DISSERTATION

CONNECTIONS BETWEEN HESSENBERG VARIETIES, CHROMATIC QUASISYMMETRIC FUNCTIONS, AND q-SERIES

Submitted by

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ABSTRACT

CONNECTIONS BETWEEN HESSENBERG VARIETIES, CHROMATIC QUASISYMMETRIC FUNCTIONS, AND q-SERIES

In many ways, the combinatorics of symmetric functions can help us understand how other mathematical objects behave. For example, the Schur functions encode information about symmetric group representations as well as intersection theory in the Grassmannian. In this dissertation, we investigate connections between chromatic symmetric functions and Hessenberg varieties, and how each one can elevate the understanding of the other. Stanley and Stembridge conjectured in 1993 that the chromatic symmetric functions for unit interval graphs expanded with positive coefficients in the basis of elementary symmetric functions. This conjecture has been proved directly for several families of graphs, and a recent full proof was proposed by Hikita in 2024. Geometrically, this corresponds to showing that the cohomology rings of Hessenberg varieties, acted on by the symmetric group, decompose into permutation modules. Again, this result has been proven for several families of Hessenberg varieties, but in general remains open.

For the Hessenberg function $h = (h(1), n, \ldots, n)$, the structure of the cohomology ring was determined by Abe, Horiguchi, and Masuda in 2017. In this dissertation, we define two new bases for this cohomology ring, one of which is a higher Specht basis, and the other of which is a permutation basis. We also examine the transpose Hessenberg variety, indexed by the Hessenberg function $h' = ((n-1)^{n-m}, n^m)$, and show that analogous results hold. Further, we give combinatorial bijections between the monomials in the new basis and sets of P-tableaux, motivated by the work of Gasharov, and use P-tableaux to find a new formula for the Poincaré polynomial of these Hessenberg varieties.

Another open problem is to determine conditions for which the chromatic quasisymmetric function is symmetric. In 2024, Aliniaeifard et al. showed that if P is a path graph, then $X_P(\mathbf{x}; q)$

is symmetric if and only if the vertices of P are labeled in increasing or decreasing order, and if S is a star graph, then $X_S(\mathbf{x};q)$ is not symmetric. In this dissertation, we extend this result, and show that if G is any tree, other than the path graph given above, then $X_G(\mathbf{x};q)$ is not symmetric. We also construct a family of graphs called mixed mountain graphs, which are similar to unit interval graphs, and show that their chromatic quasisymmetric functions are symmetric.

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Chapter 1

Introduction

Hessenberg varieties, originally defined in [1] by De Mari, Procesi, and Shayman, are subvarieties of the full flag variety with deep connections to algebraic geometry, representation theory, and symmetric function theory. To start, a flag is a sequence of nested subspaces of $\mathbb{C}^n$, denoted $V_\bullet = V_1 \subset V_2 \subset \ldots \subset V_n$, where each V_i has dimension i. The flag variety $\mathrm{Fl}(\mathbb{C}^n)$ is the set of all (complete) flags of $\mathbb{C}^n$. Hessenberg varieties are subvarieties of the flag variety restricted to certain conditions: Let $[n]$ denote the set $\{1, \ldots, n\}$. A function $h : [n] \to [n]$ is called a Hessenberg function if it is weakly increasing and $h(i) \geq i$ for all i. We will usually write a Hessenberg function as a vector $h := (h(1), \ldots, h(n))$. Given a Hessenberg function h and an $n \times n$ matrix X over $\mathbb{C}$, we define the associated Hessenberg variety to be

$$\mathrm{Hess}(X, h) := \{V_\bullet \in \mathrm{Fl}(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \text{ for all } 1 \leq i \leq n\}.$$

Different specializations of Hessenberg varieties yield well-studied varieties. For example, when $h(i) = n$ for all i, $\mathrm{Hess}(X, h)$ is the flag variety $\mathrm{Fl}(\mathbb{C}^n)$ itself. If $h(i) = i$ for all i, and X is a nilpotent matrix, then $\mathrm{Hess}(X, h)$ is the Springer fiber. If $h(i) = i + 1$ for all $i \neq n$, and X is a regular semisimple matrix, $\mathrm{Hess}(X, h)$ is the toric variety associated to the permutahedron. On the other hand, if $h(i) = i + 1$ for all $i \neq n$, and X is a principal nilpotent matrix, $\mathrm{Hess}(X, h)$ is the Peterson variety.

