THESIS

THE IMPACT OF NON-LOCAL ELASTICITY FACTORS ON NATURAL FREQUENCIES OF A RECTANGULAR CANTILEVER BEAM

Submitted By

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ABSTRACT

THE IMPACT OF NON-LOCAL ELASTICITY FACTORS ON NATURAL FREQUENCIES OF A RECTANGULAR CANTILEVER BEAM

The natural frequencies of a structural element are important factors in attaining a safe design. Natural frequency is the frequency at which an element tends to vibrate in the absence of any driving or damping force. When an object vibrates at a frequency equivalent to its natural frequency, its vibration amplitude increases significantly, which could lead to severe damage. A safe design would thus require having a different natural frequency compared to the frequency of the vibrating element.

In some cases, obtaining accurate natural frequencies is challenging. In cases in which non-local elasticity, where the stress at a point is a function of the strain at the close region around that point, provides a better solution to the mechanical problems compared to other theories, natural frequencies should be studied.

The non-local elastic solution to the non-local elastic natural frequencies of a rectangular cantilever beam problem was developed using a Fortran code, and the finite elements of non-local mesh were generated using a MATLAB code. The eigenvalue problem was solved, and the mode shapes were plotted using another MATLAB code.

The results indicate that the natural frequencies for the non-local solution have dropped 25–30 percent. The non-local factors, mesh size, and slenderness influenced the drop in the natural frequencies. The non-local natural frequencies tended to match the local natural frequencies up to the third frequency, then start diverged. The mode shapes are similar to the local elastic mode shapes in all cases.

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INTRODUCTION

The non-local elastic continuum theory was introduced by eringen1972non-local as a new theory of elasticity that better describes the behavior of materials. Unlike the local elasticity theory which assumes that the stress at a point is a function of the strain at that point, the non-local elasticity theory introduces the stress at a point as a function of the strains in the surrounding points in the close region. Eringen's assumption is that the stress at a point is a function of the strains at all the points at the close region where the function of the Euclidean distance weights the stresses in the surrounding region. The concept of long range effect was proposed by Krumhansl (1968) and Kröner (1967) to approximate the behavior of atoms. Some physical theories fail to solve some mechanical problems with a good level of accuracy if the influence of the atomic structure is significant. The failure can be discovered when the lab results vary from the mathematical calculations. The internal characteristic of materials plays an important role in giving accurate results when applying classical theories. Eringen (1973a) discussed the domain of applicability of continuum theories and lattice dynamics. Eringen sees that if the motion and distribution of the particles of the material are making an essential difference in calculations, then lattice dynamics provides the solution to the phenomenon; otherwise, the continuum theories will provide less complicated solutions.

The theory of non-local elasticity has been in development since then and many recognizable contributions have been made in refining its assumptions (Edelen, Green, and Laws (1971), Edelen (1969), eringen1972non-local). Various mechanical problems have been solved using the theory Eringen, Speziale, and Kim (1977), Eringen (1977b), Eringen (1977a). Borino and Polizzotto (2014), discussed the choices of the kernel function. The theory gained more attention after the introduction of the Non-Local Finite Elements Method by Polizzotto (2001). The boundary element method was introduced by Schwartz, Niane, and Njiwa (2012).

Eringen, Speziale, and Kim (1977), Eringen (1977a), and Eringen (1977b) successfully applied the theory of non-local elasticity to solve the crack tip and screw dislocation problems. Wang (1990b), studied the buckling and vibration of carbon nanotubes. Recent research studies were about non-local analysis of nanobeams and nanotubes by Barretta et al. (2019); Barretta and Sciarra (2019); Uzun and Civalek (2019).

The in-plane vibration of beams was the interest of the current research. The research was conducted using the non-local elasticity Fortran code in 2D and a Matlab Code to calculate the natural frequencies and plot their corresponding mode shapes.

1.1 Objectives

The focus of this research is the in-plane vibration of non-local elastic beams. The special mesh generation and the solution procedure are presented. The impact of the non-locality factors on the vibrational modes and natural frequencies are investigated. The research presents and discusses the differences in the eigenvalues and their corresponding mode shapes using NL-FEM FORTRAN and Matlab codes. The results are compared with high and low non-locality cases to understand changes in behavior. The slenderness influence is addressed. The references for the natural frequencies and mode shapes are the existing eigenvalues of Euler Bernoulli beam, Pisano's in-plane strain results and Reddy's natural frequencies. A fixed-free beam of 10x1 is used in this study.

1.2 Organization of the research

The thesis starts with an overview of the theory and the related work that has been done. The research moves forward by explaining the theory and non-local finite element method (NL-FEM) method Polizzotto (2001). In-plane vibration will be briefly mentioned and the related finite elements and eigenvalue calculations are presented to remind the readers. The development of a special mesh to indicate more variables which are related to the nature of the non-local theory is described. The research validates the Fortran code and compares the strain for a certain problem with the existing results Pisano, Sofi, and Fuschi (2009a).

After validating the results, the eigenvalue problem is solved using a Matlab code which provides the eigenvalues along with their corresponding eigenmodes. The work is compared with Abaqus results for the same problem.

LITERATURE REVIEW

Prior to the appearance of the non-local continuum mechanics theory, general theories were proposed in the sixties where the initial ideas about the effect of cohesive forces appeared. Green and Rivlin (1964), used an energy principle with velocity components to develop what is called multipolar elasticity theory. Krumhansl (1968), successfully derived a continuum theory from the latter lattice theory. This last theory deems that the resultant stresses and forces are contact forces with no cohesive forces. The principle of the long-range effect, which gives the non-locality is the basis of the non-local elasticity theory proposed by Eringen (1972) latter in the seventies. Kröner (1967), first introduced a continuum theory for elastic materials where the existence of the long-range forces represents the non-local effect. Kunin (1968), worked on Kröner's approach on a microscale.

In the seventies, most of the related work had been done by Eringen and coworkers. eringen1972non-local, using thermodynamic and balance laws, presented the non-local elasticity theory. All the work done after that used the principles that Eringen and Edelen used. Eringen (1972), worked on linearizing the relations in what had been done in the latter work. Eringen (1973b), derived electromagnetic momentum equations for his new continuum theory of non-local electromagnetic elastic solids. Eringen and Kim (1974), worked on the effect of non-local elasticity on a crack tip for a uniform load case. Edelen (1975), came up with the compatibility conditions for the stress boundary value problem for a non-local elastic member.

Eringen, Speziale, and Kim (1977), continued working on the Griffith crack tip problem for a non-local elastic plate. Stresses at a point in the non-local elasticity theory depend not only on the strains at the same point but a wide range of strains around the point. When the effect of the strain that comes from the surrounding points goes to zero, the non-locality vanishes. Eringen (1977a), used the non-local elasticity theory to solve the screw dislocation problem. non-local elasticity gives no singularity as there is in the local elasticity solution where the shear stresses along the core of the cylinder go to infinity.

Kovács and Vörös (1979), where the interaction between dilatation centers and their self-energy is obtained using non-local elastic stresses while in local elasticity theory no interaction exists. The results indicate an excellent agreement with Flinn (1962) using a discrete atomic model. Having no singularities on the non-local stresses gave solutions to many previously unsolvable problems in the local elasticity theory. Singular partial differential equations were derived from integropartial differential equations for special kernels and the dispersion relations were addressed for the transverse plane waves Eringen (1983)andEringen (1984a). More work in the non-local elasticity relations related to the linear theories for anisotropic and isotropic were obtained. Moreover, through lattice dynamics, the mathematical properties of non-local moduli were achieved Eringen (1984b). The solutions for initial boundary value problems for a homogeneous, isotropic and linear non-local elasticity were found to be unique Altan (1984) Altan (1989). In these papers, the field equations follow the mechanical displacement and strain relations, and the equations of motion and the stress and strain relations for the initial value problem are addressed. The solution is the same under the conditions they assumed. A notable example of applying the non-local elasticity theory on the quantitative properties of the lattice defect was given by Wang (1990b); Wang (1990a). The study indicated that in the atomic scale, the real stress field of dislocation appears very close to the non-local elastic field of dislocation.

Different problems in a non-local continuum relate to stress singularity in cracks and the dispersion of waves Eringen (1992). Some of these problems have been tested and confirmed experimentally. Most of the results show a high agreement with dynamical lattice calculations. Many papers have been done about the screw dislocation problem. The classical singularities and related issues have been covered differently in most of them. Pan and Fang (1992), obtained the non-local image force for the screw dislocation of elliptical and circular holes using the spatial mapping technique. An estimate has been done to get the maximum positions of the dislocation and the image force. Jun and Dhaliwal (1993), used the work and energy principle and the non-local elasticity theory for a mixed boundary value problem for a homogeneous and anisotropic material to prove the uniqueness theorem.

Pan (1995), studied two different cases, One of them with a crack and the other one with no dislocation using the non-local elasticity theory. The positions of the maximum and minimum shear stresses indicated an interaction between the crack and dislocation. The use of non-local elasticity avoids having infinite stresses at the crack tip.

Pan and Ji (1997), obtained expressions for mixed, displacement and stress boundary conditions for the non-local elasticity problem using the basic equations of balance laws and additional equations mentioned in the paper. Pan and Takeda (1998), proposed the concept of non-local Interface Residual (NIR), which is a microscopic quantity that disappears if the microstructural size of the material goes to zero. Using the latter concept and the non-local elasticity theory, the stresses of an edge dislocation were calculated for a bi-material. Unlike the classical elasticity theory, the non-local theory was able to give stresses at the dislocation's core. Özer (1999), obtained the symmetries of one and two-dimensional non-local elasticity problems. Investigations of the solutions can be done for the integrodifferential equations and some boundary value problems. Gao (1999a) and Gao (1999b), used Galileo invariance to find the constitutive equations for two theories of asymmetry of non-local elasticity for quasicontinuum and continuum fileds respectively. The results indicate that the constitutive equations for both theories of asymmetry are the same.

The non-local elasticity theory has been used in solving many problems and many contributions have been added. Silling (2000), introduced the peridynamic continuum theory where the spatial derivatives do not exist in the integrodifferential equation. The author proved that the new approach gives the same solutions for problems where discontinuities are involved no matter which equations are applied on the crack or around it. This means that this approach can be used to model problems where the cracks are not known.

Polizzotto (2001), gave the standard finite element method (FEM) solution for the non-local elasticity problems where non-locality is calculated based on local prediction in an iterative process. Another FEM solution was introduced which is called NL-FEM Polizzotto (2001). An attenuation function is used in this solution based on the internal distance between the material particles. Polizzotto replaced the use of the Euclidean distance which had been proposed by eringen1972non-local with the geoditcal distance. The effect of this function vanishes whenever the distance between the Gauss points exceeds the influence distance. The global stiffness matrix is composed of self-local stiffness matrices that come from each element and cross stiffness matrices that come from the neighbor elements of each element within the influence distance. Benvenuti, Borino, and Tralli (2002), introduced a thermodynamic non-local model using the second principle of thermodynamics for quasi-brittle materials which are the materials that lose strength after reaching a certain amount of stress. Two examples were presented along with the solution's algorithm.

Bažant and Jirásek (2002), presented a non-local model for damage. Peddieson, Buchanan, and McNitt (2003), applied Eringen's model to the Euler Bernoulli beam theory to get a non-local elastic Euler Bernoulli beam model. The effect of non-local elasticity was studied on four problems using nanoscale and microelectromechanical scale devices. non-local effects were predicted for nanoscale devices, while no effect was predicted for microelectromechanical devices. Silling, Zimmermann, and Abeyaratne (2003), modeled two problems as linear Fredholm integral equation using peridynamic continuum theory. The self-loaded bar problem was solved using the Fourier method. The concentrated load bar was solved using the Green function. Pisano and Fuschi (2003), solved a one-dimensional bar in tension using the Eringen model and the FEM solution proposed by Polizzotto (2001). The Fredholm integral equation and Volterra integral equation of the second kind were developed to get the governing equation. The concept of equivalent distance was proposed by Polizzotto, Fuschi, and Pisano (2004), where the effect of the inhomogeneity makes the difference between the proposed distance and the Euclidean eringen1972non-local and geodetical distance Polizzotto (2001).

Polizzotto, Fuschi, and Pisano (2006), solved a one-dimensional bar in tension using the equivalent distance concept for the Eringen model. The bar consists of two different materials. The results indicate the difference between using different internal lengths. Ece and Aydogdu (2007), applied the non-local elasticity to the Timoshenko-beam theory to model the in-plane simply supported beam for double-walled carbon nanotubes. The increment of the in-plane loads decreases the natural frequencies of the beam. In Emmrich and Weckner (2007), the peridynamic continuum theory and an intgrodifferential equation were applied in modeling long-range force interactions for an initial value problem. The uniqueness of solutions was also obtained for several examples.

Adali (2008) studied the buckling of a multi-walled carbon nanotube for an Euler Bernoulli beam using the non-local elasticity theory along with the effect of the internal forces between nanotubes of carbon. Aifantis (2009), several nanoscale problems proves that the non-local elasticity theory can be used in solving this kind of problem. Heireche et al. (2008), exhibited the importance of the nanoscale effect on the single walled carbon nanotubes for simply supported non-local elastic Euler Bernoulli beam and non-local elastic Timoshenko beam. The wave propagation was considerably effected by the nanoscale effect. Murmu and Pradhan (2009), worked in the same topic that Heireche et al. (2008) worked on but the new authors modeled the single-walled carbon nanotubes using Winkler and Pasternak models. The results confirm the conclusion of Heireche et al. (2008). Pisano, Sofi, and Fuschi (2009a), discussed in detail the numerical and coding issues of the NL-FEM method which was proposed by Polizzotto (2001). One and two-dimensional plane stress examples with a bar-like structure were presented. Furthermore, the coding procedure and the assembly of the global stiffness matrix were explained in detail. In the coming paper by the same authors, problems involving two different materials were solved using the NL-FEM method Pisano, Sofi, and Fuschi (2009b). Sundararaghavan and Waas (2011), used wave dispersion data found from lattice dynamics in a Single Walled Carbon Nanotubes (SWCN) to obtain the non-local kernel of nanotubes. The kernel was found to be negative for the farthest points. Ghosh et al. (2013), derived the non-local kernel for epoxy using the vibrational spectrum and velocity data. Borino and Polizzotto (2014), explained the relation between the non-local elasticity and gradient elasticity theories. Transformation approaches between the two theories were presented in the paper based on the principle of virtual power. Şimşek (2016), used the Hamiltonian method to solve the non-local frequencies of a functionally graded nanobeam. The model was designed for the non-local strain gradient theory applied to Euler Bernoulli beam. The non-local results vary from the classical results depending on the effect of the dimensionless material length and non-local parameters. Norouzzadeh and

Ansari (2017), compared the deflection of the nanoscale Timoshenko beam for the integral and differential finite element model using the non-local elasticity theory. The differential model was modified using a numerical approach to fix the behavior of this model. Studies in the field of nanotechnology and microtechnology have advanced. Engineers and scientists are pursuing this field of study to produce nanostructures and microstructures products, such as nanoprobes, nano actuators, nanowires, cantilever tips, and atomic force microscopes. In nanotechnology, for example, nanotubes have widely been studied. Nanotubes refer to tube-shaped one-dimensional nanostructures with incredible properties Uzun, Numanoglu, and Civalek (2018). The behavior of nanostructures is influenced by incredible yet varying properties, especially by considering the atomic level of the nanostructures. This observation is attributed to the fact that nanostructures have high or ultra high natural frequencies Chwał (2018). Vibrations of non-local elastic beams are an essential tool used in the analysis of sensors, oscillators, and resonators of nanotube equipment made of carbon. Various techniques have been used by researchers to analyze nanostructures and microstructures. For example, as cited in Uzun, Numanoglu, and Civalek (2018), different beam theories have been reformulated by Reddy, including the Levinson, Reddy, Timoshenko, and Euler–Bernoulli beam theories. The purpose of these theories is to simplify the analysis of vibrations of non-local elastic beams and differential constitutive relations. Most of these theories, however, have not lived up to the expectation of many scholars, for example, many scholars have used the Timoshenko beams theory; however, it does not adequately offer analysis needed for small beam-like structures. This limitation could be attributed to the fact that the Timoshenko beam theory is scale-free; hence cannot yield sufficient mechanical properties. Also, these higher-order continuum theories present a detailed investigation of carbon nanotubes (CNTs), including the analysis of boron nitride nanotube (BNNT) using the size-dependent approach Uzun, Numanoglu, and Civalek (2018). Timoshenko beam theory has been achieved through the application of various finite element formulations and cross-sections; this has also listed specific elements of a non-local matrix using parameters that are size and beam-dependent. In this case, the finite element parameters are obtained using the Galerkin weighted residual technique. Uzun et al. (2018) explain how the work of Kong et al. offer a solution to the Bernoulli-Euler beams' problems using the non-local elasticity theory. This group of researchers used both finite element and analytical methods to determine the frequency values for the boron nitride nanotube and carbon nanotube vibration behaviors. Tuna and Kirca (2017) developed finite element formulations for the analysis of free vibration of nanobeam structures using the basic Eringen non-local model eringen1972non-local. They found that it is possible to lower the number of elements without accuracy being sacrificed. The concept of non-local elasticity theory can be used to develop an elastic beam, which, in turn, is applied in the microtubes bending analysis; this is, however, defined in the Euler-Bernoulli beam theory Zargaripoor et al. (2018); Attarnejad and Ershadbakhsh (2016). Similarly, the finite element method has often been used in this theory for the vibration of non-local elastic beams analysis via the non-local constitutive equation of Eringen. Additionally, there are double-walled models of carbon nanotubes developed for the vibration of double-magneto-electro-elastic nanoplate systems. Jamalpoor et al. (2017) found that interaction between two nanoplates with magnetoelectric-elastic properties results in higher biaxial loads and frequencies achievement. Ribeiro and Thomas (2017) analyzed carbon nanotubes vibration using the finite element method. They modeled the carbon nanotubes consistent with the Timoshenko beam and Euler-Bernoulli theory focusing on the carbon nanotubes nonlinearity modes. Their results demonstrated noticeable non-local effects at greater lengths, and this was due to internal resonance Moshir and Eipakchi (2019). As postulated by Ghannadpour (2019), it is not easy to predict the non-local elastic beam's vibration using the Timoshenko beam theory mainly when the wavenumbers are large. The reason being that the microstructures significantly influence the flexural wave dispersion of the large flexible beams. Therefore, the mechanics of the non-local continuum can be used to perform the small scale effect through the stress state stipulation at a particular section of the beam, thereby creating a strain states function in all parts in the beam Eptaimeros, Koutsoumaris, and Tsamasphyros (2016). This observation explains why engineers and scientists apply the concept of non-local elasticity for the beam-like elements vibration analysis in nano and microelectromechanical equipment (Malgorzata, 2018; Jamalpoor, 2017; Ghannadpour, 2018). Although there is no satisfactory consistency in the non-local beam theory as far as the boundary conditions and governing equations variational formulation is concerned, the derivations of the non-local Timoshenko beams free vibration can be achieved using the

Hamiltonian's principle Aria and Friswell (2019); Eltaher, Khater, and Emam (2016).That is, a mechanical system's development time is a fundamental difference that exists between the potential and the kinetic energy provided the system is in a static state. Naghinejad and Ovesy (2018) and Pandeya and Singhb (2015) used non-local integral elasticity and the principle of total potential energy to develop a finite element method to study beam vibrations using different boundary conditions. The finite element method is used to determine the free vibration of nano-beams functional grading based on the first-order and third-order shear deformation theory Srividhya et al. (2018); Mehar et al. (2018). Hence, it is possible to determine the correct values of the non-local Timoshenko beams' vibration frequencies provided the rotary inertia, the deformation of transverse shear, and the small scale effects are considered. In past studies, many researchers have failed to incorporate the rotary inertia and transverse shear deformation effects when studying vibrations of non-local beams due to the adoption of the theory of Euler–Bernoulli beam Pradhan and Phadikar (2009); Nguyen, Kim, and Lee (2015); Ribeiro and Thomas (2017); Tuna and Kirca (2017). Analysis of the Euler-Bernoulli theory takes into account the finite element method efficiency and accuracy that is based on the shape functions of the beam as an interpolation function. These functions are applied in the formulation of the elements displacement field Attarnejad and Ershadbakhsh (2016). Currently, studying these novel functions, for example, the (BDFs) or primary displacements functions, are used to exploit the nanobeams' structural analyses with the help of finite element technique relative to Euler-Bernoulli and non-local elasticity Eringen theories. Therefore, essential displacement functions are determined by solving the motion of nanobeams using a governing differential equation through a series of powerful techniques. However, the flexibility of the finite element technique is different from the other displacement-based conventional methods since it is used to ensure exact equilibrium equations satisfy the interior sections of the element under study. As a result, it is easy to obtain structural matrices and shape functions based on the essential displacement functions by directly applying mechanical principles. For example, one can quickly evaluate the efficiency, accuracy, and competency of any method proposed using various boundary conditions Naghinejad and Ovesy (2018). Zakeri et al. (2017) conducted numerous numerical analyses. They found that the free transverse vibration, free longitudinal vibration, and stability analyses indicate complete correlations with the ones in the literature Challamel et al. (2015). According to Shiva et al. (2019), it is essential to include surface and non-locality effects as they have a critical impact on the buckling load. However, they conclude by saying that this significance varies depending on a variety of span and aspect ratios to thickness ratios as well as the laminated plate boundary conditions.

