

DISSERTATION

WAVE-BASED TECHNIQUES FOR DETECTION AND IMAGING

Submitted by

Cole Moore

Department of Mathematics

In partial fulfillment of the requirements

For the Degree of Doctor of Philosophy

Colorado State University

Fort Collins, Colorado

Summer 2026

Doctoral Committee:

Advisor: Margaret Cheney

Co-Advisor: Matthew Burfeindt

Louis Scharf

Jennifer Mueller

Mahmood Azimi-Sadjadi

Copyright by Cole Moore 2026

All Rights Reserved

ABSTRACT

WAVE-BASED TECHNIQUES FOR DETECTION AND IMAGING

The field of wave-based sensing covers a wide range of applications where measurements of a signal carried on propagating waves are used to learn about the environment. Sensing can be either passive, where a measured signal's source is external from the measurement system, or active, where a measurement system generates its own signal. Both passive and active sensing are challenging fields requiring steady incremental contributions in order to advance. In the pages that follow, I will present my own contributions to this challenging field. First, I detail work which addresses the problem of passively detecting an unknown signal in noisy measurements. My results allow for a quantification of the relationship between passive detection performance and the level of knowledge about the unknown measured signal. Then, I develop and analyze techniques for reducing computational cost and increasing performance of physics-based imaging techniques in radar applications. My results and technique prove to be a key ingredient in the first successful application of a technique to realistic real-world radar data. Finally, I design and test a method for improving existing radar imagery using a more accurate physics-based imaging technique. The results of my tests show a clear visual improvement in image quality.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Margaret Cheney. Dr. Cheney's patience, guidance, and kindness have been key factors in helping me to get this dissertation to completion. Next, I would like to thank my co-advisor, Dr. Matthew Burfeindt. The near-daily meetings I have had with Dr. Burfeindt have kept me focused and have provided an invaluable learning venue. A special thanks to Dr. Louis Scharf. It was a great pleasure to work with and learn from you.

I would like to thank my family. Their encouragement and support gave me the confidence to begin my graduate journey.

Thank you to my soon to be wife. Your love and support has been greatly appreciated. Finally done with school, we can start on the next exciting chapter of our lives!

DEDICATION

For Lemma and Juniper.

TABLE OF CONTENTS

ABSTRACT	ii
ACKNOWLEDGEMENTS	iii
DEDICATION	iv
LIST OF TABLES	vii
LIST OF FIGURES	viii
Chapter 1 Introduction	1
Chapter 2 Passive Multi-Channel Detection	8
2.1 Introduction	8
2.2 Signal Models	10
2.2.1 Subspace Signal Models	13
2.3 General Linear Model	14
2.3.1 The Parameters K , M , and N	15
2.4 Bistatic Likelihood Formulas	16
2.4.1 Log-Likelihood Under H_0	16
2.4.2 Log-Likelihood Under H_1	17
2.5 Bistatic Passive Detectors	19
2.5.1 Known channel case	20
2.5.2 Partially known channel case	22
2.5.3 Unknown channel case	25
2.5.4 Final detectors	27
2.6 Numerical Results	29
2.6.1 Diversity-combining detectors	29
2.6.2 Simulation details	31
2.6.3 Known Channel Case	32
2.6.4 Partially Known Channel Case	33
2.6.5 Unknown Channel Case	35
2.6.6 Channel Comparisons	37
2.7 Conclusions	40
Chapter 3 Synthetic Aperture Radar Digital Spotlighting for Linear Sampling Method	
Imaging	42
3.1 Introduction	42
3.2 Methods	45
3.2.1 The Linear Sampling Method	45
3.2.2 Digital Spotlighting	60
3.3 Results	83
3.3.1 Proof of feasibility and data reduction results	84
3.3.2 Spatial filtering of artifacts for LSM	88
3.3.3 Multi-frequency LSM	96

3.3.4	Digital Spotlight and PE-LSM processing of SAR field data	102
3.4	Conclusions	110
Chapter 4	Linear Sampling Method enhancement for Synthetic Aperture Radar Images .	112
4.1	Introduction	112
4.2	Methods	114
4.2.1	Monostatic SAR Data Model	114
4.2.2	Transforming a SAR image into synthetic data	118
4.2.3	PE-LSM imaging	120
4.3	Results	125
4.4	Conclusions	127
Chapter 5	Conclusions	129

LIST OF TABLES

2.1	Table of Detector Statistics	28
2.2	Table of Diversity-Combining Detectors	30

LIST OF FIGURES

2.1	PD vs PFA for fully known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.	32
2.2	PD vs SNR for fully known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.	33
2.3	PD vs PFA for partially known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.	34
2.4	PD vs SNR for partially known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.	35
2.5	PD vs PFA for unknown channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.	36
2.6	PD vs SNR for unknown channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.	37
2.7	PD vs SNR for three different levels of knowledge of the signal channel model when we assume that the noise variances are known.	38
2.8	PD vs SNR for three different levels of knowledge of the signal channel model. We assume the noise variances are unknown but equal at each sensor platform.	39
2.9	PD vs SNR for three different levels of knowledge of the signal channel model. We assume the noise variances are unknown at each sensor platform.	40
3.1	General Diagram of LSM imaging context.	46
3.2	Illustration depicting overlap of monostatic and bistatic spatial frequency supports. The blue region depicts Ω_{mono} for a 360° aperture and wavenumbers between k_{min} and k_{max} . The red circle depicts the support of bistatic data collected for $\theta_t = 0$ and θ_r traversing a 360° aperture at wavenumber k_{max}	57
3.3	Flow chart of digital spotlight processing which takes SAR phase history $E(\theta_r, \theta_t, k)$ and transforms it to filtered and downsampled <i>spotlit</i> data $E_s(\theta_r, \theta_t, k)$. E_{rc} denotes the <i>re-centered</i> phase history, E_{ff} denote the <i>frequency filtered</i> phase history, E_{fds} denotes the <i>frequency downsampled</i> phase history, E_{ar} the <i>aperture resampled</i> phase history, and E_{af} the <i>aperture filtered</i> phase history.	61
3.4	Diagram of bistatic imaging geometry.	62
3.5	Diagram depicting the k -space support of bistatic data collected for fixed θ_t and k and varying θ_r from 0° to 360°	64
3.6	Diagram of multistatic sensor geometry.	72
3.7	Diagram showing sensor setup.	85

3.8	PE-LSM image of the full scene. The white dashed lines illustrate the true target locations and shapes while the red square illustrates the digital spotlight ROI.	86
3.9	PE-LSM image of the digital spotlight ROI. This image uses the full data set collected to image the entire scene.	87
3.10	PE-LSM image of the digital spotlight ROI. This image uses the digitally spotlighted data.	87
3.11	This diagram depicts the sensor and target geometry used for the FDTD simulation with a pair of triangular targets. An enlarged image of the targets is provided for clarity.	89
3.12	This diagram depicts the sensor and target geometry used for the FDTD simulation with a single bar shaped target. An enlarged image of the targets is provided for clarity.	90
3.13	Single frequency LSM image of the full scene containing two triangular targets near the large circular target. The red square indicates the digital spotlight ROI.	91
3.14	Single frequency LSM image of the ROI containing two triangular targets. Image is made with data that has not been processed by the digital spotlight procedure.	92
3.15	Single frequency LSM image of the ROI containing two triangular targets. The image is made with data that has been digitally spotlighted.	93
3.16	Single frequency LSM image of the full scene containing a bar shaped target near the large circular target. The red square indicates the digital spotlight ROI.	94
3.17	Single frequency LSM image of the ROI containing the bar shaped target. Image is made with data that has not been processed by the digital spotlight procedure.	95
3.18	Single frequency LSM image of the ROI containing the bar shaped target. The image is made with data that has been digitally spotlighted.	96
3.19	Multi-frequency LSM image of the full scene containing two triangular targets near the large circular target. The red square indicates the digital spotlight ROI.	97
3.20	Multi-frequency LSM image of the ROI containing two triangular targets. Image is made with data that has not been processed by the digital spotlight procedure.	98
3.21	Multi-frequency LSM image of the ROI containing two triangular targets. The image is made with data that has been digitally spotlighted.	99
3.22	Multi-frequency LSM image of the full scene containing a bar shaped target near the large circular target. The red square indicates the digital spotlight ROI.	100
3.23	Multi-frequency LSM image of the ROI containing the bar shaped target. Image is made with data that has not been processed by the digital spotlight procedure.	101
3.24	Multi-frequency LSM image of the ROI containing the bar shaped target. The image is made with data that has been digitally spotlighted.	102
3.25	Backprojection image of full GOTCHA data scene. We have used a light blue square to indicate the region of interest within the scene.	105
3.26	Backprojection image of car within the ROI highlighted in Figure 3.25. This image is constructed with digitally spotlighted data.	106
3.27	PE-LSM image of the car inside the ROI highlighted in Figure 3.25. This image is constructed with digitally spotlighted data.	107
3.28	Backprojection image of full GOTCHA data scene. We have used a light blue square to indicate the region of interest within the scene.	108
3.29	Backprojection image of car within the ROI highlighted in Figure 3.28. This image is constructed with digitally spotlighted data.	109
3.30	PE-LSM image of car within the ROI highlighted in Figure 3.28. This image is constructed with digitally spotlighted data.	110

4.1	Diagram depicting the monostatic SAR sensing setup.	115
4.2	Diagram depicting the k -space support of monostatic SAR data collected for angles from between θ_1 and θ_2 . Note that the labeled bandwidth is equal to $\frac{4\pi}{c}$ times the bandwidth in Hz.	117
4.3	An example of the inverse Fourier transform of a SAR image showing the data annulus.	119
4.4	Diagram showing FDTD simulation setup.	125
4.5	Imaging Results for three different targets. Imaging techniques are arranged from left to right, first backprojection, then PE-LSM on the original data, and finally PE-LSM on synthesized beamspace data.	126

Chapter 1

Introduction

Wave-based sensing encompasses a wide array of sensing modalities that involve the measurement of propagating waves. These techniques are used across a range of applications including seismic exploration [1], archaeology [2], non-destructive testing [3], environmental monitoring [4], navigation [5], and many more. Wave-based sensing modalities can be broadly categorized as either active or passive. Active systems, such as radar [6], sonar [7], or ultrasound [8], feature an element which actively emits waves carrying a known signal. These methods typically measure the scattering of these emitted waves with the aim of extracting information about the object which caused the scattering. Precise models of scattering mechanics are non-linear and the corresponding inverse scattering problem are ill-posed. This makes sensing modalities that seek to extract detailed information from scattering targets difficult. Passive systems, such as radio telescopes [9], seismic sensors [10], or radiometers [11], measure waves from sources external to the sensing system. Passive systems, by their nature, have less access to information regarding the signal of interest in measurements. This makes extracting useful information in a potentially noisy environment difficult, and often some model for the expected signal must be used. Both sides of this field present their own array of difficulties, and have required many years of focused contributions in order to grow to their current state. In this dissertation, we will make our own contributions to the following problems in the challenging field of wave-based sensing.

Passive Source Localization (PSL) refers to the problem of detecting and locating a radiating source using passively collected measurements. Typically, spatially separated sensors collect passive measurements with the goal of first identifying and then locating an unknown signal source. This problem may take on many real-world forms ranging from locating microseismic events in mines [12] to locating the position of a speaker using an array of microphones for video conferencing [13]. The PSL problem can be divided into two parts. The first part is the detection of a signal common to two or more spatially separated sensors. The second is to then determine the

locations of the detected source. Our contributions will address the first part of this problem, that of detection.

The PSL detection problem can be framed as a binary hypothesis testing problem. The null hypothesis describes a state where all sensors measure only noise, and the alternative hypothesis holds that each channel measures a shared signal. Variations of this hypothesis testing problem are treated in [14–19]. We note that this is different from the statistical detection problem in passive radar, where under the null hypothesis one channel measures a reference signal while the other channel has no signal present [20–24].

In this document, Generalized Likelihood Ratios (GLRs) are developed to address the detection problem; these GLRs are composed of a ratio of likelihood functions for each hypothesis where each likelihood is maximized separately with respect to any unknown parameters. The likelihood ratio statistics are based on a linear subspace measurement model. Nine different cases are treated based on differing assumptions regarding the knowledge of the measurements' noise variances and the channel mappings for each linear signal model. Some of the nine cases are treated in the literature in [14–19]. However, this work extends these results to the case of a finite dimensional subspace signal model. Also of interest, it is shown that many of these detector statistics decompose into a combination of two terms. The first term represents a detector which can be computed independently at each sensor, and the second term uses data from both sensors. The second term will commonly be referred to as the "cross-validation" or "coherence" term.

After describing the development of the nine GLR test statistics, a series of simulated results demonstrating the effectiveness of these detectors are presented. These simulations allow for a quantitative evaluation of the detector performance. Simulation results that offer comparisons between each of the nine cases under study are given, and these results are useful in quantitatively comparing the attainable performance between these cases. Included in the simulation results are counterpart detectors to the nine detector cases of interest. These counterpart detectors non-coherently combine data from the separated sensors rather than the coherent treatment given in

our nine cases. These results allow for a quantitative evaluation of the contribution of the cross-validation terms to the overall detector performance.

The PSL results given in this document have been published in [25], where the author of this document is listed as a co-author. Specific contributions made by the author include: development of the simulation code, analysis of the resulting simulated data, and assistance in the correcting and finalizing of the detector statistic results.

Inverse scattering is an active sensing modality in which one or more sensors emit probing signals into a scene of interest and then measure the scattering of said signal. The goal of inverse scattering techniques is to then use these scattering measurements to reconstruct an image of the scene [26–37]. Inverse scattering methods are useful in a wide array of applications including, but not limited to, radar, medical imaging, and seismic exploration. Due to the non-linearity of the scattering behavior, inverse scattering problems tend to be ill-posed. The ill-posed nature makes these problems challenging, as small perturbations in the data can lead to large variation in the problem’s solution.

Synthetic Aperture Radar, or SAR, is a special case of an inverse scattering problem where a sensor traverses a "synthetic" aperture through space during data collection. Typical SAR imaging techniques such as the Polar Format Algorithm [38], Backprojection [39], and the $\omega - k$ algorithm [40], are based on approximated models which linearize scattering behavior via the Born approximation and treat targets as clouds of point-like scatterers. These approximations, and the subsequent linearization, make these algorithms both fast and stable. The speed comes from their primary calculations being performed as either inner products or Fourier transforms, and the stability stems from the fact that the linearized scattering problem is no longer ill-posed. Due to these differences, it is common to treat these traditional SAR algorithms as separate from other inverse scattering techniques. From this point forwards, we will use the term "inverse scattering" to refer to techniques based on models which do not linearize the scattering relationship.

SAR imaging has a long history of use in a wide variety of applications including environmental monitoring, resource management, and target recognition [41–47]. However, the approximations

made by traditional SAR imaging techniques place limits on their ability to accurately capture the true shape of targets. Inverse scattering techniques tend to avoid the approximations made by traditional SAR algorithms, and therefore they possess the potential to more accurately image the true shapes of targets. Thus, it is of great interest to apply these inverse scattering techniques to SAR type applications.

The Linear Sampling Method, or LSM, is a well-known and well-studied technique for solving inverse scattering problems [48–60]. The LSM reduces the scope of the non-linear inverse scattering problem by seeking to reconstruct only the shape (not the full electrical properties) of a scattering target. Often, only the target shape is required for many applications. This simplification allows the LSM to instead solve a set of linear equations rather than the full non-linear scattering problem. The LSM also avoids the point-scatterer approximation made by traditional SAR algorithms and therefore has potential to create imagery which captures target shapes with higher fidelity than those methods.

The avoidance of the point scatterer approximation combined with the simple implementation make the LSM an attractive choice for SAR imaging via inverse scattering techniques. However, although the LSM has been applied successfully to some experimental datasets (e.g. [56–58]), its application to data from a fielded SAR system has remained an open problem since the LSM’s introduction ~ 30 years ago. The first major hurdle in applying the LSM to SAR is the LSM’s need for rich multistatic data collected from a wide angular aperture. In cases where a target is viewed from a more limited aspect angle the LSM performance degrades and when only monostatic data is available the LSM breaks down. The Phase-Encoded LSM (PE-LSM) was developed and refined through [61–64] and [65] in order to address these issues. Since its first introduction in [61], the PE-LSM has been shown to significantly improve on the standard LSM’s ability to produce quality imagery from limited angular apertures. These improvements stem from the PE-LSM’s introduction of a mechanism for coherent processing of multi-frequency data that is absent from the standard LSM formulation. The capabilities of PE-LSM were extended further in [64], where a technique to use PE-LSM for monostatic data is put forward.

With the advancements of PE-LSM, SAR imaging with LSM-type techniques was theoretically feasible, but there were still practical issues that needed addressing. The first is the computational cost. PE-LSM imaging requires the solution of a new linear system for each pixel in a scene, and for the typically large SAR datasets, these linear systems are also correspondingly large, making processing the data with PE-LSM impractical. Second is the ill-posed nature of LSM-type imaging. The increased sensitivity of the ill-posed LSM problem must be taken into consideration when it is applied to new settings and processing contexts.

We make contributions which address the issues of computational cost and instability by adapting the technique of digital spotlighting to LSM-type imaging. Digital spotlighting is an established technique in SAR imaging which filters and reduces the data when only a smaller sub-region of interest is desired for imaging [66–70]. To our knowledge, all past work has focused on spotlighting monostatic data and then subsequent imaging with traditional SAR techniques. We have developed a digital spotlight technique that is appropriate for multi-sensor and multi-frequency data. This technique is applied to PE-LSM imaging and shown to be effective despite the ill-posed nature of the imaging method. We then apply the digital spotlight method to LSM imaging and demonstrate that the method can help to mitigate interference from large and bright scatterers near a target of interest. This interference mitigation has the potential to improve performance of the ill-posed method which, in principle, should be much more sensitive to interference. Finally, our digital spotlight work proved instrumental in the first successful PE-LSM imaging of field-collected SAR data, and we include these results as well [71]. Preliminary results of our digital spotlight-based contributions have also been published in [72].

Following the work applying digital spotlighting to LSM-based imaging, we then approach the question of improving SAR imagery with LSM-type techniques from a different perspective. Rather than operating directly on the raw SAR data as we did for digital spotlight-LSM processing, we instead choose to operate on fully formed SAR images. There are several reasons one may want the ability to operate directly on SAR imagery rather than the raw data. Often one may only have access to the imagery and not the raw data; this is the case for the well-known MSTAR dataset [73].

Alternatively, due to poor record keeping and/or data rot, older SAR data may be unusable but the imagery may still be available. From a practicality and efficiency standpoint, raw SAR datasets are often quite large, and operating only on the comparatively smaller files of the associated imagery may help to streamline data transfer and processing.

Our contributions draw inspiration from [74], which applies the MUSIC algorithm to Fourier transformed SAR backprojection images. MUSIC, like backprojection, assumes a point target model and thus is limited in its capacity to "enhance" an existing backprojection image. We develop a similar technique which transforms an existing backprojection image into data which is appropriate for PE-LSM imaging. We then demonstrate the effectiveness of this technique on a set of backprojection images of simulated targets. Our results show that the output PE-LSM images of this technique offer improved fidelity to the true target shapes compared to the original backprojection image.

The remainder of this document is organized as follows: Chapter 2 details the work on passive source localization. Chapter 3 discusses the work on the application of the digital spotlight technique to LSM-based imaging. Chapter 4 contains our work focusing on enhancing existing SAR images by transforming them into data for PE-LSM images. Finally, Chapter 5 provides concluding remarks and a thorough summary of the contributions of this dissertation.

For the ease of the reader, chapters 2, 3, and 4 are written to be mostly self-contained. Each chapter will introduce the work, provide details about the methods, techniques and contributions, and will present results and conclusions. Thus, if one particular chapter is of interest, then a reader should be able to read that chapter in isolation and still come away with an understanding of its contents. We note that this self-contained nature of each chapter extends to the mathematical notation. Each chapter will introduce and define its own notation, which is done to help prevent notational "bloat" as more and more terms must be defined within a long document covering numerous topics.

We provide here a bulleted list containing an overview of the primary contributions made in this document. Contributions are grouped according to the chapter in which they occur.

- Chapter 2: Passive Multi-Channel Detection

- Developed simulations to test nine different generalized likelihood ratio detector statistics
- Collected and analyzed simulation data, providing quantification of detector performance
- Assisted with correcting and finalizing mathematical derivations of detector statistics
- Chapter 3: Synthetic Aperture Radar Digital Spotlighting for Linear Sampling Method Imaging
 - Developed multi-sensor and multi-frequency digital spotlight technique
 - Applied technique successfully to both LSM and PE-LSM imaging
 - Demonstrated that the technique improved performance in the presence of strongly interfering scatterers
 - Provided a key component in first successful LSM-based imaging of real SAR data
- Chapter 4: Linear Sampling Method enhancement for Synthetic Aperture Radar Images
 - Developed a method for converting complex SAR imagery into PE-LSM appropriate data
 - Processed simulated SAR imagery with this technique to evaluate performance
 - Demonstrated improvement in ability to capture true target shape

Chapter 2

Passive Multi-Channel Detection

2.1 Introduction

In this chapter we will focus on the problem of Passive Source Localization or PSL. This is the problem of detecting sources of a radiated signal using measurements made at a set of passive sensors. The PSL problem can be divided into two primary pieces. The first is the detection of a signal common to the passive sensors, and the second is to then determine the location of the source of this common signal. This chapter focuses on the first part of this question dealing with the detection of a common signal measured at multiple sensors. We will use a multivariate normal model for the signal measurements, and will model the signal as having a linear channel which maps a basis for the transmitted signal to the measurements made on the receive sensor.

The detection problem is framed as a binary hypothesis test. Under the null hypothesis, or H_0 , we assume no common signal is present in the measurements at any sensor. Under the alternative hypothesis, or H_1 , we assume that all sensors measure a common signal. The detection statistics developed in this paper are Generalized Likelihood Ratios (GLRs). These consist of a ratio of the likelihoods for each hypothesis. The likelihood function for both H_0 and H_1 are separately maximized for any unknown model parameters. A linear subspace channel model is assumed for the signals, and nine different GLR detector statistics are developed based on differing levels of assumed knowledge of these subspace channel maps and the noise variance. We treat three cases for assumed knowledge of the subspace channel mappings:

1. The channel mapping and gain is known. This will be referred to as the known-channel case.
2. The channel mapping is known but the channel gain is unknown. This will be referred to as the partially-known-channel case.

3. The channel mapping and the channel gain are unknown are both unknown. In this case, the dimension of the channel subspace is still assumed to be known. This will be referred to as the unknown-channel case.

Mathematical definitions of the channel mapping and gain will be given in later sections. These three cases are augmented by three different cases of assumed knowledge of the noise variance:

1. The noise variance on each channel is known. This will be referred to as the "known variance" case.
2. The noise variance on each channel is unknown but assumed to be equal. This will be referred to as the "unknown but equal variance" case.
3. The noise variance on each channel is unknown and not assumed to be equal. This will be referred to as the "unknown variance" case.

Taking each possible pair of assumptions about the channel mappings and noise variance gives the nine different cases. For example, the known-channel and known variance case represents the first of the nine cases.

It is worth noting that the hypothesis problem we deal with here for PSL is distinct from the common hypothesis testing problem associated with passive radar [20–24]. In passive radar, measurements are collected from a "reference channel" and a "surveillance channel." Under H_0 there is no measured signal in the surveillance channel but the reference channel still measures a signal. This is contrasted from the PSL problem we treat here where under H_0 there is no measured signal in any of the channels.

The unknown channel cases and the unknown variance cases are new developments. The remaining cases have been treated in past work [14–19], but these are adapted these results to the case for a finite dimension subspace channel model. Many of these detector statistics are also shown to decompose into a sum of locally computed statistics that use only measurements from a specific sensor and a coherence or cross-validation term which uses measurements from all sensors.

All nine detectors are then evaluated through a series of numerical simulations. Receiver operating characteristic (ROC) curves and probability of detection versus SNR curves are used to illustrate the effectiveness of each detector statistic. Comparisons between different detectors are shown, illustrating the changes in effectiveness as more or less information about the channel mapping and noise variance are assumed to be known. Comparisons are also made between the detectors dealt with here and so called "diversity-combining" detectors which feature only non-coherent additions of locally computed detector statistics from each sensor. This allows evaluation of the contribution of the coherence or cross validation terms in overall detector performance.

The work of this chapter has been published in [25]. The primary contributions of this author to the work include the writing and execution of the simulations used to evaluate the detector performance. ROC curves are produced and show probability of detection and probability of false alarm performance across a range of detector thresholds for a fixed SNR. Probability of detection versus SNR curves are also produced, and these are used to show detector performance at a fixed probability of false alarm rate across different SNRs. In addition to the simulation work, this author also helped with the work of finalizing the derivations of the detectors and ensuring they are error free.

The remainder of this chapter is organized as follows. Section 2.2 develops our models for the radiating source signals. Then, Section 2.3 models the measured data collected from a set of passive sensors. Section 2.4 details the likelihood formulas for both H_0 and H_1 , and from this section forwards we restrict ourselves to the bistatic case of two passive sensors. In section 2.5 GLR detection statistics are developed for the nine different cases. Details regarding the simulations and their results can be found in section 2.6. Finally, we conclude in section 2.7.

2.2 Signal Models

We assume a radiating source produces a real-valued bandlimited waveform

$$s(t) := \text{Re}\{e^{i2\pi f_c t} w(t)\}. \quad (2.1)$$

Here, f_c denotes the center frequency and $w(t)$ is the unknown baseband signal. We measure a delayed and scaled version of the signal along with additive noise,

$$r(t) = Gs(t - t' - \tau(t)) + n(t). \quad (2.2)$$

Here, G denotes a real-valued channel gain. This term can encompass attenuation losses, losses due to propagation (geometric spreading), and sensor gains. We use $\tau(t)$ to denote the propagation delay, which may be time varying. This term depends on the media of propagation and the relative positions of the source and sensor. The constant t' accounts for potential clock offset between the sensor and some reference. The additive noise is given by $n(t)$. To move forward, we assume that we are somehow able to determine the center frequency, f_c , perhaps by the methods given in [75] and [76]. We then demodulate the received signal to obtain a complex baseband signal,

$$x(t) = Ge^{-i2\pi f_c(t' + \tau(t))}w(t - t' - \tau(t)) + u(t). \quad (2.3)$$

We use $u(t)$ to denote both demodulated passband noise and baseband sensor noise. We assume that $u(t)$ is wide sense stationary (WSS) in time. Next, we approximate the time-dependent delay with a first-order series expansion

$$\tau(t) \approx \tau - \frac{\nu}{f_c}t. \quad (2.4)$$

This is equivalent to approximating the Doppler shift in the signal to be $\nu = (V/c)f_c$ where V is the relative velocity between source and sensor. We assume that the signal is narrowband such that we can neglect terms t' and $\nu t/f_c$ in the baseband signal w . These terms must be kept in the exponent where we have a large carrier frequency f_c . These approximations and assumptions result in

$$x(t) = \tilde{g}e^{i2\pi\nu t}w(t - \tau) + u(t). \quad (2.5)$$

We use $\tilde{g} = Ge^{-i2\pi f_c(t' + \tau)}$ to denote a complex-valued gain. Next, we sample and 'synchronize' the data. Here, we use the term synchronize to refer to removing the delay and doppler shift. Let

$1/T_s$ denote the rate at which we sample the received signal. The resulting discrete time sequence after sampling is given by

$$x[n] := x(nT_s) = \tilde{g}e^{i2\pi\nu nT_s}w(nT_s - \tau) + u(nT_s) \quad (2.6)$$

for $n \in \mathbb{Z}$. We assume that the sampled noise is white and that this assumption holds for advanced and demodulated versions of $u(t)$ as well. Next, the sampled signal is ‘synchronized.’ We advance the signal by k' where $\tau' = k'T_s$ is our estimated time advance. We then Doppler shift the result with estimated shift ν' . This results in

$$z[n; \theta'] := e^{-i2\pi\nu' nT_s}x[n + k'] \quad (2.7)$$

$$= e^{-i2\pi\nu' nT_s}\tilde{g}e^{i2\pi\nu(n+k')T_s}w((n+k')T_s - \tau) + u((n+k')T_s). \quad (2.8)$$

Here, $\theta' = (\tau', \nu')$ denotes estimated or hypothesized synchronizing parameters and use $\theta = (\tau, \nu)$ to denote the true delay and Doppler shift. By sampling the space of possible values for θ' , one could form ‘likelihood’ images of potential radiating sources. If $\theta' = \theta$,

$$z[n; \theta] = \tilde{g}e^{i2\pi\nu kT_s}w[n] + u[n]. \quad (2.9)$$

We defined $w[n] = w(nT_s)$ and $u[n] = u(nT_s)$. We now consider the signal ‘synchronized,’ and we derive the detector under the assumption that this case holds. We do not study effects of incorrect synchronization here, they are instead explored in [77].

Suppose we collect N samples of the signal $z[n; \theta]$. We organize these into column vector $\mathbf{z} = [z[0; \theta] \dots z[N-1; \theta]]^T \in \mathbb{C}$. We similarly define $\mathbf{w}$ and $\mathbf{u}$ and obtain

$$\mathbf{z} = g\mathbf{w} + \mathbf{u}. \quad (2.10)$$

Here, $g = \tilde{g}e^{i2\pi\nu kT_s}$. We assume the additive noise vector takes the form $u \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R})$. We assume the $N \times N$ covariance matrix can be factored as

$$\mathbf{R} = \mathbb{E}[\mathbf{u}\mathbf{u}^H] = \sigma^2 \mathbf{I}_N \quad (2.11)$$

and that $\text{tr}[\mathbf{R}] = N\sigma^2$.

2.2.1 Subspace Signal Models

It is often the case that one can accurately describe the baseband signal $\mathbf{w}$ in (2.10) as a subspace signal. Mathematically, we write

$$\mathbf{w} = \mathbf{H}\mathbf{a}. \quad (2.12)$$

Recall that $\mathbf{w} \in \mathbb{C}^N$. Here, $\mathbf{a} \in \mathbb{C}^K$ is a vector of complex coefficients and $\mathbf{H} \in \mathbb{C}^{N \times K}$ is a matrix whose columns form a basis for the corresponding K dimensional subspace of $\mathbb{C}^N$. We denote the subspace with $\langle \mathbf{H} \rangle$. This subspace can be used as a model for the unknown source, the propagation channel, or both. As a simple example, consider a basis $\mathbf{H}$ constructed from K columns of $N \times N$ discrete Fourier transform (DFT) matrix. Here, we are modeling the signal as a linear combination of K Fourier modes with $\mathbf{a}$ giving the Fourier coefficients. If we take $K = N$, then this is equivalent to expanding the signal in the DFT basis.

The advantage of the subspace model is the convenient way it divides the signal and noise into separate spaces. After measuring vectors in $\mathbb{C}^N$, we assume the signal is present in only a K dimensional subspace. This opens up a $N - K$ dimensional space where we may estimate properties of the noise. This allows us to compute the ratio of signal to noise energy as

$$SNR = \frac{N}{K} \frac{|g|^2 \mathbf{a}^H \mathbf{a}}{\sigma^2 N}. \quad (2.13)$$

We note that a processing gain of N/K is achieved when we resolve onto our subspace of dimension K .

2.3 General Linear Model

In the previous section, models for signals measured at individual sensors were developed where each measurement is a complex-valued time series. Now, consider a system with L total spatially separated sensors. A source emits a signal that is measured by all L sensors. Each sensor will measure a uniquely delayed and Doppler shifted version of the source signal depending on the relative locations and velocities of the source and the sensors. We hypothesize a source location and velocity and use this to synchronize the measurements at all sensors.

After synchronization, our measurement at sensor l is

$$\mathbf{z}_l = g_l \mathbf{H}_l \mathbf{a} + \mathbf{u}_l \in \mathbb{C}^{N_l}. \quad (2.14)$$

Because the source is common to all sensors, $\mathbf{a}$ is the same for each channel. We assume that $\mathbf{a}$ is unknown and we do not constrain it to a probabilistic or deterministic model.

For all L sensors, our model is

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_L \end{bmatrix} = \begin{bmatrix} g_1 \mathbf{H}_1 \\ g_2 \mathbf{H}_2 \\ \vdots \\ g_L \mathbf{H}_L \end{bmatrix} \mathbf{a} + \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_L \end{bmatrix}. \quad (2.15)$$

We write this compactly as

$$\mathbf{z} = \mathbf{F} \mathbf{a} + \mathbf{u} \in \mathbb{C}^{N_Z} \quad (2.16)$$

where $N_Z = \sum_{l=1}^L N_l$ is the length of the composite vector $\mathbf{z}$. Each channel l has a subspace model given by matrix $\mathbf{H}_l$ and we define $\mathbf{F}_l = g_l \mathbf{H}_l$. We assume $\mathbf{u} \sim \mathcal{CN}_{N_Z}(\mathbf{0}, \mathbf{R})$. The covariance matrix $\mathbf{R}$ is assumed to take the form $\mathbf{R} = \text{blkdiag}[\{\sigma_l^2 \mathbf{I}_{N_l}\}]$. Let us assume that we then collect M measurements, each of length N_Z . Then, we define $\mathbf{Z} = [\mathbf{z}[1], \dots, \mathbf{z}[M]] \in \mathbb{C}^{N_Z \times M}$, $\mathbf{A} = [\mathbf{a}[1], \dots, \mathbf{a}[M]] \in \mathbb{C}^{K \times M}$, and $\mathbf{U} = [\mathbf{u}[1], \dots, \mathbf{u}[M]] \in \mathbb{C}^{N_Z \times M}$. Using these definitions, we

construct the linear model

$$\mathbf{Z} = \mathbf{F}\mathbf{A} + \mathbf{U} \in \mathbb{C}^{N_Z \times M}. \quad (2.17)$$

It is important to note that $\mathbf{F}$ is assumed to remain the same for all M measurements. We assume that the columns of the noise matrix $\mathbf{U}$ are a series of independent $\mathcal{CN}_{N_Z}(\mathbf{0}, \mathbf{R})$ random vectors.

2.3.1 The Parameters K , M , and N

One natural way for a signal model defined as in (2.17) to arise is when each of our l sensors is an array consisting of N_l co-located sensors. Then, each $\mathbf{z}_l[m] = g_l \mathbf{H}_l \mathbf{a}[m] + \mathbf{u}_l[m] \in \mathbb{C}^{N_l}$ is a sampled snapshot of the N_l array element measurements. We are organizing these M snapshots from L arrays into the $N \times M$ spacetime data matrix $\mathbf{Z}$. This comes with the assumption that each array component for each snapshot has the same complex gain g_l and subspace basis $\mathbf{H}_l$.

An alternate way the signal model in (2.17) may be defined is if each sensor l is a single sensor, or the complex scalar output of a beamformed array. Then, let us assume that a long measurement of length P is sampled. We divide up this measurement into M shorter sequences of length N_l such that $N_l \times M = P$. If each of these short sequences may be modeled as $\mathbf{x}_l[m] = g_l \mathbf{H}_l \mathbf{a}[m]$ with $\mathbf{H}_l \in \mathbb{C}^{N \times K}$, then each sensor will measure $\mathbf{Z}_l = g_l \mathbf{H}_l \mathbf{A} + \mathbf{U}$, which can be stacked vertically with the other sensors to obtain (2.17). The obvious question is why split up the length P sequence into length N sequences for each sensor? One could argue that modeling a long sequence as a subspace signal of dimension K requires more modeling confidence than modeling a shorter sequence with the same subspace of dimension K . However, the key idea here is that the modeler has the flexibility to choose the subspace dimension K , the sequence length N , and the number of snapshots M . This allows the model to be tailored to specific applications where appropriate.

If we do not divide up the signal of length P , then we say $M = 1$ and $N = P$. Then, if $K = N$, we can model $\mathbf{H} = \mathbf{I}_N$. The resulting signal model is $\mathbf{z} = \mathbf{g}\mathbf{a} + \mathbf{u}$. In the work that follows, we will focus on the general case where M , and N are arbitrary and we assume the subspace dimension $K \leq N$. Specialized cases where $M = 1$, and $K = N$ are also presented.

2.4 Bistatic Likelihood Formulas

From this point forwards, we will focus on the case where $L = 2$ sensors. For increased notational clarity and only a small loss in generality, we assume $N_1 = N_2 = N$.

2.4.1 Log-Likelihood Under H_0

Recall that under the null hypothesis, there is no signal present in the measurements, meaning $\mathbf{A} = \mathbf{0}$. Thus, our signal is $\mathbf{Z} = \mathbf{U}$. The Gaussian log-likelihood is thus

$$\mathcal{L}_0(\sigma_1^2, \sigma_2^2; \mathbf{Z}) = -MN \log \sigma_1^2 - MN \log \sigma_2^2 - M \text{tr} [\mathbf{R}^{-1/2} (\mathbf{Z}\mathbf{Z}^H / M) \mathbf{R}^{-1/2}]. \quad (2.18)$$

We assume noise at each separated sensor is uncorrelated. Thus, $\mathbf{R} = \text{diag}[\sigma_1^2 \mathbf{I}_N, \sigma_2^2 \mathbf{I}_N]$. If $\sigma_1^2 = \sigma_2^2 = \sigma^2$ then $\mathbf{R} = \sigma^2 \mathbf{I}_{2N}$.

We compute the maximum likelihood (ML) estimates of the noise variance under H_0 by maximizing the log likelihood with respect to the noise variance assuming the other parameters are known. First, consider the case where $\sigma_1^2 = \sigma_2^2 = \sigma^2$. We have

$$\frac{\partial \mathcal{L}_0(\sigma^2; \mathbf{Z})}{\partial \sigma^2} = 0 = -\frac{2MN}{\sigma^2} + \frac{1}{(\sigma^2)^2} M \text{tr}[(\mathbf{Z}_1 \mathbf{Z}_1^H / M) + (\mathbf{Z}_2 \mathbf{Z}_2^H / M)]. \quad (2.19)$$

Solving, we obtain the estimate

$$\hat{\sigma}^2|_0 = \frac{1}{2N} \text{tr}[(\mathbf{Z}_1 \mathbf{Z}_1^H / M) + (\mathbf{Z}_2 \mathbf{Z}_2^H / M)], \quad (2.20)$$

where we use the notation $\hat{\sigma}^2|_0$ to denote that this is the ML estimate under H_0 . The estimate only requires the fusion of scalar autocorrelations from each sensor. When we substitute the ML estimated parameters into the likelihood function we refer to the result as the compressed likelihood. In this case, the compressed likelihood is

$$\mathcal{L}_0(\hat{\sigma}^2|_0; \mathbf{Z}) = -M(2N) \log \hat{\sigma}^2|_0 - M(2N). \quad (2.21)$$

Because the set of covariance matrices $\{\mathbf{R} = \text{diag}[\sigma_1^2 \mathbf{I}_{N_1}, \sigma_2^2 \mathbf{I}_{N_2}], \sigma_1^2 > 0, \sigma_2^2 > 0\}$ is a positive cone, a lemma in [78] tells us that the ML estimate of $\mathbf{R}$ constrains the trace term to be equal to $M(2N)$.

