nevertheless it was found that after initial compilation the typical average time per forward step including all input and output operations was 0.28, 0.32, and 0.36 seconds of central processor time for the ALM, DLM, and ASM models respectively on a CDC6400 computer.

5.0 Results

5.1 Case I: Neutrally stable atmosphere

For each model proposed this neutral case was used to adjust C_{E2} to obtain optimum time behavior of k when compared to the MRS results. In Figures 2, 3, and 4, are shown the calculated values of the mean velocity u, $k = \frac{\overline{(u_i'-U_i')}}{2}$, and $\overline{u'w'}$ for four dimensionless times t = 42, 90, 135, and 180. At very small times, t < 20, there is still very little turbulence created by the mean action and most turbulence is left over from the decaying initial turbulent layer. When the forcing function, X, is removed at t = 90, the turbulent kinetic energy is a maximum. The influence of the value $\partial u/\partial z = 0$ at z = 0 on the production of new turbulence is evident. Finally at t = 180 one sees the effect after decay has set in.

By plotting the maxima of k for the various models versus time in Fig. 5 one can perceive the degree of agreement between the closure trials. One perceives the initial decay of turbulence, followed by the production of the correlations as a result of the interaction of turbulence and mean flow, and finally the decay after the forcing function is removed.

5.2 Case II: Unstable-stable-unstable atmosphere

Fgures 6, 7, and 8 display calculated values for this case as before. The mean velocity and shear are changed somewhat but not drastically by the effect of the temperature profile. In the thermally unstable regions the development of turbulence is accelerated. At large times the production by $\partial u/\partial z$ becomes significant and a maxima in k occurs at t = 90. Note the large influence on k, k_T, and w'T' in the regions where

$\partial T/\partial z < 0$ versus those regions where $\partial T/\partial z > 0$. If one insists upon an eddy diffusivity model a large variation in turbulent Prandtl number with stability is indicated. Note that in regions of high stability the existence of a single length scale in the ALM method is not sufficient to develop the necessary degrees of temperature fluctuation dissipation. Thus k_T is excessively large near $z = 0$.

The ALM and DLM models fail to keep pace with the changes recorded by the ASM and MRS relations. In addition the ALM and DLM models inherently require $\overline{w'T'}$, $\overline{u'T'}$, and $\overline{u'w'}$ be identically zero where the respective gradients, $\partial u/\partial z$ and $\partial T/\partial z$, are zero. It appears that failure to follow the time rate of change in k and other correlations is due to the inherent assumption that Production equals Dissipation in the ALM and DLM formulations. This effect is most marked in the logarithmic plot of the maxima of the turbulent kinetic energy and turbulent temperature fluctuations as shown in Figs. 9 and 10. At small times the major production of turbulence is by thermal instability. Near the turbulence maxima there is finite contribution due to mean shear, du/dz; but, after t = 90, the thermal instabilities in the outer regions dominate.

5.3 Case III: Stable-unstable-stable atmosphere

This case was utilized to specify the magnitude of constant C_{Et2}. In Cases I and IV the constants were unchanged for all models. Again in Figs. 11, 12, and 13 we do not detect major influence of stability upon the production of a shear layer. There is intense production of k, k_T, and $\overline{w'T'}$ in the center of the unstable region. Again only the ASM method can track the nonequilibrium behavior displayed by the MRS relations. Note how the outlying stable regions limit the vertical dispersion of this intense turbulence. The temperature inversion is extremely persistent at large $\pm$ z. In the time dependent plots of k and k_T, Figs. 14 and 15, we detect the initial production of turbulence by thermal instability, the modification of this by shear generated turbulence near t = 90, and the final phase where production of turbulence by thermal instability just about balances dissipation.

5.4 Case IV: Stable-unstable atmosphere

This example is marked by its asymetric appearance, the persistence of the inversion region, $\partial T/\partial z > 0$, and the rapid vertical diffusion of energy into the neutral upper regions where initially $\partial T/\partial z = 0$. No existing eddy diffusivity model would be expected to perform adequately. Figures 16, 17, and 18 compare the performance of ALM, DLM, and ASM models respectively. Figures 19 and 20 record the time rate of change of k and k_T maxima.

5.5 Discussion of Results

Of course one of the most interesting aspects of these results is the duplication of Donaldson et al. (1972) conclusion that there is a radically different behavior of the heat flux correlation $\overline{w'T'}$ depending on whether $\partial T/\partial z$ is greater or less than zero. In fact, the ASM equations of Table 12 (7), (8), and (9) indicate that for small values of $\partial T/\partial z$ and $\partial u/\partial z$ there may be transport of heat and momentum <u>up</u> the gradients due to finite values of k_T or $\overline{u'T'}$! This effect has often been observed by experimentalists in atmospheric transport. In addition if one considers only production terms and neglects pressure scrambling and dissipation terms in equations for k_T and $\overline{w'T'}$ it is not difficult to show that when $\partial T/\partial z < 0$, there [illegible] exponential development of $\overline{w'T'}$. However when the atmosphere is stable, i.e., when $\partial T/\partial z > 0$, the heat flux correlation $\overline{w'T'}$ is oscillatory about the Bront-Vassala frequency.

Utilizing a simple, time-dependent, one-dimensional example as a test case this report has discussed a solution which represents the important characteristics of a buoyancy dominated shear flow by solving four partial differential equations in addition to the mean equations of motion. This suggested model solves equations for total turbulent kinetic energy, k, total turbulent temperature fluctuations, k_t, eddy dissipation, ε, and thermal eddy dissipation, ε_t. Three separate versions of this model were discussed--an algebraic length scale version, a Prandtl-Kolmogorov eddy viscosity version, and an algebraic stress and heat flux model. The final version (requiring six partial differential equations) manages to replicate results for a much more complicated version (requiring ten partial differential equations). The advantages for two and three dimensional problems are even greater.

One must conclude on the basis of these results that:

1) An algebraic length scale version of a MTE model closure is not able to replicate the behavior of thermally stratified flow, especially in regions where production and dissipation of turbulence are not in equilibrium. A single dissipation length scale does not appear sufficient here to develop the expected degree of damping in stable regions.

2) Addition of transport equations for length scales does not suffice to solve the above problem. Such MTE models are still inadequate.

3) Addition of algebraic relations for stress and heat flux which incorporate the influence of stability do appear to incorporate the physics of the phenomena to the extent that results are similar to the MRS test case.

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Table 1: Turbulent Models: Neutral Flow Zero Equation Models

Author	2d	3d	Terms
Prandtl (1925)	0	0	Mixing length
vKarman (1930)	0	0	Mixing length
Trubchikov (1938)	0	0	ν_T for jets
Prandtl (1942)	0	0	ν_T for jets
Clauser (1954)	0	0	ν_T for wall flows
Escudier (1965)	0	0	ℓ_m for wall flows
Mellor-Herring (1968)	0	0	$\nu_T = \varepsilon\phi\frac{(\chi)}{\varepsilon} + \Phi(\chi) - \chi$ $\varepsilon = \frac{1}{R_{\delta*}}$; $\chi = \frac{y\sqrt{\tau/\rho}}{u_\infty \delta_*}$
Cebeci-Smith (1968)	0	0	ν_T = Van Driest Model $\nu_{T\infty} = k_2 u_\infty \delta^* [1 + 5.5(\frac{y}{\delta})^6]^{-1}$

Table 2: Turbulent Models: Neutral Flow One Equation Models

Author	2d	3d	Terms
Prandtl (1945)	1	1	$\nu_T = k^{\frac{1}{2}}\ell$, $\ell = f(x)$
Glushko (1965)	1	1	$\nu_T = k^{\frac{1}{2}}\ell$, $\ell = f(x)$
Bradshaw-Ferris (1967)	1	--	$\overline{u_1u_2} = \alpha k$, $\ell = f(y)$ Hyperbolic set of Equations
Beckwith-Bushnell (1968)	1	1	(developed Glushko)
Mellor-Herring (1968)	1	1	(developed Glushko)
Nee and Kovaznay (1968)	1	1	Emperical equation for $\frac{D\nu_T}{Dt}$ (ad hoc)

Table 3: Turbulent Models: Neutral Flow Two Equation Models

Author	2d	3d	Terms
Kolmogorov (1942)	2	2	k, $\frac{k^{1/2}}{\ell}$ mean frequency of energetic motions
Harlow, Nakayama (1967)	2	2	k, $\frac{k^{3/2}}{\ell} = \varepsilon$ eddy dissipation
Spalding (1969)	2	2	k, $\frac{k}{\ell^2} = w$ square of mean frequency of energetic motions
Ng, Rodi, Spalding (1970)	2	2	k, $k\ell$ chosen to avoid fact that ℓ does not diffuse $\sim \frac{d\ell}{dy}$
Launder (1971)	2	2	k, ε + algebraic stress models for $\overline{u'v'}$, $\overline{u'}$, $\overline{v'}$, $\overline{w'}$
Jones and Launder (1972)	2	2	k, ε with low Reynolds number effects

Table 4: Turbulent Models: Neutral Flow Multi Equation Models

Author	2d	3d	Terms
Hanjalic-Launder (1970)	3	--	k, ε, $\overline{u'w'}$
Donaldson (1968)	4	6	$\overline{u'w'}$, $\overline{v'}$, $\overline{w'}$, $\overline{u'}$, and ℓ = algebraic
Rotta (1951)	5	7	$\overline{u'w'}$, $\overline{v'}$, $\overline{w'}$, $\overline{u'}$ and ε
Daly and Harlow (1970)	5	7	$\overline{u'w'}$, $\overline{v'}$, $\overline{w'}$, $\overline{u'}$ and ε
Chou (1945)	9	17	model triple correlations
Daidov (1961)	10	23	model triple correlations
Kolovandin (1969)	20	28	model triple correlations

Table 5: Turbulent Models: Stratified Flow
(does not include $\overline{T}$)

Author	1d	2d	3d	Terms
Pandolfo (1969)	0	0	0	Mixing length modified by Ri#
Spalding (1971)	1	1	1	$k_t = \frac{\overline{T'^2}}{2}$; no buoyancy effects
Launder (1972)	1	1	1	$\overline{T'^2}$, ν_T = as before $-\overline{w'T'} = \frac{k}{C\varepsilon}(\overline{w'^2}\frac{\partial T}{\partial y} - g\frac{\overline{T'^2}}{T})$
Bradshaw-Ferris (1968)	1	1	--	$\overline{T'^2}$, no buoyancy effects $\ell_t = f(y)$
Donaldson (1973)	3	3	4	$\overline{u'T'}$, $\overline{w'T'}$, $\overline{T'^2}$
Daly-Harlow (1972)	4	4	5	$\overline{u'T'}$, $\overline{w'T'}$, $\overline{T'}$, ε - Rayleigh-Benard Cells
Lumley (1972)	5	5	6	$\overline{u'T'}$, $\overline{w'T'}$, $\overline{T'}$, ε, ε_t (proposed only)
Meroney				
ALM	1	1	1	k_t, ℓ algebraic
DLM	2	2	2	k_t, ε_t
ASM	2	2	2	k_t, ε_t, ASM $\overline{w'T'}$, $\overline{u'T'}$

Table 6: Exact Equations: Cartesian Coordinates
Boussinesq Approximation

Continuity

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \frac{\partial u_i'}{\partial x_i} = 0$$

Momentum

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j} - \rho\,\overline{u_i' u_j'}\right) + \frac{\overline{T}}{T_o}\frac{\partial \phi}{\partial x_j}$$

Energy

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \frac{1}{\rho C_p}\frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j} - \rho\,\overline{u_j' T'}\right)$$

Double Velocity Correlations

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} + u_k \frac{\partial \overline{u_i' u_j'}}{\partial x_k} = -\frac{\partial}{\partial x_k}\left[-\nu \frac{\partial \overline{u_i' u_j'}}{\partial x_k} + \overline{u_k' u_i' u_j'} + \overline{(\delta_{jk} u_i' + \delta_{ik} u_j')\frac{p'}{\rho}}\right]$$

$$+ \overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - \overline{u_i' u_k'}\frac{\partial u_j}{\partial x_k} - \overline{u_j' u_k'}\frac{\partial u_i}{\partial x_k}$$

$$- 2\nu\left(\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}\right) + \frac{1}{T_o}\left(\frac{\partial \phi}{\partial x_j}\overline{u_i' T'} + \frac{\partial \phi}{\partial x_i}\overline{u_j' T'}\right)$$

Table 6: (Continued)

Velocity Temperature Correlations

$$\frac{\partial \overline{u_i'T'}}{\partial t} + u_k \frac{\partial}{\partial x_k}(\overline{u_i'T'}) = - \frac{\partial}{\partial x_k}[- \frac{\nu}{Pr} \frac{\partial \overline{u_i'T'}}{\partial x_k} + \overline{u_k'u_i'T'} + \frac{\overline{p'T'}}{\rho_o} \delta_{ik}]$$

$$- \overline{u_k'u_i'} \frac{\partial T}{\partial x_k} - \overline{u_k'T'} \frac{\partial u_i}{\partial x_k} + \overline{p' \frac{\partial T'}{\partial x_i}}$$

$$- \frac{2\nu}{Pr}\overline{(\frac{\partial u_i'}{\partial x_k} \frac{\partial T'}{\partial x_k})} + \frac{1}{T_o} \frac{\partial \phi}{\partial x_i} \overline{T'^2}$$

Temperature Correlation

$$\frac{\partial \overline{T'^2}}{\partial t} + u_k \frac{\partial \overline{T'^2}}{\partial x_k} = + \frac{\partial}{\partial x_k}(\frac{\nu}{Pr} \frac{\partial \overline{T'^2}}{\partial x_k} - \overline{u_k'T'^2})$$

$$- 2 \overline{u_k'T'} \frac{\partial T}{\partial x_k} - 2 \frac{\nu}{Pr}\overline{(\frac{\partial T'}{\partial x_k})^2}$$

Dissipation

$$\frac{\partial \varepsilon}{\partial t} + u_k \frac{\partial \varepsilon}{\partial x_k} = -2\nu\overline{(\frac{\partial u_i}{\partial x_k} \frac{\partial u_i'}{\partial x_\ell} \frac{\partial u_k'}{\partial x_\ell} + \frac{\partial u_\ell'}{\partial x_i} \frac{\partial u_\ell'}{\partial x_k})} - 2\nu \frac{\partial^2 u_i}{\partial x_k \partial x_\ell} \overline{(u_k' \frac{\partial u_i'}{\partial x_\ell})}$$

$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_\ell} \frac{\partial u_k'}{\partial x_\ell}} - \nu \frac{\partial}{\partial x_k} \overline{u_k'(\frac{\partial u_i'}{\partial x_j})^2} - \frac{\nu}{\rho} \frac{\partial}{\partial x_i}\overline{(\frac{\partial p'}{\partial x_\ell} \frac{\partial u_i'}{\partial x_\ell})} + \nu \frac{\partial^2 \varepsilon}{\partial x_k^2}$$

$$- 2\overline{(\nu \frac{\partial^2 u_i'}{\partial x_k \partial x_\ell})^2} + \frac{2\nu}{T_o} \frac{\partial \phi}{\partial x_j} \overline{(\frac{\partial T'}{\partial x_\ell})(\frac{\partial u_j'}{\partial x_\ell})}$$

Table 6: (Continued)

Energy Dissipation

$$\frac{\partial \varepsilon_t}{\partial t} + u_k \frac{\partial \varepsilon_t}{\partial x_k} = -2 \frac{\nu}{Pr} \frac{\partial u_k}{\partial x_\ell} \overline{\frac{\partial T'}{\partial x_\ell} \frac{\partial T'}{\partial x_k}} - \frac{2\nu}{Pr} \frac{\partial T}{\partial x_k} \overline{\left(\frac{\partial T'}{\partial x_\ell} \frac{\partial u_k'}{\partial x_\ell}\right)}$$

$$- 2 \frac{\nu}{Pr} \frac{\partial^2 T}{\partial x_k \partial x_\ell} \overline{\left(u_k' \frac{\partial T'}{\partial x_\ell}\right)} - 2 \frac{\nu}{Pr} \overline{\frac{\partial u_k'}{\partial x_\ell} \frac{\partial T'}{\partial x_k} \frac{\partial T'}{\partial x_\ell}}$$

$$- \frac{\nu}{Pr} \frac{\partial}{\partial x_k} \left(\overline{u_k' \, t'}\right) + \frac{\nu}{Pr} \frac{\partial^2 \varepsilon_t}{\partial x_k^{2}} - 2 \overline{\left(\frac{\nu}{Pr} \frac{\partial^2 T'}{\partial x_k \partial x_\ell}\right)^2}$$

Table 7: Mixing Length Model: Stratified Flows
Pandolfo (1968)

Inversion Conditions $\quad Ri > 0 \quad Ri = \dfrac{\frac{-g}{\rho}(\frac{\partial\rho}{\partial z})}{(\frac{\partial v}{\partial z})^2}$

$$\nu_T = \alpha_T$$

$$\nu_T = k^2 z^2 |\frac{\partial \bar{v}}{\partial z}| (1 + \alpha Ri)^2$$

$$\alpha = -3.0$$

$$k = 0.4$$

Lapse Forced Convection $\quad -0.05 < Ri < 0$

$$\nu_T = k^2 z^2 |\frac{\partial \bar{v}}{\partial z}| (1 - \alpha Ri)^{-2}$$

$$\alpha_T = \nu_T [1 - \alpha Ri]^{-2}$$

Lapse Free Convection $\quad Ri \leq -0.05$

$$\nu_T = \alpha_T (\frac{3}{C})^{-\frac{1}{2}} |Ri|^{-\frac{1}{6}}$$

$$\alpha_T = h(z)^2 |\frac{g}{T} (\frac{\partial T}{\partial z} + \Gamma)|^{\frac{1}{2}}$$

$C = 3k^{\frac{4}{3}} h^{-\frac{2}{3}}$; h = Priestly Constant $\cong 0.9$

Table 8: Prandtl-Kolmogorov Viscosity Assumption

$$\underbrace{\frac{D\overline{u'w'}}{Dt}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial z}(\overline{u'w'^2} + \frac{\overline{p'u'}}{\rho})}_{\text{Diffusion}} = \underbrace{-\overline{w'^2}\frac{\partial u}{\partial z}}_{\text{Prod.}} + \underbrace{\frac{R'}{\rho}(\frac{\partial u'}{\partial z} + \frac{\partial w'}{\partial x})}_{\text{Redistribution}} - \underbrace{2\nu \sum_{\ell} (\overline{\frac{\partial u'}{\partial x_{\ell}} \frac{\partial w'}{\partial x_{\ell}}})}_{\text{Dissipation}}$$

▲ $(A - D)_{\overline{u'w'}} \equiv 0$

▲ Production $\propto k\frac{\partial u}{\partial z}$

▲ Diffusion $\propto -\frac{\partial}{\partial z}[k^{\frac{1}{2}}\ell \frac{\partial\overline{u'w'}}{\partial z}]$

▲ Dissipation $\propto$ 0 (large Re#)

▲ Redistribution $\propto k\frac{\partial u}{\partial z}$ and $(\frac{k^{\frac{1}{2}}}{\ell})\overline{u'w'}$

∴ $0 = -C(k\frac{\partial u}{\partial z} + \frac{k^{\frac{1}{2}}}{\ell}\overline{u'w'})$

$\overline{u'w'} = k^{\frac{1}{2}}\ell \frac{\partial u}{\partial z}$

$\boxed{\nu_T = k^{\frac{1}{2}}\ell}$ but $\frac{k^{\frac{3}{2}}}{\ell} \propto \hat{\varepsilon}$

∴ $\boxed{\nu_T \equiv C_D\frac{k^2}{\varepsilon}}$

Table 8: (Continued) Meroney Diffusivity Assumption

$$\underbrace{\frac{D\overline{w'T'}}{Dt}}_{\text{Advection}} + \underbrace{\frac{\partial}{\partial z}(\overline{w'^2T'} + \frac{\overline{p'T'}}{\rho})}_{\text{Diffusion}} = \underbrace{-\overline{w'^2}\,\frac{\partial T}{\partial z} - \overline{w'T'}\,\frac{\partial u}{\partial z}}_{\text{Production}} + \underbrace{\frac{\overline{p'\partial T'}}{\partial z}}_{\text{Redistribution}} - \underbrace{\frac{2u}{Pr}\sum_{\ell}\overline{(\frac{\partial u_i'}{\partial x_\ell}\,\frac{\partial T'}{\partial x_\ell})}}_{\text{Diss.}}$$