Pisano and Fuschi (2018), proposed the symmetric solution to non-local elastic symmetrical structures. Pisano and Fuschi's model needs to be enlarged based on the non-local parameters of the material. The authors checked the accuracy of the proposed solution in several examples.

Vibration analysis has been done by several researchers Reddy (2007); Wang (1990b). In Reddy (2007), the deflection, buckling and natural frequencies were obtained and compared for several beam theories. Additionally, the non-local modulus influence has been addressed. Pradhan and Phadikar (2009), studied the non-local vibration of nanoplates. Thai (2012) presents the results of bending, buckling and vibration of nanobeams.

METHODOLOGY

3.1 Overview

This chapter outlines the research methodology. The chapter exhibits the methods and theories applied to achieve the research's objectives. The chapter's main points are organized as follows: methods and theories; mesh generation; geometry of the elements and material properties. The chapter also discusses the validity of the findings. The in-plane vibration analysis including natural frequencies and mode shapes for a beam with non-local elastic material is presented for various boundary condition cases and non-local factors.

3.2 Euler Bernoulli Beam Theory

The equations of motions of the Euler Bernoulli Beam Theory can be expressed as the following:

$$U(x,y,z) = -z\frac{dw}{dx}. \tag{3.1}$$

$$V(x,y,z) = 0. \tag{3.2}$$

$$W(x,y,z) = w(x). \tag{3.3}$$

These equations yield the following strain relations:

$$\epsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{d^2 w}{dx^2}. \tag{3.4}$$

$$\epsilon_{yy} = \frac{\partial v}{\partial y} = 0. \tag{3.5}$$

$$\epsilon_{zz} = \frac{\partial w}{\partial z} = 0. \tag{3.6}$$

$$\sigma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0. \tag{3.7}$$

$$\sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0. \tag{3.8}$$

$$\sigma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0. \tag{3.9}$$

The corresponding stresses will be as the following:

$$\sigma_{yy} = \sigma_{yz} = \sigma_{yx} = 0. \tag{3.10}$$

The constitutive relation between stress and strain can be expressed by generalized Hook's law of elastic stiffness tensor C_{ijkl}.

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}. \tag{3.11}$$

3.3 Eringen Non-local Elastic Model

In non-local elasticity theory Eringen (1972),eringen1972non-local,Eringen and Kim (1974),the stresses depend not only on the strain at the current point but on a region around that point where the equation of stress depends on x' which is in the spatial formulation for all the points around the current element. The non-local elastic model Eringen (1984b) for a homogeneous isotropic elastic material is expressed as the following:

$$t_{kl,k} + \rho(f_l - \ddot{u}_l) = 0. \tag{3.12}$$

$$t_{kl}(x) = \int_v \alpha(\mid x' - x \mid, \tau)\sigma_{kl}(x')dv(x'). \tag{3.13}$$

$$\sigma_{kl}(x') = \lambda e_{rr}(x')\delta_{kl} + 2\mu e_{kl}(x'). \tag{3.14}$$

$$e_{kl}(x') = \frac{1}{2}\left(\frac{\partial u_k(x')}{\partial x'_l} + \frac{\partial u_l(x')}{\partial x'_k}\right). \tag{3.15}$$

where:

t_{kl}: Stress tensor of second order

σ_{kl}: Cauchy stress tensor of second order

ρ: Mass density

f : Body force

u : Displacement

α: non-local Modulus

$\lambda\,,\mu$: Lame parameters

e_{kl}: Strain tensor at x' of second order

In equation (2), (V) is the 3D Eculidian domain occupied by the material particles connected to each other with cohesive forces and long range forces for the farthest elements. The modulus α or the attenuation function is a scalar function of the Euclidean distance between the current point and x' which gives the non-locality effect. The non-local modulus $\alpha(|x'-x|)$ eringen1972non-local can be written as follows:

$$\alpha(\mid x'-x\mid,\tau). \tag{3.16}$$

$$\tau = e_o a/l. \tag{3.17}$$

where a and l are the lattice and external parameters respectively. The internal length 'a' is a material characteristic length, and e_o is a constant that defers from one material to another. When the Euclidean distance $|x'-x| > R$, the influence distance, $\alpha \rightarrow 0$. For $|x'-x| << R$, $\alpha \neq 0$. The normalization condition in equation (1.7) has to be imposed. When the l becomes zero, the attenuation function goes to one and non-locality vanishes rapidly.

$$\int_\Omega \alpha(|x'-x|,\tau)d\Omega = 1. \tag{3.18}$$

where Ω is the volume of an infinite domain. Many approximates for α have been proposed by Eringen (1984a),Polizzotto (2001),Bažant and Jirásek (2002). The two-dimensional Moduli can be written as follows:

$$\alpha(\mid x'-x\mid,\tau) = \frac{1}{2\pi l^2\tau^2}K_o(\sqrt{x.x}/l\tau), \tag{3.19}$$

using Bessel function for K_o

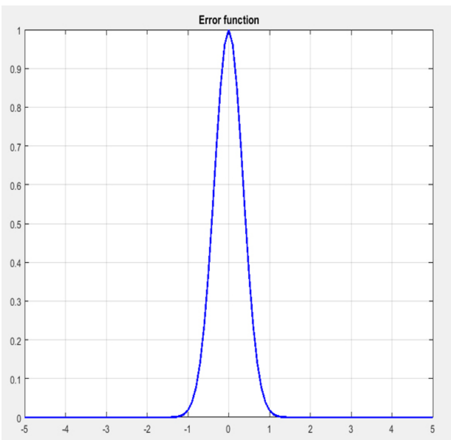
$$\alpha(\mid x'-x\mid,\tau) = \frac{1}{\pi l^2\tau}exp(-x.x/l^2\tau), \tag{3.20}$$

Pisanos' Attenuation function.

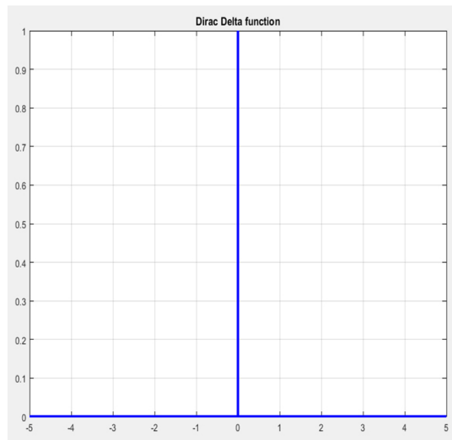
$$\alpha(\mid x' - x \mid, \tau) = \frac{1}{2\pi l^2 t} exp(-x.x/l). \tag{3.21}$$

In this study, Pisanos' and Bessel Kernels are used.

The non-local modulus becomes a Dirac delta function when the internal length 'a' goes to zero.



(a) non-local kernel



(b) Local kernel "Dirac Delta Function"

Figure 1: The non-local kernel in (a) as proposed by Eringen (1983) vs. the local kernel in (b)

The final stress constitutive equation is the summation of two phases. The classical elastic phase and the non-local elastic phase. These phases are multiplied by factors that follow the equation (1.10). When the non-locality vanishes, equation (1.10) is the summation of two local elastic phases which gives the classical results. Equation (1.11) is the final stress constitutive equation Eringen (1984b), Altan (1989)

$$\xi_1 + \xi_2 = 1. \tag{3.22}$$

$$\sigma x = \xi_1 D : \epsilon(x) + \xi_2 \int_v \alpha(x, x') D : \epsilon(x') dV'. \tag{3.23}$$

Polizzotto (2001), introduced the geodetical distance. The geodetical distance r(x,x') is the shortest distance that does not intersect with the boundaries of the body. The new concept deals with cracks and holes by taking the path around the crack, where

$r(x, x') >= |x' - x|$, is true for all the pairs in the domain. The distances may be the same when the body has no cracks and holes.

3.3.1 Non-locality factors

The non-local effect comes from the cohesive force that comes from the neighbor particles eringen1972non-local; Eringen (1972). This effect relies on several factors which are the following:

ξ_1: The weight of the local elastic contribution.

ξ_2: The weight of the non-local elastic contribution.

l: The internal length.

$Ł_R$: The range.

3.3.2 Non-local Finite Element Method (NL-FEM)

The non-local boundary value problem has been solved and discussed Polizzotto (2001). For a homogeneous non-local material in the material description, the BVP can be solved using the following equations.

$$div\sigma(x) + \bar{b}(x) = 0 \qquad in \quad V. \tag{3.24}$$

$$\sigma(x).n(x) = \bar{t}(x) \qquad on \quad St. \tag{3.25}$$

$$\epsilon(x) = \nabla^s u(x) \qquad in \quad V. \tag{3.26}$$

$$u(x) = \bar{u}(x) \qquad on \quad Su. \tag{3.27}$$

$$\sigma x = \xi_1 D : \epsilon(x) + \xi_2 \int_v \alpha(x, x') D : \epsilon(x') dV' \qquad in \quad V. \tag{3.28}$$

where:

V: The volume of the body

$\bar{b}(x)$: External volume forces

$\bar{t}(x)$: Surface forces on S_t

$n(x)$: The unit vector

$\bar{u}$: Imposed Displacements on S_u

S_u: Surface portion of the volume where displacements are imposed

S_t: Surface portion of the volume where stress tractions act

where, equation (3.24) and equation (3.25) are the field equilibrium and boundary equilibrium equations, respectively. equation (3.26) and equation (3.27) are the field compatibility and boundary compatibility equations, respectively. The final equation is the governing equation of non-local elasticity. Polizzotto (2001) proved the uniqueness of the non-local elastic solution to the problem using the difference strain and difference stress fields with no acting forces as follows:

$$\Delta\epsilon = \epsilon' - \epsilon'' \tag{3.29a}$$

$$.\Delta\sigma = \sigma' - \sigma''. \tag{3.29b}$$

According to Polizzotto (2001) the virtual work equation can be written as follows.

$$\int_v \Delta\sigma : \Delta\epsilon dV = 0. \tag{3.30}$$

By substituting equation (3.13) and equation (3.30), the final virtual work equation is:

$$\int_v \int_v \alpha(\mid x' - x \mid)\Delta\epsilon(x) : D : \Delta\epsilon(x')dV'dV = 0. \tag{3.31}$$

The total potential energy functional of a non-local elastic material can be written as follows Polizzotto (2001):

$$\begin{aligned} \Pi[u(x)] = \frac{1}{2}\int_v \int_v \alpha(\mid x' - x \mid)\nabla^s u(x) : D : \nabla^s u(x')dV'dV \\ - \int_v \bar{b}(x).u(x)dV - \int_v \bar{t}(x).u(x)dS. \end{aligned} \tag{3.32}$$

The FEM method can be used to solve for the non-local elasticity problem for the functional in equation (3.32). The domain can be discretized into a number of finite elements and the unknown displacement can be approximated using the following equations.

$$u(x) = N_n(x)d_n, \qquad \forall x \in V_n. \tag{3.33a}$$

$$\epsilon(x) = B_n(x)d_n, \qquad \forall x \in V_n. \tag{3.33b}$$

$$B_n(x) = \nabla^s N_n(x). \tag{3.34}$$

Equations(3.33a and 3.33b) are the approximations to the unknown displacements and strains. N_n is the shape function of class C^0. The node displacements vectors are stored in d_n. B_n represents the derivatives of the shape functions N_n as shown in equation (3.34). Substituting equations(3.33a,3.33b and 3.34) into equation (3.32) the functional can be rewritten as follows:

$$\begin{aligned}\Pi =& \sum_{n=1}^{Ne}\sum_{m=1}^{Ne}\frac{1}{2}d_n^T\left(\int_{V-n}\int_{V_m}\alpha(\mid x'-x\mid)B_n^T(x):D:B_n^T(x')dV'dV\right)d_m \\ & -\sum_{n=1}^{Ne}d_n^T\left(\int_{V_n}N_n^T.\bar{b}(x)dV-\int_{S_{t(n)}}N_n^T.\bar{t}(x)dS\right).\end{aligned} \tag{3.35}$$

Applying equation (3.28) on equation (3.35) gives.

$$\begin{aligned}\Pi =& \sum_{n=1}^{Ne}\frac{1}{2}\xi_1 d_n^T\left(\int_{V_n}B_n^T(x)DB_n(x)dV\right)d_n \\ & +\sum_{n=1}^{Ne}\sum_{m=1}^{Ne}\frac{1}{2}\xi_2 d_n^T\left(\int_{V_n}\int_{V_m}\alpha(\mid x'-x\mid)B_n^T(x)DB_m(x')dV'dV\right)d_m \\ & -\sum_{n=1}^{Ne}d_n^T\left(\int_{V_n}N_n^T.\bar{b}(x)dV-\int_{S_{t(n)}}N_n^T.\bar{t}(x)dS\right).\end{aligned} \tag{3.36}$$

$$k_n^{loc} = \int_{Vn} B_n^T(X)DB_n(X)dV. \tag{3.37}$$

$$k_{nm}^{nonloc} = \int_{Vn}\int_{Vm} A(X,X^{'})B_n^T(X)DB_m(X^{'})dV^{'}dV. \tag{3.38}$$

$$f_n = \int_{Vn} N_n^T(X)\bar{b}(X)dV + \int_{S_{t(n)}} N_n^T(X)\bar{t}(X)dS. \tag{3.39}$$

where:

$$d_n = C_nU. \tag{3.40}$$

U: The displacement vector for the entire mesh

C_n: The node connection matrices

By substituting Eq3.40 into Eq3.36, a discretized form of the total non-local potential energy functional can be written as the follows:

$$\hat{K}U = F. \tag{3.41}$$

$\hat{K}$ is the non-local global stiffness matrix which is symmetric.

$$\hat{K} = \sum_{n=1}^{N_e} \left[\xi_1 K_n^{loc} + \xi_2 \sum_{m=1}^{N_e} K_{nm}^{nonloc} \right]. \tag{3.42}$$

3.3.3 Mesh Generation

In order to describe the geometric domain, material properties and boundary conditions as well as the non-local effect, a special mesh has to be developed. Polizzotto (2001) introduced the concept of having neighborer elements in the finite elements non-local elastic method.

Neighbor elements are the elements that surround the current element and fall within L_R distance from each Gauss point as shown in Figure 2.

In the figure, the element n has 15 neighbors which contribute to the non-local stiffness matrix of element n. The non-local effect depends on the distance $|X_{GPi} - X_{GPj}|$. As this distance increases the non-locality vanishes.

The number of neighbor elements and the neighbor elements were identified for each element using a Matlab code. The Matlab code calculates the distance between the center of element n and element m and compares it with the influence distance L_R. The following comparison tells whether the element m is a neighbor or not.

```
nem=0;
for i=1:nel
    ne=0;
```

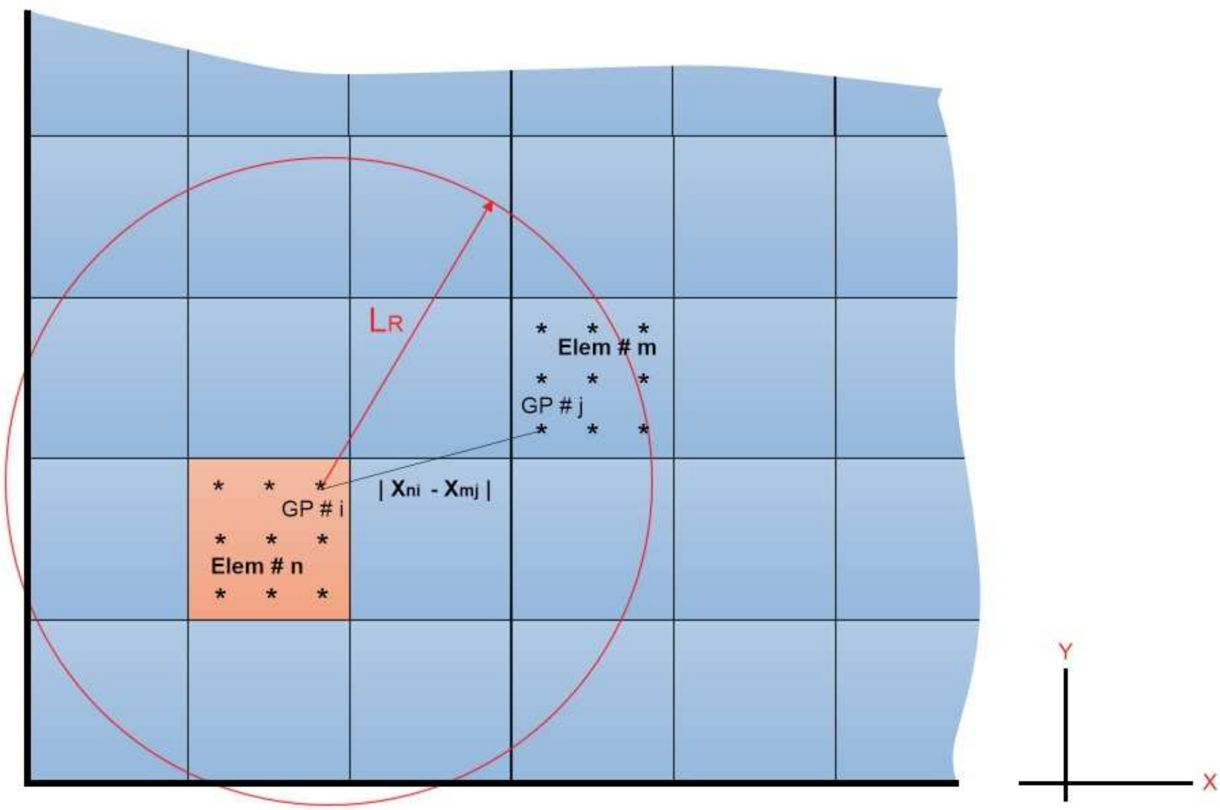


Figure 2: The neighbor elements contributing to element n using nine Gauss points evaluation

```
    nii=elm(i,1);
    nfi=elm(i,3);
    xci=(x(nii)+x(nfi))/2;
    yci=(y(nii)+y(nfi))/2;
   for j=1:nel
       if j~=i
           nij=elm(j,1);
           nfj=elm(j,3);
           xcj=(x(nij)+x(nfj))/2;
           ycj=(y(nij)+y(nfj))/2;
           dist=sqrt((xci-xcj)^2+(yci-ycj)^2);
           if dist<(lr+0.5*sqrt(hx^2+hy^2))
               ne=ne+1;
               nle(i,ne+1)=j;
           end
       end
   end
    nle(i,1)=ne;
    if ne>nem
        nem=ne;
    end
end
```

where the 8 nodes quad element was used in the NL-FEM solution with two degrees of freedom for each node. Three different element sizes were used to elucidate the relation between non-locality and the distance between particles represented by Gauss points as shown in Fig 2.