Next, we consider the case where $\sigma_1^2 \neq \sigma_2^2$. For σ_1^2 , we have

$$\frac{\partial \mathcal{L}_0(\sigma^2; \mathbf{Z})}{\partial \sigma_1^2} = 0 = -\frac{MN}{\sigma_1^2} + \frac{M}{(\sigma_1^2)^2} \text{tr}[(\mathbf{Z}_1 \mathbf{Z}_1^H / M)], \quad (2.22)$$

where we have taken advantage of the block diagonal structure of $\mathbf{R}$. An analogous form holds for σ_2^2 . This yields the estimates

$$\hat{\sigma}_1^2|_0 = \frac{1}{N} \text{tr}[(\mathbf{Z}_1 \mathbf{Z}_1^H / M)] \quad (2.23)$$

$$\hat{\sigma}_2^2|_0 = \frac{1}{N} \text{tr}[(\mathbf{Z}_2 \mathbf{Z}_2^H / M)]. \quad (2.24)$$

The resulting compressed likelihood is

$$\mathcal{L}_0(\hat{\sigma}_1^2|_0, \hat{\sigma}_2^2|_0; \mathbf{Z}) = -MN \log \hat{\sigma}_1^2|_0 - MN \log \hat{\sigma}_2^2|_0 - M(2N). \quad (2.25)$$

We note that in this case the estimates of the noise variance are computed locally at each sensor.

2.4.2 Log-Likelihood Under H_1

Under H_1 , there is a common signal present for both sensors. Thus, the Gaussian log-likelihood is

$$\mathcal{L}_1(\sigma_1^2, \sigma_2^2, \mathbf{F}, \mathbf{A}; \mathbf{Z}) = -MN \log \sigma_1^2 - MN \log \sigma_2^2 - M \text{tr}[\mathbf{R}^{-1}(\mathbf{Z} - \mathbf{F}\mathbf{A})(\mathbf{Z} - \mathbf{F}\mathbf{A})^H / M]. \quad (2.26)$$

We want to find a ML estimate for the subspace signal $\mathbf{A}$. Again, we assume all other parameters are known and maximize (2.26) with respect to $\mathbf{A}$. We have

$$\frac{\partial \mathcal{L}_1(\sigma_1^2, \sigma_2^2, \mathbf{F}, \mathbf{A}; \mathbf{Z})}{\partial \mathbf{A}} = \mathbf{0} = \text{tr}[-\mathbf{F}^H \mathbf{R}^{-1} \mathbf{Z} + \mathbf{F}^H \mathbf{R}^{-1} \mathbf{F} \mathbf{A}], \quad (2.27)$$

where we have use the invariance of the trace under the conjugate transpose operation and the cyclic property of the trace to simplify the expression. This gives the ML estimate of the signal

$$\hat{\mathbf{A}} = (\mathbf{F}^H \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{F}^H \mathbf{R}^{-1} \mathbf{Z}. \quad (2.28)$$

Expanding this gives,

$$\hat{\mathbf{A}} = \frac{1}{|g_1|^2/\sigma_1^2 + |g_2|^2/\sigma_2^2} \left[\frac{\bar{g}_1}{\sigma_1} \mathbf{H}_1^H \frac{\mathbf{Z}_1}{\sigma_1} + \frac{\bar{g}_2}{\sigma_2} \mathbf{H}_2^H \frac{\mathbf{Z}_2}{\sigma_2} \right]. \quad (2.29)$$

We see that the estimate of $\mathbf{A}$ is a linear combination of measurements from both sensors. For the compressed likelihood, we define

$$\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}} = \mathbf{R}^{-1/2} \mathbf{F} (\mathbf{F}^H \mathbf{R}^{-1} \mathbf{F})^{-1} \mathbf{F}^H \mathbf{R}^{-1/2}, \quad (2.30)$$

a projection onto the subspace $\langle \mathbf{R}^{-1/2} \mathbf{F} \rangle$. Similarly, we define $\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}}^\perp = \mathbf{I}_{2N} - \mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}}$ a projection on the $(2N - K)$ subspace $\langle \mathbf{R}^{-1/2} \mathbf{F} \rangle$. With these definitions, we can express the compressed likelihood as

$$\mathcal{L}_1(\sigma_1^2, \sigma_2^2, \mathbf{F}, \hat{\mathbf{A}}; \mathbf{Z}) = -MN \log \sigma_1^2 - MN \log \sigma_2^2 - M \text{tr} [\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}}^\perp \mathbf{R}^{-1/2} (\mathbf{Z} \mathbf{Z}^H / M) \mathbf{R}^{-1/2}] \quad (2.31)$$

Next we work towards the ML estimates for the noise variance at each sensor. Again, we consider different cases. First, assume $\sigma_1^2 = \sigma_2^2 = \sigma^2$. Note that in this case, $\mathbf{R}^{-1} = \frac{1}{\sigma^2} \mathbf{I}_{2N}$. Using this, we obtain

$$\frac{\partial \mathcal{L}_1(\sigma_1^2, \sigma_2^2, \mathbf{F}, \mathbf{A}; \mathbf{Z})}{\partial \sigma^2} = 0 = -\frac{2N}{\sigma^2} + \frac{1}{(\sigma^2)^2} \text{tr} [\mathbf{P}_{\mathbf{F}}^\perp (\mathbf{Z} \mathbf{Z}^H / M)]. \quad (2.32)$$

We defined $\mathbf{P}_{\mathbf{F}} = \mathbf{F} (\mathbf{F}^H \mathbf{F})^{-1} \mathbf{F}^H = \mathbf{F} \mathbf{F}^H$ ($\mathbf{F}$ has unitary columns) and $\mathbf{P}_{\mathbf{F}}^\perp = \mathbf{I}_{2N} - \mathbf{P}_{\mathbf{F}}$ to simplify the above expression. The resulting estimate for the variance is

$$\hat{\sigma}^2|_1 = \frac{1}{2N} \text{tr} [\mathbf{P}_{\mathbf{F}}^\perp (\mathbf{Z} \mathbf{Z}^H / M)]. \quad (2.33)$$

The corresponding compressed likelihood is

$$\mathcal{L}_1(\hat{\sigma}^2|_1, F, \hat{\mathbf{A}}; \mathbf{Z}) = -M(2N) \log \hat{\sigma}^2|_1 - M(2N). \quad (2.34)$$

As before, the final trace term evaluates to $M(2N)$ after invoking the lemma from [78].

In the case where $\sigma_1^2 \neq \sigma_2^2$ are unknown, then the variances appear in (2.31) in the leading log terms, $\mathbf{R}$, and in the matrix inverse of $\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}}^\perp$. The result is that the ML estimates of σ_1^2 and σ_2^2 are the zeros of a pair of coupled nonlinear equations. Rather than solve these equations, we instead use the "local" estimates of the variance which maximize

$$\mathcal{L}_1^{\text{loc}}(\sigma_l^2, \mathbf{F}_l, \hat{\mathbf{A}}; \mathbf{Z}_l) = -MN \log \sigma_l^2 - M \text{tr}[\mathbf{P}_{\mathbf{R}_l^{-1/2}\mathbf{F}_l}^\perp \mathbf{R}_l^{-1/2} (\mathbf{Z}_l \mathbf{Z}_l^H / M) \mathbf{R}_l^{-1/2}], \quad (2.35)$$

for $l = 1, 2$. Here, we use subscript l when we only consider the rows of a given matrix associated with sensor l . Recall that $\mathbf{R}_l^{-1} = \frac{1}{\sigma_l^2} \mathbf{I}_N$. Thus, we can follow the same process as for the case where $\sigma_1^2 = \sigma_2^2 = \sigma^2$, this gives estimates

$$\hat{\sigma}_1^2|_1 = \frac{1}{N} \text{tr}[\mathbf{P}_{\mathbf{H}_1}^\perp (\mathbf{Z}_1 \mathbf{Z}_1^H / M)] \quad (2.36)$$

$$\hat{\sigma}_2^2|_1 = \frac{1}{N} \text{tr}[\mathbf{P}_{\mathbf{H}_2}^\perp (\mathbf{Z}_2 \mathbf{Z}_2^H / M)]. \quad (2.37)$$

Note that $\mathbf{P}_{\mathbf{F}_l} = \mathbf{P}_{\mathbf{H}_l}$. The compressed likelihood is then

$$\mathcal{L}_1(\hat{\sigma}_1^2|_1, \hat{\sigma}_2^2|_1, \mathbf{F}, \hat{\mathbf{A}}; \mathbf{Z}) = -MN \log \hat{\sigma}_1^2|_1 - MN \log \hat{\sigma}_2^2|_1 - M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2}\mathbf{F}}^\perp \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z} \mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right], \quad (2.38)$$

with $\hat{\mathbf{R}}_1^{-1/2} = \text{diag}[\hat{\sigma}_1^2|_1 \mathbf{I}_N, \hat{\sigma}_2^2|_1 \mathbf{I}_N]$. Note that only the final trace term depends on data from both sensors.

2.5 Bistatic Passive Detectors

Recall that the channel matrices take the form $\mathbf{F}_l = g_l \mathbf{H}_l$. This gives rise to three cases:

1. Known channel: $\mathbf{F}_l = g_l \mathbf{H}_l$ is known. Coherent processing possible.
2. Partially known channel: $\mathbf{H}_l$ basis mapping is known but complex gain g_l is unknown. Coherent processing only partially possible.
3. Unknown channel: g_l is unknown and only the dimension of $\langle \mathbf{H}_l \rangle$ is known. Only incoherent processing possible.

Each of these three cases has three subcases:

1. Noise variances σ_1^2 and σ_2^2 are known.
2. Noise variances σ_1^2 and σ_2^2 are unknown but assumed to be equal.
3. Noise variances σ_1^2 and σ_2^2 are unknown.

This gives nine total cases for which we present detectors.

2.5.1 Known channel case

We assume here that the composite channel matrix $\mathbf{F}$ is known. Although an unrealistic assumption, the results help to establish methods used for later cases.

Noise variances σ_1^2 and σ_2^2 known

The log-likelihood ratio is given by

$$\mathcal{L} = \mathcal{L}_1(\sigma_1^2, \sigma_2^2, \mathbf{F}, \hat{\mathbf{A}}; \mathbf{Z}) - \mathcal{L}_0(\sigma_1^2, \sigma_2^2, ; \mathbf{Z}). \quad (2.39)$$

Inserting equations (2.18) and (2.31) yields

$$\mathcal{L} = M \text{tr} \left[\mathbf{P}_{\mathbf{R}^{-1/2} \mathbf{F}} \mathbf{R}^{-1/2} (\mathbf{Z} \mathbf{Z}^H / M) \mathbf{R}^{-1/2} \right]. \quad (2.40)$$

Using the cyclic property of the trace, the fact that $\mathbf{P}_F = \mathbf{H}\mathbf{H}^H$, and some algebra, we can reformulate this ratio statistic as a Rayleigh quotient

$$\mathcal{L} = \frac{\boldsymbol{\beta}^H \mathbf{M} \boldsymbol{\beta}}{|\boldsymbol{\beta}|^2}. \quad (2.41)$$

Here, we have defined

$$\boldsymbol{\beta} = \begin{bmatrix} g_1/\sigma_1 \\ g_2/\sigma_2 \end{bmatrix}, \quad (2.42)$$

and

$$\mathbf{M}(\sigma_1, \sigma_2) = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \bar{\Lambda}_{12} & \Lambda_{22} \end{bmatrix} \quad (2.43)$$

with

$$\Lambda_{ll'} = \text{tr} \left[\mathbf{H}_l^H \frac{\mathbf{Z}_l \mathbf{Z}_{l'}^H}{\sigma_l \sigma_{l'}} \mathbf{H}_{l'} \right]. \quad (2.44)$$

The terms where $l = l'$ are matched subspace detectors [17]. These are scaled by β to determine their contribution to the total ratio. The term Λ_{12} measures correlation between the data from sensors 1 and 2.

When $M = 1$, we have

$$\Lambda_{ll'} = \text{tr} \left[\mathbf{H}_l^H \frac{\mathbf{z}_l \mathbf{z}_{l'}^H}{\sigma_l \sigma_{l'}} \mathbf{H}_{l'} \right]. \quad (2.45)$$

When $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{I}$, this simplifies further to

$$\Lambda_{ll'} = \text{tr} \left[\frac{\mathbf{z}_l \mathbf{z}_{l'}^H}{\sigma_l \sigma_{l'}} \right]. \quad (2.46)$$

In this case our terms with $l = l'$ are no longer matched subspace detectors but just energy detectors. The final term still measures correlation between the measurements from each sensor.

Noise variances σ_1^2 and σ_2^2 unknown but equal

We now construct our likelihood ratio using (2.21) and (2.34). Taking $\mathcal{L} = \mathcal{L}_1 - \mathcal{L}_0$ gives

$$\mathcal{L} = -M(2N) \log \hat{\sigma}^2|_1 + M(2N) \log \hat{\sigma}^2|_0 \quad (2.47)$$

$$= -M(2N) \log \left(\frac{1}{2N} \text{tr}[(\mathbf{I} - \mathbf{P}_F)\mathbf{Z}\mathbf{Z}^H/M] \right) + M(2N) \log \left(\frac{1}{2N} \text{tr}[(\mathbf{Z}\mathbf{Z}^H/M)] \right). \quad (2.48)$$

Noting that log is a monotone function, with some algebra we can see that

$$\mathcal{L} \sim \frac{\text{tr}[\mathbf{P}_F(\mathbf{Z}\mathbf{Z}^H/M)]}{\text{tr}[(\mathbf{Z}\mathbf{Z}^H/M)]} = \frac{\beta^H \mathbf{M}(1, 1) \beta}{|\beta|^2 \text{tr}[\mathbf{Z}\mathbf{Z}^H]}. \quad (2.49)$$

We use $\mathbf{M}(1, 1) = \mathbf{M}(\sigma_1 = 1, \sigma_2 = 1)$. The case where $M = 1$ and $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{I}$ follow the cases in the previous section with the exception now that we have $\mathbf{M}(1, 1)$ rather than $\mathbf{M}(\sigma_1, \sigma_2)$ and the Rayleigh quotient is scaled by $1/\text{tr}[\mathbf{Z}\mathbf{Z}^H]$.

Noise Variances σ_1^2 and σ_2^2 unknown

In this case our likelihood ratio for H_0 is (2.25) and for H_1 is (2.38). Our likelihood ratio, neglecting additive constants, is thus

$$\mathcal{L} = MN \left(\log \frac{\hat{\sigma}_1^2|_0}{\hat{\sigma}_1^2|_1} + \log \frac{\hat{\sigma}_2^2|_0}{\hat{\sigma}_2^2|_1} \right) - M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2} \mathbf{F}}^\perp \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H/M) \hat{\mathbf{R}}_1^{-1/2} \right]. \quad (2.50)$$

The first terms are monotone functions of local detectors, and the third term is a cross validation term which depends on data from both sensors. In the case where $M = 1$, data matrices $\mathbf{Z}_l$ become vectors $\mathbf{z}_l$. When $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{I}$ then trace terms further simplify to a form like $\mathbf{z}_l^H \mathbf{z}_{l'}$.

2.5.2 Partially known channel case

Here, we assume that channel bases $\mathbf{H}_1$ and $\mathbf{H}_2$ are known but the complex gains g_1 and g_2 are unknown.

Noise Variances σ_1^2 and σ_2^2 known

Begin with (2.41). We now aim to maximize this expression with respect to the unknown complex gains g_1 and g_2 . Recalling that these gains appear in the definition of $\beta = [g_1/\sigma_1, g_2/\sigma_2]^T$, we can see that maximizing the likelihood ratio is equivalent to maximizing the Rayleigh quotient with respect to a normalized unit vector β . This solution is given by

$$\mathcal{L} = \text{maxev}(\mathbf{M}(\sigma_1, \sigma_2)), \quad (2.51)$$

where $\text{maxev}(\cdot)$ denotes the maximum eigenvalue of a matrix. As clock synchronization errors and sensor location errors are often found in an unknown multiplicative phase, the maximum eigenvector of our matrix $\mathbf{M}$ represents a passive autofocus solution.

Since $\mathbf{M}(\sigma_1, \sigma_2)$ is a 2×2 matrix, a compact analytic solution for the maximum eigenvalue can be found. This is given by

$$\mathcal{L} = A \left(1 + \sqrt{1 - \frac{G^2}{A^2}(1 - |\rho_{12}|^2)} \right), \quad (2.52)$$

with $A = \frac{1}{2}(\Lambda_{11} + \Lambda_{22})$, $G = (\Lambda_{11}\Lambda_{22})^{1/2}$, and $|\rho_{12}|^2 = \frac{|\Lambda_{12}|^2}{\Lambda_{11}\Lambda_{22}}$. See (2.44) for the definition of $\Lambda_{ll'}$. When $M = 1$, then we replace $\mathbf{Z}_l$ with $\mathbf{z}_l$ in (2.44). If $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{I}$ then this trace term becomes $\mathbf{z}_l^H \mathbf{z}_{l'}$.

Noise variances σ_1^2 and σ_2^2 unknown but equal

Now we begin with (2.49). We know that rather than maximizing the exact likelihood function with respect to the unknown complex gains, we can equivalently maximize a monotonic function of the likelihood with respect to the unknown complex gains. Thus, we maximize

$$\mathcal{L} \sim \frac{\beta^H \mathbf{M}(1, 1) \beta}{|\beta|^2 \text{tr}[\mathbf{Z}\mathbf{Z}^H]} \quad (2.53)$$

with respect to a unit vector β , just as was done for the known noise variance case. This results in

$$\mathcal{L} \sim \frac{\text{maxev}(\mathbf{M}(1, 1))}{\text{tr}[\mathbf{Z}\mathbf{Z}^H]}. \quad (2.54)$$

For the cases where $M = 1$ and the cases where $\mathbf{H}_1 = \mathbf{H}_2 = \mathbf{I}$, the same substitutions can be made as were made in the known noise variance case.

Noise variances σ_1^2 and σ_2^2 unknown

Start with the detector for the known channel and unknown noise variance case, given in (2.50). We aim to maximize this with respect to the unknown channel gains g_1 and g_2 . Note, these terms only appear in the final trace term. Focus for now on this term; we can re-write it as

$$-M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2} \mathbf{F}}^\perp \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right] = -M \text{tr} \left[\hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right] \quad (2.55)$$

$$+ M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2} \mathbf{F}} \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right] \quad (2.56)$$

$$= -M \text{tr} \left[\hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right] \quad (2.57)$$

$$+ M \frac{\beta^H \mathbf{M}(\hat{\sigma}_1|_1, \hat{\sigma}_2|_1) \beta}{|\beta|^2}. \quad (2.58)$$

Now only the Rayleigh quotient term depends on the unknown channel gains and we can maximize the full likelihood ratio as

$$\mathcal{L} = MN \left(\log \frac{\hat{\sigma}_1^2|_0}{\hat{\sigma}_1^2|_1} + \log \frac{\hat{\sigma}_2^2|_0}{\hat{\sigma}_2^2|_1} \right) \quad (2.59)$$

$$- M \text{tr} \left[\hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right] \quad (2.60)$$

$$+ M \text{maxev}(\mathbf{M}(\hat{\sigma}_1|_1, \hat{\sigma}_2|_1)). \quad (2.61)$$

2.5.3 Unknown channel case

We now assume that we only know the dimension of the space $\langle \mathbf{F} \rangle$. Thus, our goal for the following cases will be to develop expressions which are maximized with respect to these unknown channel mappings $\mathbf{F}_1$ and $\mathbf{F}_2$.

Noise variances σ_1^2 and σ_2^2 known

Begin with (2.40). We wish to maximize the expression with respect to $\mathbf{F}$ which means choosing the projection $\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}}$ which maximizes the expression with the constraint that the space must be of dimension K . Thus, we set

$$\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}} = \mathbf{U}_K \mathbf{U}_K^H \quad (2.62)$$

with $\mathbf{U}_K \in \mathbb{C}^{2N \times K}$ the matrix of the eigenvectors corresponding to the K largest eigenvalues of $\mathbf{R}^{-1/2} \mathbf{Z} \mathbf{Z}^H \mathbf{R}^{-1/2}$. Substituting this projection into the trace term gives

$$\mathcal{L} = \sum_{k=1}^K \text{ev}_k(\mathbf{R}^{-1/2} \mathbf{Z} \mathbf{Z}^H \mathbf{R}^{-1/2}), \quad (2.63)$$

where $\text{ev}_k(\cdot)$ denotes the k th largest eigenvalue. This sum represents an energy measurement in the dominant subspace of the whitened correlation matrix. Note that when $M = 1$, there is only a single eigenvalue of the matrix so the expression reduces to $\mathcal{L} = \mathbf{z}^H \mathbf{R}^{-1} \mathbf{z}$.

Noise variances σ_1^2 and σ_2^2 unknown but equal

Start with (2.54). We wish now to maximize this with respect to the unknown $\mathbf{F}$. Again, we choose the projection onto the dominant eigenspace, this time of $\mathbf{Z} \mathbf{Z}^H$. Take $\mathbf{P}_{\mathbf{F}} = \mathbf{U}_K \mathbf{U}_K^H$, where $\mathbf{U}_K$ is a matrix whose columns are the dominant eigenvectors of $\mathbf{Z} \mathbf{Z}^H$. The resulting log likelihood ratio is

$$\mathcal{L} = \frac{\sum_{k=1}^K \text{ev}(\mathbf{Z} \mathbf{Z}^H)}{\text{tr}[\mathbf{Z} \mathbf{Z}^H]} = \frac{\sum_{k=1}^K \text{ev}(\mathbf{Z} \mathbf{Z}^H)}{\sum_{k=1}^{2N} \text{ev}(\mathbf{Z} \mathbf{Z}^H)}. \quad (2.64)$$

This statistic represents the fraction of the total energy which is found in the dominant eigenspace, and as a result, when $M = 1$ there is no likelihood ratio ($\mathbf{z} \mathbf{z}^H$ only has a rank of 1).

Noise variances σ_1^2 and σ_2^2 unknown

Since (2.18) has no dependence on $g_1 \mathbf{H}_1$ or $g_2 \mathbf{H}_2$, there is no additional maximization that needs to be carried out. Thus we take (2.25), which, after dropping additive constants, can be written equivalently as

$$\max_{H_0} \mathcal{L}_0 = -MN \left(\log \sum_{k=1}^N \text{ev}_k(\mathbf{Z}_1 \mathbf{Z}_1^H) + \log \sum_{k=1}^N \text{ev}_k(\mathbf{Z}_2 \mathbf{Z}_2^H) \right). \quad (2.65)$$

Recall that in this case, we approximated the estimates of the noise under H_1 . We now maximize these approximations with respect to the unknown subspace channel matrices $\mathbf{H}_1$ and $\mathbf{H}_2$. Recall that these estimates are given by

$$\hat{\sigma}_1^2|_1 = \frac{1}{N} \text{tr}[\mathbf{P}_{\mathbf{H}_1}^\perp (\mathbf{Z}_1 \mathbf{Z}_1^H / M)] \quad (2.66)$$

$$\hat{\sigma}_2^2|_1 = \frac{1}{N} \text{tr}[\mathbf{P}_{\mathbf{H}_2}^\perp (\mathbf{Z}_2 \mathbf{Z}_2^H / M)]. \quad (2.67)$$

Maximizing with respect to the unknown basis has us choose $\mathbf{P}_{\mathbf{H}_1}^\perp = \mathbf{V}_1 \mathbf{V}_1^H$ where $\mathbf{V}_1$ is the $N \times (N - K)$ matrix of eigenvectors associated with the $N - K$ smallest eigenvalues of $\mathbf{Z}_1 \mathbf{Z}_1^H$ ($\mathbf{V}_2$ is defined analogously). As a result, the trace becomes equivalent to the sum of the $N - K$ smallest eigenvalues of the covariance matrix. Our approximate variance estimates thus becomes

$$\hat{\sigma}_1^2|_1 = \sum_{K+1}^N \text{ev}_k(\mathbf{Z}_1 \mathbf{Z}_1^H / M) \quad (2.68)$$

$$\hat{\sigma}_2^2|_1 = \sum_{K+1}^N \text{ev}_k(\mathbf{Z}_2 \mathbf{Z}_2^H / M). \quad (2.69)$$

We use these estimates to define $\hat{\mathbf{R}}_1 = \text{diag}[\hat{\sigma}_1^2|_1 \mathbf{I}_N, \hat{\sigma}_2^2|_1 \mathbf{I}_N]$. We also define $\hat{\mathbf{F}} = [\mathbf{V}_1'; \mathbf{V}_2']$, where $\mathbf{V}_1'$ is the $N \times K$ matrix of the eigenvectors associated with the K largest eigenvalues of $\mathbf{Z}_1 \mathbf{Z}_1^H$ (similarly for $\mathbf{V}_2'$). Substituting this into (2.38) and then combining with (2.65) yields

$$\mathcal{L} = MN \left(\log \frac{\hat{\sigma}_1^2|_0}{\hat{\sigma}_1^2|_1} + \log \frac{\hat{\sigma}_2^2|_0}{\hat{\sigma}_2^2|_1} \right) - M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2} \hat{\mathbf{F}}}^\perp \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z} \mathbf{Z}^H / M) \hat{\mathbf{R}}_1^{-1/2} \right]. \quad (2.70)$$

Similar to the case where the variance was unknown but assumed equal, there is no likelihood ratio statistic (not detector) when $M = 1$ as the ratio of eigenspaces is undefined for a rank 1 covariance matrix.

2.5.4 Final detectors

We include here a table organizing all nine detectors. Each column corresponds to different assumed knowledge of the subspace channel and each row corresponds to different assumed knowledge of the noise variances.

Table 2.1: Table of Detector Statistics

	Known Channel	Partially Known Channel	Unknown Channel
Known noise			
variance	$\mathcal{L} = M \text{tr} [\mathbf{P}_{\mathbf{R}^{-1/2}\mathbf{F}} \mathbf{R}^{-1/2} (\mathbf{Z}\mathbf{Z}^H/M) \mathbf{R}^{-1/2}]$	$\mathcal{L} = \text{maxev}(\mathbf{M}(\sigma_1, \sigma_2))$	$\mathcal{L} = \sum_{k=1}^K \text{ev}_k(\mathbf{R}^{-1/2} (\mathbf{Z}\mathbf{Z}^H/M) \mathbf{R}^{-1/2})$
Unknown but			
equal noise	$\mathcal{L} = \frac{\beta^H \mathbf{M}(\sigma_1 = 1, \sigma_2 = 1) \beta / \beta ^2}{\text{tr}[\mathbf{Z}\mathbf{Z}^H/M]}$	$\mathcal{L} = \frac{\text{maxev}(\mathbf{M}(\sigma_1 = 1, \sigma_2 = 1))}{\text{tr}[\mathbf{Z}\mathbf{Z}^H/M]}$	$\mathcal{L} = \frac{\sum_{k=1}^K \text{ev}_k(\mathbf{Z}\mathbf{Z}^H/M)}{\text{tr}[\mathbf{Z}\mathbf{Z}^H/M]}$
variance			
Unknown			
noise	$\mathcal{L} = MN \left(\log \frac{\hat{\sigma}_1^2 _0}{\hat{\sigma}_1^2 _1} + \log \frac{\hat{\sigma}_2^2 _0}{\hat{\sigma}_2^2 _1} \right)$	$\mathcal{L} = MN \left(\log \frac{\hat{\sigma}_1^2 _0}{\hat{\sigma}_1^2 _1} + \log \frac{\hat{\sigma}_2^2 _0}{\hat{\sigma}_2^2 _1} \right)$	$\mathcal{L} \approx MN \left(\log \frac{\hat{\sigma}_1^2 _0}{\hat{\sigma}_1^2 _1} + \log \frac{\hat{\sigma}_2^2 _0}{\hat{\sigma}_2^2 _1} \right)$
variance	$- M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2}\mathbf{F}}^\perp \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H/M) \hat{\mathbf{R}}_1^{-1/2} \right]$	$- M \text{tr}[\hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H/M) \hat{\mathbf{R}}_1^{-1/2}]$ $+ M \text{maxev}(\mathbf{M}(\hat{\sigma}_1 _1, \hat{\sigma}_2 _1))$	$- M \text{tr} \left[\mathbf{P}_{\hat{\mathbf{R}}_1^{-1/2}\hat{\mathbf{F}}}^\perp \hat{\mathbf{R}}_1^{-1/2} (\mathbf{Z}\mathbf{Z}^H/M) \hat{\mathbf{R}}_1^{-1/2} \right]$

2.6 Numerical Results

We present some numerical results for the above derived detector statistics. The effects of the differing modeling assumptions and knowledge of the channel and noise are compared. First, we introduce what we refer to as "diversity-combining" detectors. These detectors neglect coherent information transfer between platforms. These will provide an important comparison and insight into the performance of our detectors. We will then present both receiver operating characteristic (ROC) curves and probability of detection (P_D) versus input SNR curves for each detector.

2.6.1 Diversity-combining detectors

We are particularly interested in what cases and under what conditions the terms in our detectors which combine data from both detectors are of importance. Thus, we first present some detectors derived by assuming that each platform is independent. GLR detectors at each platform are derived assuming only measurements at the given platform are available and the resulting statistics are then non-coherently added.

As an example, consider the case where the noise is known but our channel model is unknown. The result for two platforms is given in (2.63). If we now consider a case where both sensors compute their test statistics independently, and then add the results together, we obtain a similar result with the exception that rather than the full covariance matrix $\mathbf{Z}\mathbf{Z}^H$, we compute the eigenvalues of each locally computed covariance matrix $\mathbf{Z}_1\mathbf{Z}_1^H$ and $\mathbf{Z}_2\mathbf{Z}_2^H$. The result is

$$\mathcal{L}_D = \sum_{k=1}^K \text{ev}_k(\mathbf{R}_1^{-1/2}(\mathbf{Z}_1\mathbf{Z}_1^H/M)\mathbf{R}_1^{-1/2}) + \sum_{k=1}^K \text{ev}_k(\mathbf{R}_2^{-1/2}(\mathbf{Z}_2\mathbf{Z}_2^H/M)\mathbf{R}_2^{-1/2}). \quad (2.71)$$

Similar equations can be developed for all nine cases and are presented in the following table.

Table 2.2: Table of Diversity-Combining Detectors

	Known Channel	Partially Known Channel	Unknown Channel
Known noise variance	$\mathcal{L}_D = \frac{\beta_1^2 \text{tr}[\mathbf{P}_{\mathbf{H}_1}(\mathbf{Z}_1 \mathbf{Z}_1^H / M \sigma_1^2)]}{ \beta ^2} + \frac{\beta_2^2 \text{tr}[\mathbf{P}_{\mathbf{H}_2}(\mathbf{Z}_2 \mathbf{Z}_2^H / M \sigma_2^2)]}{ \beta ^2}$	$\mathcal{L}_D = \text{tr}[\mathbf{P}_{\mathbf{H}_1}(\mathbf{Z}_1 \mathbf{Z}_1^H / M \sigma_1^2)] + \text{tr}[\mathbf{P}_{\mathbf{H}_2}(\mathbf{Z}_2 \mathbf{Z}_2^H / M \sigma_2^2)]$	$\mathcal{L}_D = \sum_{k=1}^K \text{ev}_k(\mathbf{R}_1^{-1/2}(\mathbf{Z}_1 \mathbf{Z}_1^H / M) \mathbf{R}_1^{-1/2}) + \sum_{k=1}^K \text{ev}_k(\mathbf{R}_2^{-1/2}(\mathbf{Z}_2 \mathbf{Z}_2^H / M) \mathbf{R}_2^{-1/2})$
Unknown but equal noise variance	$\mathcal{L}_D = \frac{\text{tr}[\mathbf{P}_{\mathbf{H}_1}(\mathbf{Z}_1 \mathbf{Z}_1^H / M)]}{\text{tr}[\mathbf{Z} \mathbf{Z}^H / M]} + \frac{\text{tr}[\mathbf{P}_{\mathbf{H}_2}(\mathbf{Z}_2 \mathbf{Z}_2^H / M)]}{\text{tr}[\mathbf{Z} \mathbf{Z}^H / M]}$	$\mathcal{L}_D = \max \left(\frac{\text{tr}[\mathbf{P}_{\mathbf{H}_1}(\mathbf{Z}_1 \mathbf{Z}_1^H / M)]}{\text{tr}[\mathbf{Z} \mathbf{Z}^H / M]}, \frac{\text{tr}[\mathbf{P}_{\mathbf{H}_2}(\mathbf{Z}_2 \mathbf{Z}_2^H / M)]}{\text{tr}[\mathbf{Z} \mathbf{Z}^H / M]} \right)$	$\mathcal{L}_D = \frac{\sum_{k=1}^K \text{ev}_k(\mathbf{Z}_1 \mathbf{Z}_1^H / M)}{\text{tr}[\mathbf{Z}_1 \mathbf{Z}_1^H / M]} + \frac{\sum_{k=1}^K \text{ev}_k(\mathbf{Z}_2 \mathbf{Z}_2^H / M)}{\text{tr}[\mathbf{Z}_2 \mathbf{Z}_2^H / M]}$
Unknown noise variance	$\mathcal{L}_D = MN \log \frac{\hat{\sigma}_1^2 _0}{\hat{\sigma}_1^2 _1} + MN \log \frac{\hat{\sigma}_2^2 _0}{\hat{\sigma}_2^2 _1}$	$\mathcal{L}_D = -M \text{tr}[\mathbf{P}_{\mathbf{H}_1}^\perp(\mathbf{Z}_1 \mathbf{Z}_1^H / M \hat{\sigma}_1^2 _1)] - M \text{tr}[\mathbf{P}_{\mathbf{H}_2}^\perp(\mathbf{Z}_2 \mathbf{Z}_2^H / M \hat{\sigma}_2^2 _1)]$	$\mathcal{L}_D = MN \left(\log \frac{\hat{\sigma}_1^2 _0}{\hat{\sigma}_1^2 _1} + \log \frac{\hat{\sigma}_2^2 _0}{\hat{\sigma}_2^2 _1} \right)$

2.6.2 Simulation details

Producing ROC curves

Here we will give the details and procedures used to produce the ROC curves that are to follow. For all experiments, we collect samples of length $N = 16$, we collect $M = 64$ samples, and use a subspace dimension of $K = 8$. Then we define our subspace $\langle \mathbf{H} \rangle$ to be a K column slice of a N dimensional DFT matrix. To create the signal, we assign random weights to each of these basis vectors. The weights are drawn from a $\mathcal{N}(0, 1)$ distribution. A multiplicative scale factor is applied to the signal and we then add $\mathcal{CN}(0, 1.3)$ noise. The scale factor is chosen such that the SNR is -10 dB. We then produce 50,000 realizations of our noisy signal as well as 50,000 realizations of data which consists of just noise. The eighteen different detectors (including the nine diversity-combining detectors) are then evaluated on both the data set containing a common signal and the data set containing only noise. For each of the nine cases, we then define 1,000 different detector thresholds equally spaced between the minimum and maximum detector output produced by the signal-free data set. Defining "detection" to mean a detector produces a value above a given threshold, we then compute probabilities of detection (PD) using the signal-containing data and probabilities of false alarm (PFA) using the signal-free data. The resulting probabilities are used to construct the ROC curves.

Producing PD vs SNR curves

To produce the probability of detection vs SNR curves, we defined the parameters N , M , and K and signal subspace in the same way as for the ROC curves. The signal and additive noise is produced in the same way as was done for the ROC curves with the exception that we produce different realizations with different signal gains calculated to produce a range of different SNR's. For each SNR value, 5,000 realizations of data containing a common signal and 5,000 realizations of data containing only noise are produced. Each of the different detectors is evaluated on both of these datasets; we use the signal-free data results to set thresholds which yield a PFA of 0.01 for each case. Then, for each dataset at each SNR, we use the computed threshold to compute the

probability of detection on our signal-containing data. The PD vs SNR curves are constructed with these results.

2.6.3 Known Channel Case

In this case the channel matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ are known. We consider cases where the noise variance is assumed to be known, unknown but equal, and unknown.

We show ROC curves for the six detectors in Fig.2.1. These detectors correspond to the first column of Tables 2.1 and 2.2. Diversity-combining detectors are displayed with dashed lines. Note that in this case, knowledge of the variance leads to better performance as expected, and the use of coherent information allows the full detectors to perform better than the diversity-combining detectors.

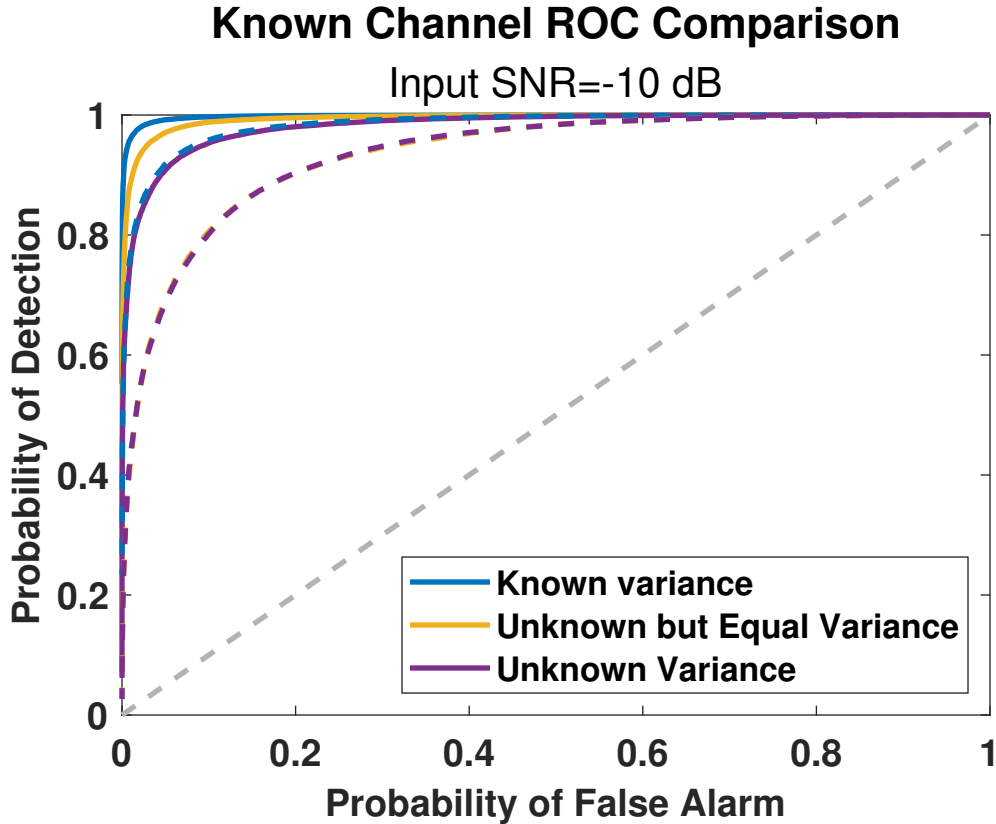


Figure 2.1: PD vs PFA for fully known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.

We show PD vs SNR curves in Fig.2.2. We can see that when we know the noise variance, diversity-combining results in about a 2 dB loss in performance when compared to the full detector. Similarly about a 4 dB loss and 3 dB loss are seen for the cases where the noise is unknown but equal on each detector and the noise is unknown respectively.