The study of Hessenberg varieties has yielded many results, including those found in [2–7]. For example, in [2], Tymoczko shows that Hessenberg varieties admit an affine paving, and that the odd-degree pieces of the cohomology rings of Hessenberg varieties vanish. In [3], the authors investigate Hessenberg varieties when the underlying matrix X is regular and nilpotent, and give a construction of generators and relations for the cohomology ring in this case. In [6], the authors give an explicit presentation for the cohomology ring in the regular nilpotent case, based on the preceding paper.

Hessenberg varieties are smooth when $X = S$ is a regular semisimple matrix, and for these matrices, the cohomology ring is known to be the quotient of a direct sum of polynomial rings as defined by [8]. Further, in [8], Tymoczko defines an action of the symmetric group $\mathfrak{S}_n$ on the cohomology ring $H^*(\mathrm{Hess}(S, h))$ in this case, using GKM theory for equivariant cohomology. While Hessenberg varieties are significant in their own right, a recent combinatorial connection has motivated more study of the geometry of these spaces. This connection comes from the study of symmetric functions and representation theory.

Given a finite graph $G = (V, E)$, a proper coloring of G is a map $\kappa : V \to \mathbb{N}$, such that if vw is an edge in E, then $\kappa(v) \neq \kappa(w)$. We say that $\mathbb{N}$ is the "set of colors" used to color the vertices of G. The chromatic polynomial $\chi_G(m)$ of a graph outputs the number of proper colorings of G where at most m colors were used. In [9], Stanley defined the chromatic symmetric function, which is a generalization of the chromatic polynomial, defined as follows:

$$X_G(\mathbf{x}) = \sum_{\kappa: V(G) \to \mathbb{N}} x_{\kappa(v_1)} \cdots x_{\kappa(v_n)}.$$

Above, we use a bold $\mathbf{x}$ to denote a countably infinite set of (commuting) indeterminates, the sum is over all proper colorings κ of the vertices of G, and the monomials record how many times each color was used in the coloring. This function is *symmetric* because swapping two indeterminates x_i and x_j fixes the function – since swapping all instances of colors i and j preserves that the coloring is proper. Notice that setting $x_i = 1$ for all $1 \leq i \leq m$ (and $x_j = 0$ otherwise) yields the number of proper colorings with at most m colors, so the chromatic symmetric function indeed generalizes the chromatic polynomial.

Often, in algebraic combinatorics, we are interested in the expansion of symmetric functions into one of the known bases of the algebra of symmetric functions. These bases include the monomial basis $\{m_\lambda\}$, the Schur basis $\{s_\lambda\}$, and the elementary basis $\{e_\lambda\}$. See Section 2.2 for definitions of these functions. In [10], Gasharov showed that the chromatic symmetric function for unit interval graphs expands in the Schur basis with nonnegative coefficients, which can be calculated using a the theory of P-tableaux, where P is a poset related to the unit interval graph. The Stanley-Stembridge conjecture [11] states that for this family of graphs, the chromatic symmetric function can be expanded with nonnegative coefficients in the elementary basis. Very recently, Hikita [12] posted a paper which proves the Stanley-Stembridge conjecture using a probabilistic interpretation of the coefficients of the elementary expansion.

