DISSERTATION

IDENTIFICATION OF PHYSICAL PROPERTIES IN GEOLOGY , HYDROLOGY AND ECOLOGY

Submitted by

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In partial fulfillment of the requirements
for the Degree of Doctor of Philosophy
Colorado State University
Fort Collins,Colorado
Fall 2001

UMI Number: 3038618

UMI Microform 3038618

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

COLORADO STATE UNIVERSITY

August 23, 2001

WE HEREBY RECOMMEND THAT THE DISSERTATION PREPARED UNDER OUR SUPERVISION BY ABDELHAMID E LMOURSI BADRAN ENTITLED IDENTIFICATION OF PHYSICAL PROPERTIES IN GEOLOGY , HYDROLOGY AND ECOLOGY BE ACCEPTED AS FULFILLING IN PARTIAL REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT OF DISSERTATION

IDENTIFICATION OF PHYSICAL PROPERTIES IN
GEOLOGY, HYDROLOGY, AND ECOLOGY

This thesis presents the details of a new method for approximating the solution to inverse problems for the identification of position dependent coefficients in differential equations from observations on the boundary of the problem domain. The method is based on the classical Backus-Gilbert method, first introduced in geophysics, but employs this idea in a new way to coefficient inverse problems. The method can be used for identifying one or more coefficients in either ordinary or partial differential equations. Inverse problems of this sort are prevalent in diverse areas including geology, hydrology and ecology where physical properties of a system must be determined indirectly from experimental data.

Several examples are provided to illustrate how the method can be applied for identification of such things as electrical, thermal and hydraulic properties.

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ACKNOWLEDGEMENTS

I would like to express my gratitude and appreciation to my advisor , professor Paul DuChateau , for his support,encouragement,guidance, and useful comments throughout the process of developing and completing this work . I also want to thank professors Rick Miranda , James Thomas , and Mohammed Siddiqui for being part of my graduate committee and for their encouragements.

DEDICATION

To my wife, Mrs. Samira Elkhouly.

CONTENTS

Introduction

Classical applied mathematics consist mainly of problems in which a given set of inputs is mapped continuously to a unique output and the problem is solved when the unique output is computed, exactly or approximately, from the inputs. Problems of this type have come to be known as direct or forward problems. Relatively recently a new type of problem has arisen in applied mathematics. These problems seek to determine some of the inputs from observations of the corresponding output and are referred to as inverse problems.

Inverse problems for differential equations is a topic of increasing interest not just to mathematicians but to scientists and engineers as well [26]. This is a result of the fact that the problems are simultaneously mathematically challenging (partly due to their characteristic instability) and of great practical importance (a result of the demands of ever increasing complexity in mathematical models). Inverse problems arise in a a tremendous variety of applications. For example, medical applications include electical impedance tomography for reconstructing images of the interior of the human body from exterior electrical measurements [21]. More traditional engineering applications include using similar techniques for non-destructive detection of interior flaws or inclusions in mechanical equipment. Other inverse problems are generated by the need to determine various internal physical properties such as hydraulic or thermal properties, from external measurements [10]. Problems of this type are particularly prevalent in hydrology, geology and ecology where one is interested in preserving and protecting the quality of groundwater.

Inverse problems have given birth to numerous mathematical techniques for the analysis of problems and the construction of approximate solutions. In this relatively new field it is not yet completely clear which approaches will turn out to be lasting and general and which will be abandoned as limited and special, [11,13,15]. Certainly the method of output least squares is the most generally applicable technique for dealing with inverse problems but its very generality limits its power in certain situations, necessitating the development of less general methods specifically tailored to a given problem. It seems clear that development of new approaches is one of the fruitful areas for additional research in this new field of inverse problems.

This thesis describes an approach to a class of inverse problems based on the use of an adjoint version of the forward problem to express a relationship between changes in the unknown quantity and corresponding changes in the measurement from which the unknown is to be determined. This relationship between input and output can be used in various ways [7] but the discussion here focuses on applying the relationship to construct an approximation to the solution to the inverse problem. The nature of this approximation approach makes it particularly suited to application to the electrical tomography problem or to the identification of spatially variable physical properties (particularly the identification of hydraulic properties).

The method is based on an idea that goes back to Backus and Gilbert, [4] who applied it in connection with problems arising in Geophysics. Maas and Louis [19] provided a mathematical context for the Backus-Gilbert idea but the application described here exploits the approach in a way that is different from either of these earlier works and is also different from other current approaches to inverse problems in this class. A particularly attractive feature of this approach is the fact that in the identification of spatially variable physical properties, refining the spatial accuracy of the reconstruction can be carried out in a simple and efficient manner.

The organization of this thesis is as follows. Chapter I is devoted to a discussion of the abstract formulation of inverse problems and the use of the abstract Green's formula to obtain integral identities relating changes in inputs to changes in outputs. Examples are given to illustrate how the adjoint problem is formed so as to select a specific boundary measurement and relate changes in this measurement to corresponding changes in the unknown ingredient that is to be identified.

Chapter II presents an elementary introduction to the B-G method, applying it in the very simple case of identification of coefficients in ordinary differential equations. This application is almost trivially simple but illustrates how the method will work in the more complicated case in which unknown coefficients in partial differential equations are to be identified from measurements.

In chapter III, the method is applied to the identification of a single unknown ingredient in a partial differential equation. The method is illustrated through examples based on the example inverse problems from chapter I.

Chapter IV discusses the extension of the method to a problem involving multiple unknowns. Although the basic idea of the approximation method does not change when it is applied to muliple unknown problems, the generalization of some of the particulars of the method is not routine.

The final chapter contains conclusions about the method and some suggestions for further research.

CHAPTER I

The Abstract Structure of Coefficient Inverse Problems

It is well known that exploitation of the abstract Green's formula provides the basic structure for the variational formulation of a large class of boundary value and initial boundary value problems in partial differential equations. This same structure lies at the root of a class of inverse problems in which unknown coefficients or other equation ingredients are to be identified from overspecified data. In the inverse problems, the abstract Green's formula leads to an integral identity relating changes in the unknown ingredient, referred to here as a structure input, to changes in the overspecification. This provides a uniform approach to analyzing such problems and suggests an adaptive algorithm for computing approximate solutions to the inverse problem.

In implementing this approach, the direct problem must be formulated in a sense that is compatible with the class of functions to which the unknown ingredient is assumed to belong. This determines the character of the solution to the direct problem which has considerable effect on how the inverse problem is to be formulated. In particular, it defines the sense in which the overspecified conditions are to be interpreted and bears on the question of identifiability of the unknown ingredient. These issues are considered in this chapter.

1. The Direct Problem

Let Ω denote a bounded open set in $\mathbf{R}^n$ having Lipschitz smooth boundary Γ composed of complementary components, Γ_1 and Γ_2. For $T > 0$, let $Q_T = \Omega \times (0,T)$ and, for i=1,2, let $S_i = \Gamma_i \times (0,T)$. Consider the following initial-boundary value problem (IBVP)

$$\begin{aligned} \partial_t C + A[x,u,D] &= F && in\ Q_T \\ u(0) &= u_o && in\ \Omega \\ \beta_1[u,D] &= f_1 && on\ S_1 \\ \beta_2[u,D] &= 0 && on\ S_2 \end{aligned} \qquad (1.1)$$

Here, u denotes the unknown dependent variable, $u = u(x,t)$, which presumably represents a state variable in some physical process for which (1.1) is the mathematical model. Then C,D, and F denote so-called "structure inputs" to the problem. These inputs represent coefficients or source terms in the equation and are related to the physical properties of the system modelled by (1.1). These coefficients may be functions of the independent variables or in some cases, functions of the dependent variables. In any case we assume that each of the functions is Lipschitz continuous on its domain.

We use the notation, $A[x,u,D]$, to denote an elliptic partial differential operator of order two. This operator may be either linear or nonlinear but in either case, we assume that we have a generalized Green's identity of the form

$$\iint_{Q_T} A[x,u,D]v\,dxdt = a_D[u,v] - \int_0^T\int_\Gamma \partial[u,D]\bullet\gamma[v]\,dS\,dt \quad for \quad u,v \in C^\infty(\bar{Q}_T)$$

where

$a_D[u,v] =$ a semi-linear or bilinear form, linear in v but not necessarily linear in u

$\partial[u,D] : H^1(\Omega) \to H^{-1/2}(\Gamma)$ and $\gamma[v] : H^1(\Omega) \to H^{1/2}(\Gamma)$ denote trace operators

The boundary conditions in the problem have been denoted by $\beta_k[u,D]$ *on* S_k $k = 1,2$, where β_k may denote either of the trace operators ∂ *or* γ composed with restriction to S_k. Usually we refer to S_1 as the "active or input boundary" to reflect the fact that we will use the inhomogeneous boundary condition to drive the process of coefficient identification. The boundary S_2 will be referred to as the "passive boundary" since we impose only a homogeneous condition on this part of the boundary.

For convenience, we will denote the structure inputs collectively by ϕ. Then for given structure inputs ϕ and state inputs f_1, u_0, the function $u = u(x,t)$ in solution space Z is a *weak solution* for the IBVP (1.1) if

$$\begin{aligned} < P[u,\phi], v > &= 0 \quad for\ all\ v\ in\ Z \\ (u(0) - u_0, v_0) &= 0 \quad for\ all\ v_0\ in\ Z_0 \\ < \beta_1[u,\phi] - f_1, w > &= 0 \quad for\ all\ w\ in\ X_1 \\ \beta_2[u,\phi] &= 0 \quad in\ X_2 \end{aligned} \qquad (1.2)$$

We distinguish here between the state inputs and the structure inputs. The state inputs are given functions which the state variable must equal initially or on the boundary of the domain, Ω, while the structure inputs characterize the structure of the physical system and are normally given as part of the input to an initial-bounary value problem. In the discussion to follow we will describe the problems that ensue when the structure inputs are not given and are, in fact, to be determined by formulating and solving an inverse problem.
We refer to (1.2) as the *direct problem* and then state conditions on the structure inputs, ϕ, (e.g., C, D, and F) and on the state inputs,(e.g.,f_1 and u_o), sufficient to imply that a unique weak solution for the IBVP, $u = u(x,t)$ exists for all such inputs. Of course statements of such conditions can only be made for specific examples. Having stated an existence theorem it may then be necessary to prove auxiliary propositions asserting that the solution has certain properties that will be needed in connection with the inverse problem. We will now describe several important special cases of the general direct problem.

Examples

1.1) Nonlinear Diffusion

Suppose $\Omega \subset R^n$ is open and bounded with smooth boundary Γ.
Suppose also,

$$C = C(u) = u,\ F = 0,\ and$$
$$D = D(u) \in Lip(\mathbf{R})\ such\ that\ d_1 \geq D(u) \geq d_0 > 0,\ \forall u$$

and $\qquad A[u,D(u)] = -div[D(u)\nabla u].$

Also, suppose
$$\beta_1[u,D(u)] = u(x,t) = f_1 \qquad on\ S_1$$
$$\beta_2[u,D(u)] = n \bullet \nabla u(x,t) = 0 \qquad on\ S_2$$

Then the direct problem becomes
$$\begin{aligned} \partial_t u(x,t) - div[D(u)\nabla u(x,t)] &= 0 && in\ Q_T \\ u(x,0) &= u_0(x) && in\ \Omega \\ u(x,t) &= f_1(x,t) && on\ S_1 \\ n \bullet \nabla u(x,t) &= 0 && on\ S_2 \end{aligned} \qquad (1.1)_1$$

Let V denote the completion of $\{\varphi \in C^\infty(\bar{\Omega}) : \varphi = 0\ on\ S_1\}$ in the norm of $H^1(\Omega)$. Then $H_0^1(\Omega) \subset V \subset H^1(\Omega)$. For $f_1 \in C^1[0,T : H^{1/2}(\Gamma_1)]$ let $\tilde{f} \in C^1[0,T : H^1(\Omega)]$ be such that the trace on S_1 of $\tilde{f}$ is f_1. We assume that Γ is smooth enough to support this extension. Then for $D = D(u)$ satisfying the condition stated above, a weak solution of the initial boundary value problem in this example is a function, $u = u(x,t)$ such that $u - \tilde{f} \in L^2[0,T : V]$ and
$$((\partial_t u, v)) + a_D[u,v] = \iint_{Q_T} [v\,\partial_t u + D(u)\nabla u \bullet \nabla v]dx\,dt = 0 \quad \forall\ v \in L^2[0,T : V]$$
$$(u(0) - u_0, v_0)_H = 0\ for\ all\ v_0\ in\ H = H^0(\Omega)$$
Then it can be shown by well known techniques that for each D satisfying the assumptions listed for this example, and for all admissible data f_1 and u_0 there is a unique weak solution having the following additional properties

$$u - \tilde{f} \in L^2[0,T : V] \cap C[0,T : H].\ \ and\ \ \partial_t(u - \tilde{f}) \in L^2[0,T : V^*]$$

In the case that $D = D(x)$ then the problem is linear and the existence-uniqueness results, although simpler to prove, are quite similar.

1.2) A Reaction Diffusion Equation

Suppose
$$C = C(u) = u,\ D = \text{constant},\ and$$
$$F = F(u) \in Lip(\mathbf{R})$$
and $\qquad A[u,D] = -D\nabla^2 u.$

Also, suppose
$$\beta_1[u,D] = u(x,t) = f_1 \qquad on\ S_1$$
$$\beta_2[u,D] = n \bullet \nabla u(x,t) = 0 \qquad on\ S_2$$

Then the direct problem becomes
$$\begin{aligned} \partial_t u(x,t) - D\nabla^2 u(x,t) &= F(u(x,t)) && in\ Q_T \\ u(x,0) &= u_0(x) && in\ \Omega \\ u(x,t) &= f_1(x,t) && on\ S_1 \\ n \bullet \nabla u(x,t) &= 0 && on\ S_2 \end{aligned} \qquad (1.1)_2$$

Let V denote the subspace of $H^1(\Omega)$ defined in the previous example and let $\{w_j\}$ denote the family of eigenfunctions for $Au = -D\nabla^2 u$, acting from $H^2(\Omega) \cap V$ into H, normalized in the $H-$ *inner product*. If $Aw_j = \lambda_j w_j$ for all j, then for $s \geq 0$, define

$$H_s = \left\{u \in H : \ \left\{\lambda_j^{s/2} u_j\right\} \in \ell_2 \ where\ u_j = (u, w_j)_H\right\}.$$

Then H_s $s \geq 0$ is a scale of Hilbert spaces with $H_0 = H\ and\ H_1 = H^2(\Omega) \cap V$ and it is well known that $H_s \subset C(\bar{\Omega})$ for $s > n/4$. The hypotheses here imply that for every $R > 0$, there exists $C_R > 0$, such that

$$|F(x) - F(y)| \leq C_R |x - y| \text{ for all } x, y \in \mathbf{R},\ |x|, |y| < R$$

Since $H_s \subset C(\bar{\Omega})$ means that for arbitrary $u \in H_s$, $\|u\|_\infty \leq c\|u\|_s$ it follows that for $u, v \in B_R(0) \subset H_s$ we have $\|u\|_\infty, \|v\|_\infty \leq R$ if $s > n/4$. Then

$$\|F(u) - F(v)\|_H^2 = \int_\Omega |Fu(x) - F(v(x))|^2 dx \leq C_R \int_\Omega |u(x) - v(x)|^2 dx$$

$$\leq C_R \|u - v\|_H^2 \leq C_R \|u - v\|_s^2 \quad \forall\ u, v \in B_R(0) \subset H_s$$

This shows that F is locally Lipschitz from H_s into H for $s > n/4$. It follows that for every $u_0 \in H_s$ $s > n/4$, the abstract initial value problem

$$u'(t) + Au(t) = F(u(t)), \qquad u(0) = u_0$$

has a unique solution $u(t)$ such that $u - \tilde{f} \in C[0, T : H_s] \cap C^1[(0, T) : H]$ where $\tilde{f}$ denotes an extension of the boundary function f_1 as in the previous example.

In the case that $F = F(x)$, then the equation is linear and the existence of a unique solution to the direct problem follows much more easily. The conclusion, however, is similar.

1.3) Steady State Conductivity

Suppose U denotes a bounded open set in R^n, $n = 2, 3$, with smooth boundary S having unit outward normal $n = n(x)$ at each point $x \in S$. Suppose also that one, but not both, of the functions $f \in H^{1/2}(S)$ and $g \in H^{-1/2}(S)$ are given. Then we can consider the problem

$$\begin{aligned} div(a(x)\nabla u(x)) &= 0 && in\ U \\ u &= f && on\ S \\ a(x)n \bullet \nabla u &= g && on\ S \end{aligned} \qquad (1.1)_3$$

where $u = u(x)$ is to satisfy the equation and one or the other (but not both) of the boundary conditions. The other condition will then be incorporated into the inverse problem as the over specification. The reason for stating the direct problem in this ambiguous fashion is to allow the possibility of either the Dirichlet or the Neumann problem as the direct problem, since in some applications it is the function value which is controlled while other applications involve the flux field is the quantity which is applied. As this is a steady state problem, the time variable does not appear.

Here we suppose $a = a(x)$ denotes a strictly positive and suitably smooth (at least Lipschitz continuous) function on $\bar{U}$. This ensures that the boundary value problem consisting of the equation and either of the two conditions listed above has a unique solution at least in a weak

sense. In fact, a classical solution may exist but for the purposes of the inverse problem, a solution which resides in a Hilbert scale of Sobolev spaces is more convenient.
The model $(1.1)_3$ describes steady state heat conduction but also models the situation relevant to electrical impedance tomography. In that setting, $u = u(x)$ denotes the electrical potential inside the conducting body U, f represents the potential on the boundary and g then is the current on the boundary. In the inverse problem associated with electrical impedance tomography, a current g, is applied to the boundary of the domain, the resulting potential, f, on the boundary is measured, and this data is used to discover the electrical conductivity, $a = a(x)$ inside U. This information then provides a picture of the interior that can provide information similar to what is provided by an X-ray picture. This has medical applications as well as applications in electrical prospecting (e.g., locating buried land mines). In the thermal application, $u = u(x)$ denotes the temperature in the conductor and it is more likely that the Dirichlet condition is imposed on the boundary.

1.4) Unsteady Groundwater Flow

The partial differential equation governing unsteady groundwater flow for an isotropic, heterogeneous confined aquifer, $U \subset R^2$ is

$$S(x,y)\,\partial_t h(x,y,t) - \nabla(T(x,y)\nabla h(x,y,t)) = -q_w(x,y) \qquad (1.1)_4$$

where

$$\begin{aligned} U &= \text{2-dim region occupied by the aquifer} \\ \Gamma &= \Gamma_1 \cup \Gamma_2 = \text{boundary of } U \\ h(x,y,t) &= \text{hydraulic head in } U_T = U \times (0,T) \\ T(x,y) &= \text{transmissivity in } U \\ S(x,y) &= \text{storage coefficient in } U \\ q_w(x,y) &= \text{withdrawl term in } U \end{aligned}$$

The unknown hydraulic head function is subject to the following initial and boundary conditions

$$\begin{aligned} h(x,y,0) &= h_0(x,y) && \text{in} && U \subset R^2 \\ h(x,y,t) &= F(x,y,t) && \text{on} && \Gamma_1 \times (0,T) \\ T\partial_N h(x,y,t) &= G(x,y,t) && \text{on} && \Gamma_2 \times (0,T) \end{aligned}$$

This linear problem is, in some sense, a composite of the first two examples (in the linear cases). Using a formulation close to what was described in examples 1 and 2, existence of a unique solution is easy to prove under reasonable hypotheses. This problem will produce an inverse problem with more than a single unknown ingredient.