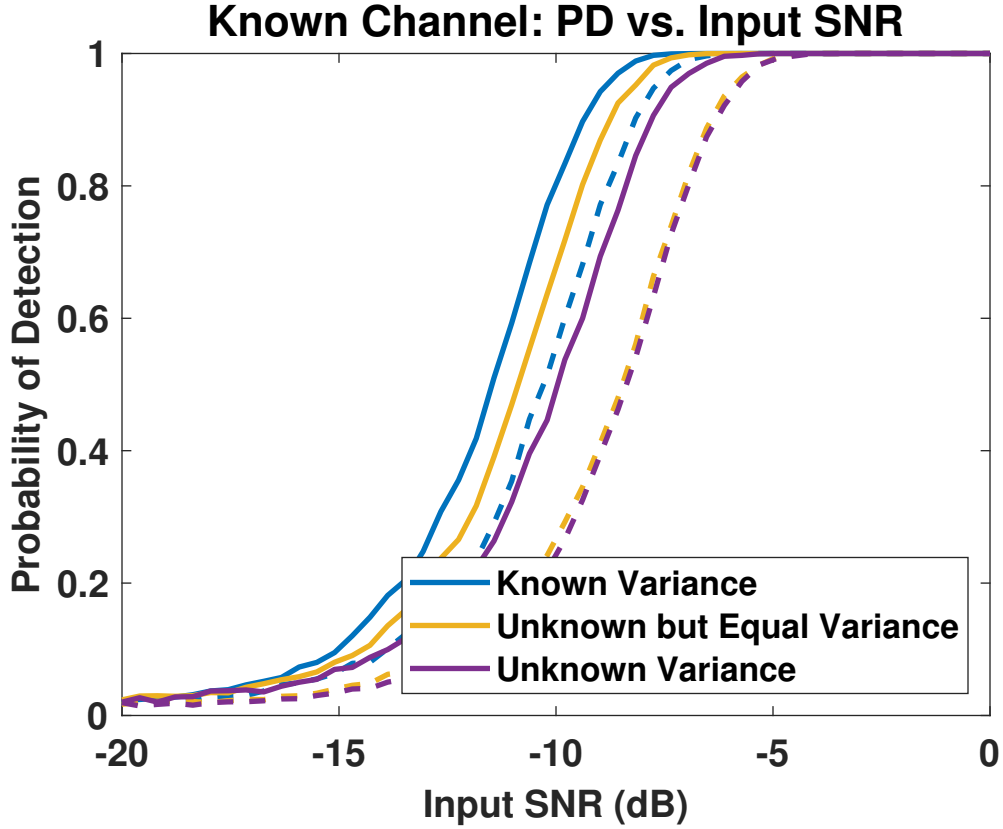


Figure 2.2: PD vs SNR for fully known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.

2.6.4 Partially Known Channel Case

In this case, we assume that the subspace basis matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ are known but the complex gains g_1 and g_2 are not. We again consider the cases where the noise variances are assumed to be known, unknown but equal, and unknown. Results for the full detectors and diversity combined detectors are shown. These correspond to the central column of Tables 2.1 and 2.2.

Fig.2.3 depicts the ROC curves for our six detectors. Note again how as more knowledge of the variance is assumed, better performance is obtained. We also note that our full detectors are, in this case, still able to leverage known information about the subspace channel and achieve better performance than the diversity-combining statistics.

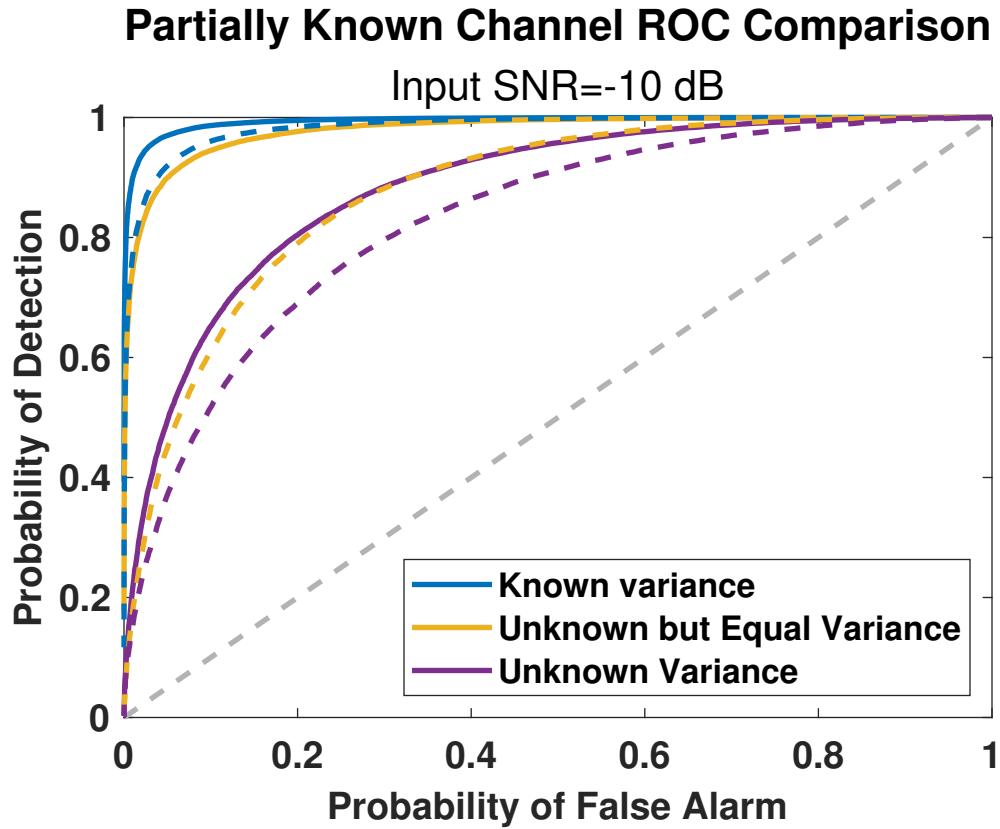


Figure 2.3: PD vs PFA for partially known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.

Fig.2.4 displays the PD vs SNR curves. When the noise variances are known, the diversity-combining detector results in a drop of about 1 dB of performance. When the noise variances are unknown but equal a 4 dB loss is observed. Finally, when the noise variances are unknown, a 2 dB loss is observed. Note that these drops in performance tend to be smaller than the loss of performance observed in the case with the fully known channel.

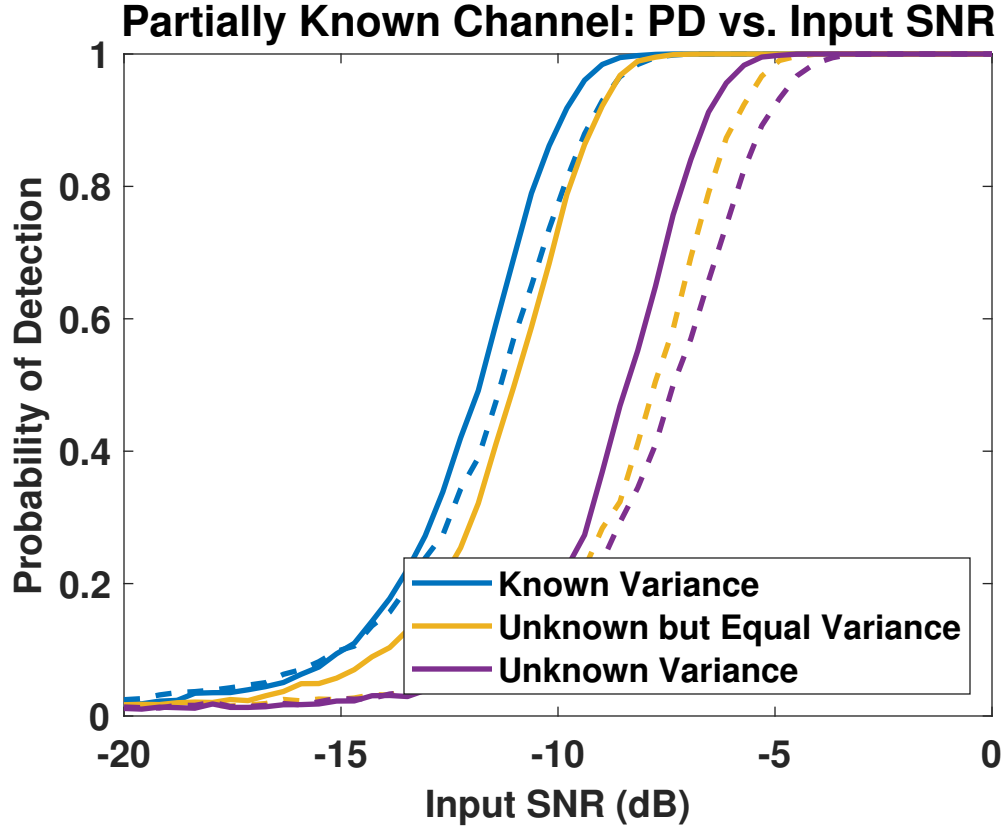


Figure 2.4: PD vs SNR for partially known channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.

2.6.5 Unknown Channel Case

In the final case, we assume that neither $\mathbf{H}_1$ and $\mathbf{H}_2$ nor g_1 and g_2 are known. We only know the dimension of the signal subspace. As before, three subcases are considered where the noise variance is assumed to be known, unknown but equal, and unknown. Full detector and diversity combined detector results are present, and these correspond to the final columns of Tables 2.1 and 2.2.

In Fig.2.5 we show the ROC curves for our six detector statistics. In this setting, we can see that unless the noise variance is known, at this SNR (-10dB) the detector performance is much lower than in our other cases. We can also see that for the case where the noise is known and the noise is unknown, the diversity combined statistics slightly outperformed the full detector statistics,

however this difference is very slight and may indicate that the performance of these detectors is essentially the same when we have no knowledge of the signal channel.

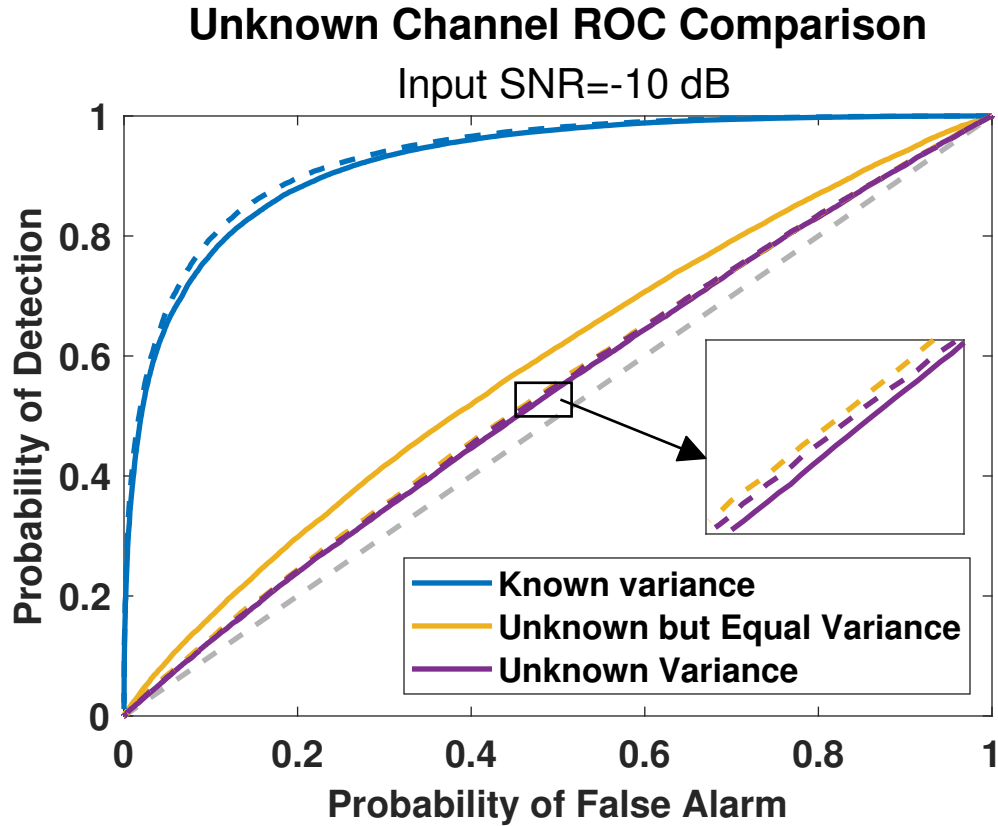


Figure 2.5: PD vs PFA for unknown channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.

In Fig.2.6 we depict the PD vs SNR curves. When the noise variances are assumed to be unknown but equal, the diversity-combining detector suffered a 3 dB loss in performance. However, for the other cases, we see essentially identical performance between the full detectors and diversity-combining ones. A possible explanation is that there is not enough information known about the subspace signal in order to effectively leverage the shared information used in the full detectors, and the result is that their performance match that of the diversity combining detectors. We take as a final note that generalized likelihood ratio detectors are detectors built on solid principles but are not claimed to be "optimal" detectors.

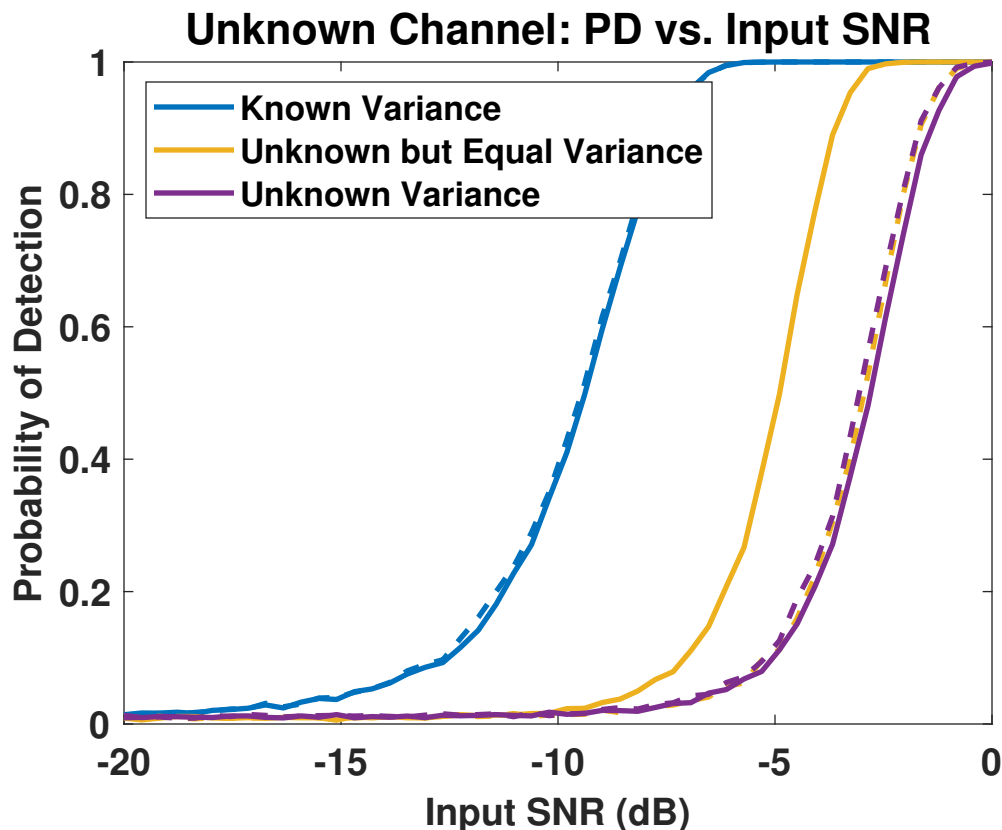


Figure 2.6: PD vs SNR for unknown channel with three different assumptions about the noise variances. The solid curves use coherent information from both sensors while the dashed curves represent diversity-combining detectors.

2.6.6 Channel Comparisons

In previous subsections we compared results for an assumed knowledge of the signal channel and different assumptions of our knowledge of the noise variances. This corresponded to comparing down the columns of Tables 2.1. Now, we instead fix our assumptions about the noise variances and compare different assumptions about the knowledge of the signal channel. This corresponds to comparisons across the rows of Table 2.1. We display results for PD vs SNR curves, which are constructed in an identical manner to those in the previous subsections. We also only show results for our full detectors, not the diversity combined detectors.

We show in Fig.2.7 the results for the case where the noise variance is assumed to be known. We observe a loss of about 1 dB performance degradation between the known and partially known

channel cases. About another 3 dB further performance loss is observed between the partially known and unknown channel cases.

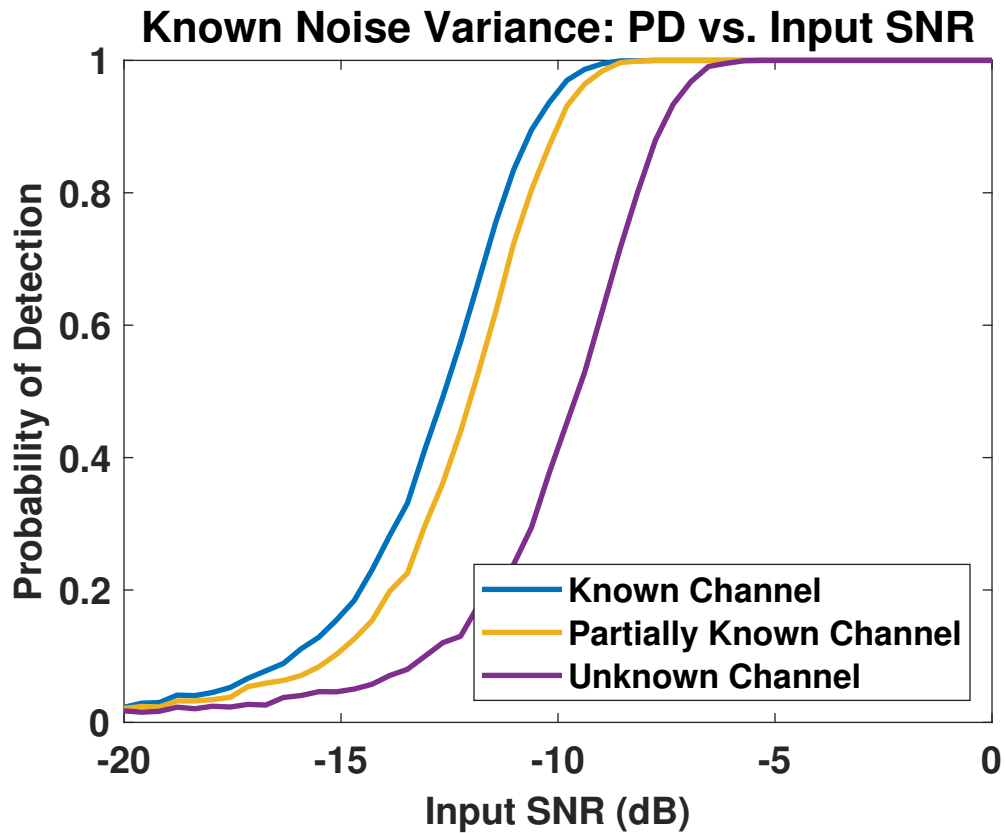


Figure 2.7: PD vs SNR for three different levels of knowledge of the signal channel model when we assume that the noise variances are known.

In Fig.2.8 we can see the results where we assume the noise variance is unknown but equal for each platform. We again see about a 1 dB loss of performance between the known and partially known channel cases, but now we observe another 6 dB loss between the partially known and unknown channel cases.

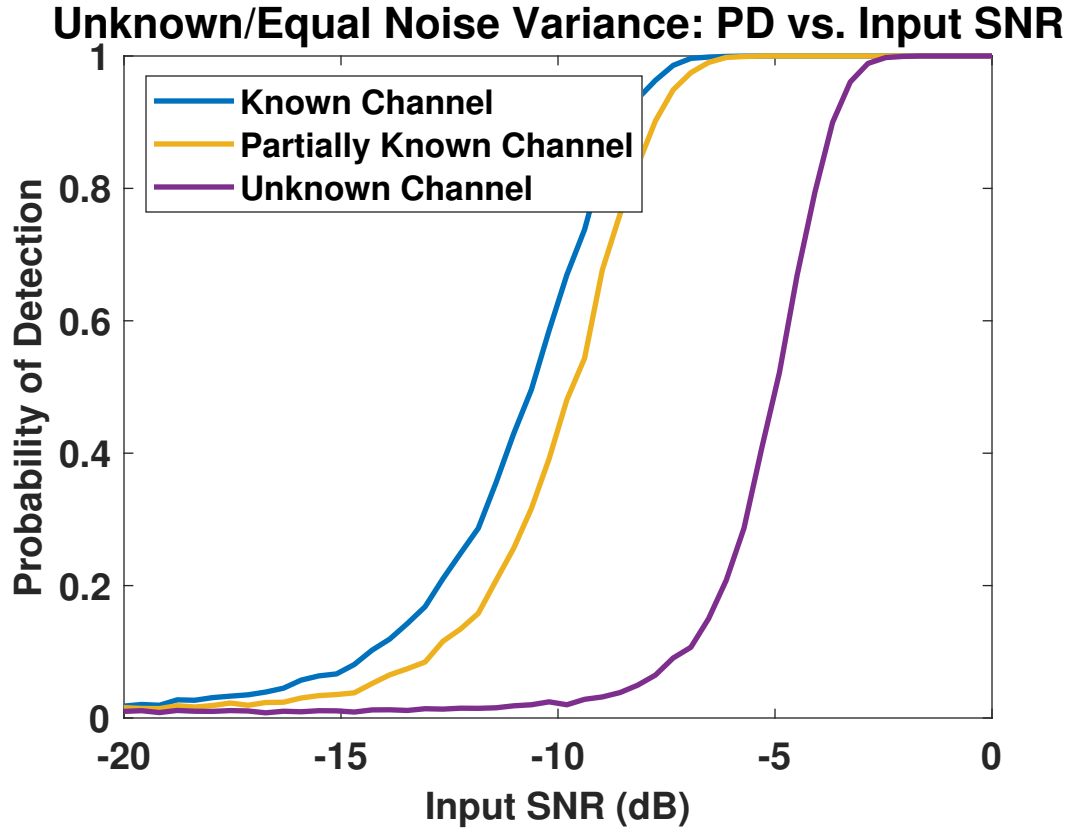


Figure 2.8: PD vs SNR for three different levels of knowledge of the signal channel model. We assume the noise variances are unknown but equal at each sensor platform.

Fig.2.9 shows the results for the case where we assume no knowledge of the noise variances. In this case, there is now about a 4 dB degradation between the known channel case and partially known channel case. Again, we observe about a 6 dB degradation between the partially known channel case and unknown channel case.

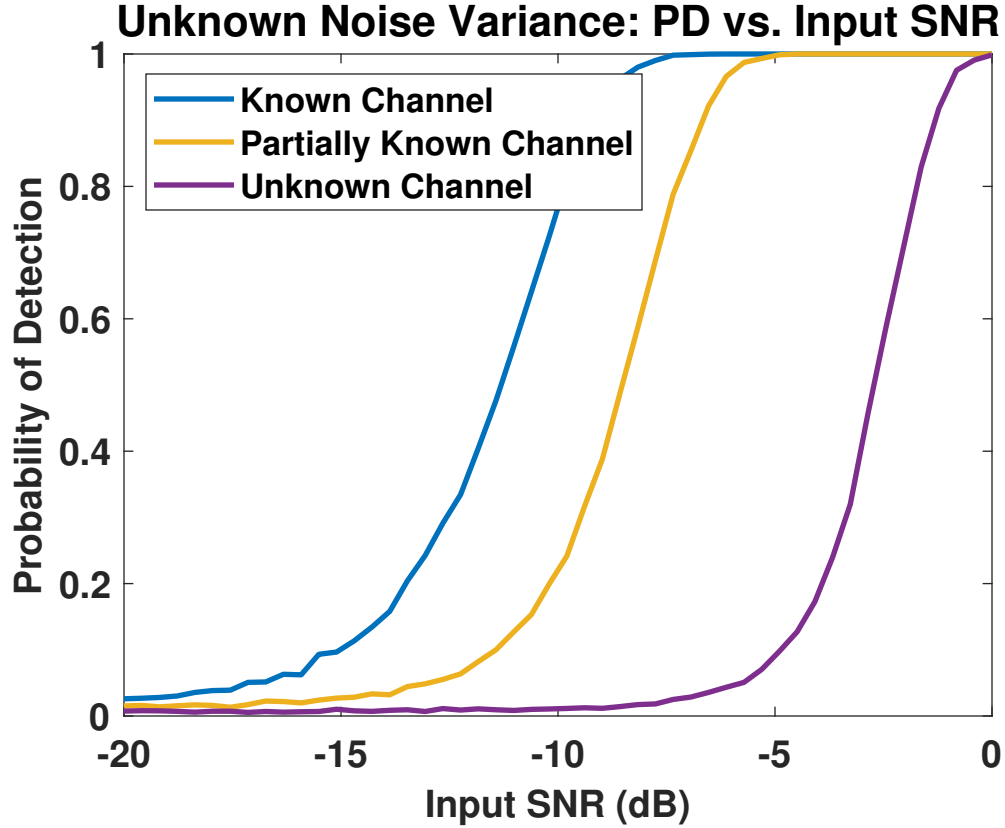


Figure 2.9: PD vs SNR for three different levels of knowledge of the signal channel model. We assume the noise variances are unknown at each sensor platform.

2.7 Conclusions

We have presented a set of nine GLR detector statistics aimed at addressing the problem of passive source localization. These likelihood ratio tests cover a range of cases for differing levels of knowledge about a given measurement's noise levels and channel mappings. The effectiveness of each of these detectors has been evaluated using simulated data. These detectors cover the first piece of the PSL problem, that of detecting a common signal measured by separated sensors. The next piece of the PSL problem is to then identify the location of the source of this shared signal.

This problem is addressed in the follow on paper [77]. The paper uses a strategy of postulating a source location, time-delaying and Doppler shifting the data to match this postulated location, and then applying a GLR detector to the resulting delayed and shifted data. A coherent sum of

these estimates as a pair of sensors move through a synthetic aperture allows for more precise estimation of the source location. Resolution and optimal filtering are also treated, and we refer the reader to [77] for further details.

Future work of interest includes the extension of the work of [77] to other cases of the nine detectors we have shown here (the paper focuses on just one case). Also of interest is the development of analogous results to the ones given here but for a model which assigns a prior statistical distribution to the unknown signal, given a "second order" statistical model for the measurements rather than the "first order" statistical model treated here.

Chapter 3

Synthetic Aperture Radar Digital Spotlighting for Linear Sampling Method Imaging

3.1 Introduction

An inverse scattering problem is the problem of determining the properties of unknown objects or targets within a scene based on the measurement of scattered fields at points outside the scene. These problems are inherently non linear and ill-posed. Inverse scattering problems are common in a wide variety of fields including non-destructive testing [27], [30], medical imaging [28], [29], [36], and radar [31], [32], [35].

Synthetic aperture radar (SAR) is a particularly well studied case of an inverse scattering problem [38], [40], [79]. In this problem, sensors typically move through space, tracing out a 'synthetic' aperture as data is collected. Commonly, SAR data is collected by a single monostatic sensor which traces an aperture which only views an imaging scene with a small span of aspect angles. Traditional SAR imaging techniques are based on several approximations, of which two closely related approximations are relevant to the work we will present here. The first approximation is the point target approximation, which assumes that targets are composed of clouds of point-like scatterers. Next is the Born approximation, which linearizes the inverse scattering problem by neglecting effects such as multiple scattering. These approximations, especially the linearizing Born approximation, eliminate the ill-posed nature of the imaging problem. They also allow traditional SAR imaging techniques to be relatively computationally efficient. These techniques work well even in the monostatic setting and with a small span of measured aspect angles. However, these approximations also fundamentally limit the ability of traditional SAR imaging techniques to fully capture the details of targets in the scene. The point target approximation places limits on the ability to

resolve certain details of the scattering targets and the reliance on the Born approximation can lead to artifacts in the scene caused by multiple scattering effects.

The Linear Sampling Method (LSM) is a well studied inverse scattering technique which aims to reconstruct only the shape of the targets in the imaging scene [48], [50], [51], [52], [53], [54]. This method avoids the point scatterer approximation and the linearizing Born approximation and thus has the potential to surpass standard SAR image reconstruction algorithms in its capacity to capture the true shape of the target. However, the LSM inverse problem is ill-posed, meaning it is much more sensitive to noise and other perturbations in the data. It has been seen that the method needs multiple sensors to surround the target from a wide span of aperture angles in order to achieve good imaging performance. This multi-sensor setup is often more difficult and expensive in practice than a setup featuring a single sensor moving through a synthetic aperture. In addition, the original form of the LSM utilizes data at only a single frequency and contains no mechanism to leverage information across a band of frequencies. So, although the LSM eliminates the reliance on the point target and Born approximations, it is limited in its utility for SAR applications.

In order to adapt the LSM to better fit SAR imaging modalities, a modified version of the technique known as the Phase Encoded Linear Sampling Method (PE-LSM) was developed [61]. The PE-LSM modifies LSM to effectively leverage multi-frequency data by adding beamforming steps and phase constraints to the LSM formulation. However, these modifications are based on the Born approximation. Thus, we can view the PE-LSM as a compromise between the Born approximated traditional SAR techniques and the LSM. Multiple studies have shown that the PE-LSM offers improved imaging performance over both traditional SAR methods and the LSM when sensors are given only a limited aperture to view the target scene [61], [62], [63]. While early results still used small sets of sensors for multistatic measurements, developments in [80] and [64] allow the PE-LSM to produce quality images using only monostatic synthetic aperture data. With this adaptation, PE-LSM can operate on data collected in the most common SAR sensing geometry. The improvements made by PE-LSM come at a cost however. The method requires the solving

of a linear system at each pixel in the scene. With large data sets and a dense set of pixels in the scene, this can lead to significant computational expense.

With the continued development of the PE-LSM, LSM-based inverse scattering imaging methods have moved progressively closer to effective imaging under more practical SAR data collection scenarios. However, the computational cost of these methods is still prohibitive, and there is still the concern that they may be more easily "contaminated" by interference since the methods are ill-posed. In this chapter, we propose a method to help on both of these fronts: Digital Spotlighting. Digital spotlighting is an established technique in SAR imaging that inputs SAR data and then filters and downsamples that data to produce a new, smaller set of data [66], [67], [68], [69], [70]. However, this new smaller data set is only suitable for imaging a small region within the original scene. Therefore, we begin with the assumption that the target of interest can be contained in a small region within the larger scene. Finding this target of interest is considered a separate problem and we focus on imaging this target after it has been identified. Once we have identified this region, we apply the digital spotlight processing to the data. The resulting data is then fed to LSM-type imaging algorithms.

The results we will present in this chapter aim to demonstrate the effectiveness of digital spotlight pre-processing for LSM-based imaging. We will first show that the digital spotlight processing and LSM-type imaging methods can successfully be used together and produce quality images. Due to the ill-posed nature of the LSM imaging techniques, it is not obvious whether or not the preprocessing steps will negatively impact the LSM imaging. In producing these digitally spotlighted LSM images, we will also see the savings that digital spotlight processing offers in terms of data reduction and therefore in computational cost. Next we will show results demonstrating that the filtering aspect of digital spotlighting actually helps the imaging process in the case of strong interference in the data. Finally, the digital spotlight pre-processing and PE-LSM are demonstrated on fielded SAR data. The resulting images, to our knowledge, constitute the first successful LSM-type imaging from data collected by a fielded SAR system. These images are only made feasible due to the performance gains allowed by the digital spotlight processing.

The remainder of the chapter is organized as follows: Section 3.2 will describe aspects of LSM and PE-LSM-based imaging and describe the procedure for digital spotlight processing, Section 3.3 will present the results, and finally Section 3.4 will offer concluding comments and remark on future work.

3.2 Methods

3.2.1 The Linear Sampling Method

Here we will present some of the basic mathematics regarding the LSM and the PE-LSM. The aim is to give the reader a clear understanding of the basics of the LSM and to show what mathematical additions give rise to the improved performance of the PE-LSM. We begin by presenting the standard LSM, originally formulated in [48]. Then we will discuss the extension put forward by the PE-LSM in [61].

Imaging Model

Consider the imaging scenario depicted in Figure 3.1. We have a scene containing a target or targets that we wish to image. A set of transceivers positioned outside the scene probe the area with radio frequency (RF) pulses and measure the resulting scattered field. The transceiver positions may completely surround the imaging scene or may only cover a limited viewing aperture, and the probing RF waves may be of a single frequency or have a certain band of frequencies.

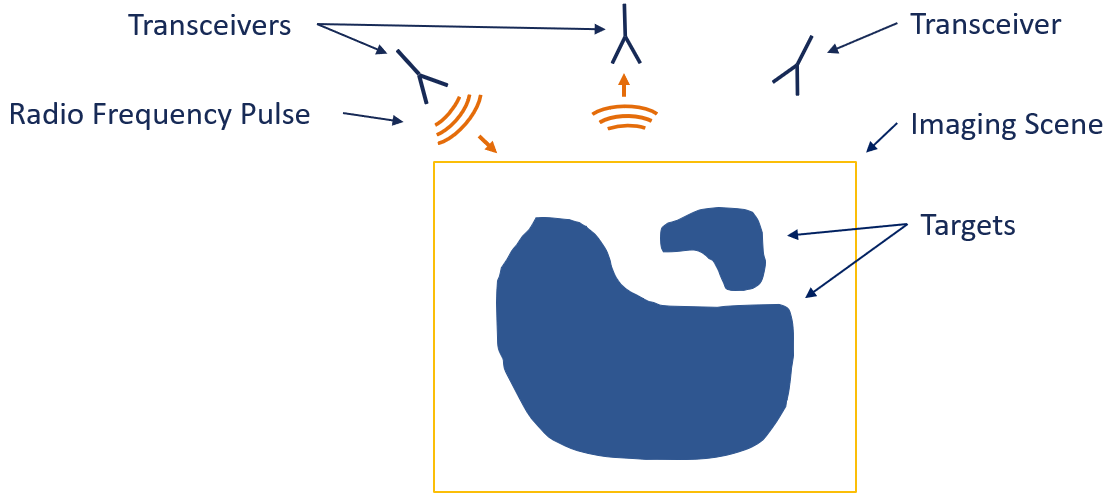


Figure 3.1: General Diagram of LSM imaging context.

In order to keep things simple, we consider the scattering problem modeled by the scalar Helmholtz equation

$$\Delta u + k^2 u = 0, \quad (3.1)$$

with $u = u^i + u^s$ is the sum of the incident and scattered field. We enforce the radiation condition at infinity

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - j k u^s \right) = 0, \quad (3.2)$$

where the wavenumber is $k = \frac{2\pi f}{c}$ and $r = \|\mathbf{x}\|$. In [48] and [52] it is argued that the LSM works regardless of the specific penetrability or boundary conditions of the targets. Let $G(\mathbf{x}, \mathbf{y}, k)$ denote the outgoing Green's function for this equation. In two dimensions this takes the form

$$G(\mathbf{x}, \mathbf{y}, k) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \quad (3.3)$$

where $H_0^{(1)}(\cdot)$ denotes the Hankel function of the first kind and order zero. In three dimensions the Green's function takes the form

$$G(\mathbf{x}, \mathbf{y}, k) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}. \quad (3.4)$$

We consider imaging data collected from N_t transmit locations and N_r receive locations. Data may be collected at a single frequency or over a frequency bandwidth B . Let

$$E(\mathbf{z}_r, \mathbf{z}_t, k) \quad (3.5)$$

denote the measurement of the scattered field phasor for a transmitter at $\mathbf{z}_t$, receiver at $\mathbf{z}_r$ and wavenumber k , and let $\mathbf{E}(k)$ denote the complex-valued $N_r \times N_t$ matrix that results from the discretization of $E(\mathbf{z}_r, \mathbf{z}_t, k)$ at wavenumber k . Since the measurements $\mathbf{E}(k)$ are often made in the far field, $\mathbf{E}(k)$ can be referred to as the far field matrix. We may also refer to $\mathbf{E}(k)$ as the multistatic response matrix since it contains measurements corresponding to different sensor locations in a multi-sensor imaging setup.

Standard LSM

Here we present the main operations needed to produce an image of a scene using the basic LSM formulation of [48]. Consider data containing only one wavenumber k . Given data organized in the form of our far field matrix $\mathbf{E}$ (we have suppressed dependence on k when only a single wavenumber is used), we solve what is referred to as the far-field equation for each point $\mathbf{x}$ in our imaging region. In matrix form, the far field equation is given by

$$\mathbf{E}\mathbf{g}(\mathbf{x}) = \phi(\mathbf{x}) \quad (3.6)$$

where $\phi(\mathbf{x})$ is a probe or test vector corresponding to the point $\mathbf{x}$ and $\mathbf{g}(\mathbf{x})$ is the LSM solution vector. The test vector $\phi(\mathbf{x})$ takes the form

$$\phi(\mathbf{x}) = \begin{bmatrix} G(\mathbf{x}, \mathbf{z}_1) \\ G(\mathbf{x}, \mathbf{z}_2) \\ \vdots \\ G(\mathbf{x}, \mathbf{z}_{N_r}). \end{bmatrix} \quad (3.7)$$

With $\mathbf{g}(\mathbf{x})$ interpreted as a vector of transmit beamforming weights, the far field equation can be interpreted as a transmit beamforming problem. The quantity $\phi(\mathbf{x})$ is the representation of the measured field from an idealized point source at $\mathbf{x}$. Solving (3.6) for $\mathbf{g}(\mathbf{x})$ is therefore equivalent to finding the set of weights to apply to each transmitter that force the scattered data to match the scattered data from a single point at $\mathbf{x}$.

In [48], it is shown in the 2-D case for a single target that the norm of the solution $\|\mathbf{g}(\mathbf{x})\|$ is bounded when $\mathbf{x}$ is inside the scattering target and becomes unbounded as $\mathbf{x}$ approaches the target boundary and stays unbounded when $\mathbf{x}$ lies outside the target. This result was extended to multiple targets in [52] and a proof for 3-D scattering can be found in [51]. For details regarding this theory we refer the readers to [48], [52], and [51]. The behavior of $\mathbf{g}(\mathbf{x})$ can be used to create an image via an indicator function. A common choice of indicator function is

$$I(\mathbf{x}) = \frac{1}{\mathbf{g}(\mathbf{x})^H \mathbf{g}(\mathbf{x})} = \frac{1}{\|\mathbf{g}(\mathbf{x})\|^2}. \quad (3.8)$$

Since $\|\mathbf{g}(\mathbf{x})\|$ is unbounded outside the boundary of the target, this indicator function will display bright pixels inside the support of the target and dark pixels outside of the target, illustrating the target shape.

Due to the ill-posed nature of the far-field equation, the linear system in (3.6) will be ill-conditioned [52]. As a result, in order to solve (3.6) in a stable manner a regularization technique is required. The LSM is often regularized using Tikhonov regularization. Rather than solving (3.6)

one instead solves

$$\min_{\mathbf{g}} \left(\|\mathbf{E}\mathbf{g}(\mathbf{x}) - \phi(\mathbf{x})\|^2 + \alpha \|\mathbf{g}(\mathbf{x})\|^2 \right) \quad (3.9)$$

where α is a positive parameter that needs to be chosen. The first term is known as the residual term while the second will be referred to as the Tikhonov term. The solution to the minimization problem (3.9) is

$$\mathbf{g}(\mathbf{x}) = \left(\mathbf{E}^H \mathbf{E} + \alpha \mathbf{I} \right)^{-1} \mathbf{E}^H \phi(\mathbf{x}). \quad (3.10)$$

A common way to express this solution is via the Singular Value Decomposition (SVD). Let $\mathbf{E} = \mathbf{U}\mathbf{S}\mathbf{V}^H$ be the SVD of $\mathbf{E}$. For simplicity, let us temporarily assume that $N_r = N_t$. Then the solution to the Tikhonov regularized far field equation is

$$\mathbf{g}(\mathbf{x}) = \sum_{i=1}^{N_r} \frac{\sigma_i}{\sigma_i^2 + \alpha} \mathbf{v}_i \mathbf{u}_i^H \phi(\mathbf{x}) \quad (3.11)$$

where σ_i is the i -th singular value of $\mathbf{E}$ and $\mathbf{u}_i$ and $\mathbf{v}_i$ are the i -th columns of $\mathbf{U}$ and $\mathbf{V}$ respectively.