A generalization of the Stanley-Stembridge conjecture remains open. In [13], Shareshian and Wachs defined the chromatic quasisymmetric function $X_G(\mathbf{x}; q)$, a generalization of $X_G(\mathbf{x})$, by including a statistic on proper colorings called the ascent number. In this function, we utilize the graph G alongside an ordering of the vertices, which yields an acyclic orientation of the edges of G. The Stanley-Stembridge conjecture can be directly generalized to these quasisymmetric functions, where now the coefficients for the decomposition into the elementary symmetric functions are polynomials in q. Shareshian and Wachs proved that $X_G(\mathbf{x}; q)$ is symmetric for unit interval graphs, and extended Gasharov's Schur-expansion result using P-tableaux to the quasisymmetric case. They conjectured that the chromatic quasisymmetric function is related to the $\mathfrak{S}_n$-representation of the cohomology rings of regular semisimple Hessenberg varieties via the graded Frobenius map. This conjecture was proven in [4] by Brosnan and Chow using geometric techniques, and separately in [5] by Guay-Paquet using a new Hopf algebra. Because of this new connection, results on the cohomology rings of Hessenberg varieties can be translated to results on chromatic (quasi)symmetric functions, and likewise in reverse. In particular, showing that the $\mathfrak{S}_n$ action on $H^*(\mathrm{Hess}(S, h))$ is a permutation representation would imply that the Stanley-Stembridge conjecture (and its generalization to the quasisymmetric case) holds. For example, Tymozcko's result in [2] that $H^*(\mathrm{Hess}(S, h))$ admits an $\mathfrak{S}_n$-module structure gives an alternative proof that $X_G(\mathbf{x})$ is Schur-positive, since the decomposition into irreducible $\mathfrak{S}_n$-modules corresponds to the decomposition of X_G into Schur functions.

For both the chromatic (quasi)symmetric function and for Hessenberg varieties, incremental progress has been made in special cases. For example, in [14], Abreu and Nigro show that when G is a unit interval graph whose complement is bipartite, $X_G(\mathbf{x})$ is e-positive and e-unimodal. In [15], Aliniaeifard, Wang, and van Willigenburg show that a certain class of *melting lollipop* graphs have e-positive $X_G(\mathbf{x}; q)$. In [16], Cho and Hong showed that $X_G(\mathbf{x}; q)$ is e-positive when G has independence number 3, meaning that the largest set of vertices whose induced subgraph contains no edges has size 3. There are a multitude of other papers with similar results for other families of graphs, including in [17–21], among many others.

After the Shareshian-Wachs conjecture was proven, which connected chromatic quasisymmetric functions to regular semisimple Hessenberg varieties, the Stanley-Stembridge conjecture could be studied from a geometric perspective as well. On this side, one piece of incremental progress came from Harada and Precup in [22]. They studied Tymoczko's dot action for abelian Hessenberg varieties, which can be defined as follows: The *index* of a Hessenberg function is the largest i such that $h(i) < n$. Then, h is an *abelian* Hessenberg function if $h(1) \geq \operatorname{index}(h)$. Harada and Precup showed geometrically that the corresponding chromatic quasisymmetric functions were e-positive, via an inductive construction of the $\mathfrak{S}_n$-representation. The authors further give an inductive formula for the Poincaré polynomial of these Hessenberg varieties.

Another piece of progress came from Abe, Horiguchi, and Masuda in [23]. They provided a new understanding of the cohomology ring $H^*(\operatorname{Hess}(S, h))$ when $h = (h(1), n, \ldots, n)$, via an isomorphism with the quotient of a single polynomial ring in two sets of variables, and calculated the Poincaré polynomial of these Hessenberg varieties. While these Hessenberg varieties are abelian, as defined above, the $\mathfrak{S}_n$-module structure on this cohomology ring is more accessible than for the cohomology ring defined by Tymozcko. Thus connections can be drawn more easily between the cohomology ring and the corresponding symmetric functions in this case.