2. The Inverse Problem

If one or more of the structure inputs are unknown then it may possible to formulate an inverse problem whose solution will result in the identification of the unknown input. This identification will be based on measured observations of the solution. These observations may be made on either of the boundary sets, S_1 *or* S_2 or they may be made on some interior subset of Q_T; in any case we will illustrate in a moment what choices are available for observations.

Let ϕ denote the unknown structure input(s) we seek to identify. For suitably fixed state inputs, f, u_0, we suppose that for each ϕ in some class, W, of admissible structure inputs there exists a unique solution $u = u(\phi)$ to the direct problem. Next we denote by $B_j[u(\phi)] = g_j$ the observations from which the unknown structure inputs, ϕ, are to be determined. Then we define a **solution for the inverse problem** to consist of a pair $\{\phi, u(\phi)\}$ such that

i) for $\phi \in W$, $u(\phi)$ is a weak solution of the direct problem

ii) *for each* j $\langle B_j[u(\phi)] - g_j, \psi \rangle = 0$ *for all* $\psi \in X$

Here X denote a suitable space of test functions. The precise definition for X will depend on the definition of the observation B_j. We say that the structure input ϕ is **identifiable** from the observation(s) B_j if, for all $\phi_1, \phi_2 \in W$, the following statements are equivalent,

(a) *for each* j $\langle B_j[u(\phi_1)] - B_j[u(\phi_2)], \psi \rangle = 0$ *for all* $\psi \in X$

(b) $\phi_1 = \phi_2$ *in* W,

Now suppose that $u_k = u(\phi_k)$ for k = 1,2 denote weak solutions to the direct problem corresponding to structure inputs $\phi_1, \phi_2 \in W$. Then

$$< P[u_1, \phi_1], v > - < P[u_2, \phi_2], v > \;= 0 \quad \textit{for all } v \textit{ in } Z$$

i.e.,

$$< P[u_1, \phi_1], v > - < P[u_2, \phi_1], v > +$$

$$< P[u_2, \phi_1], v > - < P[u_2, \phi_2], v > \;= 0 \quad \textit{for all } v \textit{ in } Z$$

or,

$$< \partial_u P[u^*, \phi_1](u_1 - u_2), v > + < \partial_\phi P[u_2, \phi^*](\phi_1 - \phi_2), v > \;= 0 \quad \textit{for all } v \textit{ in } Z$$

where we have introduced the notation

$$\partial_u P[u^*, \phi_1](u_1 - u_2) = P[u_1, \phi_1] - P[u_2, \phi_1]$$

$$\partial_\phi P[u_2, \phi^*](\phi_1 - \phi_2) = P[u_2, \phi_1] - P[u_2, \phi_2].$$

Now abstract integration by parts leads to,

$$\langle \partial_u P[u^*, \phi_1](u_1 - u_2), v \rangle = \big\langle (u_1 - u_2), \partial_u P[u^*, \phi_1]^T v \big\rangle + (\eta[u_1 - u_2, \phi_1]_{t=T}, v)$$

$$-(\eta[u_1 - u_2, \phi_1]_{t=0}, v) + \langle \partial_1[u_1 - u_2, \phi_1], \gamma_1 v \rangle_1 + \langle \partial_2[u_1 - u_2, \phi_1], \gamma_2 v \rangle_1 -$$

$$- \langle \gamma_1[u_1 - u_2, \phi_1], \partial_1 v \rangle_2 - \langle \gamma_2[u_1 - u_2, \phi_1], \partial_2 v \rangle_2$$

and

$$\langle \partial_\phi P[u_2, \phi^*](\phi_1 - \phi_2), v \rangle = \big\langle (\phi_1 - \phi_2), \partial_\phi P[u_2, \phi^*]^T v \big\rangle + (\eta[u_2, \phi_1 - \phi_2]_{t=T}, v)$$

$$-(\eta[u_2,\phi_1-\phi_2]_{t=0},v)+\langle\partial_1[u_2,\phi_1-\phi_2],\gamma_1 v\rangle_1+\langle\partial_2[u_2,\phi_1-\phi_2],\gamma_2 v\rangle_1-$$

$$-\langle\gamma_1[u_2,\phi_1-\phi_2],\partial_1 v\rangle_2-\langle\gamma_2[u_2,\phi_1-\phi_2],\partial_2 v\rangle_2$$

Then this leads to,

$$\left\langle(u_1-u_2),\partial_u P[u^*,\phi_1]^T v\right\rangle+(\eta[u_1,\phi_1]_{t=T}-\eta[u_2,\phi_2]_{t=T},v)-(\eta[u_1,\phi_1]_{t=0}-\eta[u_2,\phi_2]_{t=0},v)$$

$$+\langle\partial_1[u_1,\phi_1]-\partial_1[u_2,\phi_2],\gamma_1 v\rangle_1+\langle\partial_2[u_1,\phi_1]-\partial_2[u_2,\phi_2],\gamma_2 v\rangle_2-$$

$$-\langle\gamma_1[u_1,\phi_1]-\gamma_1[u_2,\phi_2],\partial_1 v\rangle_1-\langle\gamma_2[u_1,\phi_1]-\gamma_2[u_2,\phi_2],\partial_2 v\rangle_2=$$

$$=-\left\langle(\phi_1-\phi_2),\partial_\phi P[u_2,\phi^*]^T v\right\rangle$$

Now the initial conditions imply that

$$(\eta[u_1,\phi_1]_{t=0}-\eta[u_2,\phi_2]_{t=0},v)=0$$

and the boundary conditions impy that two of the four remaining boundary terms vanish. Which two terms vanish depends on what the boundary conditions are specified in the direct problem (1). For example, if the direct problem has boundary conditions of the form

$$\begin{aligned}
< P[u,\phi],v > &= 0 \quad \textit{for all } v \textit{ in } V\\
(u(0)-u_0,v_0) &= 0 \quad \textit{for all } v_0 \textit{ in } H\\
< \partial_1[u,\phi]-f_1,w > &= 0 \quad \textit{for all } w \textit{ in } X_1 = H^{1/2}(S_1)\\
\partial_2[u,\phi] &= 0 \quad \textit{in } X_2 = H^{-1/2}(S_2)
\end{aligned}$$

then $\partial_1[u_1,\phi_1]-\partial_1[u_2,\phi_2]=0$ *and* $\partial_2[u_1,\phi_1]-\partial_2[u_2,\phi_2]=0$, and we have

$$\left\langle(u_1-u_2),\partial_u P[u^*,\phi_1]^T v\right\rangle+(\eta[u_1,\phi_1]_{t=T}-\eta[u_2,\phi_2]_{t=T},v)-$$

$$-\langle\gamma_1[u_1,\phi_1]-\gamma_1[u_2,\phi_2],\partial_1 v\rangle_1-\langle\gamma_2[u_1,\phi_1]-\gamma_2[u_2,\phi_2],\partial_2 v\rangle_2=$$

$$=-\left\langle(\phi_1-\phi_2),\partial_\phi P[u_2,\phi^*]^T v\right\rangle$$

In this case we have the following options for observations,

Interior: $B_I[u(\phi)]=u(x,t)$ *for* $(x,t)\in U\times(t_1,t_2)\subset Q_T$

Final: $B_T[u(\phi)]=\eta[u,\phi]_{t=T}$

Boundary: $B_1[u(\phi)]=\gamma_1[u,\phi]$ *or* $B_2[u(\phi)]=\gamma_2[u,\phi]$

Suppose, for example, we want to select $B_1[u(\phi)]=\gamma_1[u,\phi]$ for the observation. In order to do this, we require that v solves the following adjoint problem

$$\partial_u P[u^*,\phi_1]^T v = 0 \quad in \;\; Q_T$$
$$v(T) = 0 \quad in \;\; \Omega$$
$$\partial_1 v = f^* \quad on \;\; S_1$$
$$\partial_2 v = 0 \quad on \;\; S_2$$

Then the integral identity above reduces to,

$$\langle \gamma_1[u_1,\phi_1] - \gamma_1[u_2,\phi_2], f^* \rangle_1 = \langle (\phi_1 - \phi_2), \partial_\phi P[u_2,\phi^*]^T v \rangle;$$

i.e.,

$$\langle \delta B_1[u(\phi)], \partial_1 v \rangle_1 = \langle \delta\phi, \partial_\phi P[u_2,\phi^*]^T v \rangle$$

Note that this equation expresses a relationship between a change $\phi_1 - \phi_2 =: \delta\phi$ in the structural input and the corresponding change $\gamma_1[u_1,\phi_1] - \gamma_1[u_2,\phi_2] =: \delta B_1[u(\phi)]$, in the observation. Note that if we define the "least squares error functional" $J[\phi]$

$$J[\phi] = 1/2 \, \langle B_1[u(\phi)] - g, B_1[u(\phi)] - g \rangle_1 = \| B_1[u(\phi)] - g \|^2_{H^{1/2}(S_1)}$$

then

$$\nabla J[\phi, \delta\phi] = \langle \delta B_1[u(\phi)], B_1[u(\phi)] - g \rangle_1$$
$$= \langle \delta\phi, \partial_\phi P[u_2,\phi^*]^T v \rangle$$

where v solves the adjoint problem shown above for the choice $f^* = B_1[u(\phi)] - g$.

We now describe several examples in which an inverse problem is formulated for the direct problem examples of example 1.1. In each of these examples, we will derive an integral identity to which an approximation technique will be applied in later chapter for the purpose of constructing an approximate solution to the inverse problem.

Examples-

2.1) Identification of an Unknown Diffusivity (Transient problem)

Consider the 1-dimensional version of the initial boundary value problem $(1.1)_1$:

$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial}{\partial x}\left[D(u) \frac{\partial}{\partial x} u(x,t) \right] \qquad 0 < x < 1, \;\; 0 < t < T,$$
$$u(x,0) = 0, \qquad 0 < x < 1,$$
$$u(0,t) = f(t), \qquad 0 < t < T,$$
$$\frac{\partial}{\partial x} u(1,t) = 0, \qquad 0 < t < T$$

This problem is a mathematical model of , for instance, an experiment involving an initially contaminant free diffusion tube in which the end at x =1 is sealed and contaminant is injected at the end x= 0 according to a prescribed schedule described by a given function f(t). The unknown position dependent diffusivity function D(u) can be determined from measurements of either of the following pieces of data:

$$Flux = -D(u)\frac{\partial}{\partial x}u(0,t) = g(t), \qquad 0 \prec t \prec T \qquad (i)$$
$$Concentration = u(1,t) = h(t), \qquad 0 \prec t \prec T \qquad (ii)$$

Introduce the notation $u(x,t) = \Psi[D;f]$ to indicate the unique solution of the direct problem corresponding to inputs D(u) and f(t) .Similarly , $g = \Gamma_0\Psi[D;f]$ and $h = \Gamma_1\Psi[D;f]$ denote the measured outputs corresponding to the inputs D and f .
Now , if $u(x,t) = \Psi[D_1;f]$ and $v(x,t) = \Psi[D_2;f]$ then by forming the difference of the two direct problems with coefficients D_1 and D_2 ,respectively, an equation is obtained which can be arranged to have differences $u - v$ on the left and differences $D_1 - D_2$ on the right . Multiplying this equation by an arbitrary test function $\eta(x,t)$ and integrating by parts leads to,

$$-\int_0^T\Big[\tfrac{\partial}{\partial x}(b_1(u)-b_2(v))\eta-(b_1(u)-b_2(v))\tfrac{\partial\eta}{\partial x}\Big]_{x=0}^{x=1}dt$$
$$-\int_0^1\int_0^T\Big[(u-v)\tfrac{\partial\eta}{\partial x}+(b_1(u)-b_2(v))\tfrac{\partial^2}{\partial x^2}\eta\Big]dxdt$$
$$+\int_0^1(u-v)\eta\ \Big|_{t=0}^{t=T}dx = \int_0^1\int_0^T\tfrac{\partial}{\partial x}(b_1(u)-b_2(v))\tfrac{\partial\eta}{\partial x}dxdt$$

where we have introduced the auxiliary variable,

$$b(u) = \int_0^u D(s)ds.$$

Let the notation $\eta = \overset{*}{\Psi}\ [D_1;\theta_0,\theta_1]$ indicate that the test function $\eta = \eta(x,t)$ solves the adjoint problem

$$\frac{\partial}{\partial t}\eta(x,t) + p(x,t)\frac{\partial^2}{\partial x^2}\eta(x,t) = 0, \quad 0 \prec x \prec 1, 0 \prec t \prec T$$
$$\eta(x,0) = 0, 0 \prec x \prec 1,$$
$$\eta(0,t) = \theta_0(t), 0 \prec t \prec T,$$
$$p(1,t)\frac{\partial}{\partial x}\eta(1,t) = \theta_1(t), 0 \prec t \prec T$$

Here ,
$$p(x,t) = \int_0^1 D_1[v(x,t) + s(u(x,t) - v(x,t))]ds$$

For such a test function the integral identity above reduces to

$$\int_0^T[\Delta g(t)\theta_0(t) + \Delta h(t)\theta_1(t)]dt = -\int_0^1\int_0^T\Delta D(v(x,t))\tfrac{\partial v}{\partial x}\tfrac{\partial\eta}{\partial x}dxdt \qquad (2.1)_1$$

For test functions $\eta_0 = \overset{*}{\Psi}\ [D_1;\theta_0,0]$ and $\eta_1 = \overset{*}{\Psi}\ [D_1;0,\theta_1]$ the identity becomes a pair of identities, each relating a difference in inputs to a difference in one or the other of the two possible measured outputs

$$\int_0^T[\Delta g(t)\theta_0(t)]dt = -\int_0^1\int_0^T\Delta D(v(x,t))\frac{\partial v}{\partial x}\frac{\partial\eta_0}{\partial x}dxdt \quad (2.1)_1(a)$$
$$\int_0^T[\Delta h(t)\theta_1(t)]dt = -\int_0^1\int_0^T\Delta D(v(x,t))\frac{\partial v}{\partial x}\frac{\partial\eta_1}{\partial x}dxdt \quad (2.1)_1(b)$$

Let us use the flux in (i) as a measured data and apply the equation (a), (the same argument will hold if we use (ii) and apply the equation (b)). In the case that the diffusivity is dependent on position but not on u, these identities reduce to

$$\int_0^T [\Delta g(t)\theta_0(t)]dt = -\int_0^1 \Delta D(x)\int_0^T \frac{\partial v}{\partial x}\frac{\partial \eta_0}{\partial x}\,dt\,dx \qquad (2.1)_1(c)$$

$$\int_0^T [\Delta h(t)\theta_1(t)]dt = -\int_0^1 \Delta D(x)\int_0^T \frac{\partial v}{\partial x}\frac{\partial \eta_1}{\partial x}\,dt\,dx \qquad (2.1)_1(d)$$

In chapter 3 we will develop an algorithm for obtaining an approximating a solution to the inverse problem based on either of the identities (c) or (d).

2.2) Identification of an Unknown Source Term

Consider the initial boundary value problem, which is a version of $(1.1)_2$:

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= \frac{\partial^2}{\partial x^2}u(x,t) + F(x), && 0 < x < 1, 0 < t < T\\
u(x,0) &= 0, && 0 < x < 1\\
\frac{\partial}{\partial x}u(0,t) &= g(t), && 0 < t < T\\
\frac{\partial}{\partial x}u(1,t) &= 0, && 0 < t < T
\end{aligned}$$

This problem models an experiment in which an initially contaminant free tube has one end sealed and a controlled rate of contaminant flow is applied at the opposite end of the tube. Contaminant diffuses in the tube but an unknown source is introducing contaminant to the tube at a constant rate. The location and strength of the source is assumed to be unknown here and is, in fact, to be determined from data measured at the ends of the tube.
The unknown source function F(x) can be determined from measurements of either of the following pieces of data:

$$\begin{aligned}
u(0,t) &= f(t), && 0 \prec t \prec T && (i)\\
u(1,t) &= h(t), && 0 \prec t \prec T && (ii)
\end{aligned}$$

Introduce the notation $u(x,t) = \Psi[F;g]$ to indicate the unique solution of the direct problem corresponding to inputs F(x) and g(t) .Similarly, $f = \Gamma_0\Psi[F;g]$ and $h = \Gamma_1\Psi[F;g]$ denote the measured outputs corresponding to the inputs F and g .

Now, if $u(x,t) = \Psi[F_1;g]$ and $v(x,t) = \Psi[F_2;g]$ then by forming the difference of the two direct problems containing terms F_1 and F_2, respectively, an equation is obtained which can be arranged to have differences $u - v$ on the left and differences F_1 -F_2on the right . Multiplying this equation by an arbitrary test function $\eta(x,t)$ and integrating by parts,then apply the boundary and initial conditions, leads to,

$$\int_{x=0}^1 [v(x,T) - u(x,T)]\eta(x,T)dx - \int_{t=0}^T [v(0,t) - u(0,t)]\eta_x(0,t)dt + \int_{t=0}^T [v(1,t) - u(1,t)]\eta_x(1,t)dt =$$
$$\int_{x=0}^1\int_{t=0}^T (v-u)[\eta_t + \eta_{xx}]dtdx + \int_{x=0}^1\int_{t=0}^T [F_2 - F_1]\eta\, dtdx$$

Let the notation $\eta \stackrel{\bullet}{=}\Psi\ [F_1;\theta_0,\theta_1]$ indicate that $\eta = \eta(x,t)$ solves the adjoint problem

$$\begin{aligned} \frac{\partial}{\partial t}\eta(x,t) + \frac{\partial^2}{\partial x^2}\eta(x,t) &= 0, && 0 \prec x \prec 1, 0 \prec t \prec T \\ \eta(x,T) &= 0, && 0 \prec x \prec 1 \\ \frac{\partial}{\partial x}\eta(0,t) &= \theta_0(t), && 0 \prec t \prec T \\ \frac{\partial}{\partial x}\eta(1,t) &= \theta_1(t), && 0 \prec t \prec T \end{aligned}$$

For such a test function the integral identity above reduces to

$$\int_0^T [\Delta f(t)\theta_0(t) + \Delta h(t)\theta_1(t)]dt = -\int_0^1\int_0^T \Delta F(x)\eta(x,t)dxdt$$

For test functions $\eta_0 \stackrel{\bullet}{=}\Psi\ [F_1;\theta_0,0]$and $\eta_1 \stackrel{\bullet}{=}\Psi\ [F_1;0,\theta_1]$ the identity becomes a pair of identities , each relating a difference in inputs to a difference in one or the other of the two possible measured outputs

$$\begin{aligned} \int_0^T [\Delta f(t)\theta_0(t)]dt &= -\int_0^1\int_0^T \Delta F(x)\eta_0 dxdt \quad (a) \\ \int_0^T [\Delta h(t)\theta_1(t)]dt &= -\int_0^1\int_0^T \Delta F(x)\eta_1 dxdt \quad (b) \end{aligned}$$

These identities are now in a form to which the Backus-Gilbert approximation technique may be applied.