The domain was divided into 10x1 element the first time. The mesh was then refined to 20x2 then to 50x5. In each of these cases, more elements fall into the Lr distance.

The accuracy of calculations is expected to ascend.

3.3.4 Imposing Boundary Conditions

The boundary conditions are a set of conditions on the boundaries of a domain that are necessary to solve the boundary value problem. In this research, the boundary conditions at one of the end faces are set to zero for all the nodes at the end face to represent the cantilever conditions. The nodes at the other end face are totally free. Euler Bernoulli Beam Theory

3.4 Vibration Analysis

The in-plane free undamped vibration analysis was applied to the beams for various boundary conditions. Using a Matlab code, the eigenvalue problem for multi degrees of freedom was solved to obtain the eigenvalues and eigenvectors. The eigenvalues give the natural frequencies, and the eigenvectors are the normal mode shapes. The system vibrates at the natural frequencies. However, in M-DOF, the system not only vibrates at a certain natural frequency, but also with a certain natural displacement configuration. Moreover, there are as many natural frequencies and associated natural configurations as the number of DOF of the system and natural modes of vibrations.

The equation of motion for free in-plane vibration of undamped beam is derived as shown:

$$[M]\ddot{x}(t) + [C]\dot{x}(t) + [K]x(t) = F. \tag{3.43}$$

3.5 Eigenvalue Problem

The eigenvalue problem for an undamped free vibration is presented in this section. Since C=0, F=0 for an undamped and free vibration system. The equation of motion can be written as follows:

$$[M]\ddot{x}(t) + [K]x(t) = 0. \tag{3.44}$$

where:

[M]: The mass matrix

[K]: The stiffness matrix

$x(t)$: The displacement vector

Assume a complex harmonic motion solution.

$$x(t) = \begin{Bmatrix} ue^{i\omega t} \\ ve^{i\omega t} \end{Bmatrix}$$

By solving the eigenvalue problem

$$([K] - \omega^2[M]) \begin{Bmatrix} \text{u} \\ \text{v.} \end{Bmatrix} = 0 \tag{3.45}$$

For a nontrivial solution, the determinant of $([K] - \omega^2[M]) = 0$.

where:

u,v: The horizontal and vertical eigenvectors

ω: The eigenvalues which are the natural frequencies

The eigenvalue problem can be solved by hand calculations easly Since the number of DOF in our beam is "53, 165, and 861" for "1x10 elements, 2x20 elements, and 5x50 elements". The calculations were done by Matlab.

The Matlab code was designed to read the K and M matrices from Fortran. The code then obtains the eigenvalues and converts them to Hz. The displacement vector contains both values of U and V which is used to plot the mode shapes.

3.6 Validation

In order to provide reliable and acceptable results, the findings of the codes need to be checked and compared with existing results. In this research, The strain results of the third problem created by Pisano, Sofi, and Fuschi (2009a) is used in a comparison to prove that the results match. Having the same strain results reflects having the exact stiffness matrices which are being extracted from the FORTRAN code. Pisano's problem is as follows.

A thin square plate with thickness of 0.5 cm and length and width of 5x5 cm is subjected to a distributed displacement. The displacement starts at zero at the ends and goes up to 0.005 at the middle of the plate. The modulus of elasticity was $2.1e^6$ with Poisson's ratio $\upsilon = 0.2$.

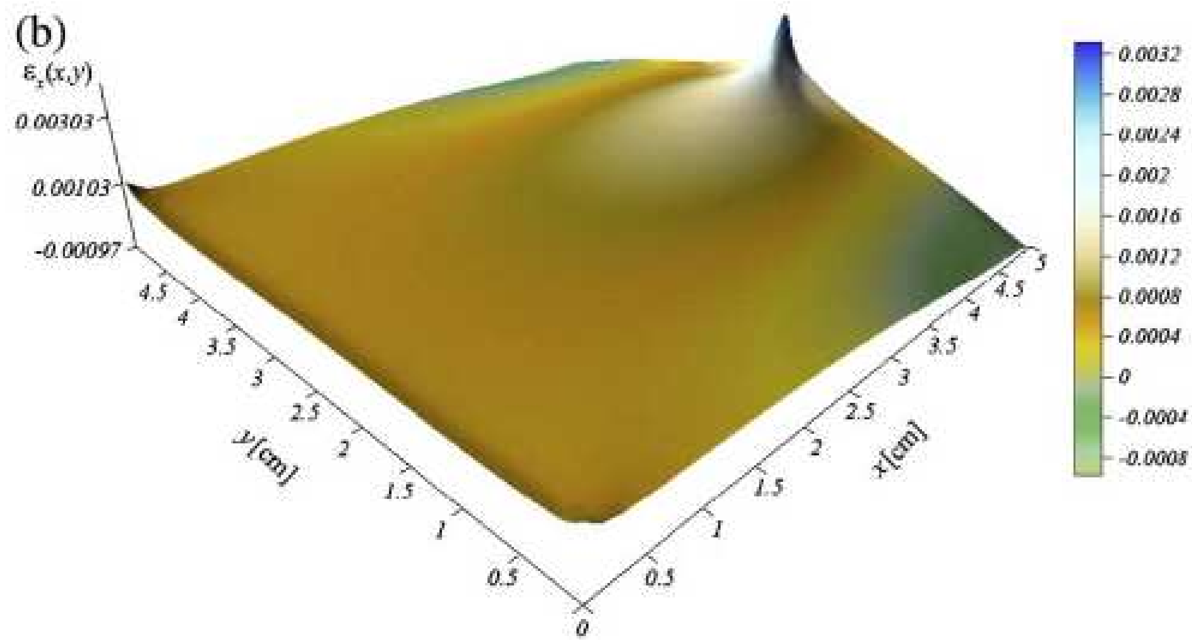


Figure 3: Example number two for case two in Pisano, Sofi, and Fuschi (2009a)

In Figure 3.3, we see that the non-local elastic strain results of the FORTRAN code, which are presented by the red solid line, matches Pissano's results, which are presented by the solid black line. The results exactly match at three different sections in the plate. This

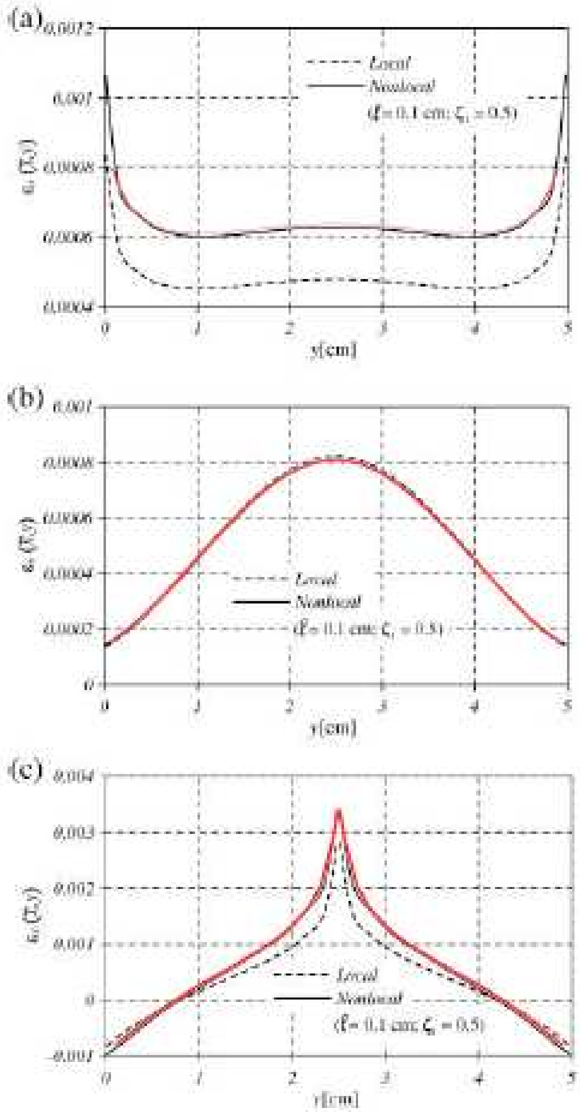


Figure 4: Comparing the strain results with Pissano's solution

match means that the stiffness matrices and the FORTRAN code give accurate results.

RESULTS AND DISCUSSION

In this chapter, the eigenvalues and mode shapes of a local elastic and non-local elastic cantilever beam are presented. Three different mesh sizes were used to study the effect and contribution of the number of neighbors of an element to the stiffness matrix. The results are compared with Euler Bernoulli beam eigenvalues and mode shapes. The case of $W_n = 0.5$ is fully discussed and then compared to the cases of $W_n = 0.9$ and $W_n = 0.1$. The findings are presented in different ways to indicate the effect of the non-local elasticity factors.

4.1 Natural Frequencies

Natural frequencies have been calculated for a local elastic model and several cases of a non-local elastic model to indicate the influence of the non-locality factors. The first four natural frequencies are presented and compared with a cantilever Euler Bernoulli beam that has the same characteristics and boundary conditions. As shown in the tables, an increment in L_R of 5 cm was considered to cover the effect of number of neighbors on the non-local contribution. In the following tables, the number of neighbors is reflected by the value of L_R.

4.2 Beam Properties

The tables indicate that the natural frequencies of the local elastic model are closer to the Euler Bernoulli natural frequencies than non-local elastic models for all mesh sizes. However, the fourth eigenvalue is evidently lower than the Euler Bernoulli value in all cases. There is a slight difference in the natural frequencies of the local elasticity case compared to the Euler Bernoulli over different mesh sizes. This difference increases as the mesh size decreases. This means that better results can be obtained from the bigger mesh size (10x1).

For the non-local elasticity case with $W_n = 0.5$ presented in tables 4-??, the natural

Table 1: The properties of the Euler Bernoulli cantilever beam

Properties	Values	Unit
L	10	cm
H	1	cm
T	1	cm
E	2.1e+4	
v	0.3	
m	8	

frequencies did not vary as the number of neighbors increased. This indicates the combined effect of the number of neighbors and the internal length, which should increase the natural frequencies as the number of neighbors increases until a certain limit where the attenuation function goes to zero.

The internal length effect is obvious when comparing the results over different mesh sizes for the same W_n. The natural frequencies slightly increase as the internal length (l) increases. In Table 4, the natural frequencies had the same values for all values of L_R.

In the Figures 9a to 11c, the graphs exhibit the first four natural frequencies for different internal lengths Lr. The number of neighbor elements is represented by Lr. As Lr increases, more neighbor elements contribute to the non-local stiffness matrix. A very small difference can be seen coming from the closer neighbors due to the non-locality effect. In cases of $l = 0.1$ and $l = 0.2$, frequencies are almost the same no matter the number of neighbors. The case of $l = 0.3$, the non-locality region is now wider and the number of neighbors increases. The difference is more noticeable as the effect is combined and comes from Lr and l.

The Figures 9a to 11c not only represent the effect of the factor Lr over the same mesh size, but also the effect of l over the same mesh size. The frequencies decrease as the number of elements increases when compared to Figure (9a).

4.3 Local Elastic Results

The natural frequencies of the 10x1 cm Euler Bernoulli beam are presented in Figure 5. In the figure, the first three frequencies tend to match Euler Bernoulli's natural frequencies. Obviously, the first frequency matches and the second and third frequencies slightly diverge. A huge drop is noticeable at the fourth, fifth and sixth frequencies. The plot shows the mesh size effect on the natural frequencies (table 2). As the mesh size gets bigger, the frequencies drop.

Table 2: First six natural frequencies of the 10x1, 20x2 and 50x5 element of the 10x1 cm local elastic Euler Bernoulli cantilever beam vs. Euler Bernoulli natural frequencies

Mode	Euler Bernoulli	Local Elastisity		
		10x1	20x2	50x5
1st	0.0828	0.0826	0.0823	0.0822
2nd	0.5187	0.4982	0.4940	0.4933
3rd	1.4523	1.2831	1.2827	1.2824
4th	2.8460	1.3205	1.3020	1.2995
5th	4.7046	2.4183	2.3679	2.3615
6th	7.0278	3.7208	3.6144	3.6016

Table 3: The differences in percent of the First six natural frequencies of the 10x1, 20x2 and 50x5 element of the 10x1 cm local elastic cantilever beam vs. Euler Bernoulli natural frequencies

Mode	Local Elastisity		
	10x1	20x2	50x5
1st	0.14%	0.56%	0.64%
2nd	3.95%	4.76%	4.89%
3rd	11.65%	11.68%	11.70%
4th	53.60%	54.25%	54.34%
5th	48.60%	49.67%	49.80%
6th	47.06%	48.57%	48.75%

4.4 Non-Local Elastic Results

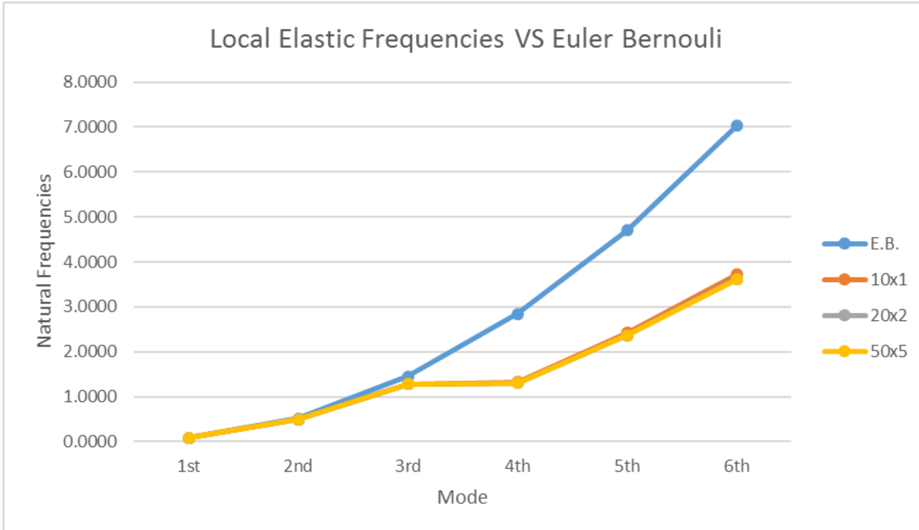


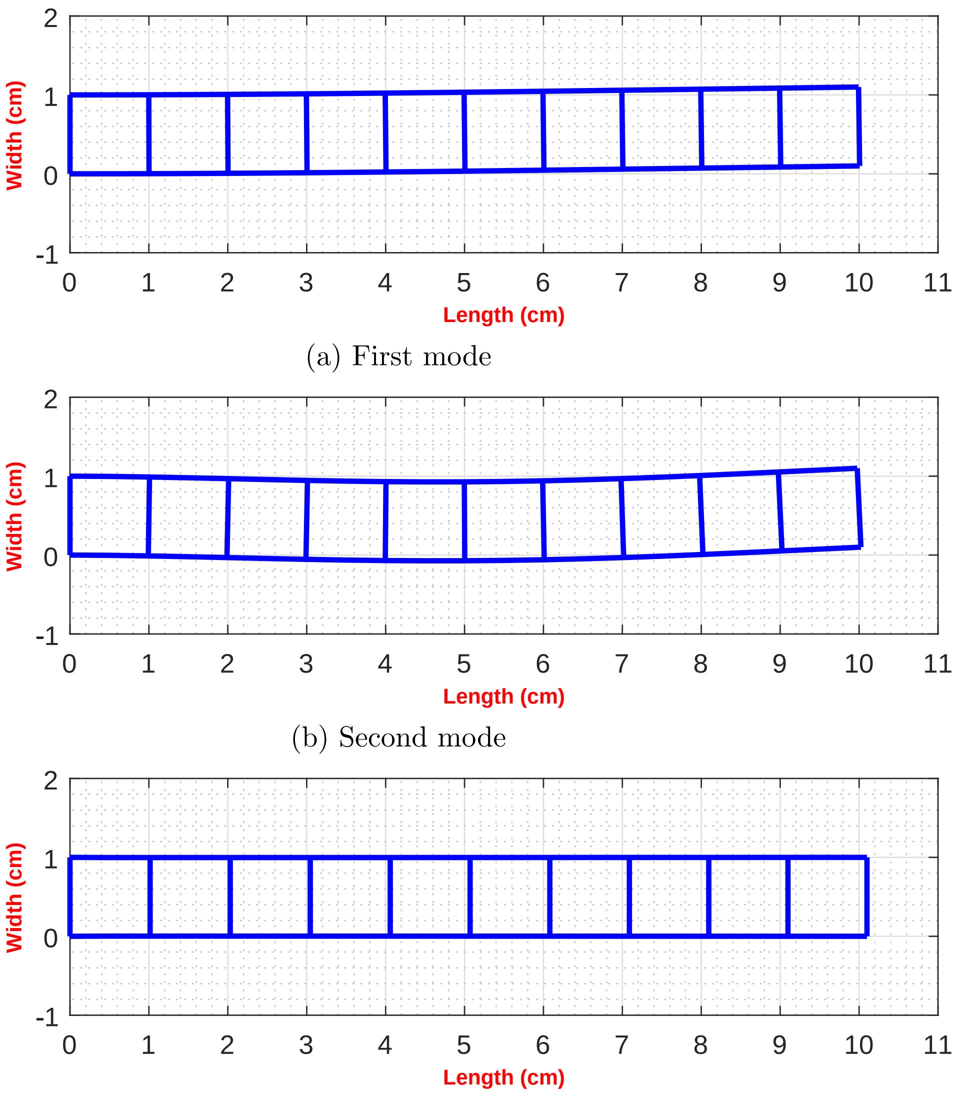
Figure 5: Local elastic natural frequencies of 10x1, 20x2 and 50x5 mesh sizes of a cantilever beam vs. Euler Bernoulli natural frequencies of 1x10 cm

Table 4: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.5$

Mode	Euler Bernoulli	Local Elasticity	Non-Local Elasticity, Wn=0.5				
			l=5	l=10	l=15	l=20	l=25
1st	0.0828	0.0826	0.0600	0.0600	0.0600	0.0600	0.0600
2nd	0.5187	0.4982	0.3626	0.3626	0.3626	0.3626	0.3626
3rd	1.4523	1.2831	0.9473	0.9473	0.9473	0.9473	0.9473
4th	2.8460	1.3205	0.9633	0.9633	0.9633	0.9633	0.9633

Table 5: First four natural frequencies of a 20x2 of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.5$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0613	0.0613	0.0613	0.0613	0.0613
2nd	0.5187	0.4982	0.3701	0.3703	0.3703	0.3703	0.3703
3rd	1.4523	1.2831	0.9825	0.9828	0.9828	0.9828	0.9828
4th	2.8460	1.3205	1.0161	1.0167	1.0167	1.0167	1.0167



