DIFFUSION FROM A LINE SOURCE IN LAMINAR FLOW OVER A WEDGE AND IN BLASIUS FLOW

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DIFFUSION FROM A LINE SOURCE IN LAMINAR FLOW OVER A WEDGE AND IN BLASIUS FLOW

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ABSTRACT

The velocity distribution in the laminar flow over a semi-infinite plate was calculated by Blasius (1908). The corresponding problem for the laminar symmetric flow over a wedge was solved by Falkner and Skan (1930), in collaboration with Hartree (1937). In the present paper, a line source of mass is considered to be situated at the leading edge of the plate or wedge, which is supposed to be nonconductive of vapor, and the resulting vapor distribution is sought. If free convection is neglected, and the velocity distribution is assumed essentially undisturbed by the variation of vapor concentration, the boundary-layer equation of diffusion for each case can be solved by certain simple substitutions and integrations, the solutions being applicable to similar problems in heat diffusion. Numerical calculations have been carried out for Blasius flow.

1. LAMINAR FLOW OVER A WEDGE

CONSIDER a wedge placed symmetrically in a uniform incident stream, with its edge perpendicular to the general direction of flow. In a plane perpendicular to the edge, let the trace of the edge be the origin, from which x is measured along the trace of the wedge, and let y be measured in a direction normal to that of x. It can be shown that the velocity in the x-direction just outside of the boundary layer is approximately

$$u_1 = cx^m \tag{1}$$

where c is a constant depending on the incident velocity, and m is connected with the included angle of the wedge, $\beta\pi$, by the relation

$$\beta = \frac{2m}{m+1} \tag{2}$$

Making the substitutions

$$\xi = \sqrt{\frac{m+1}{2}}\sqrt{\frac{c}{\nu}}\, y x^{\frac{m-1}{2}} \tag{3}$$

$$\psi = \sqrt{\frac{2}{m+1}}\sqrt{c\nu}\, x^{\frac{m+1}{2}}\, \zeta(\xi) \tag{4}$$

where ν is the kinematic viscosity, and ψ is the stream-function from which the velocity components in the x- and y-directions can be obtained, respectively, as follows:

$$u = \frac{\partial\psi}{\partial y} = cx^m\, \zeta' = u_1\, \zeta' \tag{5}$$

$$v = -\frac{\partial\psi}{\partial x} = -\sqrt{\frac{2}{m+1}}\sqrt{c\nu x^{m-1}}\left(\frac{m+1}{2}\zeta + \frac{m-1}{2}\zeta'\xi\right) \tag{6}$$

Falkner and Skan (2) transformed the boundary-layer equation of motion

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{\partial u_1}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{7}$$

to the ordinary differential equation

$$\zeta''' + \zeta\zeta'' - \beta(\zeta'^2 - 1) = 0 \tag{8}$$

where, as in Equations [5] and [6], the primes denote differentiations with respect to the variable ξ. The boundary conditions for Equation [7]

$$u = v = 0 \qquad \text{at } y = 0$$

$u = u_1$ at the outer edge of the boundary layer

can be replaced by

$$\zeta(0) = \zeta'(0) = 0, \qquad \zeta'(\infty) = 1 \tag{9}$$

Numerical solution of Equation [8] with the above boundary conditions was carried out by Hartree (3) for different values of β.

Imagine now a line source of mass situated at the edge of the wedge, which is supposed to be non-conductive of vapor. Since the vapor flux through any section perpendicular to the x-axis must then be constant, the quantity

$$M = \int_0^\infty u(c - c_o)\, dy \qquad [10]$$

where c and c_o are the concentrations (mass per unit volume) of vapor at any point and in the ambient flow, respectively, must be independent of x and indeed is a measure of half the strength of the line source.