3) Identification of an Unknown Conductivity (Steady State problem)

Consider the situation in the direct problem $(1.1)_3$. Suppose that functions $f \in H^{1/2}(S)$ *and* $g \in H^{-1/2}(S)$ are given and consider the problem of determining a function pair $\{u = u(x),\ a = a(x)\}$ such that all of the following conditions are satisfied,

$$\begin{aligned} div(a(x)\nabla u(x)) &= 0 && in\ \ U \\ u &= f && on\ S && (1.1)_3 \\ a(x)n \bullet \nabla u &= g && on\ S \end{aligned}$$

Here we suppose $a = a(x)$ denotes a strictly positive and suitably smooth (at least Lipschitz continuous) function on $\bar{U}$. This ensures that the boundary value problem consisting of the first two conditions listed above has a solution at least in a weak sense, if the coefficient $a = a(x)$ were given. Then we can view the third of the listed conditions as an additional constraint which may be used to determine $a(x)$ in addition to $u(x)$.

With $f \in H^{1/2}(S)$ fixed, suppose that the pair $\{u_1,a_1\}$ satisfies the first two conditions listed above and produces the data $u_1|_S = g_1 \in H^{-1/2}(S)$ and the pair $\{u_2,a_2\}$ is similarly associated with $u_2|_S = g_2 \in H^{-1/2}(S)$. Then

$$div(a_1(x)\nabla u_1(x) - a_2(x)\nabla u_2(x)) = 0 \qquad in \;\; U$$
$$u_1 - u_2 = 0 \qquad on \;\; S$$
$$n \bullet [a_1(x)\nabla u_1 - a_2(x)\nabla u_2(x)] = g_1 - g_2 \qquad on \;\; S$$

If we rewrite the partial differential equation in the form

$$div(a_1(x)\{\nabla u_1(x) - \nabla u_2(x)\}) = -div\{a_1(x) - a_2(x)\}\nabla u_2(x)) \qquad in \;\; U$$

then integration by parts and attention to the boundary constraints leads to the integral identity,

$$\int_U \Delta a(x) \nabla u_2(x) \bullet \nabla v(x) \; dx = \int_S \Delta g(s) \theta(s) \; ds \qquad (2.1)_3$$

where $v = v(x)$ denotes the solution of the adjoint problem,

$$div(a_1(x)\nabla v(x)) = 0 \qquad in \;\; U$$
$$v = \theta \qquad on \;\; S$$

Clearly, when the coefficient in the boundary value problem is changed from $a = a_1(x)$ to $a = a_2(x)$, there is a corresponding change in the measured output, from $g = g_1(x)$ to $g = g_2(x)$. Equation $(2.1)_3$ relates these changes in an expression of the form

$$T[\Delta a] = \Lambda[\Delta g].$$

It will be our aim in Chapter 3 to approximate T^{-1} in order to obtain an explicit (but approximate) expression of the form, $\Delta a = T^{-1} \circ \Lambda[\Delta g]$.

4) Identification of Unknown Storage and Transmissivity Coefficients

Associated with the direct problem $(1.1)_4$, consider the problem of determining the hydraulic functions S and T from extra data measured inside or on the boundary of the flow domain U.

If $h_1(x,y,t)$ denotes the solution of the direct problem corresponding to hydraulic coefficients S_1, T_1 while coefficients S_2, T_2 lead to the solution $h_2(x,y,t)$, then

$$S_1 \, \partial_t(h_1 - h_2) - \nabla(T_1 \nabla(h_1 - h_2)) = \nabla((T_1 - T_2)\nabla h_2) - (S_1 - S_2)\partial_t h_2.$$

Multiplying this equation by a test function $v(x,y,t)$ and integrating by parts, leads to

$$\int_U \Delta h(x,y,T) S_1(x,y) \; v(x,y,T) dU - \int\int_{U_T} \Delta h \, [S_1 \partial_t v + \nabla(T_1 \nabla v)] dU dt$$

$$+\int_0^T\int_{\Gamma_2}(\Delta h)T_1\partial_N v\,dSdt - \int_0^T\int_{\Gamma_1}\Delta(T\partial_N h)\,v\,dSdt =$$

$$= \int\int_{U_T}[-\Delta S\ v\partial_t h_2 + \Delta T\,\nabla v \bullet \nabla h_2]\,dUdt$$

Now suppose that the test function $v = v(x,y,t)$ satisfies the adjoint problem,

$$\begin{aligned}
S_1\partial_t v + \nabla(T_1\nabla v) &= H^*(x,y,t) && in\ U_T\\
v(x,y,T) &= 0\\
v(x,y,t) &= F^*(x,y,t) && on\ \Gamma_1\times(0,T)\\
T_1\partial_N v &= G^*(x,y,t) && on\ \Gamma_2\times(0,T)
\end{aligned}$$

We can choose the data in this adjoint problem to correspond to different choices of overspecified data from which to determine the unknown hydraulic parameters. In particular, suppose that the following data is given,

$$\begin{aligned}
h(x,y,t) &= H(x,y,t) && on\quad U_1\times(0,T) && \text{for some subset } U_1\subset U\\
h(x,y,t) &= f(x,y,t) && on\quad \Gamma_2\times(0,T)
\end{aligned}$$

Then, if we choose the data in the adjoint problem such that

$$\begin{aligned}
&H^*(x,y,t)\ is\ given\ on\ U_1\times(0,T),\ but\ H^* = 0\ \ outside\quad U_1\times(0,T)\\
&F^*(x,y,t) = 0\\
&G^*(x,y,t)\ is\ given\ on\ \Gamma_2\times(0,T)
\end{aligned}$$

then the integral identity above reduces to

$$-\int_0^T\int_{U_1}\Delta H\,H^*dUdt + \int_0^T\int_{\Gamma_2}(\Delta f)\,G^*\,dSdt = \int\int_{U_T}[-\Delta S\ v\partial_t h_2 + \Delta T\,\nabla v\bullet\nabla h_2]\,dUdt$$

In fact, if we let $v_1(x,y,t)$ denote the solution of the adjoint problem corresponding to data, $H^*(x,y,t) = F^*(x,y,t) = 0$, but nontrivial $G^*(x,y,t)$ and let $v_2(x,y,t)$ denote the solution of the adjoint problem corresponding to data, $G^*(x,y,t) = F^*(x,y,t) = 0$, but nontrivial $H^*(x,y,t)$, then we obtain the pair of identities,

$$\int\int_{U_T}[-\Delta S\ v_1\partial_t h_2 + \Delta T\,\nabla v_1\bullet\nabla h_2]\,dUdt = \int_0^T\int_{\Gamma_2}(\Delta f)\,G^*\,dSdt$$

$$\int\int_{U_T}[-\Delta S\ v_2\partial_t h_2 + \Delta T\,\nabla v_2\bullet\nabla h_2]\,dUdt = -\int_0^T\int_{U_1}\Delta H\,H^*dUdt$$

Since S and T are independent of t, we can rewrite these identities in the form

$$\begin{aligned}
\int_U[\Delta S\,K_{11} + \Delta T\,K_{12}]\,dU &= \int_0^T\int_{\Gamma_2}(\Delta f)\,G^*\,dSdt\\
\int_U[\Delta S\,K_{21} + \Delta T\,K_{22}]\,dU &= -\int_0^T\int_{U_1}\Delta H\,H^*dUdt
\end{aligned}\qquad (2.1)_4$$

where

$$K_{11}(x,y) = -\int_0^T v_1\,\partial_t h_2\,dt, \qquad K_{12}(x,y) = \int_0^T \nabla v_1 \bullet \nabla h_2\,dt,$$

$$K_{21}(x,y) = -\int_0^T v_2\,\partial_t h_2\,dt, \qquad K_{22}(x,y) = \int_0^T \nabla v_2 \bullet \nabla h_2\,dt,$$

These two integral identities are special cases of the general situation in which a pair of unknown coefficients are to be determined from overspecified data by means of integral identities which relate changes in the coefficients to changes in the overspecifications.

CHAPTER II

Backus - Gilbert Method and its Applications in the Inverse Problems of Ordinary Differential Equations

1- Backus-Gilbert Method

Consider the problem of approximating an unknown function $f \in H^s(U)$ from a finite set of
moments $\mu_i := \langle f, g_i \rangle_{H^s \times H^{-s}}$,where $g_i \in H^{-s}(U)$ are known (generalized) functions,i=1,..,N and
$U \subset R^n$. This problem can be stated as follows: Define

$$A : H^s(U) \rightarrow R^N$$
$$f \rightarrow A(f) := (\langle f, g_1 \rangle_{H^s \times H^{-s}}, \ldots, \langle f, g_N \rangle_{H^s \times H^{-s}})$$

Find $f \in H^s(U)$ such that $A(f) = (\mu_1, \ldots, \mu_N)$
Now ,for $x_o \in U$, fixed , assume that
$f(x_o) = \sum_{i=1}^N \Phi_i(x_o)\,\mu_i$,
for $\mu_i = \langle f, g_i \rangle_{H^s \times H^{-s}}$ and $\Phi_i \in H^s(U)$, $i = 1, \ldots, N$
Then , $f(x_o) = \sum_{i=1}^N \Phi_i(x_o) \langle f, g_i \rangle_{H^s \times H^{-s}} = \left\langle f, \sum_{i=1}^N \Phi_i(x_o)\, g_i \right\rangle_{H^s \times H^{-s}}$
But this implies

$$\sum_{i=1}^N \Phi_i(x_o)\, g_i(x) = \delta(x - x_o) \in H^{-s}(U) \text{ , } s \succ n/2 \qquad (1.1)$$

where $\delta(x - x_o)$ is the Dirac generalized function at x_o.
Since , $\left(Af, \vec{b}\right)_{R^N} = \left\langle f, A^T \vec{b} \right\rangle_{H^s \times H^{-s}}$ where A^T denote the transpose of A , and
$\left(Af, \vec{b}\right)_{R^N} = (\langle f, g_1 \rangle_{H^s \times H^{-s}}, \ldots, \langle f, g_N \rangle_{H^s \times H^{-s}}).\vec{b} = \sum_{i=1}^N \langle f, g_i \rangle_{H^s \times H^{-s}} b_i = \left\langle f, \sum_{i=1}^N b_i g_i \right\rangle$
Then:

$$A^T : R^N \rightarrow H^{-s}(U)$$
$$\vec{b} \rightarrow \sum_{i=1}^N b_i g_i$$

So , the equation (1.1) is equivalent to

$$A^T \overrightarrow{\Phi(x_o)} = \delta(x - x_o) \qquad (1.2)$$

where $\overrightarrow{\Phi(x_o)} = (\Phi_1(x_o), \ldots, \Phi_N(x_o))$
To solve the equation (1.2) for $\overrightarrow{\Phi(x_o)}$, we have to acknowledge that in general there will be no
solution. However, the normal equation

$$(A^T)^* A^T \overrightarrow{\Phi(x_o)} = (A^T)^* \delta(x - x_o) \qquad (1.3)$$

is always uniquely solvable (see Appendix) , where
$(A^T)^* : H^{-s}(U) \rightarrow R^N$; the adjoint of A^T is defined by
$\left\langle A^T \vec{b}, F \right\rangle_{H^{-s}} = \left(\vec{b}, (A^T)^* F\right)_{R^N}$
Note that

$\left\langle A^T\vec{b},F\right\rangle_{H^{-s}} = \left\langle\sum_{i=1}^{N} b_i g_i,F\right\rangle_{H^{-s}} = \sum_{i=1}^{N} b_i\langle g_i,F\rangle_{H^{-s}}$

which implies that

$(A^T)^*F = \left(\langle g_1,F\rangle_{H^{-s}},\ldots,\langle g_N,F\rangle_{H^{-s}}\right) \in R^N$

Now ,

$(A^T)^*A^T\overrightarrow{\Phi(x_o)} = (A^T)^*\left[\sum_{i=1}^{N} \Phi_i(x_o)g_i\right] = \sum_{i=1}^{N} \Phi_i(x_o)(A^T)^*g_i$

$= \sum_{i=1}^{N} \Phi_i(x_o)\left(\langle g_1,g_i\rangle_{H^{-s}},\ldots,\langle g_N,g_i\rangle_{H^{-s}}\right) = \sum_{i=1}^{N} \Phi_i(x_o)\left(\langle Jg_1,g_i\rangle_{H^s\times H^{-s}},\ldots,\langle Jg_N,g_i\rangle_{H^s\times H^{-s}}\right)$

and

$(A^T)^*\delta(x-x_o) = \left(\langle g_1,\delta_{x_0}\rangle_{H^{-s}},\ldots,\langle g_N,\delta_{x_0}\rangle_{H^{-s}}\right) = \left(\langle Jg_1,\delta_{x_0}\rangle_{H^s\times H^{-s}},\ldots,\langle Jg_N,\delta_{x_0}\rangle_{H^s\times H^{-s}}\right)$

where $J : H^{-s}(U) \to H^s(U)$, denote the duality isomorphism, can be defined by two different

ways , according to the set $U \in R^n$, as follow :

(1) If U is the whole space R^n :

$J : H^{-s}(R^n) \to H^s(R^n)$

$g \to J(g) := \left[a_s^{-1}\,\hat{g}\right]^{\vee}$

where $\wedge$ and $\vee$ denote the Fourier transformation and its inverse, respectively ;

$a_s^{-1} = \left(1+|y|^2\right)^{-s}$ where $y \in R^n$ is a general element .(according to the following diagram

$$\begin{array}{ccccc} R^n & \xrightarrow{A^T} & H^{-s} & \xrightarrow{\wedge} & L^2_{a_s^{-1}} \\ \updownarrow id & & \downarrow J & & \downarrow a_s^{-1} \\ R^n & \xleftarrow[A]{} & H^s & \xleftarrow[\vee]{} & L^2_{a_s} \end{array}$$

In this case ,

$\langle Jg_i,\delta_{x_0}\rangle_{H^s\times H^{-s}} = \left[a_s^{-1}\,\hat{g}_i\right]^{\vee}(x_o)$

$,\langle Jg_i,g_j\rangle_{H^s\times H^{-s}} = \left\langle\left[a_s^{-1}\,\hat{g}_i\right]^{\vee},g_j\right\rangle_{H^s\times H^{-s}} = \left\langle a_s^{-1}\,\hat{g}_i,\hat{\bar{g}}_j\right\rangle_{L^2_{a_s}\times L^2_{a_s^{-1}}} = \int_R\left(1+|y|^2\right)^{-s}\,\hat{g}_i\,(y).\hat{\bar{g}}_j\,(y)dy$

(2) If $U \subset R^n$:

$J : H^{-s}(U) \to H^s(U)$

$g(x) \to Jg(x) := \sum_{k=1}^{\infty} \lambda_k^{-s}\langle g,w_k\rangle_{H^{-s}\times H^s} w_k(x)$

where $\{w_k\}$denotes an orthonormal basis of eigenfunctions in $H^s(U)$ with eigenvalues $\{\lambda_k^2\}$.

In this case ,

$\langle Jg_i,\delta_{x_0}\rangle_{H^s\times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s}\langle g_i,w_k\rangle_{H^{-s}\times H^s} w_k(x_o),$

$,\langle Jg_i,g_j\rangle_{H^s\times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s}\langle g_i,w_k\rangle_{H^{-s}\times H^s}\langle g_i,w_k\rangle_{H^{-s}\times H^s}.$

Finally , to approximate $f(x_o)$, we generate

$M_{N\times N} = \left[\langle g_j,g_i\rangle_{H^{-s}}\right]$ and $\vec{d}_{1\times N} = [\langle Jg_i,\delta_{x_0}\rangle_{H^s\times H^{-s}}]$ and solve $M[\overrightarrow{\Phi(x_o)}] = \vec{d}$

Then

$f(x_o) = [\overrightarrow{\Phi(x_o)}].\vec{\mu} = [M^{-1}\,\vec{d}].\vec{\mu}$

$$-\frac{d}{dx}\left[K(x)\frac{d}{dx}u(x)\right] = h(x) \ , \ 0 \prec x \prec 1$$

$$u(0) = 0 \qquad (2.1)$$

$$K(1)\, u'(1) = c$$

where

$0 \prec c_0 \preceq K(x) \preceq c_1$; c_0 and c_1 are positive constant ; and $h(x) \in L^2[0,1]$

Then this (direct) problem has a unique solution in the Sobolev space $H^1[0,1]$, given by

$$u(x) = \int_0^x \frac{1}{K(s)}\left[\int_s^1 f(t)dt + c\right]ds$$

Now, we will solve the (inverse) problem , in other words, we will find the (unknown) coefficient

$K(x)$ if $u(1)$ is the only given (measured) data.

Assume $u(x)$ and $v(x)$ are the solutions of (2.1) when $K(x)$ equals $K_1(x)$ and $K_2(x)$, respectively.