▲ $(A - D)_{\overline{w'T'}} \equiv 0$

▲ Production $\propto k\frac{\partial T}{\partial z}$

▲ Dissipation $\propto 0$

▲ Redistribution $\propto \frac{k^{\frac{1}{2}}}{\ell_t}\,\overline{w'T'}$

∴ $0 = -C(k\frac{\partial T}{\partial z} + \frac{k^{\frac{1}{2}}}{\ell_t}\,\overline{w'T'})$

$$\overline{w'T'} = k^{\frac{1}{2}}\ell_t\,\frac{\partial T}{\partial z}$$

$$\boxed{\alpha_T = k^{\frac{1}{2}}\ell_t} \qquad \text{but} \qquad \varepsilon_t \propto \frac{k^{\frac{1}{2}}k_t}{\ell_t}$$

$$\boxed{\alpha_T = C_H\frac{k\ k_t}{\varepsilon_t}}$$

Table 9: Dimensional Scales

$$u = \frac{u^*}{u^*_{max}}$$

$$z = \frac{z^*}{L^*}$$

$$t = \frac{t^* u^*_{max}}{L^*}$$

$$k = \frac{k^*}{u^{*2}_{max}}$$

$$\varepsilon = \frac{\varepsilon^* L^*}{u^{*3}_{max}}$$

$$T = \frac{T^* - T^*_\infty}{(\Delta T^*)_{max}}$$

$$k_t = \frac{k^*_t}{(\Delta T^*)^2_{max}}$$

$$\varepsilon_t = \frac{\varepsilon_t L^*}{(\Delta T^*)^2_{max}\ u^*_{max}}$$

$$\nu_T = \frac{\nu^*_T}{\nu^*} = \frac{C_D k^2}{\varepsilon}$$

$$\alpha_T = \frac{\alpha^*_T}{\alpha^*} = \frac{C_H k\ k_t}{\varepsilon_t}$$

$$p = \frac{p^*}{\rho^* u^{*2}_{max}}$$

Table 10: One-dimensional Equations of Change

$$\frac{\partial u}{\partial t} = \underbrace{\frac{1}{Re}\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial z}(\overline{u'w'})}_{\text{Diffusion}} + X(t) \quad (1)$$

$$\frac{\partial T}{\partial t} = \underbrace{\frac{1}{RePr}\frac{\partial^2 T}{\partial z^2} - \frac{\partial}{\partial z}(\overline{w'T'})}_{\text{Diffusion}} \quad (2)$$

$$\frac{\partial k}{\partial t} = \underbrace{\frac{1}{Re}\frac{\partial^2 k}{\partial z^2} - \frac{\partial}{\partial z}(\overline{k'w'} + \overline{p'w'})}_{\text{Diffusion}} \underbrace{- \overline{u'w'}\frac{\partial u}{\partial z}}_{\text{Production}} - \underbrace{\varepsilon}_{\text{Diss.}} \underbrace{+ Ri\,\overline{w'T'}}_{\text{Strat.}} \quad (3)$$

$$\frac{\partial k_t}{\partial t} = \underbrace{\frac{1}{RePr}\frac{\partial^2 k_t}{\partial z^2} - \frac{\partial}{\partial z}(\overline{k_t'w'})}_{\text{Diffusion}} \underbrace{- \overline{w'T'}\frac{\partial T}{\partial z}}_{\text{Prod.}} - \underbrace{\varepsilon_t}_{\text{Diss.}} \quad (4)$$

$$\frac{\partial \varepsilon}{\partial t} = \underbrace{\frac{1}{Re}\frac{\partial^2 \varepsilon}{\partial z^2} - \frac{\partial}{\partial z}(\overline{\varepsilon' w'} + \frac{1}{Re}\overline{\frac{\partial p'}{\partial z}\frac{\partial w'}{\partial z}})}_{\text{Diffusion}} \underbrace{- \frac{2}{Re}\left[\frac{\partial u}{\partial z}(\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}}) + \frac{\partial^2 u}{\partial z^2}(\overline{w'\frac{\partial u'}{\partial z}})\right]}_{\text{Production}} \quad (5)$$

$$\underbrace{- \frac{2}{Re}\left[\overline{(\frac{\partial u'}{\partial z})^2\frac{\partial w'}{\partial z}} + \overline{(\frac{\partial v'}{\partial z})^2\frac{\partial w'}{\partial z}} + \overline{(\frac{\partial w'}{\partial z})^3}\right]}_{\text{Production}} \underbrace{- \frac{2}{Re}\left[\overline{(\frac{\partial^2 u'}{\partial z^2})^2} + \overline{(\frac{\partial^2 w'}{\partial z^2})^2} + \overline{(\frac{\partial^2 v'}{\partial z^2})^2}\right]}_{\text{Destruction}}$$

$$\underbrace{+ \frac{2}{Re}Ri(\overline{\frac{\partial u'}{\partial z}\frac{\partial w'}{\partial z}})}_{\text{Strat.}}$$

Table 10: (Continued)

$$\frac{\partial \varepsilon_t}{\partial t} = \underbrace{\frac{1}{RePr}\frac{\partial^2 \varepsilon_t}{\partial z^2} - \frac{\partial}{\partial z}(\overline{\varepsilon_t' w'})}_{\text{Diffusion}} - \underbrace{\frac{2}{Pr}\left[\frac{\partial u}{\partial z}\overline{\left(\frac{\partial T'}{\partial z}\right)^2} + \frac{\partial T}{\partial z}\overline{\left(\frac{\partial T'}{\partial z}\right)\left(\frac{\partial u'}{\partial z}\right)} + \frac{\partial^2 T}{\partial z^2}\overline{\left(w'\frac{\partial T'}{\partial z}\right)}\right]}_{\text{Production}} \quad (6)$$

$$- \underbrace{\frac{2}{Pr}\left[\overline{\frac{\partial w'}{\partial z}\left(\frac{\partial T'}{\partial z}\right)^2}\right]}_{\text{Production}} - \underbrace{\frac{2}{RePr}\overline{\left(\frac{\partial^2 T'}{\partial z^2}\right)^2}}_{\text{Destruction}}$$

$$\frac{\partial \overline{u'w'}}{\partial t} = \underbrace{\frac{1}{Re}\frac{\partial^2 \overline{u'w'}}{\partial z^2} - \frac{\partial}{\partial z}(\overline{u'w'^2} + \overline{u'p'})}_{\text{Diffusion}} - \underbrace{\overline{w'^2}\frac{\partial u}{\partial z}}_{\text{Prod.}} - \underbrace{\overline{p'\left(\frac{\partial u'}{\partial z}\right)}}_{\text{Press-Strain}} - \underbrace{\frac{2}{Re}\overline{\left(\frac{\partial u'}{\partial z}\right)\left(\frac{\partial w'}{\partial z}\right)}}_{\text{Dissipation}} \quad (7)$$

$$+ \underbrace{Ri(\overline{u'T'})}_{\text{Strat.}}$$

$$\frac{\partial \overline{w'T'}}{\partial t} = \underbrace{\frac{1}{RePr}\frac{\partial^2 \overline{w'T'}}{\partial z^2} - \frac{\partial}{\partial z}(\overline{w'^2T'} + \overline{p'T'})}_{\text{Diffusion}} - \underbrace{\overline{w'^2}\frac{\partial T}{\partial z}}_{\text{Prod.}} + \underbrace{\overline{p'\frac{\partial T'}{\partial z}}}_{\text{Press-Strain}} - \underbrace{\frac{2}{RePr}\overline{\left(\frac{\partial w'}{\partial z}\frac{\partial T'}{\partial z}\right)}}_{\text{Dissipation}} \quad (8)$$

$$+ \underbrace{2Ri\, k_t}_{\text{Strat.}}$$

$$\frac{\partial \overline{u'T'}}{\partial t} = \underbrace{\frac{1}{RePr}\frac{\partial^2 \overline{u'T'}}{\partial z^2} - \frac{\partial}{\partial z}(\overline{w'u'T'})}_{\text{Diffusion}} - \underbrace{(\overline{u'w'})\frac{\partial T}{\partial z} - (\overline{w'T'})\frac{\partial u}{\partial z}}_{\text{Production}} - \underbrace{\frac{2}{RePr}\overline{\left(\frac{\partial u'}{\partial z}\frac{\partial T'}{\partial z}\right)}}_{\text{Dissipation}} \quad (9)$$

Table 10: (Continued)

$$\frac{\partial \overline{w'^2}}{\partial t} = \underbrace{\frac{1}{Re}\frac{\partial^2 \overline{w'^2}}{\partial z^2} - \frac{\partial}{\partial z}(\overline{w'w'^2} + 2\,\overline{p'w'})}_{\text{Diffusion}} + \underbrace{2\,\overline{p'(\frac{\partial w'}{\partial z})}}_{\text{Press-Strain}} - \underbrace{\frac{2}{Re}\overline{(\frac{\partial w'}{\partial z})^2}}_{\text{Dissipation}} \tag{10}$$

$$+ \underbrace{2\,Ri(\overline{w'T'})}_{\text{Strat.}}$$

where

$$k = \tfrac{1}{2}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}) \quad \text{and} \quad \varepsilon \equiv \frac{1}{Re}(\overline{(\frac{\partial w'}{\partial z})^2} + \overline{(\frac{\partial v'}{\partial z})^2} + \overline{(\frac{\partial w'}{\partial z})^2})$$

$$k_t = \frac{\overline{T'^2}}{2} \qquad \varepsilon_t \equiv \frac{1}{RePr}\overline{(\frac{\partial T'}{\partial z})^2}$$

Table 11: Turbulent Model Equations (ALM) and (DLM)

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial z}\left(C_D \frac{k^2}{\varepsilon}\frac{\partial u}{\partial z}\right) + X(t) \tag{1}$$

$$\frac{\partial T}{\partial t} = \frac{1}{RePr}\frac{\partial^2 T}{\partial z^2} + \frac{\partial}{\partial z}\left(C_H \frac{k\, k_t}{\varepsilon_t}\frac{\partial T}{\partial z}\right) \tag{2}$$

$$\frac{\partial k}{\partial t} = \frac{1}{Re}\frac{\partial^2 k}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_D}{\sigma_k}\frac{k^2}{\varepsilon}\frac{\partial k}{\partial z}\right) + C_D \frac{k^2}{\varepsilon}\left(\frac{\partial u}{\partial z}\right)^2 - \varepsilon + RiC_H \frac{k\, k_t}{\varepsilon_t}\left(\frac{\partial T}{\partial z}\right) \tag{3}$$

$$\frac{\partial k_t}{\partial t} = \frac{1}{RePr}\frac{\partial^2 k_t}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_H}{\sigma_{kt}}\frac{k\, k_t}{\varepsilon_t}\right) + C_H \frac{k\, k_t}{\varepsilon_t}\left(\frac{\partial T}{\partial z}\right)^2 - \varepsilon_t \tag{4}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{Re}\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_D}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) + C_{E1}\, C_D\, k\left(\frac{\partial u}{\partial z}\right)^2 - C_{E2}\frac{\varepsilon^2}{k} - F \cdot Ri\, C_H\, k_t \frac{\varepsilon}{\varepsilon_t}\left(\frac{\partial T}{\partial z}\right) \tag{5}$$

$$\frac{\partial \varepsilon_t}{\partial t} = \frac{1}{RePr}\frac{\partial^2 \varepsilon_t}{\partial z^2} + \frac{\partial}{\partial z}\left(\frac{C_H}{\sigma_{\varepsilon t}}\frac{k\, k_t}{\varepsilon_t}\frac{\partial \varepsilon_t}{\partial z}\right) + C_{Et1}\, C_H\, k_t \frac{\varepsilon}{\varepsilon_t}\left(\frac{\partial T}{\partial z}\right)^2 - C_{Et2}\frac{\varepsilon\, \varepsilon_t}{k} \tag{6}$$

Table 12: Turbulent Model Equations (ASM)

$$\frac{\partial u}{\partial t} = \frac{1}{Re}\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial z}(\overline{u'w'}) + X(t) \tag{1}$$

$$\frac{\partial T}{\partial t} = \frac{1}{RePr}\frac{\partial^2 T}{\partial z^2} - \frac{\partial}{\partial z}(\overline{w'T'}) \tag{2}$$

$$\frac{\partial k}{\partial t} = \frac{1}{Re}\frac{\partial^2 k}{\partial z^2} + \frac{\partial}{\partial z}(\frac{C_D}{\sigma_k}\frac{k^2}{\varepsilon}\frac{\partial k}{\partial z}) - \overline{u'w'}\frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'} \tag{3}$$

$$\frac{\partial k_t}{\partial t} = \frac{1}{RePr}\frac{\partial^2 k_t}{\partial z^2} + \frac{\partial}{\partial z}(\frac{C_H}{\sigma_{kt}}\frac{k\ k_t}{\varepsilon_t}\frac{\partial k_t}{\partial z}) - \overline{w'T'}\frac{\partial T}{\partial z} - \varepsilon_t \tag{4}$$

$$\frac{\partial \varepsilon}{\partial t} = \frac{1}{Re}\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{\partial}{\partial z}(\frac{C_D}{\sigma_\varepsilon}\frac{k^2}{\varepsilon}\frac{\partial \varepsilon}{\partial z}) - C_{E1}\ \overline{u'w'}(\frac{\varepsilon}{k})(\frac{\partial u}{\partial z}) - C_{E2}\frac{\varepsilon^2}{k} \tag{5}$$

$$+ F\ Ri\ \overline{w'T'}\ \frac{\varepsilon}{k}$$

$$\frac{\partial \varepsilon_t}{\partial t} = \frac{1}{RePr}\frac{\partial^2 \varepsilon_t}{\partial z^2} + \frac{\partial}{\partial z}(\frac{C_H}{\sigma_{\varepsilon t}}\frac{k\ k_t}{\varepsilon_t}\frac{\partial \varepsilon_\tau}{\partial z}) - C_{Et1}\ \overline{w'T'}\ (\frac{\varepsilon}{k})(\frac{\partial T}{\partial z}) - C_{Et2}\frac{\varepsilon\ \varepsilon_t}{k} \tag{6}$$

$$\overline{u'w'} = \frac{\{-\frac{2}{3}(1 - A_1)\ k^2\ \frac{\partial u}{\partial z}\ (C_{P1}\varepsilon - \overline{u'w'}\ A_1\frac{\partial u}{\partial z} - \varepsilon + 3\ Ri\ \overline{w'T'})\}}{\{C_{P1}\varepsilon - \overline{u'w'}\frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'}\}^2} \tag{7}$$

$$+ \frac{Ri\ \overline{u'T'}\ k}{\{C_{P1}\varepsilon - \overline{u'w'}\frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'}\}}$$

Table 12: (Continued)

$$\overline{w'T'} = \frac{\{-\frac{2}{3}(1 - A_2)\ k^2\ \frac{\partial T}{\partial z}(C_{P1}\varepsilon - \overline{u'w'}\ A_1\ \frac{\partial u}{\partial z} - \varepsilon + 3\ Ri\ \overline{w'T'})\}}{\{C_{P1}\varepsilon - \overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'}\}\{C_{P2}\varepsilon - \overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'}\}} \tag{8}$$

$$+ \frac{2\ Ri\ k_t k}{\{C_{P2}\varepsilon - \overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'}\}}$$

$$\overline{u'T'} = \frac{\{(1 - A_3)(-\overline{u'w'}\ \frac{\partial T}{\partial z} - \overline{w'T'}\ \frac{\partial u}{\partial z})k\}}{\{C_{P3}\varepsilon - \overline{u'w'}\ \frac{\partial u}{\partial z} - \varepsilon + Ri\ \overline{w'T'}\}} \tag{9}$$

Table 13: Turbulence Model Constants

Investigator	C_D	C_H	C_{E1}	C_{E2}	C_{Et1}	C_{Et2}	F	σ_k	σ_ε	σ_{kt}	$\sigma_{\varepsilon t}$	A_1	A_2	A_3	C_{P1}	C_{P2}	C_{P3}
Reynolds (1970)			1.0	2.0								0.5			2.5		
Hanjalic and Launder (1972)	0.09		1.40 (1.45)	1.95 (2.00)				1.0	1.3			0.662			2.8		
Rodi (1972)	0.09		1.43 (1.2)	1.9 (1.7-2.0)				1.0				0.45 (0.4)			2.5 (2.5)		
Launder and Spalding (1972)	0.09		1.43	2.0				1.0	1.3	0.7							
Launder (1972)	0.09 $0.09g(P/\varepsilon)$		1.43 1.40	1.92 1.94				1.0 1.0	1.3 1.0	0.7 0.7							
Launder, Spalding and Whitelaw (1973)	0.09 0.09 0.09	 0.09	1.44 1.44 1.44	1.92 1.92 1.92				1.0 1.0 1.0	1.3 1.3 1.3	0.6-0.9 0.7		 0.798 0.622	 0.625		 1.3-2.8 2.8	 2.5	
Date (1973)			1.55-1.6	2.0								0.4			2.5		
Donaldson et al. (1972)	0.1	0.1	1.5	1.5	1.0-1.5	1.5	1.5	0.1		0.2		0.0	0.0	0.0	5.0	5.0	5.0
Meroney (1974)																	
ALM	0.1	0.1						0.1		0.2							
DLM	0.1	0.1	1.5	1.59	1.5	1.59	1.5	0.1	0.1	0.2	0.2						
ASM	0.1	0.1	1.5	1.55	1.5	1.5	1.5	0.1	0.1	0.2	0.2	0.0	0.0	0.0	5.0	5.0	5.0
Suggested	0.09	0.09	1.44	1.92	1.44	1.92	1.5	1.0	1.3	0.7	1.0	0.4	0.4	0.4	2.5	2.5	2.5

Table 14: MRS Equations of Donaldson, Sullivan, and Rosenbaum

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial z}(\overline{u'w'}) + X(z,t)$$

$$\frac{\partial T}{\partial t} = \frac{\nu}{Pr}\frac{\partial^2 T}{\partial z^2} - \frac{\partial}{\partial z}(\overline{w'T'})$$

$$\frac{\partial \overline{u'^2}}{\partial t} = \frac{\partial^2 \overline{u'^2}}{\partial z^2} + \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{u'^2}}{\partial z}) - 2\ \overline{u'w'}\ \frac{\partial u}{\partial z} - \frac{q}{\Lambda}(\overline{u'^2} - \frac{K}{3}) - 2\nu\frac{\overline{u'^2}}{\lambda^2}$$

$$\frac{\partial \overline{v'^2}}{\partial t} = \nu\ \frac{\partial \overline{v'^2}}{\partial z^2} + \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{v'^2}}{\partial z}) - \frac{q}{\Lambda}(\overline{v'^2} - \frac{K}{3}) - 2\nu\ \frac{\overline{v'^2}}{\lambda^2}$$

$$\frac{\partial \overline{w'^2}}{\partial t} = \nu\ \frac{\partial \overline{w'^2}}{\partial z^2} + 5\ \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{w'^2}}{\partial z}) - \frac{q}{\Lambda}(\overline{w'^2} - \frac{K}{3}) - 2\nu\ \frac{\overline{w'^2}}{\lambda 2} - \frac{2g}{T}\ \overline{w'T'}$$

$$\frac{\partial \overline{u'w'}}{\partial t} = \nu\ \frac{\partial^2 \overline{u'w'}}{\partial z^2} + 3\ \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{u'w'}}{\partial z}) - \overline{w'^2}\ \frac{\partial u}{\partial z} - \frac{q}{\Lambda}(\overline{u'w'}) - 2\nu\ \frac{\overline{u'w'}}{\lambda^2} + \frac{g}{T}\ \overline{u'T'}$$

$$\frac{\partial \overline{u'T'}}{\partial t} = \frac{\nu}{Pr}\ \frac{\partial^2 \overline{u'T'}}{\partial z^2} + \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{u'T'}}{\partial z}) - \overline{u'w'}\ \frac{\partial T}{\partial z} - \overline{w'T'}\ \frac{\partial u}{\partial z} - \frac{q}{\Lambda}(\overline{u'T'}) - 2\nu\ \frac{\overline{u'T'}}{\lambda^2}$$

$$\frac{\partial \overline{w'T'}}{\partial t} = \frac{\nu}{Pr}\ \frac{\partial^2 \overline{w'T'}}{\partial_z{}^2} + 3\ \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{w'T'}}{\partial z}) - \overline{w'^2}\ \frac{\partial T}{\partial z} - \frac{q}{\Lambda}(\overline{w'T'}) - 2\nu\ \frac{\overline{w'T'}}{\lambda^2} + \frac{g}{T}\ \overline{T'^2}$$

$$\frac{\partial \overline{T'^2}}{\partial t} = \frac{\nu}{Pr}\ \frac{\partial \overline{T'^2}}{\partial t} + \frac{\partial}{\partial z}(q\ \Lambda\ \frac{\partial \overline{T'^2}}{\partial z}) - 2\ \overline{w'T'}\ \frac{\partial T}{\partial z} - \frac{2\nu\overline{T'^2}}{\lambda^2}$$

where $K = (u_i'u_i')$, $q = K^{\frac{1}{2}}$

Table 15: Total Equations Required: Stratified Models

Author	1d	2d	3d	Terms
Donaldson, et al. (1972)	9	10	14	u, T, $\overline{u'w'}$, $\overline{w'T'}$, $\overline{u'T'}$, $\overline{T'^2}$, $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$ plus algebraic equations for λ and Λ
Meroney (1974)				
ALM	4	5	6	u, T, k, k_t plus algebraic equations ns for ℓ_1 and ℓ_2
DLM	6	7	8	u, T, k, k_t, ε, ε_t
ASM	6	7	8	u, T, k, k_t, ε, ε_t
				ASM for $\overline{u'w'}$, $\overline{u'T'}$, $\overline{w'T'}$ and $\overline{w'^2}$