Thus far, we have assumed data at a single frequency. In [50] it is reported that combining results from multiple frequencies can improve the performance of the LSM, especially when data is available from only a limited angular aperture. The LSM solution for multiple frequencies is combined noncoherently via a slightly modified indicator function

$$I(\mathbf{x}) = \sum_{i=1}^{N_f} \frac{1}{\|\mathbf{g}(\mathbf{x}, k_i)\|^2} \quad (3.12)$$

Here, $\mathbf{g}(\mathbf{x}, k_i)$ denotes the LSM solution at location $\mathbf{x}$ for given wavenumber k_i , and N_f is the total number of frequencies used in the image reconstruction.

It is now worth commenting on some the benefits of the LSM. Solving (3.9) can be accomplished by solving a simple linear equation at each pixel for each wavenumber k for which data

is collected. There is the added advantage that the matrix $\mathbf{E}$ in (3.9) contains only the measured data and is identical for each pixel $\mathbf{x}$; only $\phi(\mathbf{x})$ changes as $\mathbf{x}$ does. Thus, for each k , only a single matrix inversion or SVD computation is necessary to image the entire scene. The results is that the LSM is relatively computationally inexpensive for an inverse scattering technique. The LSM also does not depend on the Born approximation in its formulation, and thus has the potential to capture higher fidelity images than methods which do rely on this approximation.

With these benefits, there are also some downsides of the LSM. The LSM is known to struggle to produce good quality images when data is only available from a limited aperture around the target. Specifically, it is observed in [61] that standard LSM struggles to achieve high range resolution with limited apertures. This lack of range resolution stems from the lack of coherent processing across frequency, which is the primary mechanism used to achieve range resolution in traditional radar imaging techniques. Although some improvement is possible using the techniques introduced in [50], the non-coherent combination of information from multiple frequencies still results in overall poor range resolution in limited aperture scenarios. In [61], modifications to the LSM formulation are introduced. These modifications help to address the range resolution issues for limited aperture cases. This method will be explained in the following subsection.

Phase-Encoded LSM

In [61], the phase-encoded linear sampling method or PE-LSM is introduced. This method is designed with imaging from limited apertures in mind and features two main improvements over the standard multifrequency LSM formulation. The first is a beamforming procedure on the receive end, and the second is the addition of a phase delay frequency variation (PDFV) constraint. The PDFV constraint injects coherent processing across frequency into the LSM formulation, so the PE-LSM by its nature involves data sampled across a bandwidth of frequencies.

The receive beamforming improvement introduces a beamforming weight vector

$$\mathbf{w}(\mathbf{x}, k) = e^{-jk\mathbf{d}_r(\mathbf{x})}, \quad (3.13)$$

where $\mathbf{d}_r(\mathbf{x})$ is the vector of distances between sampling point $\mathbf{x}$ and the locations of the receivers. This vector is then applied to the system (3.6),

$$\mathbf{w}(\mathbf{x}, k)^H \mathbf{E}(k) \mathbf{g}(\mathbf{x}) = \mathbf{w}(\mathbf{x}, k)^H \phi(\mathbf{x}, k). \quad (3.14)$$

This beamforming step provides a form of preconditioning to the linear system by filtering out interfering signal from sources which are spatially separated from $\mathbf{x}$.

The PDFV constraint is where coherent processing across frequency is introduced to the LSM. The key idea is to force the solution at each adjacent frequency to exhibit the phase changes expected by the known frequency sample spacing. Mathematically, we require that

$$\mathbf{g}(\mathbf{x}, k_i) = \mathbf{g}(\mathbf{x}, k_1) \odot e^{-j(i-1)\Delta k \mathbf{d}_t(\mathbf{x})}. \quad (3.15)$$

Here, Δk is the spacing between adjacent sampled wavenumbers, $\mathbf{d}_t(\mathbf{x})$ is the vector of distances between a point $\mathbf{x}$ and each transmitter position, and $\odot$ denotes element-wise multiplication. This constraint can be enforced by introducing a diagonal matrix into the beamformed system in equation (3.14). Define

$$\mathbf{C}_i = \begin{bmatrix} e^{-j(i-1)\Delta k |\mathbf{x}-\mathbf{z}_1|} & 0 & \dots & 0 \\ 0 & e^{-j(i-1)\Delta k |\mathbf{x}-\mathbf{z}_2|} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{-j(i-1)\Delta k |\mathbf{x}-\mathbf{z}_{N_t}|} \end{bmatrix}. \quad (3.16)$$

This allows us to enforce (3.15) since $\mathbf{g}(\mathbf{x}, k_i) = \mathbf{C}_i \mathbf{g}(\mathbf{x}, k_1)$. The constraint is applied to (3.14) for each frequency i as

$$\mathbf{w}^H(\mathbf{x}, k_i) \mathbf{E}(k_i) \mathbf{C}_i \mathbf{g}(\mathbf{x}, k_1) = \mathbf{w}^H(\mathbf{x}, k_i) \phi(\mathbf{x}, k_i). \quad (3.17)$$

The systems in (3.17) are then solved via Tikhonov regularization. The full minimization problem is given by

$$\min_{\mathbf{g}} \sum_{i=1}^{N_f} |\mathbf{w}^H(\mathbf{x}, k_i) (\mathbf{E}(k_i) \mathbf{C}_i \mathbf{g}(\mathbf{x}, k_1) - \phi(\mathbf{x}, k_i))|^2 + \alpha \|\mathbf{g}(\mathbf{x}, k_1)\|^2. \quad (3.18)$$

Note that despite the sum over frequencies in the residual term, we still solve for $\mathbf{g}(\mathbf{x}, k_1)$ at only the first frequency corresponding to k_1 . One way to solve (3.18) is via block matrices. Define matrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 \\ \sqrt{\alpha} \mathbf{I} \end{bmatrix} \quad (3.19)$$

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_0 \\ \mathbf{0} \end{bmatrix}, \quad (3.20)$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{0}$ is a vector of zeros,

$$\mathbf{A}_0 = \begin{bmatrix} \mathbf{w}^H(\mathbf{x}, k_1) \mathbf{E}(k_1) \mathbf{C}_1 \\ \vdots \\ \mathbf{w}^H(\mathbf{x}, k_{N_f}) \mathbf{E}(k_{N_f}) \mathbf{C}_{N_f} \end{bmatrix}, \quad (3.21)$$

and

$$\mathbf{b}_0 = \begin{bmatrix} \mathbf{w}^H(\mathbf{x}, k_1) \phi(\mathbf{x}, k_1) \\ \vdots \\ \mathbf{w}^H(\mathbf{x}, k_{N_f}) \phi(\mathbf{x}, k_{N_f}) \end{bmatrix}. \quad (3.22)$$

We then find the least squares solution to the system $\mathbf{A} \mathbf{g}(\mathbf{x}, k_1) = \mathbf{b}$. The solution is thus

$$\mathbf{g}(\mathbf{x}, k_1) = \left(\mathbf{A}^H \mathbf{A} \right)^{-1} \mathbf{A}^H \mathbf{b}. \quad (3.23)$$

The PE-LSM is demonstrated in [61] and [64] to offer significant improvements in LSM imaging fidelity when a target is viewed from a limited aperture. However, this comes with some drawbacks. The standard LSM does not rely on the Born approximation, but the beamforming weight and PDFV constraint are based on the Born approximation. Thus, the PE-LSM makes a compromise between the LSM and methods built on the Born approximation.

The changes of the PE-LSM also introduce some added computational complexity. Specifically, the linear system that needs to be solved for each point $\mathbf{x}$ in our domain is no longer invariant across the domain. Instead, with this formulation a new matrix inversion operation is required for each pixel in our image. Many imaging radars feature high frequencies and bandwidths and correspondingly small wavelengths in order to provide high resolution. In order to satisfy Nyquist sampling requirements with these small wavelengths, a large number of samples are required. Thus, the matrices for PE-LSM can grow quite large, leading to significant computational burden.

Selecting the regularization parameter

Both LSM formulations require the selection of the Tikhonov regularization parameter α . The parameter α is responsible for balancing the weight of the residual term and the Tikhonov term in the minimization problem of either (3.9) or (3.18). The residual term penalizes mismatch between the solution and the predicted model behavior while the Tikhonov term penalizes large norm solutions. If α is chosen to be too small, we say that the problem is under-regularized, and similarly if α too large then the problem is over-regularized. Due to the ill-posed nature of the LSM equations, an under-regularized problem will suffer strong distorting effects due to small perturbations and noise in the data. An over-regularized problem will constrain the norm of the solution too greatly at the expense of enforcing the solution to conform to the model. In both cases, the fidelity of the image to the true target geometry can be destroyed. Therefore, the selection of a good value for α is an essential part of the reconstruction.

There are many ways to select the regularization parameter for Tikhonov regularization problems. These methods include the L-curve method, the Morozov discrepancy principle, heuristic methods, and simple trial and error. We use different approaches based on whether we are imaging

with LSM or with PE-LSM. For LSM imaging, we use either a modified version of the Morozov discrepancy principle used in [52] or we manually select the regularization parameter. For PE-LSM, we use the heuristic introduced in [61].

First, we describe our modification of the technique of [52] for selecting the regularization parameter for LSM imaging. In practice, we know that $\mathbf{E}$ will contain noisy measurements. Let $\mathbf{E}_N = \mathbf{E} + \mathbf{N}$ where $\mathbf{N}$ is a matrix which contains the additive measurement noise. Define the quantity $\delta \geq \|\mathbf{E} - \mathbf{E}_n\| = \|\mathbf{N}\|$. Next, define the discrepancy function

$$D(\alpha) = \|\mathbf{E}_N \mathbf{g}(\mathbf{x}; \alpha) - \phi(\mathbf{x})\|^2 - \delta^2 \|\mathbf{g}(\mathbf{x}; \alpha)\|^2. \quad (3.24)$$

The discrepancy function measures the match between the norm of the residual and the effect of the additive noise. If a given value of α achieves perfect balance, then $D(\alpha) = 0$. We choose α to minimize $|D(\alpha)|$ rather than searching for the exact zero of the discrepancy function. In [52], this process is repeated for each pixel location $\mathbf{x}$ resulting in potentially different values of α used for solving (3.9) at each $\mathbf{x}$. In [54], it is reported that good image quality can be obtained with a single value of α chosen for all pixels in the scene. We opt to select a single α and use this to solve (3.9) for all pixels in the scene. We select the optimal parameter α_{opt} which satisfies

$$\alpha_{opt} = \arg \min_{\alpha} \max_{\mathbf{x}} |D(\alpha)|. \quad (3.25)$$

Note that this method requires some knowledge of the additive noise $\mathbf{N}$ in order to obtain δ . We consider the problem of estimating δ outside the scope of our current work, and instead we simply assume knowledge of δ .

For PE-LSM, we select the regularization parameter via the heuristic method of [61]. First, we select a coarse grid of pixel locations $\mathbf{x}$ covering the imaging scene. We consider the quantity

$$\eta(\alpha) = \sum_{\mathbf{x}} \|\mathbf{g}(\mathbf{x}; \alpha)\| \quad (3.26)$$

where the sum is over all the pixels $\mathbf{x}$ in the coarse grid. Note that computing $\eta(\alpha)$ requires the solving of (3.18) for each pixel $\mathbf{x}$ in the grid. Since PE-LSM requires the inversion of a different matrix at each $\mathbf{x}$, this can become computationally expensive. The use of a coarse grid reduces the number matrix inversions needed to compute the regularization parameter. We have found empirically that often a 3×3 or 5×5 pixel grid covering the full imaging scene is often sufficient. We compute the derivative of this $\eta(\alpha)$ with respect to α . In our experience and in the experience of [61], the derivative curve typically has a single well defined peak corresponding the the inflection point of $\eta(\alpha)$. We choose

$$\alpha_{opt} = \arg \max_{\alpha} \frac{d\eta(\alpha)}{d\alpha}. \quad (3.27)$$

Since this is a heuristic, we make no claims to the optimality of this parameter choice. However, based on our own experience and those noted in sources such as [61], [64], and [65], this method seems to select values of α that produce quality images. A final note, in [61] the search for the inflection point of $\eta(\alpha)$ occurs after sweeping through choice of β/α until a certain convergence criterion is reached. This method was necessary for the older version of PE-LSM which contained two regularization parameters α and β . However, in the newer version of PE-LSM we use, only the single parameter α is needed. The parameter β was used to enforce a soft version of the constraint that is now enforced by the matrices $\mathbf{C}_i$ in (3.18). Enforcing this constraint with the matrices $\mathbf{C}_i$ removes the need for β .

The monostatic-to-multistatic transform

The last modification of the LSM that we will use in our imaging studies later is "monostatic-to-multistatic" transform developed in [80] and simplified in [64]. LSM-based techniques involve solving linear equations based on multistatic measurements. Measurements are collected into multistatic response matrices $\mathbf{E}(k_i)$ for frequency samples indexed by i . The rows and columns of these matrices contain data with a diversity of measurements from different transmit and receive viewing angles. Without any diversity in both the transmit and receive viewing angles, the LSM methods break down. In a practical radar setting, it is often the case that only monostatic data is

collected over a band of frequencies. In order to apply LSM-based imaging to a greater variety of radar imaging scenarios, a method for constructing virtual multistatic data from monostatic data is developed in [80]. This method is refined and used for PE-LSM imaging in [64]. This technique is required to produce the images we present in section 3.3.4, and we will describe the basics here.

The monostatic-to-multistatic transform of [80] is based on a far field and Born approximated model for the field E scattered from a target near the origin. This model, derived in [80], can be expressed as

$$E(\hat{\mathbf{y}}_r, \hat{\mathbf{y}}_t, k) = k^2 \int \rho(\mathbf{x}) e^{jk(\hat{\mathbf{y}}_r + \hat{\mathbf{y}}_t)^T \mathbf{x}} d\mathbf{x}, \quad (3.28)$$

where $\hat{\mathbf{y}}_r$ denotes a unit vector pointing from the origin towards the receiver and $\hat{\mathbf{y}}_t$ a unit vector directed towards the transmitter. The integral is evaluated over the imaging scene and $\rho(\mathbf{x})$ characterizes the reflectivity of the scene. The key observation is that under these approximations, E can be viewed as a Fourier transform of the scene reflectivity, with spatial frequencies $\mathbf{k} = k(\hat{\mathbf{y}}_r + \hat{\mathbf{y}}_t)$. For a fixed transmit direction and a 360° receive aperture, the spatial frequency support is a circle with radius k and centered at $k\hat{\mathbf{y}}_t$. This can be seen by taking

$$\hat{\mathbf{y}}_r = (\cos \phi_r, \sin \phi_r) \quad (3.29)$$

$$\hat{\mathbf{y}}_t = (\cos \phi_t, \sin \phi_t), \quad (3.30)$$

which gives

$$\mathbf{k} = k(\cos \phi_r + \cos \phi_t, \sin \phi_r + \sin \phi_t). \quad (3.31)$$

By fixing a transmit direction ϕ_t it becomes apparent that the spectrum traces a circle in the spatial frequency domain. Radially, in the spatial frequency domain, this circle will extend from 0 to $2k$. Note that the monostatic response is obtained by taking $\hat{\mathbf{y}}_t = \hat{\mathbf{y}}_r$ and thus for monostatic data. the spatial frequencies are given by $\mathbf{k}_{mono} = 2k\hat{\mathbf{y}}_r$. Parameterized by receive angle, this gives

$$\mathbf{k}_{mono} = 2k(\cos \phi_r, \sin \phi_r). \quad (3.32)$$

When collected over a bandwidth of frequencies between wavenumbers k_{min} and k_{max} , the monostatic data has an annulus shaped support in the spatial frequency space. We will refer to this support region as Ω_{mono} . The annulus shaped region, Ω_{mono} , is centered at $\mathbf{0}$ in the spatial frequency domain, and it extends from $2k_{min}$ to $2k_{max}$ in the radial direction, and from ϕ_{start} to ϕ_{end} angularly.

Consider monostatic data collected for a 360° aperture with wavenumbers ranging from k_{min} to k_{max} , and consider bistatic data collected for a fixed transmit angle ϕ_t , receive angle spanning a 360° aperture, and wavenumber $k \in [k_{min}, k_{max}]$. In this case, the support of the bistatic data must have some overlap with the monostatic data support Ω_{mono} . An illustration showing this overlap is provided in Figure 3.2.

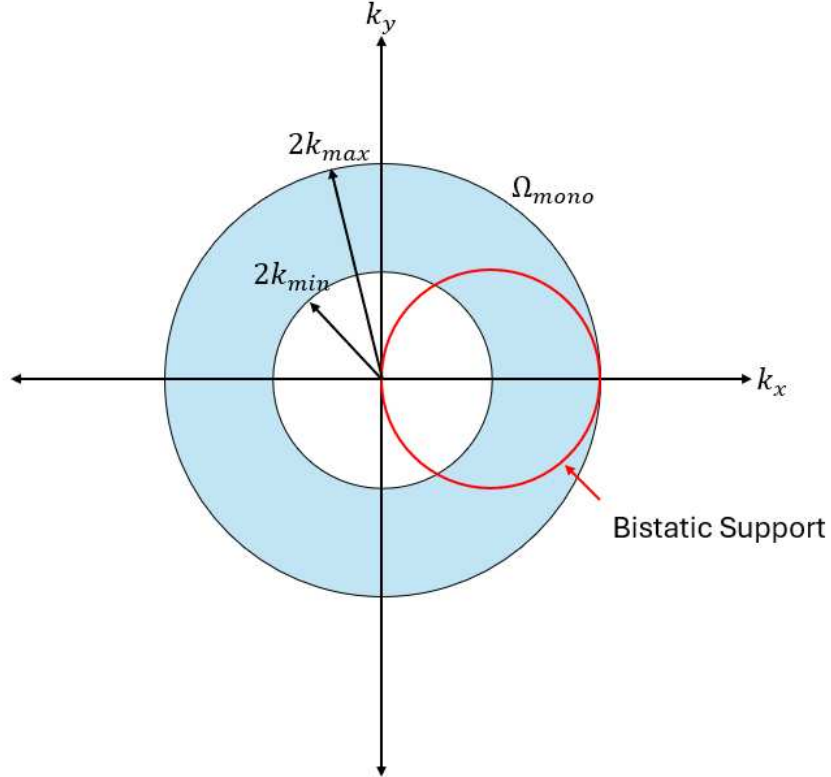


Figure 3.2: Illustration depicting overlap of monostatic and bistatic spatial frequency supports. The blue region depicts Ω_{mono} for a 360° aperture and wavenumbers between k_{min} and k_{max} . The red circle depicts the support of bistatic data collected for $\theta_t = 0$ and θ_r traversing a 360° aperture at wavenumber k_{max} .

We can use existing monostatic data to define a 2-D interpolant over Ω_{mono} . Then, due to the overlapping support regions, we can use this interpolant to synthesize virtual multistatic data. Let $F(\mathbf{k})$ denote the interpolant. Virtual multistatic data is created via

$$E(\hat{\mathbf{y}}_r, \hat{\mathbf{y}}_t, k) = \begin{cases} F(\mathbf{k}), & \mathbf{k} \in \Omega_{mono} \\ 0 & \text{otherwise.} \end{cases} \quad (3.33)$$

In [80], this interpolated data is found to provide decent imaging performance, although true multistatic data does allow for better performance. For more details regarding this method, we encourage the reader to consult [80].

The method outlined in [80] involves using monostatic data collected over a bandwidth of frequencies to generate virtual multistatic data for LSM imaging via a 2-D interpolation in the spatial frequency domain. In [64], a simplified version of this technique is used to generate virtual multistatic data for PE-LSM imaging. Let $\hat{\mathbf{y}}_t = (\cos \phi, \sin \phi)$, $\hat{\mathbf{y}}_r = (\cos(\phi + \theta), \sin(\phi + \theta))$, and $\mathbf{x} = (x, y)$. Then under the far field and Born approximations

$$E(\theta, \phi, k) = k^2 \int \int \rho(x, y) e^{-jk_x x - jk_y y} dx dy. \quad (3.34)$$

We have defined

$$k_x = -k(\cos \phi + \cos(\phi + \theta)) = -2k \cos(\phi + \theta/2) \cos(\theta/2) \quad (3.35)$$

$$k_y = -k(\sin \phi + \sin(\phi + \theta)) = -2k \sin(\phi + \theta/2) \cos(\theta/2). \quad (3.36)$$

Note that the signal for a monostatic data collect can be obtained from the above by taking $\theta = 0$. If we have monostatic data $(2k_{mono} \cos \phi_{mono}, 2k_{mono} \sin \phi_{mono})$ and wish to create virtual multi-sensor data with a sensor pair located at ϕ and $\phi + \theta$, then we attempt to match the expected phase

relationships of these two sets of data via

$$2k_{mono} \cos \phi_{mono} = 2k \cos(\phi + \theta/2) \cos(\theta/2) \quad (3.37)$$

$$2k_{mono} \sin \phi_{mono} = 2k \sin(\phi + \theta/2) \cos(\theta/2). \quad (3.38)$$

Some algebra shows that the above phase relationships are satisfied when

$$\tan \phi_{mono} = \tan(\phi + \theta/2) \quad (3.39)$$

$$k_{mono} = k \cos(\theta/2). \quad (3.40)$$

If we consider θ to be small, then this relationship simplifies to

$$\phi_{mono} \approx \phi + \theta/2 \quad (3.41)$$

$$k_{mono} \approx k. \quad (3.42)$$

This approximate relationship gives a simple way to synthesize virtual multi-sensor data for small separation angles between transmitter and receiver. Given $E(0, \phi, k)$ monostatic data, for small θ we can synthesize data $E(\theta, \phi, k)$ by interpolating the monostatic data at sample angles $\phi + \theta/2$. Due to the synthetic aperture nature of monostatic data collection, this interpolation can typically be achieved with a simple and straight forward 1-D interpolation. For receive angles that fall outside the synthetic aperture, we simply set the resulting multisensor data point to zero. This method for synthesizing virtual multistatic data uses a simpler 1-D interpolation than the 2-D interpolation of [80], but is limited to the case of small angles θ . However, since the study published in [61], it has been known that PE-LSM is able to create images of high quality with much smaller angular apertures than is needed by LSM. Thus, in [64], it is found that this 1-D interpolation method allows for high quality image reconstructions with PE-LSM.

3.2.2 Digital Spotlighting

Digital Spotlighting is an established technique used in Synthetic Aperture Radar (SAR) that extracts phase history data from a larger SAR phase history data set in order to image a smaller region of interest (ROI) within a SAR image scene [66],[67],[68],[69]. The reduced-size phase-history data for a small ROI can be useful in improving the efficiency of imaging techniques. Digital spotlighting has also been found to improve the image quality of SAR images by suppressing side lobes and aliasing in images [70]. Once an ROI is selected, digital spotlight processing can be divided into 6 different steps:

1. Re-centering of phase history to the new ROI scene center
2. Filtering of phase history in the frequency dimension
3. Downsampling of phase history in frequency dimension
4. Resampling of phase history in the aperture dimension
5. Filtering of phase history in the aperture dimension
6. Downsampling of phase history in the aperture dimension

Let $E(\theta_r, \theta_t, k)$ denote the initial SAR phase history. Figure 3.3 illustrates the order of these steps on an input phase history and concludes with the output *spotlit* data $E_s(\theta_r, \theta_t, k)$. In [66], [67], [68], and [69], only a monostatic synthetic aperture setup is used for digital spotlighting. In the remainder of this section, we will develop a multistatic version of the digital spotlight technique. We will begin with a model for simple bistatic sensing and use this to derive sampling requirements. We will then discuss how the bistatic model extends to the multistatic sensing scenario. From there, we will work through the six steps of digital spotlight processing on our multistatic data model.

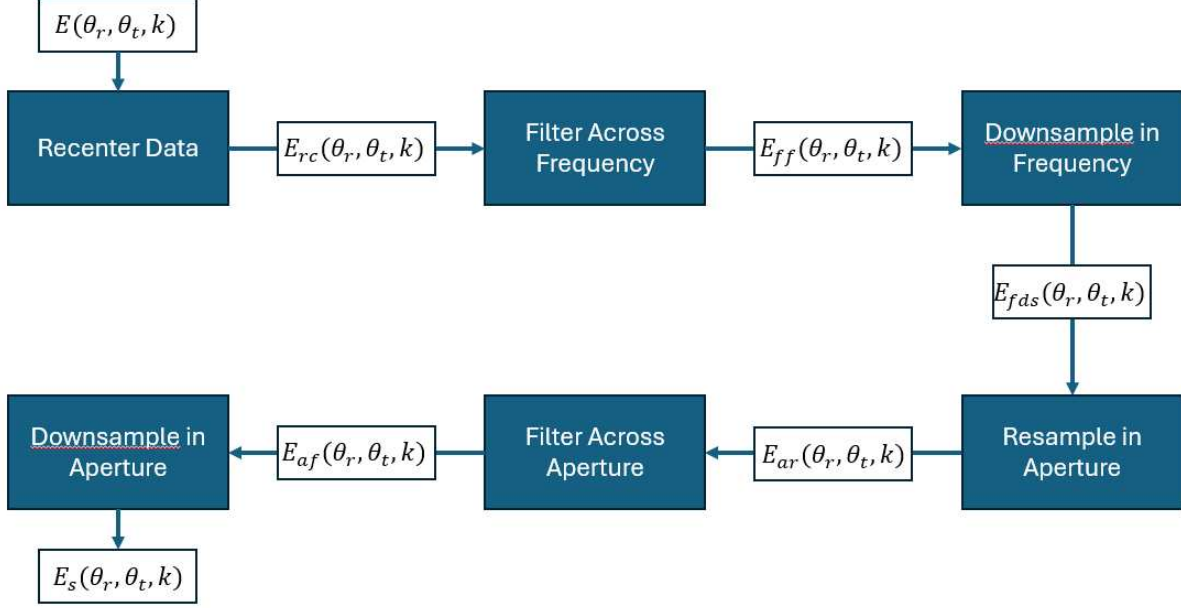


Figure 3.3: Flow chart of digital spotlight processing which takes SAR phase history $E(\theta_r, \theta_t, k)$ and transforms it to filtered and downsampled *spotlit* data $E_s(\theta_r, \theta_t, k)$. E_{rc} denotes the *re-centered* phase history, E_{ff} denote the *frequency filtered* phase history, E_{fds} denotes the *frequency downsampled* phase history, E_{ar} the *aperture resampled* phase history, and E_{af} the *aperture filtered* phase history.

Bistatic Data Model

Let Ω denote the imaging region, centered at $\mathbf{0}$. The simplest multi-sensor case is the bistatic case of two sensors. Let $\mathbf{R}_t$ and $\mathbf{R}_r$ denote the locations of the transmitter and receiver respectively. The scattered signal measured at $\mathbf{R}_r$ can be modeled by

$$E(\mathbf{R}_r, \mathbf{R}_t, k) = \int_{\Omega} a(\mathbf{R}_r, \mathbf{R}_t, k, \mathbf{x}) \rho(\mathbf{x}) e^{-jk(\|\mathbf{R}_t - \mathbf{x}\| + \|\mathbf{R}_r - \mathbf{x}\|)} d\mathbf{x} \quad (3.43)$$

where $k = 2\pi f/c$ is the wavenumber corresponding to frequency f , $a(\mathbf{R}_r, \mathbf{R}_t, k, \mathbf{x})$ is an amplitude function, and $\rho(\mathbf{x})$ is the complex valued reflectivity of the scene. It is $\rho(\mathbf{x})$ that we want to image. Note that the model in (3.43) depends on the Born approximation. Figure 3.4 illustrates this sensing geometry.

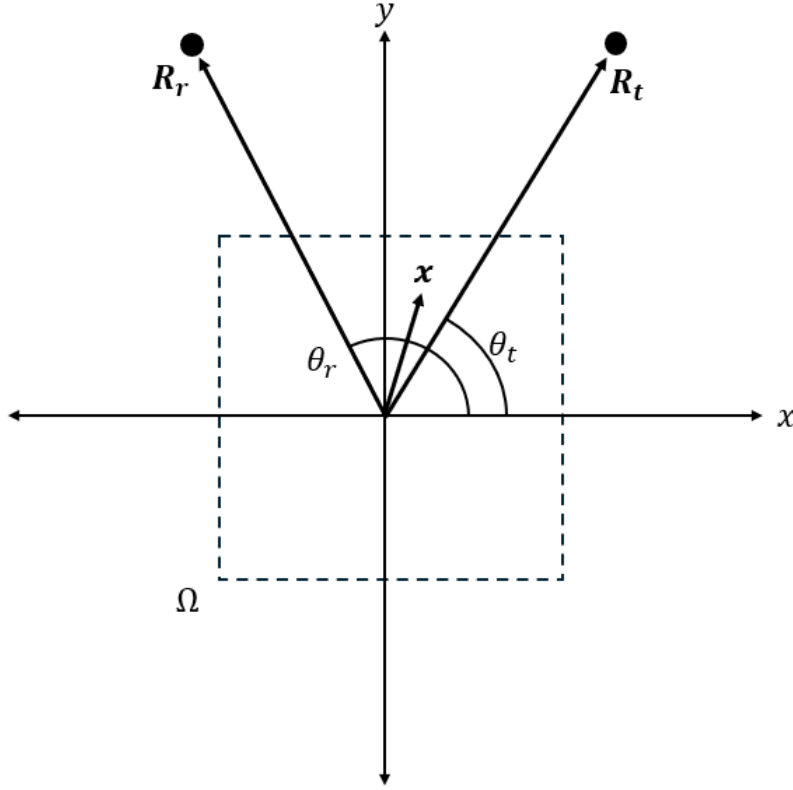


Figure 3.4: Diagram of bistatic imaging geometry.

The amplitude function $a(\mathbf{R}_r, \mathbf{R}_t, k, \mathbf{x})$ is typically slowly varying in $\mathbf{x}$ and for this reason we will ignore the dependence. Let $R_r = \|\mathbf{R}_r\|$ and $R_t = \|\mathbf{R}_t\|$, and then define $\hat{\mathbf{R}}_r = \mathbf{R}_r/R_r$ and $\hat{\mathbf{R}}_t = \mathbf{R}_t/R_t$. In the far field, the signal of (3.43) can be approximated as

$$E(\mathbf{R}_r, \mathbf{R}_t, k) \approx \int_{\Omega} a(\mathbf{R}_r, \mathbf{R}_t, k) \rho(\mathbf{x}) e^{-jk(R_t + R_r)} e^{jk(\hat{\mathbf{R}}_t + \hat{\mathbf{R}}_r)^T \mathbf{x}} d\mathbf{x}. \quad (3.44)$$

For notational simplicity, define $\tilde{a}(\mathbf{R}_r, \mathbf{R}_t, k) = a(\mathbf{R}_r, \mathbf{R}_t, k) e^{-jk(R_t + R_r)}$ which is an amplitude function which does not depend on $\mathbf{x}$. We can then express the signal as

$$E(\mathbf{R}_r, \mathbf{R}_t, k) \approx \tilde{a}(\mathbf{R}_r, \mathbf{R}_t, k) \int_{\Omega} \rho(\mathbf{x}) e^{j\mathbf{k}^T \mathbf{x}} d\mathbf{x} \quad (3.45)$$

where $\mathbf{k} = k(\hat{\mathbf{R}}_t + \hat{\mathbf{R}}_r)$. We can see that (3.45) has the form of a spatial 2-D Fourier transform of the scene reflectivity. Define

$$\hat{\mathbf{R}}_r = (\cos \theta_r, \sin \theta_r) \quad (3.46)$$

$$\hat{\mathbf{R}}_t = (\cos \theta_t, \sin \theta_t). \quad (3.47)$$

This allows us to express the components of $\mathbf{k}$ as

$$k_x = k(\cos \theta_t + \cos \theta_r) \quad (3.48)$$

$$k_y = k(\sin \theta_t + \sin \theta_r). \quad (3.49)$$

To image the scene, data is collected over a bandwidth of frequencies and sensor positions. Thus, through equations (3.48) and (3.49) we can see that sampling the scattered signal $E(\theta_r, \theta_t, k)$ corresponds to sampling the 2-D spatial Fourier transform of $\rho(\mathbf{x})$. We will refer to the 2-D spatial frequency domain as k -space. An illustration of the k -space support of $E(\theta_r, \theta_t, k)$ for a fixed transmit angle θ_t and fixed frequency k is given in Figure 3.5. For a fixed transmit angle θ_t and fixed frequency k , varying θ_r will trace a circle of radius k in k -space. The location of the center of the circle is determined by the transmit angle θ_t . If instead θ_r were fixed and θ_t were varied, the k -space support of the data would exhibit the same behavior except that the roles of θ_t and θ_r would be exchanged. In that case, θ_t would trace out the circular support in k -space and θ_r would determine the circle's center point.

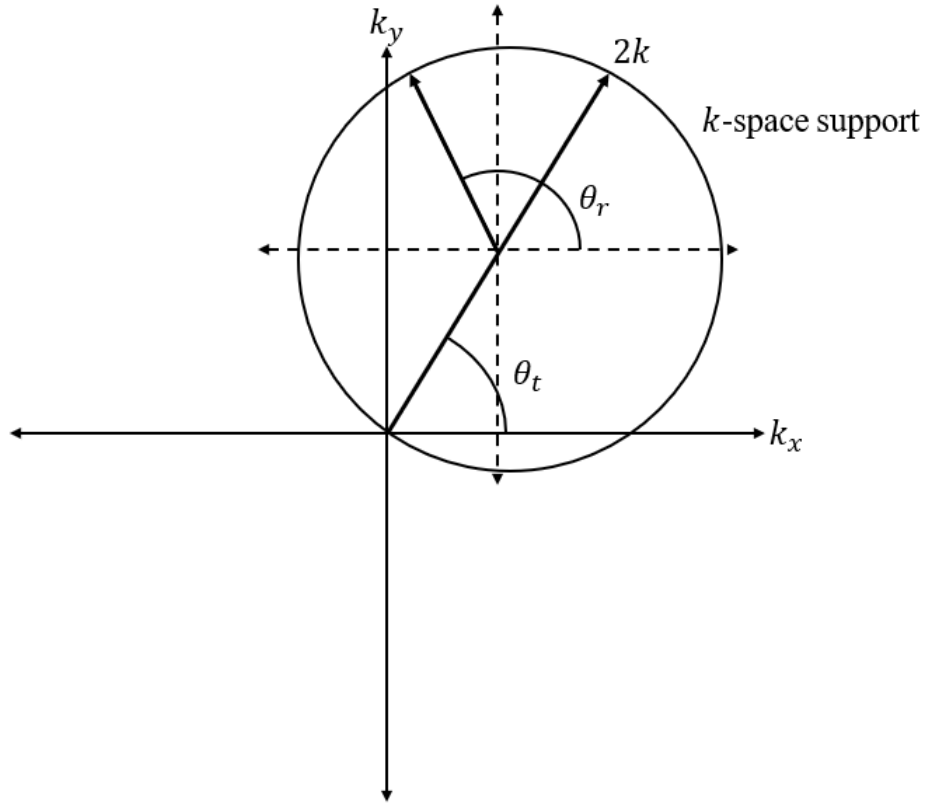


Figure 3.5: Diagram depicting the k -space support of bistatic data collected for fixed θ_t and k and varying θ_r from 0° to 360° .

The operations of digital spotlight processing are based on the sampling requirements driven by this model. In the following section we will use this model to derive the sampling requirements to image the scene Ω .

Bistatic Sampling Requirements

Equations (3.48) and (3.49) tell us where in k -space the sampled data will be located. By the Nyquist-Shannon sampling theorem, in order to prevent aliasing artifacts in our image of Ω , we

need our samples to satisfy

$$\Delta k_x \leq \frac{\pi}{W_x} \quad (3.50)$$

$$\Delta k_y \leq \frac{\pi}{W_y}, \quad (3.51)$$

where $[-W_x, W_x]$ and $[-W_y, W_y]$ give the extent of Ω in the x and y directions respectively. For simplicity, let us assume a square imaging region where $W_x = W_y = W$. The results from this section generalize in a straightforward manner to regions where $W_x \neq W_y$. From equations (3.48) and (3.49), we can see that the sampled data will not lie on a Cartesian grid in k_x and k_y space. However, let us assume that θ_r and θ_t are confined to a narrow aperture centered at $\theta_r = \theta_t = 0$. With this aperture assumption, the x coordinate direction corresponds to the range dimension in the image and the y direction corresponds to cross range. We expand (3.48) and (3.49) for small angles in both the transmit and receive dimension. This operation can be thought of as expanding around the monostatic data where $\theta_r = \theta_t$. To first order, the results are

$$k_x \approx 2k \quad (3.52)$$

$$k_y \approx k(\theta_r + \theta_t). \quad (3.53)$$

We can use these approximations as well as equations (3.50) and (3.51) to derive sampling requirements for our bistatic sensing setup. Consider first the k_x direction (in this case, the range direction). In order to satisfy Nyquist sampling in k must satisfy

$$2\Delta k \leq \frac{\pi}{W}. \quad (3.54)$$

Recalling that $k = 2\pi f/c$, this sampling requirement can be expressed as

$$\frac{4\pi\Delta f}{c} \leq \frac{\pi}{W} \quad (3.55)$$

or

$$\Delta f \leq \frac{c}{4W}. \quad (3.56)$$

Now consider the k_y or cross range direction. For a fixed k , equation (3.53) depends on both θ_r and θ_t . We fix θ_t and allow θ_r to vary, which gives the sampling requirement

$$k\Delta\theta_r \leq \frac{\pi}{W}. \quad (3.57)$$

Substituting our expression $k = 2\pi f/c$, we get

$$\frac{2\pi f \Delta\theta_r}{c} \leq \frac{\pi}{W} \quad (3.58)$$

or

$$\Delta\theta_r \leq \frac{c}{2fW}. \quad (3.59)$$

An analogous expression can be obtained for $\Delta\theta_t$ by assuming θ_r fixed. We can see that the allowed angular spacing depends on the frequency f . In order to ensure that the angular spacing satisfies these requirements across the full bandwidth of frequencies sampled, we enforce

$$\Delta\theta_r = \Delta\theta_t \leq \frac{c}{2f_{max}W} \quad (3.60)$$

where f_{max} is the maximum frequency in the sampled data.

Note that when monostatic data is collected, $\theta_r = \theta_t = \theta$. This does not change the sampling requirements placed on Δf , but for angular sampling, we note that (3.53) simplifies to $k_y \approx 2k\theta$. This change means that we now require

$$2k\Delta\theta \leq \frac{\pi}{W}. \quad (3.61)$$

After some simple rearranging, and again taking the maximum frequency, we see that this imposes the requirement that $\Delta\theta$ satisfy

$$\Delta\theta \leq \frac{c}{4f_{\max}W}. \quad (3.62)$$

Thus, monostatic data must be sampled twice as finely as the bistatic data.