Then

$$\frac{d}{dx}\left[K_1(x)\frac{d}{dx}u(x)\right]\ \eta(x) + \ \frac{d}{dx}\left[K_2(x)\frac{d}{dx}v(x)\right]\ \eta(x) = 0 \ \forall \text{ smooth function } \eta$$

By integration by parts , we can get

$$[u(1)-v(1)]\, K_2(1)\ \eta'\ (1)\ + \int_0^1 (u-v)\left[-K_2\ \eta'\right] dx = \int_0^1 (K_2 - K_1)\, \eta'\ v'dx$$

Assume

$u(1) = \beta_1$ and $v(1) = \beta_2$; let $\Delta\beta = \beta_1 - \beta_2$ and $\Delta K = K_2 - K_1$, then

$$\Delta\beta K_1(1)\ \eta'\ (1) + \int_0^1 (u-v)\left[-K_1\ \eta'\right]' dx = \int_0^1 \Delta\beta\, \eta'\ v'dx$$

Assume η satisfies the boundart value problem :

$$-\frac{d}{dx}\left[K(x)\frac{d}{dx}\ \eta'(x)\right] = 0, 0 \prec x \prec 1$$

$$\eta(0) = 0$$

$$K_1(1)\ \eta'\ (1) = \theta$$

Then

$$\Delta\beta.\ \theta \ = \ \int_0^1 \Delta K\, \eta'\ v'dx$$

This equation can be stated as follows :

$$A : H^s[0,1] \to R$$

$$\Delta K \to A(\Delta K) := \int_0^1 \Delta K\, \eta'\ v'dx = \Delta\beta.\ \theta$$

Applying Backus-Gilbert method, we get

$$\Delta K(x_0) \simeq \Delta\beta.\ \theta\ .\frac{J\left[\eta'v'\right](x_0)}{\left\|\eta'v'\right\|_{H^{-s}}^{2}} \text{ ; for any fixed point } x_0 \in (0,1)$$

3- Numerical Examples

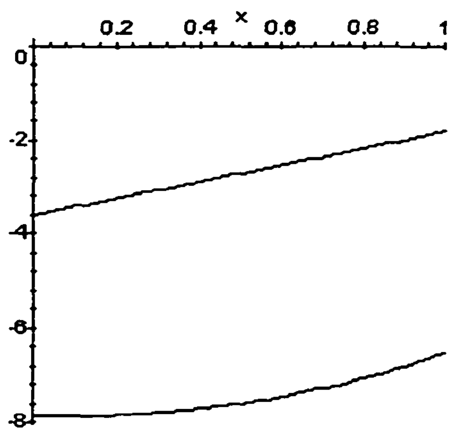
Example (1) :Consider the boundary value problem:

$$-\frac{d}{dx}\left[K(x)\frac{d}{dx}u(x)\right] = \frac{x}{2}\ e^{x/2}\ ,\ 0 \prec x \prec 1$$

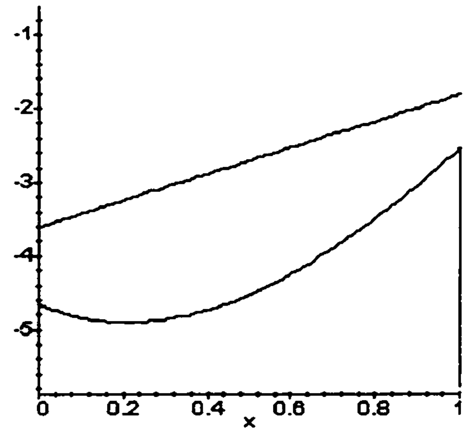
$$u(0) = 0$$

$$K(1)\ u'(1) = e^{1/2}$$

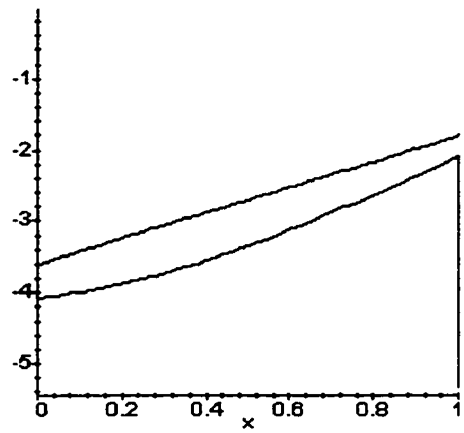
Assume $v(x)$ and $u(x)$ are the solutions when $K(x) \equiv K_1(x) = \alpha(-2x+4)$ $,K(x) \equiv K_2(x) = (-2x+4)$, respectively ;$\alpha = 0.1, 0.2, 0.9$.
Then $v(x) = \frac{1}{\alpha}\ [\exp(x/2) - 1]\ , u(x) = [\exp(x/2) - 1]\ ,\ \beta_2 \equiv v(1) = \frac{1}{\alpha}\ [\exp(1/2) - 1]$, and $\beta_1 \equiv u(1) = \exp(1/2) - 1; \Delta K = K_2 - K_1 = (-2x+4)(\alpha - 1)$. Also,when $\theta = 1, \eta(x) = -\frac{1}{2}\ln(2-x) + \frac{1}{2}\ln(2)$. Then , by applying Backus- Gilbert method to approximate
the unknown ΔK, we get the results which described in Fig $(1,i,j)$, $for\ s = i$; $\alpha = 0.j\ ,\ i = 0,1,2,3,4\ ;\ j = 1,2,9$



Fig(1,0,1)



Fig(1,1,1)



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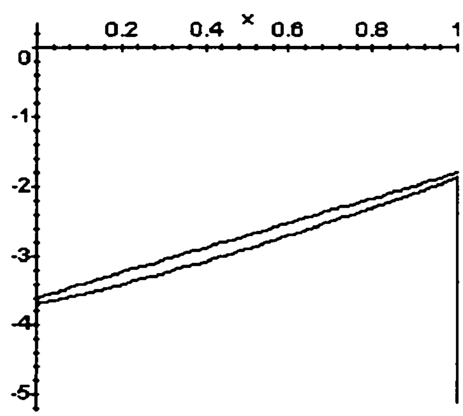

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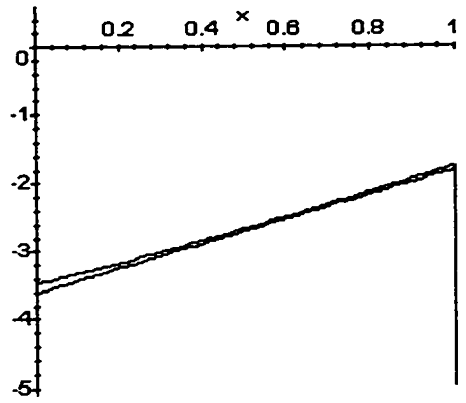
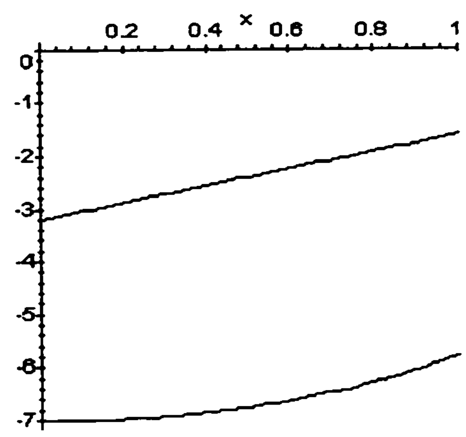
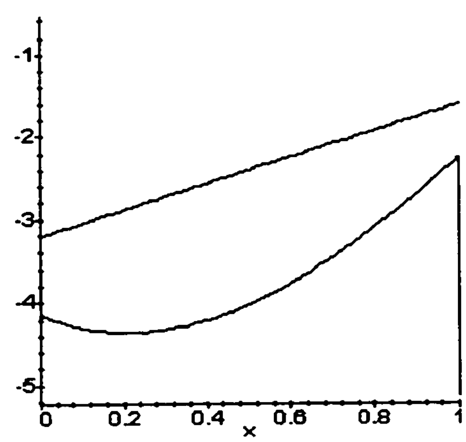


Fig(1,4,1)



Fig(1,0,2)



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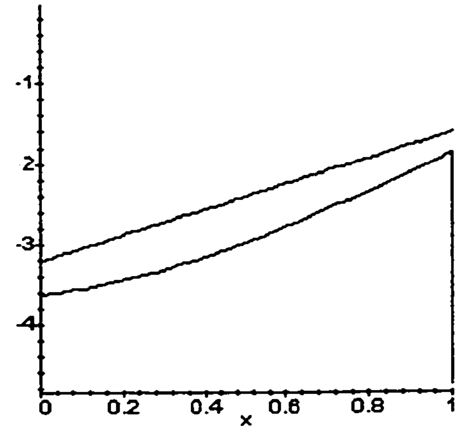

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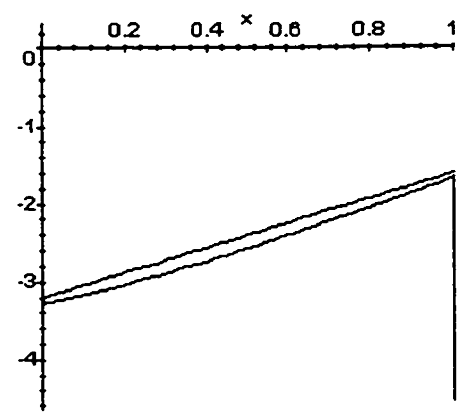

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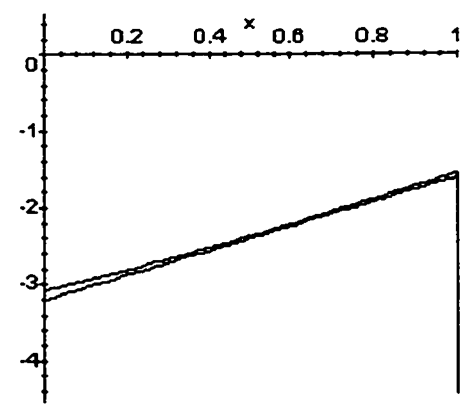
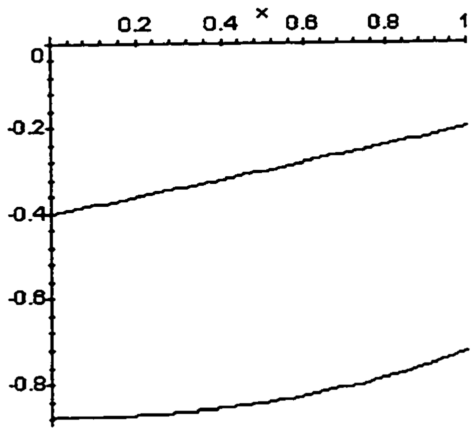
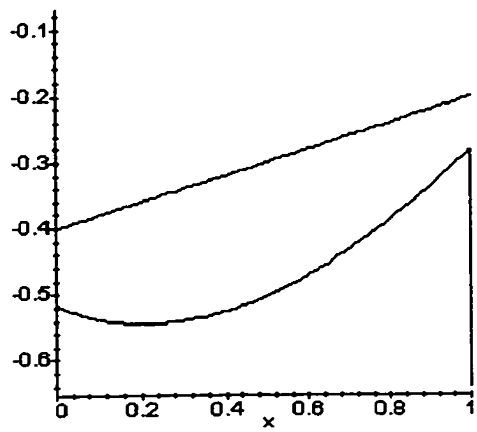


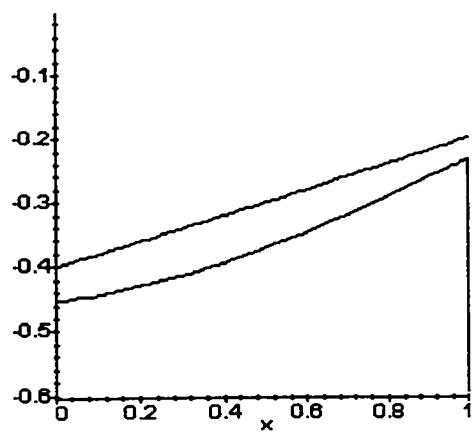
Fig(1,4,2)



Fig(1,0,9)



Fig(1,1,9)



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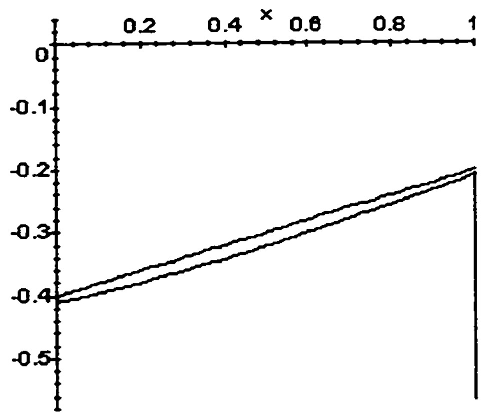
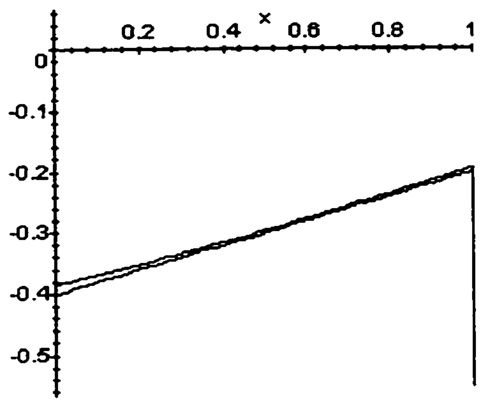
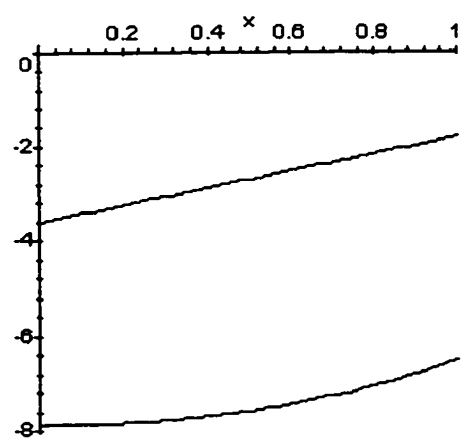


Fig(1,3,9)



Fig(1,4,9)



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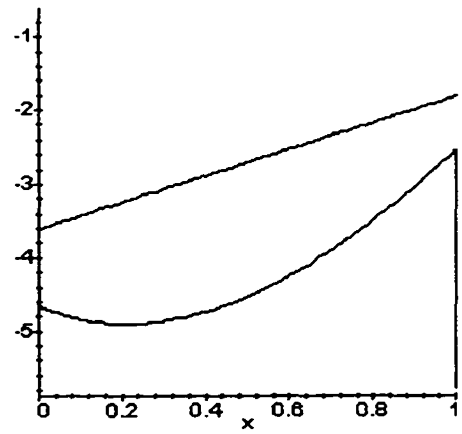
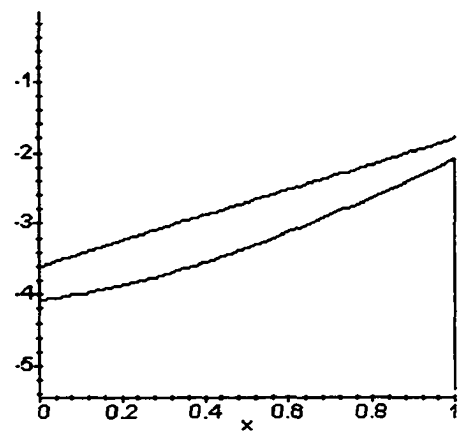


Fig(1,1,1)



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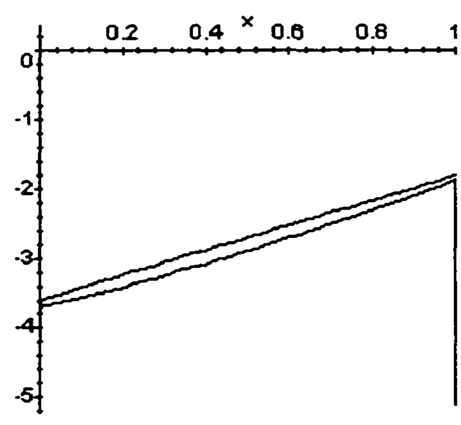
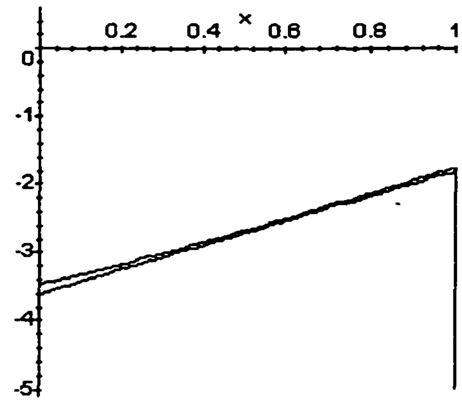
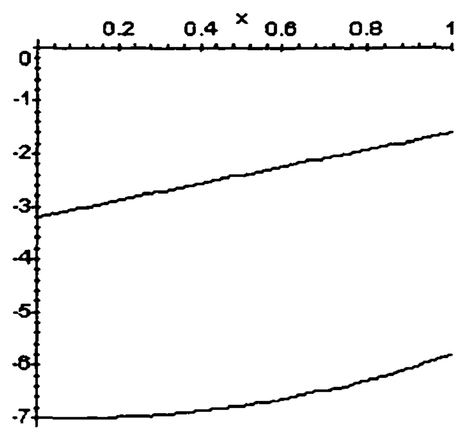


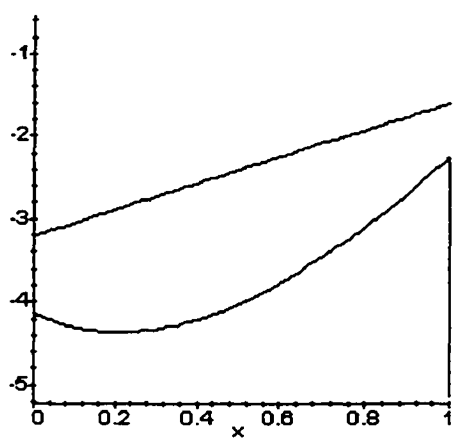
Fig(1,3,1)



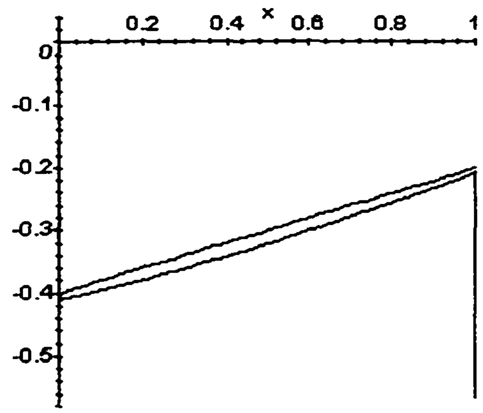
Fig(1,4,1)



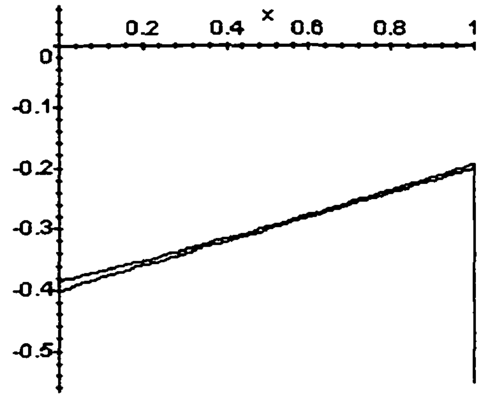
Fig(1,0,2)



Fig(1,1,2)



Fig(1,3,9)



Fig(1,4,9)

Example (2) :Consider the boundary value problem:

$$-\frac{d}{dx}\left[K(x)\frac{d}{dx}u(x)\right] = \frac{x}{2}e^{x/2}, 0 \prec x \prec 1$$

$$u(0) = 0$$

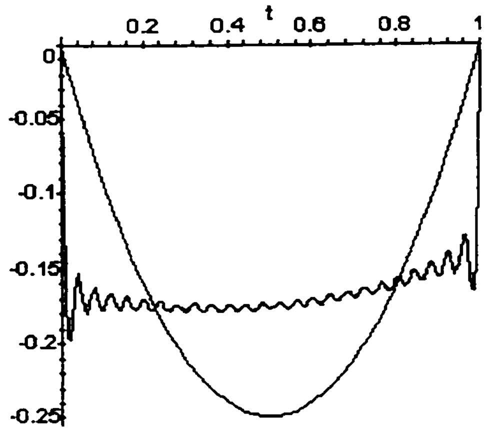
$$K(1)\, u'(1) = e^{1/2}$$

Assume $v(x)$ and $u(x)$ are the solutions when $K(x) = K_2(x) = x^2 - 3x + 4$ and