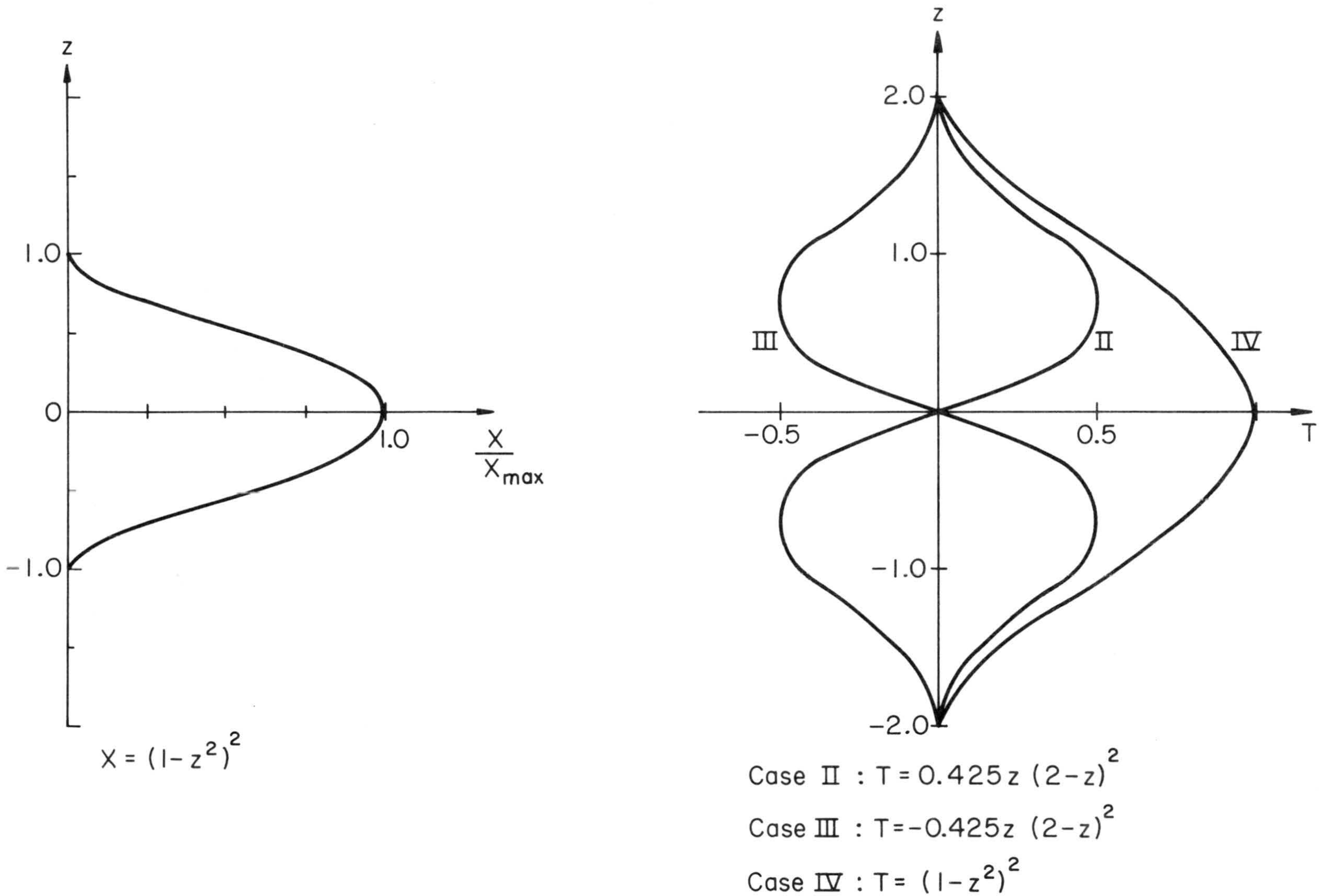

Fig. 1. Driving force function and initial temperature conditions clear-air turbulence model.

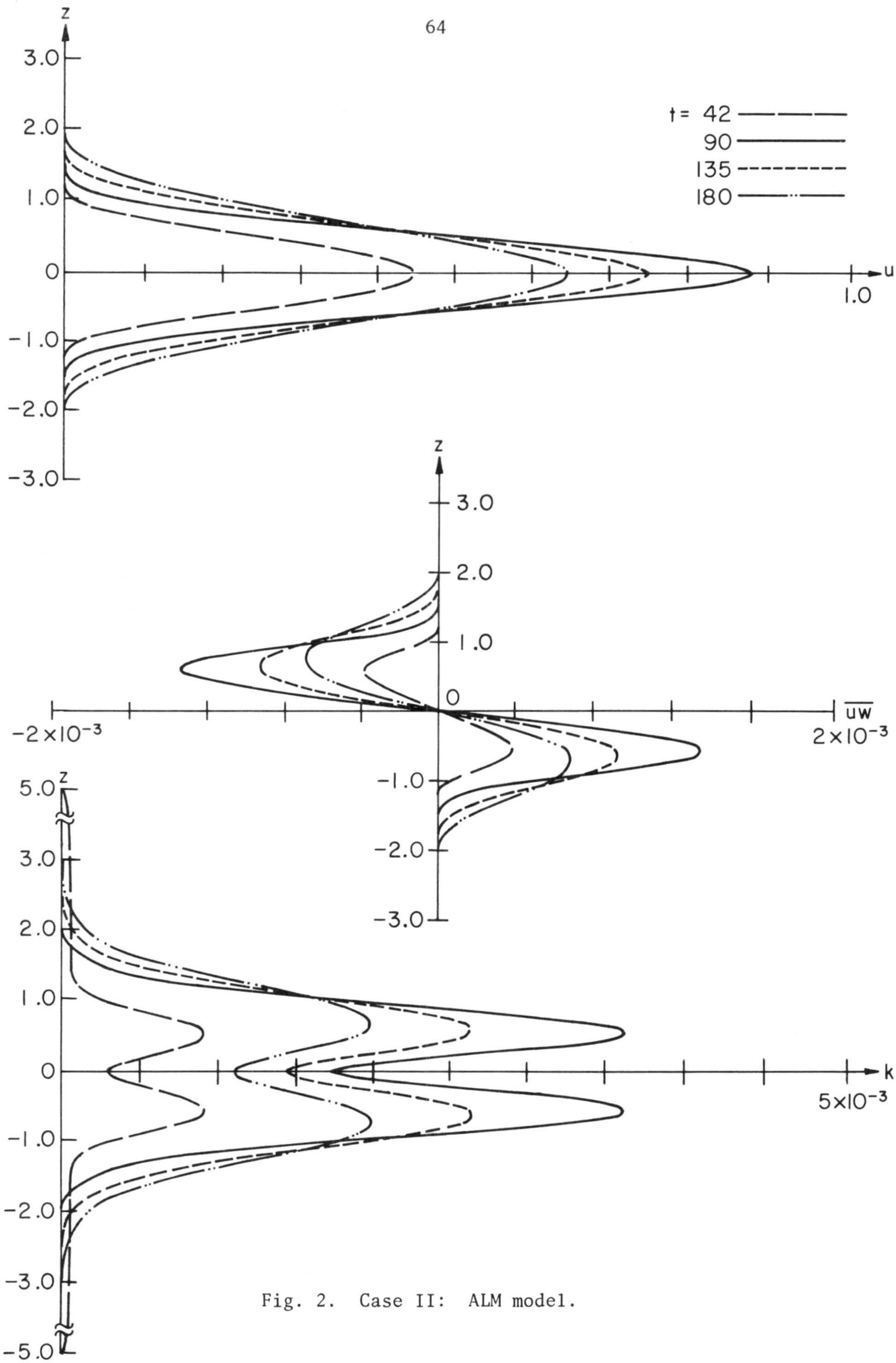

Fig. 2. Case II: ALM model.

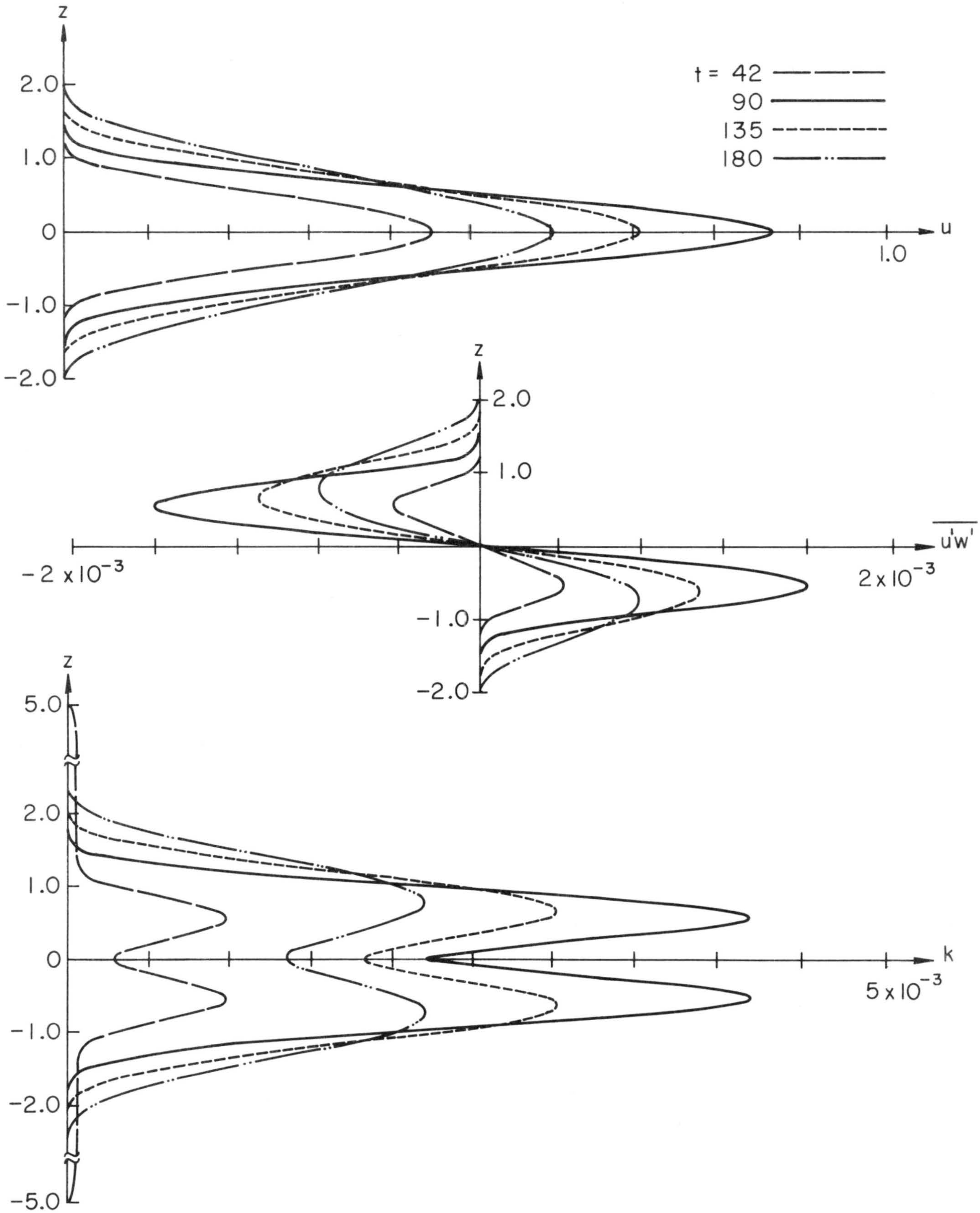

Fig. 3. Case I: DLM model.

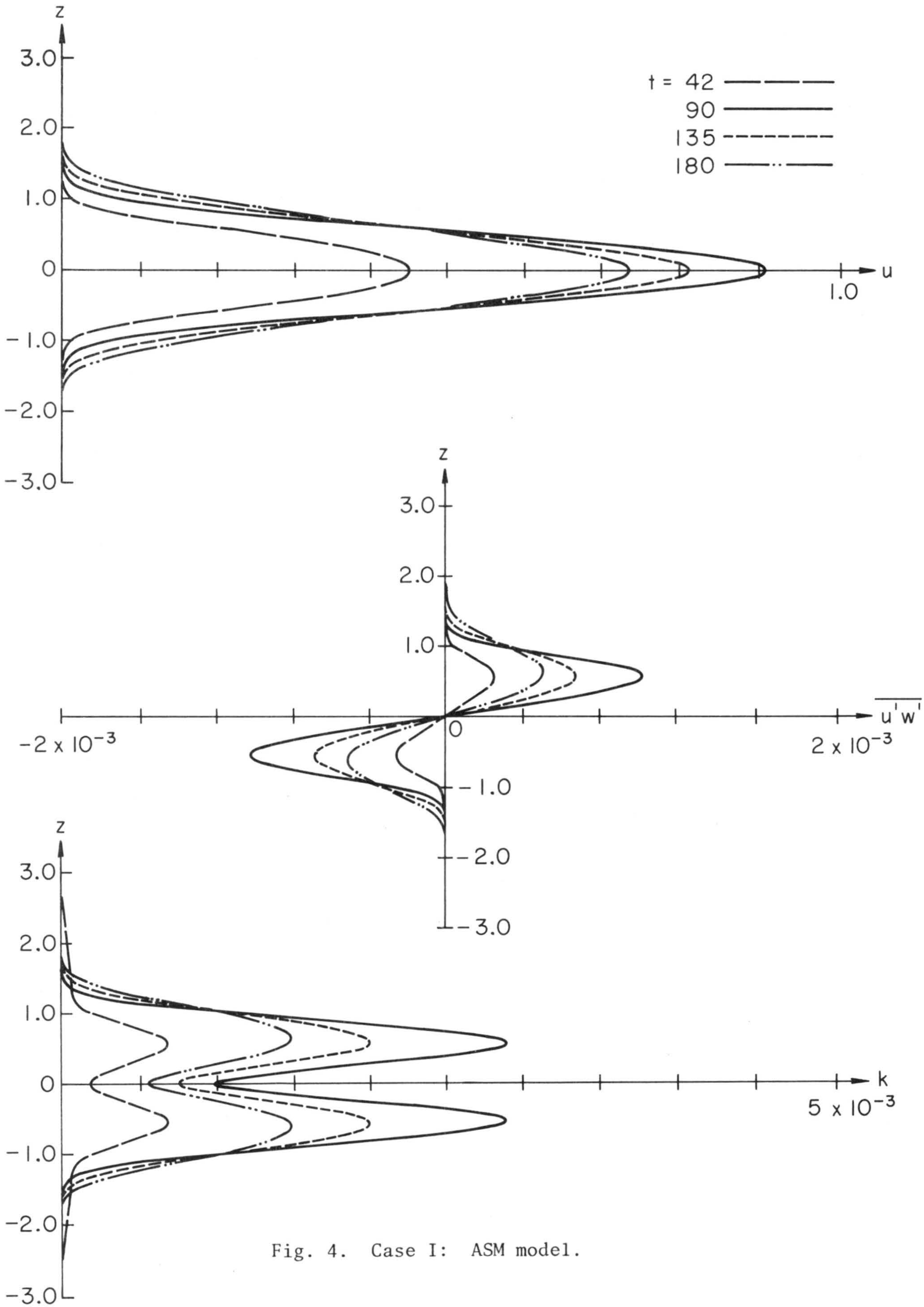

Fig. 4. Case I: ASM model.

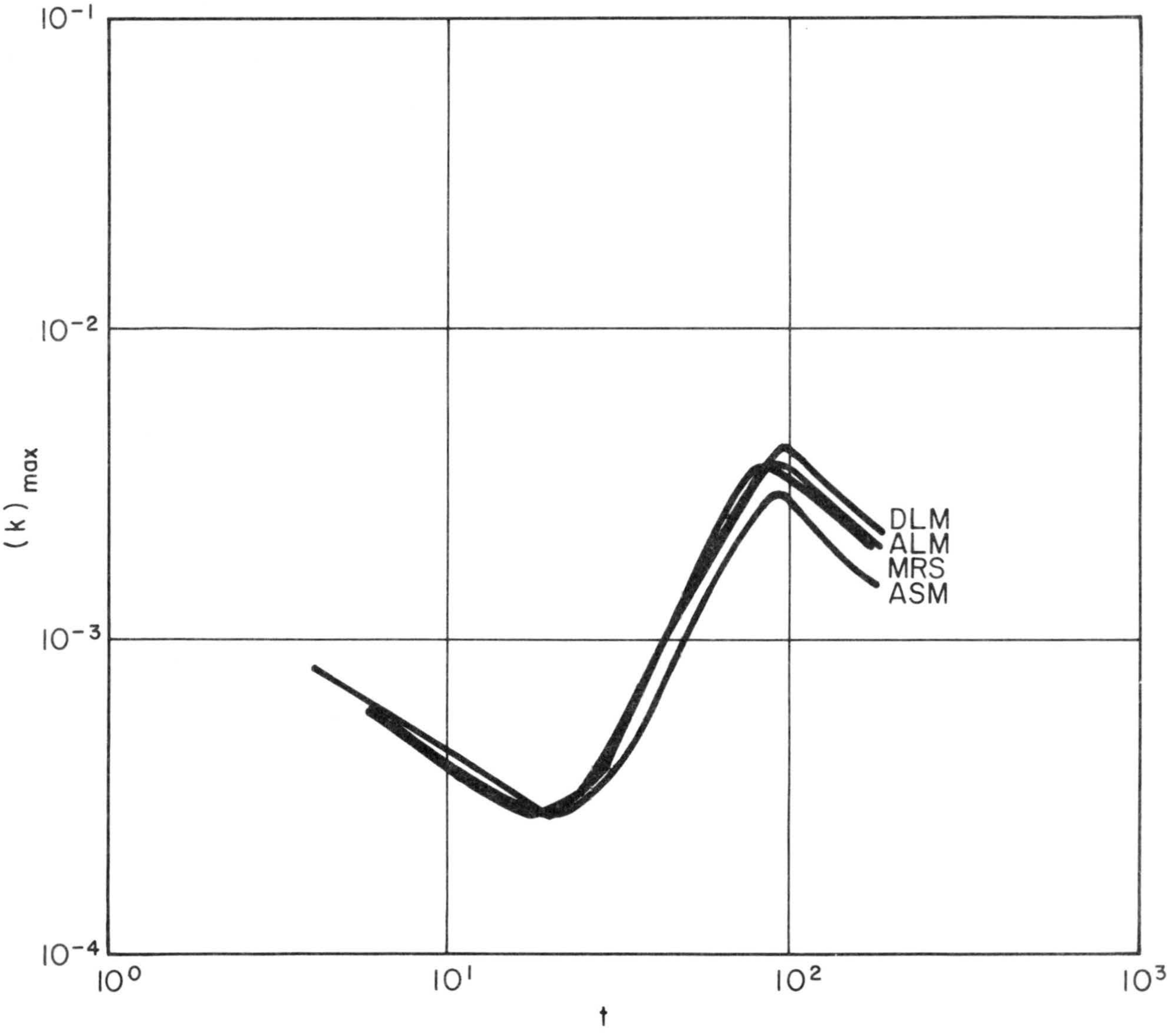

Fig. 5a. Maxima of velocity correlations: Case I.

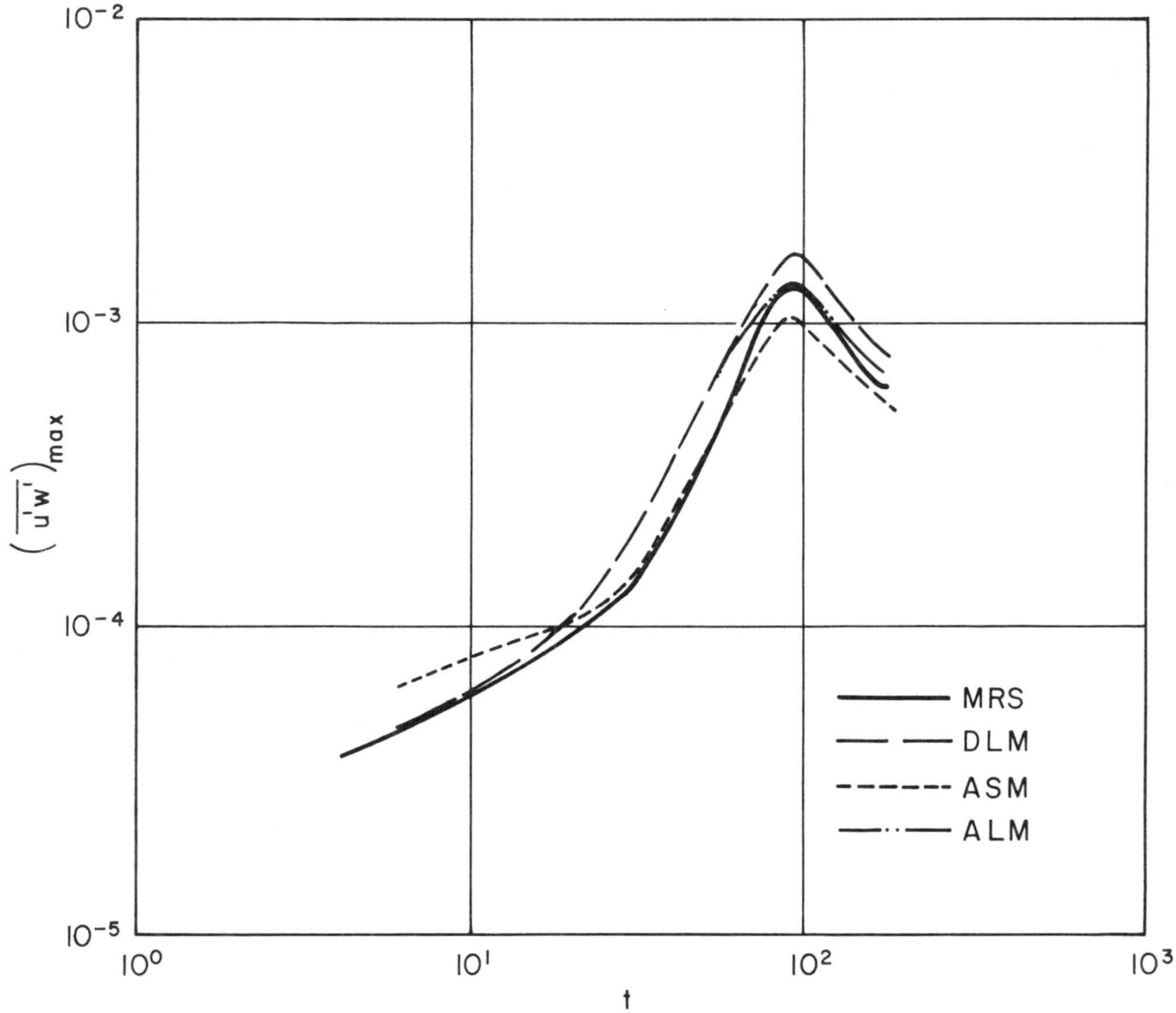

Fig. 5b. Maxima of velocity correlations: Case I.

