The Use of Intrinsic Variables in Geodetic surveying ~~Geological~~ Reconnaissances of Small Areas

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The use of the curvilinear distance s and the curvature K as intrinsic variables of a curve leads to a formula giving the area of a closed plane curve in terms of K and s. Remembering that s corresponds to the mileage and K the degree of turning of the steering wheel, this formula offers a possibility of using an airplane flying horizontally for a quick survey of an enclosed region in geodetic surveying ~~geological reconnaissance~~ where the curvature of the earth can be neglected.

The ordinary method used in plane surveying to compute the area of an enclosed region requires a knowledge of the coordinates of the controlling points. These coordinates not only completely determine the shape and the area of the approximating polygon, but also fix its orientation with respect to a point chosen as the origin--unnecessarily from the point of view of area evaluation. To study the intrinsic properties of a closed plane curve, which do not depend on its orientation, one prefers to use the intrinsic variables, namely, curvature and the distance along the curve. Let a point on the curve be chosen as the reference point, from which the curvilinear distance of any other point on the curve is measured and denoted by s. The curvature at the same point will be denoted by K. It may be noted that the reference point, being on the curve and movable with the curve, does not fix its orientation.

The proposed intrinsic variables properly describe not only any smooth plane curve, but also a polygon which can not be described by a single equation in Cartesian coordinates. The simplest curve is the circle.

$$x^2 + y^2 = r^2 \tag{1}$$

which describes a circle of radius r with its center at the origin. Using the intrinsic variables, one has

$$K = \frac{1}{r} \tag{2}$$

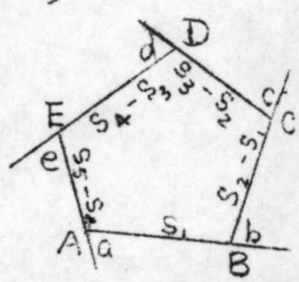
which describes all circles intrinsically identical with that represented by Equation (1), without fixing their centers anywhere. For a study of the intrinsic properties of a circle with radius r, Equation (2) is therefore more appropriate.

To describe a polygon, let the Dirac function $\delta(s_i)$ be defined as

$$\delta(s) = 0 \qquad \text{for } s \neq s_i$$

$$\delta(s_i) = \infty \qquad \text{in such a manner that} \quad \int_{-\infty}^{\infty} \delta(s)\,ds = 1$$

A pentagon ABCDEA with s-coordinates o, s_1, s_2, s_3, s_4, s_5 for the corner points in the order named, and with exterior

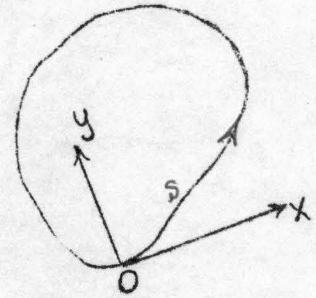


angles a, b, c, d, e corresponding to A, B, C, D, E, respectively, can be described by the single equation

$$K = b\,\delta(s_1) + c\,\delta(s_2) + d\,\delta(s_3) + e\,\delta(s_4) + a\,\delta(s_5) \tag{3}$$

with the reference point A considered situated on line AB rather than on line EA. The equation for an n-sided polygon can be similarly constructed.

To find the area of a closed plane curve, let the reference point be chosen as the origin and the tangent and the normal at the reference point be the x-axis and y-axis, respectively. These choices do not in any way fix the position of the curve and are made only for convenience of computation. One has then



$$A = \frac{1}{2}\int_0^l \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & y \\ 1 & x+dx & y+dy \end{vmatrix} = \frac{1}{2}\int_0^l \begin{vmatrix} x & y \\ dx & dy \end{vmatrix} \qquad (4)$$

where A and l are respectively the area and the perimeter of the closed plane curve.

It is easily seen that at any point s

$$dx = ds \cos \int_0^s K ds'' \qquad (5)$$

$$dy = ds \sin \int_0^s K ds \qquad (6)$$

$$x = \int_0^s \left(\cos \int_0^{s'} K ds'' \right) ds' \qquad (7)$$

$$y = \int_0^s \left(\sin \int_0^{s'} K ds'' \right) ds' \qquad (8)$$

Substituting Equations (5) to (8) in (4), one obtains

$$A = \frac{1}{2}\int_0^l \begin{vmatrix} \int_0^s \left(\cos \int_0^{s'} K ds''\right) ds' & \int_0^s \left(\sin \int_0^{s'} K ds''\right) ds' \\ \cos \int_0^s K ds'' & \sin \int_0^s K ds'' \end{vmatrix} ds$$

$$= \frac{1}{2}\int_0^l ds \int_0^s ds' \left[\left(\sin \int_0^s K ds''\right)\left(\cos \int_0^{s'} K ds''\right) - \left(\cos \int_0^s K ds''\right)\left(\sin \int_0^{s'} K ds''\right) \right]$$

$$= \frac{1}{2}\int_0^l \int_0^s \sin \left(\int_0^s K ds'' - \int_0^{s'} K ds'' \right) ds' ds$$

$$= \frac{1}{2}\int_0^l \int_0^s \left(\sin \int_{s'}^s K ds'' \right) ds' ds \qquad (9)$$

which applies not only to closed smooth plane curves, but also to plane polygons if K is properly expressed in terms of the Dirac functions. For instance, an n-sided polygon with all its sides equal to $\frac{l}{n}$ and with exterior angles $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ has an area

$$A = \frac{1}{2}\left(\frac{l}{n}\right)^2 \sum_{k=1}^{n-2} \left[\sin(\alpha_1 + \alpha_2 + \cdots + \alpha_k) + \sin(\alpha_2 + \alpha_3 + \cdots + \alpha_k) + \cdots + \sin\alpha_k\right]$$

where 1, 2, ..., (n-2) sine terms are to be used in the square bracket when K = 1, 2, ..., (n-2), respectively. Noteworthy is the fact that α_0 and α_{n-1} do not appear in Equation (10).

In the ~~geological~~ geodetic ~~reconnaissance~~ surveying of small areas where the curvature of the earth can be neglected, the use of the intrinsic variables offers a possibility of obtaining expediently the area of a lake, forest, sand bar, etc. if the functional relationship between K and s along the boundary can be quickly determined. With s corresponding to the mileage and K to the degree of turning of the steering wheel, the horizontal flight of an aircraft with proper instruments on a windless day offers a possible means of determining that relationship. With the help of a modern calculating machine, Equation (9) can then be used to obtain the area of the ~~reconnoitered~~ surveyed region.

(1932-50)