In this dissertation, we extend the results of Abe, Horiguchi, and Masuda from [23], and prove some further results on Hessenberg varieties and chromatic quasisymmetric functions. We examine the presentation of $H^*(\operatorname{Hess}(S, h))$ for the Hessenberg function $h = (h(1), n, \ldots, n)$ and its transpose $h' = ((n-1)^{n-m}, n^m)$, determine new bases for the cohomology rings, and draw connections between the basis elements and P-tableaux. One motivation is Gasharov's expansion of $X_G(\mathbf{x})$ into Schur functions using P-tableaux, and the generalization of this result by Shareshian and Wachs. Our first main result is the following theorem, which provides a so-called *higher Specht basis* for the cohomology ring.

Theorem 1.0.1. *The following sets form a basis of* $H^*(\mathrm{Hess}(S,h))$ *when* $h = (h(1), n, \ldots, n)$*:*

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \textit{which does not contain the factor } \prod_{\ell=1}^{h(1)} x_\ell$$
$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_{k+1} - y_1) \quad \textit{which does not contain the factor } \prod_{\ell=h(1)+1}^{n} x_\ell$$

running over all $0 \leq i_j \leq n-j$ *in the first equation, and over all* $0 \leq \ell_j \leq n-1-j$*, and* $1 \leq k \leq n-1$ *in the second equation. We denote these sets of monomials* B_1 *and* B_3*.*

Formally defined in Section 2.5, higher Specht bases have the advantage of more clearly displaying the decomposition of the corresponding $\mathfrak{S}_n$ representation into irreducibles, giving a deeper understanding of these cohomology rings. Since the basis elements of the irreducible Specht modules are in bijection with standard tableaux, we are motivated to find bijections between the monomials in Theorem 1.0.1 and certain sets of tableaux. These results are summarized in the following theorems.

Theorem 1.0.2. *There exists a combinatorial, weight-preserving bijection between monomial basis elements of the first kind and the set of* P_h*-tableaux of shape* (1^n)*.*

Theorem 1.0.3. *There exists a combinatorial, weight-preserving bijection between monomial basis elements of the second kind with the set of pairs* (S, T) *where* S *is a standard tableau, and* T *is a* P_h*-tableau, both of shape* $(2, 1^{n-2})$*.*

Further, in the case of regular nilpotent Hessenberg varieties, the cohomology ring is isomorphic to the $\mathfrak{S}_n$-fixed points of the regular semisimple case. Let N be a regular nilpotent matrix. We display a bijection between known basis elements of $H^*(\mathrm{Hess}(N,h))$ and the set of P_h-tableaux of shape (1^n) for any Hessenberg function h.

Theorem 1.0.4. *There exists a combinatorial, weight-preserving bijection between basis elements of* $H^*(\mathrm{Hess}(N,h))$ *and the set of* P_h*-tableaux of shape* (1^n).

As an application, we use these bijections to develop an understanding of the Poincaré polynomial via an argument counting P_h-tableaux with an inversion statistic.

In addition to the higher Specht basis for $H^*(\mathrm{Hess}(S,h))$ when $h = (h(1), n, \ldots, n)$, we show that there is a natural way of defining a basis which decomposes into a sum of permutation representations. This result directly connects to the Stanley-Stembridge conjecture, since the Frobenius character of a permutation representation (after applying the involution ω on symmetric function) yields an e-positive symmetric function.

We further show that many of the arguments for the Hessenberg function $h = (h(1), n, \ldots, n)$ also work for the transpose Hessenberg function $h' = ((n-1)^{n-m}, n^m)$. The polynomial generators are defined in a similar matter, and share many of the properties from the initial case.