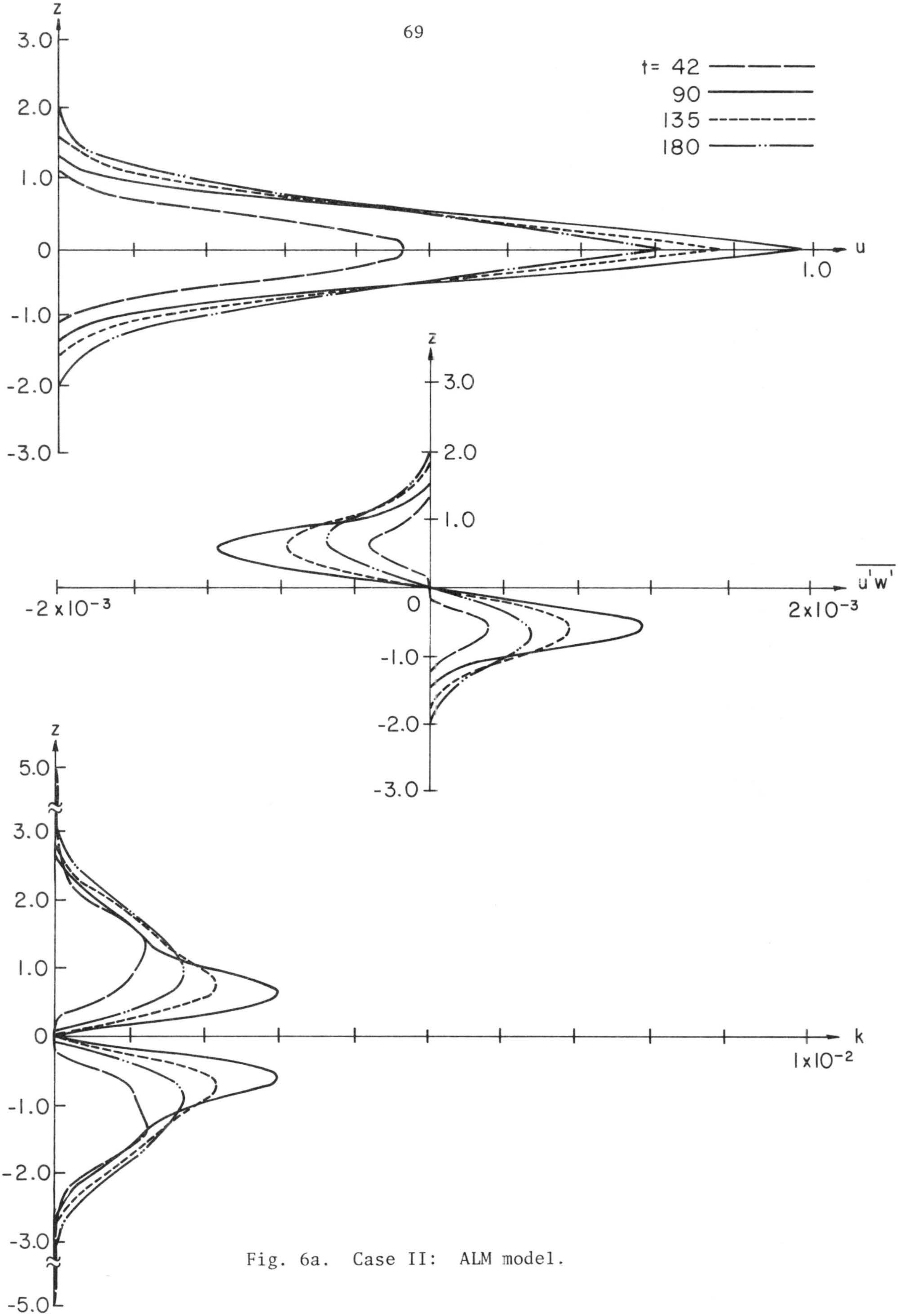

Fig. 6a. Case II: ALM model.

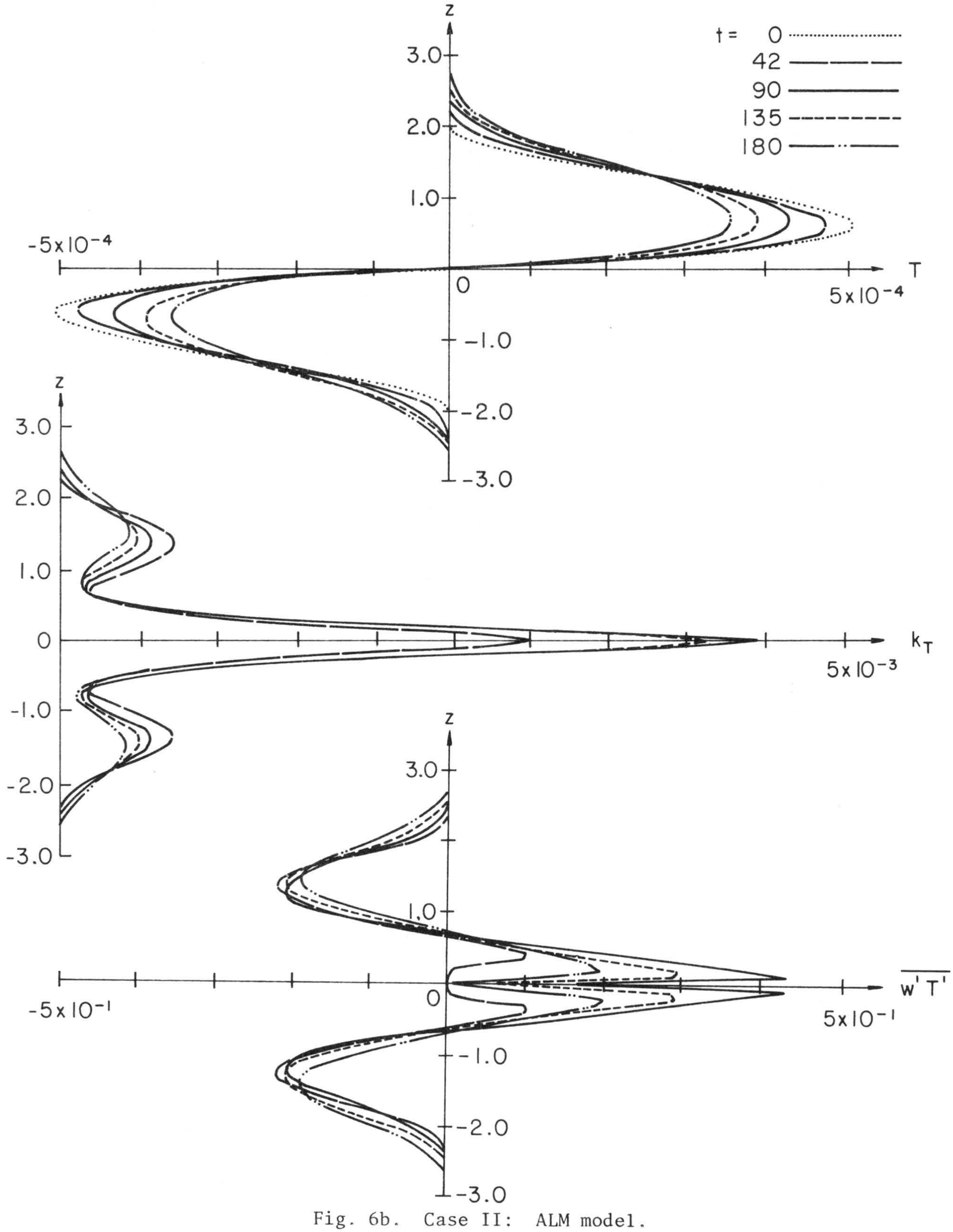

Fig. 6b. Case II: ALM model.

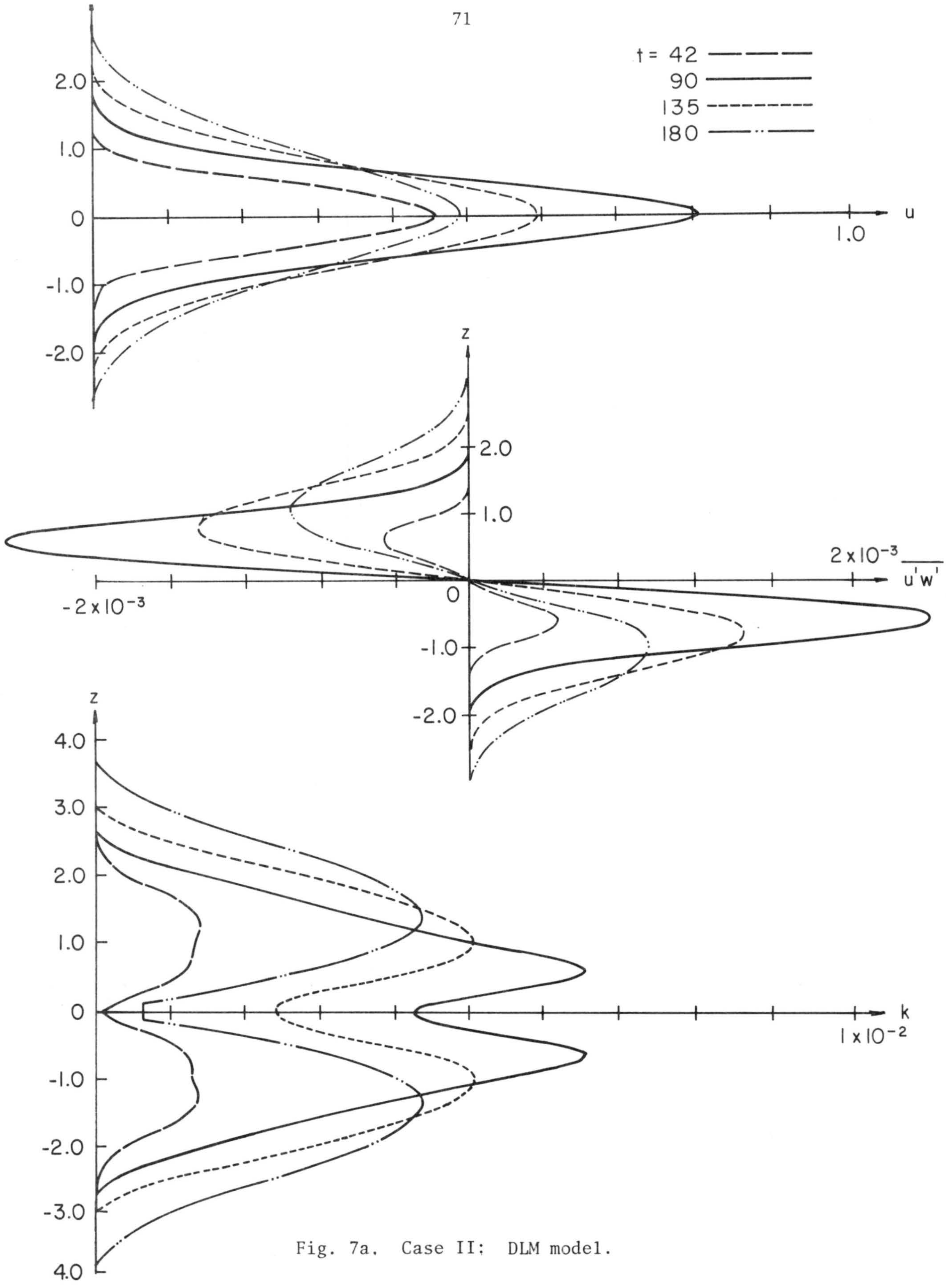

Fig. 7a. Case II: DLM model.

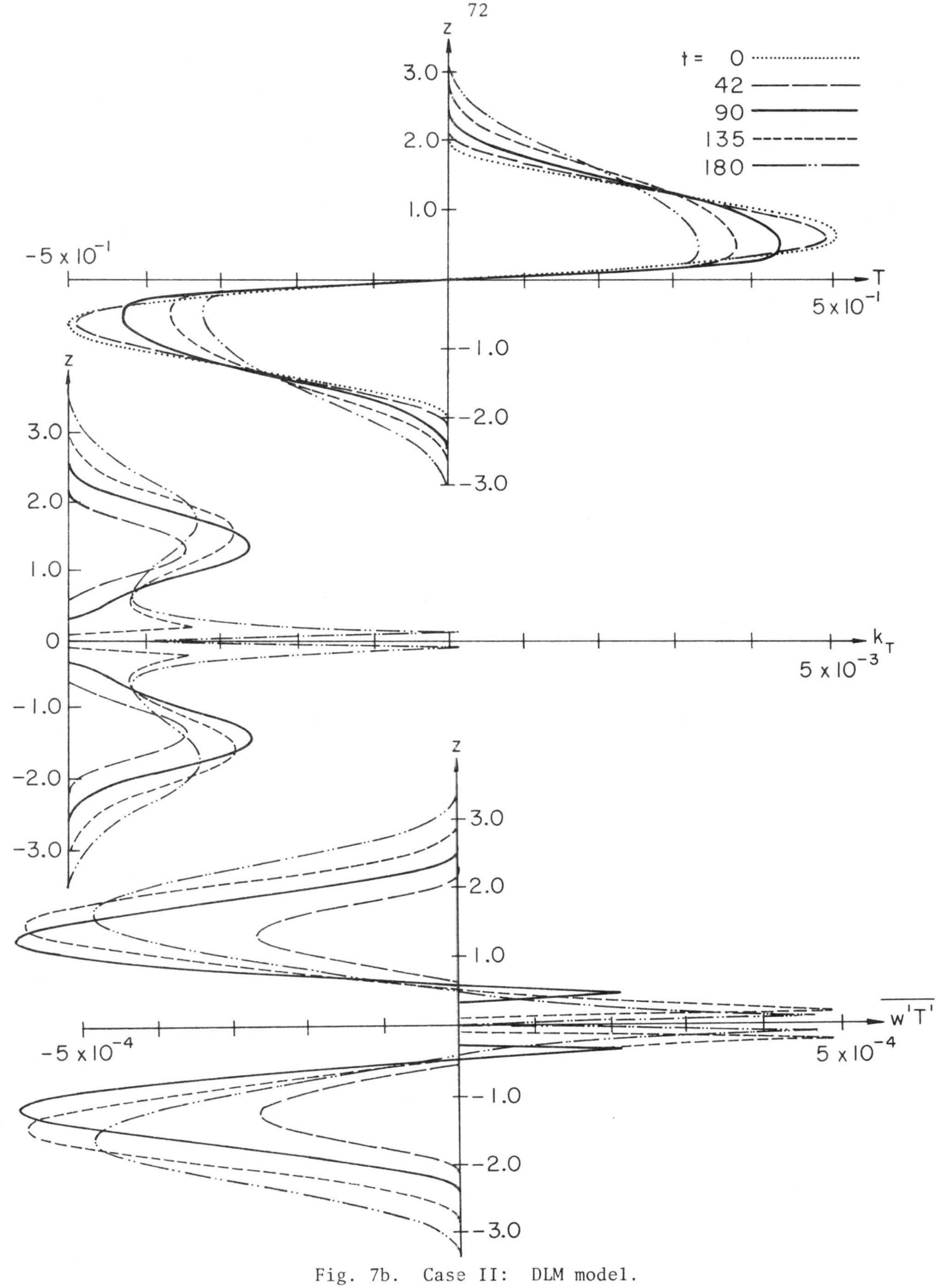

Fig. 7b. Case II: DLM model.

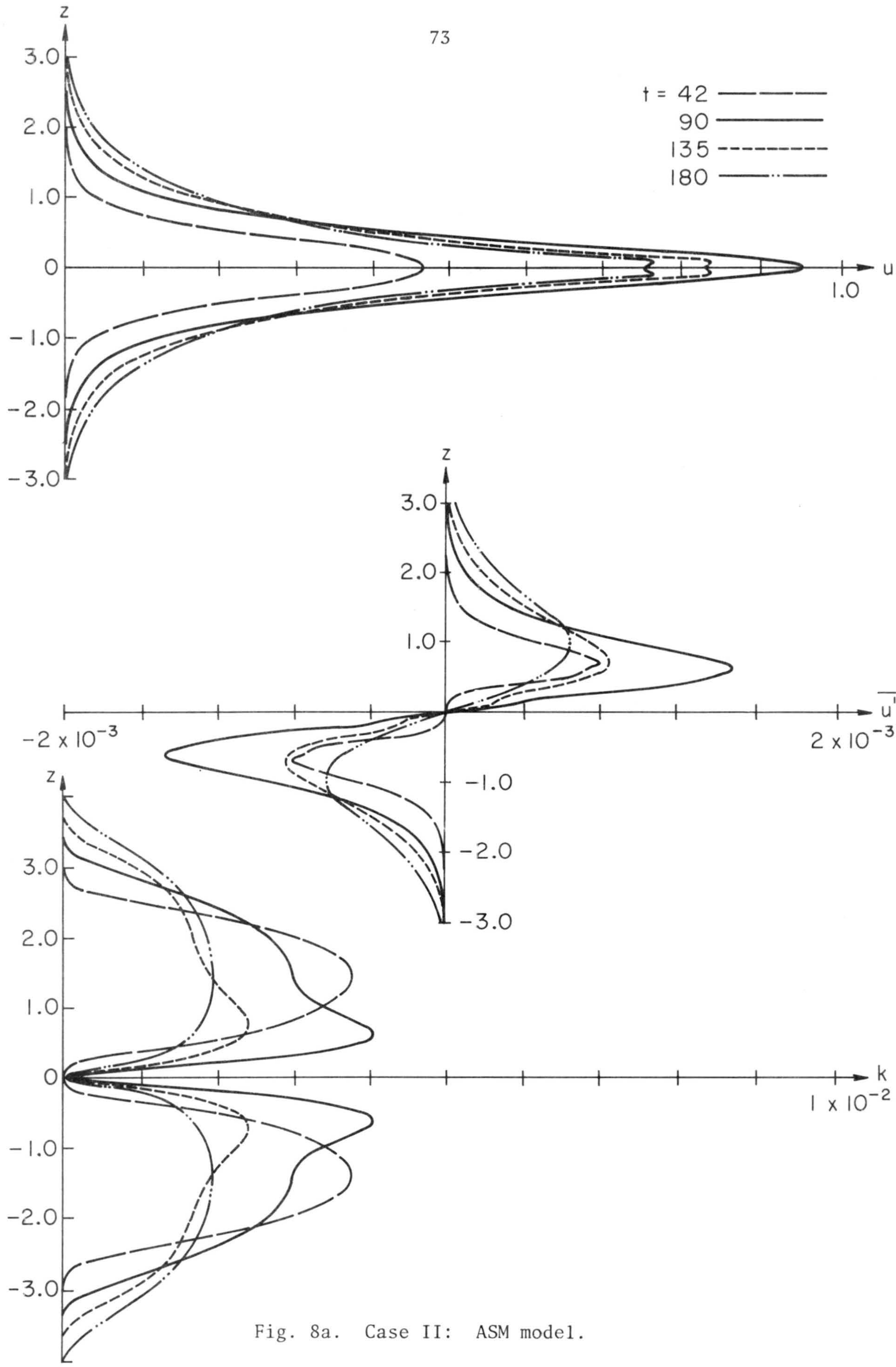

Fig. 8a. Case II: ASM model.

