EFFECTS OF BOUNDARY SHAPE ON CHANNEL SEEPAGE

by

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SUMMARY

The present work is focused on analysis of gravity flow of groundwater and particularly on problems of steady seepage from ditches, where the flow may be assumed to be two-dimensional. Calculation of the seepage from a channel of arbitrary shape is a most difficult task. However, this problem is of great interest in the design of groundwater recharge basins and unlined canals for the conveyance of water.

If one accepts as a starting point Darcy's generalized law, the mathematical formulation of this specific topic leads to a mixed boundary value problem (with a free surface) of potential theory. Several cases have been previously solved by inverse methods or hodograph methods, with the restrictions inherent to those procedures, namely, either the channel cross section has a very simple geometrical shape or is not known *a priori*. In the present paper these well-known methods are used to find the seepage from a rectangular channel. As a step toward the solution of the more general problem, the writer here presents a first order solution of a perturbation type for a channel of nearly rectangular cross section. The method would be equally applicable to the triangular or trapezoidal section, and in principle would apply for any ditch of nearly rectilinear section.

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1. INTRODUCTION

1.1 Background. In 1856 Henri Darcy, a French engineer, first formulated an analytical approach to predicting the flow of water through a porous medium in connection with the design of the water supply system for the city of Dijon. Since then the theory has undergone extensive development by scientists and engineers. A summary has been given in the recent monograph by Scheidegger (1961), and in the earlier text by Muskat (1937). A review of the important contributions of the Russian school may be found in the book by Polubarinova-Kotchina (1952) or the paper by Polubarinova-Kotchina and Falkover (1951).

The interest in the present paper is focused on gravity flow problems and particularly on problems of steady seepage from ditches. If one accepts as a starting point Darcy's generalized law, the mathematical formulation of this specific topic is a mixed boundary value problem (with a free surface) of potential theory. Several cases have been previously solved by inverse methods or hodograph methods, with the restrictions inherent to those procedures, namely, either the channel cross section has a very simple geometrical shape or is not known *a priori*. In the present paper these well-known methods are used to find the seepage from a rectangular channel. In addition, for the sake of completeness a free translation is appended of Verdernikov's solution of the corresponding problems for triangular and trapezoidal channels.

Calculation of the seepage from a ditch of arbitrary shape is a most difficult task. However, this problem is of great interest in the design of groundwater recharge basins and unlined canals for the conveyance of water. As a step toward the solution of the more general problem, the writer here presents a first order solution of a perturbation type for a channel of nearly rectangular cross section. The method would be equally applicable to the triangular or trapezoidal section, and in principle would apply for any ditch of nearly rectilinear section.

1.2 The porous medium. One may wonder what differentiates seepage flow from that of the usual hydrodynamics or hydraulics. In fact many authors have tried to reduce the porous medium to a set of tubes and porous medium flow to flow in pipes. However, the porous medium is not so much characterized by its innumerable voids of varying sizes and shapes as by the multiple interconnection of the pores. For that matter, it is necessary to differentiate between "absolute porosity", that fraction of the bulk volume not occupied by the solid framework, and "effective porosity", the interconnected fraction, for a rock may have considerable absolute porosity and yet have no conductivity to fluid for lack of pore interconnection. The porosity is thus an upper limit of the water-holding ability of the soil, which may or may not be used to its full capacity.

The moisture content of a soil, apart from many other factors, will generally vary with the depth. Roughly speaking, three regions exist: a region of constant moisture content which overlies a region of rapidly changing degree of saturation beneath which there is a saturated region. Water movement in the first two regions, called the capillary zone, is essentially different from that in the saturated region. In the capillary zone, water displaces air, while in the saturated (or groundwater) zone, water moves as a continuum, enclosed by rigid boundaries, under the action of gravity forces.

The pressure variation in the pores can be used as the basis of a quantitative differentiation between the two zones, the true capillary zone being defined as that in which the pressure of the water is less than atmospheric and the groundwater zone as that where the pressure is greater than atmospheric. The water-table, which is called a free surface when showing a large curvature under the gravitational forces, is then defined as the surface at which the fluid pressure is equal to the atmospheric pressure. It will always be overlain by a capillary layer. In the present work, however, the capillary effects will be ignored and the assumption will be made that the flow takes place in a saturated porous medium, whose free surfaces are sharply defined.

1.3 Darcy's law. The first experiments on flow in a porous medium were conducted by Darcy between 1852-1855 and the results published in 1856. The report was primarily concerned with the water services of the city of Dijon.

In these tests, Darcy used sand from the Saône river. The soil to be tested was placed in a vertical tube of diameter 35 cm, with a height varying from 58 to 171 cm. The sand was poured into the tube which had been previously filled with water to remove the air from the pores. The pressure during the test was measured with sensitive manometers at the upper and lower end of the tube. The pressure oscillated somewhat and computations were carried on the basis of their means. The range of piezometric slope was 1.50 to 18.78 and the porosity of the soil was 38 percent. The tests led their author to come to the conclusion, known as Darcy's law, which may be stated as follows: "For a sand of a given type, it is possible to assume that the filtrating discharge is proportional to the pressure and inversely proportional to the length of sand layer." This law is expressed in the formula:

$$Q = kAH/L \tag{1.1}$$

where Q is the discharge of water, L the column length, H the head loss between the two ends of the column, A the filter cross section, k a coefficient which is a function (according to Darcy) only of the type of soil. In a simpler form, Darcy's law may be written as $Q = kAJ$, where J is the appropriate piezometric slope or hydraulic gradient. As was mentioned before, Darcy investigated only sands. Later investigations have shown that the filtration coefficient k depends on the fluid as well as the medium. In spite of shortcomings of this sort, the value of his work was very great: in it a solution to the first investigated problem of filtration was given.

If (1.1) is now written in the form

$$v = Q/A = kJ \tag{1.2}$$

the ratio Q/A represents the quantity of water filtrating per second through a unit surface, or the so-called filtration velocity. This velocity yields a variable directly obtainable in experiments.

2. MATHEMATICAL FORMULATION

2.1 <u>Basis of the theory of filtration.</u> In order to analyze the flow of groundwater it is customary to assume that Darcy's law is valid at any point of the flow field. Along with the continuity equation, this will lead to a partial differential equation, the complexity of which depends on what further assumptions are made about the fluid and the porous medium. Here it will be assumed that the fluid is incompressible and that the porous medium is homogeneous, isotropic, and incompressible. Moreover, the flow is taken to be steady and two-dimensional.

Let m be the porosity for a homogeneous porous medium and let $\vec{v}$ be the filtration velocity, i.e., the discharge per unit of time which filtrates through a unit surface normal to the velocity direction. The ratio $\vec{v}/m$ gives the average velocity $\vec{u}$. (It is the true velocity in the sense that it is the velocity in the pores, the average velocity in the sense that this velocity is not uniform in the pores.)

Assuming that both the water and the soil are incompressible we obtain the continuity equation

$$\nabla \cdot \vec{v} = 0 \quad \text{or} \quad \nabla \cdot \vec{u} \tag{2.1}$$

Darcy's law may now be written in the form

$$\vec{v} = k \text{ grad } h \tag{2.2}$$

Here h is the head defined in terms of the elevation y (positive downwards), the pressure p, and the specific weight γ, by the relationship

$$h = y - p/\gamma .$$

Hence for a seepage flow with a constant filtration coefficient k, the potential $\emptyset$ is equivalent to kh, i.e.,

$$\vec{v} = \text{grad } kh = \text{grad } \emptyset \tag{2.3}$$

If this result is combined with the continuity equation (2.1) the result is

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{2.4}$$

Therefore the potential satisfies Laplace's equation.

2.2 <u>The hodograph</u>. Without loss of generality we can set $k = 1$. In the case of a soil whose filtration coefficient is different from unity, it will be sufficient to multiply both velocities and discharges by k. With this understanding Darcy's law becomes

$$\vec{j} + \nabla(p/\gamma) + \vec{v} = 0 \tag{2.5}$$

where $\vec{j}$ is the unit vector in the vertical direction.

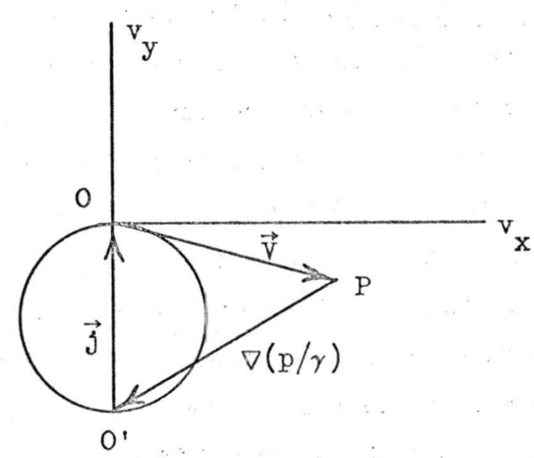


Figure 2.1

The geometry of this sum can be described with a closed triangle OPO', which we called the filtration triangle as constructed in Figure 2.1. To each point of the flow will correspond some point of the filtration triangle. For a continuous shift in the physical plane along any path, the point P describes some continuous curve in the plane

v_x, v_y (except at points where the velocity becomes infinite). A great number of points P corresponding to interior points of the flow generate a connected domain which is called the hodograph plane.

In two-dimensional flow it is convenient to introduce complex variables. Thus the complex potential W is defined in terms of the potential ϕ and the stream function ψ by $W = \phi + i\psi$ and the physical plane z is given by $z = x + iy$. The problem is then to find W as a function of z. We first note that

$$\frac{dW}{dz} = \frac{\partial}{\partial x}(\phi + i\psi) = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = v_x - iv_y \tag{2.4}$$

from the Cauchy-Riemann equations, or $v_x + iv_y = d\overline{W}/dz$. The hodograph plane appears as a reflected mapping of the plane of the derivative dW/dz about the v_x axis. Consequently, corresponding curves will be circulated in opposite directions.

2.3 Boundary conditions. If the flow, except for the free surface, is bounded with straight lines, along which one among the quantities ϕ, ψ, or p is constant, then the boundaries of the hodograph plane will be given by means of simple geometric construction as follows.

1. Free surface: If on the free surface there is neither evaporation nor absorption of water, then in steady flow the velocity direction is that of the tangent. Further, it may be assumed that on the free surface the pressure p = constant. It follows that $\nabla(p/\gamma)$ has a direction normal to the free surface. Consequently $\vec{v}$ and $\nabla(p/\gamma)$ form at their vertex P of the filtration triangle a right angle, as shown in Figure 2.2. The geometric locus of such points P will be a circle of diamter OO'.

2. Lines ϕ = constant: Such boundaries separate the soil from the waters of a stationary reservoir. From the condition ϕ = constant it results that the vector $\vec{v}$ is perpendicular to the boundary. It can vary quantitatively but not in direction. In the hodograph plane we shall have consequently a straight line, going through the origin and perpendicular to the boundary.

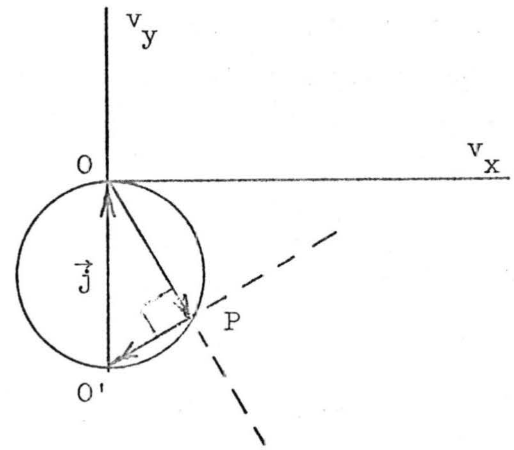


Figure 2.2

3. Lines ψ = constant: These will be boundaries along which the stream slips. Along such boundaries the velocity can vary in magnitude but not in direction. In the hodograph plane we get obviously a straight line going through O and parallel to the stream boundary.

4. Lines p = constant: On such boundaries seepage occurs into the atmosphere. The vector $\nabla(p/\gamma)$ is normal to the boundary. In the hodograph plane we get a straight line, going through O' and perpendicular to the boundary.

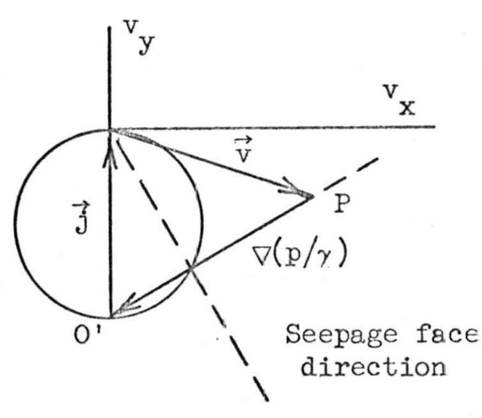


Figure 2.3

Hence the boundaries of the hodograph are known if the boundaries in the physical plane corresponding to the second, third, and fourth condition are known. Such will be the case for seepage from a channel with rectilinear sides and bottom. It is then only necessary to use the well-known methods of conformal mapping to relate the plane of complex potential W to the hodograph plane. Finally, the flow in the physical plane may be constructed by use of

$$z = \int \frac{dW}{v_x - iv_y} \tag{2.5}$$

A detailed example is given in the next section.

3. SEEPAGE FROM A RECTANGULAR CHANNEL

The problem of seepage from a rectangular channel will be analyzed as an example to show the methods which can in principle be used on any channel having rectilinear boundaries. It is assumed that the pervious layer extends infinitely far downward as indicated in Figure 3.1.

3.1 The hodograph plane method. As seen previously, according to Darcy's law:

$$\vec{v} = k \text{ grad } h = \text{grad } \phi \tag{3.1}$$

$$\phi = kh = k(y - p/\gamma \tag{3.2}$$

On the free surface, since $p = 0$, $\phi - ky = 0$. Differentiating with respect to the arc s of the free surface

$$\frac{\partial\phi}{\partial s} - k\frac{\partial\phi}{\partial s} = 0 \qquad \text{or} \qquad \left(\frac{\partial\phi}{\partial s}\right)^2 - k\frac{\partial\phi}{\partial s}\cdot\frac{\partial y}{\partial s} = 0$$

i.e.

$$v_x^{\,2} + v_y^{\,2} - kv_y = 0 \tag{3.3}$$

Using a reduced velocity v = (actual velocity)/k, the equation becomes:

$$v_x^{\,2} + v_y^{\,2} - v_y = 0,$$

the equation of a circle. The diameter of this circle is unity, as indicated in Figure 3.1. An inversion of pole f and modulus unity will transform all the boundaries into rectilinear boundaries as shown.

The plane $v/|v|^2$ can be mapped onto the upper half-plane δ with the help of the Schwarz-Christoffel transformation. Consequently, at least in principle, we can express $v/|v|^2$ as a function of δ, say $v/|v|^2 = f(\delta)$. The complex potential plane $W = \phi + i\psi$ has also a

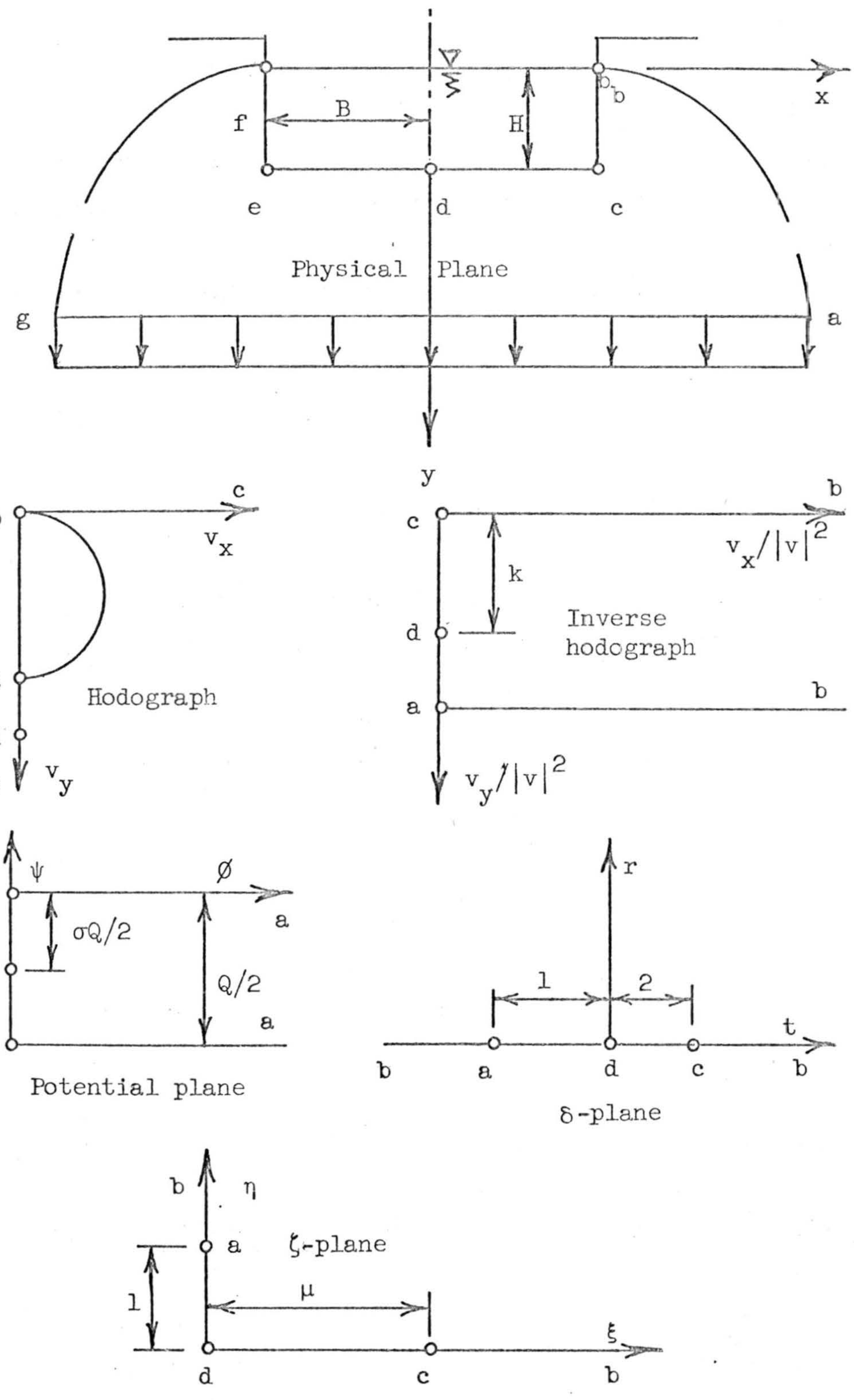

Physical Plane
B
H
x
y
Hodograph
v_x
v_y
Inverse hodograph
$v_x/|v|^2$
$v_y/|v|^2$
k
Potential plane
ψ
ø
σQ/2
Q/2
δ-plane
r
t
1
2
ζ-plane
η
ξ
μ
1

Figure 3.1

polygonal contour as shown in Figure 3.1. It can be mapped upon the δ-plane so that one obtains a relation of the form $W = g(\delta)$. Elimination of δ yields $v/|v|^2 = F(W)$. However, our interest lies in the knowledge of $\phi = \phi(x,y)$, i.e., of $W = W(z)$. We saw previously that $dW/dz = v_x - iv_y$ so that

$$\frac{dz}{dW} = \frac{1}{v_x - iv_y} = \frac{v_x + iv_y}{|v|^2} = \frac{v}{|v|^2}$$

Consequently

$$z = \int (v/|v|^2 dW = \int F(W)dW \qquad (3.4)$$

With the use of reduced velocities this expression is modified into the form

$$z = \frac{1}{k} \int \frac{v}{|v|^2} dW = \frac{1}{k} \int F(W)dW = \frac{1}{k} \int f(t)g'(t)dt,$$

whichever is most useful.

3.2 Conformal mappings. We shall take advantage of the symmetry of the problem. Aside from the usual auxiliary δ-plane we shall use another auxiliary plane, the ζ-plane; the relation between these two planes is given by the formula

$$\delta = \zeta^2 \qquad (3.5)$$

Consequently, $\mu = \sqrt{\lambda}$. Corresponding points in the several planes have the following values:

$$\text{point a:} \quad \begin{cases} x = Q/2k & v_x/|v|^2 = 0 \quad & t = -1 \quad & \xi = 0 \\ y = \infty & v_y/|v|^2 = 1 \quad & r = 0 \quad & \eta = 1 \end{cases}$$

point b: $\begin{cases} x = B & \phi = 0 \\ y = 0 & \psi = -Q/2 \end{cases}$

point c: $\begin{cases} x = B & \phi = 0 & v/|v|^2 = 0 & t = \lambda & \xi = \sqrt{\lambda} \\ y = H & \psi = Q_{dc} & & r = C & \eta = 0 \end{cases}$

point d: $\begin{cases} x = 0 & \phi = 0 & v_x/|v|^2 = 0 & t = 0 & \xi = 0 \\ y = H & \psi = 0 & v_y/|v|^2 = 1/v_d & r = 0 & \eta = 0 \end{cases}$

Mapping of the plane $v/|v|^2$ into the δ-plane can be carried out easily, using the Schwarz-Christoffel transformation. Thus

$$v/|v|^2 = A\int^{\delta} (t+1)^{-1/2}\,(t-\lambda)^{-1/2}\,dt + B$$

$$= A\int^{\delta} \frac{dt}{\sqrt{t^2 + (1-\lambda)t - \lambda}} + B$$

$$= A\int^{\delta} \frac{dt}{\frac{\lambda+1}{2}\sqrt{\left(t + \frac{1-\lambda}{2}\right)^2 \Big/ \left(\frac{\lambda+1}{2}\right)^2 - 1}} + B$$

Substituting the variable w defined by:

$$v/|v|^2 = A\int^{\delta} \frac{dw}{\sqrt{w^2 - 1}} + B = A \text{ arc cosh } w \Big|^{\delta} + B$$

$$= A \text{ arc cosh } \left[\frac{t + (1-\lambda)/2}{(1+\lambda)/2}\right]\Bigg|^{\delta} + B$$

The constants A and B must be determined by the boundary conditions. Since $v/|v|^2 = 0$ for $\delta = \lambda$, it follows that $B = 0$. Also, for $\delta = -1$, we have $v/|v|^2 = i$ and so $A = i/\text{arc cosh}\,(-1) = 1/\pi$. Finally

$$v/|v|^2 = \frac{1}{\pi} \text{ arc cosh } \frac{2\zeta^2 + 1 - \lambda}{\lambda + 1} \tag{3.6}$$

Similarly, we carry through the mapping of the W-plane as follows:

$$W(\delta) = A \int^{\delta} \frac{dt}{(t + 1)\sqrt{t}} + B = 2A \text{ arc tan } \sqrt{\delta} + B$$

Boundary conditions at c and d yield

$$W_c(\lambda) = 2A \text{ arc tan } \sqrt{\lambda} + B = -i\sigma Q/2$$

$$W_d(0) = 2A \text{ arc tan } (0) + B = 0$$

Hence $B = 0$ and $A = -i\sigma Q/4 \text{ arc tan } \sqrt{\lambda}$. At point b we find that $W_b(1) = 2A \text{ arc tan } (\infty) = -iQ/2$ so $A = -iQ/2\pi$. Consequently

$$W(\zeta) = -\frac{iQ}{\pi} \text{ arc tan } \zeta \tag{3.7}$$

and

$$\sigma = 2\frac{Q_{dc}}{Q} = \frac{2}{\pi} \text{ arc tan } \sqrt{\lambda} \tag{3.8}$$

The constants k and λ are not independent. Thus at point d we have

$$ik = \frac{1}{\pi} \text{ arc cosh } \frac{1-\lambda}{1+\lambda} = \frac{i}{\pi} \text{ arc cos } \frac{1-\lambda}{1-\lambda}$$

$$= \frac{i}{2} - \frac{i}{\pi} \text{ arc sin } \frac{1-\lambda}{1+\lambda}$$

$$\frac{1}{v_d} = \frac{1}{2} - \frac{1}{\pi} \text{ arc sin } \frac{1-\lambda}{1+\lambda} \tag{3.9}$$

From Formula (3.4) the physical plane is given by

$$z = -\frac{iQ}{\pi^2 k}\int^{\zeta} \text{arc cosh}\,\frac{2\zeta^2 + 1 - \lambda}{\lambda + 1} \cdot \frac{d\zeta}{1 + \zeta^2} \tag{3.10}$$

3.3 Evaluation of the discharge. The bottom and sides of the ditch correspond to the real axis of the ζ-plane and Formula (3.10) can be rewritten

$$z - z_d = -\frac{iQ}{\pi^2 k}\int_0^{\zeta} \text{arc cosh}\,\frac{2\zeta^2 + 1 - \lambda}{\lambda + 1}\,\frac{d\zeta}{1 + \zeta^2} \tag{3.11}$$

$$z_c - z_d = B = -\frac{iQ}{\pi^2 k}\int_0^{\sqrt{\lambda}} \text{arc cosh}\,\frac{2\zeta^2 + 1 - \lambda}{1 + \lambda}\,\frac{d\zeta}{1 + \zeta^2} \tag{3.12}$$

Here λ is a parameter depending on the dimensions of the channel. It is also related to the velocity at d by the equation

$$1/v_d = 1/2 - (1/\pi)\,\text{arc sin}\,[(1 - \lambda)/(1 + \lambda)]$$

Once the parameter λ has been chosen all the physical quantities can be determined.