Theorem 1.0.5. *The following sets form a higher Specht basis of* $H^*(\mathrm{Hess}(S,h))$ *when the Hessenberg function is* $h = ((n-1)^{n-m}, n^m)$*:*

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \textit{which does not contain the factor} \prod_{\ell=n-m+1}^{n} x_\ell$$
$$x_{n-1}^{\ell_1} x_{n-2}^{\ell_2} \cdots x_1^{\ell_{n-1}} (y_{k+1} - y_1) \quad \textit{which does not contain the factor} \prod_{\ell=1}^{n-m} x_\ell$$

running over all $0 \leq i_j \leq j-1$ *in the first equation, and over all* $0 \leq \ell_j \leq j-1$ *and* $2 \leq k \leq n$ *in the second equation.*

We show that there are similar bijections between these monomial basis elements and sets of tableaux, with shapes (1^n) and $(2, 1^{n-2})$. A future goal is to connect the combinatorics of h and h' to understand Hessenberg functions of the form $(h(1), (n-1)^m, n^{n-m-1})$.

Returning to the chromatic quasisymmetric function, it is important to note that not all chromatic quasisymmetric functions are also symmetric. While Shareshian and Wachs [13] proved that unit interval graphs have symmetric $X_G(\mathbf{x}; q)$, there are simple examples that are not symmetric (see Section 2.2 for such an example). In [24], the authors show that when G is the path graph, $X_G(\mathbf{x}; q)$ is symmetric if and only if G is naturally labeled. They further show that if $G = K_{1,n-1}$ is the star graph, then $X_G(\mathbf{x}; q)$ is never symmetric for $n \geq 4$. The authors pose the following question:

Question 1.0.6 ([24], Question 5.5)**.** For which labeled trees T is $X_T(\mathbf{x}; q)$ symmetric?

In a paper by Gillespie, Pappe, and myself [25], we answer this question and show that the only labeled tree T for which $X_T(\mathbf{x}; q)$ is symmetric is the naturally labeled path graph. Further, we construct a class of *mixed-mountain graphs*, which are similar to unit interval graphs, which always have symmetric $X_G(\mathbf{x}; q)$. The results of this paper are included in this dissertation, in Chapters 7 through 9. Our main results are given by the following theorems.

Theorem 1.0.7. *Let G be a connected, directed acyclic graph. If G has at least two sources, then $X_G(\mathbf{x}; q)$ is not symmetric.*

Theorem 1.0.8. *Let G be a (p, k)-mixed mountain graph. Then $X_G(\mathbf{x}; q)$ is symmetric.*

1.1 Outline

In Chapter 2, we give the necessary background on Hessenberg varieties, chromatic symmetric and quasisymmetric functions, and the representation theory of the symmetric group $\mathfrak{S}_n$ in order to understand our results. We also define the Specht polynomials and higher Specht bases, coming from irreducible $\mathfrak{S}_n$-modules, looking towards building a higher Specht basis for certain Hessenberg varieties.

In Chapter 3, we review the process Abe, Horiguchi, and Masuda used for finding a basis for $H^*(\text{Hess}(S, h))$ when $h = (h(1), n, \ldots, n)$. We define a new class of GKM graphs, and show that this can be extended to a basis when $h' = ((n-1)^{n-m}, n^m)$. We call this the transpose

Hessenberg function, since the corresponding Dyck path is a reflection of the original Dyck path across a diagonal.

In Chapter 4, we define an alternative basis for $H^*(\mathrm{Hess}(S,h))$ when $h=(h(1),n,\ldots,n)$, and show that this is a higher Specht basis, which directly illustrates the decomposition of $H^*(\mathrm{Hess}(S,h))$ into irreducible $\mathfrak{S}_n$ representations. We further show that the monomials from Theorem 1.0.1 can be reorganized into a permutation basis, which gives a geometric proof of the known result that the corresponding incomparability graphs have e-positive chromatic symmetric function.

In Chapter 5, we display a bijection between the fixed monomials in the higher Specht basis, which correspond to trivial representations of $\mathfrak{S}_n$, and the set of P_h-tableaux of shape (1^n), which correspond to the Schur function $s_{(n)}$ in Gasharov's expansion of $\omega X_G(\mathbf{x})$. We also display a bijection between the second set of basis elements, which correspond to standard representations of $\mathfrak{S}_n$, and the set of P_h-tableaux of shape $(2,1^{n-2})$, which correspond to the Schur function $s_{(n-1,1)}$. Building on this, in Chapter 6, we use the Shareshian-Wachs generalization of Gasharov's P-tableaux formula to give an alternative method for finding the Poincaré polynomial of $H^*(\mathrm{Hess}(S,h))$ for these Hessenberg functions.