(a) First mode

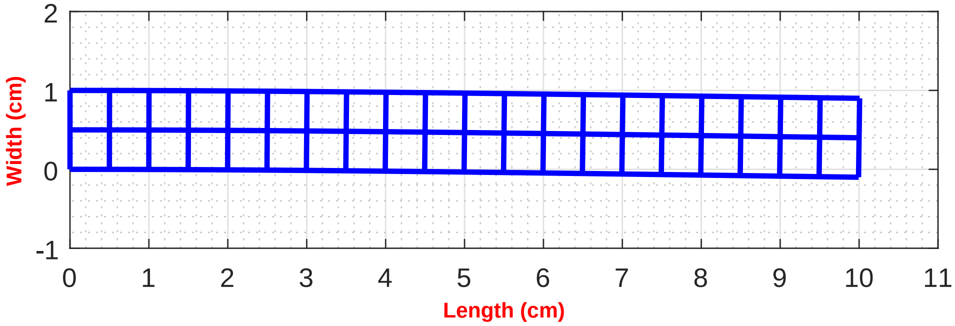
(b) Second mode

(c) Third Mode

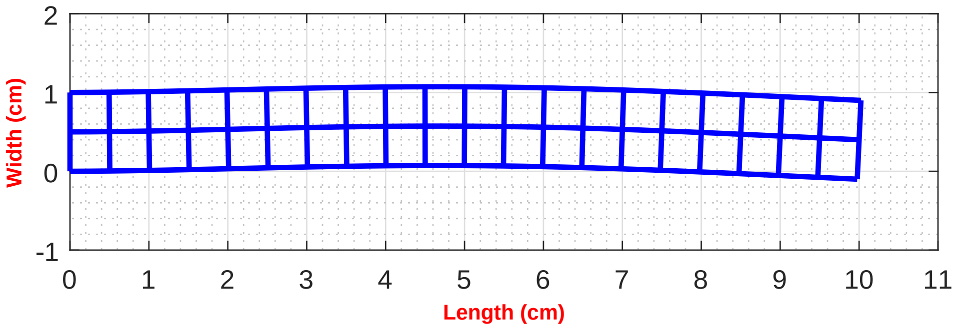
Figure 6: The first three modes of the 10x1 cantilever beam for 10x1 mesh using a local elastic solution

Table 6: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.5$

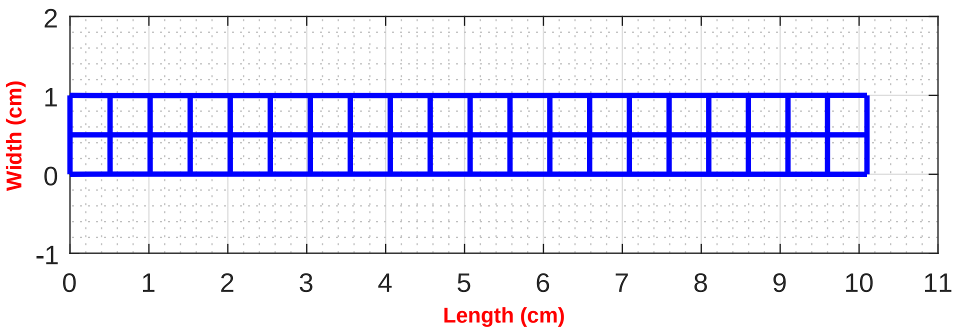
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0670	0.0670	0.0670	0.0670	0.0670
2nd	0.5187	0.4982	0.4044	0.4043	0.4043	0.4043	0.4043
3rd	1.4523	1.2831	1.0725	1.0724	1.0724	1.0724	1.0724
4th	2.8460	1.3205	1.1138	1.1151	1.1151	1.1151	1.1151



(a) First mode



(b) Second mode

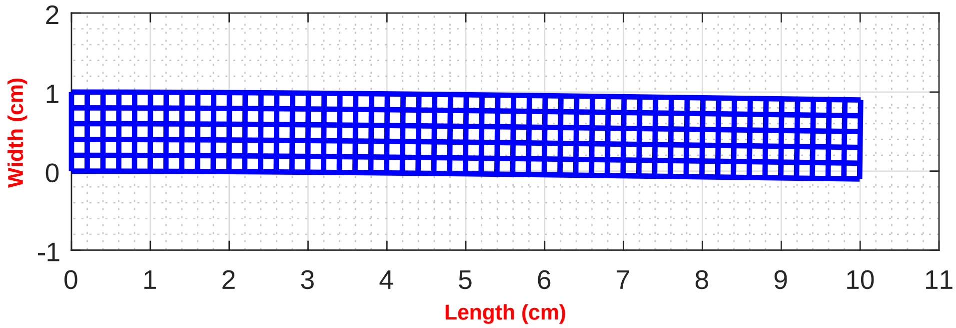


(c) Third Mode

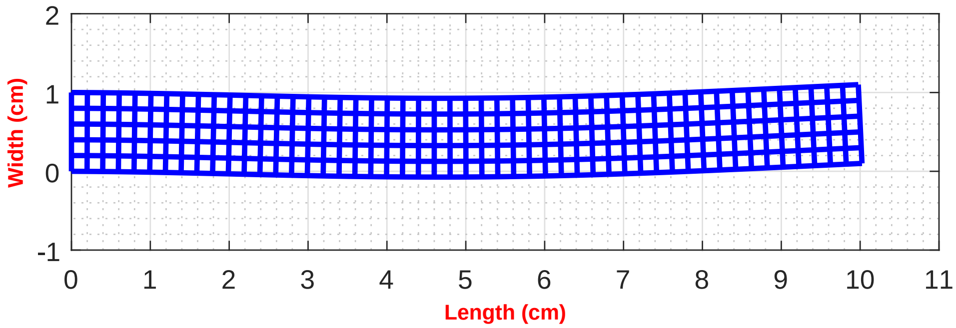
Figure 7: The first three modes of the 10x1 cantilever beam for 20x2 mesh using a local elastic solution

Table 7: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.5$

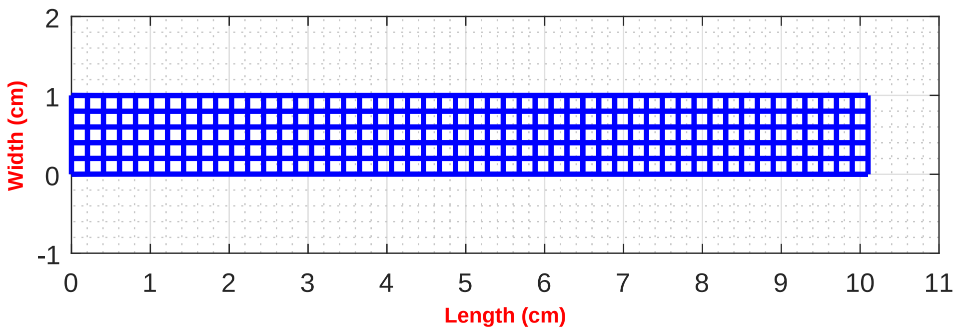
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0601	0.0601	0.0601	0.0601	0.0601
2nd	0.5187	0.4982	0.3633	0.3633	0.3633	0.3633	0.3633
3rd	1.4523	1.2831	0.9652	0.9653	0.9653	0.9653	0.9653
4th	2.8460	1.3205	0.9691	0.9696	0.9696	0.9696	0.9696



(a) First mode



(b) Second mode



(c) Third Mode

Figure 8: The first three modes of the 10x1 cantilever beam for 50x5 mesh using a local elastic solution

Table 8: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.5$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0607	0.0607	0.0607	0.0607	0.0607
2nd	0.5187	0.4982	0.3671	0.3672	0.3672	0.3672	0.3672
3rd	1.4523	1.2831	0.9754	0.9756	0.9756	0.9756	0.9756
4th	2.8460	1.3205	1.0523	1.0531	1.0531	1.0531	1.0531

Table 9: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.5$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0670	0.0648	0.0648	0.0648	0.0648
2nd	0.5187	0.4982	0.4043	0.3910	0.3910	0.3910	0.3910
3rd	1.4523	1.2831	1.0724	1.0379	1.0379	1.0379	1.0379
4th	2.8460	1.3205	1.1151	1.1215	1.1215	1.1215	1.1215

Table 10: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.5$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0598	0.0598	0.0598	0.0598	0.0598
2nd	0.5187	0.4982	0.3615	0.3615	0.3615	0.3615	0.3615
3rd	1.4523	1.2831	0.9604	0.9604	0.9604	0.9604	0.9604
4th	2.8460	1.3205	0.9799	0.9800	0.9800	0.9800	0.9800

Table 11: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.5$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0598	0.0598	0.0598	0.0598	0.0598
2nd	0.5187	0.4982	0.3615	0.3615	0.3615	0.3615	0.3615
3rd	1.4523	1.2831	0.9602	0.9602	0.9602	0.9602	0.9602
4th	2.8460	1.3205	1.0550	1.0557	1.0557	1.0557	1.0557

Table 12: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.5$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.5				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0581	0.0581	0.0581	0.0581	0.0581
2nd	0.5187	0.4982	0.3488	0.3488	0.3488	0.3488	0.3488
3rd	1.4523	1.2831	0.9068	0.9068	0.9068	0.9068	0.9068
4th	2.8460	1.3205	0.9189	0.9189	0.9189	0.9189	0.9189

Table 13: The differences in percent for the First four natural frequencies of the 10x1, 20x2 and 50x5 element of a non-local elastic cantilever beam; l=0.1, $W_n = 0.5$ compared to Euler Bernoulli

Mode	non-local Elasticity, Wn=0.5		
	10x1	20x2	50x5
1st	27.5%	25.9%	19.0%
2nd	30.1%	28.6%	22.0%
3rd	34.8%	32.4%	26.2%
4th	66.2%	64.3%	60.9%

Unlike the case of $W_n = 0.5$, the natural frequencies of the beam were reduced by 60%. The results are expected to get lower as the non-local contribution increases.

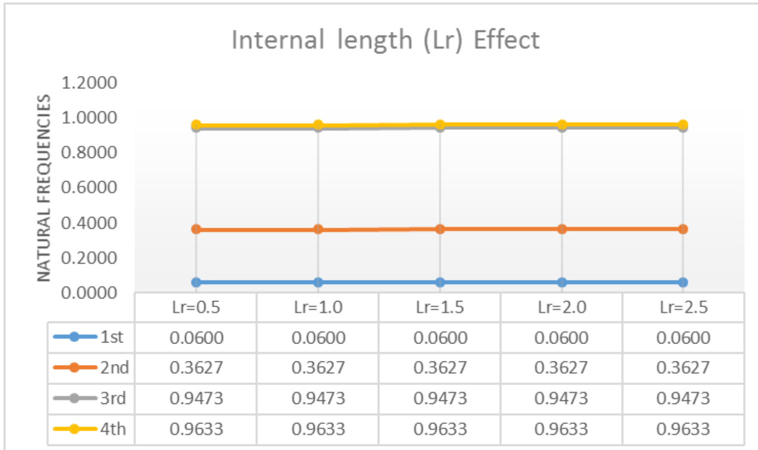
The next figures reflect the effect of non-local factors on the mode shapes of the cantilever beam. Figures 22a to 24c exhibit the convergence in the natural frequencies as the internal length increases. The natural frequencies of different numbers of neighbors are aligned to one line. The other two lines represent the Euler Bernoulli and the local elasticity results. The fourth eigenvalue is obviously higher than the local and non-local values in all cases.

Interestingly, the natural frequencies decease when comparing figures 22a to 22c with figures 23a to 23c and figures 24a to 24c.

The corresponding eigenvectors for the previous natural frequencies are presented in the following figures. The figures show in 2D the first four vibrational mode shapes. These shapes are invariable, which means that all the mode shapes for all the cases are the same. The number of elements is also represented with a scaled length and width.

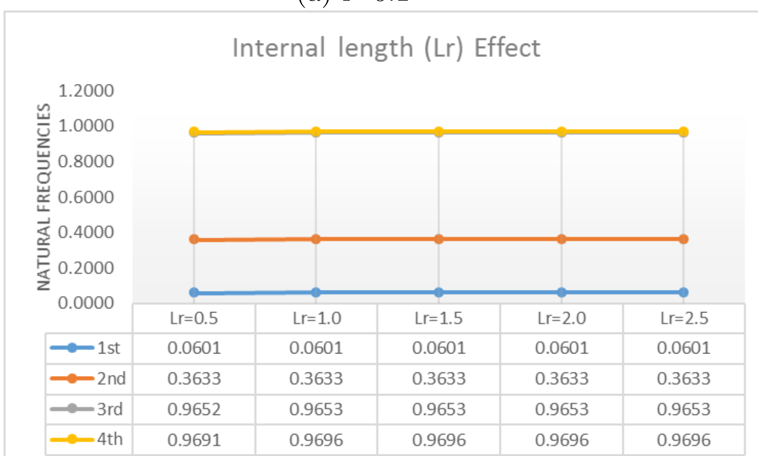
4.5 Comparison

All the previously mentioned work regarding the non-local factors effect has been done to the same beam with the same boundary conditions but changing the local and non-local stiffness contributions weight (W_n). The tables and figures listed in Appendix A present how a high



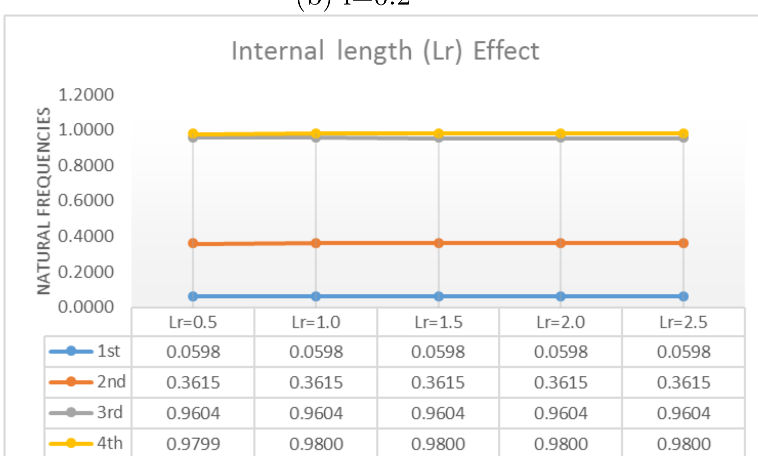
	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0600	0.0600	0.0600	0.0600	0.0600
2nd	0.3627	0.3627	0.3627	0.3627	0.3627
3rd	0.9473	0.9473	0.9473	0.9473	0.9473
4th	0.9633	0.9633	0.9633	0.9633	0.9633

(a) l=0.1



	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0601	0.0601	0.0601	0.0601	0.0601
2nd	0.3633	0.3633	0.3633	0.3633	0.3633
3rd	0.9652	0.9653	0.9653	0.9653	0.9653
4th	0.9691	0.9696	0.9696	0.9696	0.9696

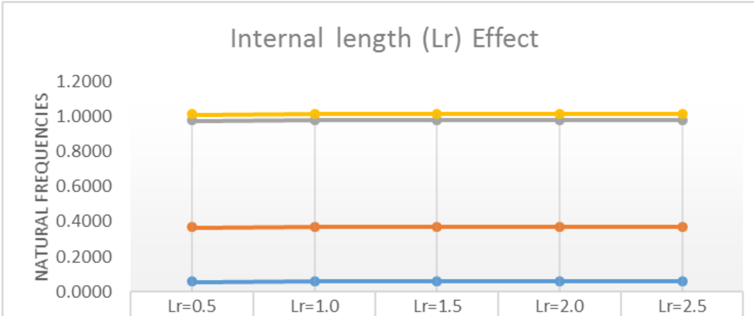
(b) l=0.2



	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0598	0.0598	0.0598	0.0598	0.0598
2nd	0.3615	0.3615	0.3615	0.3615	0.3615
3rd	0.9604	0.9604	0.9604	0.9604	0.9604
4th	0.9799	0.9800	0.9800	0.9800	0.9800

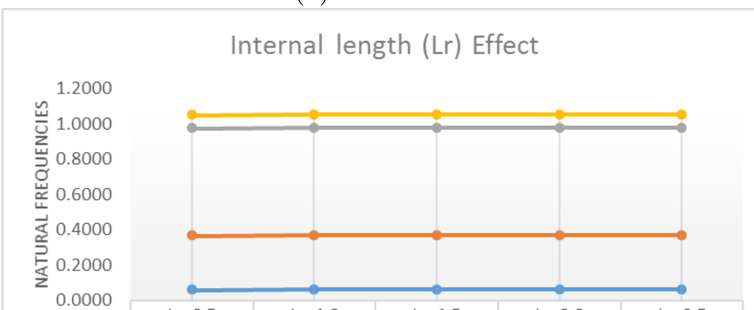
(c) l=0.3

Figure 9: The first four modes of the 10x1 cantilever beam for 10x1 mesh using the non-local elasticity theory for different numbers of neighbor elements



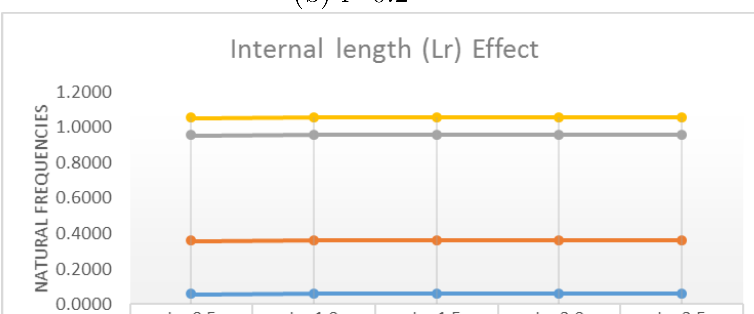
	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0613	0.0613	0.0613	0.0613	0.0613
2nd	0.3701	0.3703	0.3703	0.3703	0.3703
3rd	0.9825	0.9828	0.9828	0.9828	0.9828
4th	1.0161	1.0167	1.0167	1.0167	1.0167

(a) l=0.1



	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0607	0.0607	0.0607	0.0607	0.0607
2nd	0.3671	0.3672	0.3672	0.3672	0.3672
3rd	0.9754	0.9756	0.9756	0.9756	0.9756
4th	1.0523	1.0531	1.0531	1.0531	1.0531

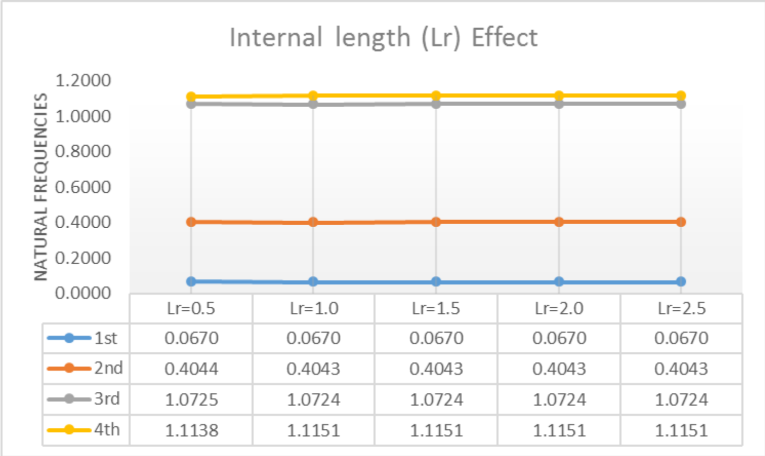
(b) l=0.2



	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0598	0.0598	0.0598	0.0598	0.0598
2nd	0.3615	0.3615	0.3615	0.3615	0.3615
3rd	0.9602	0.9602	0.9602	0.9602	0.9602
4th	1.0550	1.0557	1.0557	1.0557	1.0557