From (3.10) we also have

$$z_b - z_c = -iH = -\frac{iQ}{\pi^2 k}\int_{\sqrt{\lambda}}^{\infty} \text{arc cosh}\,\frac{2\zeta^2 + 1 - \lambda}{1 + \lambda}\,\frac{d\zeta}{1 + \zeta^2} \tag{3.13}$$

From (3.12) and (3.13) we determine Q and the ratio B/H. Explicitly:

$$B = \frac{Q}{\pi^2 k}\int_0^{\sqrt{\lambda}} \text{arc cos}\,\frac{2\zeta^2 + 1 - \lambda}{1 + \lambda}\,\frac{d\zeta}{1 + \zeta^2}$$

$$H = \frac{Q}{\pi^2 k} \int_{\sqrt{\lambda}}^{\infty} \text{arc cosh} \; \frac{2\zeta^2 + 1 - \lambda}{1 + \lambda} \; \frac{d\zeta}{1 + \zeta^2}$$

$$= \frac{Q}{\pi^2 k} \int_{\sqrt{\lambda}}^{\infty} \log_e \left[\frac{2}{\lambda + 1} \left(\zeta^2 + \frac{1 - \lambda}{2} + \sqrt{(\zeta^2 + 1)(\zeta^2 - \lambda)} \right) \right] \frac{d\zeta}{1 + \zeta^2}$$

We introduce the following notation:

$$I(\lambda) = \int_{0}^{\sqrt{\lambda}} \text{arc cos} \; \frac{2\zeta^2 + 1 - \lambda}{1 + \lambda} \; \frac{d\zeta}{1 + \zeta^2}$$

$$I^*(\lambda) = \int_{\sqrt{\lambda}}^{\infty} \log_e \left[\frac{2}{\lambda + 1} \left(\zeta^2 + \frac{1 - \lambda}{2} + \sqrt{(\zeta^2 + 1)(\zeta^2 - \lambda)} \right) \right] \frac{d\zeta}{1 + \zeta^2}$$

$$I^*(\lambda) = 2.3026 \; J^*(\lambda) \tag{3.14}$$

Then the previous results may be summarized in terms of the parameter λ:

$$B/H = I(\lambda)/I^*(\lambda)$$

$$Q = \pi^2 k \; B/I(\lambda)$$

$$Q_{dc} = (Q/\pi) \text{ arc tan } \sqrt{\lambda} \tag{3.15}$$

$$1/v_d = 1/2 - (1/\pi) \text{ arc sin } [(1 - \lambda)/(1 + \lambda)]$$

Numerical calculations have been carried out using Simpson's rule for four values of λ, covering a reasonable range of ratios H/B. The results are presented in Table 3.1.

TABLE 3.1

λ	1	4	9	36
$I(\lambda)$	1.00	2.21	2.70	3.56
$J^*(\lambda)$	1.00	0.60	0.43	0.22
Q_{dc}/Q_{bc}	1.00	2.4	4	9
Q/kB	9.86	4.46	3.65	2.78
$Q/k\sqrt{2BH}$	4.60	4.01	4.08	5.20
H/B	2.30	0.62	0.40	0.143

The dimensionless coefficient $Q/k\sqrt{2BH}$ is a measure of how the seepage varies for a given excavation area, i.e., for a given value of 2BH. It is plotted as a function of H/B in Figure 3.2. Note that the optimum hydraulic cross section coincides approximately with the minimum water-loss cross section.

3.4 Velocity distribution on the canal perimeter. The points of the wetted perimeter of the canal correspond to the real positive axis of the ζ-plane. Consequently on the canal perimeter:

$$v/|v|^2 = \frac{1}{\pi} \text{arc} \cosh \frac{2\xi^2 + 1 - \lambda}{1 + \lambda} \tag{3.16}$$

On the bottom dc of the waterway $v/|v|^2 = iv_y/v_y^2 = i/v$ and moreover $0 < \xi < \sqrt{\lambda}$. Then

$$\frac{i}{v} = \frac{i}{\pi} \text{arc} \cos \frac{2\xi^2 + 1 - \lambda}{1 + \lambda}$$

$$\frac{v_a}{k} = \frac{\pi}{\text{arc} \cos \dfrac{2\xi^2 + 1 - \lambda}{1 + \lambda}} \qquad 0 < \xi < \sqrt{\lambda} \tag{3.17}$$

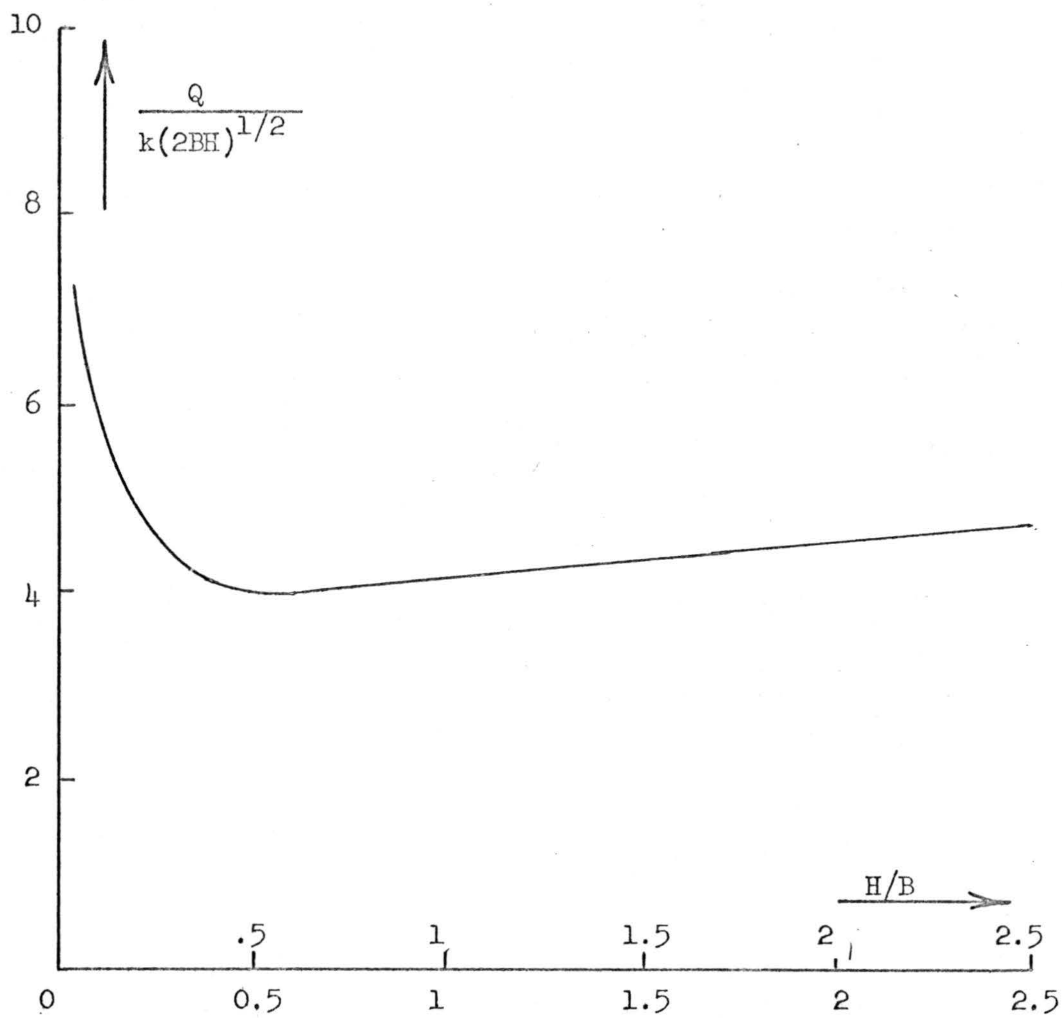


Figure 3.2

On the side bc of the waterway $v/|v|^2 = v_x/|v|^2 = 1/v$ and $\xi > \lambda$. Therefore

$$\frac{1}{v} = \frac{2.3026}{\pi} \log_{10} \left[\frac{2}{\lambda + 1} \left(\xi^2 + \frac{1 - \lambda}{2} + \sqrt{(\xi^2 + 1)(\xi^2 - \lambda)} \right) \right]$$

[Equation (3.18) is continued on the following page.]

$$\frac{v_a}{k} = \frac{\pi}{2.3026 \log_{10}\left[\frac{2}{\lambda + 1}\left(\xi^2 + \frac{1-\lambda}{2} + \sqrt{(\xi^2+1)(\xi^2-\lambda)}\right)\right]} \tag{3.18}$$

We shall evaluate the velocities on the perimeter at locations such that between two successive points a discharge $Q/2n$ flows, where $2n$ is an integer. The values of ξ corresponding to such points are derived from Formula (3.7). Explicitly:

$$p\frac{Q}{2n} = \frac{Q}{\pi}\,\text{arc tan}\,\xi \qquad \xi_p = \tan\frac{p\pi}{2n} \qquad p = 0,1,\ldots,n \tag{3.19}$$

The coordinates of the waterway perimeter are given by the Formula (3.10). On the bottom

$$x_p = \frac{B}{I(\lambda)}\int_0^{\xi_p} \text{arc cos}\,\frac{2\xi^2 + 1 - \lambda}{1+\lambda}\cdot\frac{1}{1+\xi^2}\,d\xi \tag{3.20}$$

$$y = H$$

On the side bc

$$x = B$$

$$y_p = H - \frac{H}{J^*(\lambda)}\int_{\sqrt{\lambda}}^{\xi_p} \log_{10}\left[\frac{1}{\lambda+1}\left(\xi^2 + \frac{1-\lambda}{2} + \sqrt{(\xi^2+1)(\xi^2-\lambda)}\right)\right]\frac{d\xi}{1+\xi^2} \tag{3.21}$$

For example in the case $\lambda = 1$, we have on the bottom:

$$\frac{v_a}{k} = \frac{\pi}{\text{arc cos}\,\xi^2} \qquad x_p = B\int_0^{\xi_p} \text{arc cos}\,\xi^2\,\frac{d\xi}{1+\xi^2}$$

and on the side:

$$\frac{v_a}{k} = \frac{\pi}{2.3026 \log_{10}\left[\xi^2 + \sqrt{(\xi^2 - 1)(\xi^2 + 1)}\right]}$$

$$y_p = H\left[1 - \int_{\sqrt{\lambda}}^{\xi_p} \log_{10}\left(\xi^2 + \sqrt{(\xi^2 - 1)(\xi^2 + 1)}\right)\frac{d\xi}{1 + \xi^2}\right]$$

Results are indicated in Table 3.2.

TABLE 3.2

p	$p\pi/20$	ξ_p	v_a/k	x/B	y/H
0	0	0	2.000	0	1
1	9^o	0.158	2.032	0.245	1
2	18^o	0.325	2.144	0.483	1
3	27^o	0.510	2.410	0.702	1
4	36^o	0.727	3.098	0.887	1
5	45^o	1	∞	1	1
6	54^o	1.376	2.508	1	0.949
7	63^o	1.962	1.553	1	0.837
8	72^o	3.078	1.068	1	0.669
9	81^o	6.314	0.718	1	0.425
10	90^o	∞	0	1	0

On Figure 3.3 is indicated the velocity distribution. Between each two successive arrows flows the same discharge Q/20.

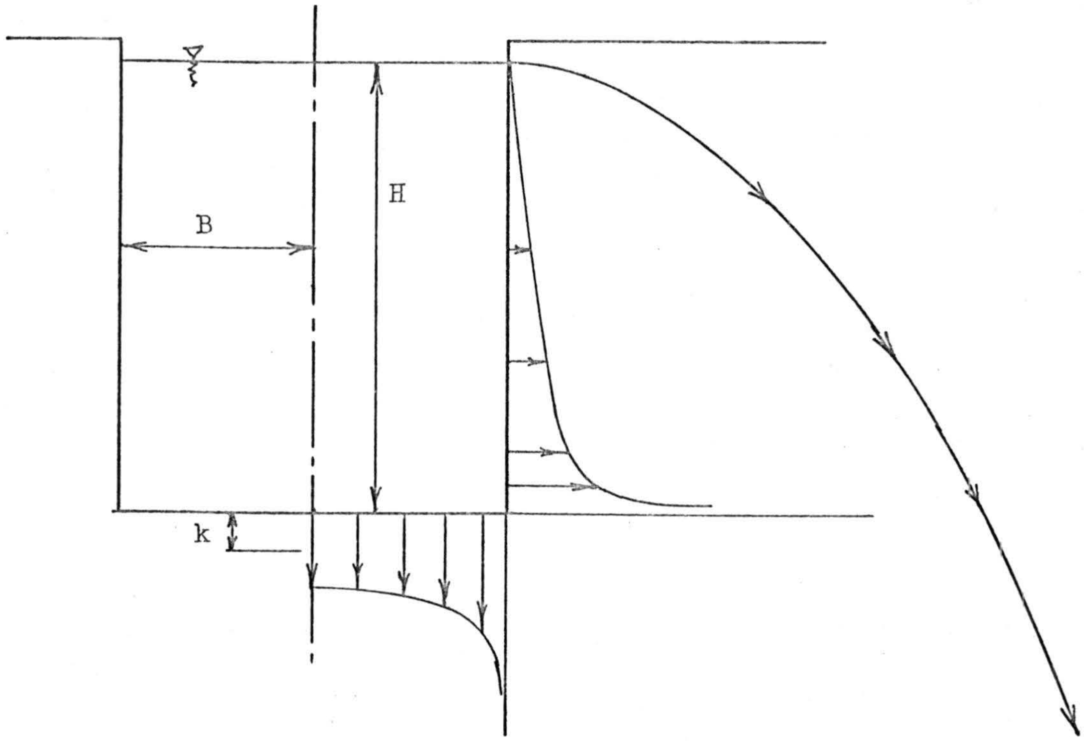


Figure 3.3

3.5 Free surface determination. The coordinates of the free surface are obtained from Formula (3.7) for the imaginary values of ζ of modulus greater than one, that is, for $\zeta = i\eta$ with $\eta > 1$. Then

$$z = -\frac{iQ}{k\pi^2}\int^{\eta} \text{arc cosh}\ \frac{-2\eta^2 + 1 - \lambda}{1 + \lambda}\ \frac{i d\eta}{1 - \eta^2} + C$$

$$= -\frac{Q}{k\pi^2}\int^{\eta} \text{arc cosh}\ \frac{-2\eta^2 + 1 - \lambda}{1 + \lambda}\ \frac{d\eta}{\eta^2 - 1} + C$$

For the case corresponding to $\lambda = 1$,

$$z = -\frac{Q}{k\pi^2}\int^{\eta} \text{arc cosh}\ (-\eta^2)\ \frac{d\eta}{\eta^2 - 1} + C$$

$$= - \frac{Q}{k\pi^2} \int^{\eta} (\text{arc cosh } \eta^2 + i\pi) \frac{d\eta}{\eta^2 - 1} + C$$

Boundary conditions are that for $\eta = \infty$, $z = B$ and consequently

$$z = \frac{Q}{k\pi^2} \int_{\infty}^{\eta} (\text{arc cosh } \eta^2 + i\pi) \frac{d\eta}{\eta^2 - 1} + B \tag{3.22}$$

or

$$x = \frac{2.3026Q}{k\pi^2} \int_{\eta}^{\infty} \log_{10} \left(\eta^2 + \sqrt{(\eta^2 - 1)(\eta^2 + 1)} \right) \frac{d\eta}{\eta^2 - 1} + B$$

$$y = \frac{2.3026Q}{2k\pi^2} \log_{10} \frac{\eta + 1}{\eta - 1} \tag{3.23}$$

or

$$x/B = 1 + 2.3026 \int_{\eta}^{\infty} \log_{10} \left(\eta^2 + \sqrt{(\eta^2 - 1)(\eta^2 + 1)} \right) \frac{d\eta}{\eta^2 - 1}$$

(3.24)

$$y/B = \frac{\pi}{2} \log_{10} \frac{\eta + 1}{\eta - 1}$$

Results are gathered in Table 3.3.

TABLE 3.3

y/H	η	x/B	v_y/v_x
0.25	5.515	2.094	0.765
0.50	2.850	2.706	1.129
0.75	2.000	3.144	1.522
1.00	1.600	3.475	1.972
1.50	1.250	3.950	3.096
2.00	1.113	4.265	4.63
3.00	1.025	4.781	9.90
5.00	1.001	4.930	50.
∞	1.000	4.930	∞

From Formula (3.6) we derive

$$(v_x + iv_y)/|v|^2 = (1/\pi)(\text{arc cosh } \eta^2 + i\pi)$$

and thus

$$v_x/v_y = (2.3026/\pi) \log_{10}\left(\eta^2 + \sqrt{(\eta^2 - 1)(\eta^2 + 1)} \right) \tag{3.25}$$

A plot of the free surface is made to scale on Figure 3.3. All numerical integrations of definite integrals have been carried out by use of Simpson's formula.

4. THE GREEN-NEUMANN FUNCTION METHOD

The previous problem is basically a problem of potential theory. The solution depends on the boundary values and on the boundary configuration. For a given configuration one may wonder whether the solution of a particular boundary value problem does entitle us to solve all the boundary value problems, according to the well-known methods of potential theory. In this section we will extend the theory of Green's functions to include the mixed boundary value problem at hand.

4.1 Green's function. Let us consider a two-dimensional domain D as indicated in Figure 4.1. Let $\phi(P)$ and $\psi(P)$ be two harmonic functions in this domain. From Green's theorem we know that:

$$0 = \int_C \left(\phi(Q) \frac{\partial \psi(Q)}{\partial n} - \psi(Q) \frac{\partial \phi(Q)}{\partial n} \right) d\sigma \tag{4.1}$$

$$\phi(P) = \int_C \left[\phi(q) \frac{\partial}{\partial n} \left(\frac{1}{2\pi} \log \frac{1}{r} \right) - \frac{1}{2\pi} \log \frac{1}{r} \frac{\partial \phi(Q)}{\partial n} \right] d\sigma \tag{4.2}$$

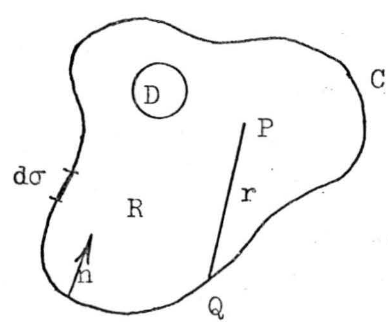


Figure 4.1

Combination of (4.1) and (4.2) yields:

$$\emptyset(P) = \int \left[\emptyset(Q) \frac{\partial}{\partial n} \frac{1}{2\pi}\left(\log \frac{1}{r} + \psi\right) - \left(\psi(Q) + \frac{1}{2\pi} \log \frac{1}{r}\right) \frac{\partial \emptyset(Q)}{\partial n} \right] d\sigma \quad (4.3)$$

The function $G(R,P) = \frac{1}{2\pi} \log \frac{1}{r} + \psi(R)$ is called a Green's function of the domain D. Such a function is however not yet defined, for it depends upon an arbitrary regular harmonic function. The Green's function will be completely determined if we prescribe its boundary values. The usual Green's function is the one which takes the value zero on the boundary, i.e., $G(Q,P) = 0$ on C. This condition determines completely $\psi(P)$, for it is a harmonic regular function of value on C:

$$\psi(Q) = - (1/2\pi) \log (1/r_{QP})$$

Assuming that such a function can be found, then Formula (4.3) reduces to

$$\emptyset(P) = \int_C \emptyset(Q) \frac{\partial G(Q,P)}{\partial n} \, d\sigma \qquad (4.4)$$

If the Green's function of the domain D is known, then any regular harmonic function in D of given boundary value $\emptyset(Q)$ will be obtained from Formula (4.4).

Most problems however are not that simple. In the case of the rectangular channel already we are faced with a mixed boundary problem. On parts of the boundary the value of the potential is given, on the other its normal derivative. A method for treating such mixed boundary value problems is that of combining the method for conjugate-function transformations with that of mixed Green's functions, which are defined so that they vanish over parts of the boundary where the velocity potential is specified and their normal derivatives vanish over those parts where the normal derivative is specified. The determination of such a function is not easy and to quote Muskat (1937), "Even the simple case in which the potential is specified over the right half of the X axis and the normal derivative over the left half, does not seem to be solvable by the methods given in the standard textbooks".

Green's theorem in its general form can be written as

$$\emptyset(P) = \int_C \emptyset(Q) \frac{\partial G(Q,P)}{\partial n} - G(Q,P) \frac{\partial \emptyset(Q)}{\partial n} \, d\sigma$$

Suppose the boundary C is divided into two parts C_1 and C_2 such that $G(Q,P) = 0$ on C_1 and $\partial G(Q,P)/\partial n = 0$ on C_2. Then we have

$$\emptyset(P) = \int_{C_1} \emptyset(Q) \frac{\partial G(Q,P)}{\partial n} \, d\sigma - \int_{C_2} G(Q,P) \frac{\partial \emptyset(Q)}{\partial n} \, d\sigma \qquad (4.5)$$

where $G(Q,P)$ is the Green-Neumann or mixed Green's function.

4.2 <u>Green-Neumann function for the rectangular channel.</u> The geometry of the domain and the boundary values for the potential and the Green-Neumann function are indicated below in the ζ-plane in Figure 4.2.

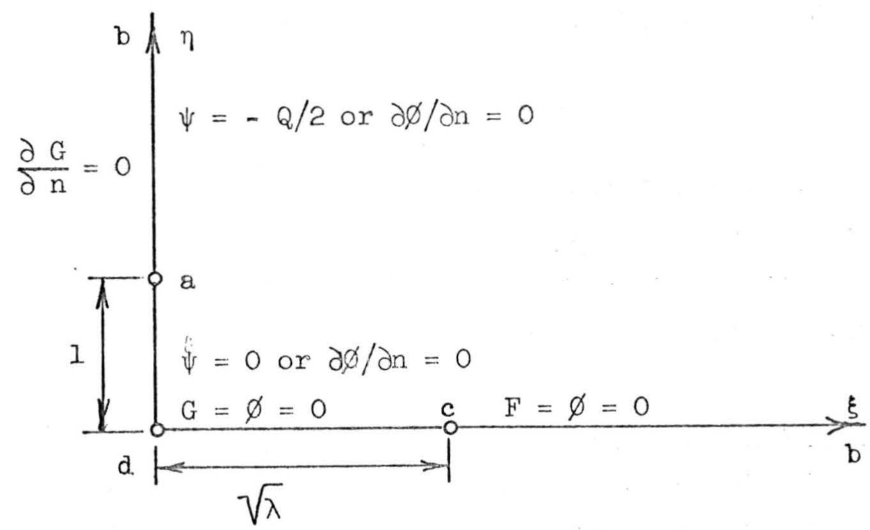


Figure 4.2

We shall obtain the Green-Neumann function by a method of images, as shown in Figure 4.3. The superposition of the four source potentials is a harmonic function singular at P, which is zero on $0\,\xi$ and whose normal derivative is zero on $0\,\eta$.

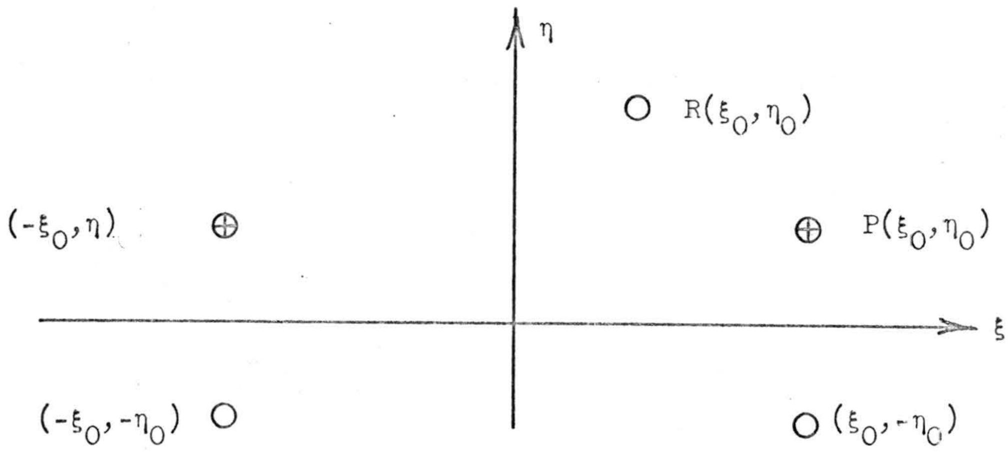


Figure 4.3

The potential due to a source of strength unity is $(1/4\pi)\log r^2$ so that we have

$$G(R,P) = -\frac{1}{4\pi}\log\frac{[(\xi-\xi_0)^2+(\eta-\eta_0)^2][(\xi+\xi_0)^2+(\eta-\eta_0)^2]}{[(\xi-\xi_0)^2+(\eta+\eta_0)^2][(\xi+\xi_0)^2+(\eta-\eta_0)^2]} \tag{4.6}$$

In theory then, any mixed boundary potential problem in the ζ-plane can be solved. Once the potential is known, the complex potential is known also from the Cauchy-Riemann conditions, and we can complete the solution of the problem by use of conformal mapping.

4.3 Singularities and the boundary value problem. The harmonic function $\phi(P)$ given by Formula (4.5) is regular over the whole domain D. In many problems the potential will not be regular everywhere but will present irregularities at a few points. The potential Ω is then the sum of a singular part and a regular harmonic function ϕ which satisfies different boundary conditions from Ω and can be obtained from (4.5).