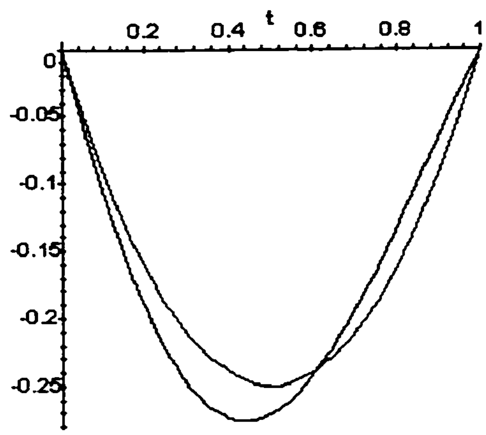
$K(x) = K_1(x) = -2x + 4$, respectively.Then $u(x) = [\exp(x/2) - 1]$, and $\beta_1 \equiv u(1) = \exp(1/2) - 1$; $\Delta K = K_2 - K_1 = x^2 - x$. Also, when $\theta = 1, \eta(x) = -\frac{1}{2}\ln(2-x) + \frac{1}{2}\ln(2)$. Then,by applying

Backus- Gilbert method to approximate the unknown ΔK, we get the results which described in

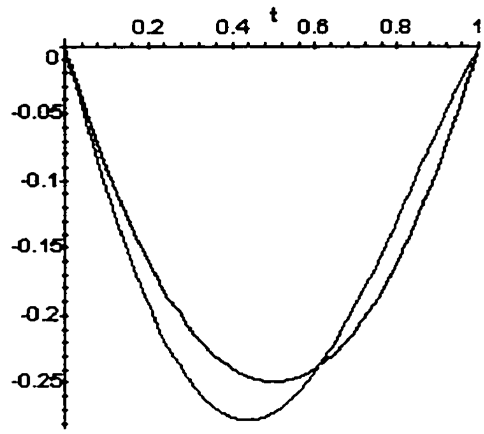
Fig $(2, i)$, *for* $s = i$, $i = 0, 1, 2, 3, 4$



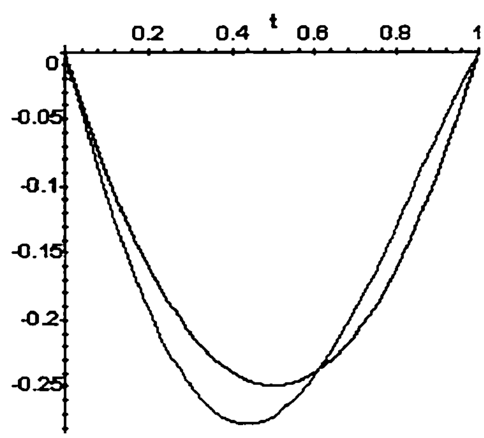
Fig(2,0)



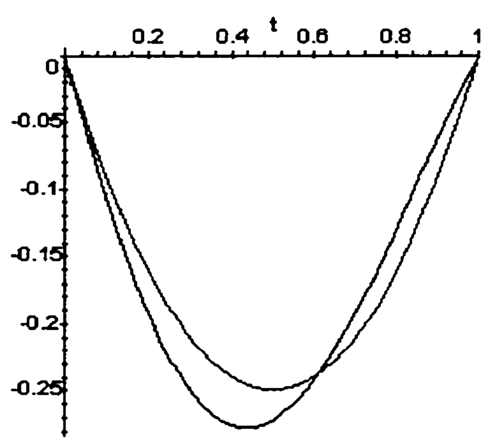
Fig(2,1)



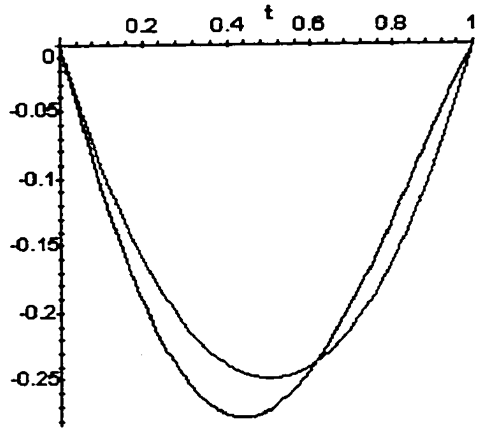
Fig(2,2)



Fig(2,3)



Fig(2,4)



Fig(2,5)

CHAPTER III

Backus-Gilbert Approximation to the Solution of Inverse Problems in Partial Differential Equations

This chapter presents the application of the Backus-Gilbert approximation technique to the solution of inverse problems for partial differential equations. In particular, the problems that are presented in this chapter will deal with the identification of a single spatially dependent coefficient/source term in a partial differential equation from various types of measured data. The examples will describe the application of the method to some of the example inverse problems that were developed in example I.2. The main focus is on examples in which the unknown coefficient is a function of the spatial variable(s) since this method provides a new way of dealing with such problems. The final example of the chapter discusses identification of a state dependent unknown to provide constrast with the other examples.

The following chapter will deal with identification of multiple unknowns.

EXAMPLES

3.1) Identification of a Spatially Dependant Diffusivity

Consider the initial boundary value problem for a one dimensional diffusion equation with a spatially dependent diffusivity:

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\left[D(x)\frac{\partial}{\partial x}u(x,t)\right], \quad 0 \prec x \prec 1, 0 \prec t \prec T$$

$$u(x,0) = 0, \qquad 0 \prec x \prec 1,$$

$$u(0,t) = f(t), \qquad 0 \prec t \prec T,$$

$$\frac{\partial}{\partial x}u(1,t) = 0, \qquad 0 \prec t \prec T$$

The unknown diffusivity function $D(x)$ can be determined from measurements of either of the following pieces of data:

$$Flux = -D(0)\frac{\partial}{\partial x}u(0,t) = g(t), 0 \prec t \prec T \qquad (i)$$

$$Concentration = u(1,t) = h(t), \qquad 0 \prec t \prec T \qquad (ii)$$

Continue to use the notation $u(x,t) = \Psi[D;f]$ to indicate the unique solution of the direct problem corresponding to inputs D(u) and f(t) . Similarly, let $g = \Gamma_0\Psi[D;f]$ and $h = \Gamma_1\Psi[D;f]$ denote the measured outputs i) and ii), respectively, corresponding to the inputs D and f .

Recall also that the notation $\eta = \overset{*}{\Psi}[D_1;\theta_0,\theta_1]$ indicates that $\eta = \eta(x,t)$ solves the adjoint problem

$$\frac{\partial}{\partial t}\eta(x,t) + \frac{\partial}{\partial x}(D_1\frac{\partial}{\partial x}\eta) = 0, \quad 0 < x < 1, 0 < t < T,$$
$$\eta(x,T) = 0, \quad 0 < x < 1,$$
$$\eta(0,t) = \theta_0(t), \quad 0 < t < T,$$
$$D_1(1)\frac{\partial}{\partial x}\eta(1,t) = \theta_1(t), \quad 0 < t < T$$

For such a test function the previously derived integral identity reduces to

$$\int_0^T [\Delta g(t)\theta_0(t) + \Delta h(t)\theta_1(t)]dt = -\int_0^1\int_0^T \Delta D(x)\frac{\partial v}{\partial x}\frac{\partial \eta}{\partial x}dxdt$$

For test functions $\eta_0 = \overset{*}{\Psi}[D_1;\theta_0,0]$ and $\eta_1 = \overset{*}{\Psi}[D_1;0,\theta_1]$ the identity becomes a pair of identities , each relating a difference in inputs to a difference in one or the other of the two possible measured outputs

$$\int_0^T [\Delta g(t)\theta_0(t)]dt = -\int_0^1\int_0^T \Delta D(x)\frac{\partial v}{\partial x}\frac{\partial \eta_0}{\partial x}dxdt \qquad (a)$$
$$\int_0^T [\Delta h(t)\theta_1(t)]dt = -\int_0^1\int_0^T \Delta D(x)\frac{\partial v}{\partial x}\frac{\partial \eta_1}{\partial x}dxdt \qquad (b)$$

Consider the case in which the flux is the measured data and apply the equation (a), (essentially the same development will ensue if we use the other piece of measured data and apply the equation (b)). We now begin to describe how an approximation to $\Delta D(x)$ can be obtained.

For a fixed number $N \geq 1$, let $\eta_0^{(i)} = \Psi^*[D_1;\theta_0^{(i)},0]$, where $\theta_0^{(i)} = \eta(0,t), i = 1,\ldots N$ denote N linearly independent but otherwise arbitrarily chosen functions. Then the identity (a) implies

$$\int_0^T [\Delta g(t)\theta_0^{(i)}(t)]dt = -\int_0^1 \Delta D(x)\int_0^T \frac{\partial v}{\partial x}\frac{\partial \eta_0^{(i)}}{\partial x}dtdx \;, \qquad i = 1,..N$$

Use this set of N equations to define a mapping, $A : H^s[0,1] \to R^N$ by

$$f \to -\left(\langle f,K^1\rangle_{H^s[0,1]\times H^{-s}[0,1]},\ldots,\langle f,K^N\rangle_{H^s[0,1]\times H^{-s}[0,1]}\right)$$

where

$$K^i(x) = \int_0^T \frac{\partial v}{\partial x}\frac{\partial \eta_0^{(i)}}{\partial x}dt, \qquad i = 1,\ldots,N.$$

We want to find $f(x) = \Delta D(x)$, such that $A[\Delta D(x)] = (\mu_1,\ldots\ldots,\mu_N)$, where

$$\mu_j = \int_0^T [\Delta g(t)\theta_0^{(j)}(t)]dt, \quad j = 1,\ldots,N.$$

For this purpose, we are going to assume that for a fixed $x_0 \in [0,1]$

$$\Delta D(x_0) = \sum_{i=1}^N \Phi_i(x_0)\mu_i. \;,$$

where $\Phi_i \in H^{-s}[0,1],\ i = 1,..N$
Then,

$$\Delta D(x_0) = \sum_{i=1}^{N} \Phi_i(x_0)\langle \Delta D(x), K^i(x)\rangle_{H^s[0,1]\times H^{-s}[0,1]}$$

$$= \left\langle \Delta D(x,t), \sum_{i=1}^{N} \Phi_i(x_0,t_0)K^i(x)\right\rangle_{H^s[0,1]\times H^{-s}[0,1]}$$

It is evident from this last equality that

$$\sum_{i=1}^{N} \Phi_i(x_0)K^i(x) = \delta(x-x_0) \in H^{-s}[0,1], \qquad s > 1/2$$

where $\delta(x-x_0)$ is the Dirac distribution concentrated at x_0. Note that the restriction $s > 1/2$ ensures that $\delta(x-x_0) \in H^{-s}[0,1]$.

Since
$$\left(Af, \vec{b}\right)_{R^N} = \left\langle f, A^T\vec{b}\right\rangle_{H^s[0,1]\times H^{-s}[0,1]},$$

where A^T denotes the transpose of A, and we have

$$\left(Af, \vec{b}\right)_{R^N} = \left(\langle f, K^1\rangle_{H^s[0,1]\times H^{-s}[0,1]}, \ldots, \langle f, K^N\rangle_{H^s[0,1]\times H^{-s}[0,1]}\right) \cdot \vec{b}$$

$$= \sum_{i=1}^{N} b_i\langle \Delta D(x), K^i\rangle_{H^s[0,1]\times H^{-s}[0,1]} = \left\langle f, \sum_{i=1}^{N} K^i b_i\right\rangle_{H^s[0,1]\times H^{-s}[0,1]}$$

it follows that

$$A^T : R^N \to H^{-s}[0,1] \text{ is given by } \vec{b} \to \sum_{i=1}^{N} K^i(x)\, b_i$$

i.e.,
$$A^T\left(\vec{\Phi}(x_0)\right) = \sum_{i=1}^{N} \Phi_i(x_0)K^i(x),$$

then the equation defining $\vec{\Phi}(x_0)$ is equivalent to:

$$A^T\left(\vec{\Phi}\,(x_0)\right) = \delta(x-x_0)$$

Since A^T is not surjective, this equation will have no solution in general. In such cases it is natural to look for a solution to the equation which minimizes the square of the residual, a so called, least squares solution. This solution can be shown to satisfy the normal equation

$$(A^T)^* A^T\vec{\phi}(x_0) = (A^T)^*\delta(x-x_0) = (A^T)^*\delta_{x_0},$$

where $(A^T)^*$ is the adjoint of A^T. This least squares problem is always uniquely solvable. Note first that

$$(A^T)^* A^T\vec{\phi}(x_0) = \left\{(K^j, A^T\vec{\phi}(x_0))_{H^{-s}[0,1]}\right\} \in \mathbf{R}^N$$

$$= \left\{\left(K^j, \sum_{i=1}^{N} \phi_i(x_0)K^i\right)_{H^{-s}[0,1]}\right\}$$

$$= \sum_{i=1}^{N} \phi_i(x_0) \left(K^i, K^j\right)_{H^{-s}[0,1]}$$

That is,

$$\left[\left(K^i, K^j\right)_{H^{-s}(U)}\right] \vec{\phi}(x_0) = (A^{\top})^* \delta_{x_0}$$

Now $\quad (A^{\top})^* \delta_{x_0} = \left\{\left(K_j, \delta_{x_0}\right)_{H^{-s}[0,1]}\right\}$

and

$$\left(K_j, \delta_{x_0}\right)_{H^{-s}(U)} = \langle JK_j, \delta_{x_0}\rangle_{H^s \times H^{-s}}$$

where

$$J : H^{-s}[0,1] \to H^s[0,1]$$

denotes the duality isomorphism from $H^{-s}[0,1]$ *to* $H^s[0,1]$. Here J is given by

$$JK_i(x) = \sum_{k=1}^{\infty} \lambda_k^{-s} (K_i, w_k)_0 w_k(x)$$

where

$\{w_k\}$ denotes a convenient choice of orthonormal basis of eigenfunctions for $H^0[0,1]$ e.g., the eigenfunctions for $-\nabla^2$ on $H_0^1[0,1]$

and,

$$\lambda_k w_k = -\nabla^2 w_k \qquad \textit{for all } k \geq 1.$$

Once the definition of J is specified, the formulas for the inner products and duality pairings become specific as well; e.g.,

$$(K_i, w_k)_0 = \int_{[0,1]} K_i(x)\ w_k(x)\, dx,$$

and

$$\langle JK_j, \delta_{x_0}\rangle_{H^s \times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s} (K_i, w_k)_0 w_k(x_0),$$

$$(K_i, K_j)_{H^{-s}(U)} = \langle JK_i, K_j\rangle_{H^s \times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s} (K_i, w_k)_0 (K_j, w_k)_0.$$

In order to approximate $\Delta D(x_0)$ we generate

$$M = \left[\left(K^i, K^j\right)_{H^{-s}[0,1]}\right] \qquad \text{and} \qquad \vec{d} = \left[\langle JK_j, \delta_{x_0, t_0}\rangle_{H^s \times H^{-s}}\right]$$

and solve

$$M[\vec{\varphi}(x_0)] = \vec{d}(x_0)$$

Then

$$\begin{aligned} \Delta D(x_0) &= \vec{\mu} \bullet [\vec{\varphi}(x_0)] \\ &= \left[\left(\Delta g(s), \theta_j(s)\right)_{H^0(S_2)}\right] \bullet [\vec{\varphi}(x_0)] \\ &= \left[\left(\Delta g(s), \theta_j(s)\right)_{H^0(S_2)}\right]^{\top} [M^{-1}] \vec{d}(x_0) \\ &= \vec{W} \bullet \vec{d}(x_0) \end{aligned}$$

This can be carried out at a grid of points x_0 in $[0,1]$ in order to provide an approximation for

$\Delta D(x_0) = D_2(x_0) - D_1(x_0)$ and, since D_2 is given , this provides an approximation for the unknown coefficient D_1. Note that $\vec{W} = \left[(\Delta g(s), \theta_j(s))_{H^0(S_2)}\right]^T [M^{-1}]$ does not depend on x_0 so that generating $\Delta D(x_0)$ values at a large number of points x_0 requires only the computation of $\vec{d}(x_0)$ and does not mean repeated solution of the linear least squares system of equations.

Of course the computation here does require solving the adjoint problem, which in turn requires knowledge of the conductivity coefficient D_1. Therefore this identification algorithm would have to be carried out iteratively. The algorithm for this iteration will be discussed in more detail in the third example.

3.2) Identification of a Spatially Dependant Source Term

Consider the initial boundary value problem :

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= \frac{\partial^2}{\partial^2 x}u(x,t) + F(x), && 0 \prec x \prec 1, 0 \prec t \prec T \\
u(x,0) &= 0, && 0 \prec x \prec 1 \\
\frac{\partial}{\partial x}u(0,t) &= g(t), && 0 \prec t \prec T \\
\frac{\partial}{\partial x}u(1,t) &= 0, && 0 \prec t \prec T
\end{aligned}$$

The unknown source function F(x) can be determined from measurements of either of the following pieces of data:

$$\begin{aligned}
u(0,t) &= f(t), && 0 \prec t \prec T && (i) \\
u(1,t) &= h(t), && 0 \prec t \prec T && (ii)
\end{aligned}$$

Recall that the notation $u(x,t) = \Psi[F;g]$ indicates the unique solution of the direct problem corresponding to inputs F(x) and g(t) .Similarly, $f = \Gamma_0\Psi[F;g]$ and $h = \Gamma_1\Psi[F;g]$ are used to denote the measured outputs corresponding to the inputs F and g .

Recall further that the notation $\eta = \overset{*}{\Psi}[F_1;\theta_0,\theta_1]$ indicates that $\eta = \eta(x,t)$ solves the adjoint problem

$$\begin{aligned}
\frac{\partial}{\partial t}\eta(x,t) + \frac{\partial^2}{\partial x^2}\eta(x,t) &= 0, && 0 \prec x \prec 1, 0 \prec t \prec T \\
\eta(x,T) &= 0, && 0 \prec x \prec 1 \\
\frac{\partial}{\partial x}\eta(0,t) &= \theta_0(t), && 0 \prec t \prec T \\
\frac{\partial}{\partial x}\eta(1,t) &= \theta_1(t), && 0 \prec t \prec T
\end{aligned}$$

For such a test function the integral identity of example I.2.2 reduces to

$$\int_0^T [\Delta f(t)\theta_0(t) + \Delta h(t)\theta_1(t)]dt = -\int_0^1\int_0^T \Delta F(x)\eta(x,t)dxdt$$

For test functions $\eta_0 = \overset{*}{\Psi}[F_1;\theta_0,0]$ and $\eta_1 = \overset{*}{\Psi}[F_1;0,\theta_1]$ the identity becomes a pair of identities , each relating a difference in inputs to a difference in one or the other of the two possible measured outputs

$$\int_0^T [\Delta f(t)\theta_0(t)]dt = -\int_0^1 \int_0^T \Delta F(x)\eta_0 dxdt \quad (a)$$

$$\int_0^T [\Delta h(t)\theta_1(t)]dt = -\int_0^1 \int_0^T \Delta F(x)\eta_1 dxdt \quad (b)$$

Now application of the approach of the previous example leads to an approximation of the form

$$\begin{aligned} \Delta F(x_0) &= \vec{\mu} \bullet [\vec{\varphi}(x_0)] \\ &= \left[(\Delta f(s), \theta_j(s))_{H^0(S_2)} \right] \bullet [\vec{\varphi}(x_0)] \\ &= \left[(\Delta f(s), \theta_j(s))_{H^0(S_2)} \right]^T [M^{-1}] \vec{d}(x_0) \\ &= \vec{W} \bullet \vec{d}(x_0) \end{aligned}$$

Since the development is so nearly the same as the previous example, the details will be omitted.