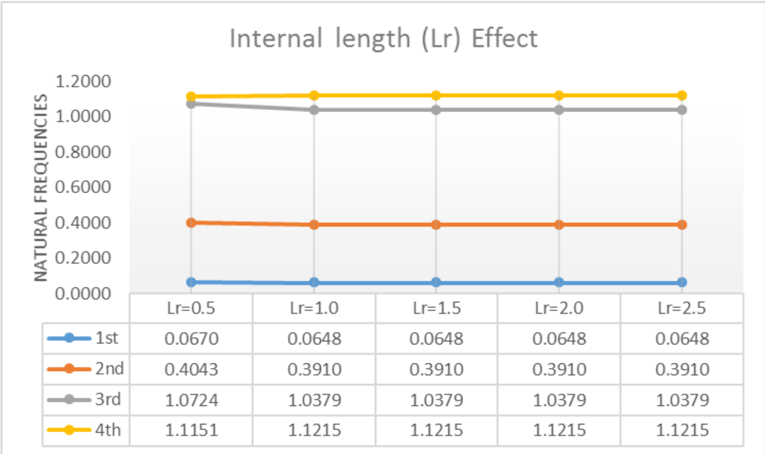
(c) l=0.3

Figure 10: The first four modes of the 10x1 cantilever beam for 20x2 mesh using the non-local elasticity theory for different numbers of neighbor elements



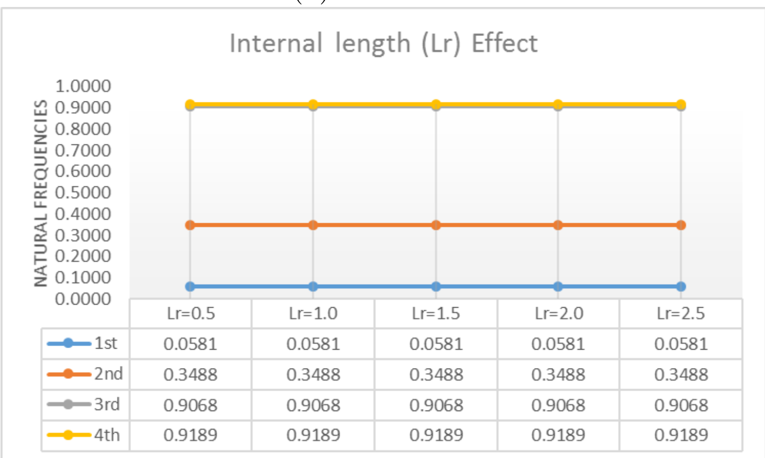
	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0670	0.0670	0.0670	0.0670	0.0670
2nd	0.4044	0.4043	0.4043	0.4043	0.4043
3rd	1.0725	1.0724	1.0724	1.0724	1.0724
4th	1.1138	1.1151	1.1151	1.1151	1.1151

(a) l=0.1



	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0670	0.0648	0.0648	0.0648	0.0648
2nd	0.4043	0.3910	0.3910	0.3910	0.3910
3rd	1.0724	1.0379	1.0379	1.0379	1.0379
4th	1.1151	1.1215	1.1215	1.1215	1.1215

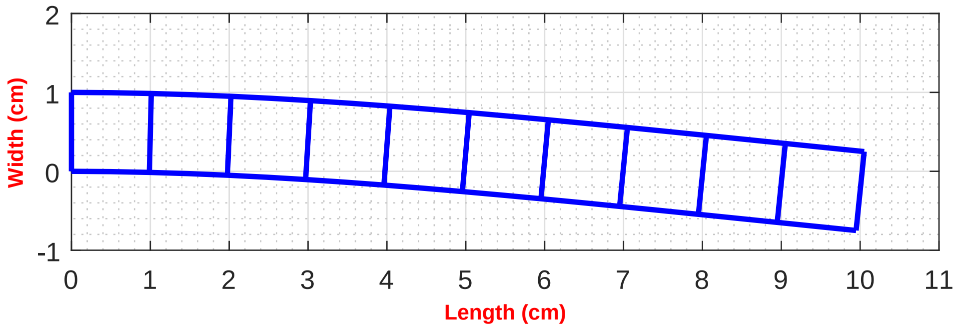
(b) l=0.2



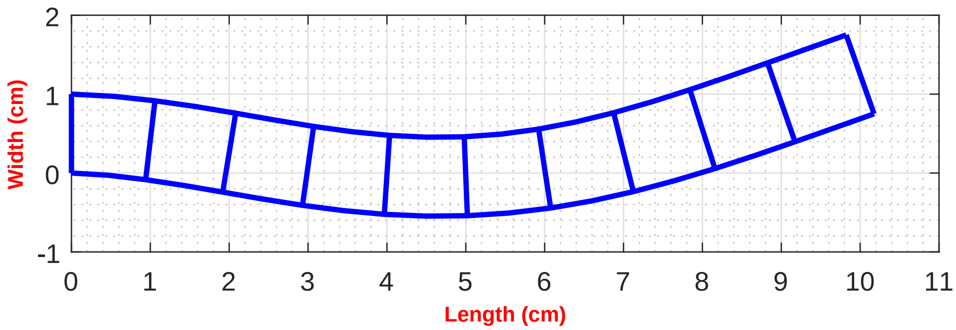
	Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0581	0.0581	0.0581	0.0581	0.0581
2nd	0.3488	0.3488	0.3488	0.3488	0.3488
3rd	0.9068	0.9068	0.9068	0.9068	0.9068
4th	0.9189	0.9189	0.9189	0.9189	0.9189

(c) l=0.3

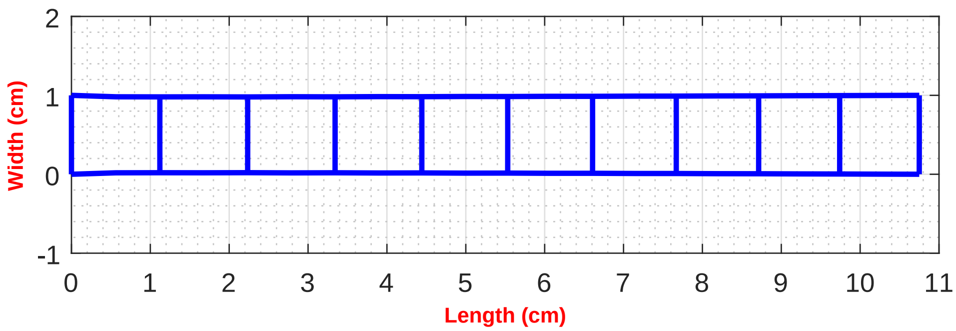
Figure 11: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory for different numbers of neighbor elements



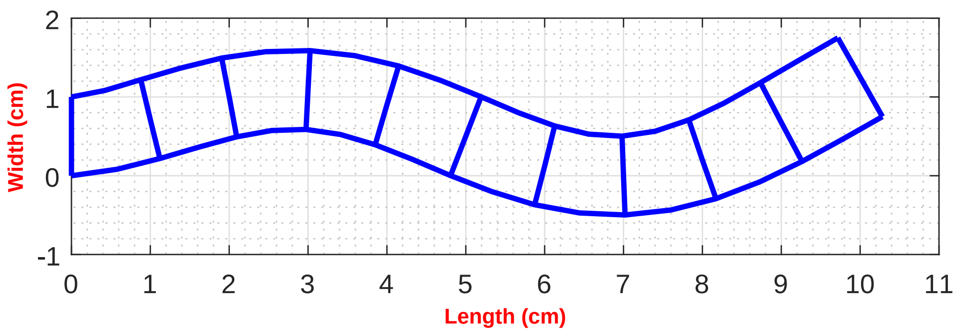
(a) First mode



(b) Second mode

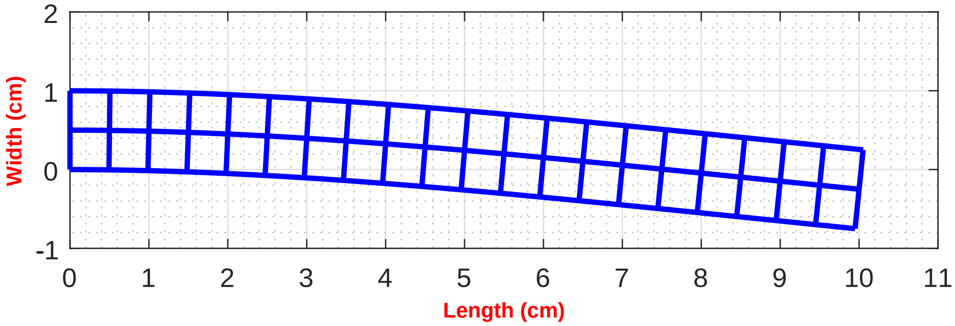


(c) Third Mode

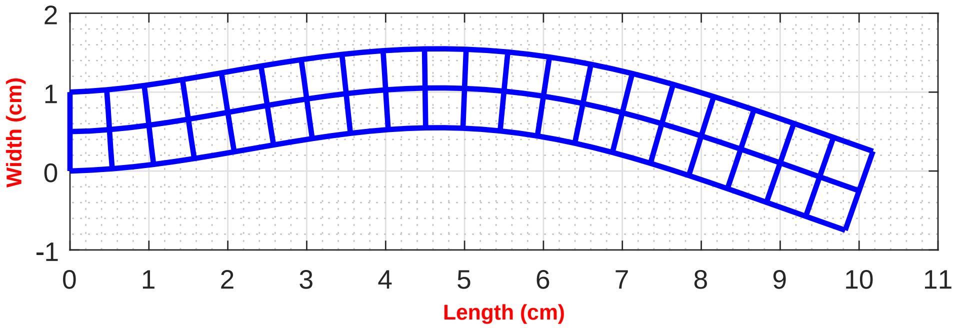


(d) Fourth mode

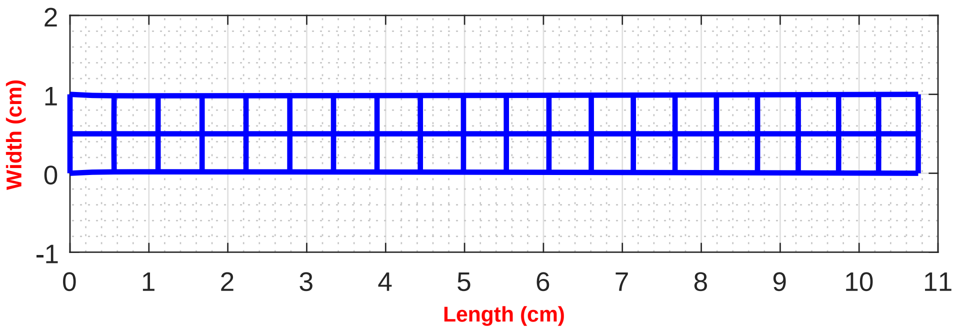
Figure 12: The first four modes of the 10x1 cantilever beam for 10x1 mesh using the non-local elasticity theory



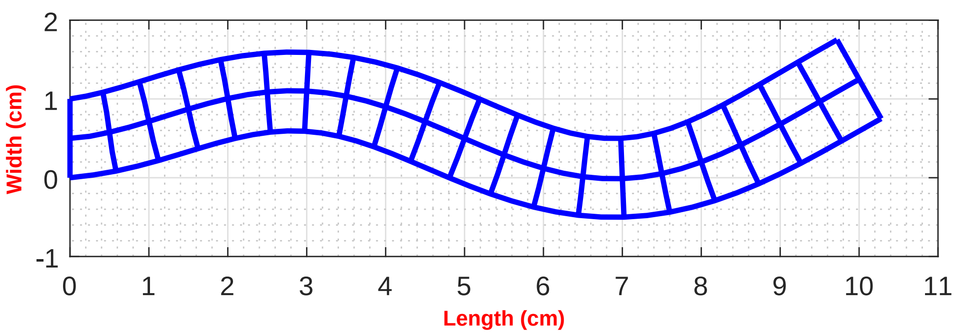
(a) First mode



(b) Second mode

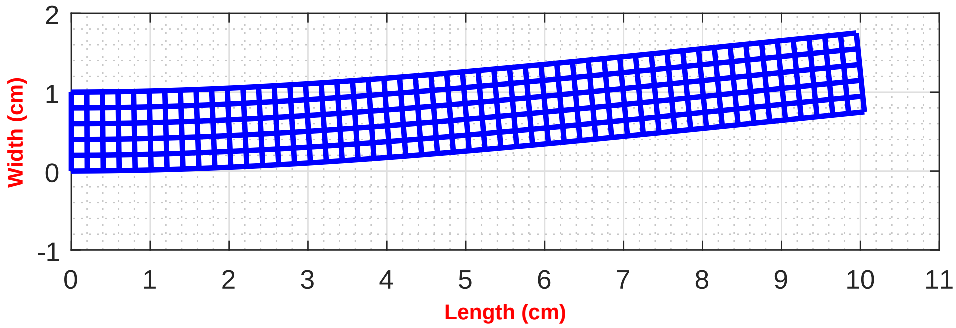


(c) Third Mode

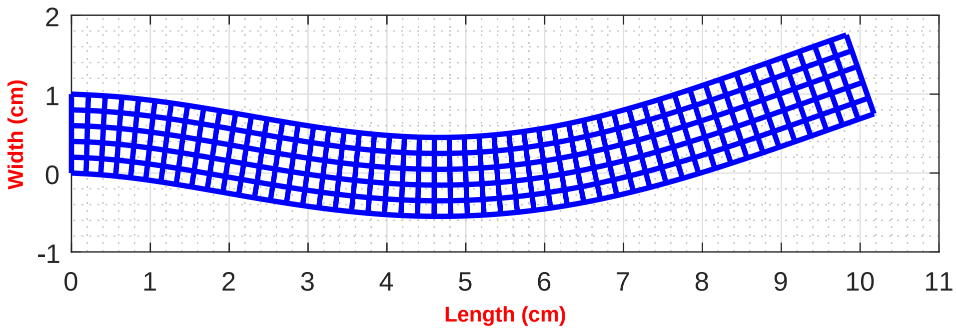


(d) Fourth mode

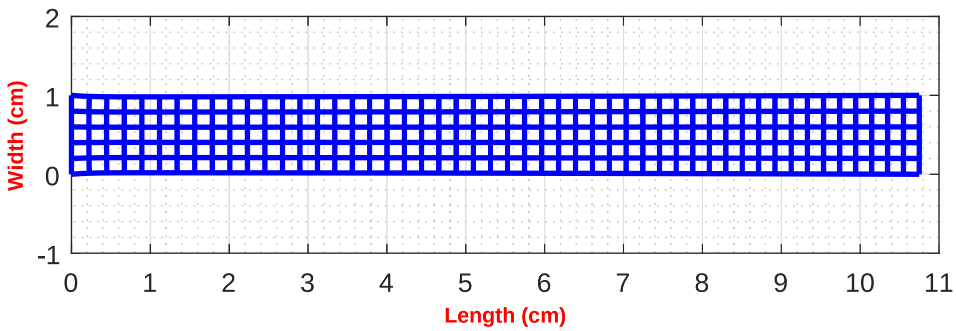
Figure 13: The first four modes of the 10x1 cantilever beam for 20x2 mesh using the non-local elasticity theory



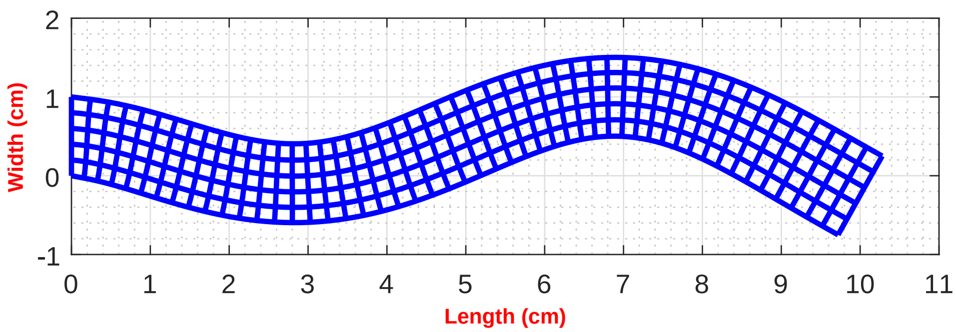
(a) First mode



(b) Second mode



(c) Third Mode



(d) Fourth mode

Figure 14: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory

non-local contribution factor ($W_n = 0.9$) can make a difference in the results. The equation $W_n = 0.1$ has been examined to demonstrate that as the non-local contribution vanishes, the results match the local elastic solutions.

4.6 Slenderness Influence

In Figure 16, the local natural frequencies where $L/h = 100$ are higher than the non-local frequencies due to stiffness influence. The highest values among the local elastic natural frequencies are the 10x1-local results. Then the 20x2 is slightly higher than the 50x5. In the non-local elastic results, the 10x1 is also the highest in the non-local elastic results. Unlike the local results, the 50x5 looks higher than 20x2 due to having the exact contribution in the non-local stiffness matrix from the neighbor elements. The mesh size 20x2 tends to give higher contributions to the matrix which makes it give lower natural frequencies results.

In Figure 17, the local natural frequencies where $L/h = 20$ are also higher than the non-local ones. The local elastic results tend to converge up to the fifth frequency. Then, the frequencies diverge with 10x1. The 10x1 mesh size gives the highest values. In the non-local elastic results, the 50x5 is the highest. Unlike the case of $L/h = 100$, the 20x2 is higher than 10x1 due to having a better or exact contribution in the non-local stiffness matrix from the neighbor elements in the cases of 50x5 then 20x2. The mesh size 10x1 tends to give higher contributions to the matrix but is not as accurate as the other two cases. In Figure 17, the frequencies drop at the fifth one so the fourth and fifth frequencies are very close to each other. At the sixth frequency, the value goes up again with a similar increment rate as the rate at the beginning.

In Figure 18, the local and non-local natural frequencies where $L/h = 10$ tend to take the same pattern as the case of $L/h = 20$. In Figure 18, the frequencies drop at the fourth one so the third and fourth frequencies are very close to each other. At the fifth frequency, the value goes up again with a similar increment rate as the rate the beginning.

Figure 15, indicates how each of the previously mentioned cases looks when comparing them together. The case of $L/h = 100$ has the highest values with a steady increment rate. In the other two cases, the frequencies look lower than the first case. Finally, the local and non-local results tend to align and converge with a small slenderness ratios.