Suppose that at S there is a source of strength unity. Then if r_{SP} is the distance from S to the point P, we have that $\Omega(P) = 1/(2\pi)\log r_{SP} + \phi(P)$. On the boundary C_1 the function $\phi(P)$ has the value $\Omega(Q) = -(1/2\pi) \log r_{SQ} = \phi(Q)$, where Q designates a point of the boundary. On the other part of the boundary C_2, the function $\phi(P)$ has the value

$$\frac{\partial\Omega(Q)}{\partial n} - \frac{1}{2\pi}\frac{1}{r_{SQ}} = \frac{\partial\phi(Q)}{\partial n}$$

The solution is then given by

$$\Omega(P) = \frac{1}{2\pi} \log r_{SP} + \int_{C_1} \left[\Omega(Q) - \frac{1}{2\pi} \log r_{SQ}\right] \frac{\partial G(QP)}{\partial n} \, d\sigma$$

$$- \int_{C_2} G(Q,P) \left[\frac{\partial\Omega(Q)}{\partial n} - \frac{1}{2\pi}\frac{1}{r_{SQ}}\right] d\sigma \tag{4.7}$$

We now proceed to work out the explicit result for the rectangular channel. In the ζ-plane of Figure 4.2, the potential shows a singularity at point a, namely, a sink. Direct application of (4.17) yields

$$\Omega(\xi_0, \eta_0) = -\frac{1}{4\pi} \log \left[\xi_0^2 + (\eta_0 - 1)^2\right] - \frac{1}{2\pi} \int_0^\infty \log (\xi^2 + 1) \frac{\partial G}{\partial \eta} \, d\xi$$

$$+ \frac{1}{2\pi} \int_0^\infty G \frac{d\eta}{\sqrt{(\eta - 1)^2}}$$

The mixed Green's function G is given by (4.6). On the η axis

$$G = -\frac{1}{4\pi} \log \frac{[\xi_0^2 + (\eta - \eta_0)^2]^2}{[\xi_0^2 + (\eta + \eta_0)^2]^2}$$

$$= -\frac{1}{2\pi} \log \left|\frac{\eta^2 - 2\eta\eta_0 - r_0^2}{\eta^2 + 2\eta\eta_0 - r_0^2}\right|$$

where $r_0^2 = \xi_0^2 + \eta_0^2$. Similarly, we evaluate $\partial G/\partial \eta$ on the ξ axis:

$$\frac{\partial G}{\partial \eta} = \frac{1}{\pi} \eta_0 \left(\frac{1}{(\xi - \xi_0)^2 + \eta_0^2} + \frac{1}{(\xi + \xi_0)^2 + \eta_0^2} \right)$$

Then

$$\Omega(\xi_0, \eta_0) = -\frac{1}{4\pi} \log \left[\xi_0^2 + (\eta_0 - 1)^2 \right]$$

$$- \frac{\eta_0}{2\pi^2} \int_0^\infty \log (\xi^2 + 1) \left(\frac{1}{(\xi - \xi_0)^2 + \eta_0^2} + \frac{1}{(\xi + \xi_0)^2 + \eta_0^2} \right) d\xi$$

$$- \frac{1}{4\pi^2} \int_0^\infty \log \left| \frac{\eta^2 - 2\eta\eta_0 - r_0^2}{\eta^2 + 2\eta\eta_0 - r_0^2} \right| \cdot \frac{d\eta}{\sqrt{(\eta - 1)^2}} \qquad (4.8)$$

Evaluation of these definite integrals would yield the searched for solution. However, here the method of images yields an immediate solution. The function:

$$\phi(\xi_0, \eta_0) = -\frac{1}{4\pi} \log \left[\xi_0^2 + (\eta_0 - 1)^2 \right] + \frac{1}{4\pi} \log \left[\xi_0^2 + (\eta_0 + 1)^2 \right]$$

shows a singularity at a is harmonic everywhere else in the ζ-plane and satisfies the boundary conditions. Consequently

$$\operatorname{Re} W(\zeta) = A \left(\log \left[\xi^2 + (\eta - 1)^2 \right] - \log \left[\xi^2 + (\eta + 1)^2 \right] \right)$$

The Cauchy-Riemann conditions yield

$$\frac{\partial \phi}{\partial \xi} = \frac{\partial \psi}{\partial \eta} = 2A\xi \left(\frac{1}{\xi^2 + (\eta - 1)^2} - \frac{1}{\xi^2 + (\eta + 1)^2} \right)$$

$$-\frac{\partial \phi}{\partial \eta} = +\frac{\partial \psi}{\partial \xi} = +2A\left(\frac{\eta - 1}{\xi^2 + (\eta - 1)^2} - \frac{\eta + 1}{\xi^2 + (\eta + 1)^2}\right)$$

Integration of these equations yields the stream function

$$\psi = 2A\left(\arctan\frac{\eta - 1}{\xi} - \arctan\frac{\eta + 1}{\xi}\right) + B$$

The constants A and B are determined by the conditions that for $\xi = 0$, $\psi = -Q/2$ when $\eta > 1$ and $\psi = 0$ when $\eta < 1$.

$$-\frac{Q}{2} = 2A\,[\arctan(+\infty) - \arctan(+\infty)] + B \qquad B = -\frac{Q}{2}$$

$$0 = 2A\,[\arctan(-\infty) - \arctan(+\infty)] - \frac{Q}{2} = -2A\pi - \frac{Q}{2}$$

$$A = -\frac{Q}{4\pi}$$

Substitution in the preceding equation shows that $A = -Q/4\pi$ and $B = -Q/2$. Then finally

$$\phi = -\frac{Q}{4\pi}\left(\log\,[\xi^2 + (\eta - 1)^2] - \log[\xi^2 + (\eta + 1)^2]\right)$$

$$\psi = -\frac{Q}{2\pi}\left(\arctan\frac{\eta - 1}{\xi} - \arctan\frac{\eta + 1}{\xi}\right)$$

The complex potential $W = \phi + i\psi$ is given by:

$$W(\zeta) = -\frac{Q}{2\pi}\left(\log\sqrt{\xi^2 + (\eta - 1)^2} + i\arctan\frac{\eta - 1}{\xi}\right)$$

$$+\frac{Q}{2\pi}\left(\log\sqrt{\xi^2 + (\eta + 1)^2} + i\arctan\frac{\eta + 1}{\xi}\right) - i\frac{Q}{2}$$

or

$$W(\zeta) = -\frac{Q}{2\pi}\log\frac{\zeta - i}{\zeta + i} - i\frac{Q}{2} \qquad (4.9)$$

This result is only apparently different from Formula (3.7) for

$$\frac{dW(\zeta)}{d\zeta} = -\frac{Q}{2\pi}\left(\frac{1}{\zeta - i} - \frac{1}{\zeta + i}\right) = -\frac{Q}{2\pi}\,\frac{2i}{1 + \zeta^2}$$

$$= -\frac{iQ}{\pi}\cdot\frac{1}{1 + \zeta^2}$$

and consequently $W(\zeta) = -(iQ/\pi)\,\mathrm{arc\,tan}\,\zeta$.

The simplicity of the domain geometry allowed us to bypass the rather tedious integration of the Green's function method with the help of the method of images. However, a minor change of the boundary slope would invalidate this procedure. It is then that the Green's function method will prove very useful as we shall see in a later section.

5. SEEPAGE FROM CHANNELS OF ARBITRARY SHAPE

The solution of Laplace's equation with mixed geometry of the boundary departs from the elementary forms. There is little hope to solve the problem completely or in simple terms, but the case when the boundary departs from an elementary form by an infinitesimal amount may be a first step to the solution. If it is possible to find a complex function which will map the perturbed boundary onto its original shape, the problem is then solved, at least in principle, if the potential function is known for the elementary geometry boundary problem. Of the three methods used in previous chapters, which are not really unrelated, the method of images seems hopeless even for a minor change. The methods of conformal mapping and of the Green's function are, however, still usable and can best be used concurrently.

5.1 Domain variation of the Green's function. Consider a domain D bounded by a smooth curve C and suppose that the Green's function $G(z,\zeta)$ of this domain is known. We will now investigate how $G(z,\zeta)$ varies with slight variations of the boundary curve C. We shall star all the quantities which refer to the new domain D^*. We shall assume that the smooth boundary curve C^* lies inside C. Thus, if $\delta\nu$ is the normal distance from C to C^*, we may write $\delta\nu = \epsilon v(s)$ where $\epsilon > 0$ is a smallness parameter and $v(s) > 0$ is a smooth function of the distance s along the curve C. The problem is now to find the Green's function $G^*(z,\zeta)$ for the new domain in terms of $G(z,\zeta)$ up to an error of the order ϵ^2.

Before going any further we shall write down several versions of Green's formula. First, by means of the Green's function we may express every function $h(z)$ which is regular harmonic in D in terms of its boundary values on C as follows:

$$h(z) = \int_C [\partial G(t,z)/\partial\nu_t]\, h(t)\, ds_t \tag{5.1}$$

where ν is the inner normal. Second, consider two harmonic functions

$u(z,\zeta)$ and $v(z,\eta)$, both of which are regular in a domain R except maybe at $z = \zeta$ and $z = \eta$. Then in a subdomain D bounded by C which does not include any singular point, we have

$$0 = \int_C \left[u(t,\zeta)\ \partial v(t,\eta)/\partial\nu - v(t,\eta)\ \partial u(t,\zeta)/\partial\nu\right] ds_t \tag{5.2a}$$

and in a subdomain D bounded by C which includes one of the singular points, say $z = \zeta$, we have

$$v(\zeta,\eta) = \frac{1}{2\pi}\int_C \left[v(z,\eta)\frac{\partial u(z,\zeta)}{\partial\nu_z} - u(z,\eta)\frac{\partial v(z,\eta)}{\partial\nu_z}\right] ds_z \tag{5.2b}$$

Furthermore, in a subdomain D bounded by C which includes both singular points, we have

$$v(\zeta,\eta) - u(\eta,\zeta) = \frac{1}{2\pi}\int_C \left[v(z,\eta)\frac{\partial u(z,\zeta)}{\partial\nu_z} - u(z,\zeta)\frac{\partial v(z,\eta)}{\partial\nu_z}\right] ds_z \tag{5.2c}$$

The factor $(1/2\pi)$ comes from the logarithmic singularity. Finally, we write Green's identity in the form

$$\iint_R u \operatorname{div}(\operatorname{grad} v)\, dx\, dy = \int_S u(\partial v/\partial\nu)\, ds - \iint_R \operatorname{grad} u \cdot \operatorname{grad} v\, dA \tag{5.2d}$$

Consider now the difference in the Green's functions $\left[G^*(z,\zeta) - G(z,\zeta)\right]$. It is a regular harmonic function in the smaller domain D^* since the singularities cancel each other. It can be shown that

$$G^*(z,\zeta) - G(z,\zeta) = -\int_C \frac{\partial G(t,z)}{\partial\nu_t}\ \frac{\partial G(t,\zeta)}{\partial\nu_t}\ \delta\nu_t\ ds_t \tag{5.3}$$

The proof of this result, which is known as Hadamard's formula, is given in the monograph by Bergman and Schiffer (1953). The formula can be extended to the case in which $v(s)$ is still twice continuously differentiable but is no longer restricted in sign. However, the limitation of (5.3) to the case of smooth boundaries is a serious one and restricts the usefulness of Hadamard's formula considerably. To alleviate this difficulty we shall use a different approach, the method of interior variation. But first we shall have to discuss in more detail the $V_{\rho q}$ variation. The following is a digest of articles on the subject by Schiffer (1943 - 1958).

Consider the complex transformation (with $\rho > 0$)

$$z^* = z + \frac{e^{2i\beta}\rho^2}{z - z_0} \qquad 0 \leq \beta \leq 2\pi \tag{5.4}$$

It transforms the circle $|z - z_0| = \rho$ into a segment in the z*-plane, as shown in Figure 5.1. More specifically, the circle $z = z_0 + \rho e^{i\alpha}$ transforms into $z^* = z_0 + 2\rho \cos(\beta - \alpha)\, e^{i\beta}$.

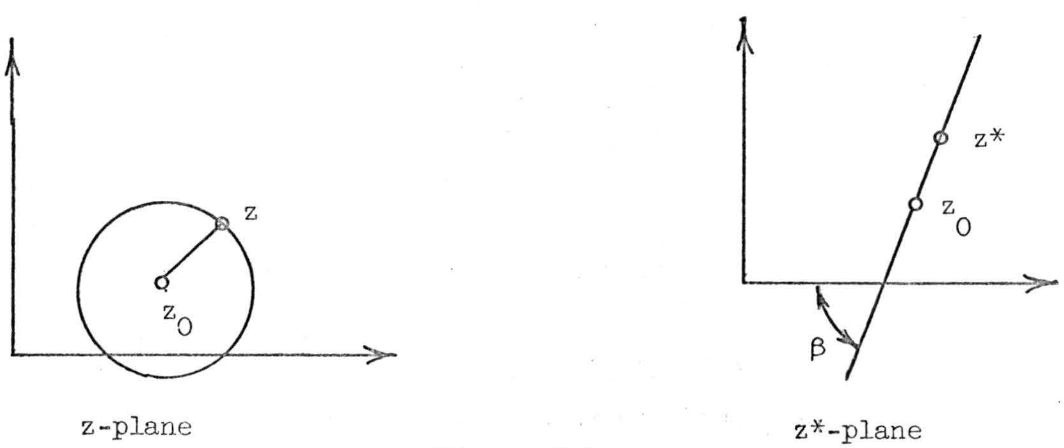


Figure 5.1

The transformation is univalent for the exterior of $|z - z_0| = \rho$. For ρ sufficiently small this representation is univalent on all curves C_n and transforms them in one-to-one manner into neighboring curves C^*_n which enclose a new domain D^* of the z-plane (note that z_0 is an interior point of D).

The transformation (5.4) is a particular case of a larger class we shall now define:

$$z^* = z + \rho q(z) \tag{5.5}$$

where $q(z)$ is a uniform function regular everywhere except at a finite number of poles z_i. Thus

$$q(z) = \frac{a_i}{z - z_i} + b_i + c_i\,(z - z_i) + \ldots$$

The function $z + \rho q(z)$ will be regular in R except at $z = \infty$ and at the points z_i. Around each point z_i we describe a circle K_i with radius r, so small that no branch point and no z_K lies in the interior $\tilde{K}_i$ of K_i. If now m is the maximum absolute value of $q(z)$, we have

$$|z - z_i| = r/2 \qquad i = 0, 1 \ldots m$$

If ρ is such that $|\rho| \frac{r}{2m}$ than $z^*(z)$ is exactly p-valued over the domain $R - \sum_{i=0}^{m} \tilde{K}_i$. In other words, the representation $z^*(z)$ transforms $R - \sum_{i=0}^{m} \tilde{K}_i$ contained in the Riemann surface R into a domain bounded by $(m + 1)$ simple curves K^*_i and covering the z^* plane p times at most. If we add to this domain in the interiors $\tilde{K}^*_i$ of the curves K^*_i, we get a closed Riemann surface. $R^*_{\rho q}$ with p sheets. Hence, $R^*_{\rho q}$ is the Riemann surface obtained from R by means of the variation $V_{\rho q}$ and this variation preserves the number of sheets and the genus of the Riemann surface.

Let D be a domain on R. If no pole z_i of $q(z)$ is situated on the boundary of D and if r is so small that no point of this boundary is situated in any $\tilde{K}_i$, then $V_{\rho q}$ determines in an unambiguous way a variation of the domain D, say $D^*_{\rho q}$. Therefore, a domain D on a closed Riemann surface R with p sheets and with genus g is transformed by means of a variation $V_{\rho q}$ into a domain D^* on a Riemann surface R^* of the same type.

5.2 Interior variation of the Green's function. Consider the domain D on a Riemann surface R of the above type, which is bounded by analytic curves and a transformation of the type (5.5) with appropriate smallness for ρ. Then the Green's function $G^*(z,\zeta)$ is defined for z and ζ in D^*. Consider the function

$$\Delta(z,\zeta) = G^*(z^*,\zeta^*) - G(z,\zeta)$$

with the points z and ζ in the domain $D - \sum \tilde{K}_i$ where $i = 0, 1, 2, \ldots, m$. Moreover, suppose all the points z_i are situated in D. Then $\Delta(z,\zeta)$ is harmonic in $D - \sum \tilde{K}_i$ and is a uniform function of z, for $V_{\rho q}$ transforms this domain into $D - \sum \tilde{K}^*_i$ and there $G^*(z^*,\zeta^*)$ is defined and harmonic.

Use of Green's formula (5.2b) yields

$$\Delta(z,\zeta) = \frac{1}{2\pi} \int_P \left[\Delta(t,\zeta) \frac{\partial G(t,z)}{\partial \nu_t} - G(t,z) \frac{\partial \Delta(t,\zeta)}{\partial \nu_t}\right] ds_t$$

Here P is the boundary of $D - \sum \tilde{K}_i$, that is, $C + \sum K_i$. But on C, $G(t,z) = 0$ and $\Delta(t,\zeta) = 0$ so that

$$\Delta(z,\zeta) = \frac{1}{2\pi} \int_{\sum K_i} \left[\Delta(t,\zeta) \frac{\partial G(t,z)}{\partial \nu_t} - G(t,z) \frac{\partial \Delta(t,z)}{\partial n}\right] ds_t = S_i$$

On K_i let $t = z_i + re^{i\theta}$. Then after some manipulation S_i takes the form:

$$S_i = \frac{1}{2\pi}\int_0^{2\pi}\left[G^*\left(t + \frac{a_i\rho}{re^{i\theta}} + \rho\,\lambda\,(re^{i\theta}),\zeta^*\right)\frac{\partial G(t,z)}{\partial r}\right.$$

$$\left. - G(t,z)\,\frac{\partial G^*}{\partial r}\left(t + \frac{a_i\rho}{re^{i\theta}} + \rho\,\lambda\,(e^{i\theta}r),\ \zeta^*\right)\right] r d\theta$$

for $t^* = t + \rho q(t) = t + \rho\,\dfrac{a_i}{t - z_i} + \rho\,\lambda_i(re^{i\theta})$ where $\lambda_i(re^{i\theta})$ iw an analytic function of its argument.

Let $p(z,\zeta)$ and $p^*(z,\zeta)$ be the analytic completion of the corresponding Green's functions and $p'(z,\zeta)$ the derivative with respect to the first argument. Taylor's development yields:

$$p(z^*,\zeta^*) = p(z,\zeta^*) + \left[\frac{a_i\rho}{z - z_i} + \rho\,\lambda_i\,(z - z_i)\right]p'(z,\zeta^*) + O(\rho^2)$$

$$G(z^*,\zeta^*) = G(z,\zeta^*) + \mathrm{Re}\left[\frac{a_i\rho}{z - z_i} + \rho\,\lambda_i\,(z - z_i)\,p'(z,\zeta^*)\right] + O(\rho^2)$$

Replacing into S_i yields and use of Green's formula (5.2a) yields:

$$S_i = \frac{1}{2\pi}\int_0^{2\pi}\left[\mathrm{Re}\left(\frac{a_i\rho}{re^{i\theta}}\,p'^*\,(t,\zeta^*)\right)\frac{\partial G(t,z)}{\partial r} - G(t,z)\,\frac{\partial}{\partial r}\,\mathrm{Re}\left(\frac{a_i\rho}{re^{i\theta}}\,p'^*(t,\zeta^*)\right)\right] r d\theta$$

and since $\mathrm{Re}\left[\left(p^{*\prime}(t,\zeta^*) - p^{*\prime}(z_i,\zeta^*)\right)\left(\dfrac{a_i\rho}{t-z_i}\right)\right]$ is regular harmonic in $\tilde{K}_i$ we find that

$$S_i = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{a_i\rho}{e^{i\theta}}\, p'^*(z_i,\zeta^*)\right)\frac{\partial G(t,z)}{\partial r}\, d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} G(t,z)\,\mathrm{Re}\left(\frac{a_i\rho}{re^{i\theta}}\, p'^*(z_i,\zeta^*)\right) d\theta + O(\rho^2)$$

Now $G(t,z)$ can be developed in a series of powers of $re^{i\theta}$:

$$G(t,z) = G(z_i,z) + \mathrm{Re}\left[\sum_{N=1}^{\infty} \frac{1}{N!}\, p^{(N)}(z_i,z)\, r^N e^{iN\theta}\right]$$

if r is sufficiently small. Hence, we have

$$S_i = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(p^{*\prime}(z_i,\zeta^*)\frac{a_i\rho}{e^{i\tau}}\right)\mathrm{Re}\left[\sum_1^{\infty} \frac{1}{(\nu-1)!}\, p^{(\nu)}(z_i,z)\, r^{\nu-1}e^{i\nu\tau}\right] d\tau$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}\left[G(z_i,z) + \mathrm{Re}\left[\sum_1^{\infty}\frac{1}{\nu!}p^{(\nu)}(z_i,z)r^{\nu}e^{i\nu\tau}\right]\mathrm{Re}\left[p^{*\prime}(z_i,\zeta^*)\,\frac{a_i\rho}{re^{i\tau}}\right]\right] d\tau$$

Let $\rho a_i p^{*\prime}(z_i,\zeta^*) = Ae^{i\mu}$ $\quad p'(z_i,z) = Be^{i\sigma}$ and remembering the orthogonality of trigonometric functions:

$$S_i = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(Ae^{i\mu}\cdot e^{-i\theta}\;\mathrm{Re}\;Be^{i\sigma}e^{i\theta}\right) d\theta + O(\rho^2)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2AB\cos(\mu-\theta)\cos(\sigma+\theta)\, d\theta + O(\rho^2)$$

$$= \frac{AB}{2\pi}\int_0^{2\pi}\left[\cos(\sigma+\mu) + \cos(\mu-\sigma-2\theta)\right] d\theta + O(\rho^2)$$

$$= AB \cos (\sigma + \mu + O(\rho^2) = \text{Re } [\rho a_i p^{*\prime} (z_i, \zeta^*) p'(z_i, z)]$$

Explicitly:

$$G^*(z^*, \zeta^*) = G(z, \zeta) + \sum_{i=0}^{m} \text{Re } \left(\rho a_i p^{*\prime}(z_i, \zeta^*) p'(z_i, z) \right) + O(\rho^2) \qquad (5.8)$$

but

$$G^*(z^*, \zeta^*) = G^*[z + \rho q(z), \zeta + \rho q(\zeta)]$$

and since $p^{*\prime}(u,v) = p'(u,v) + O(\rho)$ we finally obtain the formula:

$$G^*(z, \zeta) = G(z, \zeta)$$

$$+ \text{Re } \rho \left[\sum_{i=0}^{m} a_i p'(z_i, \zeta) \, p'(z_i, z) - q(z) p'(z, \zeta) - q(\zeta) p'(\zeta, z) \right]$$

$$+ O(\rho^2) \qquad (5.7)$$

We note that $G^*(z, \zeta)$ is expressed in terms of the original Green's function and its derivative for the variable argument or evaluated at the fixed interior points z. Therefore, the method is called the "method of interior variation". The new Green's function is obtained without any quadrature along the boundary by means of the values of the original Green's function at some fixed interior points. It is interesting to note the differentiation of the terms in the modification factor: a magnitude factor and a shape factor. In many applications Formula (5.6) will be most useful since we are concerned with the value of the Green's function on the boundary.

As an example, consider the following simple case. Let $z^* = z + \epsilon/(z - z_0)$. Then we find that

$$G^*(z, \zeta) - G(z, \zeta) = \text{Re} \left[\epsilon \left(p'(z_0, z) p'(z_0, \zeta) - p' \frac{p'(z, \zeta)}{z - z_0} - p' \frac{(\zeta, z)}{\zeta - z_0} \right) \right] + O(\epsilon^2)$$

5.3 Variation of the Green-Neumann function. We begin with the derivation of Hadamard's analogous formula. Suppose that the boundary C is divided into two parts such that $G(z,\zeta)$ is zero on C_1 and $\partial G(z,\zeta)/\partial n = 0$ on C_2. Consider again the difference $\Delta(z,\zeta) = G^*(z,\zeta) - G(z,\zeta)$. Green's identity then yields

$$\Delta(z,\zeta) = \int_{C_1^* + C_2} \Delta(t,\zeta) \frac{\partial G^*(t,z)}{\partial \nu_t} - G^*(t,z) \frac{\partial \Delta(t,\zeta)}{\partial \nu_t} \, ds_t$$

But on C_1^*, $\Delta(t,\zeta) = - G(t,\zeta)$ and $G^*(t,z) = 0$ while on C_2, $\partial G^*(t,z)/\partial \nu_t = 0$ and $\partial \Delta(t,\zeta)/\partial \nu_t = 0$. Hence, we find that

$$G^*(z,\zeta) - G(z,\zeta) = \int_{C_1^*} - G(t,\zeta) \frac{\partial G^*(t,z)}{\partial n} \, ds_t$$

to the second order in ϵ, and since $G(t,\zeta) = 0$ on C_1

$$G^*(z,\zeta) - G(z,\zeta) = \int_{C_1 + C_1^*} - G(t,\zeta) \frac{\partial G(t,z)}{\partial \nu_t} \, ds_t$$

which by (5.2d) gives

$$G^*(z,\zeta) - G(z,\zeta) = - \int_{C_1} \frac{\partial G(t,z)}{\partial \nu_t} \frac{\partial G(t,\zeta)}{\partial \nu_t} \, \delta \nu_t \, ds_t$$

$$= - \int_{C_1} \frac{\partial G(t,z)}{\partial \nu_t} \frac{\partial G(t,\zeta)}{\partial \nu_t} \, \epsilon v(s) \, ds_t \tag{5.9}$$

We now undertake to apply this result to the rectangular channel. In the ζ-plane the Green-Neumann is so defined that on the ξ-axis $G = 0$ and

$$\frac{\partial G}{\partial \eta} = \frac{\partial G}{\partial \nu_t} = -2\eta_0\left[\frac{1}{(\zeta-\zeta_0)^2+\eta_0^2}+\frac{1}{(\zeta+\zeta_0)^2+\eta_0^2}\right]$$

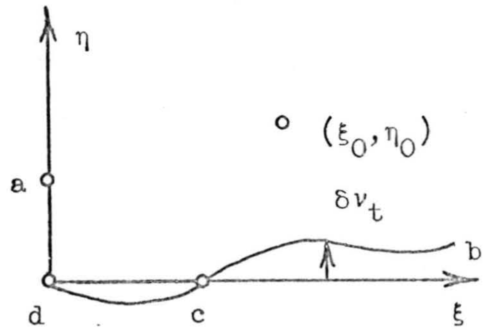


Figure 5.2

$$G^*(\zeta,\zeta_0) - G(\zeta,\zeta_0) = -\int_0^\infty 4\,\eta\eta_0\left[\frac{1}{(t-\xi_0)^2+\eta_0^2}+\frac{1}{(t+\xi_0)^2+\eta_0^2}\right] \times$$

$$\left[\frac{1}{(t-\xi)^2+\eta^2}+\frac{1}{(t+\xi)^2+\eta^2}\right]\epsilon v(t)dt \qquad (5.10$$