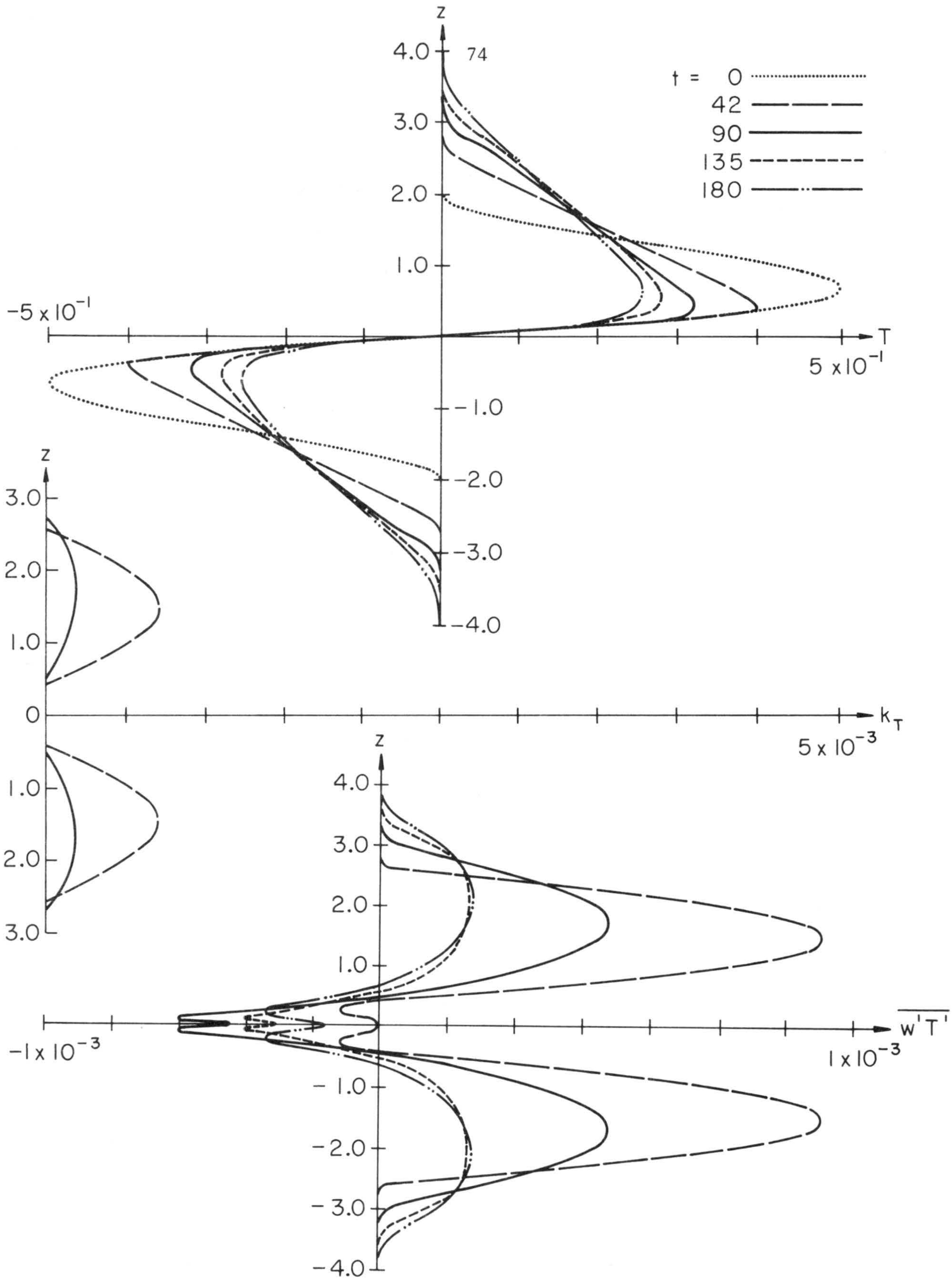

Fig. 8b. Case II: ASM model.

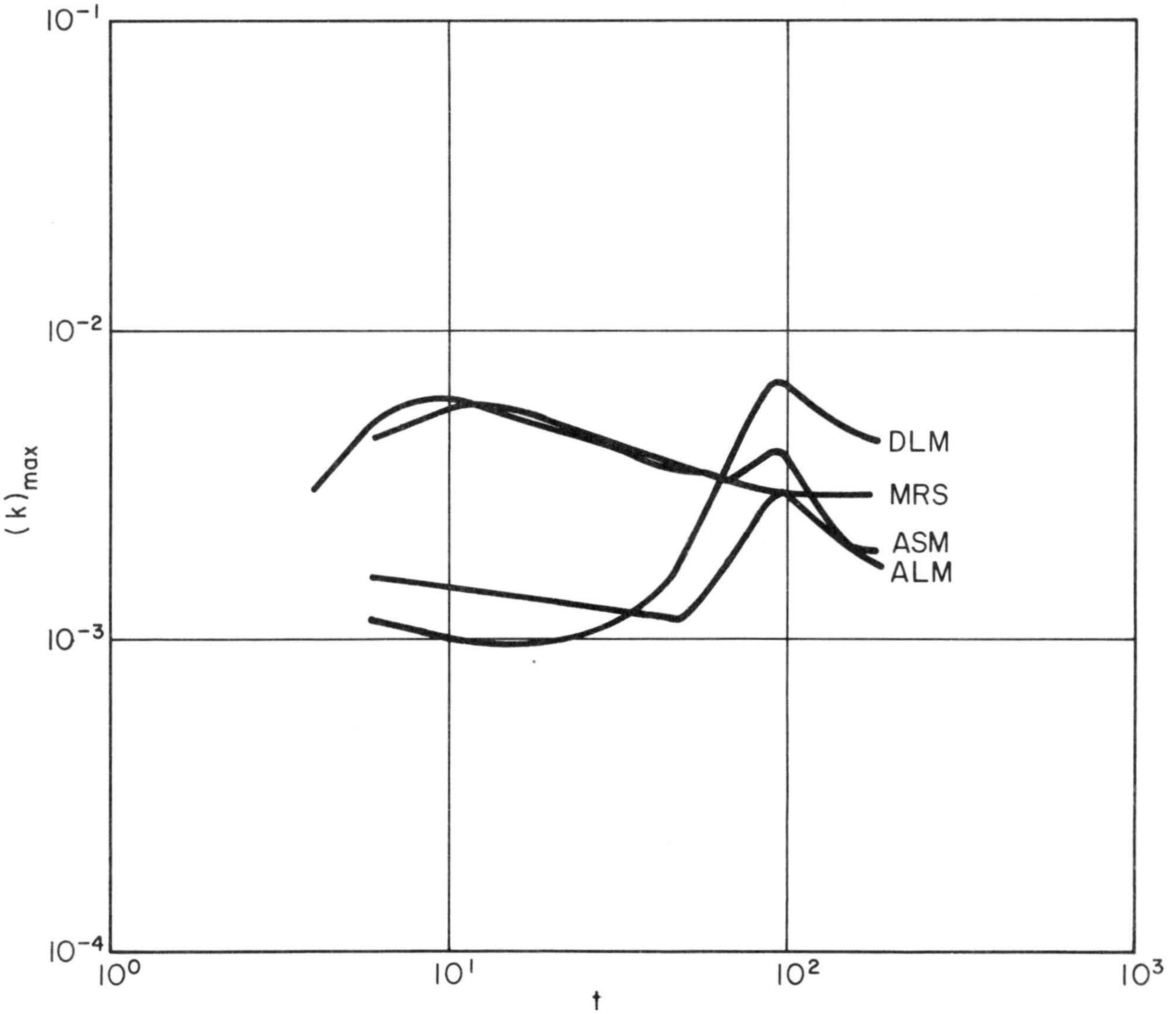

Fig. 9a. Maxima of velocity correlations; Case II.

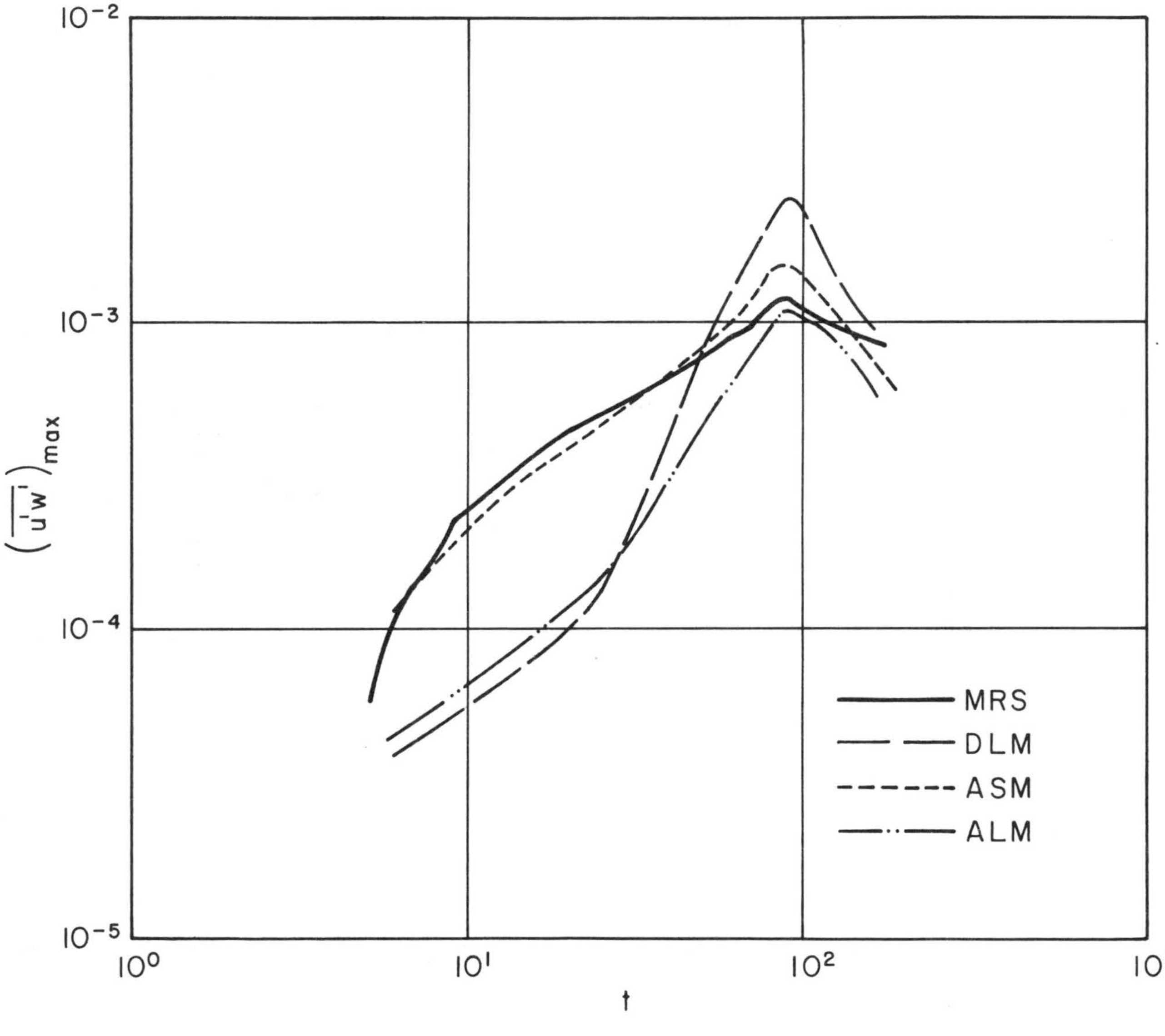

Fig. 9b. Maxima of velocity correlations: Case II.

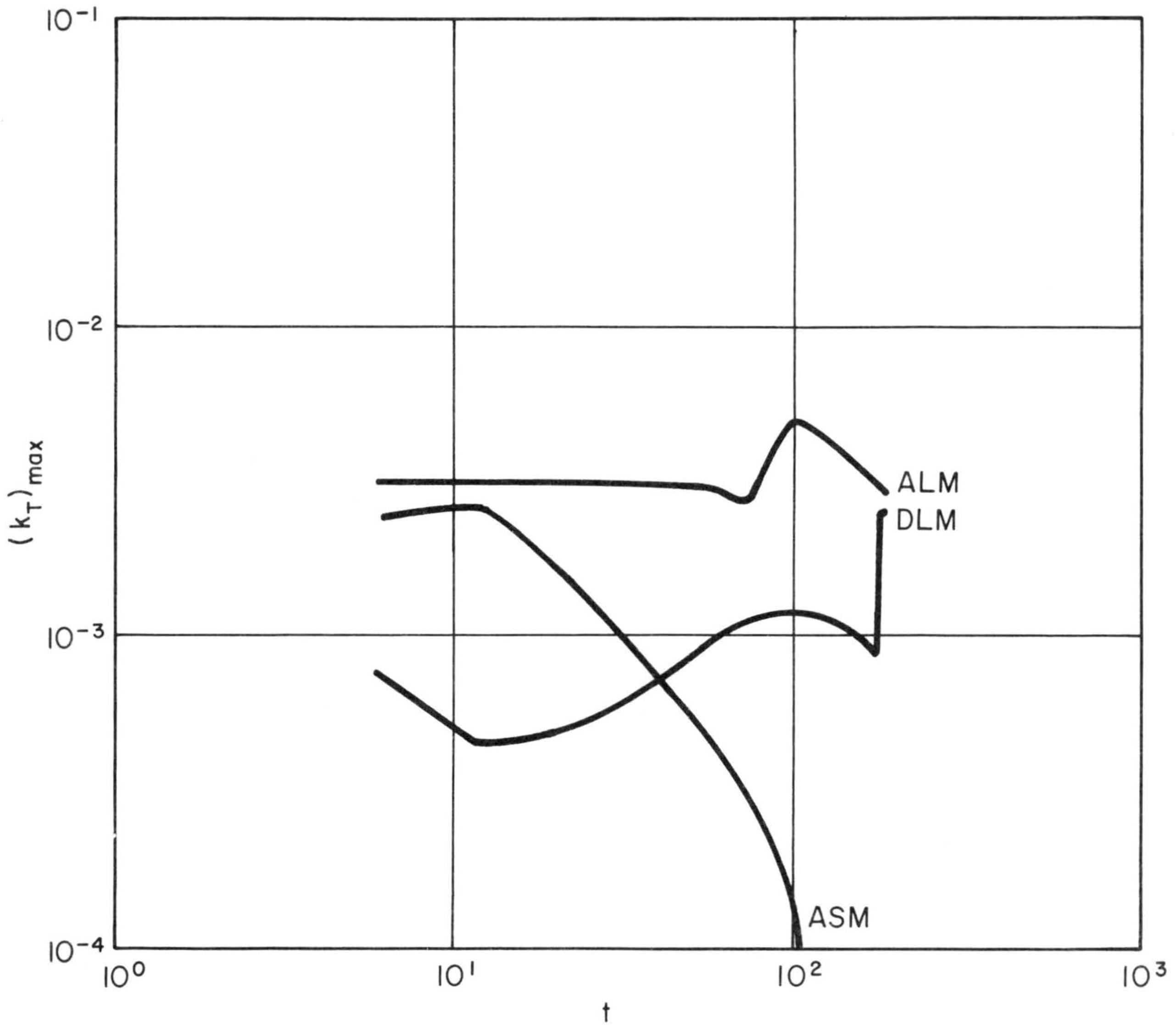

Fig. 10. Maxima of temperature correlations.

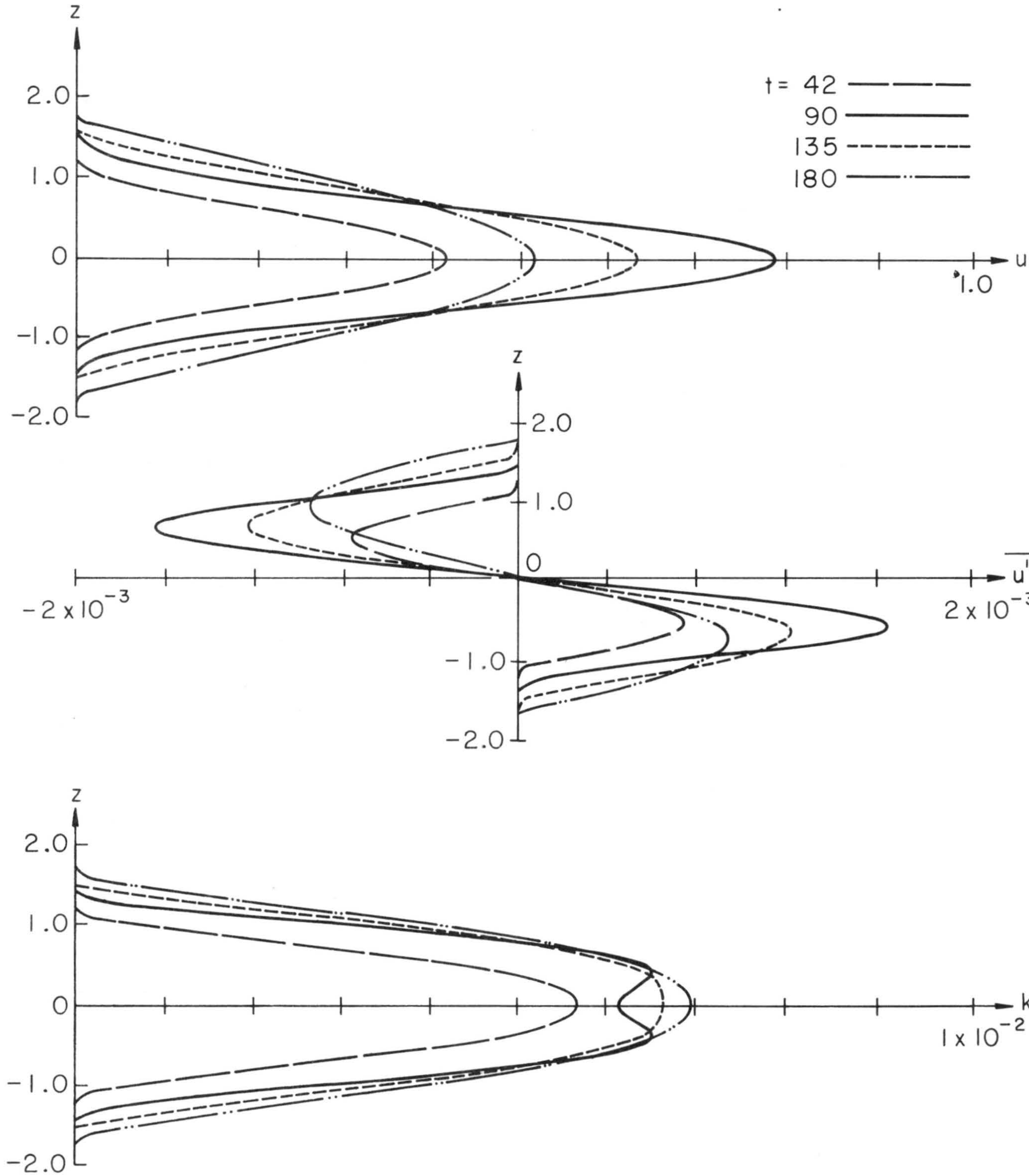

Fig. 11a. Case III: ALM model.

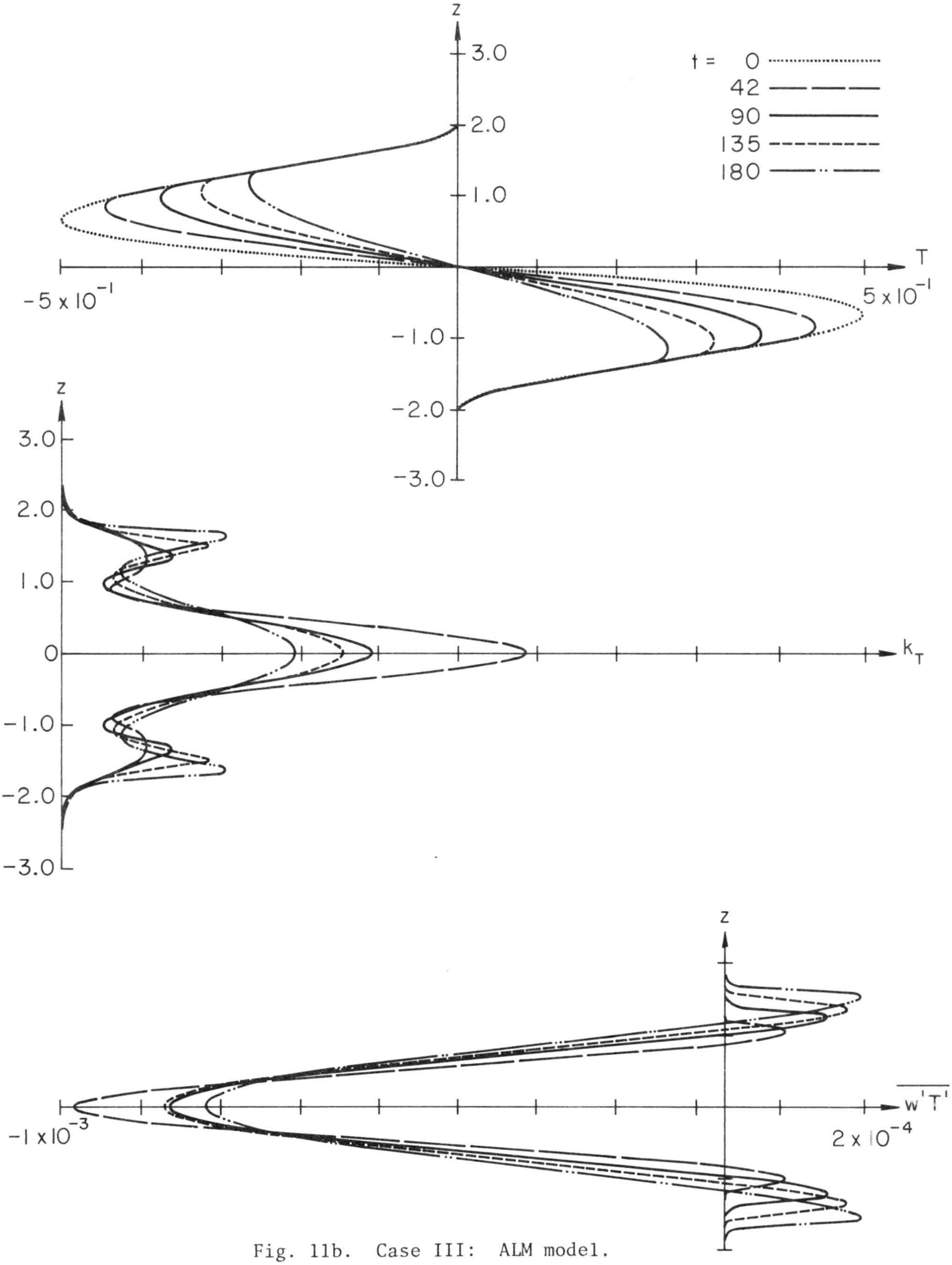

Fig. 11b. Case III: ALM model.