Table 14: Non-dimensional natural frequencies of a cantilever beam (E=2.1e4, m=8, $\nu = 0.3$) for different mesh sizes and slenderness ratios

L/h	Local			Non-Local		
	10x1	20x2	50x5	10x1	20x2	50x5
100	0.563	0.561	0.560	0.398	0.369	0.425
	3.566	3.518	3.507	2.522	2.319	2.664
	10.158	9.874	9.814	7.183	6.514	7.457
	20.426	19.418	19.215	14.443	12.825	14.602
	34.958	32.250	31.730	24.719	21.334	24.117
	54.574	48.459	47.335	38.590	32.117	35.989
20	0.562	0.559	0.559	0.402	0.399	0.433
	3.494	3.467	3.463	2.503	2.481	2.685
	9.670	9.543	9.526	6.934	6.852	7.405
	17.343	17.335	17.332	12.674	13.166	14.207
	18.662	18.260	18.215	13.391	13.833	14.955
	30.326	29.333	29.232	21.762	21.249	22.885
10	0.559	0.556	0.556	0.406	0.411	0.438
	3.368	3.335	3.335	2.456	2.482	2.644
	8.676	8.671	8.671	6.526	6.595	7.017
	8.928	8.786	8.786	6.552	7.115	7.573
	16.351	15.967	15.967	11.965	12.087	12.844
	25.157	24.351	24.351	18.390	18.553	19.689

4.7 Comparison with existing results

In order to understand the influence of the non-local modulus on the natural frequencies, a simply supported beam Reddy (2007) was used. Then, the final non-dimensional natural frequencies were compared with Reddys' results. In Chapter 3, Pissanos' attenuation function was applied in the cantilever beam analysis Pisano, Sofi, and Fuschi (2009b). In the comparison with Reddys' work, the original Kernel Eringen (1983) as proposed by Eringen was applied. Then, the attenuation function is as in equation 3.21. The non-local modulus $\mu = e_o^2 a^2$, which is a weighting factor, gives how much non-locality is applied. In this

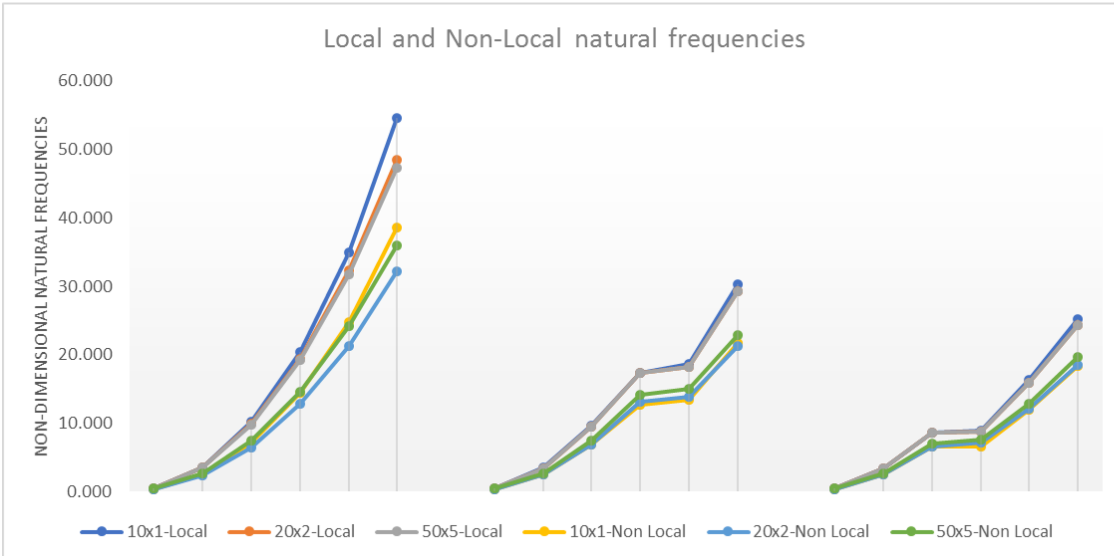


Figure 15: Non-dimensional natural frequencies of a local and non-local elastic cantilever beam (E=2.1e4, m=8, $\nu = 0.3$) for different mesh sizes and slenderness ratios

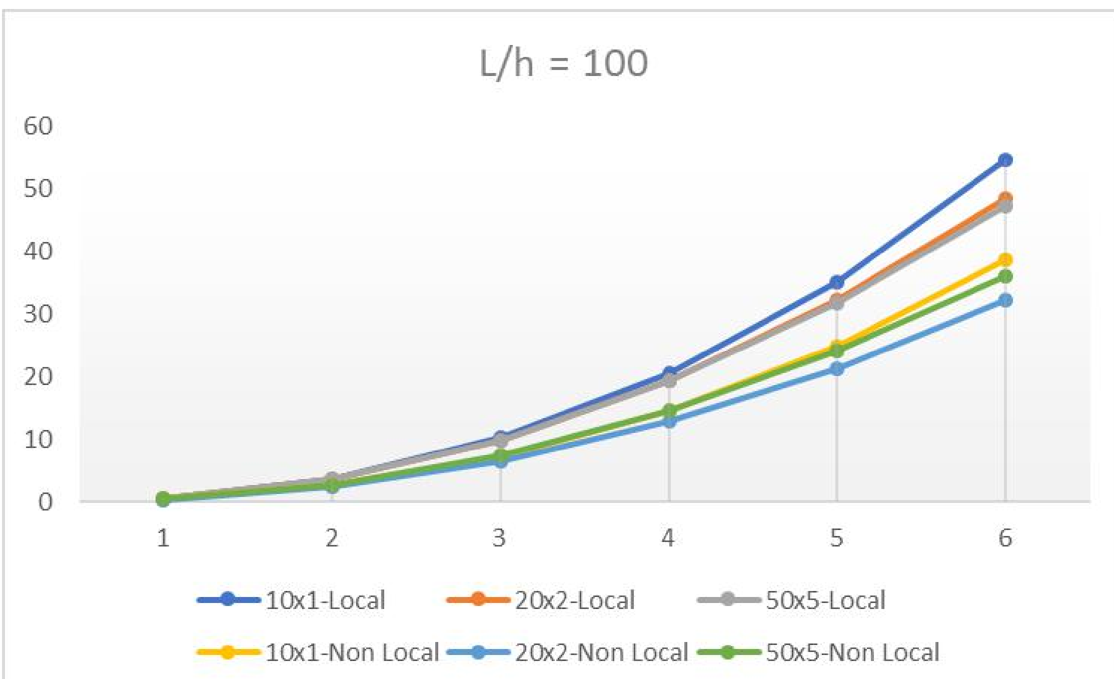


Figure 16: Non-dimensional natural frequencies of a local and non-local elastic cantilever beam (E=2.1e4, m=8, $\nu = 0.3$) for L/h = 100

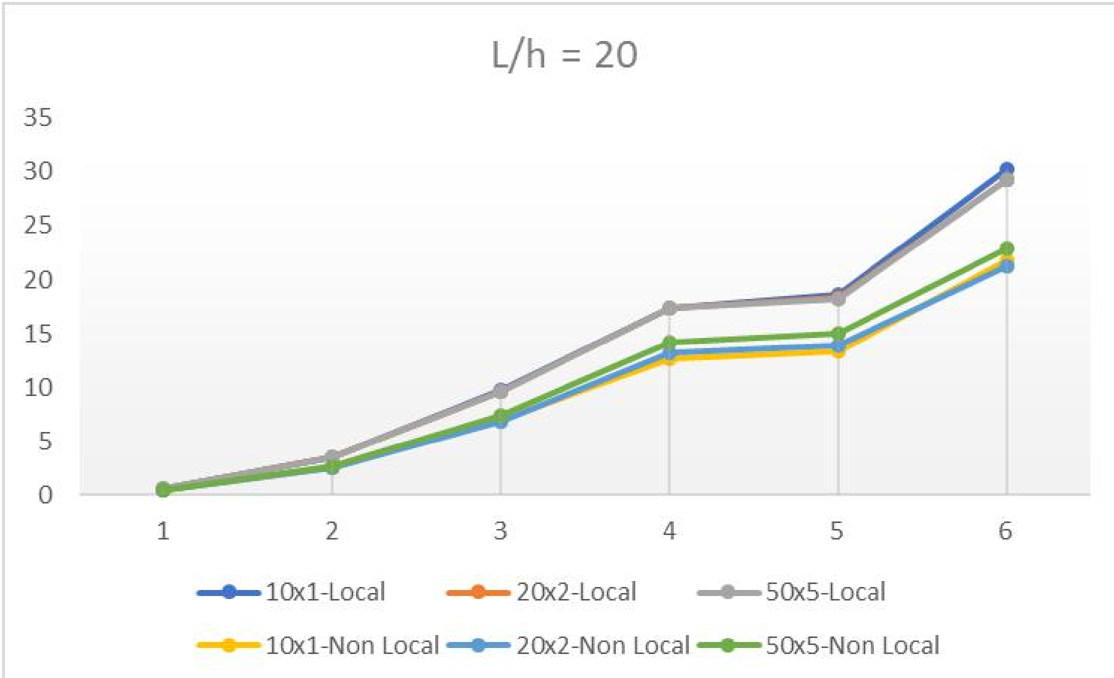


Figure 17: Non-dimensional natural frequencies of a local and non-local elastic cantilever beam (E=2.1e4, m=8, $\nu = 0.3$) for L/h = 20

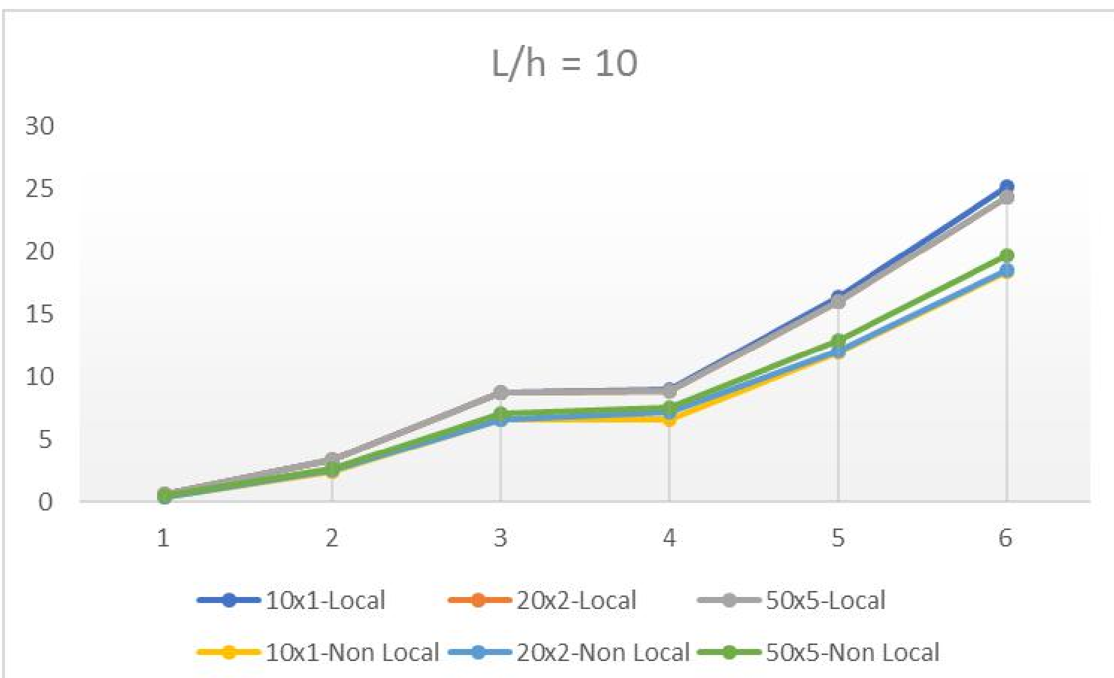


Figure 18: Non-dimensional natural frequencies of a local and non-local elastic cantilever beam (E=2.1e4, m=8, $\nu = 0.3$) for L/h = 10

comparison, mesh sizes of $(100x1, 20x1, 10x1)$ with internal length $l = 0.1$ and $Lr = 6l$ were used. Table 14 indicates how the non-local modulus influences the natural frequencies.

In Reddys' results including non-local elastic Euler Bernoulli beam (EBT),non-local elastic Timoshenko beam (TBT),non-local Reddy beam (RBT) and non-local Levinson Beam (LBT), the first non-dimensional frequency when $L/h = 100$ in the current study looks so close to their results. In Figure 19, the frequencies tend to drop linearly with 14%. This drop reflects the non-locality influence on the frequencies. When $\mu = 0$, the non-local elastic contribution is completely neglected and the natural frequency is the local elastic natural frequency. As the non-local contribution increases, the value drops slightly. The results diverge with Reddy's reults as indicated at $\mu = 0.3$.

In Figures 20 the influence on the first natural frequency $L/h = 20$ is obviously lower than Reddy's results. The other frequencies have the same behavior as in the case of $L/h = 100$. However, the difference is smaller due to the slenderness impact.

The case of $L/h = 10$ starts with a low frequency when $\mu = 0$ with 4%. The frequencies converge with Reddy's results after the third frequency.

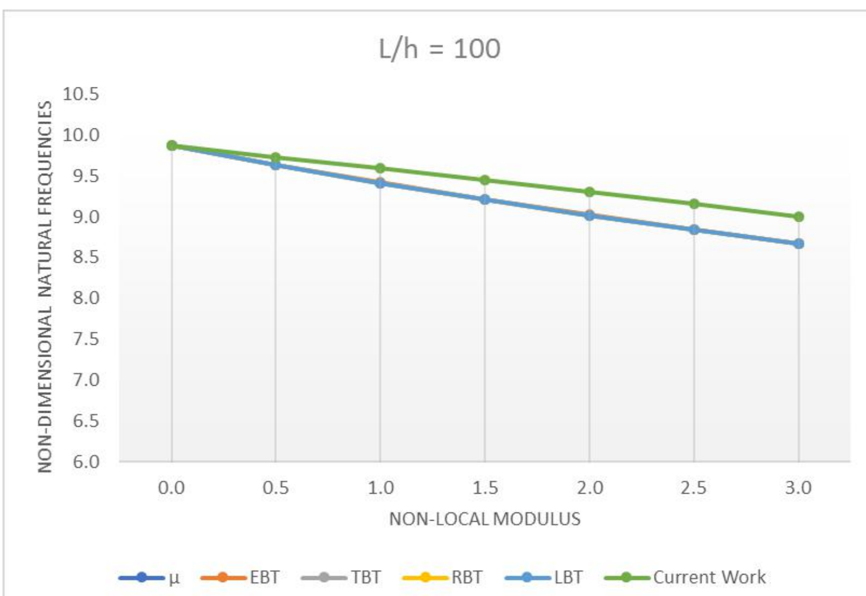


Figure 19: The influence of the non-local modulus on the first non-dimensional natural frequency of a simply supported beam (E=30e6, m=1, $\nu = 0.3$) for L/h = 100

Table 15: The influence of the non-local modulus on the first non-dimensional natural frequency of a simply supported beam (E=30e6, m=1, $\nu = 0.3$) for various slenderness ratios

L/h	μ	EBT	TBT	RBT	LBT	Current Work
100	0.0	9.8696	9.8683	9.8683	9.8685	9.8657
	0.5	9.6347	9.6335	9.6335	9.6337	9.7295
	1.0	9.4159	9.4147	9.4147	9.4149	9.5907
	1.5	9.2113	9.2101	9.2101	9.2103	9.4489
	2.0	9.0195	9.0183	9.0183	9.0185	9.3039
	2.5	8.8392	8.838	8.838	8.8382	9.1553
	3.0	8.6693	8.6682	8.6682	8.6683	9.0027
20	0.0	9.8696	9.8381	9.8381	9.8433	9.7695
	0.5	9.6347	9.604	9.604	9.6091	9.6348
	1.0	9.4159	9.3858	9.3858	9.3908	9.4975
	1.5	9.2113	9.1819	9.1819	9.1868	9.3572
	2.0	9.0195	8.9907	8.9907	8.9955	9.2137
	2.5	8.8392	8.811	8.811	8.8156	9.0667
	3.0	8.6693	8.6416	8.6416	8.6462	8.9156
10	0.0	9.8696	9.7454	9.7454	9.7657	9.4620
	0.5	9.6347	9.5135	9.5135	9.5333	9.3317
	1.0	9.4159	9.2973	9.2973	9.3168	9.1987
	1.5	9.2113	9.0953	9.0953	9.1144	9.0628
	2.0	9.0195	8.9059	8.9059	8.9246	8.9239
	2.5	8.8392	8.7279	8.7279	8.7462	8.7814
	3.0	8.6693	8.5601	8.5601	8.578	8.6349

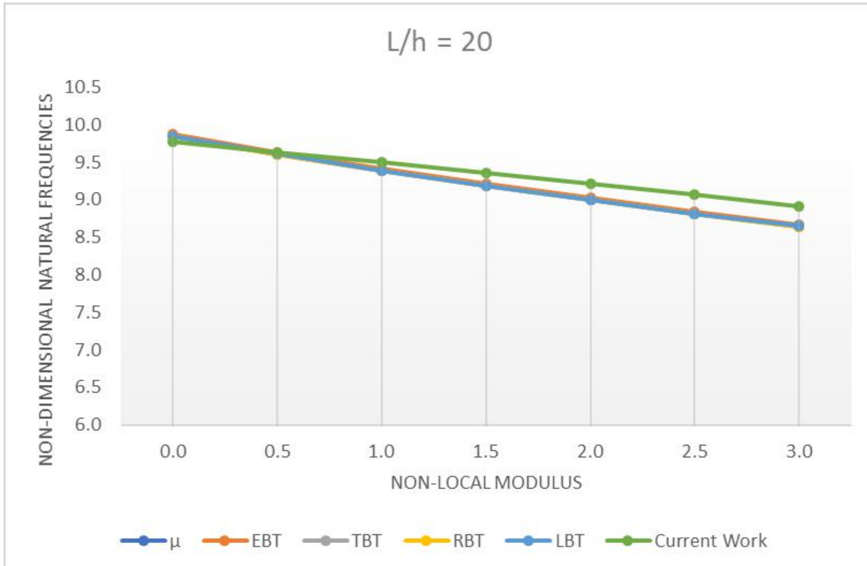


Figure 20: The influence of the non-local modulus on the first non-dimensional natural frequency of a simply supported beam (E=30e6, m=1, $\nu = 0.3$) for L/h = 20

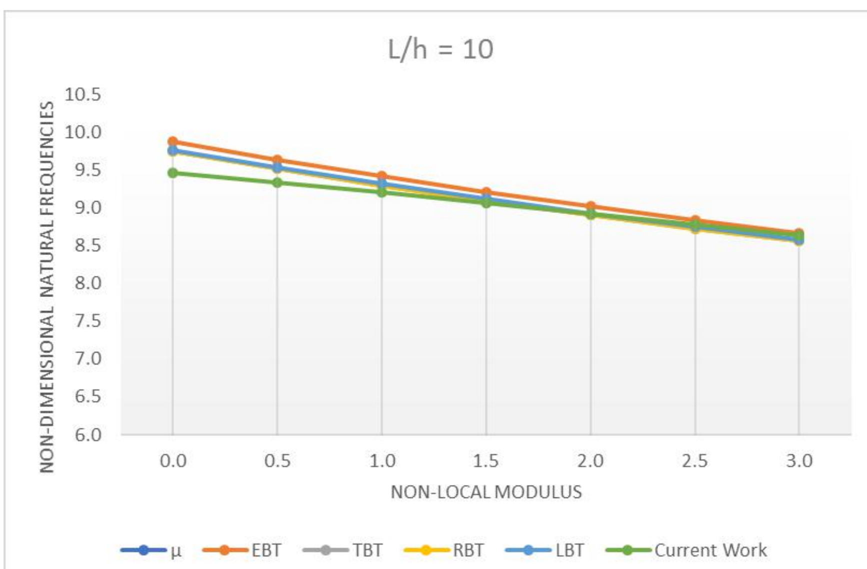


Figure 21: The influence of the non-local modulus on the first non-dimensional natural frequency of a simply supported beam (E=30e6, m=1, $\nu = 0.3$) for L/h = 10

CONCLUSION

In this research, the dynamic behavior of a two-dimensional and three-dimensional 10x1 beam was examined using local elasticity and non-local elasticity approximations. The results were compared to Euler–Bernoulli beam linear vibration results. A cantilever beam of various mesh sizes was used to understand the non-locality effect on the dynamic behavior of the beam.

5.1 Concluding Remarks

The NL-FEM method was applied to obtain the local and non-local stiffness matrices contributions Polizzotto (2001) using a Fortran 90 code. The mass matrix was also obtained to solve the Eigenvalue problem in order to get the natural frequencies and mode shapes of the 10x1 cantilever beam. Matlab code was used to obtain the natural frequencies and their corresponding mode shapes. The results have been validated by comparing the outcomes of a plain strain plate with Pisano, Sofi, and Fuschi (2009a) results. The natural frequencies have been calculated for several cases. The local and non-local elastic results are presented in Chapter 4. The internal length effect was addressed along with the effect of the neighbor elements number. In addition, the mesh size effect has been studied by testing a 10x1, 20x2 and 50x5 element.