The problem is reduced to the integration of the integral of type

$$\int_0^\infty \frac{\eta\eta_0\; v(t)}{[(t-\xi_0)^2+\eta_0^2][(t-\xi)^2+\eta^2]}\,dt = \int_0^\infty \frac{v(t)\;dr}{(t^2-2\xi_0 t+r_0^2)(t^2-2\xi t+r^2)}$$

To determine the potential in the ζ^*-plane we only need the value of $G^*(\zeta,\zeta_0)$ on the original boundary C, if we linearize the problem. This value is given by the formula:

$$G^*(\xi,\zeta_0) = -\lim_{\eta\to 0}\int_0^\infty \frac{4\eta\eta_0\epsilon v(t)}{(t-\xi)^2+\eta^2}\left[\frac{1}{(\xi-\xi_0)^2+\eta_0^2}+\frac{1}{(\xi+\xi_0)^2+\eta_0^2}\right]dt$$

(5.11)

$$G^*(\xi,\zeta_0) = -\lim_{\eta\to 0} 4\epsilon\eta_0\, v(\xi)\left(\frac{1}{(\xi-\xi_0)^2+\eta_0^2}+\frac{1}{(\xi+\xi_0)^2+\eta_0^2}\right)\int_0^\infty \frac{\eta}{(t-\xi)^2+\eta^2}\,dt$$

$$= -4\epsilon\eta_0\, v(\xi)\,\pi\left(\frac{1}{(\xi-\xi_0)^2+\eta_0^2}+\frac{1}{(\xi+\xi_0)^2+\eta_0^2}\right)$$

The Green-Neumann function for the new domain, evaluated on the ξ-axis boundary of the old domain, is (provided $\eta_0 \neq 0$):

$$G^*(\xi_1 0,\xi_{01}\eta_0) = -4\pi\epsilon\, v(\xi)\left(\frac{\eta_0}{(\xi-\xi_0)^2+\eta_0^2}+\frac{\eta_0}{(\xi+\xi_0)^2+\eta_0^2}\right)$$

$$= -4\pi\,\delta_\nu(\xi)\eta_0\left(\frac{1}{(\xi-\xi_0)^2+\eta_0^2}+\frac{1}{(\xi+\xi_0)^2+\eta_0^2}\right) \qquad (5.12)$$

and similarly, if $\eta \neq 0$,

$$G^*(\xi,\eta;\xi_0,0) = -4\pi\,\delta_\nu\,(\xi_0)\eta\left(\frac{1}{(\xi-\xi_0)^2+\eta^2}+\frac{1}{(\xi+\xi_0)^2+\eta^2}\right) \qquad (5.13)$$

We next consider the $V_{\rho q}$ variation for the Green-Neumann function. Formula (5.5) can be rewritten in a slightly different notation

$$\zeta^* = \zeta + \epsilon\,\Theta(\zeta) \qquad (5.14)$$

For the particular problem of the rectangular channel, this transformation must conserve the η-axis for $\eta > 0$. Let

$$\xi^* + i\eta^* = \xi + i\eta + \epsilon[A(\xi,\eta) + iB(\xi,\eta)]$$

The condition is $A(0,\eta) = 0$ for $\eta > 0$. A harmonic function $A(\xi,\eta)$ which is odd in ξ will satisfy this condition, say

$$A(\xi,\eta) = \int_0^\infty B(\omega)\, e^{-\omega\eta} \sin \omega\xi \, d\omega$$

$$= B(\omega)\, e^{-\omega\eta} \sin \omega\xi$$

From the Cauchy-Riemann conditions $\partial A/\partial\xi = \partial B/\partial\eta$ and $\partial A/\partial\eta = -\,\partial B/\partial\xi$. Hence we have

$$\frac{\partial B}{\partial\eta} = \omega B(\omega) \cos \omega\xi e^{-\omega\eta} \qquad \frac{\partial B}{\partial\xi} = \omega B(\omega) \sin \omega\xi \; e^{-\omega\eta}$$

or

$$B(\xi,\eta) = -\, B(\omega)\, e^{-\omega\eta} \cos \omega\xi$$

This leads to

$$\Theta(\zeta) = \int_0^\infty B(\omega)\, e^{-\omega\eta} \,(\sin \omega\Omega - i \cos \omega\xi)\, d\omega$$

$$= i \int_0^\infty C(\omega)\, e^{i\omega\zeta} \, d\omega$$

$$= i \int_0^\infty C(\omega)\, e^{i\omega\zeta} \, d\omega$$

where $C(\omega)$ is a real function. The transformation

$$\zeta^* = \zeta + \epsilon i \int_0^\infty C(\omega)\, e^{i\omega\zeta} \, d\omega \tag{5.15}$$

would satisfy the required conditions. Conservation of the origin would require $\int_0^\infty C(\omega)\, d\omega = 0.$

5.4 Interior variation of the Green-Neumann function. Here $\Theta(\zeta)$ should be meromorphic and have a pole, say at $\zeta = \zeta_p$ with coordinates (ξ_p, η_p). Such a function can be easily constructed and the transformation explicitly written by use of an image at $(-\xi_p, \eta_p)$. The result is

$$\zeta^* = \zeta + \epsilon \left(\frac{1}{\zeta - \zeta_p} + \frac{1}{\zeta + \zeta_p} + i \int_0^\infty C(\omega)\, e^{i\omega\zeta}\, d\omega \right) \tag{5.16}$$

We now proceed with the derivation of the basic formulas. In a process similar to that followed for the usual Green's function, we shall start from the function: $\Delta(z,\zeta) = G^*(z^*,\zeta^*) - G(z,\zeta)$ with z and ζ in

$D - \sum \tilde{K}_i$ and suppose all the z_i are situated in D. Then as before

suppose that $G(z,\zeta) = 0$ on C_1 and $\partial G(z,\zeta)\partial\nu_t = 0$ on C_2. Use of Green's Formula (5.2b) yields:

$$\Delta(z,\zeta) = \frac{1}{2\pi} \int_P \left(\Delta(t,\zeta) \frac{\partial G(t,z)}{\partial \nu_t} - G(t,z) \frac{\partial\, \Delta(t,\zeta)}{\partial\, \nu_t} \right) ds_t$$

Here P is the boundary of $D - \sum \tilde{K}_i$ i.e., $C_1 + C_2 + \sum K_i$.

Then on C_1 $G(t,z) = 0$ and $\Delta(t,z) = 0$; while on C_2, $\partial G(t,z)/\partial\nu_t = 0$, $\partial G^*(t,z)/\partial\nu_t = 0$ and $\partial\Delta(t,z)/\partial\nu_t = 0$. Hence

$$\Delta(z,\zeta) = \frac{1}{2\pi} \int_{\sum K_i} \left(\Delta(t,\zeta) \frac{\partial G(t,z)}{\partial \nu_t} - G(t,z) \frac{\partial\Delta(t,\zeta)}{\partial \nu_t} \right) ds_t$$

and ultimately, if terms of order ϵ^2 are neglected

$$G^*(\zeta^*,\zeta_0^*) = G(\zeta,\zeta_0) + \sum_{i=0}^{m} \mathrm{Re}\, \epsilon\, p'(\zeta_{p_i},\zeta)\, p'(\zeta_{p_i},\zeta_0) \tag{5.17}$$

or

$$G^*(\zeta,\zeta_0) = G(\zeta,\zeta_0) + \sum_{i=0}^{m} \operatorname{Re} \epsilon \Big[p'(\zeta_{pi},\zeta)\, p'(\zeta_{pi},\zeta_0) - p'(\zeta,\zeta_0)\, \Theta(\zeta)$$

$$- p'(\zeta_0,\zeta)\, \Theta(\zeta_0) \Big] \qquad (5.18)$$

Conservation of the origin of coordinates in the transformation (5.15) will be satisfied under the condition

$$2 \sum_{i=0}^{m} \frac{\eta_{pi}}{\xi_{pi}^2 + \eta_{pi}^2} + \int_0^\infty C(\omega)\, d\omega = 0 \qquad (5.19)$$

Since

$$\sum_{i=0}^{m} \frac{\eta_{pi}}{\xi_{pi}^2 + \eta_{pi}^2} > 0 \qquad \int_0^\infty C(\omega)\, d\omega < 0$$

However, we probably would like sometimes a modification of the boundary without the alternating term, still conserving the origin. According to Formula (5.19) it seems impossible. This is illusory for we have restricted too much the class of admissible functions $\Theta(\zeta)$. Let

$$\zeta^* = \zeta + \epsilon \Bigg[\sum_{i=0}^{m} \frac{1}{\zeta - \zeta_{pi}} + \frac{1}{\zeta + \bar{\zeta}_{pi}} - \sum_{j=0}^{n} \frac{1}{(\zeta - \zeta_{pj})} + \frac{1}{\zeta + \bar{\zeta}_{pj}}$$

$$+ i \int_0^\infty C(\omega)\, e^{i\omega\zeta}\, d\omega \Bigg] \qquad (5.20)$$

Conservation of the origin is then expressed as

$$2 \sum_{i=0}^{m} \frac{\eta_{pi}}{\xi_{pi}^2 + \eta_{pi}^2} - 2 \sum_{j=0}^{n} \frac{\eta_{pj}}{\xi_{pj}^2 + \eta_{pj}^2} + \int_0^\infty C(\omega)\, d\omega = 0 \qquad (5.21)$$

Any sum of two points on a circle passing through the origin with center on the η-axis will satisfy the condition (5.21). The simplest transformation will be

$$\zeta^* = \zeta + \epsilon \left[\frac{1}{\zeta - \zeta_p} + \frac{1}{\zeta + \bar{\zeta}_p} - \frac{1}{\zeta - \zeta_q} - \frac{1}{\zeta + \bar{\zeta}_q} \right] \qquad (5.22)$$

where ζ_p and ζ_q are related by the formula $\eta_p/r_p^{\,2} = \eta_q/r_q^{\,2}$. For simplicity's sake we can even choose $\xi_p = \xi_q$. Then if R is the radius of the circle we have

$$\frac{\eta_p}{r_p^{\,2}} = \frac{\eta_q}{r_q^{\,2}} = \frac{1}{2R}$$

which corresponds to a well-known geometrical property for a right triangle.

5.5 Shape of the modified boundary. We have

$$\zeta^* = \zeta + \epsilon \left[\frac{1}{(\xi - \xi_p) + i(\eta - \eta_p)} + \frac{1}{(\xi + \xi_p) + i(\eta - \eta_p)} \right.$$

$$\left. - \frac{1}{(\xi - \xi_q) + i(\eta - \eta_q)} - \frac{1}{(\xi + \xi_q) + i(\eta - \eta_q)} \right]$$

$$\zeta^* = \zeta + \epsilon \left[\frac{(\xi - \xi_p) - i(\eta - \eta_p)}{(\xi - \xi_p)^2 + (\eta - \eta_p)^2} + \frac{(\xi + \xi_p) - i(\eta - \eta_p)}{(\xi + \xi_p)^2 + (\eta - \eta_p)^2} \right.$$

$$\left. - \frac{(\xi - \xi_q) - i(\eta - \eta_q)}{(\xi - \xi_q)^2 + (\eta - \eta_q)^2} - \frac{(\xi + \xi_q) - i(\eta - \eta_q)}{(\xi + \xi_q)^2 + (\eta - \eta_q)^2} \right]$$

This transformation leaves the η-axis unaltered, but the ξ-axis ($\eta = 0$) is modified. Then

$$\xi^* + i\eta^* = \xi + \epsilon \left[\frac{(\xi - \xi_p) + i\eta_p}{(\xi - \xi_p)^2 + \eta_p^2} + \frac{(\xi + \xi_p) + i\eta_p}{(\xi + \xi_p)^2 + \eta_p^2} - \frac{(\xi - \xi_p) + i\eta_q}{(\xi - \xi_p)^2 + \eta_q^2} \right.$$

$$\left. - \frac{(\xi + \xi_p) + i\eta_q}{(\xi + \xi_p)^2 + \eta_q^2} \right]$$

When separated into real and imaginary parts this becomes

$$\xi^*_{pq} = \xi + \epsilon \left[\frac{(\xi - \xi_p)}{(\xi - \xi_p)^2 + \eta_p^2} + \frac{(\xi + \xi_p)}{(\xi + \xi_p)^2 + \eta_p^2} - \frac{(\xi - \xi_p)}{(\xi - \xi_p)^2 + \eta_q^2} \right.$$

$$\left. - \frac{(\xi + \xi_p)}{(\xi + \xi_p)^2 + \eta_q^2} \right] \tag{5.23}$$

$$\eta^*_{pq} = \epsilon \left[\frac{\eta_p}{(\xi - \xi_p)^2 + \eta_p^2} + \frac{\eta_p}{(\xi + \xi_p)^2 + \eta_p^2} - \frac{\eta_q}{(\xi - \xi_p) + \eta_q^2} \right.$$

$$\left. - \frac{\eta_q}{(\xi + \xi_p)^2 + \eta_q^2} \right]$$

As a numerical example let $\xi_p = 1$ and $\eta_q = 1/2$. Consequently

$$\xi^* = \xi + \epsilon \left[\frac{\xi - 1}{(\xi - 1)^2 + 4} + \frac{\xi + 1}{(\xi + 1)^2 + 4} - \frac{4(\xi - 1)}{4(\xi - 1)^2 + 1} - \frac{4(\xi - 1)}{4(\xi + 1)^2 + 1} \right]$$

$$\eta^* = 2\epsilon \left[\frac{1}{(\xi - 1)^2 + 4} + \frac{1}{(\xi + 1)^2 + 4} - \frac{1}{4(\xi + 1)^2 + 1} - \frac{1}{4(\xi + 1)^2 + 1} \right]$$

TABLE 5.1

ξ	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.50	3.00	4.00
$\frac{\xi^*-\xi}{\epsilon}$	0.2940	0.5223	0.4556	-0.2204	-0.9141	-1.0233	-0.8728	-0.6933	-0.4247	-0.2668	-0.1188
$\frac{\eta^*}{2\epsilon}$	-0.0466	-0.2047	-0.4880	-0.4906	-0.4906	-0.2056	-0.0340	0.0499	0.1015	0.1008	0.0745
					$\epsilon = 0.2$						
ξ	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.50	3.00	4.00
ξ^*	0.31	0.60	0.84	0.96	1.07	1.30	1.58	1.86	2.42	2.95	3.98
η^*	-0.02	-0.08	-0.20	-0.27	-0.20	-0.08	-0.01	0.02	0.04	0.04	0.03

Numerical values may now be tabulated as in Table 5.1. A superposition of functions η^*_{pq}, allowing a fair number of parameters, enables one to fit any curve $\eta^*(\xi)$, at least approximately as

$$\eta^*(\xi) = \sum_{p,q} \eta^*_{pq}(\xi)$$

6. SEEPAGE FROM A NEARLY RECTANGULAR CHANNEL

In Section 4 we had derived the Green-Neumann function for the ζ-plane, but ultimately solved the potential problem by a method of images, a simpler method for such simple boundary shape and boundary conditions. In the last section we studied the Green-Neumann function for the ζ-domain. Such a study led us to discover the mapping function which conserves the origin and the imaginary axis and modifies the real axis in an arbitrary fashion. The Green-Neumann function method would be the only available method if, together with the modification of the boundary curve, the boundary conditions were modified, too. This is not the case and the problem can be solved by conformal mapping transformations, now that the perturbation mapping is known.

6.1 Perturbation mappings and assumptions. First of all, we assume that the symmetry of the problem is conserved, i.e., ad is a streamline and v_d remains finite as indicated in Figure 6.1. The new boundary curve is tangent to dc at point d. Second, we assume that the velocity at b remains zero. The new boundary curve is therefore tangent to bc at b. Third, we assume that points b and b* and also d and d* coincide.

Using the result of Equation (3.7) we have the relations between the potential W^* and ζ^*

$$W^*(\zeta^*) = -\frac{iQ^*}{\pi} \text{arc tan } \zeta^* \qquad (6.1)$$

$$W^* = \zeta^* + i \int_0^\infty C(\omega)\, e^{i\omega\zeta^*}\, d\omega \qquad (6.2)$$

Introduce the following definitions for f_1 and f_2:

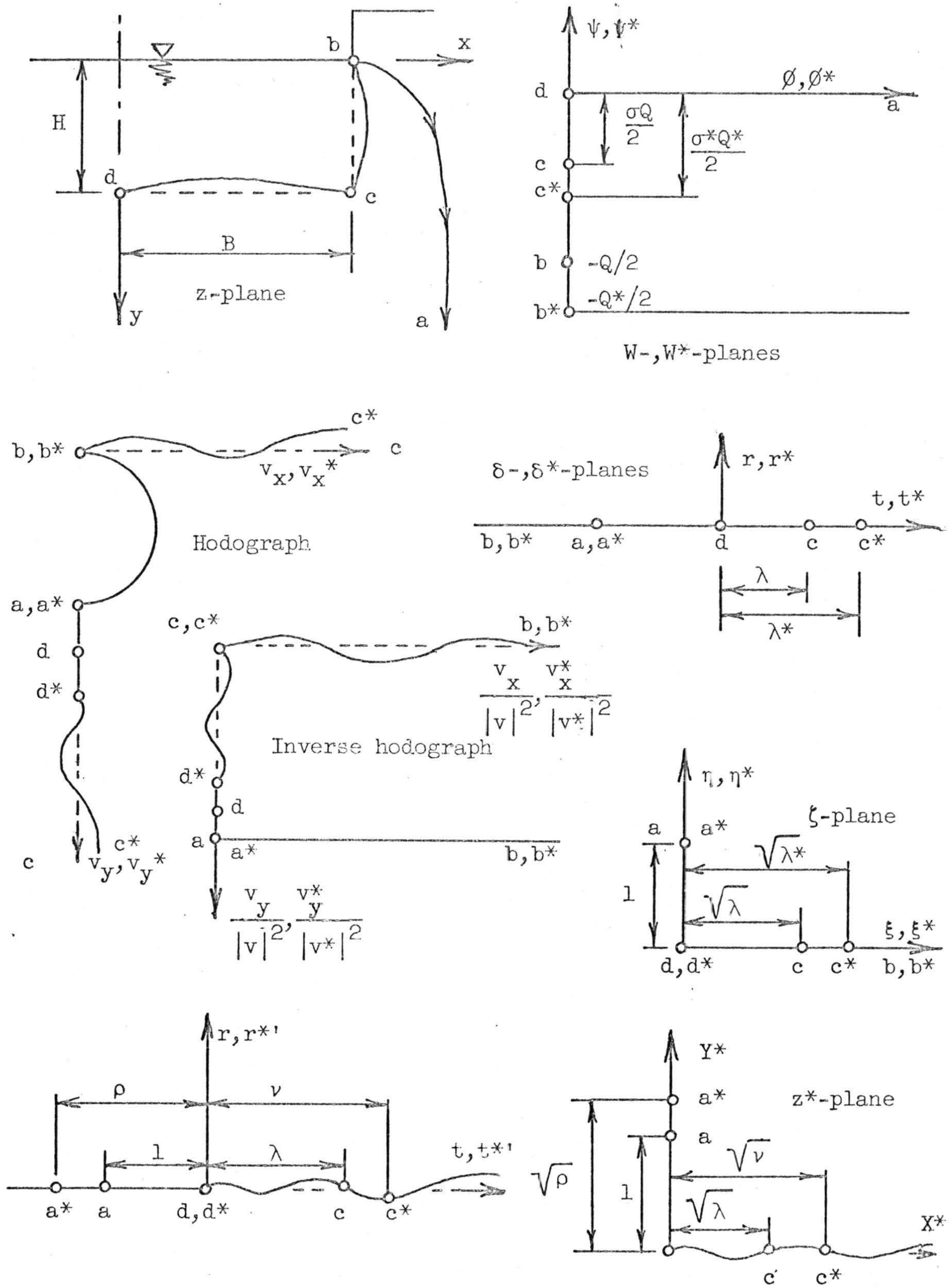

z-plane
H
B
W-, W*-planes
σQ/2
σ*Q*/2
-Q/2
-Q*/2
Hodograph
δ-, δ*-planes
Inverse hodograph
ζ-plane
z*-plane

Figure 6.1

$$\int_0^\infty C(\omega) \cos \omega\zeta^* \, d\omega = f_1(\zeta^*)$$

$$\int_0^\infty C(\omega) \sin \omega\zeta^* \, d\omega = f_2(\zeta^*) \tag{6.3}$$

$$\int_0^\infty C(\omega) \, e^{i\omega\zeta^*} \, d\omega = f(\zeta^*) = f_1(\zeta^*) + if_2(\zeta^*)$$

Conservation of the origin in the perturbation mapping is expressed through the condition

$$\int_0^\infty C(\omega) \, d\omega = 0 \qquad \text{or} \qquad f_1(0) = 0 \tag{6.4}$$

After perturbation the point c^* must remain on the real axis. Now

$$\begin{aligned} \sqrt{\nu} &= \sqrt{\lambda^*} + i\epsilon f\,(\sqrt{\lambda^*}) \\ &= \sqrt{\lambda^*} + i\epsilon f_1\,(\sqrt{\lambda^*}) - \epsilon f_2(\sqrt{\lambda^*}) \\ &= \sqrt{\lambda^*} - \epsilon f_2(\sqrt{\lambda^*}) \end{aligned} \tag{6.5}$$

Consequently the condition for c^* to remain on the real axis is

$$f_1(\sqrt{\lambda^*}) = 0 \tag{6.6}$$

Similarly we find:

$$\sqrt{\rho} = 1 + \epsilon f(i) \tag{6.7}$$

As shown in Figure 6.1 the next transformation is $Z^{*2} = \delta'^*$. From Equation (3.6) we find the relation between the $v^*/|v^*|^2$-plane and the Z^*-plane to be

$$v^*/|v^*|^2 = \frac{1}{\pi} \text{ arc cosh } \frac{2}{\rho + \nu}\left(Z^{*2} + \frac{\rho - \nu}{2}\right) \tag{6.8}$$

Relation (6.5) can be rewritten

$$\nu = \lambda^* - 2\epsilon\sqrt{\lambda^*}\; f_2(\sqrt{\lambda^*}) + O(\epsilon^2)$$

$$= \lambda^* - 2\epsilon\sqrt{\lambda}\; f_2(\sqrt{\lambda}) + O(\epsilon^2)$$

Let $\lambda^* = \lambda + \epsilon\alpha(C(\omega)) = \lambda + \epsilon\;\alpha\;(f)$ where α is a scalar functional of the domain boundary, i.e., of $C(\omega)$ or $f(\zeta^*)$. Hence

$$\nu = \lambda + \epsilon\left[\alpha - 2\sqrt{\lambda}\, f_2(\sqrt{\lambda})\right] \tag{6.9}$$

Similarly

$$\rho = 1 + 2\epsilon\; f(i) \tag{6.10}$$

The discharge Q^* will be modified according to

$$Q^* = Q(1 + \beta\;\epsilon) \tag{6.11}$$

where $\beta = \beta(f)$ and similarly to (3.10) we have the physical plane

$$z^*(\zeta^*) = \frac{1}{k}\int^{\zeta^*} \frac{1}{\pi} \text{ arc cosh } \frac{2}{\rho + \nu}\left[\zeta^{*2} + \frac{\rho - \nu}{2} + 2i\epsilon\zeta^*\; f(\zeta^*)\right] x$$

$$\left(- \frac{iQ^*}{\pi}\,\frac{d\zeta^*}{1 + \zeta^{*2}}\right)$$

Let us develop the arc cosh term in a series of powers of ϵ and drop higher order terms. Since there is no ambiguity we shall now write ζ instead of ζ^*. The end result is:

$$z^*(\zeta) = z(\zeta) - \frac{iQ}{k\pi^2}\,\epsilon \int^{\zeta} \beta \text{ arc cosh } \frac{2\zeta^2 + 1 - \lambda}{1 + \lambda} \quad \frac{d\zeta}{1 + \zeta^2}$$

$$- \frac{iQ}{k\pi^2}\,\epsilon \int^{\zeta} \frac{2i\ \zeta\ f(\zeta)}{[(\zeta^2 - \lambda)(\zeta^2 + 1)]^{1/2}} \quad \frac{d\zeta}{1 + \zeta^2}$$

$$+ \frac{iQ}{k\pi^2}\,\epsilon \int^{\zeta} \frac{1}{1 + \lambda} \quad \frac{[\alpha + 2f(i) - 2\sqrt{\lambda}\ f_2(\sqrt{\lambda})]}{[(\zeta^2 - \lambda)(\zeta^2 + 1)]^{1/2}} \quad \frac{\zeta^2 d\zeta}{1 + \zeta^2}$$

$$- \frac{iQ}{k\pi^2}\,\epsilon \int^{\zeta} \frac{[2\lambda\ f(i) + 2\sqrt{\lambda}\ f_2(\sqrt{\lambda}) - \alpha(1 - \lambda)/2]}{[(\zeta^2 - \lambda)(\zeta^2 + 1)]^{1/2}} \quad \frac{d\zeta}{1 + \zeta^2} \tag{6.12}$$

6.2 Hadamard's normal variation. In writing Hadamard's formula two cases must be distinguished. First, on the channel bottom dc we have $0 \le \zeta \le \sqrt{\lambda}$ and on dc

$$z(\zeta) = z(\zeta_0) + (\zeta - \zeta_0)\ z'(\zeta_0)$$

$$z^*(\zeta) - z(\zeta) = z^*(\zeta) - z(\zeta_0) + (\zeta_0 - \zeta)\ z'(\zeta_0)$$

$$= - i\ \epsilon\ \delta(\zeta) + \text{real terms}$$

Equating the imaginary parts we obtain

$$- i\ \epsilon\ \delta(\zeta) = \frac{2Q}{k\pi^2}\,\epsilon \int^{\zeta} \frac{- i\zeta f_1(\zeta)}{[(\lambda - \zeta^2)(\zeta^2 + 1)]^{1/2}} \quad \frac{d\zeta}{1 + \zeta^2}$$

$$\delta(\zeta) = \frac{2Q}{k\pi^2} \int^{\zeta} \frac{\zeta f_1(\zeta)}{[(\lambda - \zeta^2)(\zeta^1 + 1)]^{1/2}} \quad \frac{d\zeta}{1 + \zeta^2}$$

[Formula (6.13) is continued on the following page.]

$$\delta'(\zeta) = \frac{2Q}{k\pi^2} \frac{f_1(\zeta)}{[(\lambda - \zeta^2)(\zeta^2 + 1)]^{1/2}} \frac{\zeta}{1 + \zeta^2}$$

$$f_1(\zeta) = \frac{k\pi^2}{2Q} \frac{\sqrt{(\lambda - \zeta^2)(\zeta^2 + 1)} \cdot (1 + \zeta^2) \cdot \delta'(\zeta)}{\zeta} \qquad (6.13)$$

Second, on the channel side bc, we have $\zeta > \sqrt{\lambda}$

$$z(\zeta) = z(\zeta_0) + (\zeta - \zeta_0) z'(\zeta_0)$$

$$z^*(\zeta) - z(\zeta) = z^*(\zeta) - z^*(\zeta_0) + (\zeta_0 - \zeta)\, z'(\zeta_0)$$

$$= - \epsilon\, \delta(\zeta) + \text{imaginary terms}$$

Equating the real parts we obtain

$$- \epsilon\, \delta(\zeta) = \frac{2Q}{k\pi^2} \epsilon \int^{\zeta} \frac{\zeta\, f_1(\zeta)}{(\zeta^2 - \lambda)(\zeta^2 + 1)} \frac{d\zeta}{1 + \zeta^2}$$

$$f_1(\zeta) = \frac{k\pi^2}{2Q} \cdot \frac{1 + \zeta^2}{\zeta} \sqrt{(\zeta^2 - \lambda)(\zeta^2 + 1)} \quad \delta'(\zeta) \qquad (6.14)$$

6.3 Boundary conditions. In addition to the above results, we must take proper account of the boundary conditions. First, conservation of the origin implies that $f_1(0) = 0$ or that

$$\lim_{\xi \to 0} \frac{\delta'(\xi)}{\xi} = 0$$

Second, since the point c* is on the real axis of ζ^* we have to the order ϵ^2:

$$\delta'(\sqrt{\lambda}) = 0 \rightarrow \text{or} \rightarrow \delta'(\sqrt{\lambda}) = 0$$

Third, since the new boundary curve is tangent to dc at point d, we must have

$$\frac{d\,\delta(\xi)}{d\xi} \quad \frac{d \operatorname{Re} z^*(\xi)}{d\xi} = 0$$

Hence, $\delta'(\xi)/z'(\xi) = 0$ for $\xi = 0$. Then

$$\lim_{\xi \to 0} \left\{ \frac{\dfrac{2Q}{k\pi^2} \quad \dfrac{f_1(\xi)\,\xi}{[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}} \quad \dfrac{1}{1 + \xi^2}}{\dfrac{-iQ}{k\pi^2} \operatorname{arc\,cosh} \dfrac{2\xi^2 + 1 - \lambda}{\lambda + 1} \quad \dfrac{1}{1 + \xi^2}} \right\} = 0$$

So finally

$$\lim_{\xi \to 0} \xi f_1(\xi) = 0$$

Since $f_1(0) = 0$ this is always true.