Bistatic Sampling Requirements: Circular Apertures

In the previous section, we used small-angle expansions of (3.48) and (3.49) around $\theta_r = \theta_t = 0$ to arrive at the sampling requirements (3.56) and (3.60). This method for deriving the sampling requirements provides a simple and intuitive way to derive the bistatic sampling requirements. However, we wish to apply digital spotlighting to imaging geometries with up to fully circular apertures. In these wide angle and circular apertures, the small-angle approximation we made in order to obtain the expressions (3.52) and (3.53) breaks down. In this section, we will derive bounds on the sample spacing Δf and $\Delta\theta$ which ensure that (3.50) and (3.51) are satisfied without making the same small angle assumption.

Begin with the expressions (3.48) and (3.49) relating the data collected in (θ_r, θ_t, k) to spatial frequencies k_x and k_y . First, we will derive bounds on the sample spacing Δf . Fix θ_r and θ_t . In terms of $\Delta k = k_n - k_{n-1}$, we have

$$\Delta k_x = \Delta k (\cos \theta_r + \cos \theta_t) \quad (3.63)$$

$$\Delta k_y = \Delta k (\sin \theta_r + \sin \theta_t). \quad (3.64)$$

These expressions can be written equivalently using a trigonometric identity as

$$\Delta k_x = 2\Delta k \cos\left(\frac{\theta_r + \theta_t}{2}\right) \cos\left(\frac{\theta_r - \theta_t}{2}\right) \quad (3.65)$$

$$\Delta k_y = 2\Delta k \sin\left(\frac{\theta_r + \theta_t}{2}\right) \cos\left(\frac{\theta_r - \theta_t}{2}\right). \quad (3.66)$$

We can see that

$$|\Delta k_x| \leq 2\Delta k. \quad (3.67)$$

By inspection, we can see that this bound is achieved when $\theta_r = \theta_t = 0$ and $\theta_r = \theta_t = \pi$. Similarly,

$$|\Delta k_y| \leq 2\Delta k, \quad (3.68)$$

and this bound is achieved when $\theta_r = \theta_t = \frac{\pi}{2}$ and $\theta_r = \theta_t = \frac{3\pi}{2}$. In order to ensure that (3.50) is satisfied regardless of viewing angles, we will bound the worst case scenario where $\Delta k_x = 2\Delta k$ (or $\Delta k_y = 2\Delta k$). Thus, we require that

$$2\Delta k \leq \frac{\pi}{W}. \quad (3.69)$$

Substituting $\Delta k = 2\pi\Delta f/c$ and rearranging, we obtain

$$\Delta f \leq \frac{c}{4W}. \quad (3.70)$$

Note that (3.70) gives the same bound as (3.56). The small-angle expansion used to obtain (3.56) turns out to in fact be an expansion at the point where the sample spacing in k_x (or with rotated coordinate system, k_y) is the largest, and therefore satisfying the Nyquist requirement at this point will ensure that it is satisfied for the entirety of a circular aperture.

Next we will derive bound on the angular sample spacing $\Delta\theta$. Fix θ_t and k . Results would proceed analogously if we had instead fixed θ_r , so we will drop the subscripts and refer to the non-fixed angle as simply θ . In this case we have

$$\Delta k_x = k(\cos \theta_m - \cos \theta_{m-1}) \quad (3.71)$$

$$\Delta k_y = k(\sin \theta_m - \sin \theta_{m-1}), \quad (3.72)$$

Where θ_m and θ_{m-1} are consecutive samples in θ . After applying a trigonometric identity these can be equivalently expressed as

$$\Delta k_x = -2k \sin\left(\frac{\theta_m + \theta_{m-1}}{2}\right) \sin\left(\frac{\theta_m - \theta_{m-1}}{2}\right) \quad (3.73)$$

$$\Delta k_y = 2k \cos\left(\frac{\theta_m + \theta_{m-1}}{2}\right) \sin\left(\frac{\theta_m - \theta_{m-1}}{2}\right). \quad (3.74)$$

We can rewrite these expressions equivalently in terms of $\Delta\theta = \theta_m - \theta_{m-1}$ as

$$\Delta k_x = -2k \sin\left(\theta_{m-1} + \frac{\Delta\theta}{2}\right) \sin\left(\frac{\Delta\theta}{2}\right) \quad (3.75)$$

$$\Delta k_y = 2k \cos\left(\theta_{m-1} + \frac{\Delta\theta}{2}\right) \sin\left(\frac{\Delta\theta}{2}\right). \quad (3.76)$$

We can see that

$$|\Delta k_x| \leq 2k \sin\left(\frac{\Delta\theta}{2}\right), \quad (3.77)$$

and Δk_x achieves this bound when $\theta_{m-1} + \frac{\Delta\theta}{2} = \frac{\pi}{2}$, or $\theta_{m-1} + \frac{\Delta\theta}{2} = \frac{3\pi}{2}$. Similarly, We can see that

$$|\Delta k_y| \leq 2k \sin\left(\frac{\Delta\theta}{2}\right), \quad (3.78)$$

and Δk_y achieves this bound when $\theta_{m-1} + \frac{\Delta\theta}{2} = 0$, or $\theta_{m-1} + \frac{\Delta\theta}{2} = \pi$. As before, we want to ensure that (3.50) and (3.51) are satisfied across a fully circular aperture. To do this, we bound the worst case scenarios where

$$\Delta k_x = 2k \sin\left(\frac{\Delta\theta}{2}\right) \quad (3.79)$$

$$\Delta k_y = 2k \sin\left(\frac{\Delta\theta}{2}\right). \quad (3.80)$$

We assume that $\Delta\theta$ is small, giving

$$\Delta k_x \approx k\Delta\theta \quad (3.81)$$

$$\Delta k_y \approx k\Delta\theta. \quad (3.82)$$

Then, applying either (3.50) or (3.51) as appropriate, we obtain

$$k\Delta\theta \leq \frac{\pi}{W}. \quad (3.83)$$

Substituting $k = 2\pi f/c$ and rearranging yields

$$\Delta\theta \leq \frac{c}{2fW}. \quad (3.84)$$

To ensure that the angular spacing is appropriate for all sampled frequencies, we require that

$$\Delta\theta \leq \frac{c}{2f_{max}W}. \quad (3.85)$$

As for the case of the frequency sample spacing, the bound (3.85) matches (3.60). This is again due to the small-angle expansion of (3.53) occurring at angles which maximized Δk_y across a full circular aperture.

Multistatic Data

Consider now, instead of a single transmitter and single receiver forming a bistatic pair, the sensing geometry depicted in Figure 3.6, which consists of a set of N_p sensors spanning some angular aperture from θ_{min} to θ_{max} . Each of these sensors transmits electromagnetic waves to probe the scene and all sensors measure the resulting scattered field. The measured waves are assumed to have bandwidth B and we collect N_f frequency samples. Let $E(\theta_r, \theta_t, k)$ denote the measured field phasor for receive angle θ_r , transmit angle θ_t and wavenumber k . Samples of this function will lead to a "data cube" of phase history data. Consider a slice of this data cube for

fixed transmit angle θ_t . This slice is equivalent to the data collected by a bistatic pair with fixed transmit location and receiver moving across the aperture from θ_{min} to θ_{max} . If we instead fixed the receiver location θ_r , then the slice through the data is equivalent to a bistatic pair with fixed receiver and transmitter moving from θ_{min} to θ_{max} . Thus, the data cube $E(\theta_r, \theta_t, k)$ can be seen to be a collection of $N_p \times N_p$ bistatic data collections. Therefore, we enforce bistatic sampling requirements on the samples of $E(\theta_r, \theta_t, k)$. For sensors spanning a small angular aperture near the monostatic regime, the sampling requirements of (3.56) and (3.60) apply.

Without a small loss in generality in exchange for simplified notation, assume that data is collected for a square imaging region, $\Omega = \{(x, y) | x \in [-W, W], y \in [-W, W]\}$. We will also assume that data is collected from a narrow angular aperture centered at $\theta = 0$, and that the sensors are sufficiently far from Ω that the far field approximation applies. With these assumptions, the sampled data can be modeled as

$$E(\theta_r, \theta_t, k) = \int_{\Omega} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) \rho(x, y) e^{-jk(R_r(\theta_r) + R_t(\theta_t))} e^{j2kx + jk(\theta_r + \theta_t)y} dx dy. \quad (3.86)$$

We will use this model in our development of the digital spotlight processing.

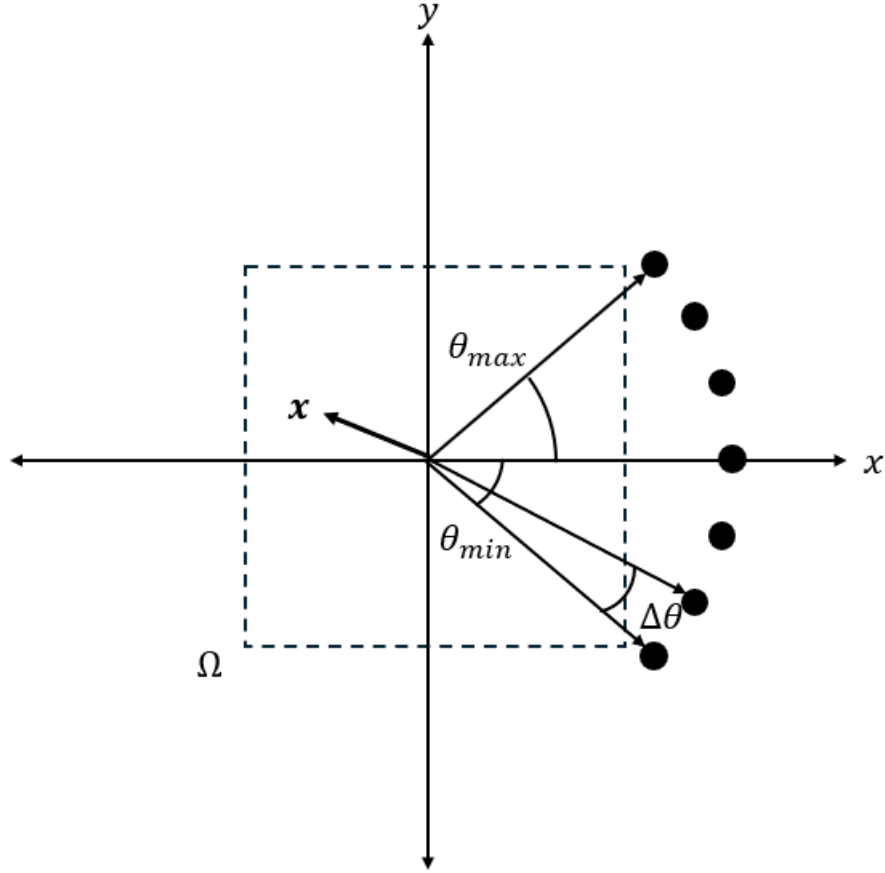


Figure 3.6: Diagram of multistatic sensor geometry.

Digital Spotlight ROI

Before applying digital spotlight processing to the measured phase history data $E(\theta_r, \theta_t, k)$, an ROI which falls within the original imaging region must be defined. Let $\Omega'_0 = \{(x, y) | x \in [c_x - W_0, c_x + W_0], y \in [c_y - W_0, c_y + W_0]\}$ be the digital spotlight ROI with center located at $\mathbf{c} = (c_x, c_y)$. We have assumed that the ROI is a square for simplicity in explication. We make the assumption that $\Omega'_0 \subseteq \Omega$ and $W_0 \leq W$. In practical scenarios, the ROI will likely be selected to contain a certain target of interest. Choosing this ROI then requires the identification and localization of a target of interest within the scene Ω . While this is a potentially challenging problem in its own right, it is an independent problem from that of the digital spotlight processing

which comes after it. For our purposes, we leave the selection of the digital spotlight ROI outside the scope of our work. Instead, we will take Ω'_0 as being given.

Re-centering phase history

Consider the data modeled by (3.86). Imaging this data will result in a image which is centered at $\mathbf{0}$ in the spatial domain. The first step in the digital spotlighting process is to apply a phase shift to each sample of the measured data to re-center the data to the new center of the spotlight ROI.

Define the quantities

$$d_r(\theta_r) = \|\mathbf{R}_r(\theta_r) - \mathbf{c}\| \quad (3.87)$$

$$d_t(\theta_t) = \|\mathbf{R}_t(\theta_t) - \mathbf{c}\|, \quad (3.88)$$

which measure the distance between each sensor position and the ROI center. To re-center the phase history data, we multiply each (θ_r, θ_t, k) sample by

$$e^{jk(R_r(\theta_r)+R_t(\theta_t)-d_r(\theta_r)-d_t(\theta_t))}. \quad (3.89)$$

The result is

$$E_{rc}(\theta_r, \theta_t, k) = \int_{\Omega_{rc}} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) \rho(x, y) e^{-jk(d_r(\theta_r)+d_t(\theta_t))} e^{j2kx+jk(\theta_r+\theta_t)y} dx dy. \quad (3.90)$$

We now operate in a new coordinate system with the origin at $\mathbf{0}$ redefined to coincide with the center of the digital spotlight ROI (previously located at $\mathbf{c}$). To reflect this new coordinate system, we define the regions $\Omega_{rc} = \{(x, y) | x \in [-W - c_x, W - c - x], y \in [-W - c_y, W - c_y]\}$ and $\Omega_0 = \{(x, y) | x \in [-W_0, W_0], y \in [-W_0, W_0]\}$ which represent the re-centered versions of Ω and Ω'_0 respectively.

Filter across frequency

Due to the smaller size of the spotlight ROI, fewer frequency samples are needed in order to image without aliasing artifacts. Using (3.56), if we sample at the Nyquist limit we need a frequency sample spacing of

$$\Delta f = \frac{c}{4W} \quad (3.91)$$

to image Ω and a sample spacing of

$$\Delta f' = \frac{c}{4W_0} \quad (3.92)$$

to image Ω_0 . Since $W_0 \leq W$, $\Delta f' \geq \Delta f$, and this means that fewer samples will be required to image Ω_0 alias-free for the same measured bandwidth.

If we sample in frequency with sample spacing $\Delta f'$ or finer, then self aliasing artifacts in the image of Ω_0 will be avoided. However, (3.90) contains contributions from all of Ω_{rc} , not just Ω_0 . If we downsample the frequency data to the new reduced rate of $\Delta f'$, then the signal contributions from points (x, y) outside Ω_0 will alias into our scene. In order to prevent these aliases from affecting the eventual image of the digital spotlight ROI, we must filter out these signal contributions before downsampling.

We will filter out these outside reflections in two independent parts, first in the range dimension and then again in the cross range dimension. Define the window function

$$H_f(x) = \begin{cases} 1, & x \in [-W_0, W_0] \\ 0, & \text{otherwise,} \end{cases} \quad (3.93)$$

and apply this to the recentered data as

$$E_{ff}(\theta_r, \theta_t, k) = \int_{\Omega_{rc}} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) H_f(x) \rho(x, y) e^{-jk(d_r(\theta_r) + d_t(\theta_t))} e^{j2kx + jk(\theta_r + \theta_t)y} dx dy. \quad (3.94)$$

This can be equivalently written as

$$E_{ff}(\theta_r, \theta_t, k) = \int_{-W-c_y}^{W-c_y} \int_{-W-c_x}^{W-c_x} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) e^{-jk(d_r(\theta_r)+d_t(\theta_t))} e^{jk(\theta_r+\theta_t)y} \rho(x, y) H_f(x) e^{j2kx} dx dy \quad (3.95)$$

$$= \int_{-W-c_y}^{W-c_y} \int_{-W_0}^{W_0} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) e^{-jk(d_r(\theta_r)+d_t(\theta_t))} e^{jk(\theta_r+\theta_t)y} \rho(x, y) e^{j2kx} dx dy \quad (3.96)$$

We can see that applying this window eliminates the contributions from all points in the scene outside of the ROI in the range (or x) direction. However, our data is sampled in frequency k and not spatially, and we would like to be able to eliminate these contributions by operating directly upon the sampled data. To do this, we again turn to the Fourier relationship present between x and $k_x = 2k$. We can equivalently express the windowing in (3.95) using the convolution theorem as

$$E_{ff}(\theta_r, \theta_t, k) = E_{rc}(\theta_r, \theta_t, k) * h_f(k) \quad (3.97)$$

where $*$ denotes convolution and

$$h_f(k) = \mathcal{F}_{k_x}(H_f(x)) \quad (3.98)$$

is the Fourier transform of $H_f(x)$ with respect to the Fourier variable $k_x = 2k$. This convolution allows us to operate directly on the measured data, $E_{rc}(\theta_r, \theta_t, k)$ while still eliminating the unwanted signal contributions from outside of the desired region. When performing this convolution on sampled data, we convolve the filter response $h_f(k)$ with the data for each transmit and receive angle pair independently.

Note that the filter defined in (3.93) represents the *ideal anti-aliasing* filter. In practice, this filter is unrealizable as it leads to an infinitely long response $h_f(k)$. Simple truncation of the filter response $h_f(k)$ can lead to unwanted artifacts appearing in the filtered signal. Instead, one should use one of the many well known window functions such as a Hamming Window or Hann Window to filter the signal. These different window functions each come with different sets of

advantages and disadvantages. However, for many applications, any of these choices can offer good performance. In the results we will present later, we have used a Hamming-window-based FIR filter.

Downsample in Frequency

Having filtered out the responses from outside the ROI in the range direction, we are now free to downsample the measured data without the risk of aliasing. There are two ways we can accomplish the downsampling processes. The first is to downsample by an integer factor in which case the resulting data will contain some of the original samples. The second is to interpolate and resample the data at the coarsest allowed sample spacing. We will describe both processes here, and provide some comments on the benefits and drawbacks of each.

Define the integer downsample factor

$$D_{f,int} = \left\lfloor \frac{\Delta f'}{\Delta f} \right\rfloor \quad (3.99)$$

where $\lfloor \cdot \rfloor$ denotes the floor function. The integer downsampling can then be achieved by simply taking every $D_{f,int}$ -th sample and discarding the rest. Mathematically,

$$E_{fds}(\theta_r, \theta_t, k_{n'}) = E_{ff}(\theta_r, \theta_t, k_{min} + n'D_{f,int}\Delta k) \quad (3.100)$$

where k_{min} is the lowest sampled wavenumber and $n' = 0, 1, \dots, \lceil N_f/D_{f,int} \rceil - 1$. We use $\lceil \cdot \rceil$ to denote the ceiling function and define $\Delta k = 2\pi\Delta f/c$. This downsampling technique has the advantage that each sample $k_{n'}$ in $E_{fds}(\theta_r, \theta_t, k)$ is taken from the original samples of $E_{ff}(\theta_r, \theta_t, k)$. However, due to the floor function in (3.99), we are not maximizing the number of samples that may be eliminated.

If we want to reduce the number of data samples by the maximum amount, consider instead the downsample factor

$$D_f = \frac{\Delta f'}{\Delta f}. \quad (3.101)$$

We can see that $D_f \geq D_{f,int}$ with equality achieved only in the rare circumstance of D_f being an integer. To achieve this greater downsample factor, we first define the new set of samples $k'_{n'} = k_{min} + n'\Delta k'$ for $n' = 0, 1, \dots, \lceil N_f/D_f \rceil - 1$, where $\Delta k' = 2\pi\Delta f'/c$. This set of samples is spaced exactly at the Nyquist limit. Then, for each transmit and receive pair (θ_r, θ_t) , construct an interpolant $L_{r,t}(k)$ using the original samples $E_{ff}(\theta_r, \theta_t, k_{min} + n\Delta k)$, $n = 0, 1, \dots, N_f$. We then use these interpolants to resample $E_{ff}(\theta_r, \theta_t, k)$ at the lower sample rate via

$$E_{fds} = (\theta_r, \theta_t, k'_{n'}) = L_{r,t}(k'_{n'}). \quad (3.102)$$

Since this non-integer downsampling results in samples with spacing equal to $\Delta k'$, by (3.56), this method achieves the lowest possible number of samples spanning our frequency bandwidth while still avoiding aliasing our image of Ω_0 . However the construction of $L_{r,t}(k)$ has the possibility of introducing some error into the resulting resampled measurements.

Resampling in Aperture

We have now filtered and downsampled the data in the frequency dimension, which corresponds to reducing our imaging region in the range direction. We next want to repeat this process in the aperture or angular dimensions of the data and reduce the imaging region in the cross range direction as well. However, while in most radar data collection scenarios it is typical to sample frequency with regular sample spacing, this is not the case for angular sampling. For example, a set of sensors spaced uniformly along a linear aperture will not be spaced regularly in angle. Having regular sample spacing is important more for practical reasons than theoretical. Most optimized routines for computing FFT's and performing filtering operations are written with the assumption of regularly spaced samples, and they cannot be guaranteed to produce correct outputs on data that does not fit these assumptions. Therefore, in order to leverage these optimized filtering routines, we resample the aperture data onto grid with regular angular spacing.

We assume that the data has been sampled at a rate which satisfies (3.60). We then define a regular sample spacing

$$\Delta\theta = \frac{\theta_{max} - \theta_{min}}{N_p}. \quad (3.103)$$

Let θ_{m_r} and θ_{m_t} , $m_r, m_t = 0, 1, N_p - 1$ denote the original sensor angular locations and then define the new regularly spaced sensor locations to be

$$\tilde{\theta}_m = \theta_{min} + m\Delta\theta. \quad (3.104)$$

Then, for each k , construct the interpolant $L_k(\theta_r, \theta_t)$ using the samples of $E_{fds}(\theta_{m_r}, \theta_{m_t}, k)$. The aperture resampled data is then computed via

$$E_{ar}(\tilde{\theta}_{m_r}, \tilde{\theta}_{m_t}, k) = L_k(\tilde{\theta}_{m_r}, \tilde{\theta}_{m_t}). \quad (3.105)$$

With $E_{ar}(\theta_r, \theta_t, k)$, we now have data that has regular sample spacing in the angular/aperture dimensions. We need to update our records of the sensor positions to match the newly resampled data. Define an interpolant $L_r(\theta)$ using the record of sensor locations $d_r(\theta_m)$. We then resample the record of the receive positions via

$$\tilde{d}_r(\theta_r) = L_r(\tilde{\theta}_m). \quad (3.106)$$

This process is repeated identically for the record of transmit positions $d_t(\theta_t)$. With this resampling complete and our data sampled at regular angular spacing, we are ready to filter and downsample the data in the angular/aperture dimensions. In order to reduce notational clutter, from this point on, we drop the tilde's from the resampled angles and distances.

Filter Across Aperture

Based on (3.60), the angular sample spacing at the Nyquist limit would be

$$\Delta\theta = \frac{c}{2f_{max}W}. \quad (3.107)$$

For imaging only Ω_0 , we only need the coarser angular sample spacing

$$\Delta\theta' = \frac{c}{2f_{max}W_0} \quad (3.108)$$

in order to prevent self aliasing artifacts. However, as was the case for downsampling in frequency, signal components coming from outside of Ω_0 may still alias into the image of Ω_0 , so we must filter out these signal components prior to downsampling.

The filtering in the frequency dimension of the data has already eliminated the signal components coming from outside Ω_0 in the range or x direction, so our data is modeled by

$$E_{ar}(\theta_r, \theta_t, k) = \int_{-W-c_y}^{W-c_y} \int_{-W_0}^{W_0} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) \rho(x, y) e^{-jk(d_r(\theta_r)+d_t(\theta_t))} e^{jk(\theta_r+\theta_t)y} e^{j2kx} dx dy. \quad (3.109)$$

To eliminate any signal components coming from objects located at cross range values $|y| \geq W_0$ we introduce the window function

$$H_a(y) = \begin{cases} 1, & y \in [-W_0, W_0] \\ 0, & \text{otherwise,} \end{cases} \quad (3.110)$$

and apply it to the data as

$$E_{af}(\theta_r, \theta_t, k) = \int_{-W-c_y}^{W-c_y} \int_{-W_0}^{W_0} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) \rho(x, y) H_a(y) e^{-jk(d_r(\theta_r)+d_t(\theta_t))} e^{jk(\theta_r+\theta_t)y} e^{j2kx} dx dy. \quad (3.111)$$

As was the case for windowing in the range or x direction, we would like to express this windowing operation in a way which operates directly on the data which is sampled in transmit and receive

angle. As before, we leverage the Fourier relationship in our data model, but this time we use the relationship between y and $k_y = k(\theta_r + \theta_t)$. Let

$$h_a(k_y) = \mathcal{F}_{k_y}(H_a(Y)) \quad (3.112)$$

be the Fourier transform of the window function $H_a(y)$. Since $k_y = k(\theta_r + \theta_t)$, we can express this filter kernel in terms of θ_r , θ_t , and k as $h_a(k(\theta_r + \theta_t)) = h_a(\theta_r, \theta_t, k)$. The convolution theorem allows us to express $E_{af}(\theta_r, \theta_t, k)$ as

$$E_{af}(\theta_r, \theta_t, k) = E_{ar}(\theta_r, \theta_t, k) * h_a(\theta_r, \theta_t, k). \quad (3.113)$$

We choose to approximate this convolution with a series of 1-D convolutions. Fix values for θ_r and k . We then perform the convolution

$$E'_{af} = E_{ar}(\theta_r, \theta_t, k) * h_a(\theta_t). \quad (3.114)$$

Fixing θ_r and k yields data that is equivalent to a bistatic data collection for one receiver location and a moving transmit location at a single frequency. The operation in (3.114) is equivalent to the filtering process which would remove the unwanted signal components from this bistatic data. We repeat (3.114) for each value of θ_r . Practically, for fixed k , the data takes the form of a matrix $\mathbf{E}(\theta_r, \theta_t)$, so this series of 1-D convolutions is equivalent to convolving along each row of $\mathbf{E}(\theta_r, \theta_t)$. Next, we fix values for θ_t and k and perform

$$E_{af}(\theta_r, \theta_t, k) = E'_{af}(\theta_r, \theta_t, k) * h_a(\theta_r) \quad (3.115)$$

For each value of θ_t . Similar to (3.114), the operation of (3.115) filters the unwanted signal components from the single frequency bistatic component of the data. However, this is now a fixed transmitter location and moving receiver location. Practically, we are now filtering along each column of $\mathbf{E}(\theta_r, \theta_t)$. The filtering process of (3.114) and (3.115) eliminate the unwanted signal

components from outside of Ω_0 for one frequency k . We repeat these convolutions for each value of k to fully filter the data. After this filtering process is complete, our data can be modeled as

$$E_{af}(\theta_r, \theta_t, k) = \int_{-W_0}^{W_0} \int_{-W_0}^{W_0} a(\mathbf{R}_r(\theta_r), \mathbf{R}_t(\theta_t), k) \rho(x, y) e^{-jk(d_r(\theta_r) + d_t(\theta_t))} e^{jk(\theta_r + \theta_t)y} e^{j2kx} dx dy. \quad (3.116)$$

Again, note that $H_a(y)$ represents an ideal anti-aliasing filter. In practice, one should use well established window functions to perform the filtering just as we do for the filtering across the frequency domain of the data. As before, we use a Hamming window based FIR filter in the results shown later.

Downsample in Aperture

With (3.116), we have data where all contributions from outside Ω_0 have been removed and which has been downsampled in the frequency dimension. The final stage in the digital spotlight process is to then downsample again in the angular or aperture dimensions of the data. As before, this can be done with an integer downsample factor or we can push the downsampling further by interpolating and resampling. Both techniques will be described.

Let

$$D_{a,int} = \left\lfloor \frac{\Delta\theta'}{\Delta\theta} \right\rfloor \quad (3.117)$$

be the integer angular downsample factor. Define indices $m'_r = 0, 1, \dots, \lceil N_p/D_{a,int} \rceil - 1$ and $m'_t = 0, 1, \dots, \lceil N_p/D_{a,int} \rceil - 1$ for the receive positions and transmit positions respectively. The downsampled data is then

$$E_s(\theta_{m'_r}, \theta_{m'_t}, k) = E_{af}(\theta_{min} + m'_r D_{a,int} \Delta\theta, \theta_{min} + m'_t D_{a,int} \Delta\theta, k). \quad (3.118)$$

Since this downsampling is akin to removing sensor positions, we must repeat the downsampling process on our record of sensor positions. Mathematically,

$$d'_r(\theta_{m'_r}) = d_r(\theta_{min} + m'_r D_{a,int} \Delta\theta) \quad (3.119)$$

$$d'_t(\theta_{m'_t}) = d_t(\theta_{min} + m'_t D_{a,int} \Delta\theta). \quad (3.120)$$

As was the case for the downsampling process in frequency, downsampling by an integer factor in this manner results in data that is pulled directly from the original data samples. However, the floor function in (3.117) means that this downsampling technique is suboptimal in terms of eliminating the greatest possible number of angular samples.

If we want to reduce the number of angular samples by the greatest amount possible, we must resample with a non integer downsample factor via interpolation. This resampling will follow the same pattern as the previous aperture resampling step that we performed, but this time we will use the coarser angular sample spacing $\Delta\theta'$. Define the new sets of angular sample locations to be

$$\theta_{m'_r} = \theta_{min} + m'_r \Delta\theta' \quad (3.121)$$

$$\theta_{m'_t} = \theta_{min} + m'_t \Delta\theta' \quad (3.122)$$

for $m'_r, m'_t = 0, 1, \dots, \lceil N_p / D_{a,int} \rceil - 1$. For each k , we construct the interpolant $F_k(\theta_r, \theta_t)$ using the samples of the filtered data $E_{af}(\theta_{m_r}, \theta_{m_t})$. The downsampled data is then

$$E_s(\theta_{m'_r}, \theta_{m'_t}, k) = F_k(\theta_{m'_r}, \theta_{m'_t}). \quad (3.123)$$

We must also resample the recorded sensor locations, so we define the interpolants of sensor ranges $F_r(\theta_r)$ and $F_t(\theta_t)$ using the samples $d_r(\theta_{m_r})$ and $d_t(\theta_{m_t})$. The resampled sensor locations are then

computed as

$$d'_r(\theta_{m'_r}) = F_r(\theta_{m'_r}) \quad (3.124)$$

$$d'_t(\theta_{m'_t}) = F_t(\theta_{m'_t}). \quad (3.125)$$

Much the same as the frequency downsampling, using this method allows that maximum reduction in total number of angular samples, obtaining a downsample factor of

$$D_a = \frac{\Delta\theta'}{\Delta\theta}. \quad (3.126)$$

This downsample factor is achieved while retaining the same aperture span. However, these samples will in general not be drawn from the original samples and thus there is the potential for interpolation error to be introduced.

At this stage, with the angular downsampling finished, the digital spotlight processing is complete. The new data, $E_s(\theta_r, \theta_t, k)$ has had signal components from outside the spotlight ROI Ω_0 filtered out, and the data has been downsampled along each data dimension θ_r , θ_t , and k . With this reduced data set, more efficient imaging of Ω_0 can be accomplished.

3.3 Results

Digital spotlighting has two primary benefits for LSM-based imaging. The first is that it can dramatically reduce the amount of data needed when we are interested in only a small subset of a larger scene. This leads to much smaller matrices and more efficient operations from LSM-based techniques. The second benefit comes from the spatial filtering process. This filtering is designed to reduce aliasing before downsampling, but it also seems to help reduce interference from targets outside the spotlight ROI as well.

Recall that LSM-based approaches are ill-posed. This means that small perturbations in the data can lead to large changes in the resulting solutions. Therefore, it is not immediately apparent that LSM-based techniques will work well with digital spotlight preprocessing. In this section, we

present results which illustrate both the feasibility of a combined digital spotlight and LSM-based imaging algorithm as well the claimed benefits.

3.3.1 Proof of feasibility and data reduction results

Digital spotlighting has the potential to greatly reduce the size of phase history data needed by the LSM imaging algorithms when the ROI is small compared to original scene. This leads to benefits in the efficiency of these methods. In (3.10), we see that we need to invert a matrix in order to compute the LSM solution at a given pixel. This has computational complexity of $O(N^3)$, and for large data sets this can result in significant computational expense. For PE-LSM the cost is even greater as in (3.23), the matrix $\mathbf{A}$ is different for each pixel. This means that for N_{pix} pixels, we must compute N_{pix} matrix inversions. For large matrices, this can lead to significant computational expense.

To demonstrate the feasibility of the digital spotlight pre-processing as well as to illustrate the cost savings, we show images from a Finite Difference Time Domain (FDTD) simulated dataset. The simulated data is collected for $N_t = N_r = 90$ sensor positions spanning a 90° aperture. A frequency band spanning 1.5 to 2 GHz is used, and $N_f = 101$ frequency samples are used. Targets are modeled as conductors with conductivity of $\sigma = 5.7 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$. We add noise to the data giving a Signal to Noise Ratio (SNR) of 20 dB. Figure 3.7 illustrates the simulation and target setup.

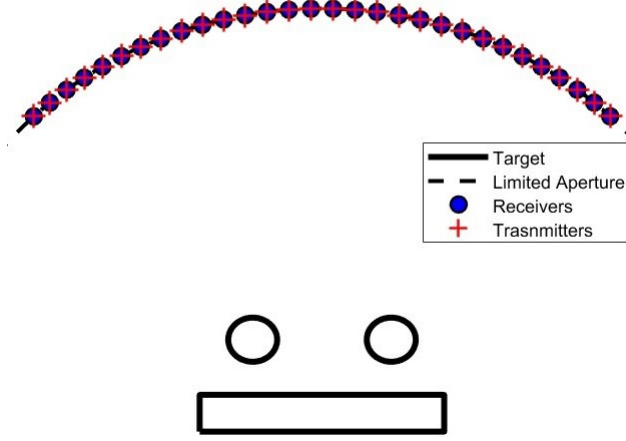


Figure 3.7: Diagram showing sensor setup.

We select a ROI containing the upper right circular target. We note that since the scene is illuminated from above, only the top surfaces of the targets are captured in the images. We choose the ROI such that the illuminated surface of the chosen target of interest is centered in the scene. Figure 3.8 depicts a PE-LSM image of the full scene as well as the ROI location. The ROI measures $0.8 \text{ m} \times 0.8 \text{ m}$ and we select $R_0 = \sqrt{2}(0.8) \text{ m}$ as the diameter of the spotlight region. This value for R_0 is the diameter of a circle inscribing the square ROI, and ensures that the width of the ROI in range and cross range is the same regardless of the perspective of a specific sensor.

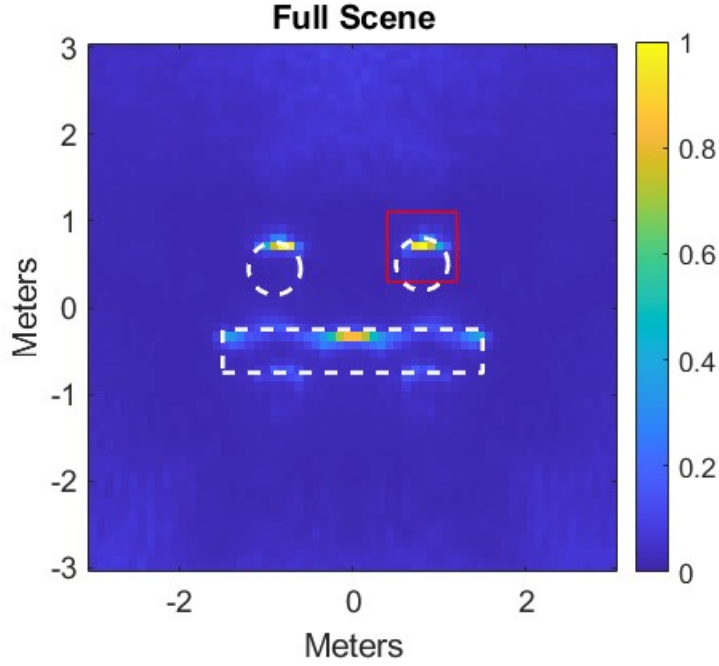


Figure 3.8: PE-LSM image of the full scene. The white dashed lines illustrate the true target locations and shapes while the red square illustrates the digital spotlight ROI.

The PE-LSM data in this case uses multiple frequencies and multiple transmit/receive positions and thus is a good test of our full multi-sensor/ multi-frequency digital spotlight routine. We elect to use non-integer downsample factors to achieve the greatest possible reduction in data size and to demonstrate that any errors introduced in the interpolation steps do not compromise the image quality of the PE-LSM images. For this specific image scene and ROI, after applying the digital spotlight routine to the data, a reduction factor of 59.25 is achieved. Figure 3.9 depicts a PE-LSM image of the ROI using the full, non-spotlit, data. In contrast, Figure 3.10 depicts a PE-LSM image of the ROI using the digitally spotlit data. We can see that although not identical, each image does a good job at capturing the illuminated surface of the target.

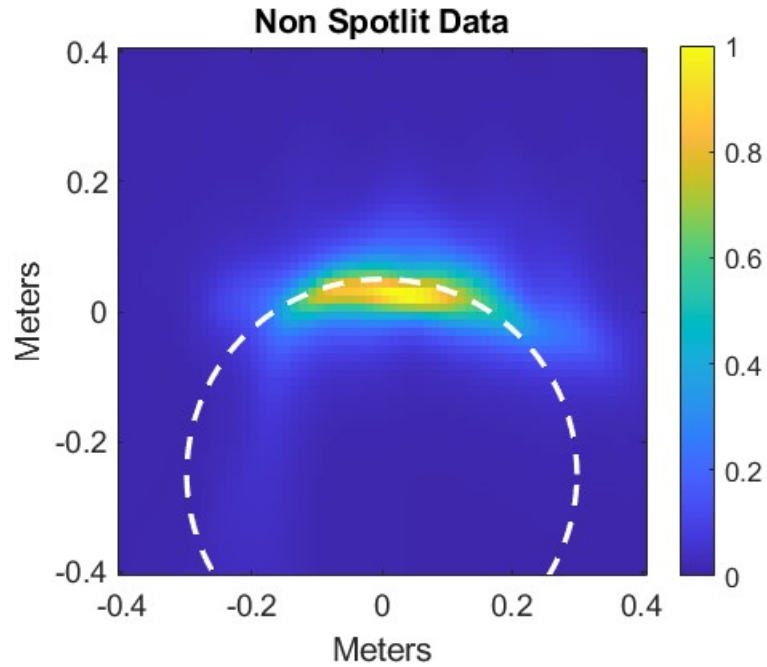


Figure 3.9: PE-LSM image of the digital spotlight ROI. This image uses the full data set collected to image the entire scene.

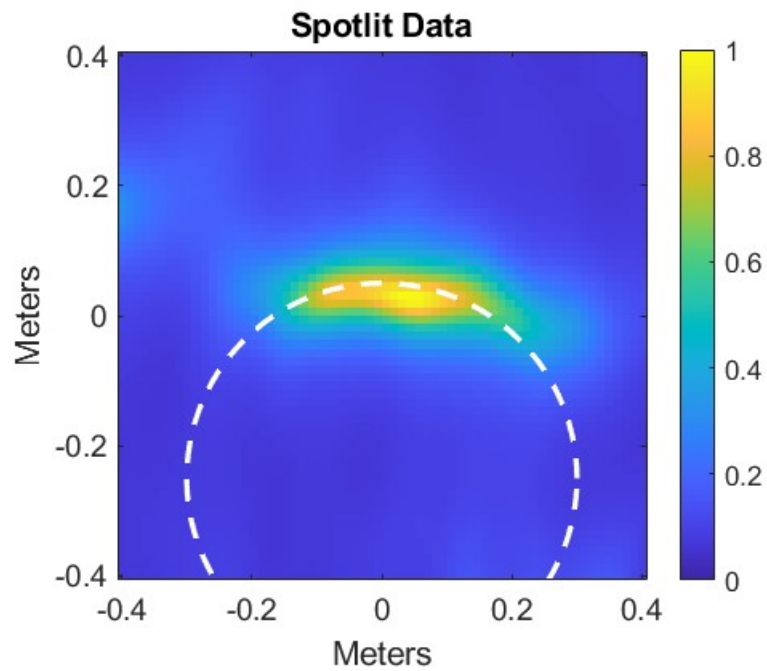


Figure 3.10: PE-LSM image of the digital spotlight ROI. This image uses the digitally spotlighted data.