Let ρ denote the density of the ambient fluid, and μ its dynamic viscosity. Taking M, x, y, c_o, ρ, u_1 and μ as the independent variables for a certain β or m, and $c - c_o$ as the dependent variable, a dimensional analysis shows that the parameter

$$\theta = \frac{c - c_o}{c_o} \qquad [11]$$

must be a function of the dimensionless parameters:

$$\frac{M}{c_o \nu}, \quad \frac{u_1 x}{\nu}, \quad \frac{c_o}{\rho}, \quad \xi$$

where ξ is given by Equation [3] and is chosen instead of $\frac{y}{x}$ at the suggestion of Falkner and Skan's solution. In order that M may be independent of x and Equation [10] may be identically satisfied, it can be easily verified that θ must be of the form

$$\theta = \frac{M}{c_o \nu} \sqrt{\frac{\nu}{u_1 x}}\; t(\xi) = \frac{M}{c_o \nu} \sqrt{\frac{\nu}{c x^{m+1}}}\; t(\xi) \qquad [12]$$

where $t(\xi)$ must satisfy the integral condition

$$\int_0^\infty t(\xi)\, \zeta'(\xi)\, d\xi = 1 \qquad [13]$$

Noteworthy is the fact that $\frac{c_o}{\rho}$ does not appear in Equation [13]. It should be remembered, however, that the quantities ρ, μ, and ν are taken as those of the ambient fluid. The effect of $\frac{c_o}{\rho}$ is therefore reflected in the quantity ν in Equation [12]. The variation of ν due to that of c is neglected. This is justified at sufficient distances from the line source, where $c - c_o$ is not excessively large.

From Equation [12] one obtains

$$\frac{\partial \theta}{\partial x} = -\frac{1}{2}\, x^{-\frac{m+3}{2}} \left[(m+1)\, t - (m-1)\, \xi\, t'\right] \qquad [14]$$

$$\frac{\partial \theta}{\partial y} = x^{-1} \sqrt{\frac{(m+1)c}{2\nu}}\; t' \qquad [15]$$

$$\frac{\partial^2 \theta}{\partial y^2} = \frac{1}{2}\, x^{-\frac{m+3}{2}}\, \frac{(m+1)\, c}{\nu}\, t'' \qquad [16]$$

The boundary-layer equation of diffusion (K = diffusivity)

$$u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} = K \frac{\partial^2 c}{\partial y^2} \qquad [17]$$

is to be solved with the boundary conditions

$$\frac{\partial c}{\partial y} = 0 \qquad \text{at } y = 0$$

$$c = c_o \qquad \text{at the outer edge of the boundary layer.}$$

Substituting Equations [5], [6], [11], [14], [15] and [16] in [17], one obtains (σ = Prandtl number = ν/K)

$$t'' = -\sigma\,(t\,\zeta' + \zeta\, t') \qquad [18]$$

to be solved with the transformed boundary conditions

$$t'(0) = 0 \qquad [19]$$

$$t(\infty) = 0 \qquad [20]$$

and the integral condition expressed by Equation [13].

A first integration of Equation [18] yields

$$t' = -\sigma\, t\, \zeta \qquad [21]$$

the constant of integration being zero because of Equations [9] and [19]. A second integration yields

$$t(\xi) = C e^{-\sigma \int_0^\xi \zeta\, d\xi} \qquad [22]$$

where C is determined from Equation [13] to be

$$C = \frac{1}{\int_0^\infty e^{-\sigma \int_0^\xi \zeta\, d\xi}\, \zeta'(\xi)\, d\xi} \qquad [23]$$

Equation [20] being satisfied by Equation [22], and the the values of ζ and ζ' for different ξ being given by Hartree, Equations [22] and [23], together with Equations [11] and [12], constitute the desired solution.

2. BLASIUS FLOW

When $\beta = m = 0$, Equation [8] becomes

$$\zeta''' + \zeta\,\zeta'' = 0 \qquad [24]$$

for Blasius flow. Blasius' original equation, however, was slightly different, due to the removal of two constant multipliers occurring in Equations [3] and [4]. Assuming

$$\eta = \sqrt{\frac{U}{\nu x}}\; y \qquad [25]$$

$$\psi = \sqrt{\nu U x}\; f(\eta) \qquad [26]$$

where U is the ambient velocity and ψ the stream-function from which

$$u = \frac{\partial \psi}{\partial y} = U f' \qquad [27]$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,(\eta f' - f) \qquad [28]$$

he obtained

$$f f'' + 2 f''' = 0 \qquad [29]$$

where the primes indicate differentiation with respect to η. Blasius' solution of Equation [29] yielded the tabulation of f, f', and f'', which can be found in (1) or (4).