Fourth, the new boundary curve is tangent to bc at b, and so

$$\lim_{\xi \to \infty} \left[\frac{d\,\delta(\xi)}{d\xi} \quad \frac{d \operatorname{Im} z^*(\xi)}{d\xi} \right] = 0$$

$$\lim_{\xi \to \infty} \left[\frac{\xi\, f_1(\xi)}{(1 + \xi^2)\, [(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}} \Bigg/ \frac{\operatorname{arc\,cosh}}{(1 + \xi^2)} \quad \frac{2\xi^2 + 1 - \lambda}{1 + \lambda} \right] = 0$$

$$\lim_{\xi \to \infty} \left[\frac{f_1(\xi)}{\xi} \cdot \frac{1}{\log \xi} \right] = 0$$

or

$$f_1(\xi) = O(\xi \log \xi) \qquad \text{for} \qquad \xi = \infty.$$

Fifth, the points d and d* coincide. This leads to the condition that

$$\delta(0) = 0$$

Sixth, the points b and b* coincide. But

$$\delta(\infty) - \delta(\sqrt{\lambda}) = -\frac{2Q}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty} \frac{\xi\, f_1(\xi)}{[(\xi^2-\lambda)(\xi^2+1)]^{1/2}} \frac{d\xi}{1+\xi^2}$$

This integral makes sense if, as $\xi \to \infty$, the ratio $f_1(\xi)/\xi^3$ goes to zero at least as fast as $1/\xi$. This implies that $f_1(\xi) = O(\xi^2)$ for $\xi \to \infty$ and for $\xi \to \sqrt{\lambda}$

$$\frac{f_1(\xi)}{(\xi^2-\lambda)^{1/2}} = O\left(\frac{1}{(\xi - \sqrt{\lambda})}\right)$$

So,

$$f_1(\xi) = O(\xi^2) \qquad \text{when} \qquad \xi \to \infty$$

$$f_1(\xi) = O\left(\frac{1}{(\xi - \sqrt{\lambda})^{1/2}}\right) \text{when} \qquad \xi \to \sqrt{\lambda}$$

The fifth condition can be written

$$\delta(\sqrt{\lambda}) - \delta(0) = \frac{2Q}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{\xi\, f_1(\xi)}{[\lambda - \xi^2)(\xi^1 + 1)]^{1/2}} \qquad d\xi$$

$$v = \frac{2Q}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{\xi\, f_1(\xi)}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} \qquad f_1(\xi) = O\left(\frac{1}{(\sqrt{\lambda}-\xi)^{1/2}}\right) \text{ as } \xi \to \sqrt{\lambda}.$$

Summary:

$$\left.\begin{array}{lll}
\lim\limits_{\xi \to 0} \dfrac{\delta'(\xi)}{\xi} = 0 & \text{or} & f_1(0) = 0 \\[2ex]
\delta'(\sqrt{\lambda}) = 0 & \text{or} & f_1(\sqrt{\lambda}) = 0 \\[2ex]
f_1(\xi) = O(\xi \log \xi) & \text{for} & \xi \to \infty
\end{array}\right\} \qquad (6.15)$$

6.4 <u>Evaluation of the discharge.</u> In this section we will derive the formulas necessary for calculating the discharge. Referring to Figure 6.2, we see that at the point d, $z^*_0(0) = 0 + iH$ and $z_0(0) = iH$. Also at points c^* and c we find $z^*_{c^*}(\sqrt{\lambda}) = B - h\epsilon + i(H - \nu\epsilon)$ and $z_c(\sqrt{\lambda}) = B + iH_0$.

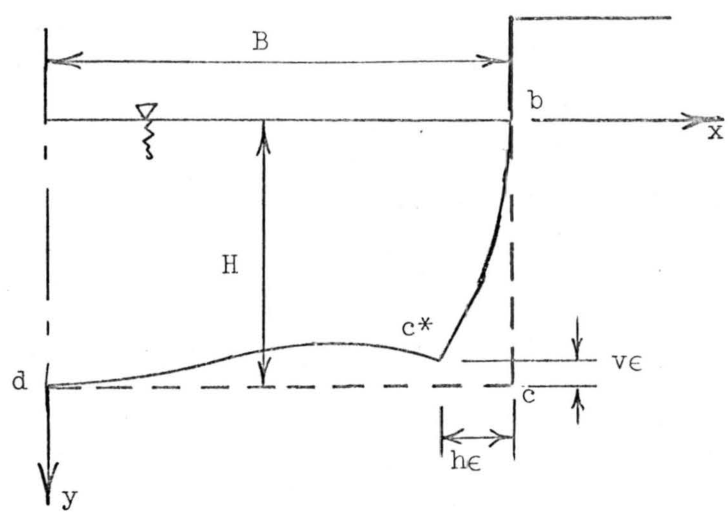


Figure 6.2

From (6.12) there results:

$$z^*(\sqrt{\lambda}) - z^*(0) = z(\sqrt{\lambda}) - z(0) - \frac{i\,\delta Q}{k\pi^2}\int_0^{\sqrt{\lambda}} i \text{ arc cos } \frac{2\xi^2 + 1 - \lambda}{1 + \lambda}\,\frac{d\xi}{1 + \xi^2}$$

$$- \frac{2iQ\epsilon}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{\xi\, f(\xi)}{[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}}\,\frac{d\xi}{1 + \xi^2}$$

$$+ \frac{Q\epsilon}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{1}{1 + \lambda}\,\frac{[\alpha + 2f(i) - 2\sqrt{\lambda}\, f_2(\sqrt{\lambda})]}{[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}} \cdot \frac{\xi^2 d\xi}{1 + \xi^2}$$

$$- \frac{Q\epsilon}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{1}{1 + \lambda}\,\frac{[2\lambda f(i) + 2\sqrt{\lambda}\, f_2(\sqrt{\lambda}) - \alpha(\frac{1 - \lambda}{2})]}{[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}}\,\frac{d\xi}{1 + \xi^2}$$

Let us introduce the notations:

$$\left[z^*(\sqrt{\lambda}) - z_0^*(0)\right] - \left[z(\sqrt{\lambda}) - z(0)\right] = -(h + iv)\epsilon$$

$$I(\lambda) = \int_0^{\sqrt{\lambda}} \text{arc cos } \frac{2\xi^2 + 1 - \lambda}{1 + \lambda}\,\frac{d\xi}{1 + \xi^2}$$

Then separation into real and imaginary parts results in

$$-h\epsilon = \frac{\delta Q}{k\pi^2} I(\lambda) + 2\,\frac{Q\epsilon}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{\xi\, f_2(\xi)}{[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}}\,\frac{d\xi}{1 + \xi^2}$$

$$+ \frac{2Q\epsilon}{k\pi^2}\,\frac{[(f(i) - \sqrt{\lambda}\, f_2(\sqrt{\lambda})]}{1 + \lambda}\int_0 \frac{d\xi}{[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}}$$

$$- 2f(i)\,\frac{Q\epsilon}{k\pi^2}\int_0^{\sqrt{\lambda}} \frac{d\xi}{(1 + \xi^2)[(\lambda - \xi^2)(\xi^2 + 1)]^{1/2}}$$

[Formula (6.16) is continued on the following page.]

$$+\frac{Q\,\delta\lambda}{k\pi^2}\int_0^{\sqrt{\lambda}}\frac{d\xi}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}}-\frac{Q\,\delta\,\lambda}{2k\pi^2}\int_0^{\sqrt{\lambda}}\frac{d\xi}{(1+\xi^2)[(\lambda-\xi^2)(\xi^2+1)]^{1/2}}$$

$$(6.16)$$

$$v=\frac{2Q}{k\pi^2}\int_0^{\sqrt{\lambda}}\frac{\xi\,f_1(\xi)}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}}\;\frac{d\xi}{1+\xi^2}$$

The last equation checks the result we found before. Equation (6.16) yields a relation between δQ and $\delta\lambda$. The coefficients $f(i)$ and $f_2(\sqrt{\lambda})$ and the function $f_2(\xi)$ have to be evaluated in terms of $f_1(\xi)$ which is known.

Now $f_2(\xi)$ is the Hilbert transform of $f_1(\xi)$, that is,

$$f_2(\xi)=-\frac{1}{\pi}\int_{-\infty}^{*\infty}\frac{f_1(t)}{t-\xi}\,dt \qquad (6.17)$$

and consequently

$$f(\zeta)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_1(t)+if_2(t)}{t-\zeta}\,dt \qquad (6.18)$$

i.e.

$$f(i)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{f_1(t)+if_2(t)}{t-i}\,dt \qquad (6.19)$$

Then

$$f(i)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{[f_1(t)+if_2(t)](t+i)}{t^2+1}\,dt$$

and since $f(i)$ is real, we have

$$f(i)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{f_1(t)+tf_2(t)}{t^2+1}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f_i(t)}{t^2+1}\, dt + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\frac{t}{t^2+1}\right)\left(-\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f_1(x)}{x-t}\, dx\right) dt$$

Interverting the order of integration leads to

$$f(i) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f_1(t)}{t^2+1}\, dt + \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} f_1(x)\left(\int_{-\infty}^{*+\infty} \frac{t\, dt}{(t^2+1)(x-t)}\right) dx$$

We now proceed to evaluate the integral $g(x)$, where

$$\int_{-\infty}^{*\infty} \frac{t\, dt}{(t^2+1)(x-t)} = g(x)$$

$$g(x) = \int_{-\infty}^{*\infty} \frac{(t-x)\, dt}{(t^2+1)(x-t)} + x\int_{-\infty}^{+\infty} \frac{dt}{(t^2+1)(x-t)}$$

$$= -\pi + \frac{x^2}{1+x^2}\int_{-\infty}^{+\infty} \frac{dt}{t^2+1} + \frac{x}{1+x^2}\int_{-\infty}^{+\infty} \frac{tdt}{t^2+1} + \frac{x}{1+x^2}\int_{-\infty}^{*+\infty} \frac{dt}{x-t}$$

$$= -\pi + \frac{x^2\pi}{x^2+1} + \left[\frac{x}{1+x^2}\, \frac{1}{2}\,(1+t^2) - \frac{x}{1+x^2}\log|x-t|\right]_{-\infty}^{+\infty}$$

$$= \pi\left(\frac{x^2}{x^2+1} - 1\right) = -\frac{\pi}{1+x^2}$$

Therefore

$$f(i) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f_1(t)}{t^2+1}\, dt + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{f_1(x)}{1+x^2}\, dt$$

$$f(i) = \frac{2}{\pi}\int_{-\infty}^{+\infty} \frac{f_1(t)}{1+t^2}\, dt \qquad (6.20)$$

We need a second equation in δQ and δt to solve the problem entirely. It is provided by considering the situation at points b and c. Thus, $z^*_b(\infty) = B$ and $z_b(\infty) = B$ while $z^*_{c*}(\sqrt{\lambda}) = B - h\epsilon + i(H - v\epsilon)$ and $z_c(\sqrt{\lambda}) = B + iH$. Consequently

$$\epsilon(h + iv) = -\frac{i\,\delta Q}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty} \operatorname{arc\,cosh}\frac{2\xi^2 + 1 - \lambda}{1 + \lambda}\cdot\frac{d\xi}{1+\xi^2}$$

$$+ \frac{2Q\epsilon}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty}\frac{\xi\, f(\xi)}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}\,\frac{d\xi}{1+\xi^2}$$

$$+ \frac{iQ\epsilon}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty}\frac{\alpha + 2f(i) - 2\sqrt{\lambda}f_2(\sqrt{\lambda})}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}\cdot\frac{\xi^2 d\xi}{1+\xi^2}$$

$$+ \frac{iQ\epsilon}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty}\frac{1}{1+\lambda}\,\frac{[2\lambda f(i) + 2\sqrt{\lambda}\, f_2(\sqrt{\lambda}) - \alpha(\frac{1-\lambda}{2})]}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}\cdot\frac{d\xi}{1+\xi^2}$$

Taking only the imaginary parts of this complex equality, we find that

$$\epsilon v = \frac{\delta Q}{k\pi^2}\, I^*(\lambda) + \frac{2Q\epsilon}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty}\frac{\xi\, f_2(\xi)}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}\cdot\frac{d\xi}{1+\xi^2}$$

$$\frac{Q\epsilon}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty}\frac{1}{1+\lambda}\,\frac{\alpha + 2f(i) - 2\sqrt{\lambda}\, f^2(\sqrt{\lambda})}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}\,\frac{\xi^2 d\xi}{1+\xi^2}$$

$$-\frac{Q\epsilon}{k\pi^2}\int_{\sqrt{\lambda}}^{\infty}\frac{1}{1+\lambda}\left[\frac{2\lambda f(i) + 2\sqrt{\lambda}\, f_2(\sqrt{\lambda}) - \alpha(\frac{1-\lambda}{2})}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}\right]\frac{d\xi}{1+\xi^2}$$

[Formula (6.21) is continued on the following page.]

$$\epsilon v = \frac{\delta Q}{k\pi^2} I^*(\lambda) + \frac{2Q\epsilon}{k\pi^2} \int_{\sqrt{\lambda}}^{\infty} \frac{\xi\, f_2(\xi)}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}} \frac{d\xi}{1 + \xi^2}$$

$$+ \frac{2Q\epsilon}{k\pi^2} \cdot \frac{f(i) - \sqrt{\lambda}\, f_2(\sqrt{\lambda})}{1 + \lambda} \int_{\sqrt{\lambda}}^{\infty} \frac{d\xi}{(\xi^2 - \lambda)(\xi^2 + 1)} \tag{6.21}$$

$$- 2f(i) \frac{Q\epsilon}{k\pi^2} \int_{\sqrt{\lambda}}^{\infty} \frac{d\xi}{(1 + \xi^2)[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}}$$

$$+ \frac{Q\delta\lambda}{k\pi^2} \int_{\sqrt{\lambda}}^{\infty} \frac{d\xi}{[(\xi^2 - \lambda)(\xi^2 + 1)]^{1/2}} - \frac{Q\delta\lambda}{2k\pi^2} \int_{\sqrt{\lambda}}^{\infty} \frac{d\xi}{(1 + \xi^2)[(\xi^2 - \lambda)(\xi^2+1)]^{1/2}}$$

Let

$$\int_0^{\sqrt{\lambda}} \frac{d\xi}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} = E(\lambda) \qquad \int_{\sqrt{\lambda}}^{\infty} \frac{d\xi}{[(\xi^2-\lambda)(\xi^2+1)]^{1/2}} = E^*(\lambda)$$

$$\int_0^{\sqrt{\lambda}} \frac{d\xi}{(1+\xi^2)[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} = H(\lambda) \qquad \int_{\sqrt{\lambda}}^{\infty} \frac{d\xi}{(\xi^2+1)[(\xi^2-\lambda)(\xi^2+1)]^{1/2}} = H^*(\lambda)$$

$$\int_0^{\sqrt{\lambda}} \frac{\xi\, f_2(\xi)}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} \frac{d\xi}{1 + \xi^2} = F_2(\lambda)$$

$$\int_{\sqrt{\lambda}}^{\infty} \frac{\xi\, f_2(\xi)}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} \frac{d\xi}{1 + \xi^2} = F_2^*(\lambda)$$

$$F_2(\lambda) = \int_0^{\sqrt{\lambda}} \frac{\xi}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} \frac{d\xi}{1 + \xi^2} \left(- \frac{1}{\pi} \int_{\infty}^{\infty} \frac{f_1(t)}{t - \xi}\, dt \right)$$

$$F^*_2(\lambda) = \int_{\sqrt{\lambda}}^{\infty} \frac{\xi}{[(\lambda-\xi^2)(\xi^2+1)]^{1/2}} \frac{1}{1+\xi^2} \left(-\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f_1(t)}{t-\xi}\, dt \right) d\xi$$

With these notations the previous equations can be written as

$$-h\epsilon = \frac{\delta Q}{k\pi^2} I(\lambda) + \frac{2Q\epsilon}{k\pi^2} F_2(\lambda) + \frac{2Q\epsilon}{k\pi^2} \frac{[f(i) - \sqrt{\lambda}\, f_2(\sqrt{\lambda})]}{1+\lambda} E(\lambda)$$

$$- 2f(i) \frac{Q\epsilon}{k\pi^2} H(\lambda) + \frac{Q\delta\lambda}{k\pi^2} E(\lambda) - \frac{Q\delta\lambda}{2k\pi^2} H(\lambda) \tag{6.22}$$

$$v\epsilon = \frac{\delta Q I^*(\lambda)}{k\pi^2} + \frac{2Q\epsilon}{k\pi^2} F^*_2(\lambda) + \frac{2Q\epsilon}{k\pi^2} \frac{[f(i) - \sqrt{\lambda}\, f_2(\sqrt{\lambda})]}{1+\lambda} E^*(\lambda)$$

$$- 2f(i) \frac{Q\epsilon}{k\pi^2} H^*(\lambda) + \frac{Q\delta\lambda}{k\pi^2} E^*(\lambda) - \frac{Q\delta\lambda}{2k\pi^2} \dot{H}^*(\lambda) \tag{6.23}$$

In other words, we have two equations for the two unknowns δQ and $\delta\lambda$. In Appendix A a numerical example is given to illustrate the method.

APPENDIX A. NUMERICAL EXAMPLE

We consider perturbations for the ditch corresponding to $\lambda = 4$ and select an arbitrary function $\delta(\xi)$. The formulas for the velocity distribution on the perimeter are:

$$\frac{V_a}{k} = \frac{\pi}{\pi - \text{arc cos}\left(\frac{1.5 - \xi^2}{2.5}\right)} \qquad 0 < \xi < 1.2249$$

$$\frac{V_a}{k} = \frac{\pi}{\text{arc cos}\left(\frac{\xi^2 - 1.5}{2.5}\right)} \qquad 1.2249 < \xi < 2$$

$$\frac{V_a}{k} = 1.3647/\log_{10}\left[\frac{\xi^2 - 1.5 + [\xi^4 - 3\xi^2 - 4]^{1/2}}{2.5}\right] \qquad 2 < \xi \tag{A.1}$$

$$\frac{X_b}{B} = \frac{1}{2.208}\left[0.24674\ b - \int_0^{\xi_b} \text{arc cos}\left(\frac{1.5 - \xi^2}{2.5}\right)\frac{d\xi}{1 + \xi^2}\right] \qquad 0 < \xi < 1.2249$$

$$\frac{X_b}{B} = \frac{1}{2.208}\left[1.30808 + \int_{1.2249}^{\xi_b} \text{arc cos}\left(\frac{\xi^2 + 1.5}{2.5}\right)\cdot\frac{d\xi}{1 + \xi^2}\right] \qquad 1.2249 < \xi < 2$$

$$\frac{Y_b}{H} = 1 - \frac{1}{0.597}\int_2^{\xi_b} \log_{10}\left[\frac{\xi^2 - 1.5 + [\xi^4 - 3\xi^2 - 4]^{1/2}}{2.5}\right]\cdot\frac{d\xi}{1 + \xi^2} \qquad \xi > 2$$

In the Table A.1 are given the coordinates of the points on the perimeter, and the velocity at these points, such that between any two successive points flows a discharge $Q/40$. The case $\lambda = 4$ is characterized by the discharge ratio H/B formulas:

TABLE A.1. Case H/W = 0.31

p	$p\pi/40$	ξp	v_a/k_f	x/B.	y/H
0	0	0	1.419	0	1.000
1	4.5°	0.079	1.421	0.079	1.000
2	9.0	0.158	1.427	0.157	1.000
3	13.5	0.240	1.437	0.235	1.000
4	18.0	0.325	1.453	0.313	1.000
5	22.5	0.414	1.474	0.389	1.000
6	27.0	0.510	1.503	0.464	1.000
7	31.5	0.613	1.542	0.537	1.000
8	36.0	0.727	1.595	0.609	1.000
9	40.5	0.854	1.667	0.678	1.000
10	45.0	1.000	1.773	0.743	1.000
11	49.5	1.171	1.937	0.803	1.000
12	54.0	1.376	2.224	0.857	1.000
13	58.5	1.632	2.891	0.909	1.000
14	63.0	1.962	9.009	0.959	1.00
15	67.5	2.414	2.743	1.000	0.964
16	72.0	3.078	1.719	1.000	0.879
17	76.5	4.165	1.240	1.000	0.755
18	81.0	6.314	0.918	1.000	0.589
19	85.5	12.706	0.647	1.000	0.351
20	90.0	∞	0	1.000	0

$$Q/\pi^2 k_f = B/2.21 = H/1.38 \qquad H/B = 0.62 \qquad (A.3)$$

A velocity distribution diagram is indicated on Figure A.1. Between two successive arrows flows a discharge $Q/40$.

Let us plot $\pi k^2 \delta(\xi)2Q$, $\delta'(\xi)$, and $f_1(\xi)$ as shown in Figure A.2.