The final results of the local and non-local elastic solution tell that the mesh size is an important factor. The natural frequencies gradually drop as the mesh size decreases (table 13).

The results were compared with a high and low non-local contribution K_n using $W_n = 0.9$ and $W_n = 0.1$, respectively. Obviously, as the local elastic contribution goes up, the natural frequencies get closer to the Euler Bernoulli solution.

The non-local results show that the best approximation can be achieved from large mesh size elements. However, the most accurate results were obtained from small mesh size elements. The large size elements gave a higher contribution to the global stiffness matrix, unlike the small elements. On the other hand, the small elements better cover the exact region of neighbors, which gives an accurate approximation. Furthermore, Bessel's attenuation function (3.20) has an effect on the very close region. Additionally, this function converts to a Dirac Delta function when the internal length goes to zero, which indicates its limited effect on the non-local contribution. In Figures 9a to 11c, the number of neighbor elements is unnoticeable in the cases of 50x5 element for all internal length cases. Only the case of 20x2 shows a slight difference between the region of $L_r = 5$ and $L_r = 10$. The rest of the regions didn't experience any differences. Regardless of all the previously mentioned differences and impacts, the mode shapes are exactly the same in all cases no matter the number of elements Figures 12a to 14d.

The impact of the internal length can be seen in the Figures 9a to 11c when comparing different cases of L for the same size of mesh. ($L = 0.1$ vs. $L = 0.2$ vs. $L = 0.3$). The factors impact decreases as L goes to zero. The better approximations can be obtained from the $L = 0.1$.

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Appendices

Table 16: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0322	0.0322	0.0321	0.0321	0.0321
2nd	0.5187	0.4982	0.1863	0.1863	0.1868	0.1868	0.1868
3rd	1.4523	1.2831	0.3596	0.3596	0.3595	0.3595	0.3595
4th	2.8460	1.3205	0.5164	0.5164	0.5164	0.5164	0.5164

Table 17: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0346	0.0347	0.0347	0.0347	0.0347
2nd	0.5187	0.4982	0.2154	0.2156	0.2156	0.2156	0.2156
3rd	1.4523	1.2831	0.5940	0.5949	0.5950	0.5950	0.5950
4th	2.8460	1.3205	0.6507	0.6514	0.6514	0.6514	0.6514

Table 18: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0434	0.0430	0.0430	0.0430	0.0430
2nd	0.5187	0.4982	0.2661	0.2632	0.2632	0.2632	0.2632
3rd	1.4523	1.2831	0.6640	0.6855	0.6855	0.6855	0.6855
4th	2.8460	1.3205	0.7205	0.7135	0.7136	0.7136	0.7136

Table 19: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0311	0.0312	0.0312	0.0312	0.0312
2nd	0.5187	0.4982	0.1864	0.1863	0.1863	0.1863	0.1863
3rd	1.4523	1.2831	0.4862	0.4856	0.4856	0.4856	0.4856
4th	2.8460	1.3205	0.5667	0.5674	0.5674	0.5674	0.5674

Table 20: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0328	0.0330	0.0330	0.0330	0.0330
2nd	0.5187	0.4982	0.1997	0.2006	0.2006	0.2006	0.2006
3rd	1.4523	1.2831	0.5284	0.5316	0.5316	0.5316	0.5316
4th	2.8460	1.3205	0.6782	0.6811	0.6811	0.6811	0.6811

Table 21: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0416	0.0416	0.0416	0.0416	0.0416
2nd	0.5187	0.4982	0.2540	0.2540	0.2540	0.2540	0.2540
3rd	1.4523	1.2831	0.6838	0.6824	0.6824	0.6824	0.6824
4th	2.8460	1.3205	0.6843	0.6843	0.6843	0.6843	0.6843

Table 22: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0305	0.0305	0.0305	0.0305	0.0305
2nd	0.5187	0.4982	0.1862	0.1862	0.1862	0.1862	0.1862
3rd	1.4523	1.2831	0.5015	0.5014	0.5015	0.5015	0.5015
4th	2.8460	1.3205	0.6003	0.6003	0.6003	0.6003	0.6003

Table 23: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.9$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0308	0.0307	0.0307	0.0307	0.0307
2nd	0.5187	0.4982	0.1881	0.1877	0.1877	0.1877	0.1877
3rd	1.4523	1.2831	0.5036	0.5017	0.5017	0.5017	0.5017
4th	2.8460	1.3205	0.6884	0.6858	0.6857	0.6857	0.6857

Table 24: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.9$

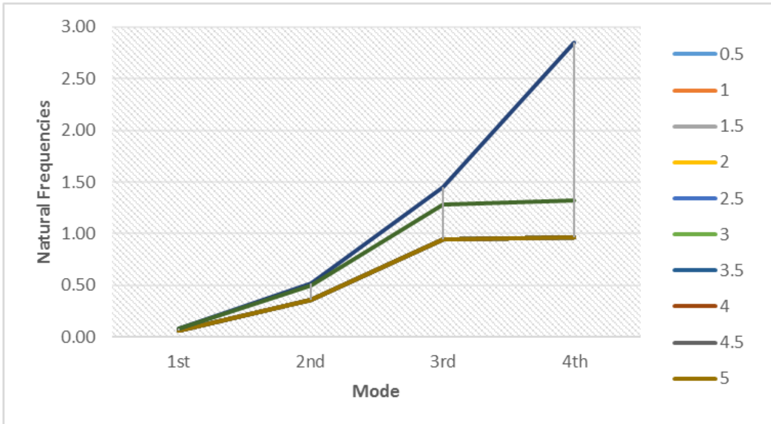
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.9				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0381	0.0381	0.0381	0.0381	0.0381
2nd	0.5187	0.4982	0.2322	0.2322	0.2322	0.2322	0.2322
3rd	1.4523	1.2831	0.6227	0.6226	0.6226	0.6226	0.6226
4th	2.8460	1.3205	0.6999	0.6995	0.6995	0.6995	0.6995

Table 25: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.1$

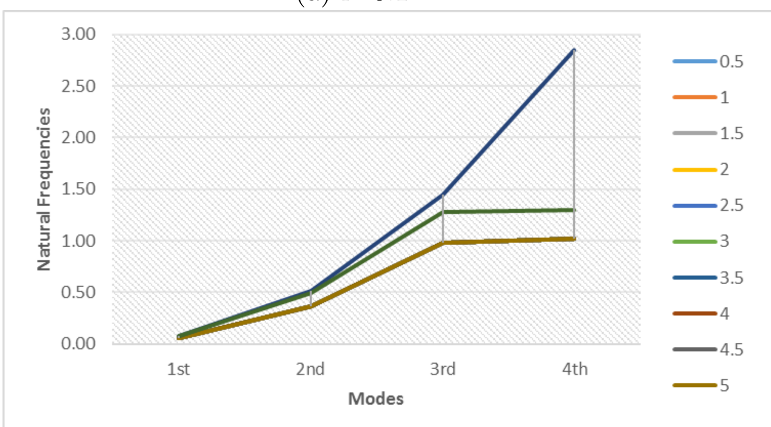
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0787	0.0787	0.0787	0.0787	0.0787
2nd	0.5187	0.4982	0.4742	0.4742	0.4742	0.4742	0.4742
3rd	1.4523	1.2831	1.2235	1.2235	1.2235	1.2235	1.2235
4th	2.8460	1.3205	1.2574	1.2574	1.2574	1.2574	1.2574

Table 26: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.1$

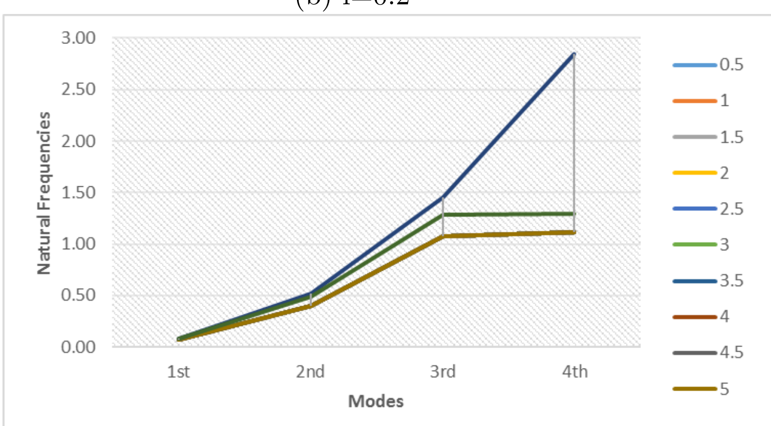
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0786	0.0786	0.0786	0.0786	0.0786
2nd	0.5187	0.4982	0.4720	0.4720	0.4720	0.4720	0.4720
3rd	1.4523	1.2831	1.2347	1.2348	1.2348	1.2348	1.2348
4th	2.8460	1.3205	1.2453	1.2453	1.2453	1.2453	1.2453



(a) l=0.1

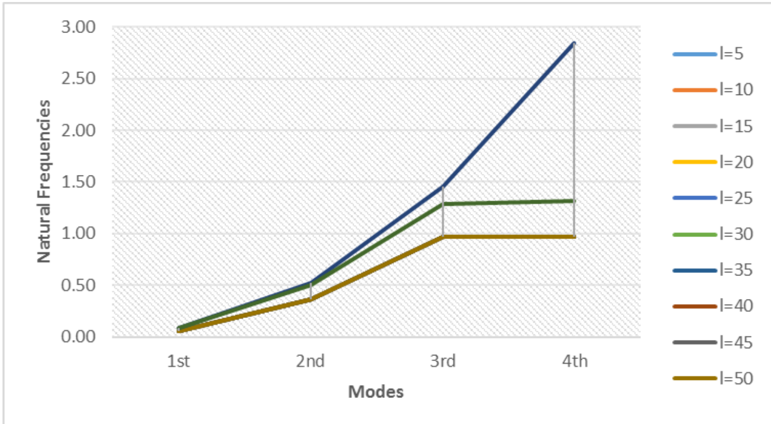


(b) l=0.2

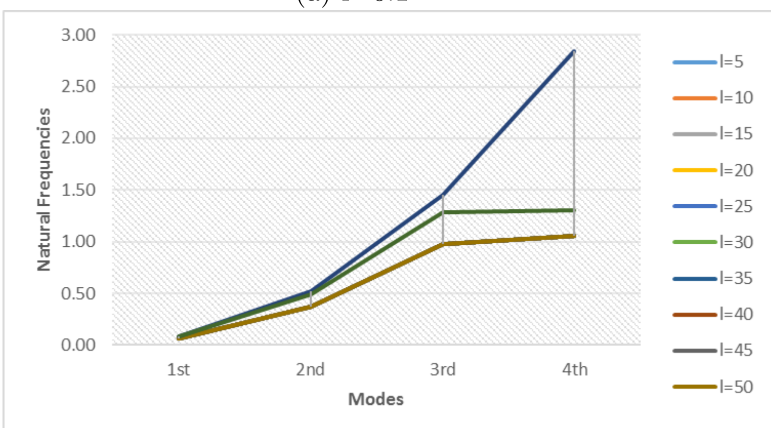


(c) l=0.3

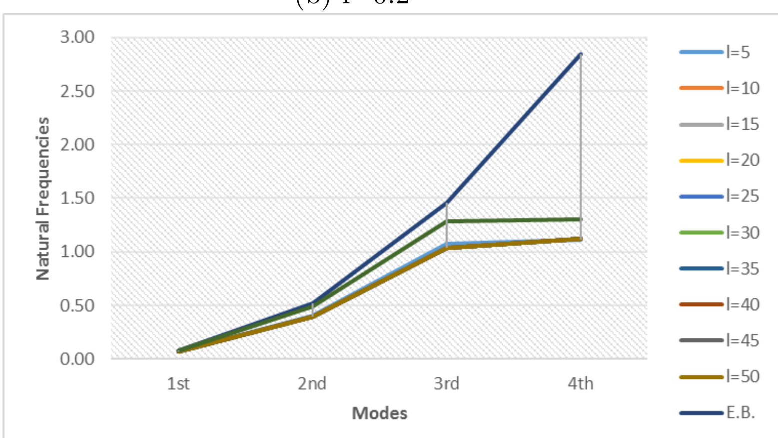
Figure 22: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory



(a) l=0.1

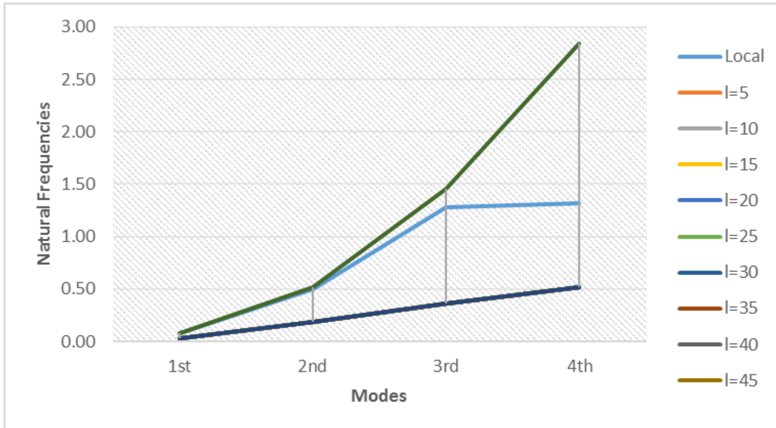


(b) l=0.2

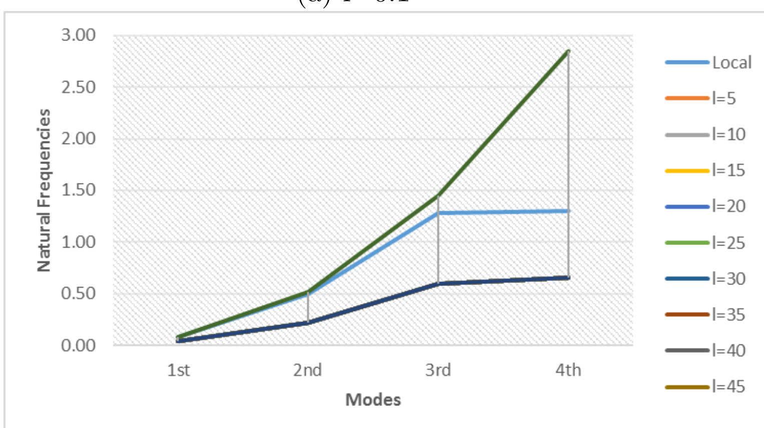


(c) l=0.3

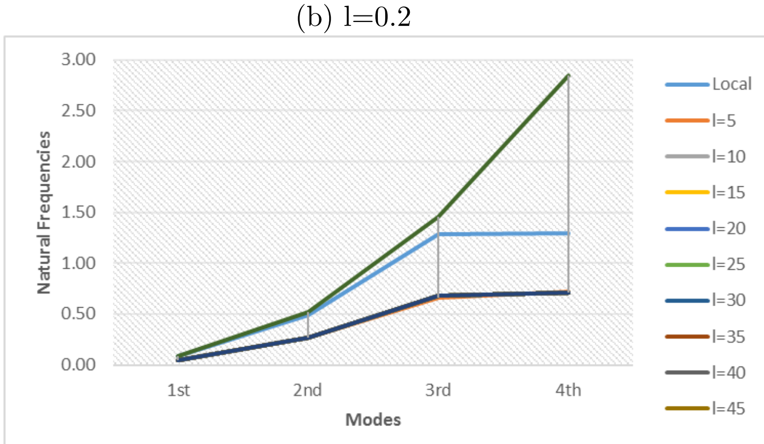
Figure 23: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory



(a) l=0.1



(b) l=0.2



(c) l=0.3

Figure 24: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory

Table 27: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.1, $W_n = 0.1$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0795	0.0795	0.0795	0.0795	0.0795
2nd	0.5187	0.4982	0.4773	0.4773	0.4773	0.4773	0.4773
3rd	1.4523	1.2831	1.2528	1.2530	1.2531	1.2531	1.2531
4th	2.8460	1.3205	1.2586	1.2586	1.2586	1.2586	1.2586

Table 28: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.1$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0787	0.0787	0.0787	0.0787	0.0787
2nd	0.5187	0.4982	0.4743	0.4743	0.4743	0.4743	0.4743
3rd	1.4523	1.2831	1.2271	1.2271	1.2272	1.2272	1.2272
4th	2.8460	1.3205	1.2577	1.2577	1.2577	1.2577	1.2577

Table 29: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.1$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0785	0.0785	0.0785	0.0785	0.0785
2nd	0.5187	0.4982	0.4715	0.4715	0.4715	0.4715	0.4715
3rd	1.4523	1.2831	1.2412	1.2413	1.2413	1.2413	1.2413
4th	2.8460	1.3205	1.2443	1.2443	1.2443	1.2443	1.2443

Table 30: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.2, $W_n = 0.1$

Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0791	0.0791	0.0791	0.0791	0.0791
2nd	0.5187	0.4982	0.4750	0.4750	0.4750	0.4750	0.4750
3rd	1.4523	1.2831	1.2527	1.2527	1.2527	1.2527	1.2527
4th	2.8460	1.3205	1.2540	1.2543	1.2543	1.2543	1.2543

Table 31: First four natural frequencies of the 10x1 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.1$

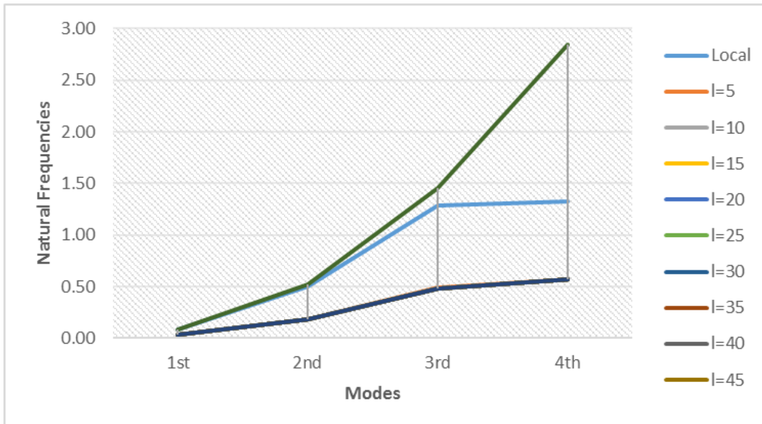
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0786	0.0786	0.0786	0.0786	0.0786
2nd	0.5187	0.4982	0.4741	0.4741	0.4741	0.4741	0.4741
3rd	1.4523	1.2831	1.2289	1.2289	1.2289	1.2289	1.2289
4th	2.8460	1.3205	1.2570	1.2570	1.2570	1.2570	1.2570

Table 32: First four natural frequencies of a 20x2 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.1$

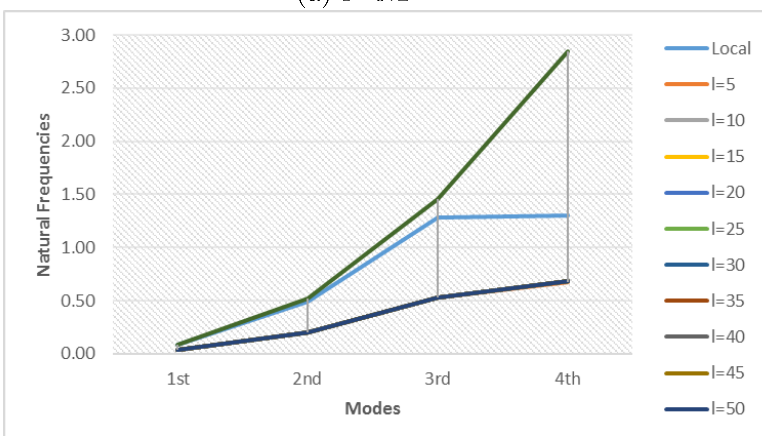
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0783	0.0783	0.0783	0.0783	0.0783
2nd	0.5187	0.4982	0.4707	0.4707	0.4707	0.4707	0.4707
3rd	1.4523	1.2831	1.2416	1.2418	1.2418	1.2418	1.2418
4th	2.8460	1.3205	1.2419	1.2419	1.2419	1.2419	1.2419