3.3.2 Spatial filtering of artifacts for LSM

Here, we present results illustrating the benefits provided by the filtering aspects of digital spotlighting. To do this, we consider a scene where a strongly scattering disc lies near smaller and more weakly scattering targets. An FDTD simulation is used to generate synthetic data of these scenes. We then define the spotlight ROI around the smaller target near the disc and image this region with and without applying the digital spotlight processing to the data. We perform these tests with single-frequency data and standard LSM and then repeat them with multi-frequency data and use the non-coherent multi-frequency LSM indicator function. We will show results from both cases using a few different targets and geometries in the ROI.

For these simulations the same FDTD simulation code as was used in the simulations depicted in Figure 3.7 with changes to the collection parameters and to the targets. We now use a 360 degree aperture which surrounds the scene. Sensors are still placed at 1° increments. For single frequency results, a frequency of 1.018 GHz is used. For the multi-frequency data, we now sample $N_f = 101$ frequencies spanning 1 to 1.2 GHz. The large disc in the scene is a conductor with conductivity $\sigma = 5.7 \times 10^7$ S/m, relative permittivity $\epsilon_r = 1$, and radius $r = 30$ cm. The smaller nearby targets of interest are less conductive and have conductivity $\sigma = 5.7 \times 10^{-3}$ S/m and relative permittivity of $\epsilon_r = 1.3$. Noise is added to the data to achieve an SNR of 20 dB. We chose a selection of different targets of interest near the larger and more strongly scattering disc. First, we show a pair of equilateral triangles with height $h = 10$ cm. This case is illustrated in Figure 3.11. Next, we show a bar with length $l = 20$ cm and width $w = 5$ cm, and this target geometry can be seen in Figure 3.12.

We note that the color scale of the LSM images shown in this section have each been normalized with respect to the brightest pixel in the image.

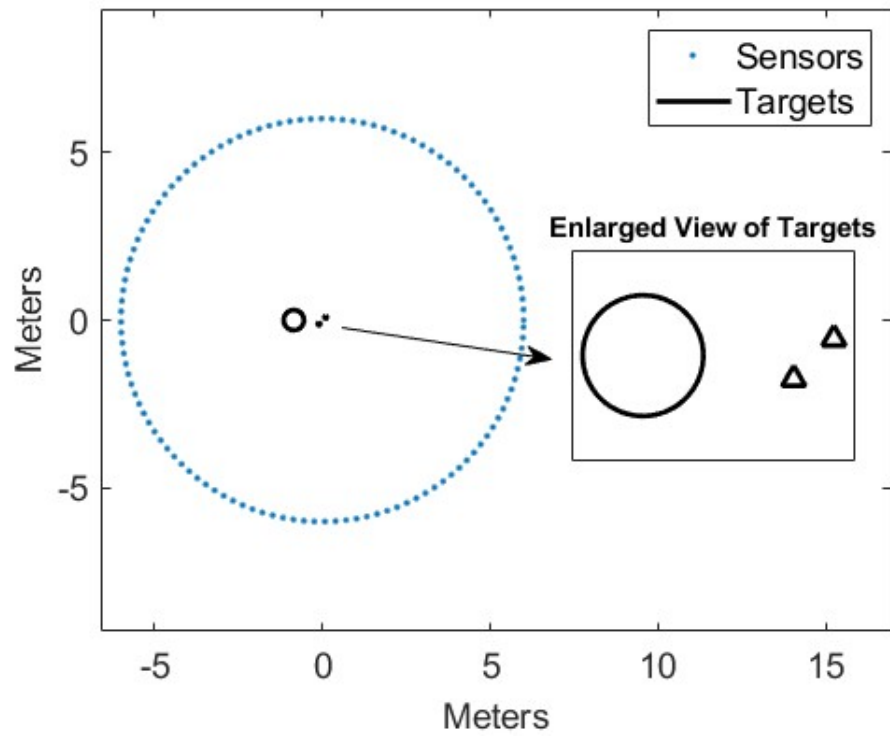


Figure 3.11: This diagram depicts the sensor and target geometry used for the FDTD simulation with a pair of triangular targets. An enlarged image of the targets is provided for clarity.

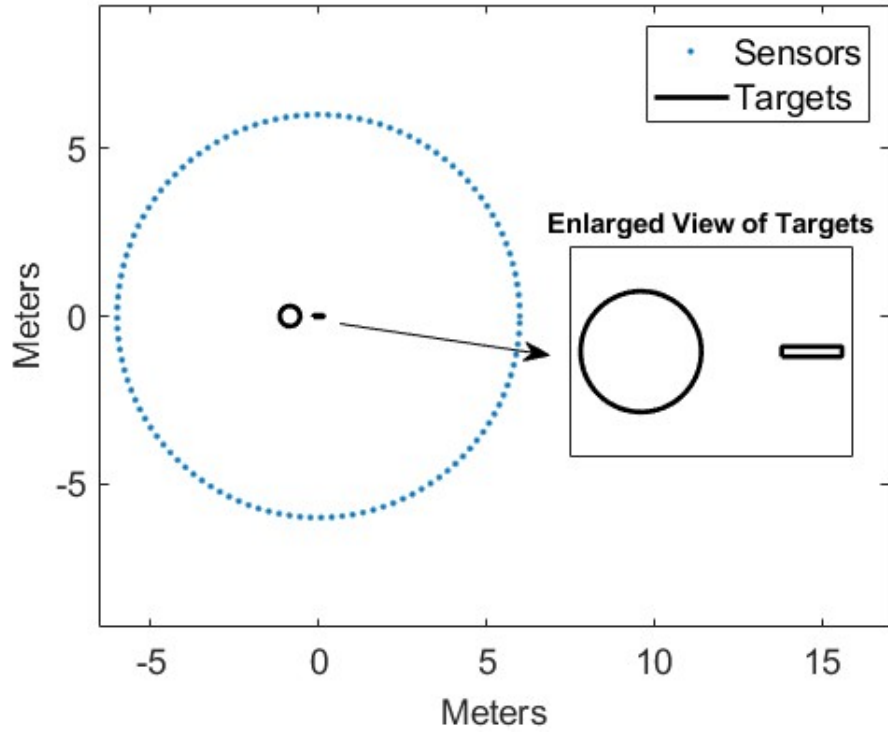


Figure 3.12: This diagram depicts the sensor and target geometry used for the FDTD simulation with a single bar shaped target. An enlarged image of the targets is provided for clarity.

Single-Frequency LSM

First we show results taken for LSM only at a single frequency. We will see that the digital spot-light processing improves the imaging of the weakly scattering targets. Later, we will repeat these tests for multi-frequency LSM and see that the use of additional frequency information provides some improvement in the image fidelity to the true target shapes.

Triangle Targets

First, we consider the scene with two 10-cm-tall triangles within the ROI, depicted in Figure 3.11. One triangle is located 10 cm above and 10 cm to the right of the ROI center, while the second triangle is 10 cm to the left and 10cm below the ROI center. In Figure 3.13 we have an LSM image of the full scene. A red square illustrates the location of the ROI. We can see from this

image the large discrepancy between the scattering strength of the disc, which appears bright, and the small triangular targets within the ROI, which are barely visible.

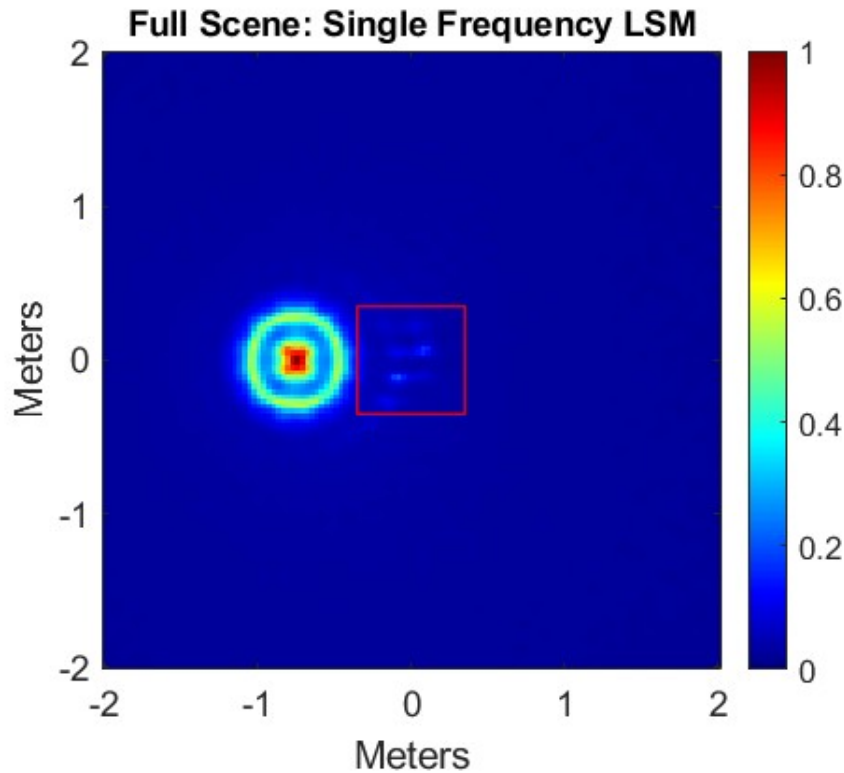


Figure 3.13: Single frequency LSM image of the full scene containing two triangular targets near the large circular target. The red square indicates the digital spotlight ROI.

To showcase the filtering effects of digital spotlighting, we first show an LSM image of the ROI produced from non-digitally spotlit data. This is displayed in Figure 3.14. We can see the two triangles are the brightest objects in the scene but there are several strong artifacts present as well. Significantly there is a very bright artifact directly above the lower triangle and the the left of the upper one. This artifact is strong enough that it may potentially hinder interpretation of this image.

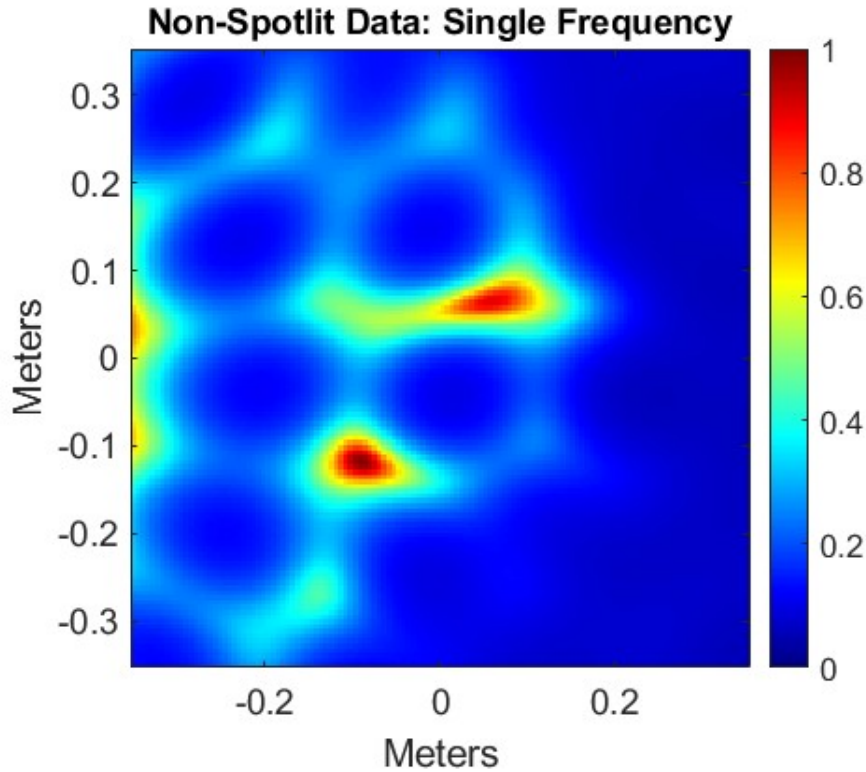


Figure 3.14: Single frequency LSM image of the ROI containing two triangular targets. Image is made with data that has not been processed by the digital spotlight procedure.

In contrast, the image constructed from digitally spotlighted data, shown in Figure 3.15, has significantly reduced these artifacts. Since the colorscale in the images in this section are normalized to the brightest pixel intensity, we can conclude that the intensity of the artifacts in Figure 3.15 has been significantly reduced compared to the intensity of the triangular targets. Although the artifacts are not completely removed, they are much weaker relative to the true targets when we have used the digital spotlight processing (Figure 3.15) compared to when we have not (Figure 3.14).

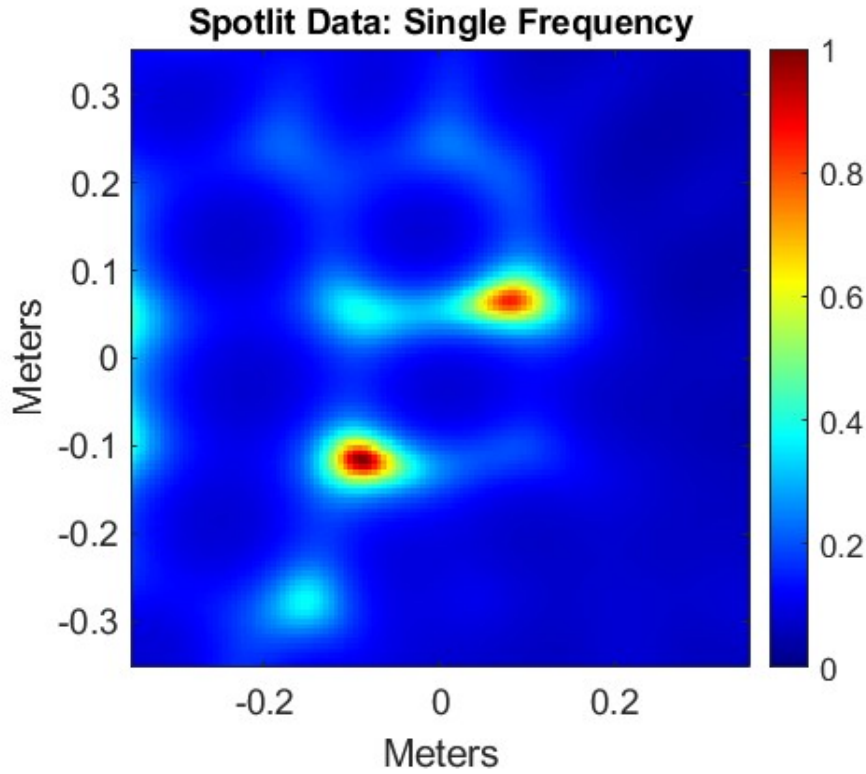


Figure 3.15: Single frequency LSM image of the ROI containing two triangular targets. The image is made with data that has been digitally spotlit.

Bar target

We repeat the previous experiment with the bar shaped target geometry depicted in Figure 3.12. The 20 cm long bar is centered in the ROI. Figure 3.16 depicts an LSM image of the full scene with the ROI highlighted by the red square. We can see that, similar to the case with the two triangles, the target of interest is very dim, compared to the much larger and more strongly scattering nearby disc.

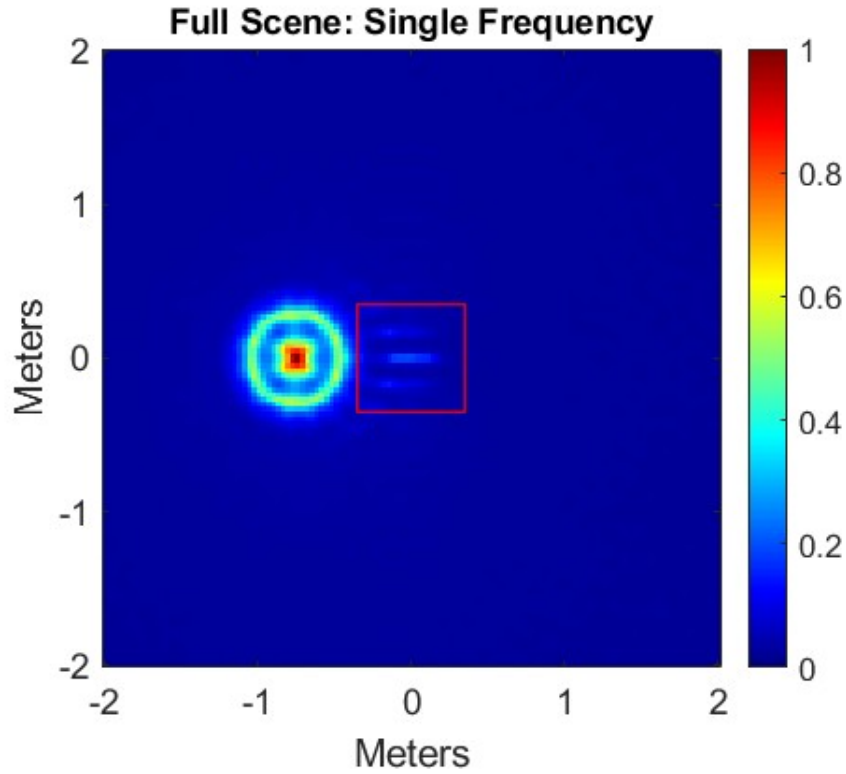


Figure 3.16: Single frequency LSM image of the full scene containing a bar shaped target near the large circular target. The red square indicates the digital spotlight ROI.

Figure 3.17 depicts the ROI using an image reconstruction based on the full, non-spotlit, data. There are two strong artifacts located symmetrically above and below the bar, as well as some contamination of the left side of the image from the strongly scattering disc.

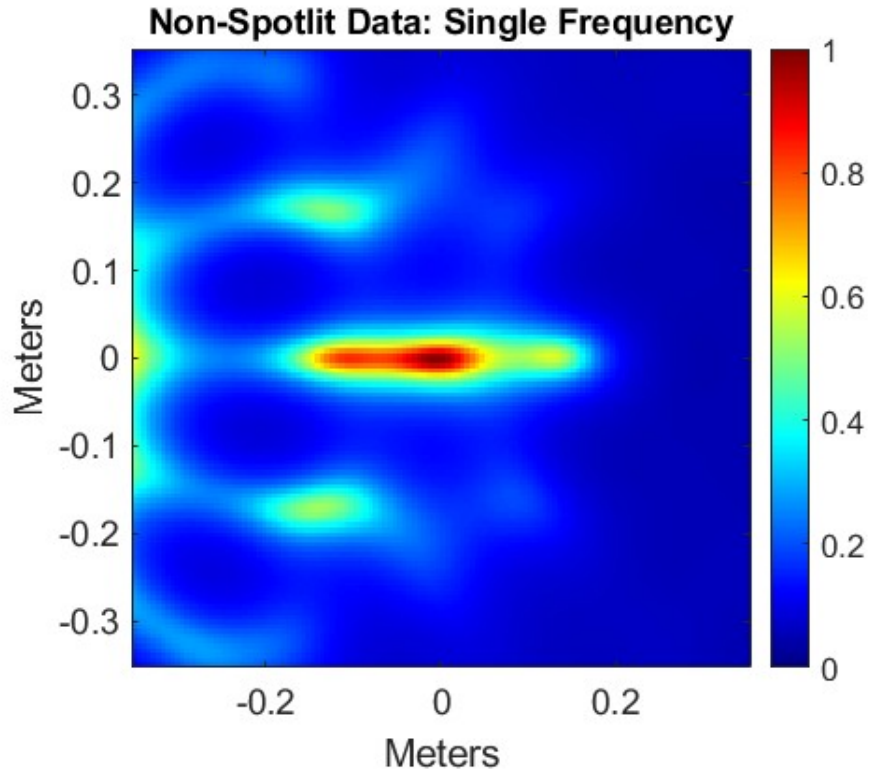


Figure 3.17: Single frequency LSM image of the ROI containing the bar shaped target. Image is made with data that has not been processed by the digital spotlight procedure.

In contrast with Figure 3.17, Figure 3.18 uses data that has been digitally spotlighted. The artifacts above and below the true bar are very noticeably reduced in intensity compared to the non-spotlit data. In addition, the leakage of the strongly scattering disc on the left is greatly reduced.

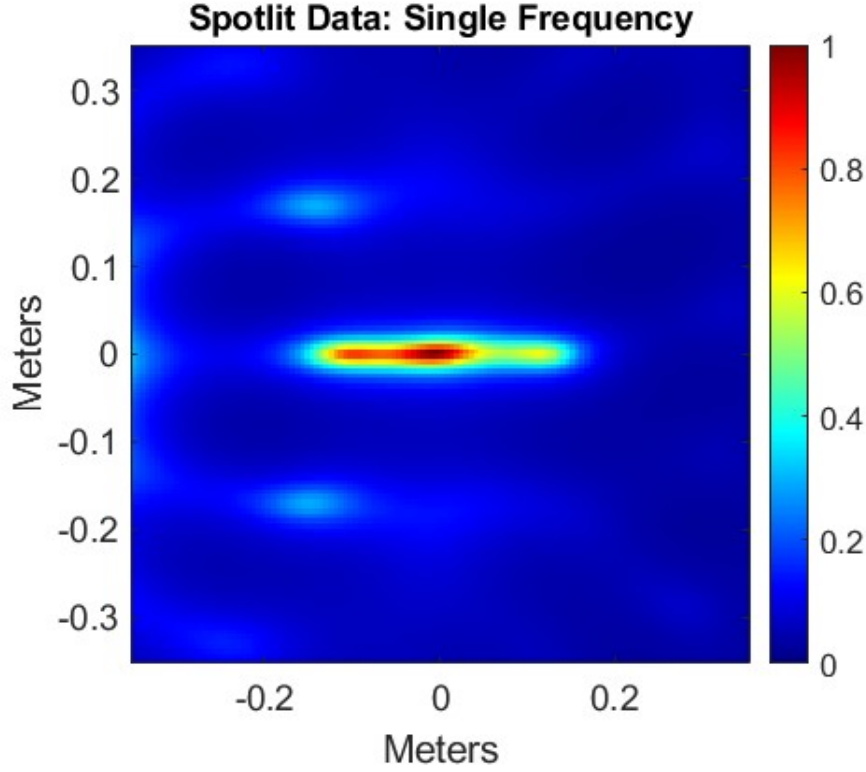


Figure 3.18: Single frequency LSM image of the ROI containing the bar shaped target. The image is made with data that has been digitally spotlighted.

3.3.3 Multi-frequency LSM

We repeat the previous experiments using a bandwidth of frequencies now. The same two target-of-interest cases, that of the pair of triangles and the bar, are used. In [50], it is demonstrated that even with non-coherent summation of LSM solutions across frequency, some benefit to image quality may be achieved. As we will see, using this technique in conjunction with digital spotlighting has comparable results to the benefits seen from the single frequency LSM in reducing artifacts in the scene. This method does improve on the reconstruction's ability to capture the true shape of the targets when compared to the single-frequency LSM.

Triangle Targets

We depict the full scene imaged by the multi-frequency LSM in Figure 3.19. The major difference that we note between this and Figure 3.13 is that the two triangle shaped targets show up much more strongly against the background and artifacts in the multi-frequency case.

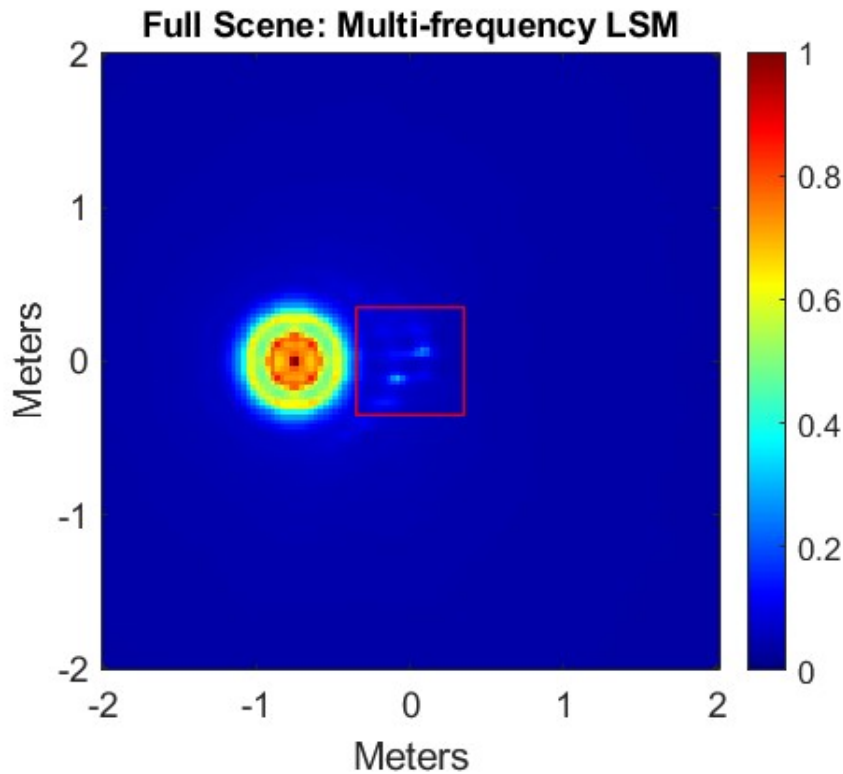


Figure 3.19: Multi-frequency LSM image of the full scene containing two triangular targets near the large circular target. The red square indicates the digital spotlight ROI.

To provide a point of comparison, we first show the ROI with non-spotlit data. We can see that, in comparison to Figure 3.14, the multi-frequency image of Figure 3.20 more strongly captures the triangular shape of the targets. However, the strong artifacts that we observed in 3.14 still contaminate the image.

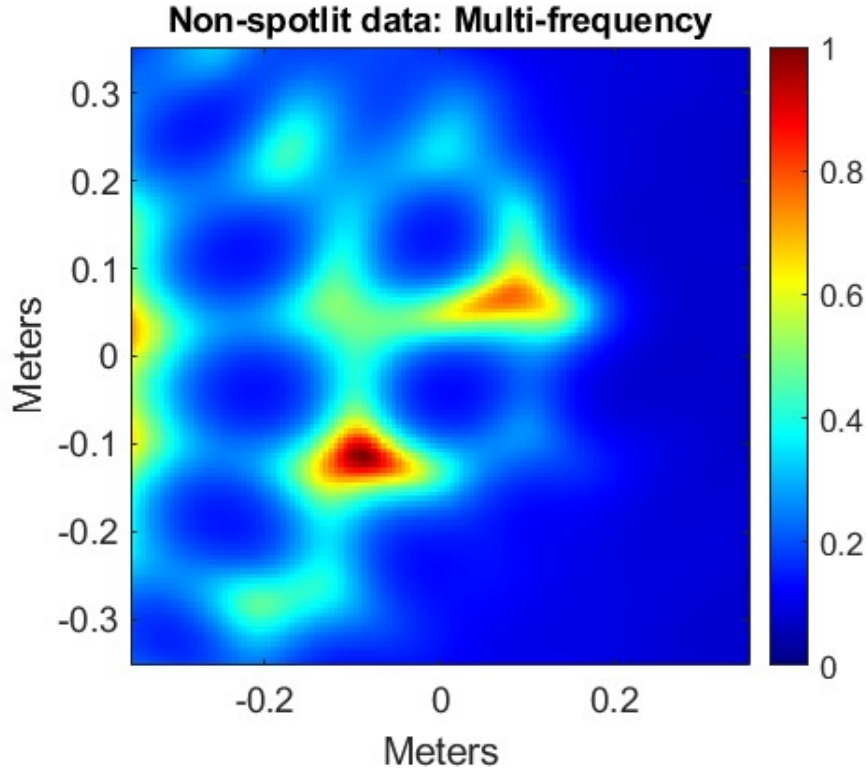


Figure 3.20: Multi-frequency LSM image of the ROI containing two triangular targets. Image is made with data that has not been processed by the digital spotlight procedure.

In Figure 3.21 we display the digitally spotlit LSM image of the ROI. Visually, the artifacts present in Figure 3.20 are significantly reduced in Figure 3.21, and the two true triangular targets are much more pronounced. In addition, while the targets of Figure 3.15 stand out well against the background and any remaining artifacts in the scene, they appear more rounded than the true triangular shape. In contrast, the image of Figure 3.21 gives good contrast between the true targets and the artifacts while also capturing the triangular shape of the targets with a much higher degree of fidelity.

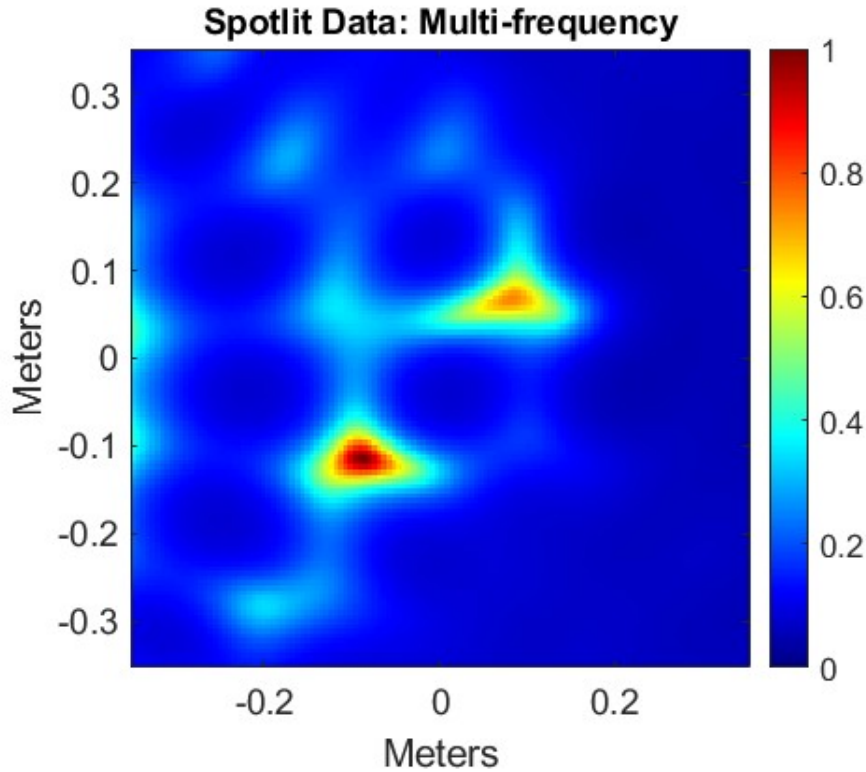


Figure 3.21: Multi-frequency LSM image of the ROI containing two triangular targets. The image is made with data that has been digitally spotlit.

Bar target

The shape of the bar target was captured with much higher fidelity in the single frequency case than were the shapes of the triangular targets. We attribute this simply to the difference in shape of the targets, as a long narrow bar shape does not require as sophisticated of an imaging method in order to be recognizable. Like the triangle case, we can see in Figure 3.22 that the bar target shows up much more clearly against the background and artifacts in our LSM image of the full scene when compared to the single frequency case in Figure 3.16.

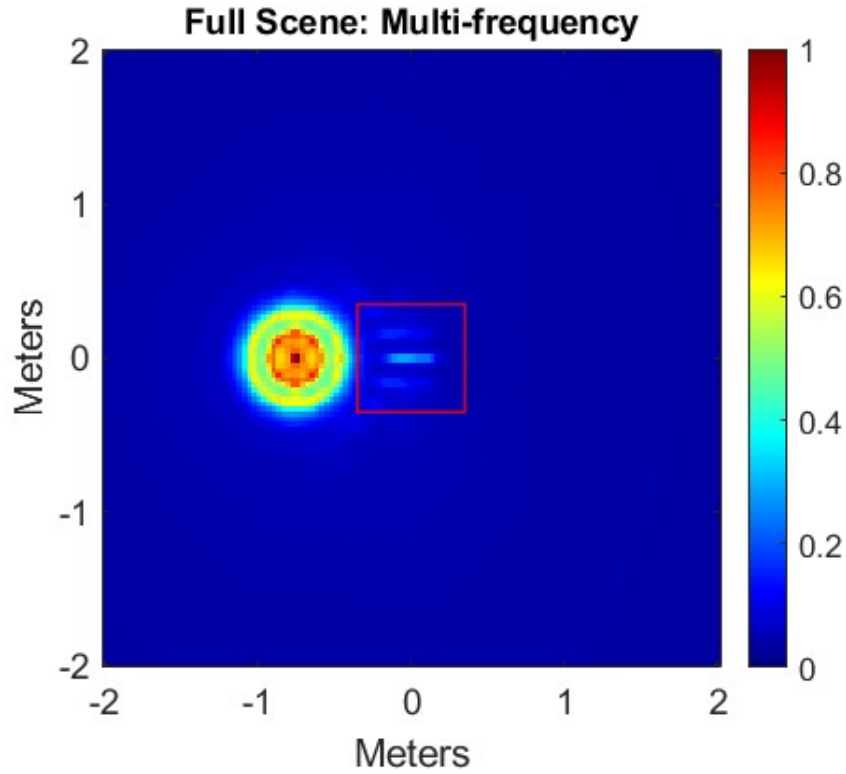


Figure 3.22: Multi-frequency LSM image of the full scene containing a bar shaped target near the large circular target. The red square indicates the digital spotlight ROI.

Figure 3.23 is very similar to the single frequency image in Figure 3.17. Both figures have strong artifacts present above and below the true bar shape and, at least visually, seem to do comparable jobs capturing the true shape of the bar. Again, this similarity between the single-frequency and multi-frequency images may be due to the nature of the bar shape when compared to the triangular shape rather than an indication of the performance of one method vs the other.

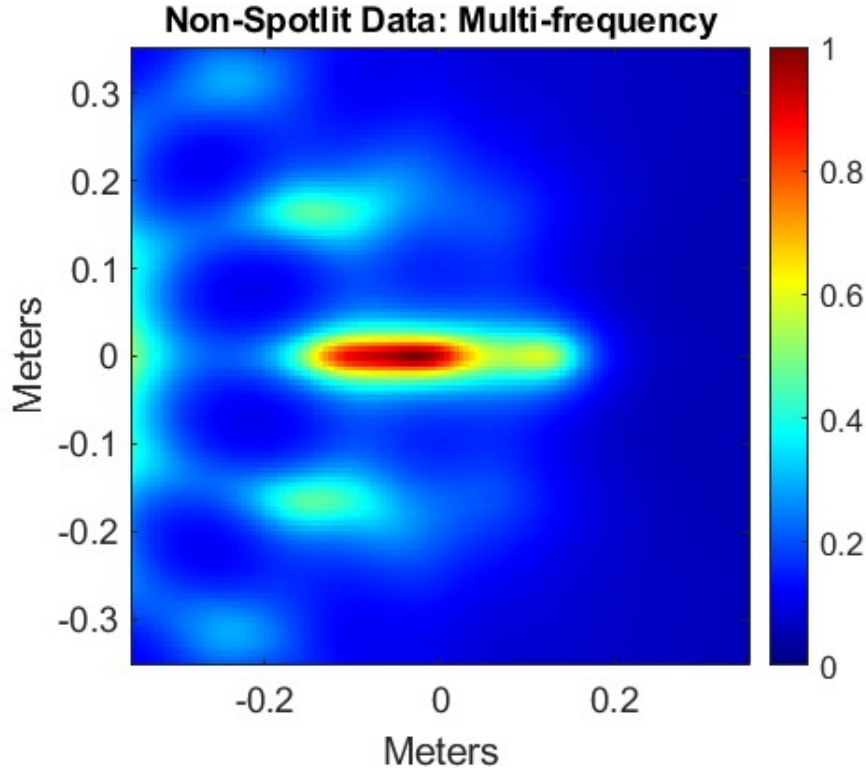


Figure 3.23: Multi-frequency LSM image of the ROI containing the bar shaped target. Image is made with data that has not been processed by the digital spotlight procedure.

Finally, in Figure 3.24 we can see the multi-frequency digitally spotlight LSM reconstruction of the ROI. As was observed in the single frequency case, the artifacts in the scene are significantly reduced, improving the interpretability of the image of the bar. Also, as before, due to the shape of the bar, no significant difference is visually apparent between the multi-frequency spotlight case and the single-frequency spotlight case in Figure 3.18. Both methods produce clear reconstructions with reduced contamination by artifacts introduced by the strongly scattering disc.

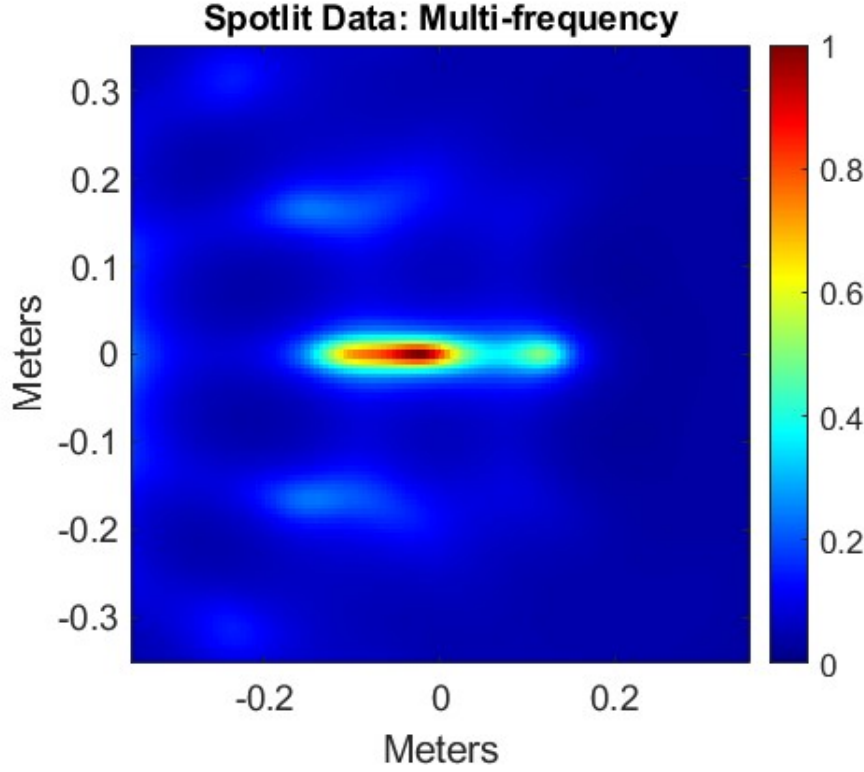


Figure 3.24: Multi-frequency LSM image of the ROI containing the bar shaped target. The image is made with data that has been digitally spotlighted.

3.3.4 Digital Spotlight and PE-LSM processing of SAR field data

As a final demonstration of digital spotlight processing for PE-LSM, we consider the GOTCHA dataset. This is a SAR data set which provides full 360° of coverage over a $50\text{ m} \times 50\text{ m}$ scene of interest. The data set also includes multiple elevation angles and multiple polarizations, although we only use a single elevation angle and horizontal transmit and horizontal receive (HH) polarized data in all the results that follow. For our images, we use 620 MHz of frequency bandwidth in the data. Since certain reflectors only return signal to the radar when viewed from certain angles (such as one side or the other of a car), full 360 degree views of the GOTCHA imaging scene are typically constructed via a sub aperture approach. This approach involves splitting the 360 degree aperture into smaller pieces and using these pieces to image the scene on a fixed pixel grid. The final image is constructed by combining the images from each subaperture. A common way to

combine these subaperture images is to plot the maximum value each pixel obtains across all the subaperture images.