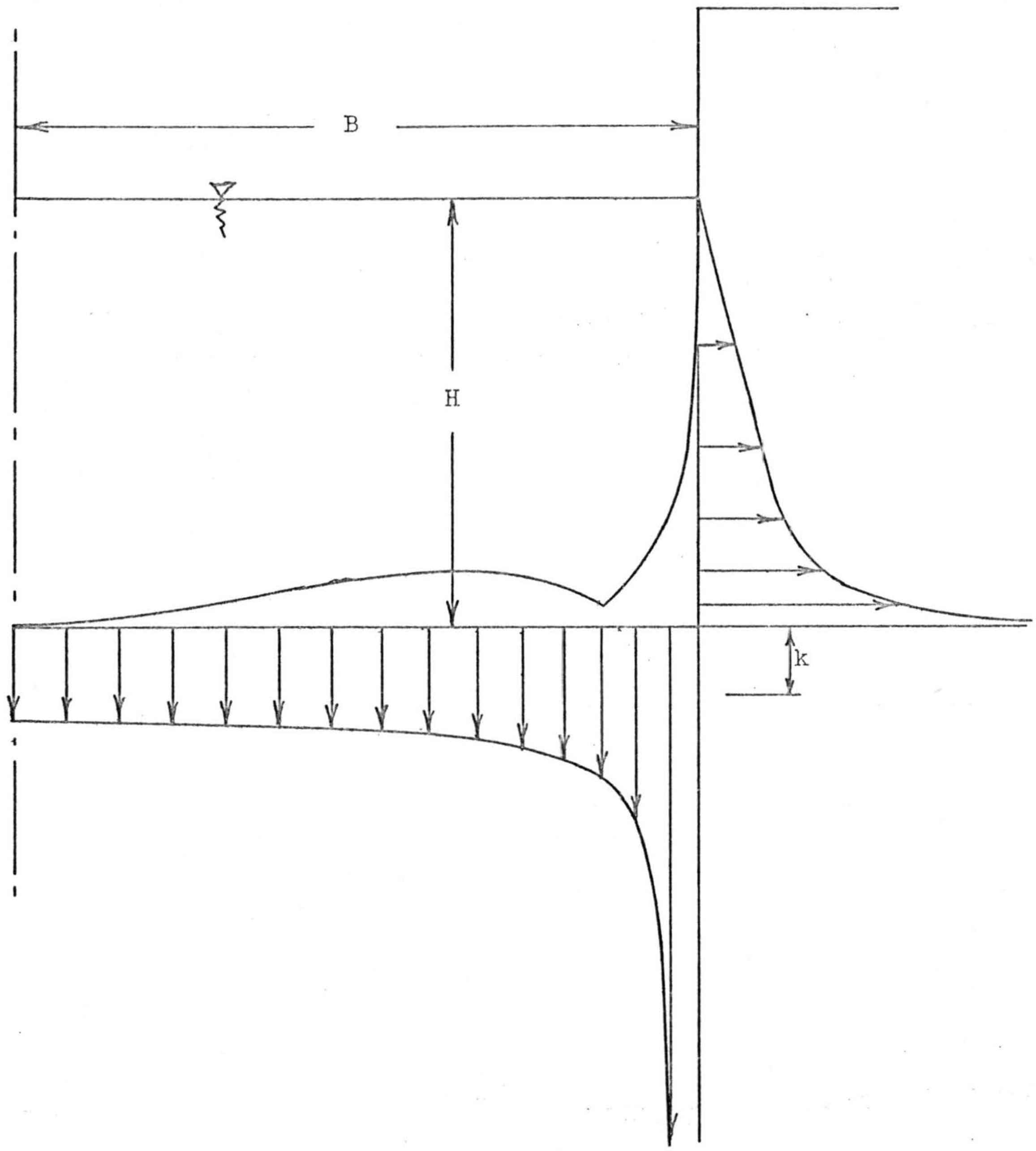
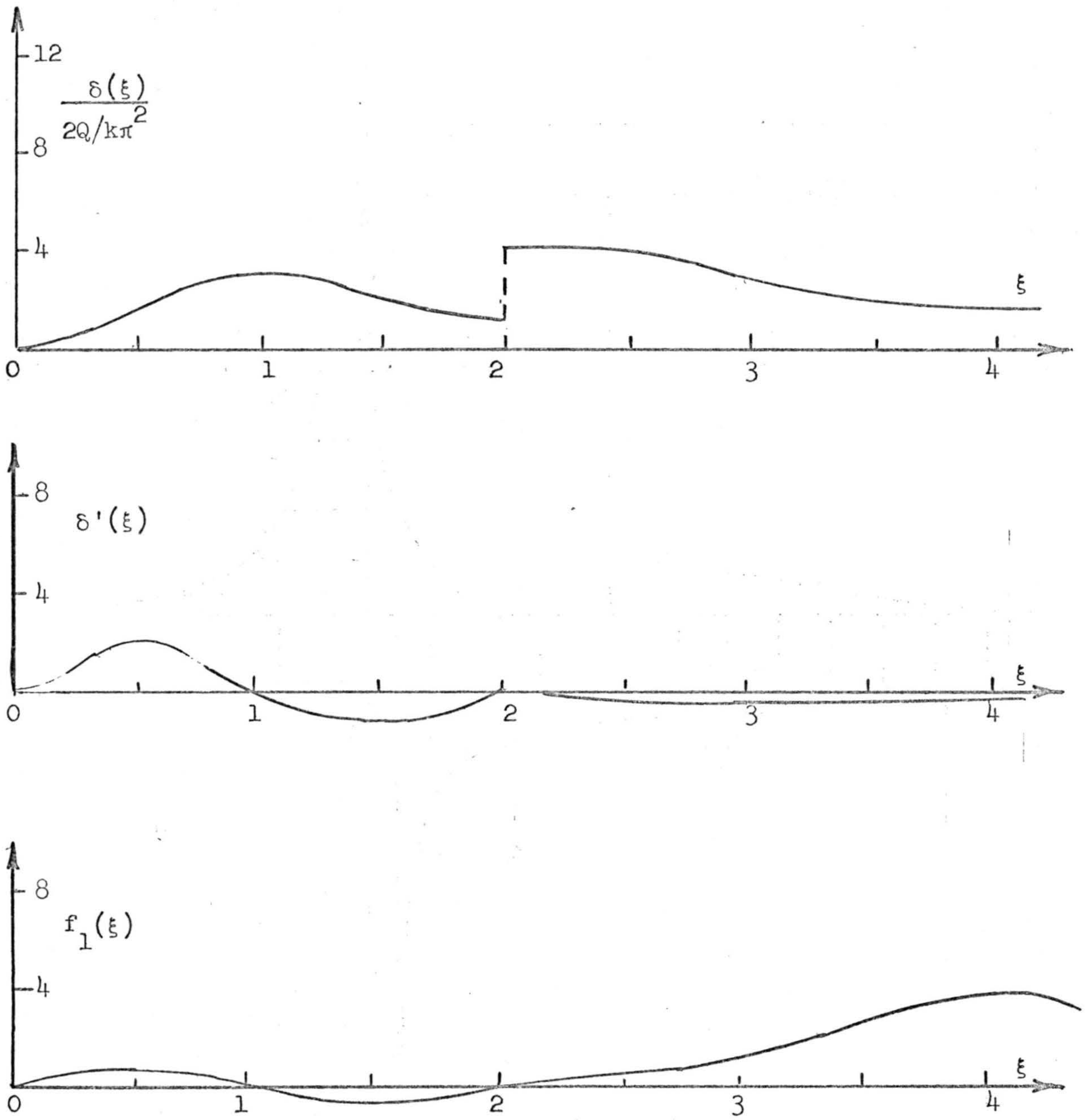


Figure A.1



12
$\frac{\delta(\xi)}{2Q/k\pi^2}$
8
4
ξ
0
1
2
3
4
8
$\delta'(\xi)$
4
ξ
0
1
2
3
4
8
$f_1(\xi)$
4
ξ
0
1
2
3
4

Figure A.2

We approximate $f_1(\xi)$ by a polynomial times a damping exponential

$$f_1(\xi)/120 = (\xi^3 - 3\xi^2 + 2\xi)\, e^{-\xi} \qquad \text{for} \quad \xi > 0 \tag{A.4}$$

The function $f_2(\xi)$ is defined by the formula

$$f_2(\xi) = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{f_1(t)}{t - \xi}\, dt = - H\left[f_1(t)\right]$$

Here $f_1(\xi)$ is an even function and is defined for all values of ξ by the formula

$$f_1(\xi)/120 = (\text{sgn}\ \xi)\xi^3\, e^{-|\xi|} - 3\xi^2\, e^{-|\xi|} + 2(\text{sgn}\ \xi)\xi\, e^{-|\xi|} \tag{A.5}$$

The Hilbert transform is then easily obtained, knowing that:

$$H\left[e^{-|\xi|}\right] = \frac{1}{\pi}\, \text{sgn}\ \xi \left[e^{|\xi|}\, \text{Ei}(-|\xi|) - e^{-|\xi|}\, \overline{\text{Ei}}(|\xi|)\right]$$

and

$$H\left[\text{sgn}\ \xi\, e^{-|\xi|}\right] = -\frac{1}{\pi}\left[e^{|\xi|}\, \text{Ei}(-|\xi|) + e^{-|\xi|}\, \overline{\text{Ei}}(|\xi|)\right]$$

by repeated use of the formula:

$$H\left[\xi f(\xi)\right] = \xi H\left[f(\xi)\right] + \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi) d\xi .$$

Ultimately:

$$f_2(\xi)/120 = (\xi/\pi)\,(\xi^2 + 2)\left[e^{|\xi|}\, \text{Ei}(-|\xi|) + e^{-|\xi|}\, \overline{\text{Ei}}(|\xi|)\right] + 4\xi/\pi$$

$$+ (3\xi^2/\pi)\, \text{sgn}\ \xi \left[e^{|\xi|}\, \text{Ei}(-|\xi|) - e^{-|\xi|}\, \overline{\text{Ei}}(|\xi|)\right] \tag{A.6}$$

which becomes for $\xi > 0$:

$$f_2(\xi)/120 = (\xi^3 + 2\xi)\left[e^{\xi}\, Ei(-\xi) + e^{-\xi}\, \overline{Ei}(\xi)\right] + 3\xi^2 \left[e^{\xi}\, Ei(-\xi) - e^{-\xi}\, \overline{Ei}(\xi)\right] + 4\xi \qquad (A.7)$$

In the Table A.2 we compare the values of $\delta(\xi)$ as arbitrarily selected with the values calculated from the chosen approximate $f_1(\xi)$ and $\delta^*(\xi)$.

TABLE A.2

ξ	$k\pi^2\delta/2Q$	$k\pi^2\delta^*/2Q$
0	0.00	0.00
0.20	-	0.25
0.50	1.50	1.58
0.70	-	2.43
1.00	3.00	2.91
1.20	-	2.76
1.50	2.00	2.22
1.70	-	1.86
2.00	1.50 ; 4.00	1.53 ; 3.73
2.50	-	3.17
3.00	2.50	2.37
4.00	1.00	1.17
5.00	0.50	0.56
6.00	-	0.29
16.00	-	< 0.002

Numerical integrations yield for the definite integrals E,E*, H and H*, the following values:

$$E(4) = \int_0^2 \frac{d\xi}{[(4-\xi^2)(\xi^2+1)]^{1/2}} = 1.010 \qquad E^*(4) = \int_2^\infty \frac{d\xi}{[(\xi^2-4)(\xi^2+1)]^{1/2}} = 0.755$$

$$H(r) = \int_0^2 \frac{d\xi}{(1 + \xi^2)[(4 - \xi^2)(\xi^2 + 1)]^{1/2}} = 0.528$$

$$H^*(4) = \int_2^\infty \frac{d\xi}{(1 + \xi^2)[(\xi^2 - 4)(\xi^2 + 1)]^{1/2}} = 0.129$$

and it was calculated earlier that

$$I(4) = \int_0^2 \text{arc cos} \left(\frac{x^2 - 1.5}{2.5}\right) \frac{dx}{1 + x^2} = 2.208$$

$$I^*(4) = 2.3026 \int_2^\infty \log_{10} \left[\frac{x^2 - 1.5 + [x^4 - 3x^2 - 4]^{1/2}}{2.5}\right] \frac{dx}{1 + x^2} = 1.375$$

Two more definite integrals must be calculated by numerical integration:

$$F_2(4) = \int_0^2 \frac{\xi\, f_2(\xi)}{[(4 - \xi^2)(\xi^2 + 1)]^{1/2}} \frac{d\xi}{1 + \xi^2} = \int_0^2 h(\xi)\xi f_2(\xi) d\xi$$

$$F_2^*(4) = \int_2^\infty \frac{\xi\, f_2(\xi)}{[(\xi^2 - 4)(\xi^2 + 1)]^{1/2}} \frac{d\xi}{1 + \xi^2} = \int_2^\infty h^*(\xi)\xi f_2(\xi) d\xi$$

Hence

$$(\pi/120)F_2(4) = \int_0^2 h(\xi) \left[\xi^2(\xi^2 + 2)\left(e^{\xi}\, Ei(-\xi) + e^{-\xi}\, \overline{Ei}(\xi)\right)\right.$$

$$\left. + 3\xi^3 \left(e^{\xi}\, Ei(-\xi) - e^{-\xi}\, \overline{Ei}(\xi)\right) + 4\xi^2\right] d\xi$$

$$(\pi/120)F_2(4) = \int_0^2 h(\xi)\left[\xi^2(\xi^2+2)\left(e^{\xi}Ei(-\xi) + e^{-\xi}\,\overline{Ei}(\xi)\right)\right.$$

$$\left. + 3\xi^3\left(e^{\xi}\,Ei(-\xi) - e^{-\xi}\,\overline{Ei}(\xi)\right)\right] d\xi + 4\,E(4) - H(4)$$

$$(\pi/120)F_2^*(4) = \int_2^{\infty} h^*(\xi)\left[\xi^2(\xi^2+2)\left(e^{\xi}\,Ei(-\xi) + e^{-\xi}\,\overline{Ei}(\xi)\right)\right.$$

$$\left. + 3\xi^2\left(e^{\xi}\,Ei(-\xi) - e^{-\xi}\,\overline{Ei}(\xi)\right)\right] d\xi + 4\,E^*(4) - H^*(4)$$

A word needs to be said about the convergence of the integral $F_2^*(4)$. For large values of ξ, $h^*(\xi) = 1/\ 1 + \xi^2[(\xi^2-4)(\xi^2+1)]^{1/2}$ is asymptotically equivalent to ξ^{-4}. The integral will be convergent provided

$$\int_2^{\infty} e^{\xi}\,Ei(-\xi) + e^{-\xi}\,\overline{Ei}(\xi) + (3/\xi)[e^{\xi}\,Ei(-\xi) - e^{-\xi}\,\overline{Ei}(\xi)] \quad d\xi$$

is convergent. The asymptotic expansions of $Ei(-\xi)$ and $\overline{Ei}(\xi)$ are:

$$Ei(-\xi) = -e^{-\xi}\,\frac{1}{\xi} - \frac{1}{\xi^2} + \frac{2}{\xi^3} + \ldots \qquad e^{\xi}Ei(-\xi) = -\frac{1}{\xi} + \frac{1}{\xi^2} + \frac{2}{\xi^3} + \ldots$$

$$\overline{Ei}(\xi) = e^{\xi}\,\frac{1}{\xi} + \frac{1}{\xi^2} + \frac{2}{\xi^3} + \ldots \qquad e^{-\xi}\,\overline{Ei}(\xi) = \frac{1}{\xi} + \frac{1}{\xi^2} + \frac{2}{\xi^3} + \ldots$$

and the integral for large values of ξ is equivalent to:

$$\int_X^{\infty}\left[\frac{2}{\xi^2} + \frac{12}{\xi^4} + \ldots + \frac{3}{\xi}\left(-\frac{2}{\xi} - \frac{4}{\xi^3} + \ldots\right)\right] d\xi$$

$$= \int_X^{\infty}\left[-\frac{4}{\xi^2} + 0\left(\frac{1}{\xi^6}\right)\right] d\xi \approx -\frac{4}{X}$$

Numerical integration then yields $(\pi/120)F_2(4) = -0.01$ and $(\pi/120)F_2^*(4) = -0.55$. It is easy to evaluate the constant $f_2(2)$. Thus, we find $(\pi/120)f_2(2) = -0.96$. The constant $f(i)$ is calculated numerically since

$$f(i) = \frac{2}{\pi}\int_0^\infty \frac{f_1(t)}{1+t^2} = \frac{2}{\pi} G(4) \frac{\pi f(i)}{240} = 0.21$$

The two equations for δQ and $\delta\lambda$ become

$$2.21\,\delta Q/Q + 0.75\,\delta\lambda = -310\epsilon \qquad 1.38\,\delta Q/Q + 0.69\,\delta\lambda = 22\epsilon$$

or

$$\delta Q/Q = -461\epsilon \qquad \delta\lambda = 950\epsilon \tag{A.8}$$

This is an interesting result but valid only for very small values of ϵ. Qualitatively, it shows very well that the dimensionless parameter $q = Q/k\sqrt{A}$, where A is the cross-sectional area, increases with the perturbation and $\delta\lambda > 0$. Since

$$\sigma = \frac{2Q_{dc}}{Q} = \frac{\text{bottom discharge}}{\text{total discharge}} = \frac{2}{\pi}\text{ arc tan }\sqrt{2}$$

$$\frac{Q_{bc}}{Q_{dc}} = \frac{\text{side discharge}}{\text{bottom discharge}} = \frac{\text{arc tan }\frac{1}{\sqrt{\lambda^*}}}{\text{arc tan }(\sqrt{\lambda^*})},$$

$$\delta\frac{Q_{bc}}{Q_{dc}} = -\frac{\pi}{2}\frac{d\lambda}{(1+\lambda^2)(\text{arc tan }\sqrt{\lambda})^2}$$

More water flows from the bottom than from the side as compared to the original case.

Formulas (A.9) are valid for relatively very small values of ϵ

because a boundary condition $f_1(\sqrt{\lambda^*}) = 0$ was approximated as $f_1(\sqrt{\lambda}) = 0$, which is a good approximation only for $\delta\lambda << \lambda$.

Taylor's development of $f_1(\ \lambda^*)$ yields

$$f_1(\sqrt{\lambda^*}) = f_1(\sqrt{\lambda}) + \frac{\delta\lambda}{2\sqrt{\lambda}} f_1'(\sqrt{\lambda}) + \frac{\delta\lambda^2}{4\lambda} f_1''(\sqrt{\lambda}) + \dots$$

A better approximation is then obtained, replacing the condition $f_1(\sqrt{\lambda^*}) = 0$ by $f_1(\sqrt{\lambda}) = 0$ and $f_1'(\sqrt{\lambda}) = 0$. We select for $f_1(\xi)$ the function

$$f_1(\xi) = 2(10^8)(\xi - 2)^3 e^{-3\xi} (e^{(1-\xi)/20} - 1)(e^{-\xi/5} - 1)^5 \qquad (A.9)$$

Indeed, $f_1(\xi)$ satisfies the condition $f_1''(2) = 0$. In the Table A.3 we compare the values of $\delta(\xi)$ as arbitrarily selected with the values calculated from the chosen approximate $f_1(\xi)$, $\delta^*(\xi)$.

TABLE A.3

ξ	$k\pi^2\delta/2\xi$	$k\pi^2\delta^*/2Q$
0	0.00	0.00
0.20	-	0.0082
0.50	1.50	0.84
0.70	-	2.13
1.00	3.00	3.14
1.20	-	2.84
1.50	2.00	2.17
1.70	-	1.97
2.00	1.50 ; 4.00	1.93 ; 3.63
2.50	-	3.50
3.00	2.50	2.80
4.00	1.00	0.80
5.00	0.50	0.20
6.00	-	0.06
11.00	-	< 0.001

The Hilbert transform is then easily obtained. Calling

$$e^{a\xi}\,Ei(-a\xi) + e^{-a\xi}\,\overline{Ei}(a\xi) = A(a)$$

$$e^{a\xi}\,Ei(-a\xi) - e^{-a\xi}\,\overline{Ei}(a\xi) = D(a)$$

we obtain

$$\frac{\pi f_2(\xi)}{2.1026 \times 10^8} = \xi \left[(\xi^2 + 10)\,\Sigma A(a) S(a) - 2\Sigma \frac{A(a)}{a^2} + 12\Sigma \frac{A(a)}{a} \right]$$

$$+ (6\xi^2 + 8)\,\Sigma A(a)\,D(a) \qquad \text{(A.10)}$$

where the summations are to be taken for a discrete set of values as indicated in Table A.4.

TABLE A.4

a	4.05	4.00	3.85	3.80	3.65	3.60
A(a)	1	- 0.9512	- 5	4.756	10	- 9.512
a	3.45	3.40	3.25	3.20	3.05	3.00
A(a)	- 10	9.512	5	4.756	- 1	0.9512

APPENDIX B. SEEPAGE FROM TRIANGULAR AND TRAPEZOIDAL CHANNELS

This appendix is a free translation of the major part of a paper in Russian by B. B. Vedernikov (1936). Substantially the same theory was also published in German by Vedernikov (1937).

B.1 Introduction. Problems of seepage of water through soils have much importance in hydrology and hydraulic engineering. Questions of seepage from canals, especially seepage with formation of a free surface appeared first with the study of water filtration for irrigation. There is available a method of analysis applicable to the solution of such problems. This method may be used in cases of filtration with a free surface, when in the domain of the complex potential the boundaries of the problem consist only of straight lines, on which either the real or imaginary part of the complex potential is constant. In the physical plane the boundary must also consist only of straight lines except, of course, for the free surface. Introducing now the vector of reduced velocity, we have the possibility of further necessary transformations because in the domain $1/v'$ (where $v' = v_x - iv_y$, the complex conjugate of the velocity vector) the boundaries will be rectilinear because of the properties of the inversion transformation. Thus, by mapping the domain $1/v'$ upon the half plane ζ through the Schwarz-Christoffel formula and mapping again this half plane upon the domain of the complex potential, through the same formula, we get the relations of the form

$$1/v' = v/|v|^2 = f_1(\zeta); \qquad W = \emptyset + i\psi = -\, i(X + iY)\, k_f = f_2(\zeta)$$

$$1/v' = f(W)$$

Further, we can obtain the relation between the complex potential and the physical domain in the formula

$$z = (1/k_f)\int v\, dW/|v|^2 = F(W) \tag{B.1}$$

In such a form, which is modified from Kirchhoff's method, problems of filtration can be solved without distortion of the boundary conditions on the free surface.

On the basis of the above method we give in the present work the solution of seepage from triangular and trapezoidal canals with linear slopes and without influx. It is further assumed that there is an extremely deep water table or, in other words, the filtration does down to infinity. The y-axis is directed downwards and consequently the angles of the velocity directions with the x-axis will be read clockwise.

B.2. Filtration from triangular canals. In the domain z the profile of the waterway is represented by a triangle with an angle of the slopes with the horizontal, of value α ($m = \text{ctn}\, \alpha$). The depth of water in the waterway is H and the width of the water surface B as indicated in Figure B.1. The profile of the free surface is not known beforehand. The velocity directions on the slopes on the right and left arc, respectively,

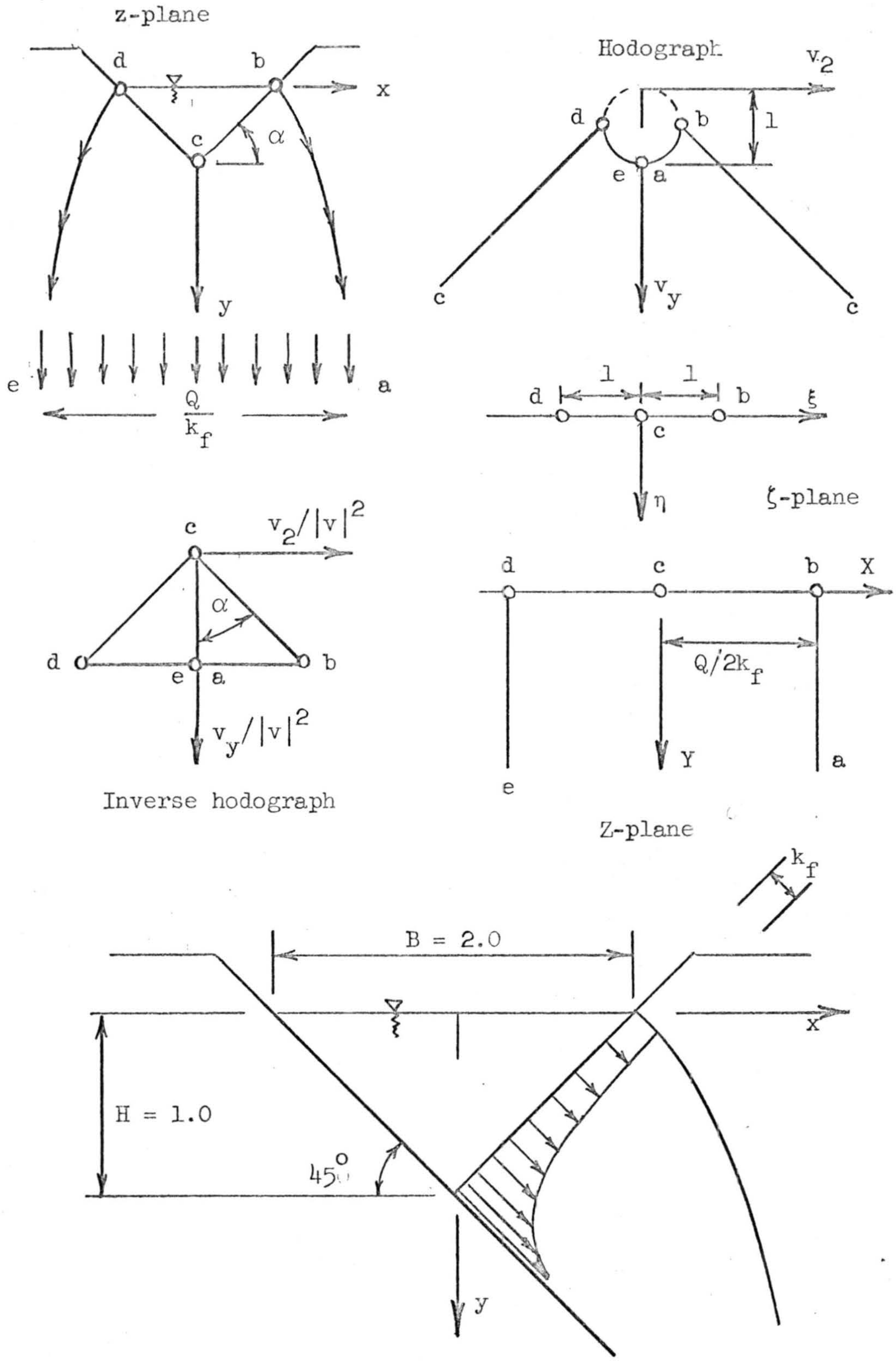

z-plane
d
b
x
c
α
y
e
a
Q
k_f
Hodograph
v_2
l
v_y
ξ
η
ζ-plane
$v_2/|v|^2$
$v_y/|v|^2$
Inverse hodograph
X
Q/2k_f
Y
Z-plane
k_f
B = 2.0
H = 1.0
45°

Figure B.1

$$\arctan (v_y/v_x) = \pi/2 - \alpha$$

$$\arctan (v_y/v_x) = \pi/2 + \alpha$$

On the free surface the pressure is constant, which leads to the condition:

$$v_x^2 + [v_y - (1/2)]^2 = (1/2)^2$$

or

$$|v|^2 = v_x^2 + v_y^2 = v_y.$$

Far from the bottom of the canal (i.e., for $y = \infty$) the filtration velocity has the uniform direction $\arctan (v_y/v_x) = \pi/2$ and the reduced value $|v| = 1$. At points b and d

$$v_x = \pm \cos \alpha \cdot \sin \alpha$$

$$v_y = \cos^2 \alpha$$

or

$$v/|v|^2 = (v_x + iv_y)/v_y = \pm \tan \alpha + i$$

In the domain of the complex velocity vector, to the reduced velocity on the right bank corresponds to line bc, making an angle $(\pi/2 - \alpha)$, and on the left side dc making an angle $(\pi/2 + \alpha)$ with the horizontal axis. To the point c of the contour corresponds the whole infinity of the domain v between the lines bc and dc, but to the whole line ae at infinity corresponds the point (a,e) in the hodograph plane.