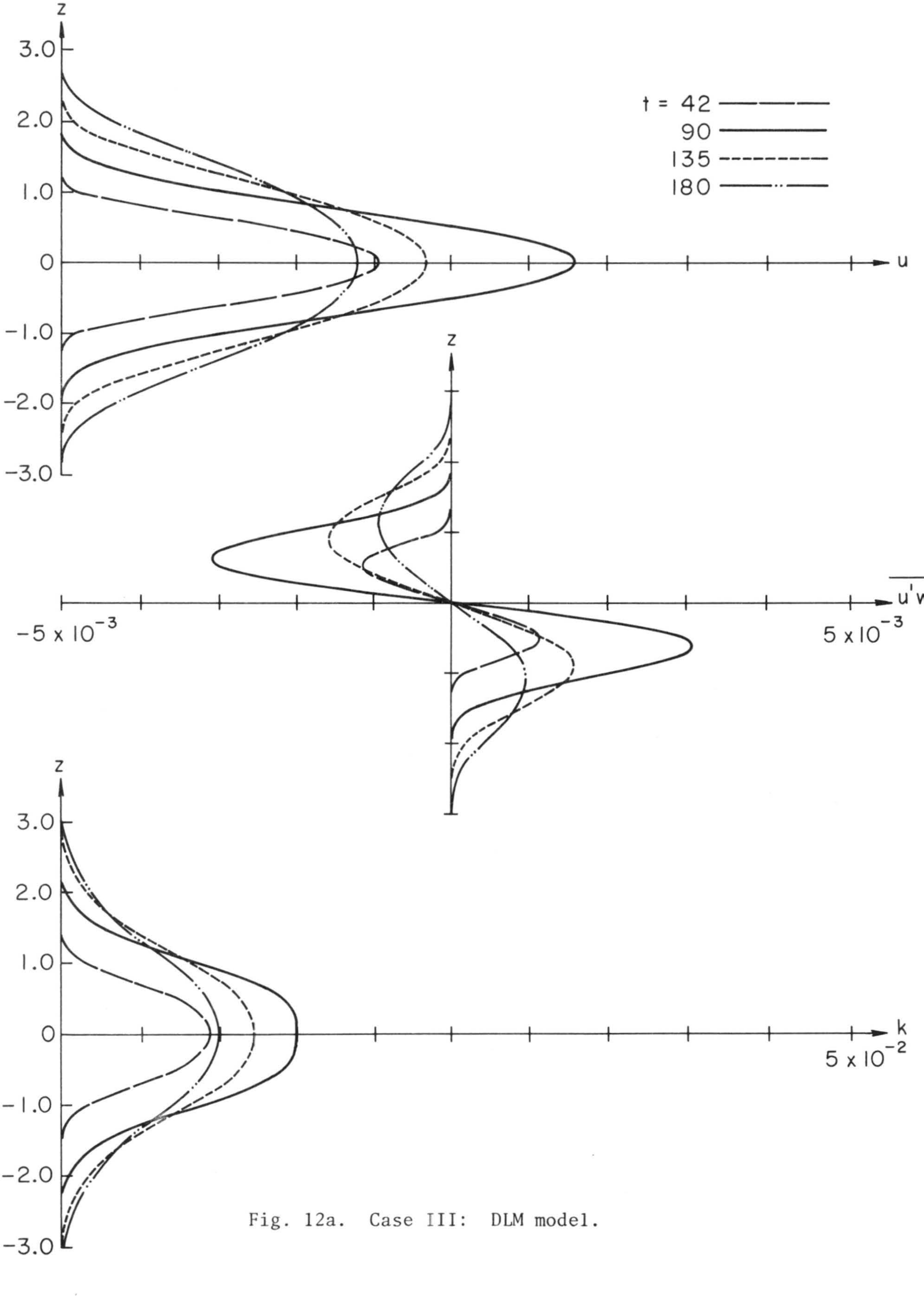

Fig. 12a. Case III: DLM model.

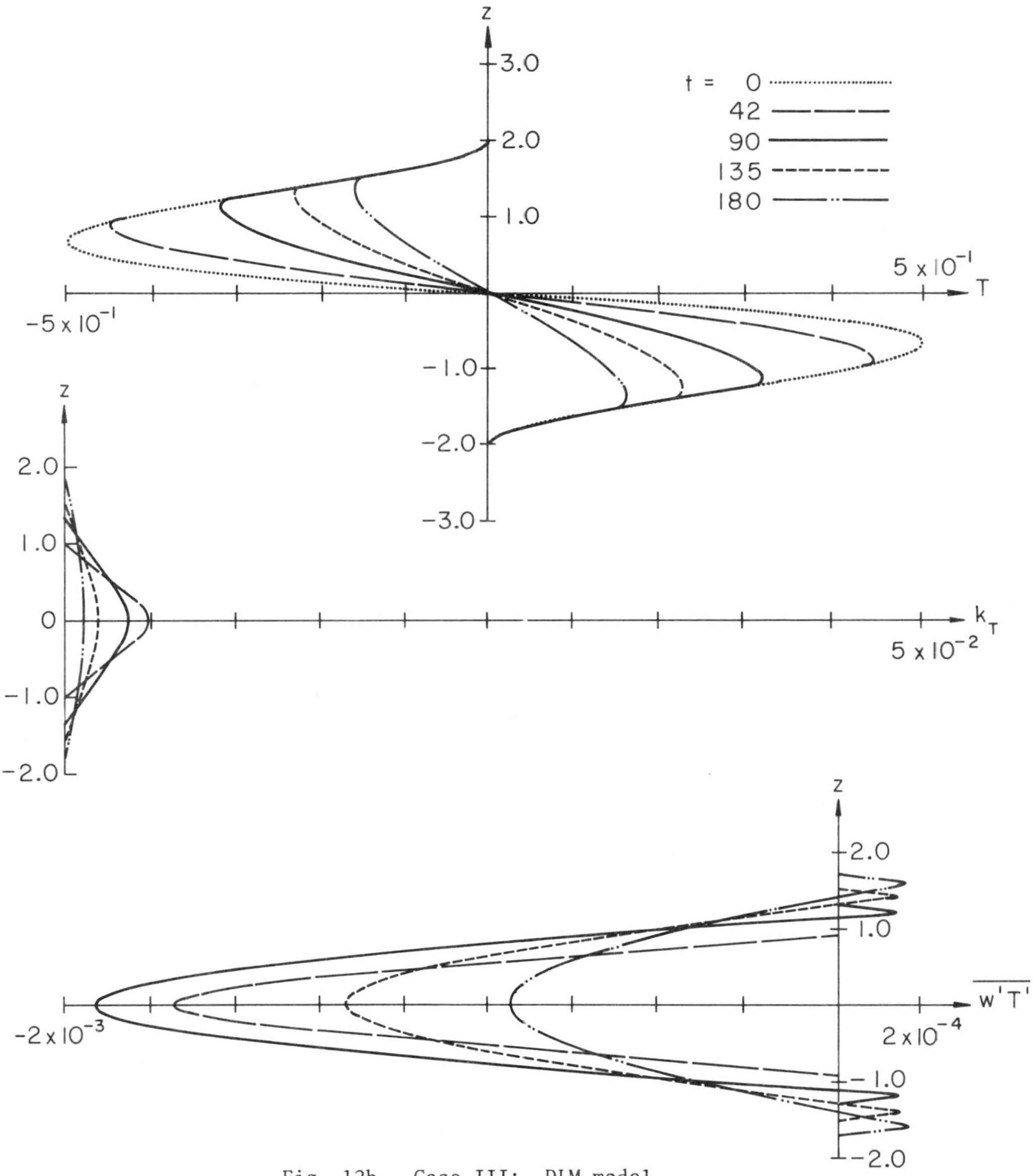

Fig. 12b. Case III: DLM model.

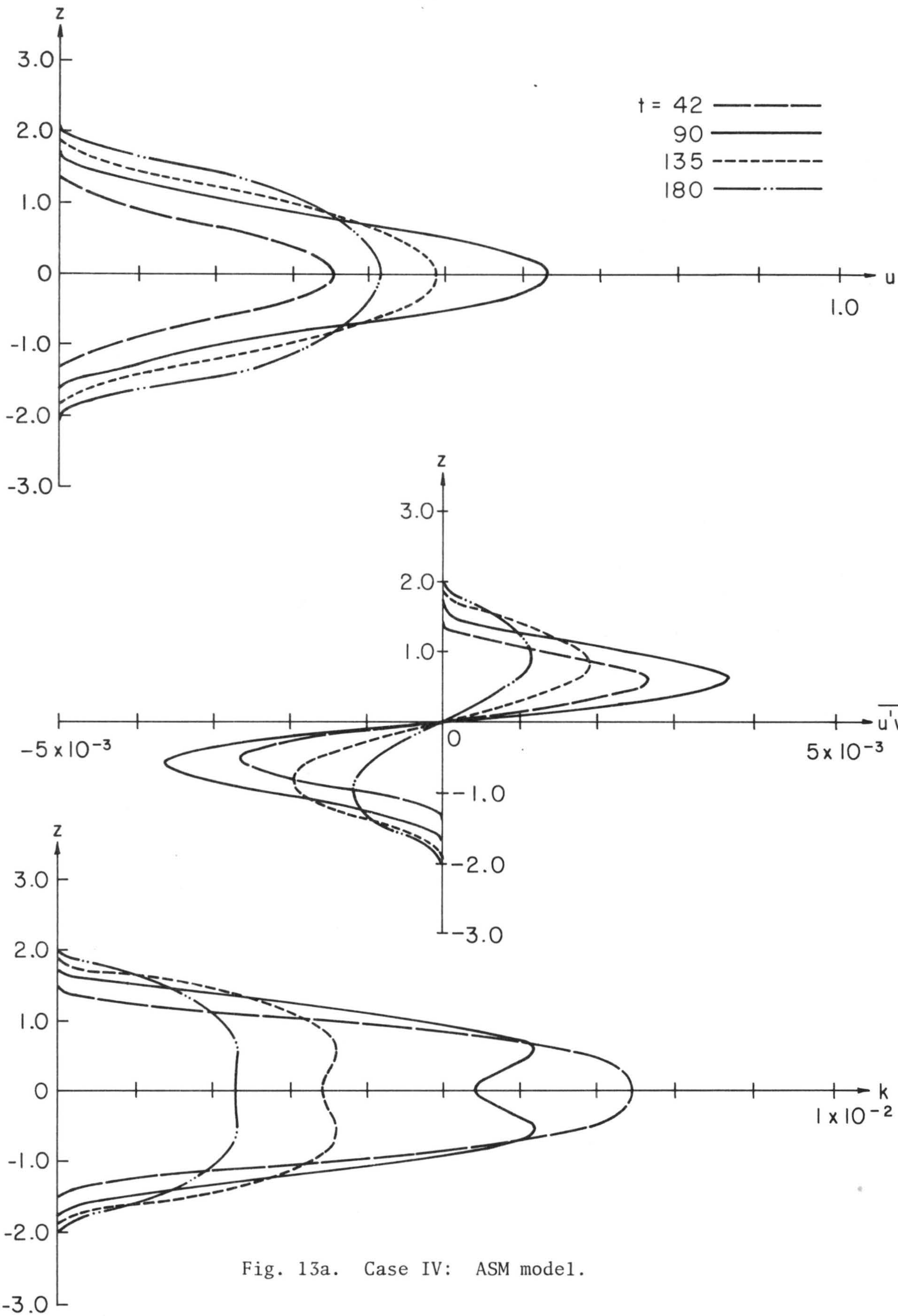

Fig. 13a. Case IV: ASM model.

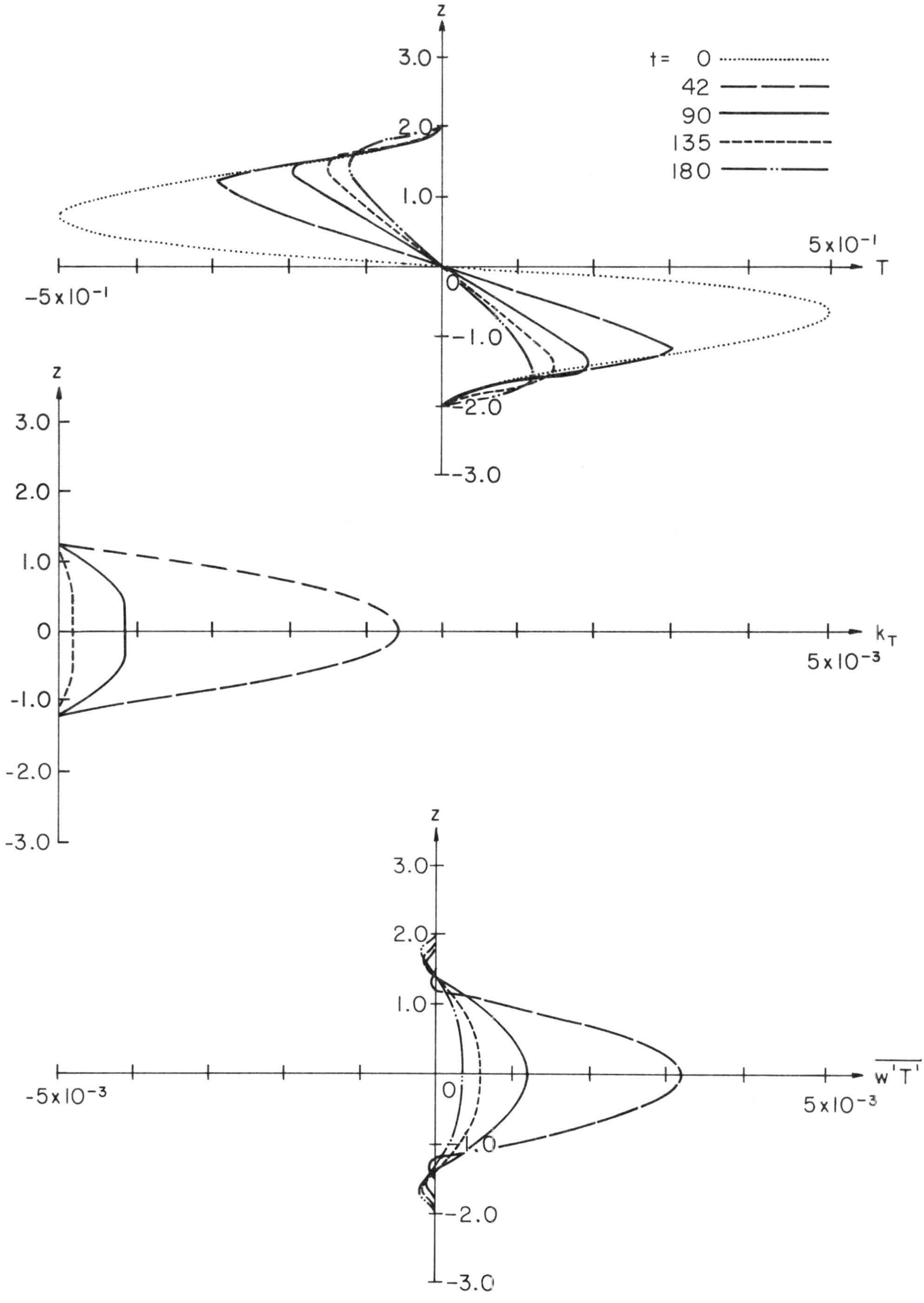

Fig. 13b. Case III: ASM model.

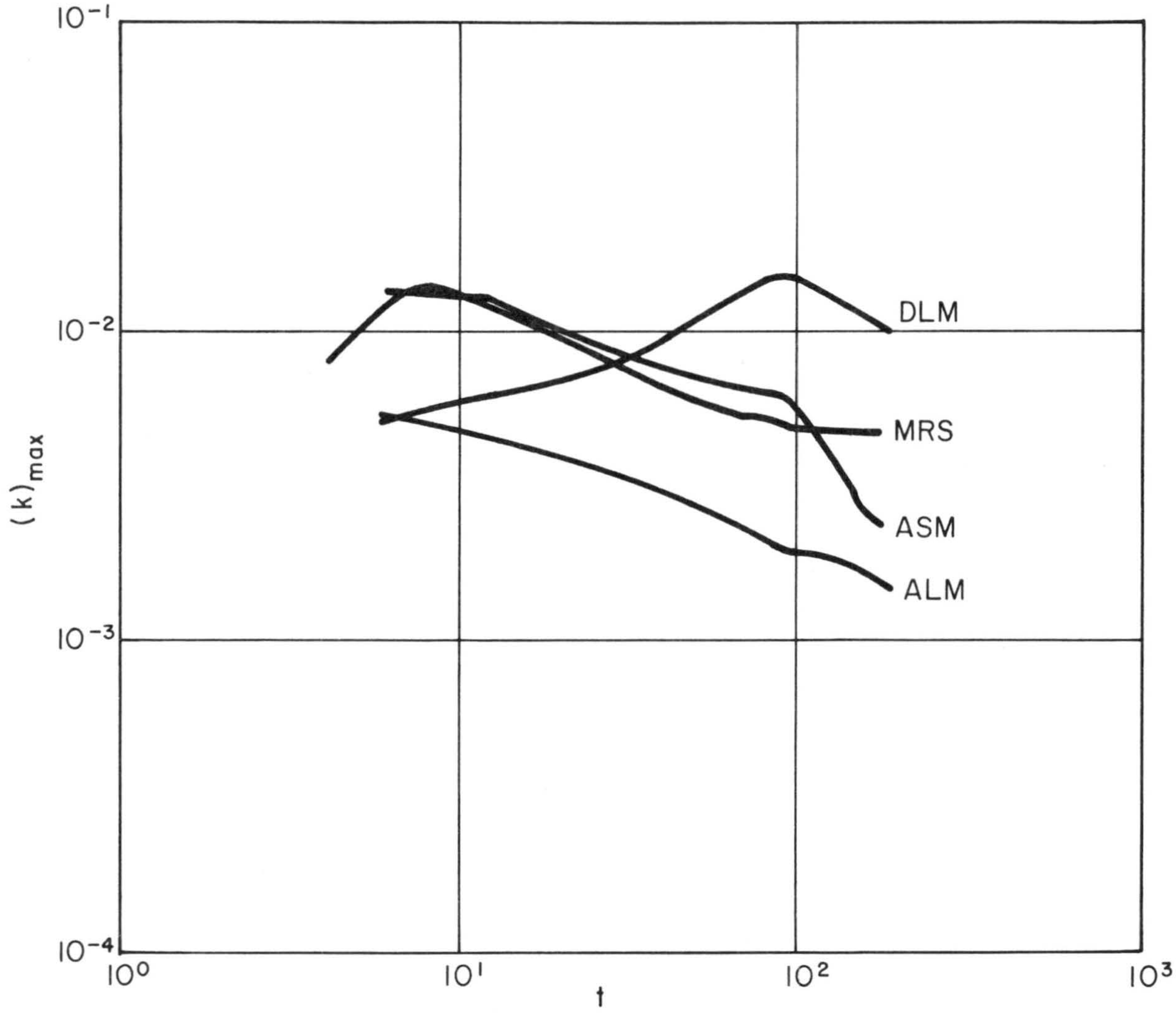

Fig. 14a. Maxima of velocity correlations: Case III.