The GOTCHA data is monostatic, so to make it fit in an LSM framework, we apply the monostatic to multistatic transform in [64]. This transform synthesized the data response of a single moving transmitter with a set of nearby receivers. This is accomplished via an interpolation process on the monostatic data. For an explanation of this process, we refer the reader to Section 3.2.1.

Up until this collaborative work at NRL, no LSM-type imaging had been accomplished for data collected by a fielded SAR system. These images were only made possible by a combination of the improvements in limited aperture imaging offered by PE-LSM, the development of the monostatic-to-multistatic transform, and the application of digital spotlighting to the SAR data in order to image small ROI's in the scene.

Specifically, digital spotlighting plays a significant role in making PE-LSM imaging on fielded SAR data computationally feasible. To image the GOTCHA scene with 10 degrees of aperture and the full bandwidth, 424 frequency samples are collected for 1170 sensor position along the synthetic aperture. This results in a block matrix $\mathbf{A}$ using in (3.23) of size 1594×1170 and therefore, a matrix inversion of the 1170×1170 matrix $\mathbf{A}^H \mathbf{A}$ is required for each pixel. If we want to image the scene for the total 360 degree aperture using the subaperture methods, this pixel by pixel inversion must be completed 36 times for each 10° degree subaperture. This creates an extremely prohibitive cost in imaging the entire scene using PE-LSM.

If we instead focus our interest to a smaller part of the scene, perhaps a single vehicle, we can reduce the data size using the digital spotlight technique. We can then obtain a PE-LSM image of the vehicle of interest. This PE-LSM image should better capture the vehicle shape than standard backprojection imaging. Consider the same 10-degree aperture as above, but focus only on an ROI contained inside a circle with 16.6 m diameter. After digital spotlighting, only 71 frequency samples for 161 sensor positions are needed. This is a data reduction factor of 43.4, and we now only need to invert a 161×161 matrix for each pixel to construct our subaperture image rather than

1170×1170 . This is a huge computational savings, without which, applying PE-LSM to SAR data is much less feasible.

We present here a demonstration of the digital spotlight imaging of the GOTCHA data using PE-LSM. First, Figure 3.25 depicts a backprojection image of the full $50 \text{ m} \times 50 \text{ m}$ scene constructed from 10° subapertures. A light blue square indicates an $11.8 \text{ m} \times 11.8 \text{ m}$ square containing the target of interest, in this case, a car. For the digital spotlight processing, we consider a circle with diameter 16.6 m which circumscribes this square when computing appropriate downsample factors when reducing the data. We use the monostatic digital spotlight procedure and then, for PE-LSM imaging, we apply the monostatic-to-multistatic transform to the resulting spotlight data. Figure 3.26 illustrates a digitally spotlighted backprojection image of the car contained in the ROI. The outer ring of scattering points surrounding the car shape are either multipath or layover artifacts. The PE-LSM image is presented in Figure 3.27. Compared to the backprojection image, the PE-LSM image captures the sides of the car as a more continuous surface rather than a collection of points. It also presents finer range resolution, with the sides of the car much tighter in the range direction than in the backprojection image. This improved range resolution allows us to make out more details of the car. Notably, on the right half of the car we can see a vertical line roughly 1 m to the left of the end of the vehicle. We believe this line shows where the windshield of the car meets the metal of the body. Inspection of the backprojection image shows a bright scatterer at this location but without the improved range resolution it is difficult to interpret what this might be.

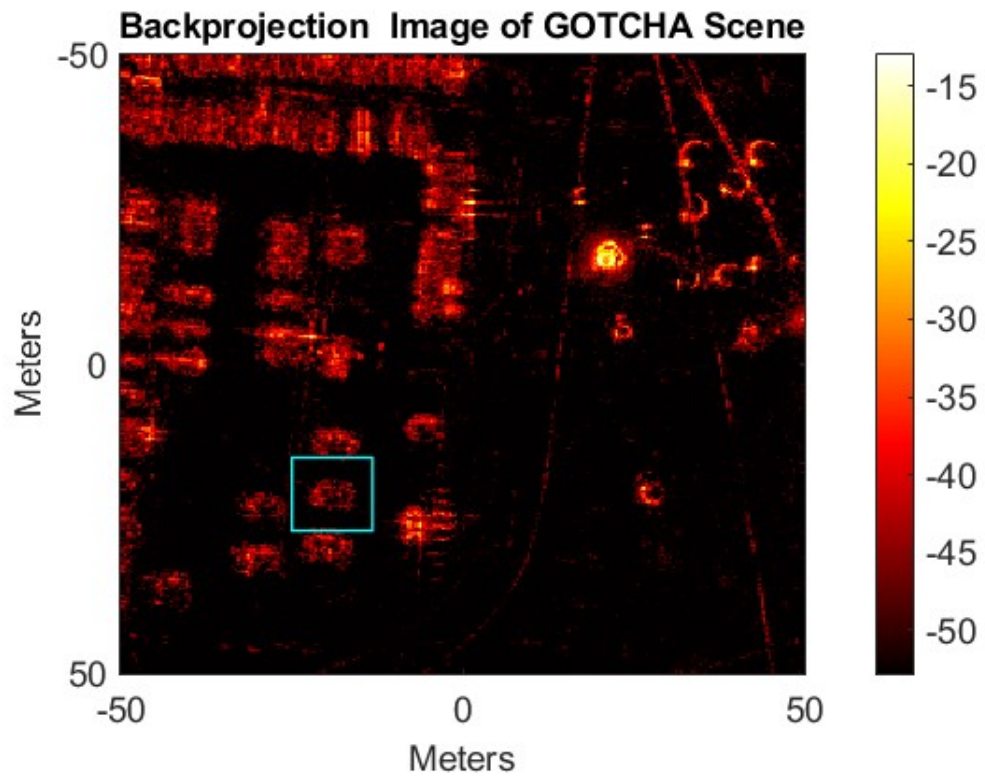


Figure 3.25: Backprojection image of full GOTCHA data scene. We have used a light blue square to indicate the region of interest within the scene.

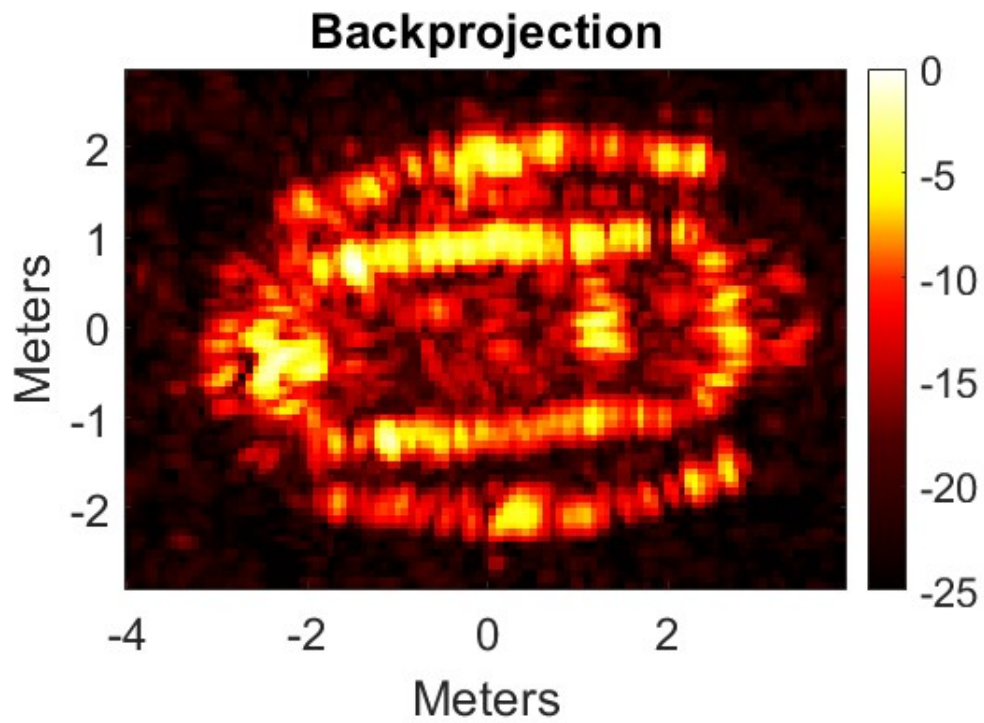


Figure 3.26: Backprojection image of car within the ROI highlighted in Figure 3.25. This image is constructed with digitally spotlighted data.

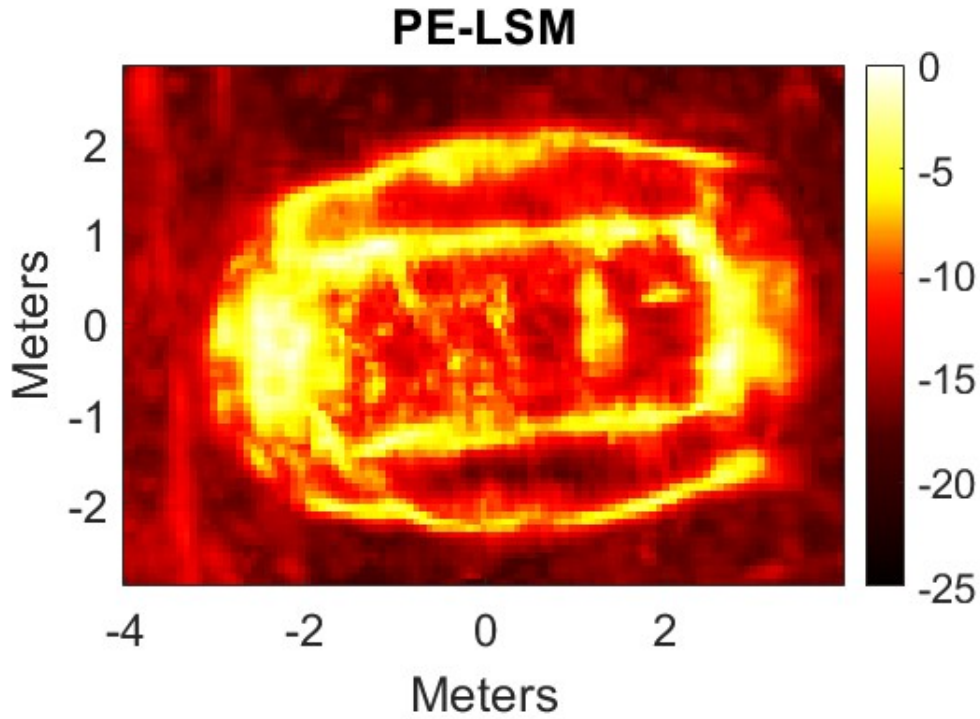


Figure 3.27: PE-LSM image of the car inside the ROI highlighted in Figure 3.25. This image is constructed with digitally spotlight data.

To further reinforce that our method is working well on the GOTCHA data, we consider a second vehicle in a different location. This location is highlighted by the light blue square in Figure 3.28 (the vehicle is located next to the original vehicle in Figure 3.25). Figures 3.29 and 3.30 depict digital spotlight images of the car using backprojection and PE-LSM respectively. Even more so than Figure 3.27, Figure 3.30 illustrates the ability of PE-LSM to tightly capture the outline of the car being imaged. The method is able to produce an image with a smooth and continuous boundary while the backprojection is not.

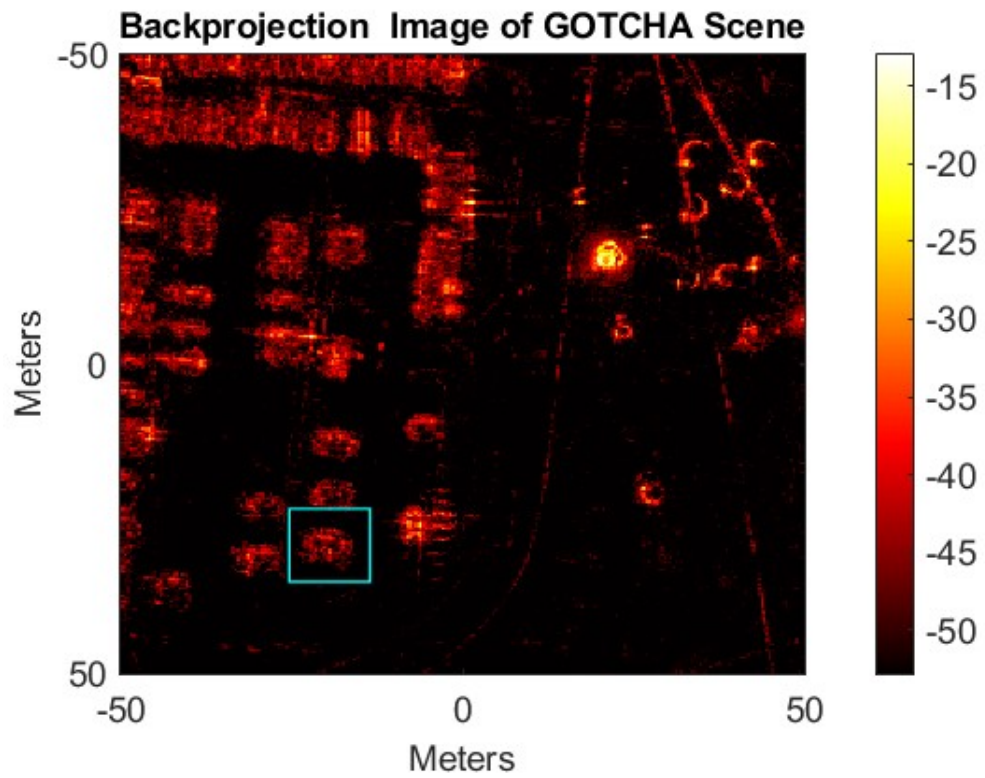


Figure 3.28: Backprojection image of full GOTCHA data scene. We have used a light blue square to indicate the region of interest within the scene.

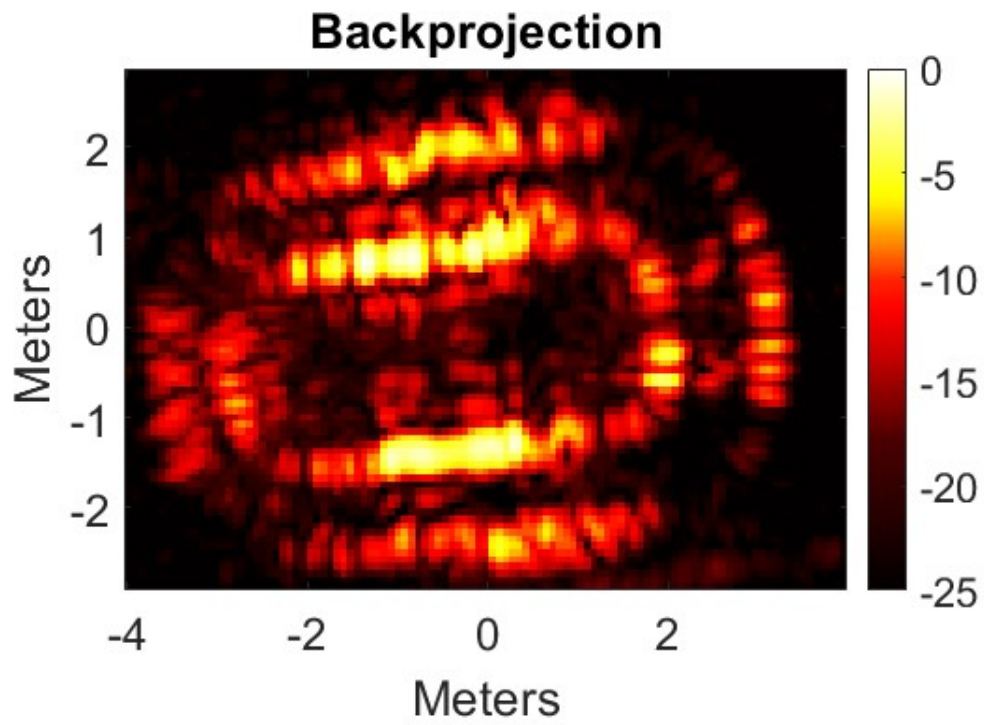


Figure 3.29: Backprojection image of car within the ROI highlighted in Figure 3.28. This image is constructed with digitally spotlighted data.

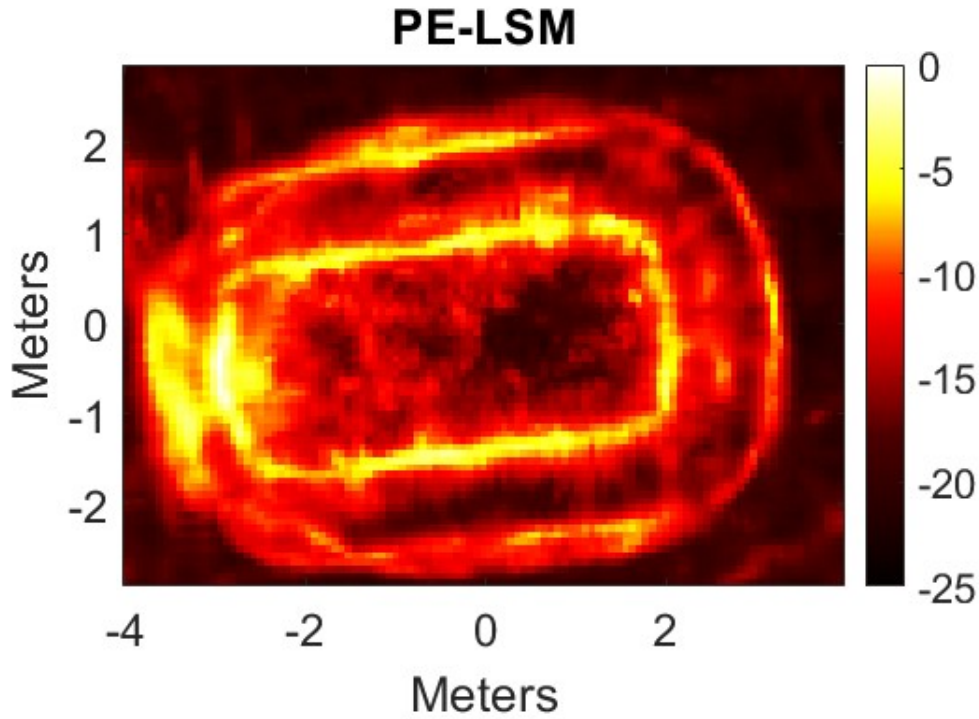


Figure 3.30: PE-LSM image of car within the ROI highlighted in Figure 3.28. This image is constructed with digitally spotlighted data.

We wish to restate to the reader the significance of the results shown in Figures 3.27 and 3.30. These images, represent the first, to our knowledge, LSM-type images created using SAR data from a fielded SAR system. It is visually apparent that the PE-LSM images do a better job at capturing the true outline the cars featured than do the backprojection images of the same scene. The PE-LSM images would not be computationally feasible for a large scene such as that of the full GOTCHA data, and the digital spotlighting is a key piece in the successful imaging displayed here.

3.4 Conclusions

We have successfully applied SAR digital spotlight processing to LSM and PE-LSM imaging. The results show clear computational savings while preserving image quality. In settings with large and bright scatterers outside the digital spotlight ROI, it has been demonstrated that the spotlight

processing reduces the artifacts produced by these scatterers. Finally, digital spotlight processing made it possible to produce the first images of fielded SAR data with an LSM-based technique.

Although we have demonstrated that the filtering aspect of digital spotlighting yields benefits for LSM imaging when the target is near a strong source of potential interference, there is still more that can be explored. One specific case of interest is Ground Penetrating Radar or GPR. In this imaging regime, the reflections from the ground can often overwhelm the signal measured from objects just under the ground surface. Digital spotlight processing may offer some benefit in reducing this contamination, improving the ability to image shallowly buried targets. All of our results demonstrating the benefits of filtering the data in reducing image artifacts have been qualitative in nature. In future studies, it may be of interest to explore mathematically these filtering operations and to then be able to make quantitative statements about the extent of these artifact reductions.

Chapter 4

Linear Sampling Method enhancement for Synthetic Aperture Radar Images

4.1 Introduction

Synthetic aperture radar (SAR) is an active imaging technique which is useful in a variety of applications including environmental monitoring [41] [42], resource management [43] [44], and target recognition [45] [46]. Traditional SAR imaging algorithms, such as the backprojection algorithm [39], are based on approximate data models which assume that a target is a collection of point-like scatterers and use the Born approximation to simplify the scattering physics of the targets.

The LSM of [48] is an imaging technique notable for its avoidance of point-scatterer and the Born approximations in its conventional formulation. Since the algorithm is based upon a model which accounts for more complex scattering behavior, the LSM has the potential to produce images which better capture the true target shapes than traditional SAR techniques. However, the LSM requires multistatic-multiview data and typically works best with apertures that fully surround the imaging scene. This type of data is not available in common SAR data collections, which are typically monostatic and often view a scene from a more limited aperture. PE-LSM is a modification of the standard LSM which is designed to improve on the shortcomings of the LSM for SAR-like limited aperture data collection geometries. Since its introduction in [61], PE-LSM has been demonstrated to offer improvements over both traditional backprojection and the standard LSM imaging [63], [62], [65]. However, LSM-based methods solve what is known as an ill-posed problem. This means that the technique is more sensitive to perturbations and noise in the data than more stable methods such as backprojection.

Due to the potential for improved imaging performance, it is of interest to apply LSM-based techniques to SAR data and imagery. The application of the PE-LSM to data collected from a monostatic SAR geometry was successfully accomplished in [62]. However, it would also be of interest to gain the PE-LSM's improved imaging performance in cases where SAR imagery is available but not the original data. Data sets, such as the well known MSTAR data [73], may only contain the complex-valued SAR imagery. Due to poor record keeping and/or data rot, raw SAR data may be unavailable or unreadable, but the accompanying imagery may still be available. In addition, fielded SAR systems produce phase history data which can be extremely large, even on the order of Gigabytes to Terabytes. The accompanying imagery is typically much smaller, making operating only on the imagery itself more attractive.

In this paper, we put forward a two-stage method for generating PE-LSM images from previously-focused backprojection images. First, we use Fourier relationships to transform the backprojection images to the spatial frequency domain. Then, we apply the PE-LSM to the transformed data. This strategy, which is referred to as beamspace processing the past, has previously been used for techniques such as MUSIC [74].

We will present the procedure and mathematics of the technique, and we will then evaluate its performance using several simulated datasets. It is critical to empirically evaluate imaging performance as the data synthesis process could, in principle, introduce perturbations to the data due to approximations and filtering operations. Due to the ill-posed nature of the PE-LSM, these perturbations have the potential to cause imaging artifacts. Our results show that while there is some degradation caused by the beamspace processing, the resulting images maintain important advantages of the PE-LSM over backprojection.

The remainder of this work will be organized as follows. In Section 4.2 we will develop a model for SAR images, outline our technique based on this model to synthesize data fit for LSM imaging, and then explain the PE-LSM imaging procedure we will use on the resulting data. Section 4.3 will display our preliminary results using this method, and finally, Section 4.4 will offer concluding remarks.

4.2 Methods

In this section we will provide the details of our method. We will first lay out a model of monostatic SAR data and imagery. Then, we will use this model to develop a technique to synthesize SAR-like data from an existing SAR image. Finally, the mechanics of LSM and PE-LSM imaging will be discussed.

4.2.1 Monostatic SAR Data Model

Consider an imaging region Ω , centered at $\mathbf{0}$, which is investigated by a monostatic SAR sensor. Under the start-stop and Born approximations, the measured data collected by the sensor can be modeled by

$$E(\theta, k) = \int_{\Omega} a(\mathbf{x}, \theta, k) \rho(\mathbf{x}) e^{-2jkR(\mathbf{x}, \theta)} d\mathbf{x}. \quad (4.1)$$

Here, $k = 2\pi f/c$ is the wavenumber associated with frequency f , $a(\mathbf{x}, \theta, k)$ is a slowly varying amplitude function, $\rho(\mathbf{x})$ is the complex reflectivity of the scene, and $R(\mathbf{x}, \theta)$ measures the distance from $\mathbf{x} = (x, y)$ to the sensor's position at aperture angle θ . Figure 4.1 depicts this imaging scenario.

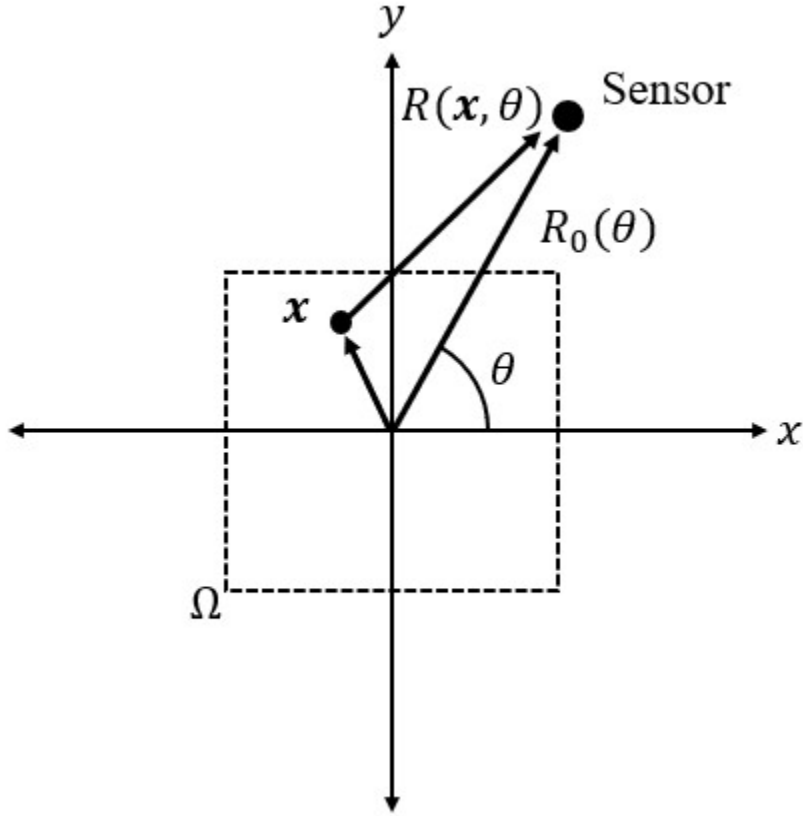


Figure 4.1: Diagram depicting the monostatic SAR sensing setup.

Typically, $a(\mathbf{x}, \theta, k)$ varies slowly enough over the domain Ω that this variation can be ignored, so we simplify the amplitude function to $a(\theta, k)$. For data collected in the far field, we can approximate the data model as

$$E(\theta, k) = \int_{\Omega} a(\theta, k) \rho(\mathbf{x}) e^{-2jkR_0(\theta)} e^{2jk\mathbf{x}^T \hat{\mathbf{r}}(\theta)} d\mathbf{x}, \quad (4.2)$$

where $\hat{\mathbf{r}}(\theta) = (\cos \theta, \sin \theta)$ and $R_0(\theta)$ is the distance from the sensor at aperture angle θ to the scene center at $\mathbf{0}$. Letting $\tilde{a}(\theta, k) = a(\theta, k) e^{-2jkR_0(\theta)}$, we can re-write the data model as

$$E(\theta, k) = \tilde{a}(\theta, k) \int_{\Omega} \rho(\mathbf{x}) e^{2jk\mathbf{x}^T \hat{\mathbf{r}}(\theta)} d\mathbf{x}. \quad (4.3)$$

Upon inspection of (4.3), we can see that according to our data model, $\rho(\mathbf{x})$ can be approximated via a scaled 2-D spatial Fourier transform of the SAR data. We will refer to this 2-D spatial Fourier domain as k -space. The Fourier relationship becomes more clear noting that $\mathbf{x}^T \hat{\mathbf{r}}(\theta) = x \cos \theta + y \sin \theta$, and by defining the quantities

$$k_x = 2k \cos \theta \quad (4.4)$$

$$k_y = 2k \sin \theta, \quad (4.5)$$

in which case our SAR data can be expressed as

$$E(\theta, k) = \tilde{a}(\theta, k) \int_{\Omega} \rho(x, y) e^{jk_x x + jk_y y} dx dy. \quad (4.6)$$

For a sensor tracing out a set of aperture angles between θ_1 and θ_2 and collecting data over a bandwidth of frequencies, (4.4) and (4.5) show that the monostatic SAR data represents a sampling of an annulus shape in the k -space of the scene [40]. An illustration of an example annulus of SAR data in k -space is illustrated in Figure 4.2.

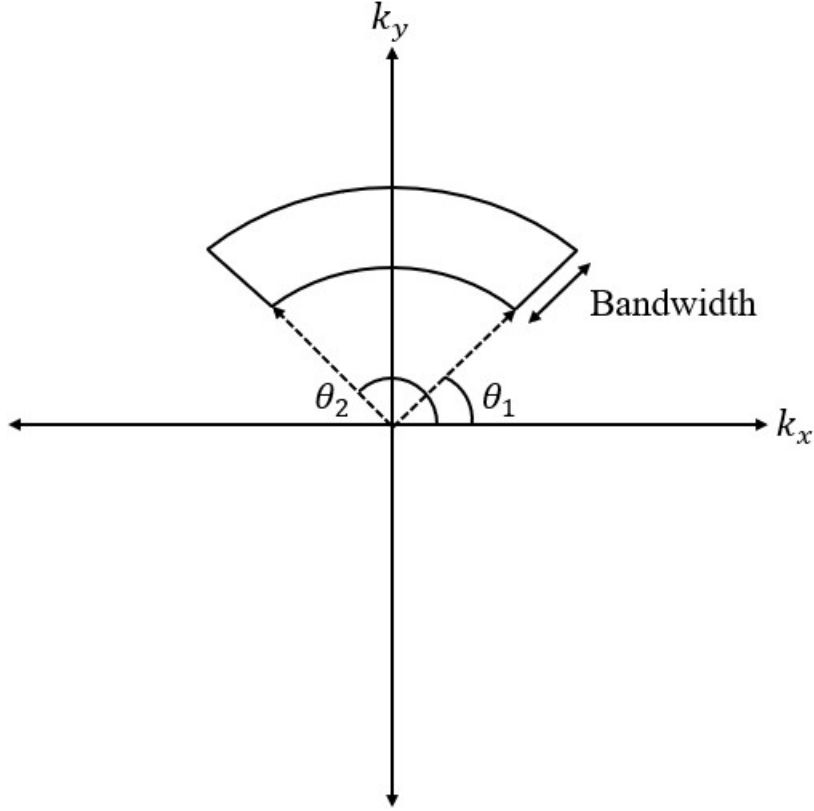


Figure 4.2: Diagram depicting the k -space support of monostatic SAR data collected for angles from between θ_1 and θ_2 . Note that the labeled bandwidth is equal to $\frac{4\pi}{c}$ times the bandwidth in Hz.

In order to image Ω , a spatial-matched filtering approach is often used. Using the far field approximation, this operation can be expressed mathematically via

$$I(\mathbf{x}) = \int \int E(\theta, k) e^{-2jk\mathbf{x}^T \hat{\mathbf{r}}(\theta)} dk d\theta, \quad (4.7)$$

where the integral is evaluated over the available angles and frequencies measured in the data. Backprojection is another spatially-matched filtered technique which does not use the far field approximation [39]. This operation can be expressed as

$$I(\mathbf{x}) = \int \int E(\theta, k) e^{2jkR(\mathbf{x}, \theta)} dk d\theta, \quad (4.8)$$

where we again evaluate the integral over all angles and frequencies in the data.

The data model outlined in (4.3) will be used by our method to transform a complex SAR image of $\rho(x, y)$ into synthetic data for use in PE-LSM imaging. In the next section, we will provide details on how this procedure is carried out.

4.2.2 Transforming a SAR image into synthetic data

We consider a given complex-valued SAR image $I(x, y)$ of a scene Ω . We assume knowledge of the center frequency f_c of the data that the image is constructed from. We also assume knowledge of the orientation of the central angle of the synthetic aperture, which is analogous to knowing which direction is the "range" direction and which is the "cross-range" direction. However, we assume no other knowledge of the frequencies sampled or the sensor locations during data collection that are typically available when producing a SAR image. It may be possible to relax the need for knowledge of the orientation of the range direction. Taking the Fourier transform of the data to reveal the data annulus could provide enough information to automatically determine the range direction. However, we leave such a technique for future work.

If we take $I(x, y)$ to be an image of the complex reflectivity, $\rho(x, y)$, of the scene, then (4.3) implies that the inverse Fourier transform of $I(x, y)$ should be closely related to the SAR data needed to produce $I(x, y)$. However, since $I(x, y)$ is assumed to be formed on a rectangularly spaced pixel grid, the resulting inverse transform will also place samples on a rectangular grid. PE-LSM can be posed operated on data in the frequency-angle domain, so in order to use $I(x, y)$ to create data in the expected "SAR-like" form, we must resample this data onto a polar grid. In the remainder of this section, we will detail the steps needed to transform $I(x, y)$ into useful data for other imaging algorithms.

The first step in processing the data is to perform a baseband conversion of the image $I(x, y)$. This is accomplished via a simple complex multiplication as

$$I_{bb} = I(x, y)e^{j2k_c y}, \quad (4.9)$$

where the subscript bb refers to the baseband image and $k_c = 2\pi f_c/c$. This baseband conversion ensures that we avoid aliasing when the Fourier transform of the baseband image $I_{bb}(x, y)$ is computed. Note that the operation in (4.9) assumes that the "y"-direction corresponds to the range direction.

After baseband conversion, the next step is to take the inverse Fourier transform of the baseband converted image. This is achieved by

$$\hat{I}(k_x, k_y) = \int I_{bb}(x, y) e^{-jk_x x - jk_y y} dx dy. \quad (4.10)$$

The result should feature an annulus, and the shape of said annulus reflects the polar sampling of the original SAR defined by (4.4) and (4.5). An example is shown in Figure 4.3.

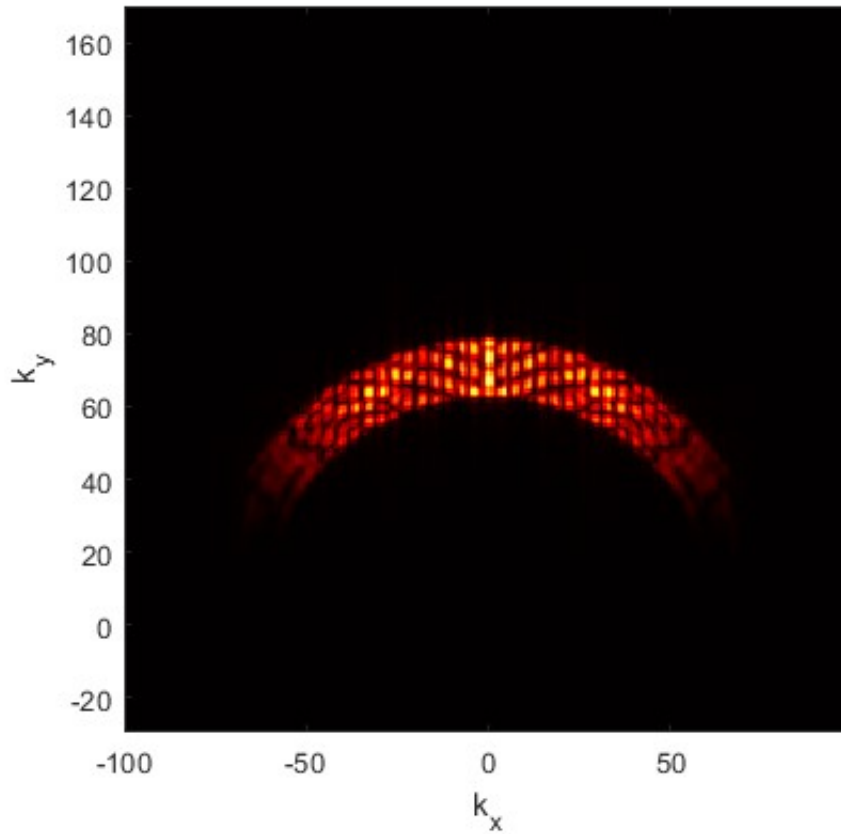


Figure 4.3: An example of the inverse Fourier transform of a SAR image showing the data annulus.

At this stage, we have $\hat{I}(x, y)$ sampled on a Cartesian grid, the next step is to define an appropriate polar grid and then re-sample the data onto this new grid. The first step in defining the new polar grid is to establish its boundary. We use a thresholding operation on $|\hat{I}(k_x, k_y)|$ to capture the boundary of the annulus. Let T denote the threshold, and consider all points (k'_x, k'_y) which satisfy

$$|\hat{I}(k'_x, k'_y)| \geq T. \quad (4.11)$$

We convert these (k'_x, k'_y) points into (θ', k') coordinates using (4.4) and (4.5). To construct a grid of equally spaced (θ, k) coordinates in k -space, we find the extreme values of k and θ from the points (θ', k') . Let these values be denoted by θ_{min} , θ_{max} , k_{min} , and k_{max} . For a desired number of samples N_θ and N_k we then define the set of coordinates $\theta_{n_\theta} = \theta_{min} + n_\theta \Delta\theta$ for $n_\theta = 0, \dots, N_\theta - 1$ and $k_{n_k} = k_{min} + n_k \Delta k$ for $n_k = 0, \dots, N_k - 1$. We have defined $\Delta\theta = \frac{\theta_{max} - \theta_{min}}{N_\theta}$ and $\Delta k = \frac{k_{max} - k_{min}}{N_k}$.

In order to create data on this new set of grid points, we define the 2-D interpolant $L(k_x, k_y)$ over the inverse transformed image $\hat{I}(k_x, k_y)$. We then compute the synthetic SAR data as

$$E_s(\theta_{n_\theta}, k_{n_k}) = L(2k_{n_k} \cos \theta_{n_\theta}, 2k_{n_k} \sin \theta_{n_\theta}). \quad (4.12)$$

This interpolated data follows the model

$$E_s(\theta_{n_\theta}, k_{n_k}) = \int_{\Omega} I(x, y) e^{2jk_{n_k} x \cos \theta_{n_\theta} + 2jk_{n_k} y \sin \theta_{n_\theta}} dx dy \quad (4.13)$$

and it is this data that we will use to image Ω with the PE-LSM.

4.2.3 PE-LSM imaging

In order to discuss the PE-LSM, it is useful to have a brief explanation of the basic mechanics of the standard LSM. Consider scattered data collected from a multistatic sensing geometry. For multistatic data sampled at N_r receive angles and N_t transmit angles, let $\mathbf{E}(k)$ denote the $N_r \times N_t$ matrix of scattered data. For a point $\mathbf{x}$ within Ω , LSM imaging is accomplished by solving

the linear system

$$\mathbf{E}(k)\mathbf{g}(\mathbf{x}, k) = \phi(\mathbf{x}, k) \quad (4.14)$$

for the LSM solution $\mathbf{g}(\mathbf{x}, k)$. The term $\phi(\mathbf{x})$ is a vector that takes the form

$$\phi(\mathbf{x}, k) = \begin{bmatrix} G(\theta_1, \mathbf{x}, k) \\ \vdots \\ G(\theta_{N_r}, \mathbf{x}, k) \end{bmatrix}, \quad (4.15)$$

Where $G(\theta_r, \mathbf{x}, k)$ is the Green's function for a wave originating at point $\mathbf{x}$ propagating in the receive direction θ_r at wavenumber k . In [48], [52], and [51], it is shown for a varied set of cases that $\|\mathbf{g}(\mathbf{x}, k)\|$ is bounded when $\mathbf{x}$ is inside the scattering target and becomes unbounded as $\mathbf{x}$ approaches the target boundary. This behavior can be used to construct an image of the target shape via an indicator function. A typical indicator function is given by

$$I_{LSM}(\mathbf{x}) = \sum_{i=1}^{N_f} \frac{1}{\|\mathbf{g}(\mathbf{x}, k_i)\|^2} \quad (4.16)$$

where the sum is over N_f total sampled frequencies. Since $\|\mathbf{g}(\mathbf{x}, k)\|$ remains bounded inside the target, an indicator function of the form (4.16) will result in an image where the target appears bright against a dark background.