To the free surface corresponds the arc of circumference bacd. In the inverse velocity domain, to the channel profile corresponds the triangle cbd with vertex at c, corresponding to the bottom of the canal. In the domain Z, to the boundary of the problem, corresponds the semi-infinite strip abda. Let us assign the following correspondence of points in the several domains:

1. $x = \pm \frac{B}{2}$ $y = 0;$ $X = \pm \frac{Q}{2k_f},$ $Y = 0$ $\frac{v}{|v|^2} = \pm \tan \alpha + i$

$$\xi = \pm 1, \quad \eta = 0$$

2. $x = 0$ $y = H;$ $X = 0$ $Y = 0;$ $\frac{v}{|v|^2} = 0;$ $\xi = \eta = 0$

3. $x = \pm \frac{0}{2k_f}$ $y = \infty;$ $x = \pm \frac{0}{2k_f}$ $Y = \infty;$ $\frac{v}{|v|^2} = i$

$$\xi = \pm \infty \quad \eta = 0$$

Let Q be the discharge per unit length of canal. Thus to the bottom, corresponds the point $\xi = 0$. The sides are distributed on the axis ξ from $\xi = 1$ to $\xi = 0$ and from $\xi = -1$ to $\xi = 0$ and the free surface $\xi = \infty$ to $\xi = 1$ and from $\xi = -\infty$ to $\xi = -1$.

Let us map on the half-plane ζ the semi-infinite strip Z. Schwarz-Christoffel formula yields:

$$X + iY = B \int_0^{\zeta} \frac{d\zeta}{(\zeta^2 - 1)^{1/2}} = q \int_0^{\zeta} \frac{d\zeta}{(1 - \zeta^2)^{1/2}} = q \text{ arc sin } \zeta \qquad \text{(B.2)}$$

For $\zeta = 1$ we obtain $Q/2k_f = \pi\, q/_2$ or $q = Q/\pi k_f$. Finally:

$$Z = (Q/\pi k_f) \text{ arc sin } \zeta \qquad \text{(B.3)}$$

or

$$W = -\, i\, (Q/\pi) \text{ arc sin } \zeta \qquad \text{(B.4)}$$

Let us map on the upper half plane ζ the triangle cbd of the inverse velocity domain. Then application of Schwarz-Christoffel theorem yields:

$$v/|v|^2 = D \int_0^{\zeta} \zeta^{2\alpha/\pi - 1} (\zeta^2 - 1)^{(\pi/2-\alpha)/\pi - 1}\, d\zeta + C \qquad \text{(B.5)}$$

For $\zeta = 0$ we have $v/|v|^2 = 0$ and hence $C = 0$. The preceding equation then becomes, after some reduction,

$$\frac{v}{|v|^2} = D(\sin \alpha + i \cos \alpha) \int_0^{\zeta} \frac{d\zeta}{\zeta^{1-2\alpha/\pi} (1-\zeta^2)^{1/2+\alpha/\pi}} \qquad \text{(B.6)}$$

The constant D is determined from the condition on the free surface for $\zeta = 1$. Let us denote:

$$I = \int_0^1 \frac{d\zeta}{\zeta^{1-2\alpha/\pi}\, (1-\zeta^2)^{1/2+\alpha/\pi}} \qquad \text{(B.7)}$$

This definite integral can be easily expressed in terms of Beta function B and Gamma function Γ. As is well known, these functions are related by

$$\int_0^1 x^m (1-x^n)^p\, dx = (1/n)\, B\, [(m+1)/n,\ p+1]$$

[Formula (B.8) is continued on the following page.]

$$= \frac{1}{n} \frac{\Gamma[(m+1)/n]\,\Gamma[p+1]}{\Gamma[(m+1)/n + p + 1]} \tag{B.8}$$

where $(m+1) > 0$, $n > 0$ and $(p+1) > 0$. We then obtain

$$I = (1/2)\, B[\alpha/\pi, (1/2 - \alpha/\pi)]$$

$$= \frac{1}{2} \frac{\Gamma[\alpha/\pi]\,\Gamma[1/2 - \alpha/\pi]}{\Gamma[1/2]} = \frac{\Pi[1/2 - \alpha/\pi]\,\Pi[\alpha/\pi]}{\alpha(1 - 2\alpha/\pi)/\sqrt{\pi}} \tag{B.9}$$

From this formula one can see immediately that $I(\alpha/\pi) = I(1/2 - \alpha/\pi)$, i.e., I takes the same value for two values of α/π related by the condition

$$(\alpha/\pi)_1 = 1/2 - (\alpha/\pi)_2 \tag{B.10}$$

For instance, for $\alpha/\pi = 1/8$ and $\alpha/\pi = 3/8$, the integral I has the same value. This character of I will be necessary for the derivation of the formula of discharge. For $\zeta = 1$ Equation (B.6) takes the form

$$v/|v|^2 = \tan\alpha + i = -D.(\sin\alpha + i\cos\alpha)\, I$$

Therefore:

$$D = -1/I\cos\alpha \tag{B.11}$$

Thus in a final form:

$$\frac{v}{|v|^2} = \frac{1}{I}(\tan\alpha + i)\int_0^\zeta \frac{d}{\zeta^{1-2\alpha/\pi}(1-\zeta^2)^{1/2+\alpha/\pi}} \tag{B.12}$$

This formula can be rewritten, in conformity with relation (B.4), in the form:

$$\frac{v}{|v|^2} = \frac{(\tan\alpha + i)}{I} \frac{i\pi}{Q} \int_0^W \frac{dW}{\sin^{1-2\alpha/\pi}(i\pi W/Q)\cos^{2\alpha/\pi}(i\pi W/Q} \tag{B.13}$$

Let us switch to the established relation between the coordinates of the domain z and the values of the complex potential or the coordinates of the half-plane ζ, since the relation between the latter and the values of the complex potential is known and is determined by relation (B.4). By Formula (B.1) we get:

$$z = \frac{1}{k_f}\int \frac{v dW}{|v|^2} = \frac{(\tan\alpha + i)(-iQ)}{\pi k_f I} \int_0^\zeta \frac{d\zeta}{\zeta^{1-2\alpha/\pi}(1-\zeta^2)^{1/2+\alpha/\pi}} \frac{d\zeta}{(1-\zeta^2)^{1/2}} + C \tag{B.14}$$

Note that for $\zeta = 0$ $z = iH$ and consequently that $C = iH$. The integral of Formula (B.14) can be integrated by parts. We get then for the coordinates of the cross-section

$$z = \frac{(1-i \tan \alpha)Q}{I\pi k_f}\left[\text{arc sin } \zeta \int_0^\zeta \frac{d\zeta}{\zeta^{L-2\alpha/\pi}(1-\zeta^2)^{1/2+\alpha/\pi}} - \int_0^\zeta \text{arc sin } \zeta \frac{d\zeta}{\zeta^{1-2\alpha/\pi}(1-\zeta^2)^{1/2+\alpha/\pi}}\right] + iH \qquad \text{(B.15)}$$

For the coordinate of the free surface, i.e., for $\zeta > 1$ we get

$$z = \frac{B}{2} + \frac{1}{I \cos \alpha}\left(\frac{Q}{\pi k_f}\right)\left[\int_1^\zeta \frac{\text{arc cosh } \zeta \, d\zeta}{\zeta^{1-2\alpha/\pi}(\zeta^2-1)^{1/2+\alpha/\pi}}\right.$$

$$\left. + \text{ arc cosh } \zeta \left(I \cos \alpha - \int_1^\zeta \frac{d\zeta}{\zeta^{1-2\alpha/\pi}(\zeta^2-1)^{1/2+\alpha/\pi}}\right)\right] + \frac{iQ}{\pi k_f} \text{ arc cosh } \zeta \qquad \text{(B.16)}$$

B.3. Discharge formula for a triangular waterway. Let us turn now to the determination of the discharge of filtration from a triangular canal. We define:

$$\int_0^1 \text{arc sin } \zeta \frac{d\zeta}{\zeta^{1-2\alpha/\pi}(1-\zeta^2)^{1/2+\alpha/\pi}} = f\left(\frac{\alpha}{\pi}\right) \qquad \text{(B.17)}$$

for $\zeta = 1$ Formula (B.15) takes the form

$$\frac{B}{2} = \frac{Q}{2k_f}\,[1 - 2f(\alpha/\pi/\pi I] \qquad \text{(B.18)}$$

Further we also have:

$$H = (Q/2k_f) \tan \alpha[1 - 2f(\alpha/\pi)/\pi I] \qquad \text{(B.19)}$$

Solving jointly relations (B.18) and (B.19), we get the formula for the discharge per unit length of canal:

$$Q = k_f\left[B + \frac{2}{\tan \alpha}\left(\frac{1}{1 - f(\frac{d}{\pi}) \,/\, \frac{\pi I}{2}} - 1\right) \cdot H\right] \qquad \text{(B.20)}$$

We define A such that

$$A = \frac{2}{\tan\alpha}\left[\frac{1}{1 - f(\frac{d}{\pi}) \,/\, \frac{\pi I}{2}} - 1\right] \qquad (B.21)$$

and then we obtain:

$$Q = k_f(B + AH) \qquad (B.22)$$

where A is a function of the angle of the slope of the channel side with the horizontal, or in other words, of the ratio of the width of the water surface to the water depth. From relation (B.17) we see that:

$$f(\alpha/\pi) = \pi I/2 - f(1/2 - \alpha/\pi) \qquad (B.23)$$

Using this and knowing that $\tan\alpha = 1/\tan(\pi/2-\alpha)$ we can easily express the coefficient A for the canal with angle of slopes of value $(\pi/2 - \alpha)$ through the coefficient A for a canal of slope angle of value α. We obtain on the basis of Formula (B.10)

$$A(\pi/2 - \alpha) = 4/A(\alpha) \qquad (B.24)$$

Therefrom we can easily evaluate the value of A for $\alpha = \pi/4$ (for a canal with slopes of unity), namely, $A(\pi/4) = 2$. We compute the value of A for canals with slope angles $\alpha/\pi = 1/6$, $\alpha/\pi = 1/8$, and $\alpha/\pi = 1/20$, the latter being the lower limit of practical significance. At the same time, knowing the value of A for those angles, we can compute from Formula (B.24) the values of A for the angles $\alpha/\pi = 1/3$, $\alpha/\pi = 3/8$ and $\alpha/\pi = 9/20$. For this it is necessary to compute the function $f(\alpha/\pi)$. We calculate using the formula of Gauss. For this it is first of all necessary that the integral expressing $f(\alpha/\pi)$ in such a form that the integrand does not become infinite. For $\alpha/\pi = 1/6$ we have

$$f(1/6) = \int_0^1 (\text{arc sin}\ \zeta)\ d\zeta/\zeta^{2/3}\ (1 - \zeta^2)^{2/3}$$

We make a change of variables such that $\zeta = (1 + x^3)^{-1/2}$. Then

$$f(1/6) = \frac{3}{2}\int_0^1 \text{arc sin}\left(\frac{1}{(1+x^3)^{1/2}}\right)\frac{dx}{(1+x^3)^{1/2}} + \frac{3}{2}\int_0^1 \text{arc sin}\left(\frac{1}{(1+x^3)^{1/2}}\right)\frac{dx}{(1+x^3)^{1/2}}$$

To get rid of the infinite limit in the second of these integrals we employ the change of variables $x = z^{-2}$ and we obtain:

$$f(\tfrac{1}{6}) = \frac{3}{2}\int_0^1 \text{arc sin}\left(\frac{1}{(1+x^3)^{1/2}}\right)\frac{dx}{(1+x^3)^{1/2}} + 3\int_0^1 \text{arc cos}\left(\frac{1}{(1+z^6)^{1/2}}\right)\frac{dz}{(1+z^6)^{1/2}}$$

Treating similarly the cases $\alpha/\pi = 1/8$ and $\alpha/\pi = 1/20$, we ultimately obtain

$$f(\tfrac{1}{8}) = 4\left[\int_0^1 \text{arc sin}\left(\frac{1}{(1+x^8)^{1/2}}\right) + \int_0^1 \text{arc cos}\left(\frac{1}{(1+z^8)^{1/2}}\right)\frac{dz}{(1+z^8)^{1/2}}\right]$$

$$f(\tfrac{1}{20}) = 10\left[\int_0^1 \text{arc sin}\left(\frac{1}{(1+x^{20})^{1/2}}\right)\frac{dx}{(1+x^{20})^{1/2}}\right.$$

$$\left. + \int_0^1 \text{arc cos}\left(\frac{1}{(1+z^{20})^{1/2}}\right)\frac{dz}{(1+z^{20})^{1/2}}\right]$$

These forms are suitable for the evaluation of those integrals by the formula of Gauss. Such computation of $f(\alpha/\pi)$ for $\alpha/\pi = 1/4$ gives the result $f(1/4) = 2.9123666$. The exact value of $f(1/4)$ by Formula (B.23) is

$$f(1/4) = (\pi/4)I \qquad \text{or} \qquad f(1/4) = 2\sqrt{\pi}\;\Pi^2(1/4)$$

Using tables of Jahnke and Emde we obtain $f(1/4) = 2.912\ldots$. For practical purposes, it is quite sufficient to have for A the first three significant figures. The values of $\Pi(x)$ for the determination of $\pi I/2$, which are necessary for the computation of A, are found in the tables of Jahnke and Emde, where $\Pi(x)$ is given with four digits. The results of the computations are collected in Table B.1.

TABLE B.1

α/π	α	ctn α	$f(\alpha/5)$	A
1/20	9°	6.31375	1.887	1.579
1/8	$22^\circ\ 30'$	2.41421	2.092	1.735
1/6	30°	1.73205	2.276	1.821
1/4	45°	1.00000	2.912	2.000
1/3	60°	0.57735	---	2.197
3/8	$67^\circ\ 30'$	0.41421	---	2.306
9/20	81°	0.15838	---	2.533

With the aid of those data a plot can be made of the coefficient A as a function of the slope angle. In the region $\alpha/\pi = 1/20$ to $\alpha/\pi = 1/4$ the function A can be expressed approximately by the formula

$$A = 2.123\ (\alpha/\pi) + 1.47 \tag{B.25}$$

For the commonly encountered slopes $m = 1.0$, $m = 1.5$ and $m = 2.0$ we get $A = 2.000$, $A = 1.864$ and $A = 1.781$.

B.4. Diagram of filtration from a triangular waterway. We can now give a picture of the velocity distribution on the wetted perimeter of a triangular canal and the configuration of the free surface. As an example we analyze a canal with slopes $m = 1.0$ and depth $H = 1.0$ meter. In that case on the side slopes the horizontal and vertical components of the velocity are equal and the value of the reduced filtration velocity on the perimeter can be derived from the relation

$$|v| = \sqrt{2}\, v_x = \frac{1}{\sqrt{2}} \frac{1}{v_x/|v|^2} = \frac{1}{\sqrt{2}} \frac{1}{v_y|v|^2} \tag{B.26}$$

The value of $v_x/|v|^2$ is determined for $m = 1$ and $\alpha = 45^{\circ}$ from Formula (B.12) in the relation

$$(1+i)\frac{v_x}{|v|^2} = \frac{v}{|v|^2} = \frac{1}{I}(1+i)\int_0^{\zeta} \frac{d\zeta}{(1-\zeta^2)^{3/4}\,\zeta^{1/2}} \tag{B.27}$$

We set $\zeta^2 = 1/(1+x^4)$ and we obtain

$$v/|v|^2 = [2(1+i)/I] \int_x^{\infty} dx/(1+x^4)^{1/2}$$

For $x > 1$ $(\zeta < 1/\sqrt{2})$ we carry through the change of variable $x = 1/z$ and get

$$v/|v|^2 = [2(1+i)/I] \int_0^{z} dz/(1+z^4)^{1/2} \tag{B.28}$$

For $x < 1$ $(\zeta > 1/\sqrt{2})$ we get

$$v/|v|^2 = [(1+i)/I]\left[I - 2\int_0^{x} dx/(1+x^4)^{1/2}\right] \tag{B.29}$$

The value of the integrals in (B.28) and (B.29) can be found by decomposition in series:

$$\int_0^{x} \frac{dx}{(1+x^4)^{1/2}} = x - \frac{1}{2}\,\frac{x^5}{5} + \frac{1.3}{2.4}\,\frac{x^9}{9} - \dots + R_n \tag{B.30}$$

Here the residual term R_n can be estimated by the formula:

$$|R_n| < \frac{1.\ 3.\ 5\ \ldots\ (2n+1)}{2.\ 4.\ 6\ \ldots\ 2n} \cdot \frac{x^{4n+1}}{4n+1} \qquad (B.31)$$

where n is the rank number of terms, starting with 2, and further fixing the value $|R_n| < 0.00005$ we arrive at the value of both integrals.

Substituting in Formula (B.15) the value ζ from Formula (B.3) we obtain for $n = 1$ the following expression for the waterway coordinates:

$$x = X \frac{v}{|v|^2} - \frac{Q}{I\pi k_f} f\left(\frac{\alpha}{\pi}, \zeta\right) \qquad (B.32)$$

$$y = H - x = \frac{B}{2} - x \qquad (B.33)$$

Here

$$f(\alpha/\pi, \zeta) = \int_0^{\zeta} (\text{arc sin } \zeta)\ d\zeta/\zeta^{1/2}\ (1 - \zeta^2)^{3/4}$$

For $\zeta < 1/\sqrt{2}$ we get

$$f(\alpha/\pi, \zeta) = 2 \int_0^{z} \text{arc cos} \frac{1}{(1 + z^4)^{1/2}} \cdot \frac{dz}{(1 + z^4)^{1/2}}$$

$$= 2\ a \int_0^{1} \text{arc cos} \frac{1}{(1 + a^4)^{1/2}} \cdot \frac{dz}{(1 + z^4 z^4)^{1/2}} \qquad (B.34)$$

Here, for convenience of application of the formula of Gauss, the upper limit of the integral is made unity, whereupon

$$a^4 = \zeta^2/(1 - \zeta^2)$$

where ζ is the upper limit of the integral $f(\pi/\alpha), \zeta)$. For

$$\zeta > 1/\ 2,\ f(\alpha/\pi, \zeta) = \frac{1}{2}\frac{\pi}{2} I - (\text{arc cos } \zeta) I$$

$$+ 2\ a \int_0^{1} \text{arc cos} \frac{1}{(1+a^4 z^4)^{1/2}}\ \frac{dz}{(1+a^4 z^4)^{1/2}}$$

(B.35)

provided $a^4 = (1 - \zeta^2)/\zeta^2$ where ζ is the upper limit of the integral $f(\alpha/\pi, \zeta)$. Setting intervals between the value of X equal to $Q/20\ \pi\ k_f$

we find the value ζ corresponding to this value of X, and after this the value of the velocity v and the coordinates x and y. From this same value we derive the coordinates of the points of the waterway perimeter, between which seeps 1/20 of the discharge and the value of the velocity at those points. On the free surface for $m = 1.0$, from relation (B.5) we have:

$$\frac{v}{|v|^2} = \frac{v_x}{v_y} + i = 1 - \frac{1}{I \cos \alpha} \int_1^{\zeta} \frac{d\zeta}{\zeta(\zeta^2 - 1)^{3/4}} + i \qquad (B.36)$$

Making the substitution $\zeta^2 = 1/(1 - x^4)^{1/2}$ we obtain

$$\frac{v}{|v|^2} = i + 1 - \frac{2}{I \cos \alpha} \int_0^x \frac{dx}{1 - x^4} = i + 1$$

$$- \frac{2}{I \cos \alpha} \left(x + \frac{1}{2} \frac{x^5}{5} + \frac{1.3}{2.4} \frac{x^9}{9} + \ldots + R_n \right) \qquad (B.37)$$

The residual term of the series can be written in the form

$$R_n < x^{4n+1} / (4n+1)(1-x^4)^{1/2} \qquad (B.38)$$

For values of x close to unity the series converges extremely slowly. Consequently, we introduce the change of variables $x^4 = 1 - z^4$ and we get

$$2 \int_0^x \frac{dx}{(1 - x^4)^{1/2}} = I \cos \alpha - 2 \int_0^z \frac{z\,dz}{(1 - z^4)^{3/4}} = I \cos \alpha - 2 \left(\frac{z^2}{2} \right.$$

$$\left. + \frac{3}{4} \frac{z^6}{6} + \frac{3}{4} \cdot \frac{7}{8} \frac{z^{10}}{10} + \ldots + R_n \right) \qquad (B.39)$$

where the residual term of the series can be determined by the formula

$$R_n < z^{4n+2} / (4n+2)(1-z)^{3/4} \qquad (B.40)$$

Let us set $R_n < 0.0005$.

The coordinates of the free surface are determined by Formula (B.16) with the relations:

$$y = (Q/\pi k_f) \text{ arc cosh } \zeta$$

$$x = y \frac{v_x}{v_y} - \frac{Q}{\pi k_f I \cos \alpha} \int_1^{\zeta} \text{arc cosh } \zeta \frac{d\zeta}{\sqrt{\zeta} (\zeta^2 - 1)^{3/4}}$$

There we introduce the same substitution as for the calculation of $\dot{v}/|v|^2$ and we obtain

$$x = y\,\frac{v_x}{v_y} - \frac{Q}{\pi\, k_f} \cdot \frac{2a}{I \cos \alpha} \int_0^1 \text{arc cosh} \left(\frac{1}{[1-a^4 z^4]^{1/2}} \right) \frac{dz}{[1-a^4 z^4]^{1/2}} \qquad \text{(B.41)}$$

where $a^4 = (\zeta^2 - 1)/\zeta^2$ provided ζ is the upper limit of the integral.

For the given problem we take H = 1.0 meter. Then B = 2.0 meters and the discharge of filtration per unit length of canal is $Q = k_f(B+2H) = 4\, k_f$. For the calculation of $f(\alpha/\pi, \zeta)$ and of the integral (B.41) by the formula of Gauss as before, we use five values of ζ. The results of calculation of the coordinates of the points of the canal perimeter, between which 1/20 of the discharge seeps, of the filtration velocities of those points and the coordinates of the free surface are illustrated in Table B.2.

TABLE B.2

Waterway Parameters				
Xk_f/Q	α = arc sin ζ	v	x	y
0	0	∞	0	1.000
0.05	9°	3.302	0.029	0.971
0.10	18°	2.323	0.081	0.919
0.15	27°	1.881	0.149	0.351
0.20	36°	1.608	0.231	0.769
0.25	45°	1.414	0.325	0.675
0.30	54°	1.262	0.431	0.569
0.35	63°	1.333	0.549	0.451
0.40	72°	1.017	0.681	0.319
0.45	81°	0.900	0.829	0.171
0.50	90°	0.707	1.000	0.000

(Table B.2 is continued on the following page.)

TABLE B.2 (Continued)

Free Surface Coordinates			
Y	v_x/v_y	x	y
0.0	1.000	1.000	0.0
0.4	0.575	1.286	0.4
0.8	0.410	1.481	0.8
1.2	0.293	1.621	1.2
1.6	0.271	1.722	1.6

The filtration picture and the velocity distribution on the wetted perimeter is presented in Figure B.1, where the points between which 1/20 of the discharge seeps are indicated with lines.

B.5 Filtration from a trapezoidal canal. In the case of filtration from a trapezoidal canal the boundaries of the problem, in addition to the sides and the free surface, will include also the bottom of the canal of width b on which arc tan $(v_y/v_x) = \pi/2$.