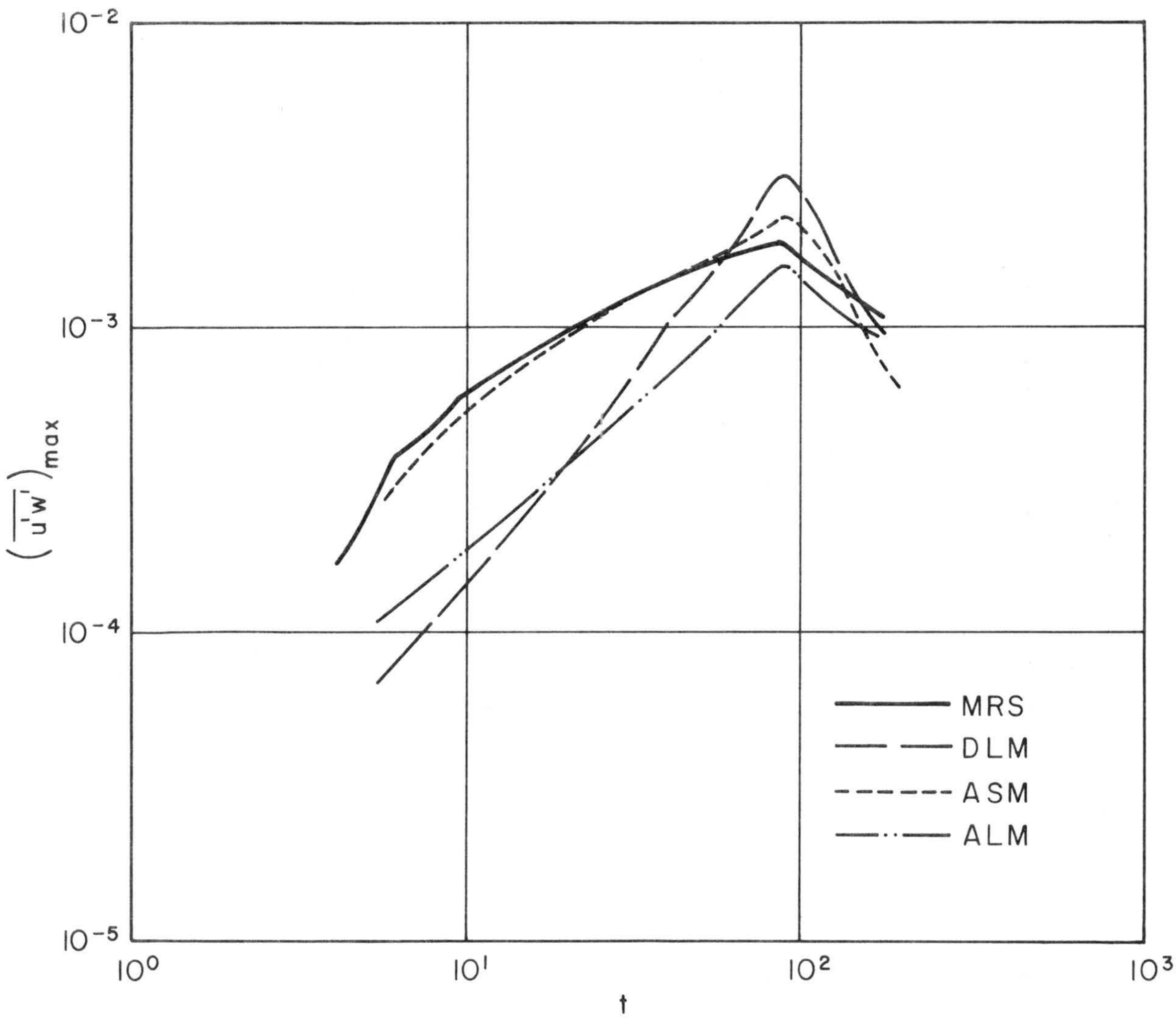

Fig. 14b. Maxima of velocity correlations: Case III.

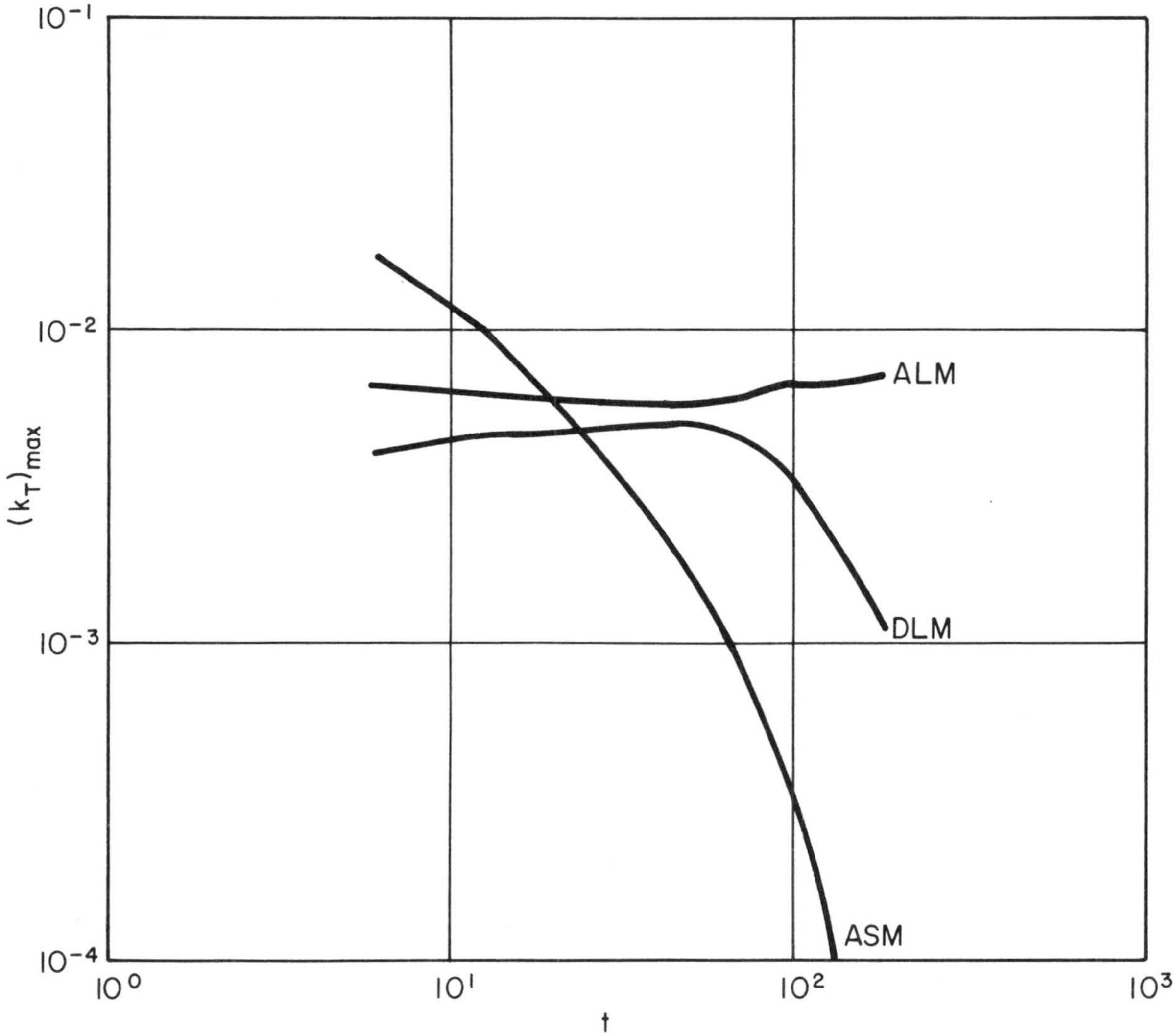

Fig. 15. Maxima of temperature correlations: Case III.

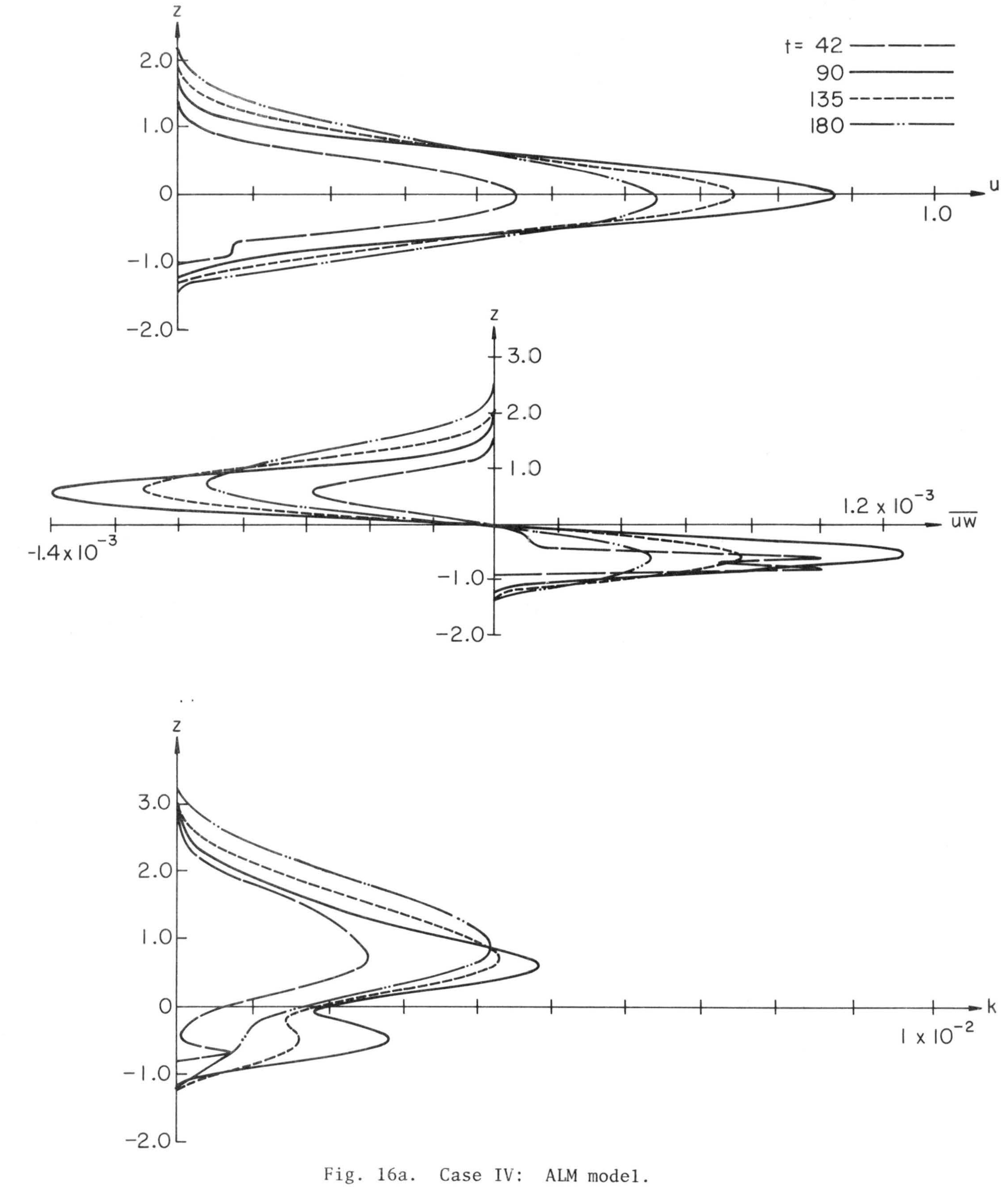

Fig. 16a. Case IV: ALM model.

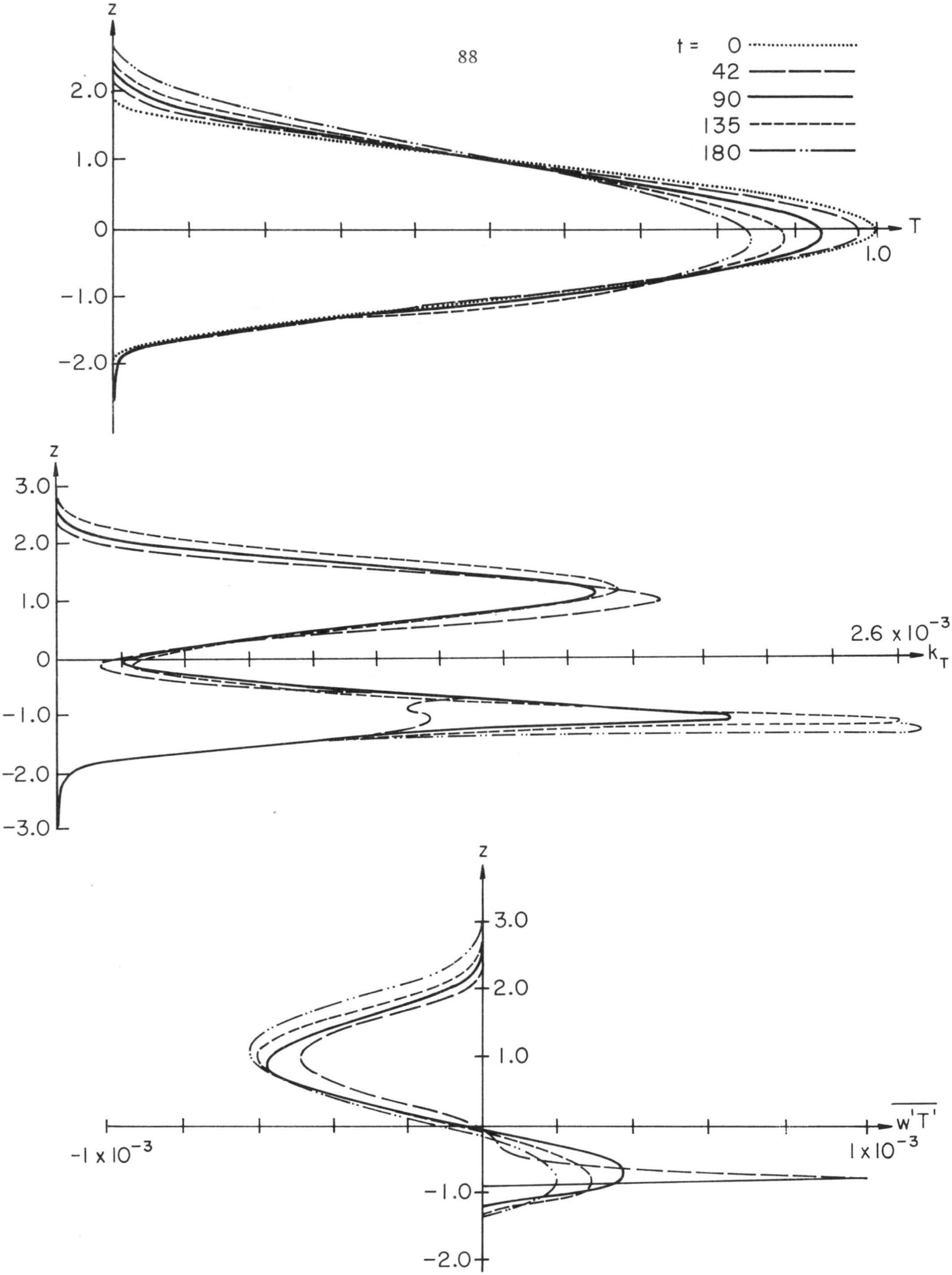

Fig. 16b. Case IV: ALM model.

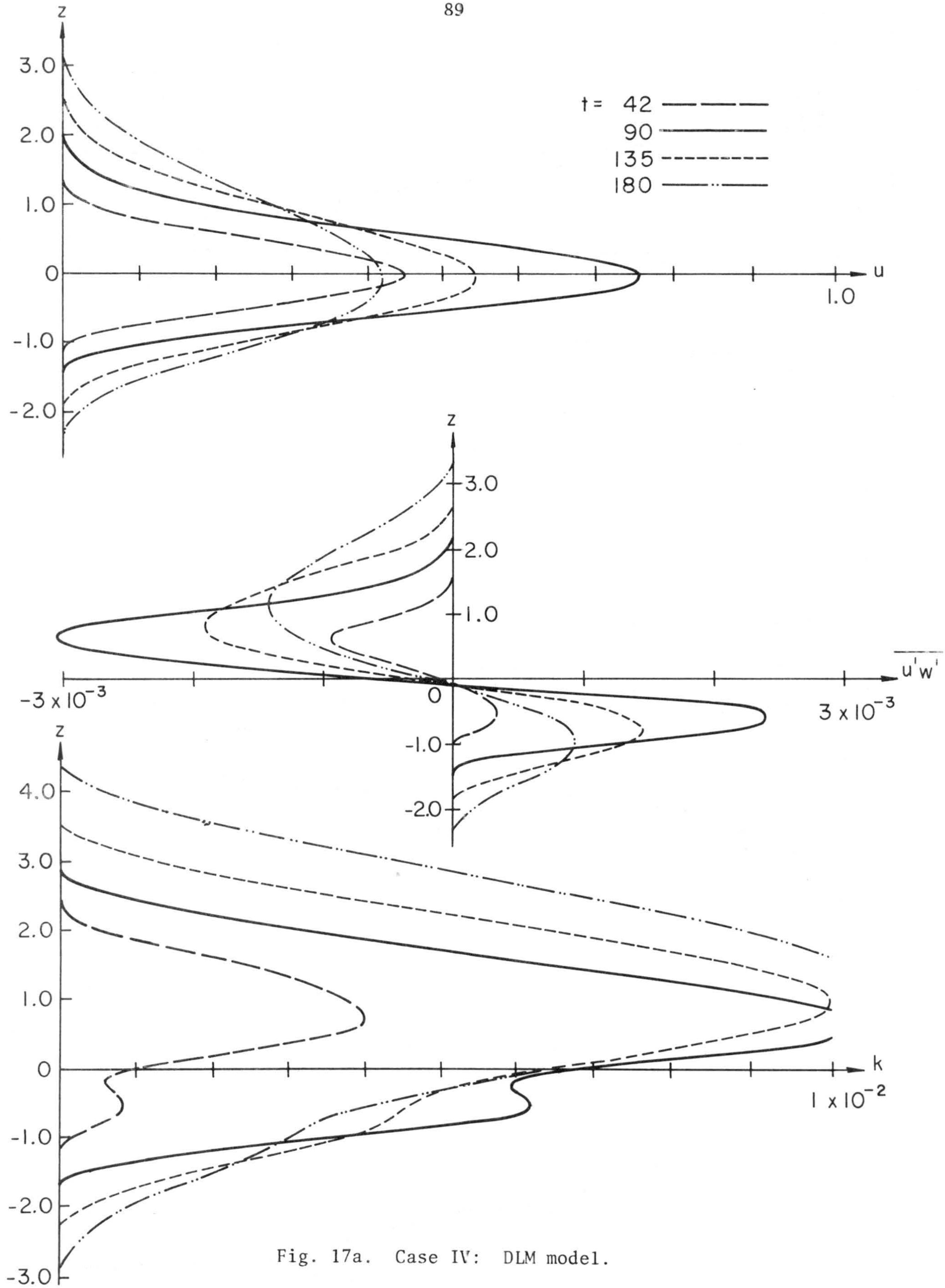

Fig. 17a. Case IV: DLM model.

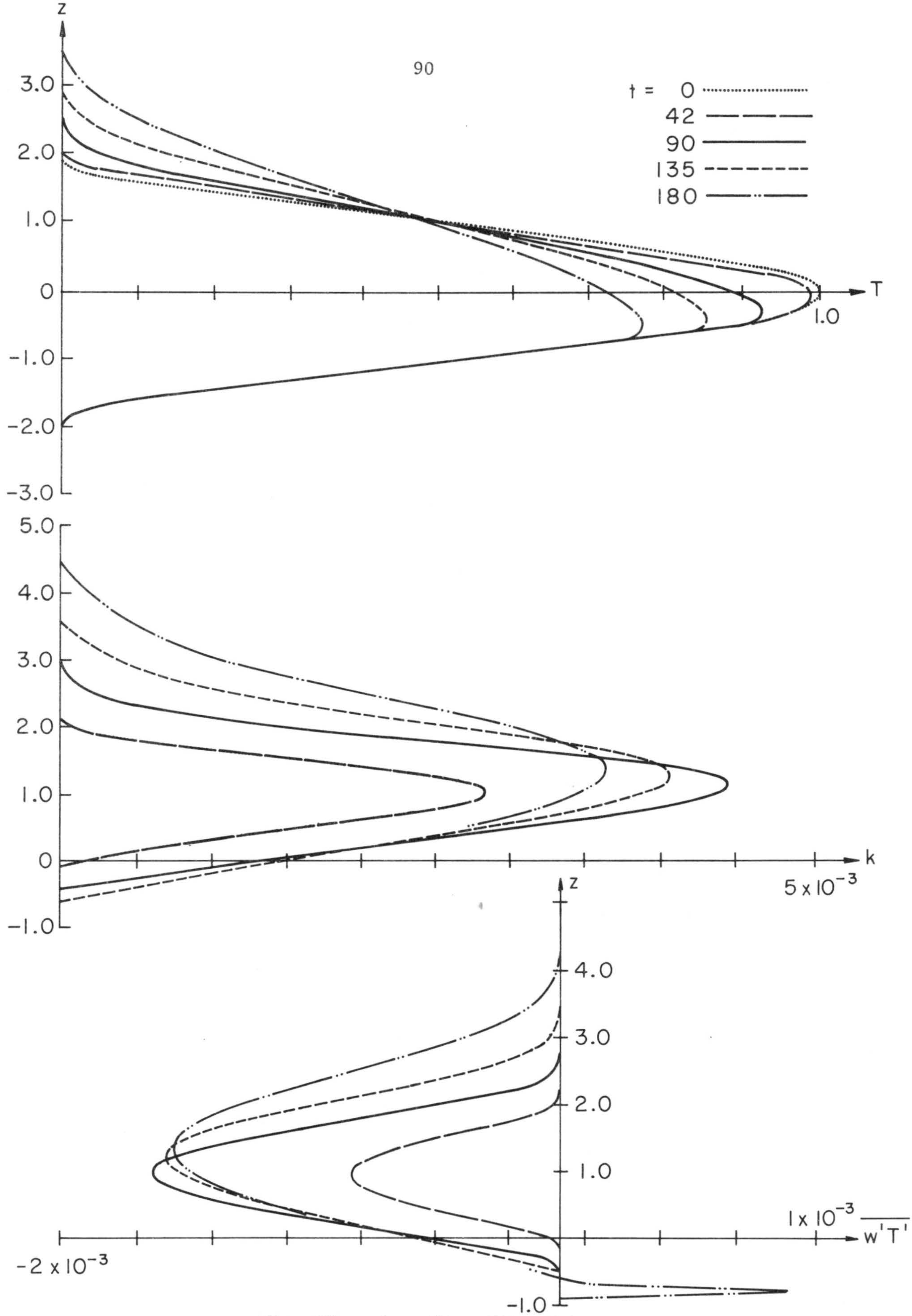

Fig. 17b. Case IV: DLM model.