As a final technical note, the equation (4.14) is ill-conditioned due to the ill-posed nature of the inverse scattering problem that the LSM solves. Thus, to find $\mathbf{g}(\mathbf{x}, k)$ in a stable manner, a regularization method is used. Tikhonov regularization is a common choice, and it is accomplished by introducing a parameter α and then solving

$$\min_{\mathbf{g}(\mathbf{x}, k)} \left(\|\mathbf{E}(k)\mathbf{g}(\mathbf{x}, k) - \phi(\mathbf{x}, k)\|^2 + \alpha \|\mathbf{g}(\mathbf{x}, k)\|^2 \right). \quad (4.17)$$

For a well-chosen value of α , this method can stabilize the ill-conditioned linear system (4.14).

We will comment on the selection of α after our discussion of the PE-LSM.

When the span of receive angles θ_r and transmit angles θ_t fully surround Ω , the LSM can produce high quality images of the targets within the scene. However, in the case where Ω is viewed from a more limited aperture, the LSM image quality significantly degrades. In real SAR-type applications, one almost never has access to multistatic data from a full circular aperture around a scene. The PE-LSM modifies the standard LSM in order to improve performance in limited aperture settings. Two changes are made to the standard LSM in the PE-LSM. The first is a receive beamforming step which helps to focus the data around the point $\mathbf{x}$. The second is a phase-delay frequency constraint which allows the PE-LSM to leverage coherent processing between sampled frequencies in order to achieve better limited aperture range resolution.

The receive beamforming operation of the PE-LSM introduces the weight vector

$$\mathbf{w}(\mathbf{x}, k) = \begin{bmatrix} e^{-jk\mathbf{x}^T \hat{\mathbf{r}}_1} \\ \vdots \\ e^{-jk\mathbf{x}^T \hat{\mathbf{r}}_{N_r}} \end{bmatrix}, \quad (4.18)$$

where $\hat{\mathbf{r}}_r = (\cos \theta_r, \sin \theta_r)$. We apply this to (4.14) as

$$\mathbf{w}(\mathbf{x}, k)^H \mathbf{E}(k) \mathbf{g}(\mathbf{x}, k) = \mathbf{w}(\mathbf{x}, k)^H \phi(\mathbf{x}, k). \quad (4.19)$$

The receive beamforming operation focuses the data around $\mathbf{x}$ which serves as a form of preconditioning that mitigates influence of data contributions away from $\mathbf{x}$ when computing the PE-LSM solution.

The phase-delay frequency constraint can be expressed as

$$\mathbf{g}(\mathbf{x}, k_i) = \mathbf{g}(\mathbf{x}, k_1) \odot e^{-j(i-1)\Delta k \mathbf{d}_t(\mathbf{x})}, \quad (4.20)$$

with

$$\mathbf{d}_t(\mathbf{x}) = \begin{bmatrix} \mathbf{x}^T \hat{\mathbf{r}}_1 \\ \vdots \\ \mathbf{x}^T \hat{\mathbf{r}}_{N_t} \end{bmatrix} \quad (4.21)$$

and $\hat{\mathbf{r}}_t = (\cos \theta_t, \sin \theta_t)$. For data collected at multiple frequencies, we expect the phase of the solution at adjacent frequencies to vary in proportion to the spacing between frequency samples. The phase-delay frequency constraint forces the solution to match this expected behavior. The result is an injection of coherent processing across frequencies, which the standard LSM has no mechanism to accomplish. To incorporate this constraint into the minimization, we introduce the matrices

$$\mathbf{C}_i = \begin{bmatrix} e^{-j(i-1)\Delta k \mathbf{x}^T \hat{\mathbf{r}}_1} & 0 & \dots & 0 \\ 0 & e^{-j(i-1)\Delta k \mathbf{x}^T \hat{\mathbf{r}}_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & e^{-j(i-1)\Delta k \mathbf{x}^T \hat{\mathbf{r}}_{N_t}} \end{bmatrix}. \quad (4.22)$$

These matrices are inserted into a modified multi-frequency version of (4.19), resulting in the minimization problem

$$\min_{\mathbf{g}(\mathbf{x}, k_1)} \sum_{i=1}^{N_f} |\mathbf{w}(\mathbf{x}, k_i)^H \mathbf{E}(k_i) \mathbf{C}_i \mathbf{g}(\mathbf{x}, k_1) - \mathbf{w}(\mathbf{x}, k_i)^H \phi(\mathbf{x}, k_i)|^2 + \alpha \|\mathbf{g}(\mathbf{x}, k_1)\|^2. \quad (4.23)$$

The PE-LSM solution is then used again to produce an image via indicator function. We use

$$I_{PE}(\mathbf{x}) = \frac{1}{\|\mathbf{g}(\mathbf{x}, k_1)\|^2}. \quad (4.24)$$

Note that we only need the solution at the first frequency k_1 as the phase delay frequency constraint controls the subsequent solutions for all other frequencies.

Although the standard LSM does not rely on the Born approximation, both the receive beam-forming and the phase delay frequency constraint of the PE-LSM are based on a Born-approximated scattering model. In this way, the PE-LSM can be viewed as a compromise where we place

Born-approximated preconditioners and constraints on (4.14), which is not Born-approximated and leverages scattering relationships in a more nuanced way compared to backprojection.

One aspect of LSM and PE-LSM imaging that we have not yet addressed is the need for multistatic diversity in the data. The synthetic data resulting from our transformation of the SAR backprojection image and modeled by (4.13) is purely monostatic. In order to use PE-LSM on this synthetic data, we must transform it into synthetic multistatic data. We do this using the technique of [64]. This technique involves interpolating the monostatic data to synthesize multistatic responses by leveraging angular redundancy in the monostatic data. For more detail regarding this monostatic to multistatic operation, we refer the reader to [64].

The final piece of LSM and PE-LSM imaging is the process of selecting a regularization parameter α . The selection of the regularization parameter can have a significant impact on the quality of the final LSM/PE-LSM image. The problem of selecting a suitable value for α can be a challenge in its own right, with techniques such as the Morozov discrepancy principle [52], the L-curve method [81], and heuristic methods [61] being used in the literature. Our results in the following section are presented with regularization parameters tuned manually by selecting the value that maximizes the subjective contrast between the targets and the background of the image. A parameter selection method based on maximizing the contrast, or on one of the established techniques such as the L-curve method or Morozov's discrepancy principle would be an interesting avenue of future exploration but is left outside the scope of this work.

We offer here a final step-by-step summary of our imaging method:

1. Begin with complex-valued backprojection image $I(x, y)$.
2. Take the Fourier transform of the baseband converted image as given by (4.10).
3. Establish a polar grid using the thresholded image Fourier transform.
4. Define an interpolant over the transformed image and interpolate data onto the polar grid via (4.12).
5. Transform this data into multi-static data using the technique of [62].

6. Image the data using PE-LSM by solving (4.23).

4.3 Results

In this section we will present our preliminary results using this technique. To produce these results, we first use an FDTD simulation to generate data based on the sensor geometry shown in Figure 4.4. The simulated data is collected for 90 sensor positions spanning a 90° aperture. One hundred frequency samples in a band from 1.5 GHz to 2 GHz are collected. Targets are modeled as conductors with conductivity of $\sigma = 5.7 \times 10^7$ S/m and relative permittivity $\epsilon_r = 1$. We add noise to the data giving a Signal-to-Noise Ratio (SNR) of 20 dB.

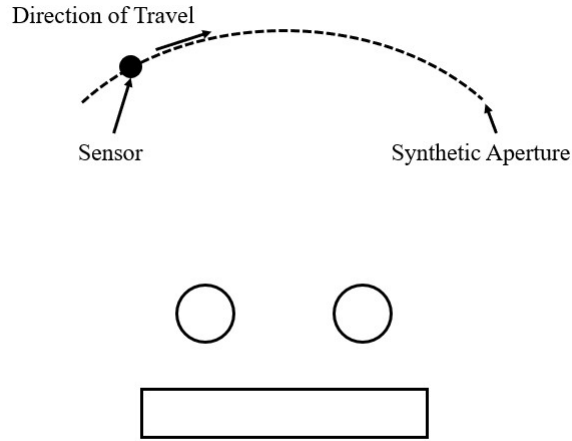


Figure 4.4: Diagram showing FDTD simulation setup.

The FDTD simulation data is then used to produce SAR backprojection images of the scene. These images are then transformed into synthetic SAR data using the method outlined in Section 4.2. This synthetic data is then imaged using PE-LSM.

Our results for three different targets are displayed in Figure 4.5. Each row corresponds to a different target, where the true target outline is illustrated with a dashed line. Each column corresponds to a different imaging method. The left-most column contains back projection images, the

central column contains PE-LSM images created from the simulation data, and the right-most column contains PE-LSM images created using beamspace data synthesized from the back projection images of the left-most column.

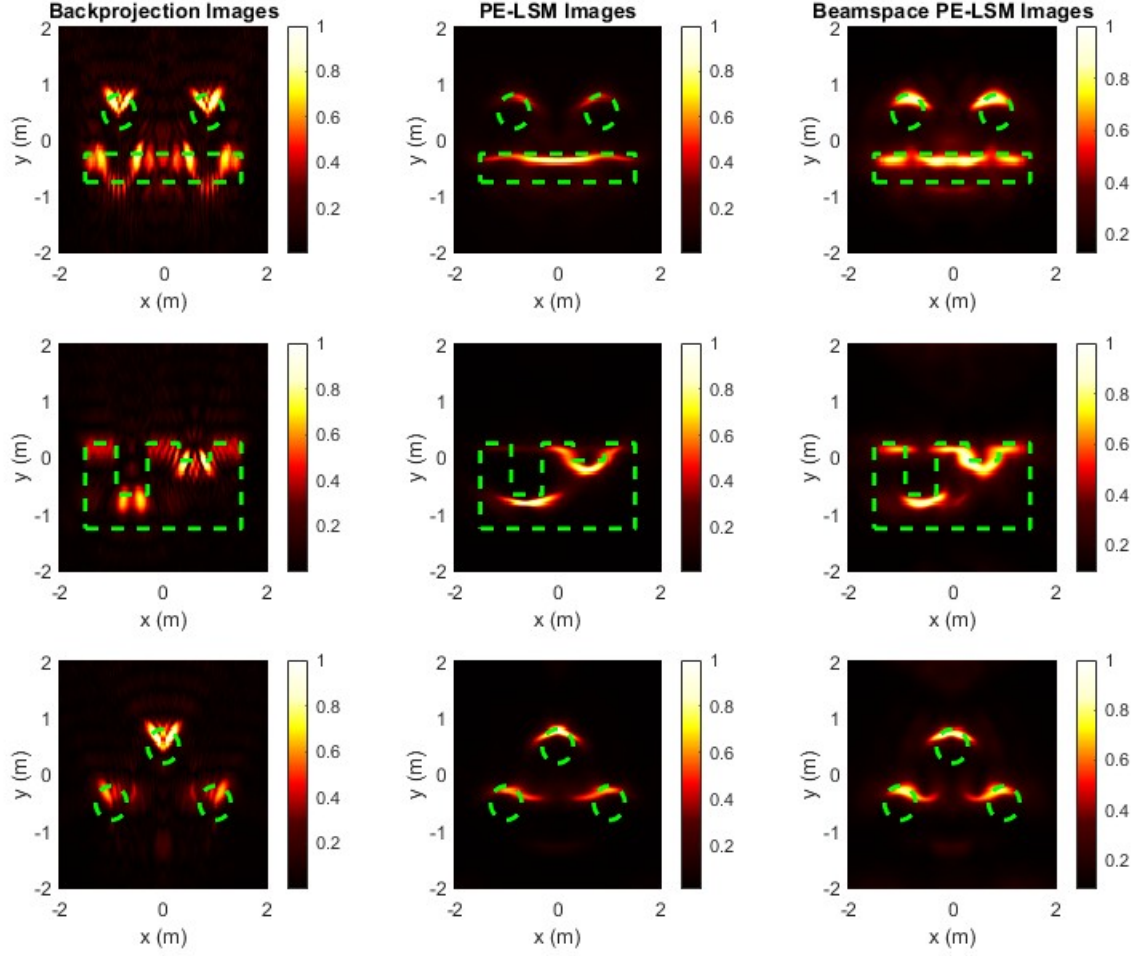


Figure 4.5: Imaging Results for three different targets. Imaging techniques are arranged from left to right, first backprojection, then PE-LSM on the original data, and finally PE-LSM on synthesized beamspace data.

We can see that the PE-LSM images of the center column more tightly capture the true shape of the illuminated upper surfaces of the targets than the backprojection images of the left-most column. The observed range resolution of the PE-LSM images is also finer than that of the back projection images (recall that the range direction corresponds to the "down" direction). The beamspace

PE-LSM images of the right-most column, when compared to the backprojection images of the left-most, also show improvement in range resolution and ability to capture the true shape of the illuminated surface. However, the beamspace PE-LSM images of the rightmost column exhibit some artifacts that are not present in the PE-LSM images made using the true simulated data.

Overall, our results indicate that the PE-LSM images based on our data synthesis technique do provide benefit over the original backprojection images. However, they also suffer from more significant artifacts than PE-LSM images based on the raw SAR data. These artifacts may be due to the ill-posed PE-LSM problem amplifying perturbations introduced to the data during the synthesis process.

4.4 Conclusions

The Phase-Encoded Linear Sampling Method is a promising technique in microwave imaging. It has been demonstrated to more effectively capture a target's true shape than traditional backprojection imagery. However due to the PE-LSM's increased complexity over backprojection and its ill-posed nature, applying the method to synthetic aperture radar imaging has been difficult. In order to expand the applicability of this technique for SAR applications, we have developed a method which synthesizes PE-LSM appropriate data using a pre-existing SAR backprojection image. We have demonstrated early results which suggest that despite being ill-posed, PE-LSM can be used to form images using data synthesized from this technique, and these images provide enhancements in capturing the true shapes of targets over the original backprojection image.

The results motivate further investigation with experimental datasets. Relevant points of investigation include the effect of clutter and other forms of experimental imperfections on the method. The effects of finer PE-LSM range resolution on the robustness of automatic target recognition are also of interest.

In the future, we plan to apply this technique to a two-step mode of operation in SAR imaging where a large scene is first imaged with backprojection or another traditional SAR method and targets of interest are subsequently identified. Image chips of these targets can then be extracted

and processed using our method to obtain enhanced PE-LSM images. Such a two-stage process is necessary due to the higher computational cost of the PE-LSM [65].

Chapter 5

Conclusions

I will conclude this dissertation with a very brief overview of each of the chapters and provide a description of the contributions made within each to the challenging field of wave-based sensing. I will also include comments on interesting avenues for future work.

In Chapter 2, I detailed the derivation of nine different generalized likelihood ratio detector statistics. These nine detectors cover a variety of cases depending on knowledge of the signal channel and signal variance. In order to quantitatively evaluate the performance of these detectors, a series of numerical tests are then performed. The work of this chapter has been published in [25].

I contributed to the passive detection effort by writing and performing all numerical simulations. These simulations cover the eighteen different detector cases outlined by Tables 2.1 and 2.2. ROC and PD vs SNR curves are produced, with results shown in Figures 2.1 - 2.9. I then analyzed the results of these tests. The results of my tests provided many important and interesting quantitative and qualitative results about the detectors of Tables 2.1 and 2.2. Significantly, my results show in which cases it is advantageous to share coherent information between sensors and in which cases not enough is known to effectively use this information to increase detector performance.

In addition to the work producing and analyzing simulation results, I participated in an effort to rework each detector from the beginning to ensure correctness. During this work I discovered and fixed errors in some of the detector statistics.

These detectors offer a framework to address the "detection" piece of the passive source localization problem. In [77], the "localization" piece is treated using one of the nine detector cases given here. This is done via a Synthetic Aperture Source Localization (SASL) approach. Further developments of this SASL approach, such as extending to the detector cases where less is known about the source signal or testing the technique on experimental data are being pursued by others.

Chapter 3 covers my development of multi-sensor multi-frequency digital spotlighting for LSM-based imaging. A series of simulated and experimental results are then shown to illustrate differing aspects of the digital spotlight technique and its interaction with LSM-based imaging.

One of my primary contributions is the mathematical development of a multistatic and multi-frequency digital spotlight technique. This technique is based on a k -space representation for bistatic SAR data illustrated by equations (3.45), (3.48), and (3.49). The k -space representation of the data also leads to equation (3.60), which implies that the sampling requirements of the multistatic data are lower than those for more common monostatic datasets. I use equations (3.96) and (3.116) to argue that this method should help filter out signal contributions from scatterers located outside the spotlight region of interest during the digital spotlight processing.

In order to thoroughly test the digital spotlight method and its utility for LSM-based imaging, my contributions also include a series of simulated and experimental results. First, I use simulated multi-sensor and multi-frequency data to produce spotlight PE-LSM images, as illustrated by Figure 3.10. Published in [72], these results demonstrate that despite the ill-posed nature of PE-LSM processing, the digital spotlight pre-processing is compatible with this imaging modality. Next, I show that the method can help improve performance in cases where a strong scatterer lying outside the region of interest causes interference through a series of simulated results. These results are seen in Figures 3.24, 3.21, 3.18, and 3.15. Finally, my spotlight code was used to help produce the first successful LSM-type imaging of fielded SAR data, and these images are included in Figures 3.30, and 3.27. These images represent a significant breakthrough for LSM-based imaging nearly 30 years after the initial development of the LSM. A detailed study of PE-LSM imaging of SAR data has been submitted for publication in [71].

The images shown in Figures 3.30, and 3.27 represent a significant step forward in using LSM-type imaging for field-collected monostatic SAR data. All other results presented here use simulated data. In the future, it would be of interest to conduct a follow up study which applies this technique to a wider array of experimental datasets. The uncertainties associated with experimental

data make this type of study an important test of the robustness and feasibility of our technique, and positive results here would be significant in moving the technique closer to real-world applicability.

Chapter 4 contains my development of an LSM-based enhancement technique for SAR imagery. A mathematical model for SAR data and SAR imagery is given, and an algorithm for enhancing SAR imagery via transformation into data usable by LSM-based techniques and subsequent PE-LSM imaging is then laid out. This algorithm development is then followed up with a simulation-based study evaluating the performance of the enhancement technique. A publication detailing my contributions in this chapter has been accepted in [82].

My contributions in this chapter include the development of the technique for transforming SAR imagery into data appropriate for LSM-type imaging use. This development is based on the k -space representation for SAR data. Equations (4.4) - (4.6) and (4.13) highlight this Fourier relationship between SAR data and the complex reflectivity of the imaging scene. I treat a SAR image as a representation of this complex reflectivity, and then I use the k -space mapping from these equations to transform the image back to the data domain. Subsequently, I apply PE-LSM to the data, generating an enhanced image. To my knowledge, no other techniques leveraging LSM imaging methods have been used to enhance existing imagery in this way.

In addition to the theoretical development of the SAR image enhancement technique, I also contribute a study of the technique's performance using simulated data. My study takes data scattered from simulated targets and uses it to form SAR backprojection images. These images are then processed with the PE-LSM-based enhancement technique, and results are given in Figure 4.5. My results demonstrate the feasibility of my technique despite the ill-posed nature of LSM-type imaging. Additionally, my results show clear visual improvement in our method's ability to capture the true target shape compared to the initial backprojection image.

While the results shown in Figure 4.5 are promising, my method requires knowledge of the center frequency and central angle of the synthetic aperture of the original data. In the future, developing techniques to automatically estimate these parameters directly from the SAR imagery would expand the range of use cases of my method. Additionally, a future study focusing on

the application of this technique to real experimental data would also represent an important step towards bringing this method closer to potential real-world applications.

Finally, for the ease of the reader, I provide here a summarized list of the chapter-by-chapter contributions made in this dissertation:

- Chapter 2: Passive Multi-Channel Detection
 - Developed simulation code for quantitative evaluation of detector statistics
 - * Treated 18 different cases, 9 features coherent processing between sensors (see Table 2.1), and 9 share only incoherent data between sensors (see Table 2.2)
 - * Produced series of ROC curves which quantify detector performance across range of threshold values at a fixed SNR (see Figures 2.1, 2.3, and 2.5)
 - * Produced series of PD vs SNR curves quantifying detector performance across range of SNR values at fixed PFA (see Figures 2.2, 2.4, and 2.6 - 2.9)
 - * Analyzed simulation results and evaluated quantitative differences in detector performance
 - Participated in verification and finalization of detector statistics
 - * Re-derived each detector and corresponding non-coherent counterpart, leading to corrections in mathematical form of some detectors
- Chapter 3: Synthetic Aperture Radar Digital Spotlighting for Linear Sampling Method Imaging
 - Developed multistatic and multi-frequency digital spotlight technique
 - * Used k -space representation of bistatic SAR data to develop theoretical foundation for digital spotlight technique (see Equations (3.45), (3.48), and (3.49))
 - * Conducted analysis of sampling requirements show that multistatic data accommodates coarser spatial sampling, allowing greater data reduction than monostatic spotlighting (see Equation (3.60))

- * Used model to argue for expected aid in filtering interference from strong outside scatterers (see Equations (3.96) and (3.116))
- Demonstrated effectiveness on multistatic multi-frequency data using PE-LSM
 - * Showed significant reduction in data and corresponding speedup
 - * Showed image quality not significantly impacted by spotlight pre-processing - ill-posed method not seen to be an issue (see Figure 3.10)
- Investigated filtering effects for LSM imaging near large scatterer
 - * Conducted numerical simulations with several small targets imaged with single and multi-frequency LSM in close proximity to large target outside region of interest
 - * Showed that across all cases, digital spotlight processing reduces the severity of contaminating artifacts (see Figures 3.24, 3.21, 3.18, and 3.15)
- Provided vital component in the first successful LSM-type imaging of real fielded SAR data
 - * Used monostatic digital spotlight code for PE-LSM imaging of GOTCHA dataset (see Figures 3.30, and 3.27)
 - * My digital spotlight code was a key ingredient to make technique computationally practical
- Chapter 4: Linear Sampling Method enhancement for Synthetic Aperture Radar Images
 - Developed technique for synthesizing SAR-like data from complex-valued SAR imagery
 - * Designed k -space remapping method using Fourier domain representation of SAR data (see Equations (4.4) - (4.6) and (4.13))
 - Demonstrated effective image enhancement using PE-LSM processing on synthesized data

- * Tested technique using backprojection images of simulated targets
- * Showed that PELSM provides clear visual improvement in output image's ability to capture target shape over initial backprojection image (see Figure 4.5)

Bibliography

- [1] Z.-C. Li and Y.-M. Qu, “Research progress on seismic imaging technology,” *Petroleum Science*, vol. 19, 2022.
- [2] L. B. Conyers, *Ground-penetrating radar for archaeology*. Bloomsbury Publishing PLC, 2023.
- [3] M. Gupta, M. A. Khan, R. Butola, and R. M. Singari, “Advances in applications of Non-Destructive Testing (NDT): A review,” *Advances in Materials and Processing Technologies*, vol. 8, 2022.
- [4] R. Bindshadler, “Monitoring ice sheet behavior from space,” *Reviews of Geophysics*, vol. 36, 1998.
- [5] S. Clark and H. Durrant-Whyte, “Autonomous land vehicle navigation using millimeter wave radar,” *Proceedings. 1998 IEEE International Conference on Robotics and Automation (Cat. No. 98CH36146)*, 1998.
- [6] M. A. Richards, J. Scheer, W. A. Holm, and W. L. Melvin, *Principles of modern radar*. Scitech publishing Raleigh, NC, USA, 2010.
- [7] N. Kolev, *Sonar systems*. BoD – Books on Demand, 2011.
- [8] T. G. Leighton, “What is ultrasound,” *Progress in Biophysics and Molecular Biology*, vol. 93, 2007.
- [9] B. F. Burke, F. Graham-Smith, and P. N. Wilkinson, *An introduction to radio astronomy*. Cambridge University Press, 2019.
- [10] L. Eisner, P. M. Duncan, W. M. Heigl, and W. R. Keller, “Uncertainties in passive seismic monitoring,” *The Leading Edge*, vol. 28, 2009.
- [11] D. Torrieri, “The radiometer and its practical implementation,” *2010 - MILCOM 2010 MILITARY COMMUNICATIONS CONFERENCE*, 2010.

- [12] J. Cheng, G. Song, X. Sun, L. Wen, and F. Li, "Research developments and prospects on microseismic source location in mines," *Engineering*, vol. 4, 2018.
- [13] H. Wang and P. Chu, "Voice source localization for automatic camera pointing system in videoconferencing," *1997 IEEE International Conference on Acoustics, Speech, and Signal Processing*, 1997.
- [14] D. E. Hack, L. K. Patton, and B. Himed, "Multichannel detection of an unknown rank-one signal with uncalibrated receivers," *2014 IEEE International Conference on Acoustics, Speech and Signal Processing*, 2014.
- [15] A. J. Weiss and A. Amar, "Direct geolocation of stationary wideband radio signal based on time delays and doppler shifts," *Proceedings IEEE/SP 15th Workshop on Statistical Signal Processing*, 2009.
- [16] N. Vankayalapati and S. Kay, "Asymptotically optimal detection of low probability of intercept signals using distributed sensors," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 48, 2012.
- [17] L. L. Scharf and B. Friedlander, "Matched subspace detectors," *IEEE Transactions on Signal Processing*, vol. 42, 1991.
- [18] K. S. Bialkowski, I. V. L. Clarkson, and S. D. Howard, "Generalized canonical correlation for passive multistatic radar detection," *2011 IEEE Statistical Signal Processing Workshop (SSP)*, 2011.
- [19] J. Liu, H. Li, and B. Himed, "Two target detection algorithms for passive multistatic radar," *IEEE Transactions on Signal Processing*, vol. 62, 2014.
- [20] S. Gogineni, P. Setlur, M. Rangaswamy, and R. R. Nadakuditi, "Passive radar detection with noisy reference channel using principal subspace similarity," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 54, 2018.
- [21] X. Zhang, H. Li, and B. Himed, "A direct-path interference resistant passive detector," *IEEE Signal Processing Letters*, vol. 24, 2017.

- [22] D. E. Hack, L. K. Patton, B. Himed, and M. A. Saville, "On the applicability of source localization techniques to passive multistatic radar," *2012 Conference Record of the Forty Sixth Asilomar Conference on Signals, Systems, and Computers (ASILOMAR)*, 2012.
- [23] D. E. Hack, L. K. Patton, and B. Himed, "A unified detection framework for distributed active and passive rf sensing," *2013 Asilomar Conference on Signals, Systems, and Computers*, 2013.
- [24] D. E. Hack, L. K. Patton, B. Himed, and M. Saville, "Centralized passive MIMO radar detection without direct-path reference signals," *IEEE Transactions on Signal Processing*, vol. 62, 2014.
- [25] T. McWhorter, L. Scharf, C. Moore, and M. Cheney, "Passive multi-channel detection: A general first-order statistical theory," *IEEE Open Journal of Signal Processing*, vol. 4, 2023.
- [26] E. A. Marengo, F. K. Gruber, and F. Simonetti, "Time-reversal MUSIC imaging of extended targets," *IEEE Transactions on Image Processing*, vol. 16, 2007.
- [27] R. Marklein, J. Miao, M.-u. Rahman, and K. J. Langenberg, "Inverse scattering and imaging in NDT: Recent applications and advances," *9th European Conference on NDT: ECNDT Berlin 2006*, 2006.
- [28] J. M. Blackledge and L. Zapalowski, "Quantitative solutions to the inverse scattering problem with applications to medical imaging," *Inverse Problems*, 1985.
- [29] D. Colton, P. Monk, and R. Kress, "A new algorithm in electromagnetic inverse scattering theory with an application to medical imaging," *Mathematical methods in the Applied Sciences*, 1998.
- [30] F. Pourahmadian and H. Yue, "Laboratory application of sampling approaches to inverse scattering," *Inverse Problems*, 2021.
- [31] B. Borden, "Mathematical problems in radar inverse scattering," *Inverse Problems*, 2001.

- [32] I. Catapano, A. Affinito, A. D. Moro, G. Alli, and F. Soldovieri, "Forward-looking ground-penetrating radar via a linear inverse scattering approach," *IEEE Transactions on Geoscience and Remote Sensing*, 2015.
- [33] J. Shen, Y. Zhong, X. Chen, and L. Ran, "Inverse scattering problems of reconstructing perfectly electric conductors with TE illumination," *IEEE Transactions on Antennas and Propagation*, vol. 61, 2013.
- [34] S. C. Pavone, G. Sorbello, and L. D. Donato, "Forward and inverse scattering of metallic objects through focused Bessel-shaped fields," *IEEE Open Journal of Antennas and Propagation*, vol. 3, 2022.
- [35] F. Soldovieri, R. Solimene, and F. Ahmad, "Sparse tomographic inverse scattering approach for through-the-wall radar imaging," *IEEE Transactions on Instrumentation and Measurement*, 2012.
- [36] M. T. Bevacqua, S. D. Meo, L. Crocco, T. Isernia, and M. Pasian, "Millimeter-waves breast cancer imaging via inverse scattering techniques," *IEEE Journal of Electromagnetics, RF and Microwaves in Medicine and Biology*, 2021.
- [37] A. B. Weglein et al., "Inverse scattering series and seismic exploration," *Inverse Problems*, vol. 19, 2003.
- [38] A. W. Doerry, "Basics of polar-format algorithm for processing synthetic aperture radar images," Sandia National Laboratories, Tech. Rep. SAND2012-3369, 2012.
- [39] L. A. Gorham and L. J. Moore, "SAR image formation toolbox for MATLAB," *SPIE 7699, Algorithms for Synthetic Aperture Radar Imagery XVII*, vol. 769906, 2010.
- [40] M. Soumekh, *Synthetic Aperture Radar Signal Processing with MATLAB Algorithms*. Wiley-Interscience, 1999.
- [41] E. Rignot, C. Williams, J. Way, and L. Viereck, "Mapping of forest types in Alaskan boreal forests using SAR imagery," *IEEE Transactions on Geoscience and Remote Sensing*, vol. 32, 1994.

- [42] M. Tanase, J. de la Riva, M. Santoro, F. Pérez-Cabello, and E. Kasischke, "Sensitivity of SAR data to post-fire forest regrowth in Mediterranean and boreal forests," *Remote Sensing of Environment*, vol. 115, 2011.
- [43] G. Lemoine, J. Chesworth, G. Schwartz-Juste, N. Kourti, and I. Shepherd, "Near real time vessel detection using spaceborne SAR imagery in support of fisheries monitoring and control operations," *IGARSS 2004. 2004 IEEE International Geoscience and Remote Sensing Symposium*, 2004.
- [44] G. Wiseman, H. McNairn, S. Homayouni, and J. Shang, "RADARSAT-2 polarimetric SAR response to crop biomass for agricultural production monitoring," *IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing*, vol. 7, 2004.
- [45] R. Schumacher and J. Schiller, "Non-cooperative target identification of battlefield targets - classification results based on SAR images," *IEEE International Radar Conference*, 2005.
- [46] S. Wong, "High range resolution profiles as motion-invariant features for moving ground targets identification in SAR-based automatic target recognition," *IEEE Transaction on Aerospace and Electronic Systems*, vol. 45, 2009.
- [47] S. W. Lasswell, "History of SAR at Lockheed Martin (previously Goodyear Aerospace)," *Proc. SPIE 5788, Radar Sensor Technology IX*, 2005.
- [48] D. Colton and A. Kirsch, "A simple method for solving inverse scattering problems in the resonance region," *Inverse Problems*, vol. 12, 1996.
- [49] M. Cheney, "The linear sampling method and the MUSIC algorithm," *Inverse Problems*, vol. 17, 2001.
- [50] I. Catapano, L. Crocco, and T. Isernia, "Improved sampling methods for shape reconstruction of 3-D buried targets," *IEEE Transaction on Geoscience and Remote Sensing*, vol. 46, 2008.
- [51] F. Cakoni, D. Colton, and P. Monk, *The Linear Sampling method in Inverse Electromagnetic Scattering*. SIAM, 2011.

- [52] D. Colton, M. Piana, and R. Potthast, “A simple method using Morozov’s discrepancy principle for solving inverse scattering problems,” *Inverse Problems*, vol. 13, 1997.
- [53] M. Brignone, G. Bozza, R. Aramini, M. Pastorino, and M. Piana, “A fully no-sampling formulation of the linear sampling method for three-dimensional inverse electromagnetic scattering problems,” *Inverse Problems*, vol. 25, 2008.
- [54] M. Piana, R. Aramini, M. Brignone, and J. Coyle, “A new formulation of the linear sampling method: Spatial resolution and post processing,” *Journal of Physics: Conference Series*, vol. 124, 2008.
- [55] I. Catapano, L. Crocco, and T. Isernia, “On *Simple Methods* for shape reconstruction of unknown scatterers,” *IEEE Transactions on antennas and propagation*, vol. 55, 2007.
- [56] P. Monk, M. Pena, and V. Selgas, “Multifrequency Linear Sampling Method on experimental datasets,” *IEEE Transactions on Antennas and Propagation*, vol. 71, 2023.
- [57] S. Doğu, M. N. Akıncı, and E. Gose, “Experimental moving target imaging in a nonanechoic environment with Linear Sampling Method,” *IEEE Geoscience and Remote Sensing Letters*, vol. 18, 2020.
- [58] M. Ambrosanio, M. T. Bevacqua, T. Isernia, and V. Pascazio, “Experimental multistatic imaging VIA the Linear Sampling Method,” *IGARSS 2019 - 2019 IEEE International Geoscience and Remote Sensing Symposium*, 2019.
- [59] Y. Guo, P. Monk, and D. Colton, “The linear sampling method for sparse small aperture data,” *Applicable Analysis*, vol. 95, 2015.
- [60] B. B. Guzina, F. Cakoni, and C. Bellis, “On the multi-frequency obstacle reconstruction via the linear sampling method,” *Inverse Problems*, vol. 26, 2010.
- [61] M. J. Burfeindt and H. F. Alqadah, “Phase-encoded linear sampling method imaging of conducting surfaces from full and limited synthetic apertures,” *IEEE Open Journal of Antennas and Propagation*, vol. 3, 2022.

- [62] M. J. Burfeindt and H. F. Alqadah, "Phase-encoded linear sampling method imaging of 3D targets from circular synthetic aperture data," *2024 IEEE Radar Conference (RadarConf24)*, 2024.
- [63] M. J. Burfeindt, S. Ziegler, and H. F. Alqadah, "Fidelity evaluation of the phase-encoded linear sampling method and backprojection imaging," *2023 IEEE International Symposium on Antennas and Propagation and USNC-URSI Radio Science Meeting (USNC-URSI)*, 2023.
- [64] M. Burfeindt, H. F. Alqadah, and S. Ziegler, "Experimental phase-encoded linear sampling method imaging with a single transmitter and receiver," *IEEE Open Journal of Antennas and Propagation*, vol. 5, 2024.
- [65] M. Burfeindt, "Monostatic qualitative inverse scattering for conducting targets with improved efficiency," *IEEE Transactions on Antennas and Propagation*, vol. 74, 2026.
- [66] K. Dungan, L. Gorham, and L. Moore, "SAR digital spotlight implementation in MATLAB," *SPIE Defense, Security, and Sensing 2013*, 2013.
- [67] M. Soumekh, "Digital spotlighting and coherent subaperture image formation for stripmap synthetic aperture radar," *Proceedings of 1st International Conference on Image Processing*, 1994.
- [68] E. J. Balster, D. B. Mundy, A. M. Kordik, and K. L. Hill, "Digital spotlighting filtering optimization for SAR imaging," *EURASIP Journal on Image and Video Processing*, no. 84, 2019.
- [69] M. Soumekh, D. Nobles, M. Wicks, and G. Genello, "Signal processing of wide bandwidth and wide beamwidth P-3 SAR data," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 37, 2001.
- [70] L. Nguyen, M. Ressler, D. Wong, and M. Soumekh, "Enhancement of backprojection SAR imagery using digital spotlighting preprocessing," *Proceedings of 2004 IEEE Radar Conference (IEEE Cat. No. 04CH37509)*, 2004.

- [71] M. Burfeindt, H. Alqadah, C. Moore, and M. Cheney, “Qualitative inverse scattering for monostatic radar imaging of passenger vehicles,” *submitted to IEEE Transactions on Geoscience and Remote Sensing*, 2026.
- [72] C. Moore, M. Burfeindt, and M. Cheney, “Demonstration of synthetic aperture radar digital spotlight technique for qualitative inverse scattering,” *2025 IEEE Research and Applications of Photonics in Defense Conference (RAPID)*, 2025.
- [73] E. R. Keydel, S. W. Lee, and J. T. Moore, “MSTAR extended operating conditions: A tutorial,” *Proc. SPIE 2757, Algorithms for Synthetic Aperture Radar Imagery III*, vol. 2757, 1996.
- [74] Y.-S. Yoon and M. G. Amin, “High-resolution through-the-wall radar imaging using beamspace MUSIC,” *IEEE Transactions on Antennas and Propagation*, vol. 56, 2008.
- [75] S. Li, H. Wan, and Z. Wang, “Accurate measurement of radar carrier frequency,” *2007 International Conference on Microwave and Millimeter Wave Technology*, 2007.
- [76] A. Yildirim, “Method for estimating the central frequency of phase-coded radar signals,” *IET Signal Processing*, vol. 10, 2016.
- [77] M. Cheney, L. Scharf, M. Rhilinger, C. Moore, and A. Celestin, “Image analysis and resolution for detection-based synthetic-aperture passive source localization,” *Inverse Problems*, vol. 40, 2024.
- [78] I. Santamaria, L. Scharf, J. Via, H. Wang, and Y. Wang, “Passive detection of correlated subspace signals in two MIMO channels,” *IEEE Transactions on Signal Processing*, vol. 65, 2017.
- [79] C. V. Jakowatz, D. E. Wahl, P. H. Eichel, D. C. Ghiglia, and P. A. Thompson, *Spotlight-Mode Synthetic Aperture Radar: A Signal Processing Approach*. Springer, 1996.
- [80] H. F. Alqadah and M. J. Burfeindt, “An adaptive monostatic inverse scattering approach using virtual multistatic geometries,” *2023 IEEE Radar Conference (RadarConf23)*, 2023.

- [81] P. C. Hansen, “The l-curve and its use in the numerical treatment of inverse problems,” English, in *Computational Inverse Problems in Electrocardiology*, P. Johnston, Ed. WIT Press, 2001, pp. 119–142.
- [82] C. Moore, M. Burfeindt, and M. Cheney, “A two-part phase encoded linear sampling technique for synthetic aperture radar,” *accepted at 2026 SPIE Defense + Security*, 2026.