In Chapters 7, 8, and 9, we investigate the question of when the chromatic quasisymmetric function is symmetric. We examine some small examples, and then show that if G is a directed tree, then $X_G(\mathbf{x};q)$ is symmetric if and only if G is a directed path. We then identify another class of graphs, called mixed mountain graphs, and prove that their chromatic quasisymmetric function is symmetric.

Chapter 2

Background

2.1 Hessenberg varieties

In this paper, we use the notation for Hessenberg varieties which can be found in [23]. Define $[n]$ to be the set $\{1, 2, \dots, n\}$. A function $h : [n] \to [n]$ is a **Hessenberg function** of length n if

- For all $i \in [n]$, $i \leq h(i)$, and
- For all $i \in [n-1]$, $h(i) \leq h(i+1)$.

We often write a Hessenberg function as a tuple $h = (h(1), h(2), \dots, h(n))$. One can associate to each Hessenberg function a lattice path in the first quadrant from $(0, 0)$ to (n, n), called a Dyck path, by associating $h(i)$ with the height of the i-th horizontal step. Dyck paths stay weakly above the diagonal $y = x$, and contain only “up” and “right” steps, and these properties correspond to the $h(i) \geq i$ and weakly-increasing conditions of Hessenberg functions. See Figure 2.1 for an example.

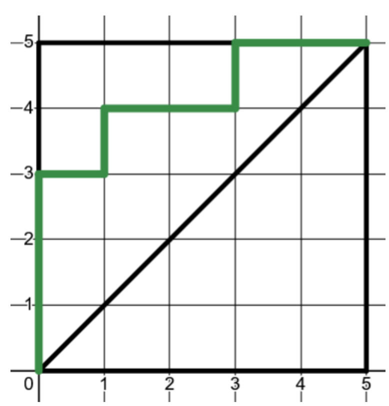


Figure 2.1: The Dyck path for the Hessenberg function $h = (3, 4, 4, 5, 5)$.

It is well-known that Dyck paths are enumerated by the Catalan numbers. In other words, the number of Dyck paths from $(0, 0)$ to (n, n), or Hessenberg functions of length n, is $\frac{1}{n+1}\binom{2n}{n}$.

To define the Hessenberg variety, we need to first define the flag variety. Formally, a (complete) flag is a sequence of nested subspaces of $\mathbb{C}^n$, denoted $V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_n$, where each V_i has dimension i. We define the **flag variety of** $\mathbb{C}^n$ to be the set of all complete flags of $\mathbb{C}^n$:

$$\mathrm{Fl}(\mathbb{C}^n) = \{V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_n \mid \dim(V_i) = i\}.$$

To describe this variety in another way, we examine a particular group action. For each flag $V_\bullet$, there exists some basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathbb{C}^n$ such that $V_i = \mathrm{span}(v_1, \ldots, v_i)$. Recall that the general linear group $\mathrm{GL}_n(\mathbb{C})$ is the group of all invertible $n \times n$ matrices over $\mathbb{C}$. We find that $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{Fl}(\mathbb{C}^n)$ in the following way: For $g \in \mathrm{GL}_n(\mathbb{C})$,

$$g \cdot V_i = g \cdot \mathrm{span}(v_1, \ldots, v_i) = \mathrm{span}(gv_1, \ldots, gv_i).$$