Table 33: First four natural frequencies of a 50x5 element of a local elastic and non-local elastic cantilever beam; l=0.3, $W_n = 0.1$

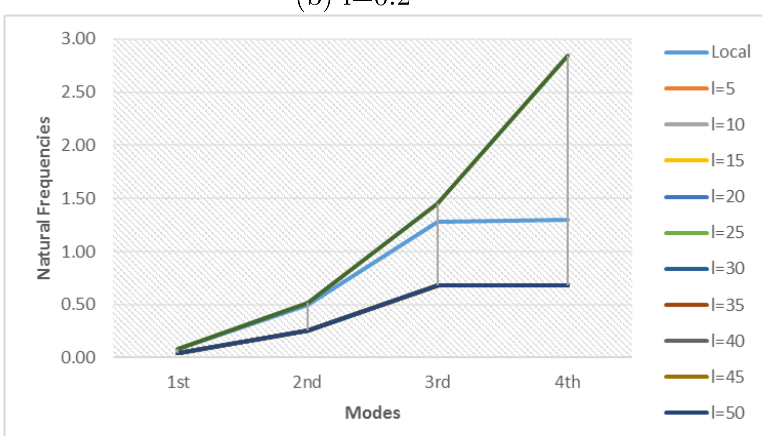
Mode	Euler Bernoulli	Local	non-local Elasticity, Wn=0.1				
			Lr=0.5	Lr=1.0	Lr=1.5	Lr=2.0	Lr=2.5
1st	0.0828	0.0826	0.0787	0.0787	0.0787	0.0787	0.0787
2nd	0.5187	0.4982	0.4727	0.4727	0.4727	0.4727	0.4727
3rd	1.4523	1.2831	1.2465	1.2465	1.2465	1.2465	1.2465
4th	2.8460	1.3205	1.2502	1.2504	1.2504	1.2504	1.2504



(a) l=0.1

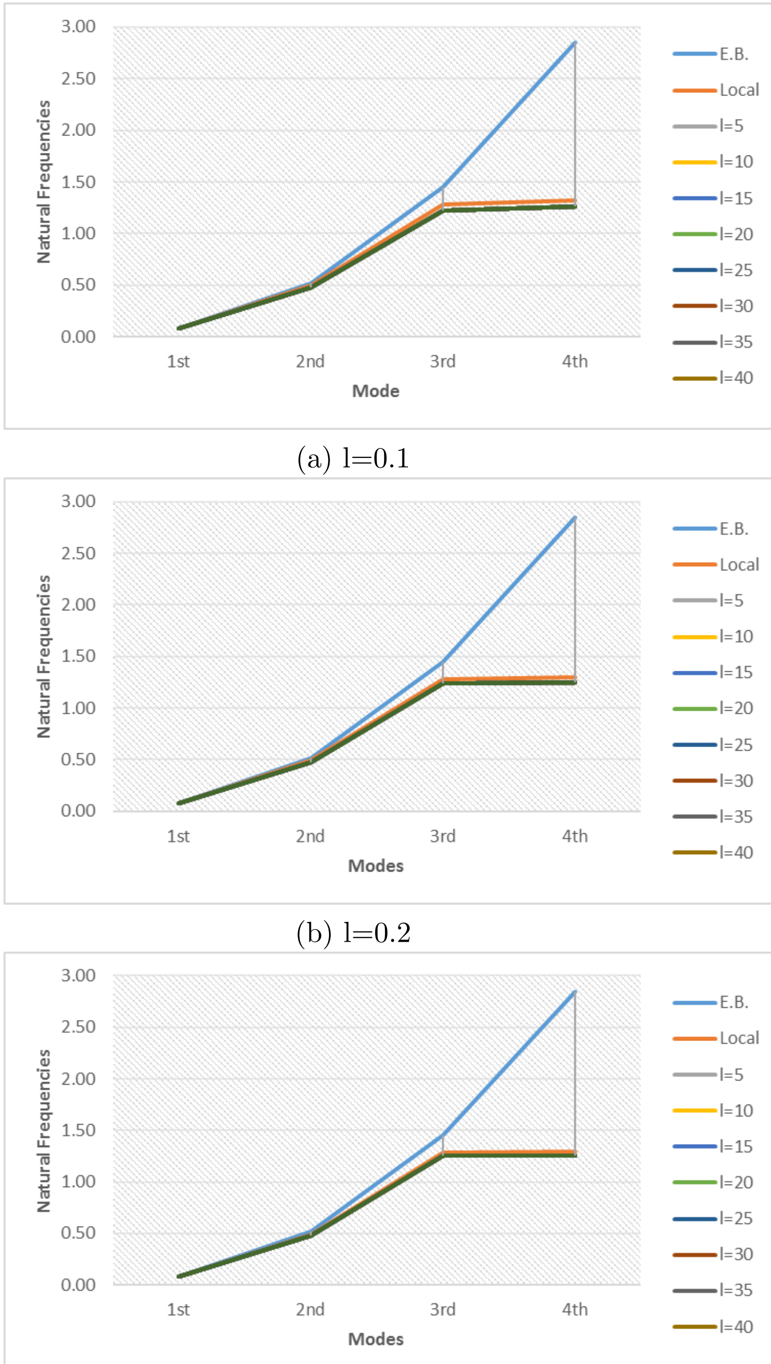


(b) l=0.2



(c) l=0.3

Figure 25: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory

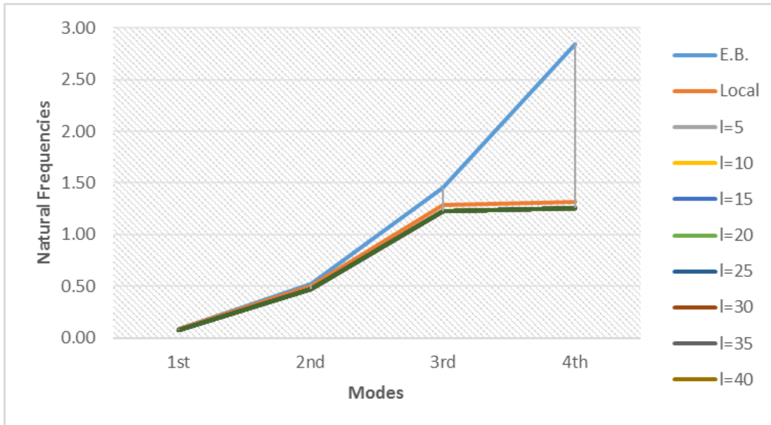


(a) l=0.1

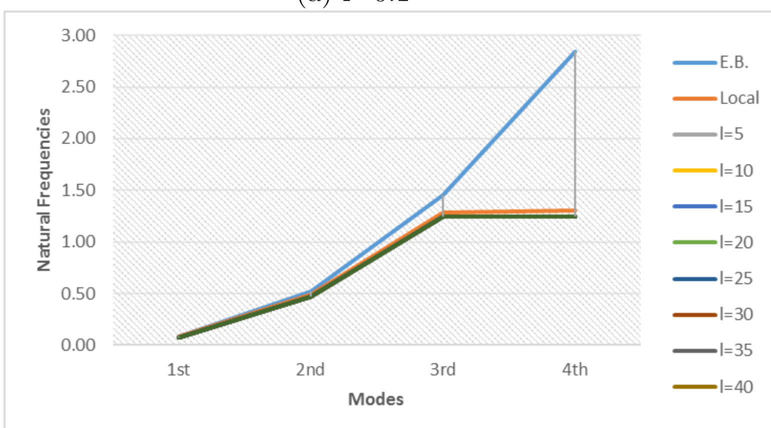
(b) l=0.2

(c) l=0.3

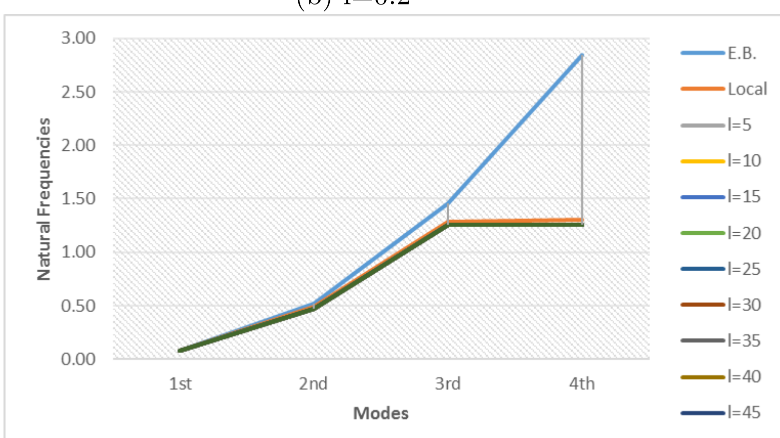
Figure 26: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory



(a) l=0.1

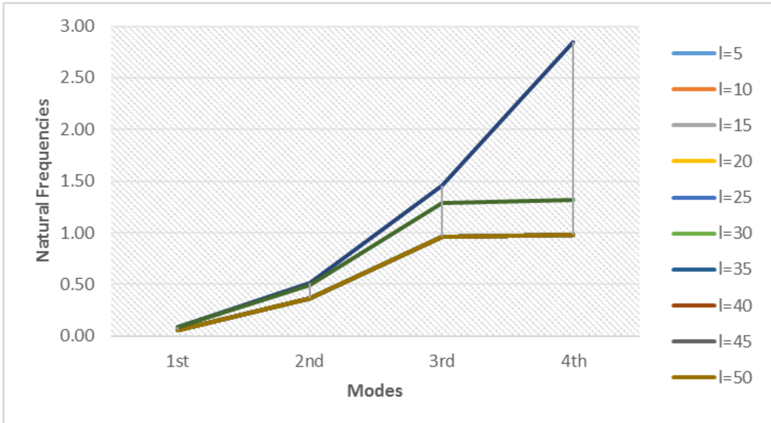


(b) l=0.2

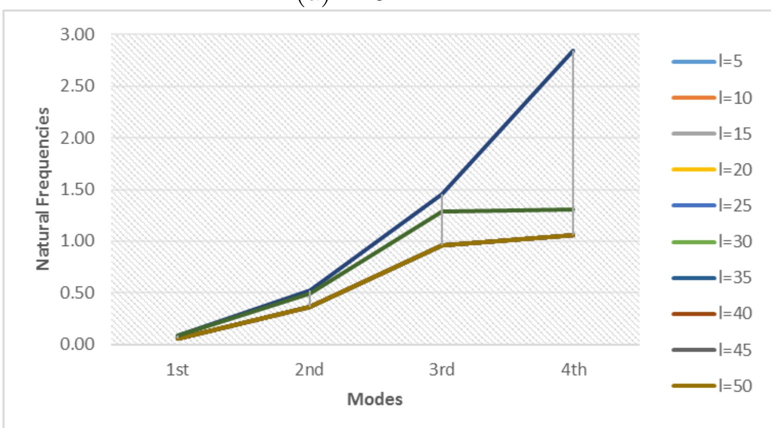


(c) l=0.3

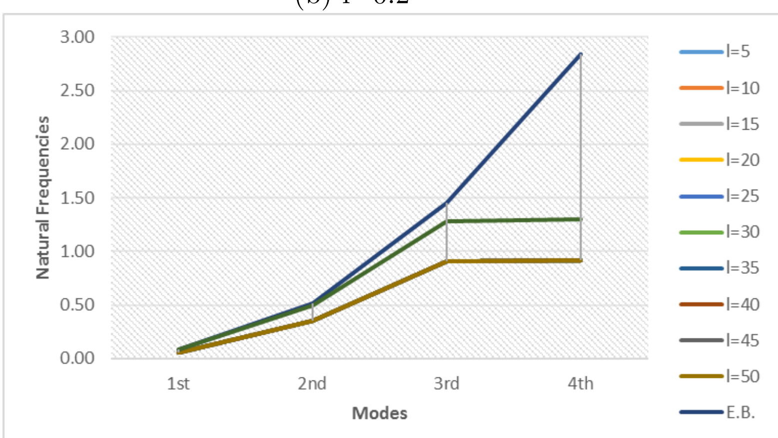
Figure 27: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory



(a) l=0.1

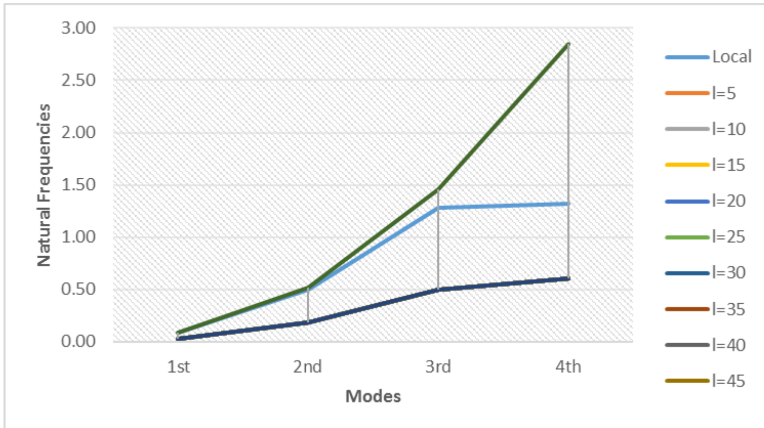


(b) l=0.2

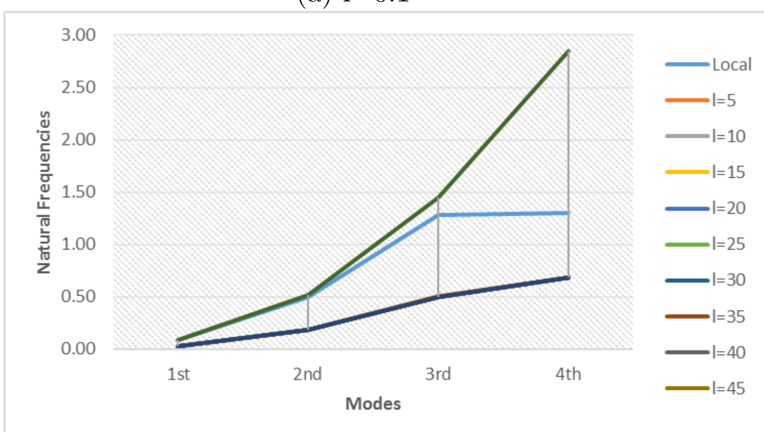


(c) l=0.3

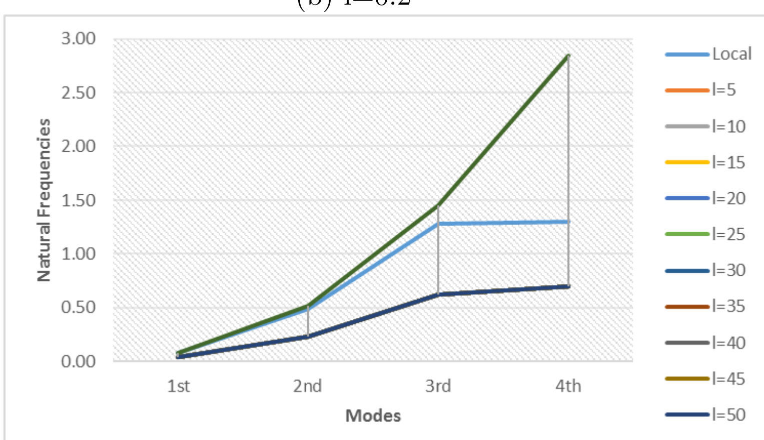
Figure 28: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory



(a) l=0.1

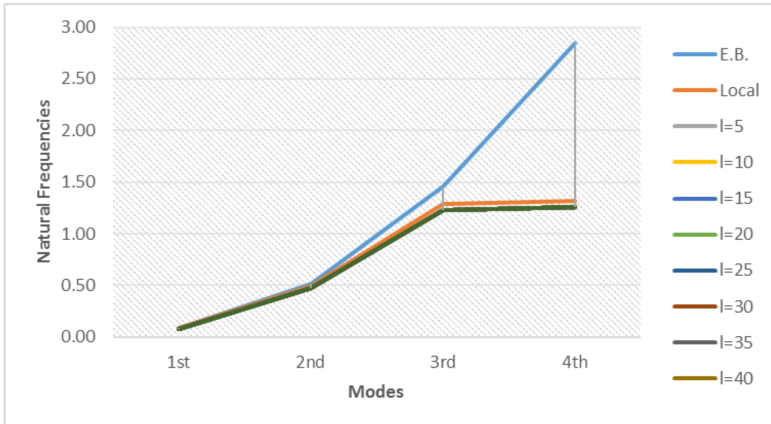


(b) l=0.2

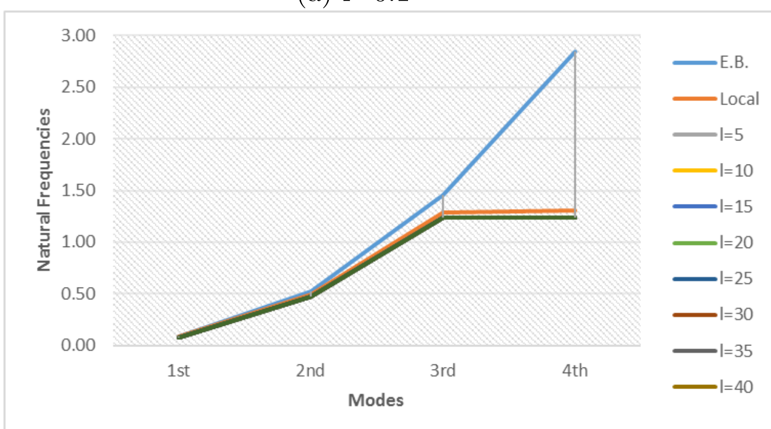


(c) l=0.3

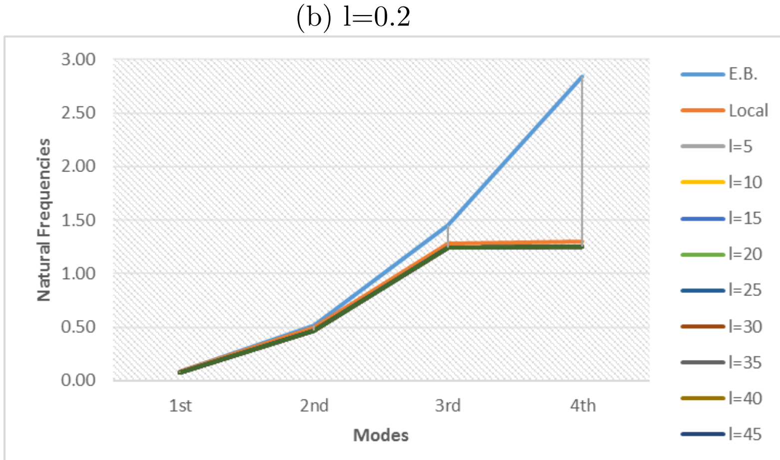
Figure 29: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory



(a) l=0.1



(b) l=0.2



(c) l=0.3

Figure 30: The first four modes of the 10x1 cantilever beam for 50x5 mesh using the non-local elasticity theory