3.3) Identification of a Spatially Dependant Conductivity

Suppose U denotes a bounded open set in $R^n, n = 2,3,$ with smooth boundary S. Suppose also that functions $f \in H^{1/2}(S)$ *and* $g \in H^{-1/2}(S)$ are given and consider the problem of determining a function pair $\{u = u(x),\ a = a(x)\}$ such that all of the following conditions are satisfied,

$$\begin{aligned} div(a(x)\nabla u(x)) &= 0 \quad && in\ \ U \\ u &= f && on\ S \\ a(x)n \bullet \nabla u &= g && on\ S \end{aligned}$$

Here we suppose $a = a(x)$ denotes a strictly positive and suitably smooth (at least Lipschitz continuous) function on $\bar{U}$.

With $f \in H^{1/2}(S)$ assumed to be imposed, suppose that the pair $\{u_1, a_1\}$ satisfies the first two conditions listed above and produces the data $u_1|_S = g_1 \in H^{-1/2}(S)$ and the pair $\{u_2, a_2\}$ is similarly associated with $u_2|_S = g_2 \in H^{-1/2}(S)$. Then we obtain the integral identity,

$$\int_U \Delta a(x) \nabla u_2(x) \bullet \nabla v(x)\ dx = \int_S \Delta g(s) \theta(s)\ ds$$

where $v = v(x)$ denotes the solution of the adjoint problem,

$$\begin{aligned} div(a_1(x)\nabla v(x)) &= 0 \quad && in\ \ U \\ v &= \theta && on\ S \end{aligned}$$

Now we describe a method to construct an approximation to the coefficient $a_1 = a_1(x)$

associated with data f *and* g_1. We suppose that $a_2 = a_2(x)$ is some **given** coefficient and that the pair $\{u_2, a_2\}$ satisfies the Dirichlet problem stated above with data f. We suppose that solving this Dirichlet problem generates the overspecified data $u_2 = g_2$ *on S*. In particular, let us suppose that a_2 is chosen to be a constant. Since we are solving a Dirichlet problem with a homogeneous equation, it is not necessary to specify the constant in order to solve the Dirichlet problem. The constant can then be chosen later to satisfy some other condition.

Begin with the integral identity derived in example I.2.3,

$$\begin{array}{ll}\text{given} & \Delta g(s) \in H^{-1/2}(S) \text{ and } K_j(x) = \nabla u_2(x) \bullet \nabla v_j(x),\ j = 1,\dots,N \\ \text{suppose} & \int_U \Delta a(x) K_j(x)\, dx = \int_S \Delta g(s)\theta_j(s)\, ds \qquad 1 \le j \le N\end{array}$$

Here $v_j(x)$ denotes the N independent solutions of the adjoint BVP corresponding to N linearly independent choices for the data, $\theta_j \in H^{1/2}(S)$. The integral identity can be expressed as

$$\langle \Delta a(x), K_j(x) \rangle_{H^s \times H^{-s}} = (\Delta g(s), \theta_j(s))_{H^0(S)} \qquad 1 \le j \le N$$

where we are assuming that

$(a)\ \ K_j \in H^0(U) \subset H^{-s}(U) \quad s \ge n/2$

$(b)\ \ \Delta a \in H^s(U), \qquad s \ge n/2$

Now, for $x_0 \in U$, fixed, assume that

$$\Delta a(x_0) = \sum_{j=1}^N \phi_j(x_0)\ \mu_j$$

for $\quad \mu_j = (\Delta g(s), \theta_j(s))_{H^0(S_2)} \qquad 1 \le j \le N$

Here $\phi_j(x_0)$ denotes a scalar, but as x_0 is going to be allowed to vary over $U \subset R^n$ $\phi_j(x_0)$ will eventually denote some type of function (or generalized function). Now we have

$$\begin{aligned}\Delta a(x_0) &= \sum_{j=1}^N \phi_j(x_0)\ \mu_j = \sum_{j=1}^N \phi_j(x_0)\ \langle \Delta a(x), K_j(x) \rangle_{H^s \times H^{-s}} \\ &= \left\langle \Delta a(x), \sum_{j=1}^N \phi_j(x_0) K_j(x) \right\rangle_{H^s \times H^{-s}}\end{aligned}$$

But this implies

$$\sum_{j=1}^N \phi_j(x_0) K_j(x) = \delta(x - x_0) \in H^{-s}(U)$$

Now define

$$A : H^s(U) \rightarrow \mathbf{R}^N$$

$$A(\Delta a) = \{\langle \Delta a, K_1 \rangle, \ldots, \langle \Delta a, K_N \rangle\} = \vec{\mu}$$

and

$$A^{\top} : \mathbf{R}^N \rightarrow H^{-s}(U)$$

where

$$(A(\Delta a), \vec{b})_{R^N} = \left\langle \Delta a, A^{\top}\vec{b} \right\rangle_{H^s \times H^{-s}}$$

But

$$(A(\Delta a), \vec{b})_{R^N} = \{\langle \Delta a, K_1 \rangle, \ldots, \langle \Delta a, K_N \rangle\} \bullet \vec{b} = \vec{\mu} \bullet \vec{b}$$

$$= \sum_{j=1}^{N} \langle \Delta a, K_j \rangle b_j = \left\langle \Delta a, \sum_{j=1}^{N} K_j b_j \right\rangle_{H^s \times H^{-s}}$$

from which it follows that for arbitrary $\vec{b} \in \mathbf{R}^N$,

$$A^{\top}\vec{b} = \sum_{j=1}^{N} K_j b_j \in H^{-s}(U)$$

Now define

$$(A^{\top})^{*} : H^{-s}(U) \rightarrow \mathbf{R}^N$$

by

$$\left(A^{\top}\vec{b}, F\right)_{H^{-s}(U)} = \left\langle \vec{b}, (A^{\top})^{*} F \right\rangle_{R^N}$$

Note that

$$\left(A^{\top}\vec{b}, F\right)_{H^{-s}(U)} = \left(\sum_{j=1}^{N} K_j b_j, F\right)_{H^{-s}(U)}$$

$$= \sum_{j=1}^{N} b_j (K_j, F)_{H^{-s}(U)}$$

which implies that

$$(A^{\top})^{*} F = \left\{(K_j, F)_{H^{-s}(U)}\right\} \in \mathbf{R}^N$$

Note here that while $A^{\top}$ denotes the **transpose** of A, $(A^{\top})^{*}$ denotes the **adjoint** of $A^{\top}$. In particular, $\left(A^{\top}\vec{b}, F\right)_{H^{-s}(U)}$ is the inner product on the Hilbert space $H^s(U)$ and $\left\langle \Delta a, A^{\top}\vec{b} \right\rangle_{H^s \times H^{-s}}$ denotes the duality pairing for $H^s(U) \times H^{-s}(U)$

Now, in order to solve the equation

$$A^{\top}\vec{\phi}(x_0) = \delta(\bullet - x_0) = \delta_{x_0}$$

we have to acknowledge that in general there will be no solution. However, the normal equation

$$(A^{\top})^{*} A^{\top}\vec{\phi}(x_0) = (A^{\top})^{*}\delta(\bullet - x_0) = (A^{\top})^{*}\delta_{x_0}$$

is always uniquely solvable. Note first that

$$A^{\top}\vec{\phi}(x_0) = \sum_{j=1}^{N} \phi_j(x_0) K_j \in H^{-s}(U)$$

and

$$(A^{\top})^{*}A^{\top}\vec{\phi}(x_0) = \left\{(K_i, A^{\top}\vec{\phi}(x_0))_{H^{-s}(U)}\right\} \in \mathbf{R}^N$$

$$= \left\{\left(K_i, \sum_{j=1}^{N} \phi_j(x_0) K_j\right)_{H^{-s}(U)}\right\}$$

$$= \sum_{j=1}^{N} \phi_j(x_0)\,(K_i, K_j)_{H^{-s}(U)}$$

That is,

$$\left[(K_i, K_j)_{H^{-s}(U)}\right]\vec{\phi}(x_0) = (A^{\top})^{*}\delta_{x_0}$$

Now

$$(A^{\top})^{*}\delta_{x_0} = \left\{(K_j, \delta_{x_0})_{H^{-s}(U)}\right\}$$

and

$$(K_j, \delta_{x_0})_{H^{-s}(U)} = \langle JK_j, \delta_{x_0}\rangle_{H^{s}\times H^{-s}}$$

where

$$J : H^{-s}(U) \to H^{s}(U)$$

denotes the duality isomorphism. Here J is given by

$$JK_i(x) = \sum_{k=1}^{\infty} \lambda_k^{-s}(K_i, w_k)_0 w_k(x)$$

where

$\{w_k\}$ denotes an orthonormal basis of eigenfunctions for $-\nabla^2$ on $H_0^1(U)$

$$\lambda_k w_k = -\nabla^2 w_k \qquad \textit{for all } k \geq 1,$$

and

$$(K_i, w_k)_0 = \int_U K_i(x)\; w_k(x)\,dx$$

The choice of the eigenfunctions for $-\nabla^2$ on $H_0^1(U)$ as the orthonormal basis for defining the spaces $H^s(U)$ is one of convenience. The eigenvalues are all strictly positive and the eigenfunctions can be easily constructed or approximated. Other choices are possible here, and advantages of alternative choices is something that should be considered.

Then

$$\langle JK_j, \delta_{x_0}\rangle_{H^{s}\times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s}(K_i, w_k)_0 w_k(x_0)$$

Similarly,

$$(K_i, K_j)_{H^{-s}(U)} = \langle JK_i, K_j\rangle_{H^{s}\times H^{-s}} = \sum_{k=1}^{\infty} \lambda_k^{-s}(K_i, w_k)_0\,(K_j, w_k)_0$$

In order to approximate $\Delta a(x_0)$ we generate

$$M = \left[(K_i, K_j)_{H^{-s}(U)}\right] \quad \text{and} \quad \vec{d} = \left[\langle JK_j, \delta_{x_0}\rangle_{H^{s}\times H^{-s}}\right]$$

and solve

$$M[\vec{\varphi}(x_0)] = \vec{d}$$

Then

$$\begin{aligned}\Delta a(x_0) &= [\vec{\varphi}(x_0)] \bullet \vec{\mu} \\ &= [\vec{\varphi}(x_0)] \bullet \left[(\Delta g(s), \theta_j(s))_{H^0(S_2)}\right] \\ &= \left[M^{-1}\vec{d}\right] \bullet \left[(\Delta g(s), \theta_j(s))_{H^0(S_2)}\right]\end{aligned}$$

This can be carried out at a grid of points x_0 in U in order to provide an approximation for $\Delta a(x) = a_1(x) - a_2(x)$ and, since $a_2(x)$ is given, this provides an approximation for the unknown conductivity coefficient $a_1(x)$. Of course the computation here requires solving the adjoint problem, which in turn requires knowledge of the conductivity coefficient $a_1(x)$. Therefore this identification algorithm would have to be carried out iteratively. A possible design for such an iteration scheme follows:

Algorithm

Given data: $f = f^*$ $\quad g = g^*$

Initial guess: $a = a_0$

1. **Direct Problem**

$$a(x) = a_n(x)$$
$$u_2(x) = u(f^*, a_n) \longmapsto g_2 = g_2(a_n)$$

$$\textit{if} \quad \|g^* - g_2\| \le \varepsilon \quad \textit{then} \quad \textit{stop}$$

2. **Adjoint Problem**

$$v_j = v(\theta_j, a_n)$$
$$K_j = \nabla u_2 \bullet \nabla v_j$$
$$\mu_j = (g^* - g_2, \theta_j)$$
$$1 \le j \le N$$

3. **Least Squares Problem**

$$[K_{ij}]$$
$$[b_j(x_0)]$$
$$\Delta a(x_0) = \mu^T [K_{ij}]^{-1} \vec{b}(x_0)$$

4. **Iterate:** $\quad a_{n+1}(x) = a_n(x) + \Delta a(x) =: T[a_n]$

return to 1

This algorithm can be viewed as producing successive refinements to the initial guess for the coefficient a(x). Note that if the initial guess a_0 is chosen to be a constant, then the corresponding solution of the direct problem $u_2(x) = u(f^*, a_0)$ is independent of the value of the constant. Then $g_2 = g_2(a_0) = a_0 \partial_N u_2$ and we can choose a_0 such that $\|g^* - g_2\| = 0$; i.e.,

$$a_0^2 \|\partial_N u_2\|^2 - 2a_0(\partial_N u_2, g^*) + \|g^*\|^2 = 0.$$

Then a_0 can be viewed as a first approximation to the unknown coefficient $a(x)$ and the iterates produced by this algorithm can be interpreted as refinements of the approximation.

Step 2 of the algorithm requires the selection of N linearly independent but otherwise arbitrary functions, θ_j, to serve as input to the adjoint problem. These functions are defined on the boundary of the domain, and the most natural choice would be to divide the boundary into N disjoint segments and choose each data function to have its support on one of the segments. These functions would, of course, be linearly independent but it remains to be seen whether this choice is most effective in terms of leading to an accurate approximate solution.

The solutions to the adjoint problem corresponding to the N chosen input functions θ_j provide the input for the next phase of the algorithm, solving the least squares system of equations. Once this system is solved, the unknown coefficient can be evaluated at any point in the coefiicient domain U. At the n-th iteration of the algorithm, the domain U has been divided into a number of cells, M(n), and the coefficient is treated as constant on each cell. For the next iteration the division of U could be refined by increasing the number of cells to M(n+1) and the values of the piecewise constant coefficient would then be generated by using the output of step 4 to evaluate the coefficient at the center of each new cell.

One of the advantages of this approximation procedure, is that it is possible to give a clear interpretation for what the approximate solution represents.

Interpretation of Δa_N

We have that,

$$\Delta a_N(x_0) = \vec{\mu} \bullet \vec{\Phi}(x_0) = \vec{\mu} \bullet [K]^{-1}\vec{b}(x_0) = \vec{W} \bullet \vec{b}(x_0)$$

Since $x_0 \in \Omega$ is arbitrary, and $\vec{W}$ does not depend on x_0, $\vec{W} \bullet \vec{b}(x_0)$ is just a linear combination of the functions $b_j(x_0)$, $1 \le j \le N$. i.e.,

$$\Delta a_N(x) \in H_N^s = span\{b_1(x), ..., b_N(x)\} \subset H^s(\Omega)$$

Lemma 3.1.1 $\Delta a_N(x)$ is the Hilbert space projection of $\Delta a(x)$ from $H^s(\Omega)$ into H_N^s

proof- Let $$P[\Delta a] = \sum_{i=1}^{N} \alpha_i b_i$$

where the α_i are chosen such that $(\Delta a, b_j)_{H^s} = (P[\Delta a], b_j)_{H^s}$
Note that

$$(P[\Delta a], b_j)_{H^s} = \sum_{i=1}^{N} \alpha_i (b_i, b_j)_{H^s}$$

and

$$(\Delta a, b_j)_{H^s} = \sum_m (K_j, w_m)_0 (\Delta a, w_m)_0 = \langle \Delta a, K_j \rangle = \mu_j$$

Also $$(b_i, b_j)_{H^s} = (K_i, K_j)_{H^{-s}} = \langle J^{-1} K_i, K_j \rangle = K_{ij}$$

so that $$\sum_{i=1}^{N} K_{ij} \alpha_i = \mu_j; \qquad i.e., \qquad [K]\vec{\alpha} = \vec{\mu}$$

Then $$\vec{\alpha} = [K]^{-1} \vec{\mu}$$

and $$P[\Delta a] = \vec{\alpha}^{\mathsf{T}} \vec{b} = \left([K]^{-1} \vec{\mu}\right)^{\mathsf{T}} \vec{b}$$

$$= \mu^{\mathsf{T}} [K]^{-1} \vec{b} = \vec{W} \bullet \vec{b} = \Delta a_N$$

When s=0 this projection is just the L^2 projection of $\Delta a(x)$ into the subspace spanned by the N vectors $b_j(x)$. When $s > 0$, then the projection is a smoothing projection into the subspace H_N^s.

Lemma 3.1.2 $\quad A[\Delta a_N] = A[\Delta a]$

proof- Note that $$A[\Delta a_N] = \{\langle \Delta a_N, K_1 \rangle, \ldots, \langle \Delta a_N, K_N \rangle\}$$

and $$\langle \Delta a_N, K_j \rangle = A[\Delta a_N]_j$$

$$= \left\langle \sum_{j=1}^{N} \mu_j \Phi_j(y), K_j(y) \right\rangle = \sum_{j=1}^{N} \mu_j \langle \Phi_j(y), K_j(y) \rangle$$

But $$\Phi_j(y) = \left([K]^{-1} \vec{b}(y)\right)_j$$

hence $$A[\Delta a_N]_j = \sum_{j=1}^{N} \mu_j \left\langle \left([K]^{-1} \vec{b}(y)\right)_j, K_j(y) \right\rangle = \sum_{j=1}^{N} \mu_j C_{ij}$$

Now $$C_{ij} = \left\langle \left([K]^{-1} \vec{b}(y)\right)_j, K_j(y) \right\rangle = \left\langle \sum_{k=1}^{N} K_{jk}^{-1} b_k(y), K_j(y) \right\rangle$$

$$= \sum_{k=1}^{N} K_{jk}^{-1} \langle b_k(y), K_j(y) \rangle$$

and

$$\langle b_k(y), K_j(y)\rangle = \sum_m \lambda_m^{-s}\,(K_k, w_m)_0\,(K_i, w_m)_0 = (K_k, K_i)_{H^{-s}} = K_{ki}$$

Then $$C_{ij} = \sum_{k=1}^{N} K_{jk}^{-1} K_{ki} = \delta_{ji}$$

and $$A[\Delta a_N]_j = \sum_{j=1}^{N} \mu_j C_{ij} = \mu_j = A[\Delta a]_j$$

Evidently, A maps Δa and Δa_N to the same N-tuple of data values.

3.4) Identification of a State Dependant Conductivity

For purposes of comparing the application of this approach to an inverse problem involving a position dependent unknown to an inverse problem with a state dependent unknown, we consider the following situation. Consider the boundary value problem :

$$\begin{aligned} div[a(u(x))\nabla u(x) &= 0 \quad && in \;\; U \\ u &= f \quad && on\; S \\ a(u)n \bullet \nabla u &= g \quad && on\; S \end{aligned}$$

Where $U \subset R^n$, $f \in H^{1/2}(S)$ *and* $g \in H^{-1/2}(S)$ are as in the previous example. Here the conductivity is assumed to be a function of the state variable u rather than a function of the position x.