Assuming

$$\theta = \frac{c - c_o}{c_o} = \frac{M}{c_o\,\nu}\sqrt{\frac{\nu}{Ux}}\; h\,(\eta) \qquad [30]$$

a similar procedure as before yields

$$h\,(\eta) = C_1\, e^{-\frac{\sigma}{2}\int_o^{\eta} f d\eta} \qquad [31]$$

where C_1 is to be determined by

$$\int_o^{\infty} h\,(\eta)\, f'\,(\eta)\, d\eta = 1 \qquad [32]$$

corresponding to Equation [13].

By means of Equation [29] and the result $f''(0) = 0.33206$ given by Blasius.

$$\int_o^{\eta} f d\eta = -2\int_o^{\eta}\frac{f'''}{f''}\,d\eta = -2\,ln\,f''\Big]_o^{\eta}$$

$$= -2\,ln\left(\frac{f''}{0.33206}\right) \qquad [33]$$

Substitution of the above in Equation [31] gives

$$h\,(\eta) = C_1\left(\frac{f''}{0.33206}\right)^{\sigma} \qquad [34]$$

where from Equation [32]

$$C_1 = \frac{1}{\int_o^{\infty}\left(\frac{f''}{0.33206}\right)^{\sigma} f'\,d\eta} \qquad [35]$$

TABLE 1 VALUES OF $h(\eta)$ FOR DIFFERENT PRANDTL NUMBERS

η/σ	0.6	0.7	1	2.	4	8	16	32	64	128	256	512	1024
0	0.4921	0.5363	0.6639	1.0153	1.5743	2.4625	3.8773	6.1516	9.7800	15.5111	24.6853	39.1696	59.9710
0.2	0.4921	0.5632	0.6637	1.0149	1.5729	2.4583	3.8643	6.1103	9.6492	15.0988	23.3905	35.1684	46.7322
0.4	0.4916	0.5357	0.6627	1.0117	1.5631	2.4276	3.7684	5.8107	8.7262	12.3485	15.6453	15.7340	9.3536
0.6	0.4904	0.5341	0.6599	1.0032	1.5371	2.3475	3.5238	5.0809	6.6719	7.2189	5.3466	1.8375	0.1275
0.8	0.4880	0.5310	0.6545	0.9869	1.4876	2.1987	3.0914	3.9104	3.9519	2.5327	0.6581	0.0278	
1.0	0.4840	0.5302	0.6458	0.9607	1.4095	1.9741	2.4920	2.5410	1.6687	0.4515	0.0210		
1.2	0.4782	0.5187	0.6330	0.9229	1.3008	1.6811	1.8072	1.3364	0.4615	0.0346			
1.4	0.4703	0.5087	0.6155	0.8727	1.1633	1.3446	1.1560	0.5468	0.0773	0.0009			
1.6	0.4623	0.4986	0.5981	0.8241	1.0372	1.0690	0.7307	0.2184	0.0123				
1.8	0.4477	0.4803	0.5657	0.7371	0.8297	0.6840	0.2991	0.0366	0.0004				
2.0	0.4411	0.4720	0.5333	0.6552	0.6556	0.4270	0.1166	0.0055					
2.2	0.4134	0.4376	0.4965	0.5679	0.4926	0.2411	0.0371	0.0006					
2.4	0.3928	0.4122	0.4560	0.4790	0.3505	0.1220	0.0095	0.0001					
2.6	0.3701	0.3845	0.4128	0.3925	0.2353	0.0550	0.0019						
2.8	0.3452	0.3547	0.3679	0.3118	0.1485	0.0219	0.0003						
3.0	0.3194	0.3239	0.3226	0.2398	0.0878	0.0077							
3.2	0.2922	0.2918	0.2782	0.1782	0.0485	0.0023							
3.4	0.2732	0.2701	0.2357	0.1280	0.0250	0.0006							
3.6	0.2369	0.2285	0.1961	0.0886	0.0120	0.0002							
3.8	0.2096	0.1886	0.1602	0.0582	0.0053								
4.0	0.1734	0.1695	0.1284	0.0380	0.0022								
4.2	0.1590	0.1432	0.1010	0.0235	0.0009								
4.4	0.1360	0.1196	0.0779	0.0140	0.0003								
4.6	0.1150	0,0985	0.0589	0.0080	0.0001								
4.8	0.0960	0.0796	0.0437	0.0044									
5.0	0.0795	0.0640	0.0318	0.0023									
5.2	0.0651	0.0507	0.0227	0.0012									
5.4	0.0524	0.0395	0.0159	0.0006									
5.6	0.0423	0.0308	0.0109	0.0003									
5.8	0.0327	0.0228	0.0073	0.0001									
6.0	0.0257	0.0171	0.0048	0.0001_{-5}									
6.2	0.0195	0.0125	0.0031										
6.4	0.0150	0.0091	0.0020										
6.6	0.0112	0.0065	0.0012										
6.8	0.0083	0.0046	0.0007										
7.0	0.0061	0.0032	0.0004										
7.2	0.0044	0.0022	0.0003										
7.4	0.0031	0.0014	0.0001										
7.6	0.0022	0.0010	0.0001										
7.8	0.0014	0.0006											
8.0	0.0009	0.0004											
8.2	0.0008	0.0003											