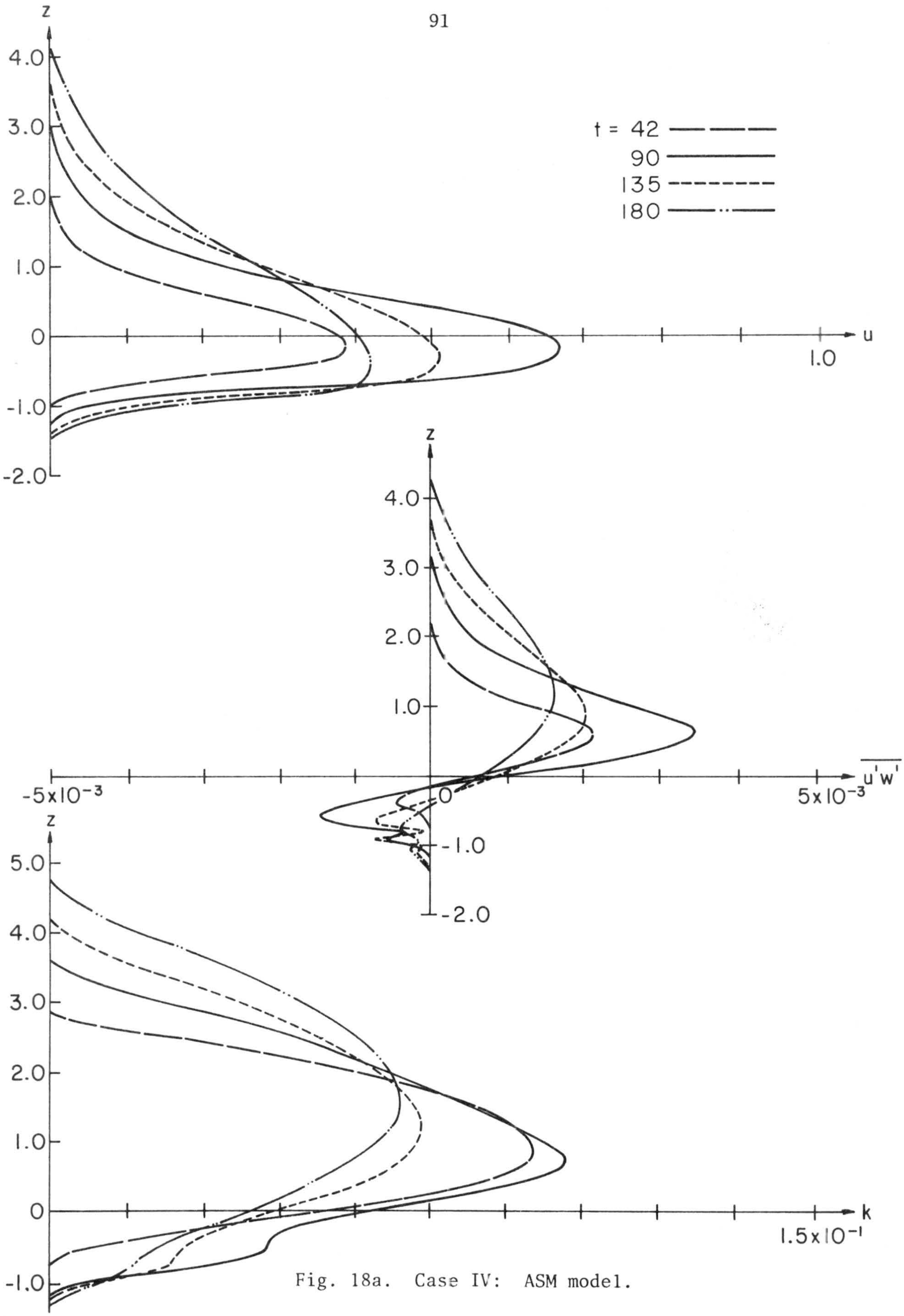

Fig. 18a. Case IV: ASM model.

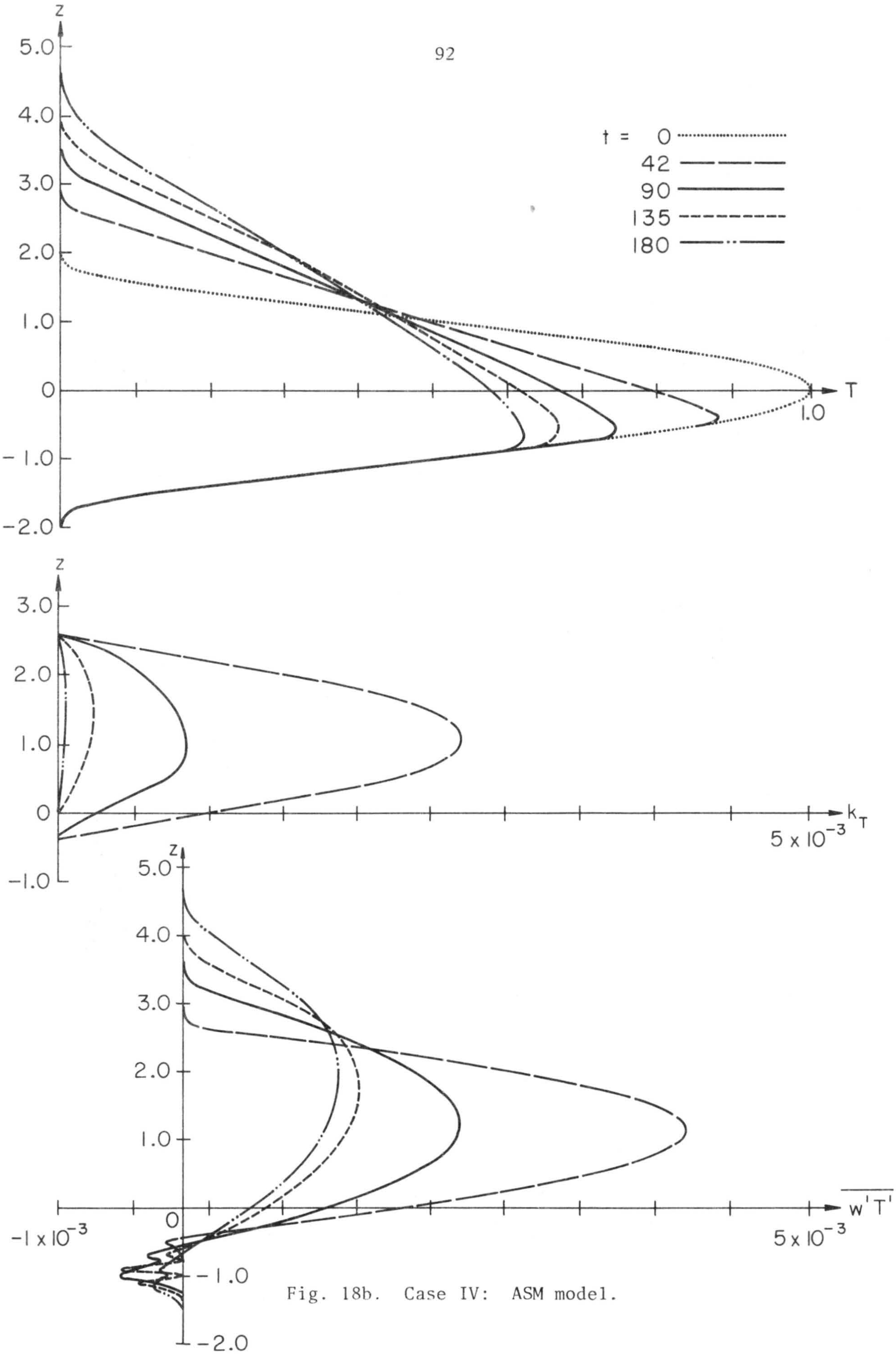

Fig. 18b. Case IV: ASM model.

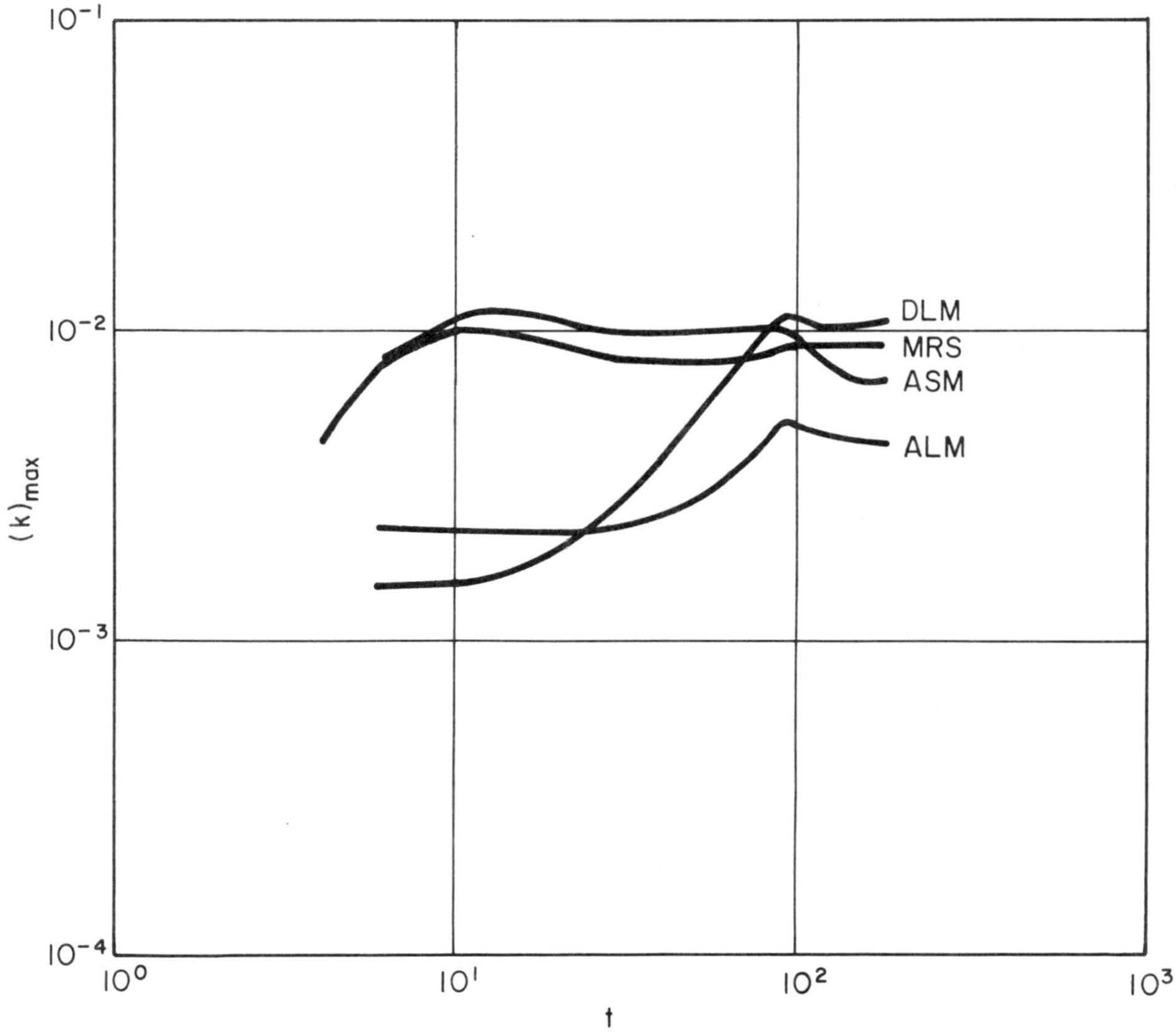

Fig. 19a. Maxima of velocity correlations: Case IV.

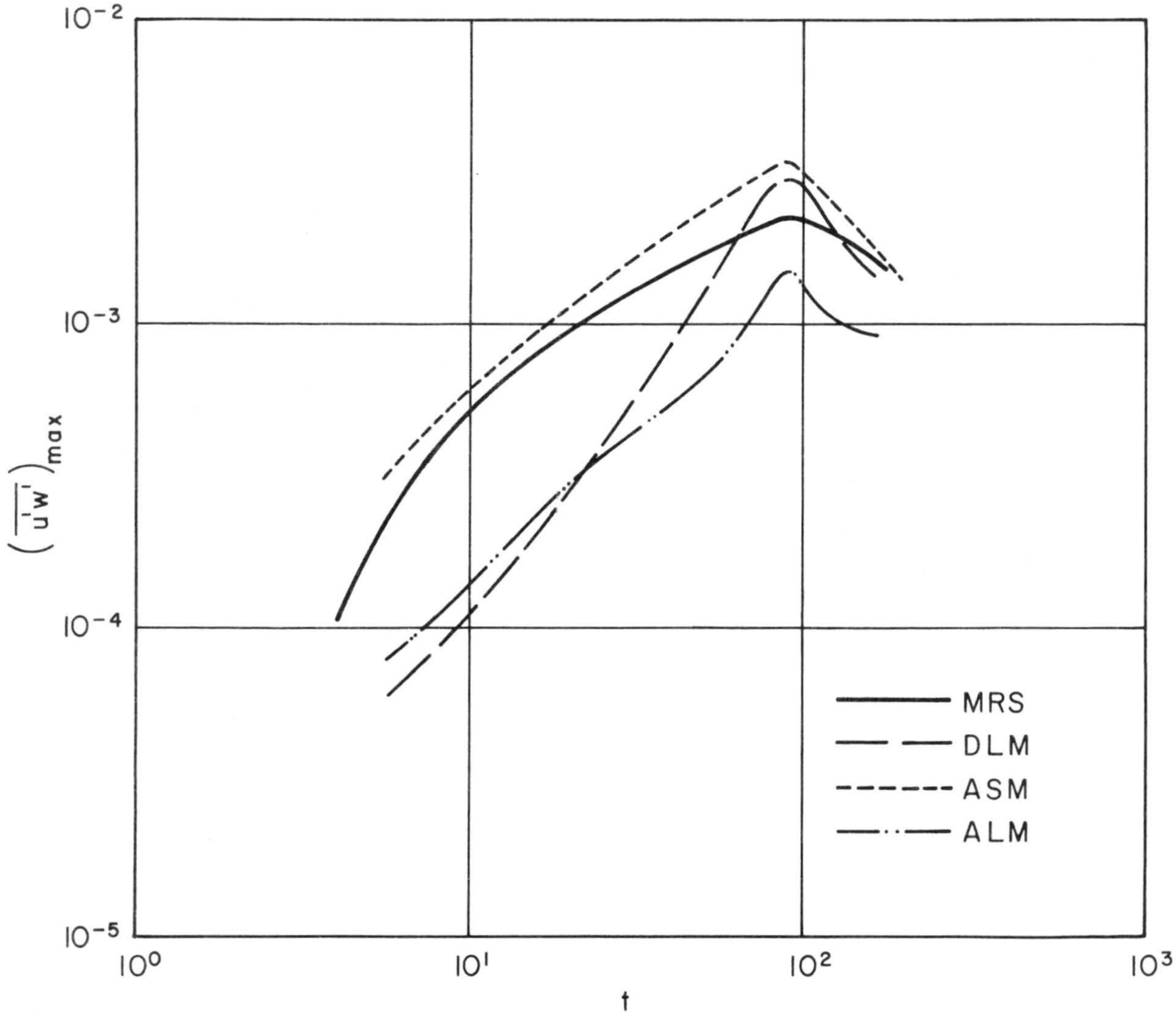

Fig. 19b. Maxima of velocity correlations: Case IV.

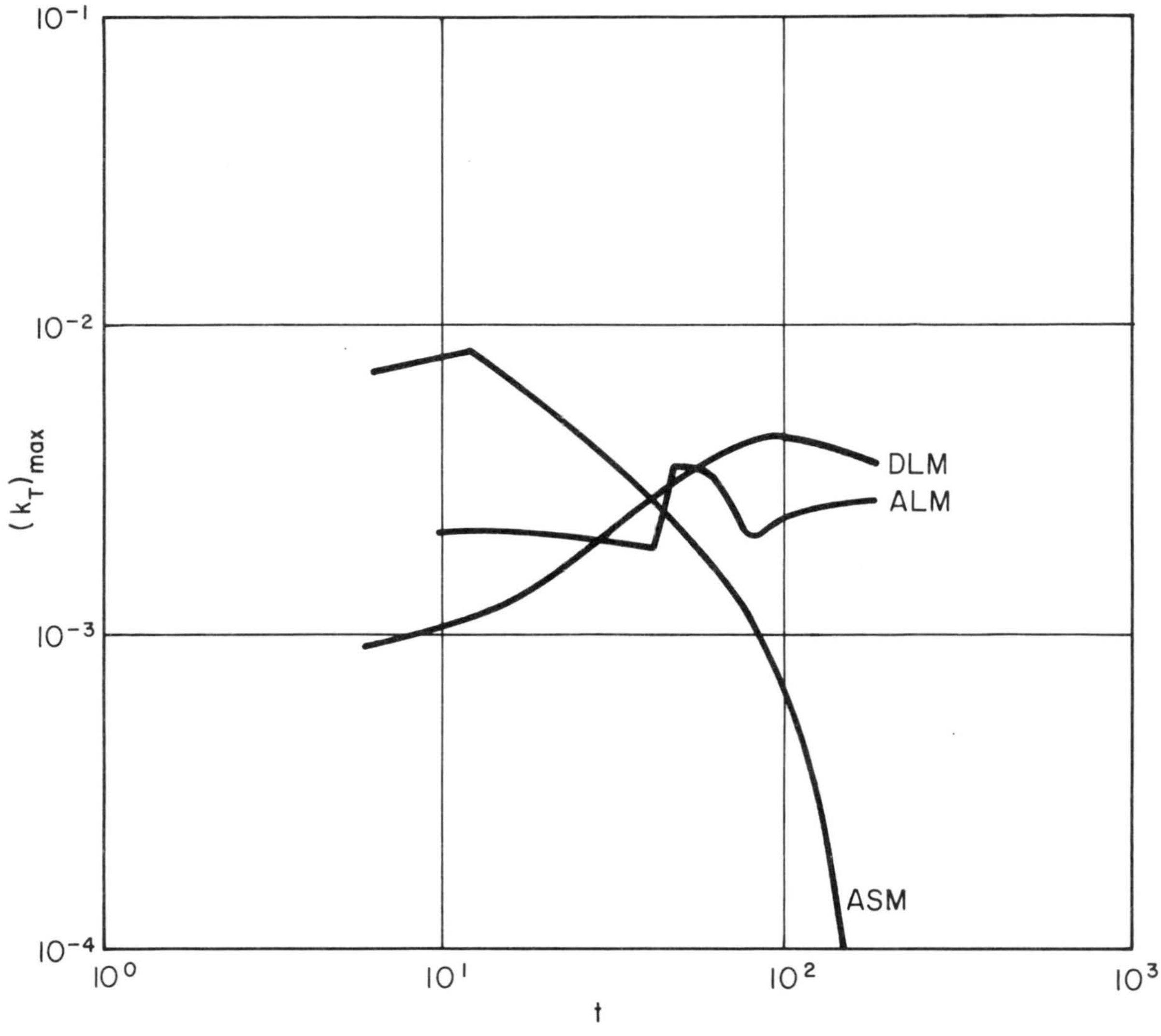

Fig. 20. Maxima of temperature correlations: Case IV.

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Utilizing a simple time dependent one-dimensional example as a test case this report discusses a solution which represents the important characteristics of a buoyancy dominated shear flow by solving four partial differential equations in addition to the mean equations of motion. This suggested model solves equations for total turbulent kinetic energy, k, total turbulent temperature fluctuations, k_t, eddy dissipation, ε, and thermal eddy dissipation, ε_t. Three separate versions of this model are discussed--an algebraic length scale version, a Prandtl-Kolmogorov eddy viscosity version, and an algebraic stress and heat flux model. The final version (requiring six partial differential equations) manages to replicate results for a much more complicated version (requiring ten partial differential equation). The advantages for two and three dimensional problems are even greater.

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