In the domain Z, the filtration pattern is mapped as in the case of a triangular canal upon a semi-infinite strip as shown in Figure B.2. In the domain of the reduced velocity vector the filtration pattern is mapped onto a pattern analogous to that of the triangular canal, but with an interior cut on line c d e, i.e., on the vertical axis corresponding to the bottom of the canal. In the domain of inverse velocity the figure will have the form of a triangle abcefg with also an interior cut on line c d e. We designate the discharge through the bottom Q_b and the velocity on the axis on the bottom v_d. We map upon the semi-plane ζ the contour of the problem in the domain $1/v'$, and in the domain Z. Let us set the following corresponding points:

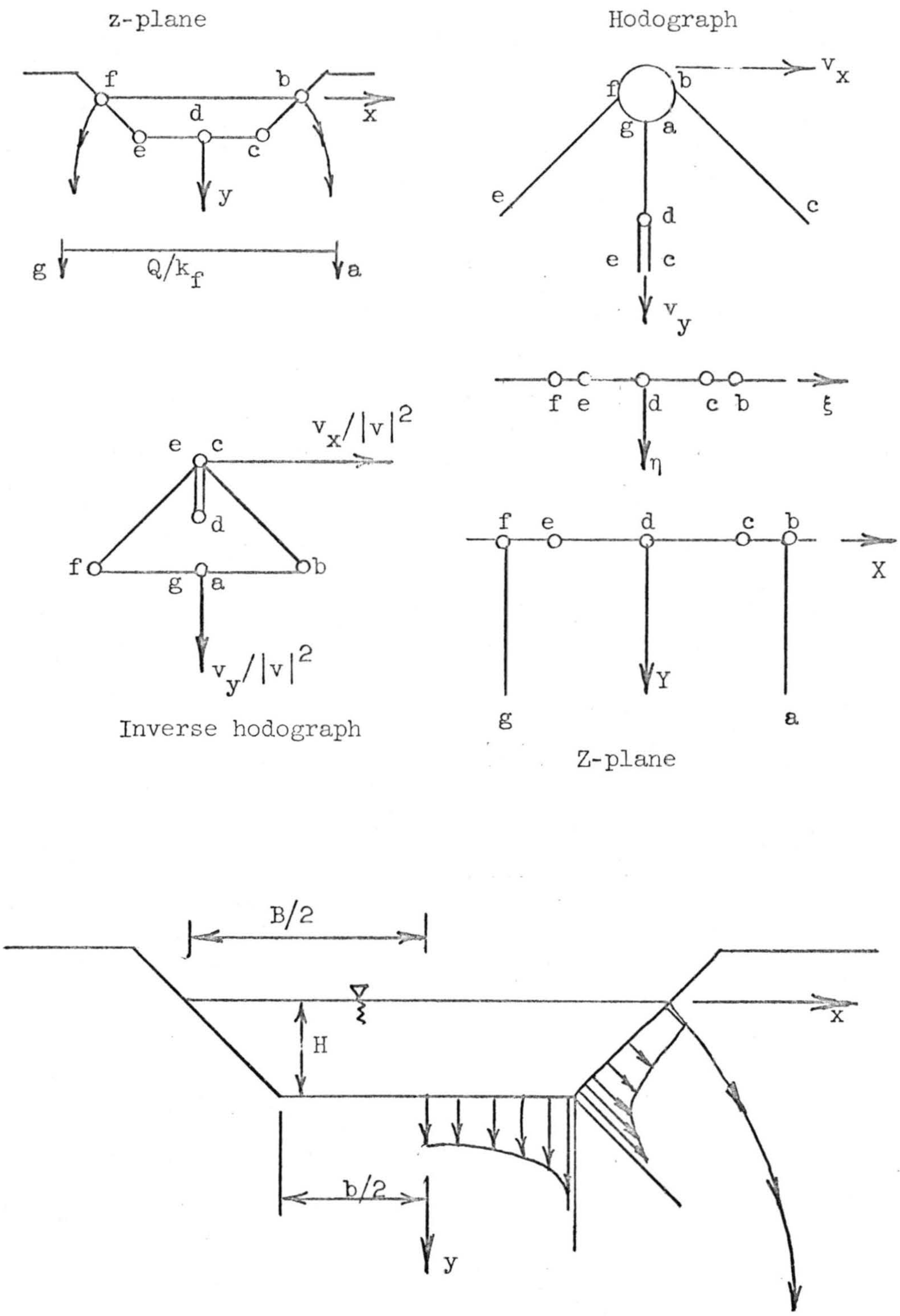
1. $x = 0 \quad y = H; \quad X = Y = 0; \quad \dfrac{v}{|v|^2} = \dfrac{i}{v_d}; \quad \xi = \eta = 0$

2. $x = \pm \dfrac{B}{2}; \quad X = \pm \dfrac{Q}{2k_f}; \quad \dfrac{v}{|v|^2} = \pm \tan \alpha + i; \quad \xi = \pm 1$

   $y = 0 \qquad Y = 0 \qquad \eta = 0$

3. $x = \pm \dfrac{Q}{2k_f}; \quad X = \pm \dfrac{Q}{2k_f}; \quad \dfrac{v}{|v|^2} = i; \quad \xi = \pm \infty \quad \eta = 0$

   $y = \infty \qquad Y = \infty$


z-plane
f
b
d
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e
c
y
g
Q/k_f
a
Hodograph
v_x
f
b
g
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c
v_y
f e
d
c b
ξ
η
e c
v_x/|v|^2
d
f
b
g
a
v_y/|v|^2
Inverse hodograph
f e d c b
X
Y
g
a
Z-plane
B/2
H
x
b/2
y

Figure B.2

The relation between Z and ζ will have the same form as before (B.3)

$$X + iY = (Q/\pi k_f) \text{ arc sin } \zeta$$

or $\zeta = \sin(iW\pi/Q)$. The relation between the vector of the domain $1/v'$ and the vector of the upper half-plane ζ we get by use of the Schwarz-Christoffel formula in the form:

$$\frac{v}{|v|^2} = D \int_0^{\zeta} \frac{\zeta d\zeta}{(\zeta^2 - 1)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} + C$$

$$= iD \int_0^{\zeta} \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (k^2 - \zeta^2)^{1-\alpha/\pi}} + \frac{i}{v_d} \qquad (B.42)$$

For $k < \zeta < 1$ we have

$$\frac{v}{|v|^2} = - D (\sin \alpha + i \cos \alpha) \int_k^{\zeta} \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} \qquad (B.43)$$

We designate

$$\int_0^{k} \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (k^2 - \zeta^2)^{1-\alpha/\pi}} = I_1$$

$$\int_k^{1} \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} = I_2$$

After the substitutions $k^2 - \zeta^2 = k^2 u$ in I_1 and $\zeta^2 - k^2 = k_1^2 t$ in I_2 we get

$$I_1 = \frac{1}{2k_1} \left(\frac{k}{k_1}\right)^{2\alpha/\pi} \int_0^1 \frac{du}{\left(1 + \frac{k^2}{k_1^2} u\right)^{1/2+\alpha/\pi} (u)^{1-\alpha/\pi}} \qquad (B.44)$$

$$I_2 = \frac{1}{2k_1} \int_0^1 \frac{dt}{(1 - t)^{1/2+\alpha/\pi} t^{1-\alpha/\pi}} \qquad (B.45)$$

The latter integral can be easily expressed through Gamma functions. We arrive to

$$I_2 = \frac{\Pi(1/2 - \alpha/\pi)\ \Pi(\alpha/\pi)}{k_1(\alpha/\pi)\ (1/2 - \alpha/\pi)\sqrt{\pi}} \qquad \text{(B.46)}$$

$$I_2 = I/k_1 \qquad \text{(B.47)}$$

where I is defined in Formula (B.9). Hence I_2 possesses the same characteristic as I, expressed in Formula (B.10). We now determine the constant D. For $\zeta = 1$ we have:

$$v/|v|^2 = \tan\alpha\ i = -D(\sin\alpha + i\cos\alpha)\ I_2$$

and hence

$$D = -1/I_2 \cos\alpha = k_1/I \cos\alpha.$$

The velocity v_d on the axis of the bottom of the canal by (B.42) is equal to:

$$v_d = -1/DI_1 = I_2 \cos\alpha/I_1 \qquad \text{(B.48)}$$

The relation between the coordinates of the plane z and of the complex potential plane or of the ζ plane [the relation between those last two planes is known and is determined by Formula (B.4)] can be derived with the help of (B.1) in the form:

$$z = \frac{-i}{I_2\cos\alpha}\ \frac{(-i)Q}{\pi k_f}\int_0^{\zeta}\left(\int_0^{\zeta}\frac{\zeta d\zeta}{(1-\zeta^2)^{1/2+\alpha/\pi}\ (k^2-\zeta^2)^{1-\alpha/\pi}}\right)\frac{d\zeta}{\sqrt{1-\zeta^2}}$$

$$+\frac{1}{v_d}\ \frac{Q}{\pi k_f}\ \text{arc sin}\ \zeta + iH \qquad \text{(B.49)}$$

For the coordinates of the bottom $(0 < \zeta < k)$ we obtain:

$$z = \frac{1}{I_2\cos\alpha}\cdot\frac{Q}{\pi k_f}\left[\text{arc sin}\ \zeta\left(I_1 - \int_0^{\zeta}\frac{\zeta d\zeta}{(1-\zeta^2)^{1/2+\alpha/\pi}\ (k^2-\zeta^2)^{1-\alpha/\pi}}\right)\right.$$

$$\left.+\int_0^{\zeta}\text{arc sin}\ \zeta\cdot\frac{\zeta d\zeta}{(1-\zeta^2)^{1/2+\alpha/\pi}\ (k^2-\zeta^2)^{1-\alpha/\pi}}\right] + iH \qquad \text{(B.50)}$$

For the coordinates of the sides $(k < \zeta < 1)$ we have:

$$z = \frac{b}{2} + iH$$

$$+ (1 - i \tan \alpha) \frac{1}{I_2} \frac{Q}{\pi k_f} \left[\text{arc sin } \zeta \cdot \int_k^\zeta \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} \right.$$

$$\left. - \int_k^\zeta \text{arc sin } \zeta \cdot \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} \right] \tag{B.51}$$

and for the coordinates of the free surface $(\zeta > 1)$:

$$z = \frac{B}{2} + \frac{1}{I_2 \cos \alpha} \cdot \frac{Q}{\pi k_f} \left[\int_1^\zeta \frac{(\text{arc cosh } \zeta)\, \zeta d\zeta}{(\zeta^2 - 1)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} \right.$$

$$\left. + \text{arc cosh } \zeta \left(I_2 \cos \alpha - \int_1^\zeta \frac{\zeta d\zeta}{(\zeta^2 - 1)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} \right) \right]$$

$$+ i \frac{Q}{\pi k_f} \text{arc cosh } \zeta. \tag{B.52}$$

B.6 Formula of discharge for a trapezoidal canal. Let us introduce the following notation:

$$\int_0^k \text{arc sin } \zeta \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (k^2 - \zeta^2)^{1-\alpha/\pi}} = f_1(\alpha/\pi, k) \tag{B.53}$$

$$\int_k^1 \text{arc sin } \zeta \frac{\zeta d\zeta}{(1 - \zeta^2)^{1/2+\alpha/\pi} (\zeta^2 - k^2)^{1-\alpha/\pi}} = f_2(\alpha/\pi, k) \tag{B.54}$$

The width of the waterway on the bottom will be determined by the relation

$$b = \frac{2}{I_2 \cos \alpha} \cdot \frac{Q}{\pi k_f} f_1(\alpha/\pi, k) \tag{B.55}$$

The waterway width on the surface and the depth will be given by

$$B = b + \frac{Q}{k_f} \left[1 - \frac{f_2(\alpha/\pi, k)}{\pi I_2/2} \right] \tag{B.56}$$

$$H = \frac{Q}{2k_f} \tan \alpha \left[1 - \frac{f_2(\alpha/\pi, k)}{\pi I_2/2} \right] \tag{B.57}$$

The discharge formula has the form $Q = k_f(B + AH)$ (B.58) where

$$A = \frac{2}{\tan\alpha} \, \frac{f_2(\alpha/\pi,k) - \frac{1}{\cos\alpha} f_1(\alpha/\pi,k)}{\pi I_2/2 - f_2(\alpha/\pi,k)} \tag{B.59}$$

Let us turn now to the calculation of the value of A. For this purpose we calculate the values of $f_1(\alpha/\pi,k)$ and $f_2(\alpha/\pi,k)$ by the formula of Gauss. We figure out the necessary integrals for values of α corresponding to $\alpha = \pi/4$, $\alpha = \pi/6$, and $\alpha = \pi/8$. For those angles $f_1(\alpha/\pi,k)$ and $f_2(\alpha/\pi,k)$ can be determined by way of substitutions analogous to those for the calculation of $f(\alpha/\pi)$ in Section B.3. For $\alpha/\pi = 1/4$ we have for $\zeta = [1 - k_1^2/(1 - k^2z^4)]^{1/2}$ where $(0 < \zeta < k)$:

$$f_1(\alpha/\pi,k) = \frac{2k}{k_1}\int_0^1 \operatorname{arc\,cos} \frac{k_1}{\sqrt{1 - k^2z^4}} \, \frac{dz}{\sqrt{1 - k^2z^4}}$$

and for $\zeta = [1 - k_1^2(1 + x^4)]^{1/2}$ and $x = 1/z$ where $(k < \zeta < 1)$:

$$f_2(\alpha/\pi,k) = \frac{2}{k_1}\left[\int_0^1 \operatorname{arc\,cos} \frac{k_1}{\sqrt{1 + x^4}} \, \frac{dx}{\sqrt{1 + x^4}} + \int_0^1 \operatorname{arc\,cos} \frac{k_1z^2}{\sqrt{1 + z^4}} \cdot \frac{dz}{\sqrt{1 + z^4}}\right]$$

For $\alpha/\pi = 1/6$ we obtain for $\zeta = [1 - k_1^2/(1 - k^2z^6)]^{1/2}$ where $(0 < \zeta < k)$:

$$f_1(\alpha/\pi,k) = \frac{3\sqrt{k}}{k_1}\int_0^1 \operatorname{arc\,cos} \frac{k_1}{\sqrt{1 - k^2z^6}} \, \frac{dz}{\sqrt{1 - k^2z^6}} \, ;$$

and for $\zeta = [1 - k_1^2x^2/(1 + x^3)]^{1/2}$ with $x = 1/z^2$ where $(k < \zeta < 1)$:

$$f_2(\alpha/\pi,k) = \frac{3}{2k_1}\int_0^1 \operatorname{arc\,cos} \frac{k_1x^{3/2}}{\sqrt{1 + x^3}} \, \frac{dx}{\sqrt{1 + x^3}} + \frac{3}{k_1}\int_0^1 \operatorname{arc\,cos} \frac{k_1}{\sqrt{1 + z^6}} \, \frac{dz}{\sqrt{1 + z^6}}$$

For $\alpha/\pi = 1/8$ we have for $\zeta = [1 - k_1^2/(1 - k^2z^8)]^{1/2}$ where $(0 < \zeta < k)$:

$$f_1(\alpha/\pi,k) = 4 \ \frac{4\sqrt{k}}{k_1} \int_0^1 \text{arc cos} \frac{k_1}{\sqrt{1 - k^2 z^8}} \ \frac{dz}{\sqrt{1 - k^2 z^8}}$$

and for $\zeta = [1 - k_1/(1 + x^8)]^{1/2}$ with $x = 1/z$, where $(k < \zeta < 1)$:

$$f_2(\alpha/\pi,k) = \frac{4}{k_1}\left[\int_0^1 \text{arc cos} \frac{k_1}{\sqrt{1 + x^8}} \ \frac{dx}{\sqrt{1 + x^8}} + \int_0^1 \text{arc cos} \frac{k_1 z^4}{\sqrt{1 + z^8}} \ \frac{z^2\,dz}{\sqrt{1 + z^8}}\right]$$

Results of the calculations are illustrated in Table B.3.

TABLE B.3

k^2	bk_f/Q	Q_b/Q	Bk_f/Q	Hk_f/Q	A	B/H
			$\alpha = 45^o$ $m = 1.0$			
0	0	0	0.5000	0.2500	2.0000	2.000
0.2500	0.1668	0.3333	0.5449	0.1891	2.407	2.882
0.5000	0.3035	0.5000	0.5947	0.1456	2.785	4.084
0.7500	0.4788	0.6667	0.6760	0.09860	3.286	6.856
0.8750	0.6093	0.7699	0.7456	0.06817	3.732	10.94
0.9375	0.7061	0.8392	0.8021	0.04801	4.123	16.71
1.000	1.0000	1.0000	1.0000	0	∞	∞
			$\alpha = 30^o$ $m = 1.732$			
0	0	0	0.6555	0.1892	1.821	3.464
0.2500	0.2036	0.3333	0.6840	0.1387	2.279	4.932
0.5000	0.3526	0.5000	0.7203	0.1062	2.634	6.785
0.7500	0.5299	0.6667	0.7769	0.07129	3.130	10.90
0.875	0.6533	0.7699	0.8245	0.04841	3.552	16.69
1.000	1.0000	1.0000	1.0000	0	∞	∞

(Table B.3 is continued on the following page.)

TABLE B.3 (Continued)

k^2	bk_f/Q	Q_b/Q	Bk_f/Q	Hk_f/Q	A	B/H
			$\alpha = 22°30'$ $n = 2.414$			
0	0	0	0.7357	0.15240	1.735	4.828
0.250	0.2289	0.3333	0.7572	0.10940	2.220	6.921
0.500	0.3840	0.5000	0.7856	0.08316	2.579	9.446
0.750	0.5592	0.6667	0.8292	0.05592	3.054	14.83
0.875	0.6782	0.7699	0.8652	0.03872	3.482	22.34
1.000	1.0000	1.0000	1.0000	0	∞	∞

In this table are given the values of b, B, H in portions of Q/k_f, i.e., of the width of the flow strip filtrating under the bottom, and corresponding to the values of the modules k and the quantities A and B/H, directly of interest to us.

The portion Q_b/Q of the discharge filtrating through the bottom of the canal is determined according to relation (B.4):

$$Q_b/Q = (2/\pi) \text{ arc sin } k.$$

and consequently the portion seeping through the side is

$$1 - Q_b/Q = 1 - (2/\pi) \text{ arc sin } k$$

B.7 Filtration pattern from a trapezoidal channel. We give now the picture of the velocity distribution on the wetted perimeter and the configuration of the free surface. We figure it out in detail for a canal of one-on-one slopes. Let us take the value arc sin $k = 52°30'$. With this we get, as we shall see later, the cross section of a canal sufficiently typical for the average irrigation canal, and the ratio (width of bottom)/(depth) will be $b/H \approx 3.16$.

The values of the filtration velocities on the bottom $(-k < \zeta < k)$ of the canal are determined by the relation: $|v| = v_y = i|v|^2/v$ where with use of (B.42)

$$\frac{v}{|v|^2} = \frac{i}{I_2 \cos \alpha} \left[I_1 - \int_0^{\zeta} \frac{\zeta \, d\zeta}{(1 - \zeta^2)^{3/4}(k^2 - \zeta^2)^{3/4}} \right]$$

Carrying out the change of variables $\zeta^2 = k^2(1 - z^4)/(1 - k^2z^4)$ we get

$$\int_0^{\zeta} \frac{\zeta \, d\,\zeta}{(1-\zeta^2)^{3/4}\,(k^2-\zeta^2)^{3/4}} = 2\,\frac{\sqrt{k}}{k_1}\int_0^{z}\frac{dz}{\sqrt{1-k^2z^4}}$$

$$= 2\,\frac{\sqrt{k}}{k_1}\left(z + \frac{1}{2.5}\,k^2z^5 + \frac{1.3}{2.4}\,\frac{k^4z^9}{9} + \ldots + R_n\right) \qquad \text{(B.60)}$$

where R_n can be put in the form:

$$R_n < k^{2n}\,\frac{z^{4n+1}}{4n+1}\cdot\frac{1}{\sqrt{1-k^2z^4}} \qquad \text{(B.61)}$$

The value of the coordinates of the bottom in (B.50) and (B.3) can be derived from the relation:

$$x = X\,\frac{v_y}{|v|^2} + \frac{1}{I_2\cos\alpha}\cdot\frac{Q}{\pi k_f}\;f_1(\alpha/\pi,k,\zeta) \qquad \text{(B.62)}$$

$$y = H \qquad \text{(B.63)}$$

Here

$$f_1(\alpha/\pi,k,\zeta) = \int_0^{\zeta}\text{arc sin}\,\zeta\,\frac{\zeta d\zeta}{(1-\zeta^2)^{3/4}\,(k^2-\zeta^2)^{3/4}}$$

$$= 2\,\frac{\sqrt{k}}{k_1}\,a\int_0^{1}\text{arc cos}\,\frac{k_1}{\sqrt{1-k^2a^4z^4}}\,\frac{dz}{\sqrt{1-k^2a^4z^4}}$$

(B.64)

provided $a^4 = (k^2-\zeta^2)/k^2(1-\zeta^2)$ where ζ is the upper limit of the integral $f_1(\alpha/\pi,k,\zeta)$. On the side we have:

$$v = \sqrt{2}\,v_x = \sqrt{2}\,v_y$$

$$\frac{v_x}{|v|^2} = \frac{k_1}{I}\int_k^{\zeta}\frac{\zeta\,d\,\zeta}{(1-\zeta^2)^{3/4}\,(\zeta^2-k^2)^{3/4}} \qquad \text{(B.65)}$$

Substituting $\zeta^2 = 1 - k_1^{\,2}/(1+x^4$, but for $\zeta > \sqrt{1-k_1^{\,2}/2}$ the substitution $\zeta^2 = 1 - k_1^{\,2}z^4/(1+z^4)$ we obtain for $k < \zeta < \sqrt{1-k_1^{\,2}/2}$

$$\frac{v_x}{|v|^2} = \frac{2}{I} \int_0^x \frac{dx}{\sqrt{1 + x^4}} \tag{B.66}$$

and for $\sqrt{1 - k_1^2/2} < \zeta < 1$

$$\frac{v_x}{|v|^2} = 1 - \frac{2}{I} \int_0^z \frac{dz}{\sqrt{1 + z^4}} \tag{B.67}$$

The sequence of calculation of integrals $\int_0^x (1 + x^4)^{-1/2} dx$

explained in Section B.4.

The coordinates of the side points using (B.51) and (B.3) are determined by the relation:

$$z = \frac{b}{2} + iH + (1 - i) \left[X \frac{v_x}{|v|^2} - \frac{Q}{I_2 \pi k_f} f_2(\alpha/\pi, k, \zeta) \right] \tag{B.68}$$

Here for $k < \zeta < (1 - k^2/2)^{1/2}$

$$f_2(\alpha/\pi, k, \zeta) = \int_k^\zeta \text{arc sin } \zeta \frac{d\zeta}{(1 - \zeta^2)^{3/4} (\zeta^2 - k^2)^{3/4}}$$

$$= 2 \frac{a}{k_1} \int_0^1 \text{arc cos } \frac{k_1}{\sqrt{1 + a^4 x^4}} \frac{dx}{\sqrt{1 + a^4 x^4}} \tag{B.69}$$

provided $a^4 = (\zeta^2 - k^2)/(1 - \zeta^2)$ where ζ is the upper limit of the integral f_2.

For $\sqrt{1 - k_1^2/2} < \zeta < 1$

$$f_2(\alpha/\pi, k, \zeta) = \frac{2}{k_1} \left[\int_0^1 \text{arc cos } \frac{k_1}{\sqrt{1 + x^4}} \frac{dx}{\sqrt{1 + x^4}} + \int_0^1 \text{arc cos } \frac{k_1 z^2}{\sqrt{1 + z^4}} \frac{dz}{\sqrt{1 + z^4}} \right]$$

$$- \frac{2a}{k_1} \int_0^1 \text{arc cos } \frac{k_1 a^2 z^2}{\sqrt{1 + a^4 z^4}} \frac{dz}{\sqrt{1 + a^4 z^4}} \tag{B.70}$$

provided $a^4 = (1 - \zeta^2)/(\zeta^2 - k^2)$, where ζ is the upper limit of the integral $f_2(\alpha/\pi, k, \zeta)$ for $\zeta > \sqrt{1 - k_1^2/2}$.

On the free surface using (B.42) we have

$$\frac{v_x}{v_y} = 1 - \frac{k_1}{I \cos \alpha} \int_1^{\zeta} \frac{\zeta \, d\zeta}{(\zeta^2 - 1)^{3/4} (\zeta^2 - k^2)^{3/4}}$$

$$= 1 - \frac{2}{k_1} \int_0^x \frac{dx}{\sqrt{1 - x^4}} \tag{B.71}$$

where $x^4 = (\zeta^2 - 1)/(\zeta^2 - k^2)$. The method of calculation of $\int_0^x (1 - x^4)^{-1/2} dx$ is shown in (B.4).

The coordinates of the free surface we derive using (B.52) and (B.3) from the relations:

$$y = (Q/\pi k_f) \text{ arc cosh } \zeta$$

$$x = \frac{B}{2} + y \frac{v_x}{v_y} + \frac{k_1}{I \cos \alpha} \frac{Q}{\pi k_f} \int_1^{\zeta} \text{arc cosh } \zeta \frac{\zeta \, d\zeta}{(\zeta^2 - 1)^{3/4} (\zeta^2 - k^2)^{3/4}}$$

$$= \frac{B}{2} + y \frac{v_x}{v_y} + \frac{2}{I \cos \alpha} \frac{Q}{\pi k_f} a \int_0^1 \text{arc sinh } \frac{ka^2x^2}{\sqrt{1 - a^4x^4}} \frac{dx}{\sqrt{1 - a^4x^4}} \tag{B.72}$$

provided $a^4 = (\zeta^2 - 1)/(\zeta^2 - k^2)$, where ζ is the upper limit of the integral.

The values $f_1(\alpha/\pi, k, \zeta)$, $f_2(\alpha/\pi, k, \zeta)$, and $\int_0^1 \text{arc sinh } \frac{ka^2x^2}{\sqrt{1 - a^4x^4}} \times \frac{dx}{\sqrt{1 - a^4x^4}}$ were computed using the formula of Gauss for five ordinates. On the waterway perimeter we set intervals between values of $X = Q/18\pi k_f$. We derive the coordinates of perimeter points between which 1/8 of the discharge seeps and the values of the filtration velocity of those points. In addition a few complementary points are given. Results of the calculation are indicated in Table B.4.

TABLE B.4

Xk_f/Q	α = arc sin ζ	v	xk_f/Q	yk_f/Q
		Bottom of Canal		
0	0	1.350	0	0.1226
1/18	10°	1.359	0.0413	0.1226
2/18	20°	1.391	0.0820	0.1226
3/18	30°	1.459	0.1213	0.1226
4/18	40°	1.612	0.1578	0.1226
5/18	50°	2.886	0.1883	0.1226
---	52° 30'	∞	0.1934	0.1226
		Side Slope of Canal		
---	52° 30'	∞	0.1934	0.1226
---	55°	2.099	0.1971	0.1189
6/18	60°	1.640	0.2073	0.1087
---	64° 30' 11".8	1.414	0.2191	0.0968
7/18	70°	1.224	0.2375	0.0784
8/18	80°	0.998	0.2711	0.0448
1/2	90°	0.707	0.3159	0

Free Surface

Yk_f/Q	v_x/v_y	xk_f/Q	yk_f/Q
0	1.000	0.3159	0
0.1	0.705	0.3799	0.1
0.2	0.295	0.4172	0.2
0.3	0.199	0.4412	0.3
0.4	0.139	0.4558	0.4
0.5	0.099	0.4636	0.5

The waterway width at the bottom is $b = 2b/2 = 0.3867\ Q/k_f$, and the depth is $H = 0.1226\ Q/k_f$. The width on the surface is $B = 0.6319\ Q/k_f$. Consequently, $B/H = 5.158$, $b/H = 3.158$, and $A = 3.005$. All the figures in the table are given in ratios of Q/k_f. Setting the same value of Q/k_f we get the absolute dimensions of the waterway. Conversely, fixing one of the quantities b, B, or H we derive the two others and the discharge Q. Let us take for example $b = 5.0$ meters, then $H = 1.58$ meters, $B = 8.16$ meters, and the seepage discharge per unit length of canal is

$$Q = k_f[(8.16) + (3.005)(1.58)] = 12.91\ k_f$$

The filtration pattern is indicated on Figure B.2. Arrows indicate the points between which 1/18 of the discharge seeps.

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