and is a function of σ only.

Equation [34] could have been written as

$$h(\eta) = C_2 (f'')^{\sigma} \qquad [36]$$

and Equation [35] as

$$C_2^{-1} = \int_o^{\infty} (f'')^{\sigma} f' \, d\eta \qquad [37]$$

But since f'' varies from 0.33206 to zero, use of Equations [34] and [35] for numerical calculation yields more accurate results for large values of σ. Therefore, and because of the advantages of systematic computation, they are used throughout in computing $h(\eta)$ for values of σ ranging from 0.6 to 1024. The results are shown in Table 1. For convenience, $\frac{h(\eta)}{C_1}$ instead of $h(\eta)$ is plotted in Fig. 1 for different values of σ, the values of C_1 corresponding to different values of σ being the same as those of $h(o)$ given in Table 1.

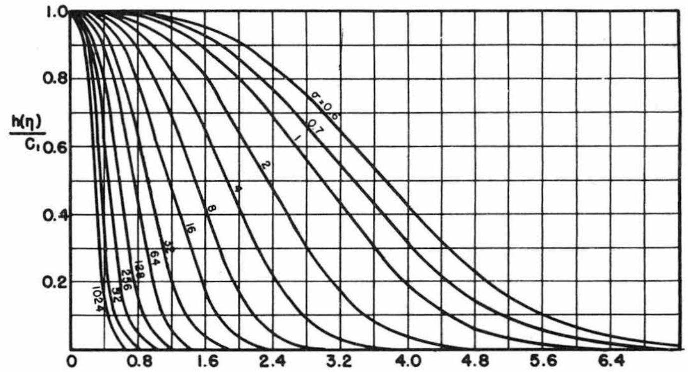

Fig. 1

3. ACKNOWLEDGMENT

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REFERENCES

1. "Grenzschichten in Flüssigkeiten mit kleiner Reibung," by H. Blasius, *Zs. Math. u. Phys.* Bd. 56, p. 1, 1908.

2. "Some Approximate Solutions of the Boundary-Layer Equations," by V. M. Falkner and S. W. Skan, R. & M. No. 1314, British A.R.C., 1930.

3. "On an Equation Occurring in Falkner and Skan's Approximate Treatment of the Equations of the Boundary Layer," by D. R. Hartree, *Cambridge Phil. Soc.*, Vol. 33, 1937.

4. "Boundary Layer Theory," Lecture Series, by H. Schlichting, Part I, *NACA Tech Mem.*, No. 1217, p. 121, 1942 (original) and 1949 (translation).