With $f \in H^{1/2}(S)$ fixed, suppose that the pair $\{u_1, a_1\}$ satisfies the first two conditions listed above and produces the data $u_1|_S = g_1 \in H^{-1/2}(S)$ and the pair $\{u_2, a_2\}$ is similarly associated with $u_2|_S = g_2 \in H^{-1/2}(S)$. Then

$$\begin{aligned} div(a_1(u_1)\nabla u_1(x) - a_2(u_2)\nabla u_2(x)) &= 0 \quad && in \;\; U \\ u_1 - u_2 &= 0 \quad && on\; S \\ n \bullet [a_1(u_1)\nabla u_1 - a_2(u_2)\nabla u_2(x)] &= g_1 - g_2 \quad && on\; S \end{aligned}$$

Let
$$b_i(u) = \int_0^u a_i(s)ds$$

Integrate the differential equation by parts and attention to the boundary constraints leads to the integral identity,

$$\int_U \Delta a(u_2)\,\nabla u_2(x) \bullet \nabla v(x)\; dx = \int_S \Delta g(s)\,\theta(s)\; ds$$

where $v = v(x)$ denotes the solution of the adjoint problem,

$$P(x)\Delta v(x) = 0 \quad in \;\; U$$
$$v = \theta \quad on \;\; S$$

Where

$$P(x) = \frac{b_1(u_2) - b_1(u_1)}{u_2 - u_1}$$

Now the algorithm to construct an approximation to the coefficient $a_1 = a_1(u)$ associated with data f *and* g_1 proceeds as in the previous example.

We suppose that a_2 is some **given** coefficient and that the pair $\{u_2, a_2\}$ satisfies the BVP above with data f and generates the over-specified data $u_2 = g_2 \;\; on \;\; S$. Then we proceed to apply the algorithm as before but now applying the method to the integral identity derived in this example.

In particular, to approximate $\Delta a(u_2(x_0))$ we generate

$$M = \left[(K_i, K_j)_{H^{-s}(U)}\right] \qquad \text{and} \qquad \vec{d} = [\langle JK_j, \delta_{x_0}\rangle_{H^s \times H^{-s}}]$$

and solve

$$M[\vec{\varphi}(x_0)] = \vec{d}$$

Then

$$\begin{aligned}\Delta a(u_2(x_0)) &= [\vec{\varphi}(x_0)] \bullet \vec{\mu} \\ &= [\vec{\varphi}(x_0)] \bullet \left[(\Delta g(s), \theta_j(s))_{H^0(S_2)}\right] \\ &= \left[M^{-1}\vec{d}\right] \bullet \left[(\Delta g(s), \theta_j(s))_{H^0(S_2)}\right]\end{aligned}$$

This can be carried out at a grid of points x_0 in U in order to provide an approximation for

$$\Delta a(u_2(x)) = a_2(u_2(x)) - a_1(u_2(x))$$

Since a_2 is given, this provides an approximation for the unknown coefficient $a_1(u_2(x))$. In this situation, however, $a_1(u_2(x))$ is determined on the interval of values between, $f_{\min}$ and $f_{\max}$, the data values on the boundary of U. Therefore grid of points x_0 in U must be chosen very close to the boundary of U, in which case, $a_1(u_2(x) \approx a_1(f(x))$. In addition, the experiment must be constructed in such a way that the boundary function, $f(t)$, is neither constant nor nearly constant, since this would limit the identification of the coefficient to a single point or small sub interval in its domain. This example illustrates that the method described in this thesis is more suited to the treatment of inverse problems with spatially dependent unknowns than to problems with state dependent unknowns.

CHAPTER IV

Application of the Backus-Gilbert Method to the Simultaneous Identification of 2 Coefficients

This chapter describes the application of the Backus-Gilbert method to an inverse problem involving the simultaneous identification of two unknown functions from overspecified data. Superficially, the treatment of the identification of mulitple unknowns is identical to the identification of a single unknown. However, the treatment of multiple unknowns requires that the single unknown techniques be extended in an appropriate way. An example of a physical problem which leads to an inverse problem in two unknown functions will be used to illustrate how this extension must proceed.

1. Identification of Unknown Storage and Transmissivity Coefficients

The partial differential equation governing unsteady groundwater flow for an isotropic, heterogeneous confined aquifer, $U \subset R^2$ is

$$S(x,y)\,\partial_t h(x,y,t) - \nabla(T(x,y)\nabla h(x,y,t)) = -q_w(x,y)$$

where

$$\begin{aligned} U &= \text{2-dim region occupied by the aquifer} \\ \Gamma &= \Gamma_1 \cup \Gamma_2 = \text{boundary of } U \\ h(x,y,t) &= \text{hydraulic head in } U_T = U \times (0,T) \\ T(x,y) &= \text{transmissivity in } U \\ S(x,y) &= \text{storage coefficient in } U \\ q_w(x,y) &= \text{withdrawl term in } U \end{aligned}$$

The unknown hydraulic head function is subject to the following initial and boundary conditions

$$\begin{aligned} h(x,y,0) &= h_0(x,y) && in \quad U \subset R^2 \\ h(x,y,t) &= F(x,y,t) && on \quad \Gamma_1 \times (0,T) \\ T\partial_N h(x,y,t) &= G(x,y,t) && on \quad \Gamma_2 \times (0,T) \end{aligned}$$

The partial differential equation together with these three auxiliary conditions, comprise the "direct problem".

Now consider the problem of determining the hydraulic functions S and T from extra data measured inside or on the boundary of the the flow domain U. If $h_1(x,y,t)$ denotes the solution of the direct problem corresponding to hydraulic coefficients $S_1,\ T_1$ while coefficients $S_2,\ T_2$ lead to the solution $h_2(x,y,t)$, then

$$S_1\,\partial_t(h_1 - h_2) - \nabla(T_1\nabla(h_1 - h_2)) = \nabla((T_1 - T_2)\nabla h_2) - (S_1 - S_2)\,\partial_t h_2.$$

Multiplying this equation by a test function $v(x,y,t)$ and integrating by parts, leads to

$$\int_U \Delta h(x,y,T) S_1(x,y)\, v(x,y,T)\, dU - \int\int_{U_T} \Delta h[S_1 \partial_t v + \nabla(T_1 \nabla v)] dU dt$$

$$+ \int_0^T \int_{\Gamma_2} (\Delta h) T_1 \partial_N v\, dS dt - \int_0^T \int_{\Gamma_1} \Delta(T \partial_N h)\, v\, dS dt =$$

$$= \int\int_{U_T} [-\Delta S\ v \partial_t h_2 + \Delta T \nabla v \bullet \nabla h_2] dU dt$$

Now suppose that the test function $v = v(x,y,t)$ satisfies the adjoint problem,

$$\begin{aligned} S_1 \partial_t v + \nabla(T_1 \nabla v) &= H^*(x,y,t) && \text{in } U_T \\ v(x,y,T) &= 0 \\ v(x,y,t) &= F^*(x,y,t) && \text{on } \Gamma_1 \times (0,T) \\ T_1 \partial_N v &= G^*(x,y,t) && \text{on } \Gamma_2 \times (0,T) \end{aligned}$$

We can choose the data in this adjoint problem to correspond to different choices of overspecified data from which to determine the unknown hydraulic parameters. For example, suppose that the following data is given,

$$\begin{aligned} h(x,y,t) &= H(x,y,t) && \text{on} \quad U_1 \times (0,T) \quad \text{for some subset } U_1 \subset U \\ h(x,y,t) &= f(x,y,t) && \text{on} \quad \Gamma_2 \times (0,T) \end{aligned}$$

Then, if we choose the data in the adjoint problem such that

$$\begin{aligned} &H^*(x,y,t) \text{ is given on } U_1 \times (0,T), \text{ but } H^* = 0 \text{ outside } \quad U_1 \times (0,T) \\ &F^*(x,y,t) = 0 \\ &G^*(x,y,t) \text{ is given on } \Gamma_2 \times (0,T) \end{aligned}$$

the integral identity above reduces to

$$-\int_0^T \int_{U_1} \Delta H\, H^* dU dt + \int_0^T \int_{\Gamma_2} (\Delta f)\, G^*\, dS dt = \int\int_{U_T} [-\Delta S\ v \partial_t h_2 + \Delta T \nabla v \bullet \nabla h_2] dU dt$$

In fact, if we let $v_1(x,y,t)$ denote the solution of the adjoint problem corresponding to data, $H^*(x,y,t) = F^*(x,y,t) = 0$, but nontrivial $G^*(x,y,t)$ and let $v_2(x,y,t)$ denote the solution of the adjoint problem corresponding to data, $G^*(x,y,t) = F^*(x,y,t) = 0$, but nontrivial $H^*(x,y,t)$, then we obtain the pair of identities,

$$\int\int_{U_T} [-\Delta S\ v_1 \partial_t h_2 + \Delta T \nabla v_1 \bullet \nabla h_2] dU dt = \int_0^T \int_{\Gamma_2} (\Delta f)\, G^*\, dS dt$$

$$\iint_{U_T}[-\Delta S\, v_2\,\partial_t h_2 + \Delta T\,\nabla v_2 \bullet \nabla h_2]\,dU dt = -\int_0^T\int_{U_1} \Delta H\, H^* dU dt.$$

It is evident that the choice of data in the adjoint problem has the effect of selecting the coefficient difference which will appear in the integral identity.

Since S and T are independent of t, we can rewrite these identities in the form

$$\int_U [\Delta S\, K_{11} + \Delta T\, K_{12}]\, dU = \int_0^T\int_{\Gamma_2} (\Delta f)\, G^*\, dS dt$$

$$\int_U [\Delta S\, K_{21} + \Delta T\, K_{22}]\, dU = -\int_0^T\int_{U_1} \Delta H\, H^* dU dt$$

where

$$K_{11}(x,y) = -\int_0^T v_1\,\partial_t h_2\, dt, \qquad K_{12}(x,y) = \int_0^T \nabla v_1 \bullet \nabla h_2\, dt,$$

$$K_{21}(x,y) = -\int_0^T v_2\,\partial_t h_2\, dt, \qquad K_{22}(x,y) = \int_0^T \nabla v_2 \bullet \nabla h_2\, dt.$$

These two integral identities are special cases of the general situation in which a pair of unknown coefficients are to be determined from overspecified data by means of integral identities which relate changes in the coefficients to changes in the overspecifications. Identities involving other choices for the measured data from which the coefficients are to be determined can be obtained by altering the structure of the adjoint problem.

2. Approximating a Solution to the Inverse Problem

For the discussion to follow we will denote the unknown coefficients by $a(p)$ and $b(p)$ where p will denote an generic point in the domain of the coefficients. For example, $a(p) = S(p),\ b(p) = T(p)$, with $p = (x,y) \in U.$

The identities and the resulting algorithm involve differences $\Delta a(p) = a_1(p) - a_2(p)$, and $\Delta b(p) = b_1(p) - b_2(p)$, where we assume $a_2(p),\ b_2(p)$ are assumed to be known coefficients and the unknown coefficients are denoted by $a_1(p),\ b_1(p)$. Obviously, determining the differences $\Delta a(p)$ and $\Delta b(p)$ is then equivalent to determining $a_1(p),\ b_1(p)$.

We will suppose that we solve the adjoint problem for N different choices of the data G^* and H^* and we let

$$\mu_{1,j} = \int_0^T\int_{\Gamma_2} (\Delta f)\, G_j^*\, dS dt \qquad \text{and} \qquad \mu_{2,j} = -\int_0^T\int_{U_1} \Delta H\, H^* dU dt.$$

Also, we make use of the previously defined notation,

$$\iint_{U_T}[-\Delta S\, v_{1,j}\,\partial_t h_2 + \Delta T\,\nabla v_{1,j} \bullet \nabla h_2]\,dU dt = \int_U \Delta a(p) K_{11,j}(p)\, dp + \int_U \Delta b(p) K_{12,j}(p)\, dp$$

$$\iint_{U_T}[-\Delta S\ v_{2,j}\,\partial_t h_2 + \Delta T\,\nabla v_{2,j}\bullet\nabla h_2]\,dU\,dt = \int_U \Delta a(p)K_{21,j}(p)\ dp + \int_U \Delta b(p)K_{22,j}(p)\ dp$$

where $v_{1,j}$ denotes the solution of the adjoint problem corresponding to data G_j^*, and $v_{2,j}$ is the solution of the adjoint problem corresponding to data H_j^*, $j = 1,\dots,N$. Then we have the integral identities

$$\iint_{U_T}\Delta a(p)K_{11,j}(p)\ dp + \iint_{U_T}\Delta b(p)K_{12,j}(p)\ dp = \mu_{1,j} \qquad 1 \le j \le N$$

$$\iint_{U_T}\Delta a(p)K_{21,j}(p)\ dp + \iint_{U_T}\Delta b(p)K_{22,j}(p)\ dp = \mu_{2,j} \qquad 1 \le j \le N$$

That is,

$$\iint_{U_T}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}_j(p)\begin{bmatrix}\Delta a(p) \\ \Delta b(p)\end{bmatrix}dp = \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}_j \qquad 1 \le j \le N$$

i.e., $\iint_{U_T}[K_j(p)]\,\Delta\vec{a}(p)\,dp = \vec{\mu}_j$, for $1 \le j \le N$

Note that if the data functions $G_j^*, j = 1,\dots,N$, form a set of linearly independent functions, then $K_{11,j}(p)$, and $K_{12,j}(p), j = 1,\dots,N$, are also linearly independent. A similar statement is true of the data functions H_j^*, $j = 1,\dots,N$, and the corresponding functions $K_{21,j}(p)$ and $K_{22,j}(p)$.

Here we are going to assume that

i) $\Delta\vec{a}(p) \in H^s(U_T)^{2\times 1} \qquad s \ge 2$
ii) $[K_j(p)] \in H^0(U_T)^{2\times 2} \subset H^{-s}(U_T)^{2\times 2}$

Now for $p \in U$ fixed, suppose we can find functions $\Phi_{ij}(p)$ such that

$$\Delta\vec{a}(p) = \begin{bmatrix}\Delta a(p) \\ \Delta b(p)\end{bmatrix} = \sum_{j=1}^{N}\begin{bmatrix}\Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22}\end{bmatrix}_j(p)\begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix}_j = \sum_{j=1}^{N}[\Phi_j(p)]\,\vec{\mu}_j$$

$$= \sum_{j=1}^{N}[\Phi]_j(p)\iint_{U_T}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22}\end{bmatrix}_j(q)\begin{bmatrix}\Delta a(q) \\ \Delta b(q)\end{bmatrix}dq$$

$$= \iint_{U_T}\sum_{j=1}^{N}[\Phi]_j(p)[K]_j(q)[\Delta\vec{a}(q)]\,dq$$

This is evidently equivalent to the statement,

$$\sum_{j=1}^{N}[\Phi]_j(p)[K]_j(q) = \begin{bmatrix} \delta(p-q) & 0 \\ 0 & \delta(p-q) \end{bmatrix}$$

or,

$$\sum_{j=1}^{N}[\Phi_j(p)][K_j(q)] = \delta(p-q)\left[\vec{E}_1, \vec{E}_2\right]$$

where $\vec{E}_1 = [1,0]$ and $\vec{E}_2 = [0,1]$.

Definitions of Abstract Mappings

1. The mapping A

Define

$$A : H^s(\Gamma)^{2\times 1} \to R^{2\times N}$$

by

$$A\begin{bmatrix} \Delta a(p) \\ \Delta b(p) \end{bmatrix} = \left\{\int\int_{U_T}\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}_j(p)\begin{bmatrix} \Delta a(p) \\ \Delta b(p) \end{bmatrix}dp\right\}$$

$$= \left\{\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_j\right\}_{1\le j\le N},$$

i.e.,

$$A[\Delta\vec{a}] = \{\langle[K_1],\Delta\vec{a}\rangle,\langle[K_2],\Delta\vec{a}\rangle,\ldots,\langle[K_N],\Delta\vec{a}\rangle\} = \{\vec{\mu}_1,\ldots,\vec{\mu}_N\}$$

where $$\langle[K_j],\Delta\vec{a}\rangle = \int\int_{U_T}[K]_j(p)\,\Delta\vec{a}(p)\,dp.$$

i.e., $\langle\bullet,\bullet\rangle$ denotes the duality pairing on $H^{-s}\times H^s$.

Duality pairing on R^{2N}

Define

$$\{\vec{\mu}_1,\ldots,\vec{\mu}_N\} \bullet \begin{Bmatrix} \vec{b}_1 \\ \vdots \\ \vec{b}_N \end{Bmatrix} = \sum_{j=1}^{N}\vec{\mu}_j \bullet \vec{b}_j = \sum_{j=1}^{N}(\mu_{1,j}b_{1,j} + \mu_{2,j}b_{2,j})$$

for the duality pairing between $R^{2\times N}$ and its dual, which we denote $R_{N\times 2}$. Note that

$$\sum_{j=1}^{N}(\mu_{1,j}b_{1,j} + \mu_{2,j}b_{2,j}) = tr\left\{\vec{b}_1,\ldots,\vec{b}_N\right\} \bullet \{\vec{\mu}_1,\ldots,\vec{\mu}_N\}$$

where, $tr\left\{\vec{b}_1,\ldots,\vec{b}_N\right\} \bullet \{\vec{\mu}_1,\ldots,\vec{\mu}_N\}$ denotes the trace of the N by N matrix that is the result of the product $\left\{\vec{b}_1,\ldots,\vec{b}_N\right\} \bullet \{\vec{\mu}_1,\ldots,\vec{\mu}_N\} = \left[\vec{b}_i^T\vec{\mu}_j\right]_{1\le i,j\le N}$.

2. Transpose of A

Let A^{T} denote the transpose of the mapping A. Then

$$A^{\mathsf{T}} : R_{N\times 2} \rightarrow (H^s(I)^{2\times 1})^* = H^{-s}(I)^{1\times 2}$$

is defined by $$A[\Delta\vec{a}] \bullet \left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}} = \left\langle \Delta\vec{a}, A^{\mathsf{T}}\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}}\right\rangle_{H^s\times H^{-s}}$$

But $$A[\Delta\vec{a}] \bullet \left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}} = \{\vec{\mu}_{1,\dots,}\vec{\mu}_N\} \bullet \left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}}$$

$$= tr\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\} \bullet \{\vec{\mu}_{1,\dots,}\vec{\mu}_N\}$$

$$= tr\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\} \bullet \{\langle [K_1], \Delta\vec{a}\rangle, \langle [K_2], \Delta\vec{a}\rangle, \dots, \langle [K_N], \Delta\vec{a}\rangle\}$$

$$= \sum_{j=1}^{N} \langle [K_j]\Delta\vec{a}\rangle \bullet \vec{b}_j = \left\langle \sum_{j=1}^{N} \vec{b}_j[K_j], \Delta\vec{a}\right\rangle.$$

It is evident that $$A^{\mathsf{T}}\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}} = \sum_{j=1}^{N} \vec{b}_j[K_j] \in H^{-s}(I)^{1\times 2}$$

3. Adjoint of A^{T}

Next, define $(A^{\mathsf{T}})^* : H^{-s}(I)^{1\times 2} \rightarrow R_{N\times 2}$

by $$\left((A^{\mathsf{T}})^*(\vec{F}), \left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}\right)_{R_{N\times 2}} = \left(\vec{F}, A^{\mathsf{T}}\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}}\right)_{H^{-s}(I)^{1\times 2}}$$

Since $$\left(\vec{F}, A^{\mathsf{T}}\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}}\right)_{H^{-s}(I)} = \left(\vec{F}, \sum_{j=1}^{N} \vec{b}_j[K_j]\right)_{H^{-s}(I)^{1\times 2}}$$

$$= \sum_{j=1}^{N} \vec{b}_j\left(\vec{F}, [K_j]\right)_{H^{-s}(I)^{1\times 2}}$$

$$= tr\left\{\vec{b}_{1,\dots,}\vec{b}_N\right\} \bullet \left\{\left(\vec{F}, [K_1]\right), \dots, \left(\vec{F}, [K_N]\right)\right\}$$

$$= \left\{\left(\vec{F}, [K_1]\right)_{H^{-s}(I)^{1\times 2}}, \dots, \left(\vec{F}, [K_N]\right)_{H^{-s}(I)^{1\times 2}}\right\} \bullet \left\{\vec{b}_{1,\dots,}\vec{b}_N\right\}^{\mathsf{T}}$$

it follows that

$$(A^{\mathsf{T}})^*\vec{F} = \left\{\left(\vec{F}, [K_1]\right)_{H^{-s}(I)^{1\times 2}}, \dots, \left(\vec{F}, [K_N]\right)_{H^{-s}(I)^{1\times 2}}\right\}^{\mathsf{T}} \in R_{N\times 2}$$

Note that $(A^{\mathsf{T}})^*$ denotes the adjoint of A^{T} as opposed to the transpose of A^{T} (which is, of course, just A). This is a consequence of the fact that $(A^{\mathsf{T}})^*$ is defined in terms of the H^{-s} inner product instead of the $H^s \times H^{-s}$ duality pairing used to define A^{T}.

Least Squares Solution to the Equation
Now we wish to solve the equation

$$\sum_{j=1}^{N}[\Phi_j(p)][K_j(q)] = \delta(p-q)\left[\vec{E}_1,\vec{E}_2\right]$$

In order to express this equation in terms of the mappings just defined, note that the equation is equivalent to the two equations,

$$A^{\mathsf{T}}[R_k(p)] = \delta(p-q)\vec{E}_k, \qquad k = 1,2 \tag{4.1}$$

where $R_k(p)$ denotes the N by 2 array whose rows are the k-th rows of the N two by two matrices $\Phi_j;\ 1 \leq j \leq N.$

$$\text{i.e., } R_k(p) = \left\{\left[\text{k-th Row of } \Phi_j\right]\right\} = \left\{\begin{matrix} \left[\begin{matrix}\Phi_{k,1} & \Phi_{k,2}\end{matrix}\right]_1 \\ \vdots \\ \left[\begin{matrix}\Phi_{k,1} & \Phi_{k,2}\end{matrix}\right]_N \end{matrix}\right\} \qquad k = 1,2$$

Evidently, the first column of $R_1(p)$ contains the (1,1) entry of all N of the matrices $\Phi_j(p)$, while the second column contains the (1,2) entries of the N matrices $\Phi_j(p)$. Similarly, the 2 columns of $R_2(p)$ contain the (2,1) and (2,2) entries of the N $\Phi-\mathit{matrices}$. For future reference, let us denote the i-th column of R_j by C_{ji}.

In general, the equation (4.1) will have no solution R_k. Therefore, we will solve instead the associated least squares equations which are always uniquely solvable,

$$(A^{\mathsf{T}})^{*}A^{\mathsf{T}}[R_k(p)] = (A^{\mathsf{T}})^{*}\left(\delta(p-q)\vec{E}_k\right), \qquad k = 1,2. \tag{4.2}$$

Now $\quad A^{\mathsf{T}}[R_k(p)] = \sum_{j=1}^{N}[R_k(p)]_j[K_j]$

and $\quad (A^{\mathsf{T}})^{*}\vec{F} = \left\{\left(\vec{F},[K_1]\right)_{H^{-s}(\Gamma)^{1\times 2}},\ldots,\left(\vec{F},[K_N]\right)_{H^{-s}(\Gamma)^{1\times 2}}\right\}$

so $\quad (A^{\mathsf{T}})^{*}A^{\mathsf{T}}[R_k(p)] = \left\{\sum_{j=1}^{N}[R_k(p)]_j([K_j]^{\mathsf{T}},[K_i])_{H^{-s}}\right\}_{i=1,\ldots,N}$

This may be expressed as

$$(A^{\mathsf{T}})^{*}A^{\mathsf{T}}[R_k(p)] = \left\{[M_{ij}]\begin{bmatrix}\Phi_{k,1}^{(1)} \\ \vdots \\ \Phi_{k,1}^{(N)}\end{bmatrix},[M_{ij}]\begin{bmatrix}\Phi_{k,2}^{(1)} \\ \vdots \\ \Phi_{k,2}^{(N)}\end{bmatrix}\right\}$$

where

$$[M_{ij}] = [([K_i]^{\top},[K_j])_{H^{-s}}] = N \text{ by } N \text{ matrix (symmetric)}$$

Also $\quad (A^{\top})^{*}\left(\delta(p-q)\vec{E}_k\right) = \left\{\left(\delta(p-q)\vec{E}_k,[K_1]\right)_{H^{-s}},\ldots,\left(\delta(p-q)\vec{E}_k,[K_N]\right)_{H^{-s}}\right\}^{\top}$

But

$$\vec{E}_k[K_j] = (K_{k1},K_{k2})_j = \text{k-th row of } [K_j]$$

hence

$$(A^{\top})^{*}\left(\delta(p-q)\vec{E}_k\right) = \left\{\left(\left(\delta(p-q),(K_{k1},K_{k2})_j(q)\right)_{H^{-s}}\right)\right\}_{1\le j\le N}$$

Now $\quad \left(\delta(p-q),(K_{k1},K_{k2})_j(q)\right)_{H^{-s}} = \left\langle \delta(p-q),(J^{-1}K_{k1},JK_{k2})_j(q)\right\rangle_{H^{-s}\times H^{s}}$

where $J : H^{s} \to H^{-s}$ denotes the duality isomorphism. Here J^{-1} is given by

$$J^{-1}F(q) = \sum_{n=1}^{\infty} \lambda_n^{-s}(F,w_n)_0 w_n(q)$$

where $\{w_n(q)\}$ denotes a convenient orthonormal basis for $H^0(U_T)$. Then

$$(F,w_n)_0 = \int_{U_T} F(q)w_n(q)\,dq$$

and

$$\left\langle \delta(p-q),(J^{-1}K_{k1},JK_{k2})_j(q)\right\rangle_{H^{-s}\times H^{s}} = \sum_{n=1}^{\infty} \lambda_n^{-s}((K_{k1},w_n)_0,(K_{k1},w_n)_0)_j w_n(p)$$

$$= \{d_{k1},d_{k2}\}_j$$

Similarly $\quad ([K_i]^{\top},[K_j])_{H^{-s}} = \langle [K_i]^{\top},J^{-1}[K_j]\rangle_{H^{-s}\times H^{s}}$

$$= \sum_{n=1}^{\infty} \lambda_n^{-s}([K_i]^{\top},w_n)_0([K_j],w_n)_0$$

Therefore, to solve

$$(A^{\top})^{*}A^{\top}[R_k(p)] = (A^{\top})^{*}\left(\delta(p-q)\vec{E}_k\right), \qquad k = 1,2.$$

we form $\quad [M_{ij}] = [([K_i]^{\top},[K_j])_{H^{-s}}] = \left[\sum_{n=1}^{\infty} \lambda_n^{-s}([K_i]^{\top},w_n)_0([K_j],w_n)_0\right]$

and $\quad \left[\vec{d}_1,\vec{d}_2\right]_k = \left\langle \delta(p-q),(JK_{k1},JK_{k2})_j(q)\right\rangle_{H^{-s}\times H^{s}}$

$$= \sum_{n=1}^{\infty} \lambda_n^{-s}((K_{k1},w_n)_0,(K_{k2},w_n)_0)_j w_n(p)$$

$$= \left\{ \begin{bmatrix} d_1^{(1)} \\ \vdots \\ d_1^{(N)} \end{bmatrix}, \begin{bmatrix} d_2^{(1)} \\ \vdots \\ d_2^{(N)} \end{bmatrix} \right\}_k$$

Then we solve

$$[M_{ji}] \begin{bmatrix} \Phi_{k,1}^{(1)} \\ \vdots \\ \Phi_{k,1}^{(N)} \end{bmatrix} = \begin{bmatrix} d_1^{(1)} \\ \vdots \\ d_1^{(N)} \end{bmatrix}_k, \qquad [M_{ji}] \begin{bmatrix} \Phi_{k,2}^{(1)} \\ \vdots \\ \Phi_{k,2}^{(N)} \end{bmatrix} = \begin{bmatrix} d_2^{(1)} \\ \vdots \\ d_2^{(N)} \end{bmatrix}_k$$

for k=1,2. This is four systems of N equations in N unknowns, in which the N unknowns are the N $(i,j)-entries$ of the Φ matrices, for i,j=1,2. In terms of the $\vec{C}'_{i,j}s$ this is

$$[M]\vec{C}_{i,j} = \vec{d}_{j,i}, \qquad or \qquad \vec{C}_{i,j} = [M]^{-1}\vec{d}_{j,i}, \qquad 1 \le i,j \le 2$$

Once these solutions are obtained then we have

$$\Delta\vec{a}(p) = \sum_{j=1}^{N} [\Phi_j(p)]\vec{\mu}_j$$

$$\begin{bmatrix} \Delta a(p) \\ \Delta b(p) \end{bmatrix} = \sum_{j=1}^{N} \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}_j (p) \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_j$$

where

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}_j = \begin{bmatrix} \Phi_{1,1}^{(j)} & \Phi_{1,2}^{(j)} \\ \Phi_{2,1}^{(j)} & \Phi_{2,2}^{(j)} \end{bmatrix}$$

In terms of the $\vec{C}'_{i,j}s$ this is

$$\begin{bmatrix} \Delta a(p) \\ \Delta b(p) \end{bmatrix} = \begin{bmatrix} \vec{C}_{1,1} \bullet \vec{\mu}_1 + \vec{C}_{1,2} \bullet \vec{\mu}_2 \\ \vec{C}_{2,1} \bullet \vec{\mu}_1 + \vec{C}_{2,2} \bullet \vec{\mu}_2 \end{bmatrix}$$

or

$$\Delta a(p) = \vec{\mu}_1[M]^{-1}\vec{d}_{1,1}(p) + \vec{\mu}_2[M]^{-1}\vec{d}_{1,2}(p)$$

$$\Delta b(p) = \vec{\mu}_1[M]^{-1}\vec{d}_{2,1}(p) + \vec{\mu}_2[M]^{-1}\vec{d}_{2,2}(p)$$

Here $\vec{\mu}_k$ denotes the N-vector whose entries are $(\mu_k)_j$, $j = 1,\ldots,N$, and $[M]^{-1}$ denotes the inverse of the nonsingular matrix

$$[M_{ij}] = [([K_i]^{\top},[K_j])_{H^{-s}}] = \left[\sum_{n=1}^{\infty} \lambda_n^{-s}([K_i]^{\top},w_n)_0([K_j],w_n)_0\right]$$

Neither of these depend on the point $p \in U$. All that is required in order to evaluate the coefficients at a new point $p \in U$ is to recompute the four p-dependent vectors $\left[\vec{d}_1,\vec{d}_2\right]_k$ and solve again the 4 systems of equations for the 4 entries in the N $\Phi - matrices$.

Conclusions

This thesis describes the combination of a Backus-Gilbert like approach with the integral identities developed in Chapter I. The result is an algorithm for constructing approximate solutions for a class of inverse problems that is different from existing algorithms for problems of this type. This algorithm has the evident advantage that refining the reconstructed unknown does not require additional solutions of the direct problem but can be accomplished with relatively simple additional calculations. Whether this method will, in the end, represent an improvement on known algorithms such as output least squares and others, will only be seen from extensive numerical testing.

In addition to numerical testing, other points that will require additional study include:

1. The role of the eigenfunctions used for the construction of the spaces $H^s(U)$. Is the choice of eigenfunctions significant to the effectiveness of the method?

2. The choice of data in the N adjoint problems solved to generate the functions $v_j(x) = v(x,\theta_j)$ which are, in turn, used to form the functions $K_j(x) = \nabla u_2(x) \bullet \nabla v_j(x)$. Is there some optimal way of choosing the adjoint data to enhance the stability of the inversion?

3. The role of iteration versus refinement; each time a new coefficient is obtained as a result of one complete cycle of the algorithm, it is possible to evaluate this coefficient at any point x_0 in the domain. Is there a limit to how fine this refinement can be made without additional iterations of the algorithm? In particular, is there some limit to the resollution that can be obtained from a given set of measurements. Common sense suggests that there should be a limit, the question then is, how is this limit manifested in this approximation technique?

4. The parameter, s, indicating the smoothness of the space $H^s(U)$ to which the unknown coefficient belongs, plays the role of a "regularizer" in the solution of this ill-posed inverse problem. This role needs clarification.

5. Can this algorithm be compared to already well known algorithms like output least squares?

Appendix

A least squares solution

Consider the equation :

$$Kx = y \qquad \text{(A.1)}$$

where

$$K : H_1 \to H_2$$

is a bounded linear operator on a real Hilbert space H_1, taking values in a real Hilbert space, H_2.
A solution x of (A.1) exists if and only if y $\in R(K)$. Since K is linear , $R(K)$ is a subspace of H_2, however , it generally does not exhaust H_2 . Therefore , we introduce the idea of a ***least squares solution*** . A function $x \in H_1$ is called a ***least squares solution*** of (A.1) if

$$\|Kx - y\| = inf\{\|Ku - y\| : u \in H_1\}.$$

This is equivalent to saying that $Py \in R(K)$, where P is the orthogonal projector of H_2 onto $\bar{R}(K)$, the closure of the range of K . Now , $Py \in R(K)$ if and only if

$$y = Py + (I - P)y \in R(K) + R(K)^{\perp}. \qquad \text{(A.2)}$$

Therefore , a least squares solution exists if and only if y lies in the dense subspace $R(K) + R(K)^{\perp}$ *of* H_2.
Equation (A.2) is equivalent to the condition

$$Kx - y \in R(K)^{\perp} = N(K^*), \qquad \text{dition}$$

that is , ,

$$K^*Kx = K^*y,$$

where $K^*$is the adjoint of K .
where $K^*$is the adjoint of K .

BIBLIOGRAPHY

(1) R.A.Adams , Sobolev Spaces , Academic Press , New York , 1975

(2) G.Allesandrini , Singular solutions of elliptical equations and the determination of conductivity by boundary measurements, J. Differential Equations , 84,pp.252-273,1990

(3) J.Aubin , Applied Functional Analysis ,, :⁾..

(4) G.Backus and F.Gilbert , The resolving power of gross earth data , Geophys.J.R.astr.Soc. 16 , pp. 169-205 , 1968

(5) D.Colton , R.Ewing , and W.Rundell , Inverse Problems in partial differential equations, SIAM,Philadelphia , PA , 1990

(6) J.Christensen-Dalsgaard , J. Schou and M.J. Thompson , A comparison of methods for inverting helioseismic data , Mon.Not.R.Astr.Soc.242, pp. 353-369 , 1990

(7) P.DuChateau , Monotonicity and uniqueness results in identifying an unknown coefficient in a nonlinear diffusion equation , SIAM J.Appl.M,vol 41 , pp 310 - 323 , 1981

(8) P.DuChateau , Introduction to inverse problems in partial differential equations for engineers, scientists and mathematicians Proc.Conf. on Inverse Problems in Geology (Dordrecht : Kluwer) pp 1-50 , 1995

(9) P.DuChateau , Monotonicity and invertibility of coefficient to data mappings in parabolic inverse problems SIAM 26 1473 - 87

(10) P.DuChateau , An inverse problem for hydraulic properties of porous media , SIAM J. Math. Anal. , 28 , pp.611-632 , 1997

(11) H.W. Engl and W.Rundell , Inverse Problems in Diffusion Processes , SIAM,Philadelphia , PA , 1995

(12) L.C. Evans , Partial differential equations , 1999

(13) C.W.Groetsch , Inverse problems in the mathematical sciences , Vieweg 1993

(14) H. Haario and E.Somersalo , The The Backus- Gilbert method revisited : background, implementation and examples, Numer. Funct.Anal.Optimiz.9 , pp.917-943 , 1987

(15) V.Isakov and J.Powell ,On the inverse conductivity problem with one measurement , Inverse Problems , 6 pp.311 - 318 , 1990

(16) V.Isakov , Uniqueness and stability in inverse parabolic problems, in Inverse problems in Diffusion Processes , Proc. GAMM-SIA Symposium,SIAM,Philadelphia , PA , pp.21-42 , 1995

(17) A.Kirsch , B.Schomburg , G.Berendt, The Backus- Gilbert method, Inverse Problems 4, 1988

(18) K. Kunisch , A review of some recent results on the output least squares formulation of parameter estimation problem , Automatica 24 ,pp.531-539 , 1988

(19) A.K.Louis and P. Maass , Smoothed projection methods for the moment problem , Numer.Math.59 , pp. 277-294 , 1991

(20) A.K.Louis and P. Maass , A mollifier method for linear operator equations of the first kind, Inverse Problems 6 , pp. 427-440 , 1990

(21) F.Natterer , The Mathematics of Computerized Tomography , 1986

(22) F.P.Pijpers and M.J. Thompson , The SOLA method for helioseismic inversion , Astron. Astrophys.281 , pp.231-240, 1994

(23) G.R.Richter , An inverse problem for steady state diffusion equation , SIAM J. Appl.Math. 41 , pp. 210-221 , 1981

(24) B.Schomburg , G.Berendt, On the convergence of the Backus- Gilbert algorithm , Inverse Problems 3 , pp.341-346 , 1987

(25) J.K.Seo , A uniqueness result on inverse conductivity problem with two measurements , J. Fourier Anal.Appl. ,2 pp. 227-235 , 1996

(26) G.Vainikko ,Inverse problem of ground water filtration : Identifiability , discretization and regularization, in Inverse Problems in Diffusion Processes , Proc. GAMM-SIAM Symposium, H.W. Engl and W.Rundell,eds.,SIAM,Philadelphia , PA , pp. 90-107 , 1995

(27) J. Wloka , Partial differential equations , 1987