Notably, this group action is transitive, since there always exists a change-of-basis matrix g to transform any flag $V_\bullet$ into any other flag $W_\bullet$. We define $E_\bullet$ to be the **standard flag**, where E_i is generated by the standard unit basis vectors $e_1, \ldots, e_i$. The stabilizer subgroup of this action on $E_\bullet$ is the set of invertible upper-triangular matrices, which we will denote B (called the Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$). Since the group action is transitive, we get a well-defined bijection between the set of cosets $\mathrm{GL}_n(\mathbb{C})/B$ and the flag variety $\mathrm{Fl}(\mathbb{C}^n)$, which sends the coset gB to the flag $g \cdot E_\bullet$. In other words, we have $\mathrm{GL}_n(\mathbb{C})/B \cong \mathrm{Fl}(\mathbb{C}^n)$ as sets. Hence, we can write down the **coset model** for the flag variety as:

$$\mathrm{Fl}(\mathbb{C}^n) = \{gB \in \mathrm{GL}_n(\mathbb{C})/B\}.$$

We now define the Hessenberg variety, which is a certain subvariety of the flag variety, and was originally presented in [1]. Given a Hessenberg function $h : [n] \to [n]$ and an $n \times n$ matrix X, we define the **Hessenberg variety** as follows:

$$\mathrm{Hess}(X, h) := \{V_\bullet \in \mathrm{Fl}(\mathbb{C}^n) \mid X(V_i) \subseteq V_{h(i)} \text{ for all } 1 \leq i \leq n\}.$$

Similarly to the flag variety, we can define Hessenberg varieties equivalently as a subspace of $\mathrm{GL}_n(\mathbb{C})/B$. For a Hessenberg function h of length n, let $H(h)$ be the set of matrices M such that $M_{i,j} = 0$ whenever $i > h(j)$. We call this the **Hessenberg space** for h; see Figure 2.2 for an example of a Hessenberg space. For an $n \times n$ matrix X, we can use the coset model for the flag variety to express the Hessenberg variety as the following set of cosets:

$$\mathrm{Hess}(X,h) = \{gB \in \mathrm{Gl}_n(\mathbb{C})/B \mid g^{-1}Xg \in H(h)\}.$$

Notice that, in the coset model for the Hessenberg variety, there is a natural invertible map from $\mathrm{Hess}(X,h) \to \mathrm{Hess}(m^{-1}Xm, h)$ for any $m \in \mathrm{Gl}_n(\mathbb{C})$, given by $gB \mapsto mgB$. Hence

$$\mathrm{Hess}(X,h) \cong \mathrm{Hess}(m^{-1}Xm, h), \tag{2.1.1}$$

which is a fact we will use in Corollary 2.1.3.

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

Figure 2.2: The form of any matrix in the Hessenberg space $H(h)$ for $h = (3,4,4,5,5)$.

In [2], Tymoczko proved that Hessenberg varieties behave nicely as varieties:

Proposition 2.1.1. Every Hessenberg variety $\mathrm{Hess}(X,h)$ admits an affine paving, that is, can be expressed as the disjoint union of a countable number of affine algebraic subvarieties. In particular, the cohomology ring $H^*(\mathrm{Hess}(X,h))$ is torsion-free and vanishes in the odd degree.

As a result, we can write the Poincaré polynomial of $\mathrm{Hess}(X,h)$ in the following way:

$$\mathrm{Poin}(\mathrm{Hess}(X,h)), q) = \sum_{i=0}^{d} \dim(H^{2i}(\mathrm{Hess}(X,h))\, q^i$$

where d is the dimension of $\mathrm{Hess}(X, h)$, and $\deg(q) = 2$. This expression will be important for connecting the combinatorics of chromatic symmetric functions to the geometry of Hessenberg varieties.

If h is a Hessenberg function of length n, we define the **transpose** Hessenberg function h' to be the Hessenberg function obtained by reflecting the corresponding Dyck path for h across the anti-diagonal ($y = n - x$). In other words, $h'(i) = |h^{-1}(\{n, n-1, \ldots, n+1-i\})|$. For an example of a Hessenberg function and its transpose, see Figure 2.3.

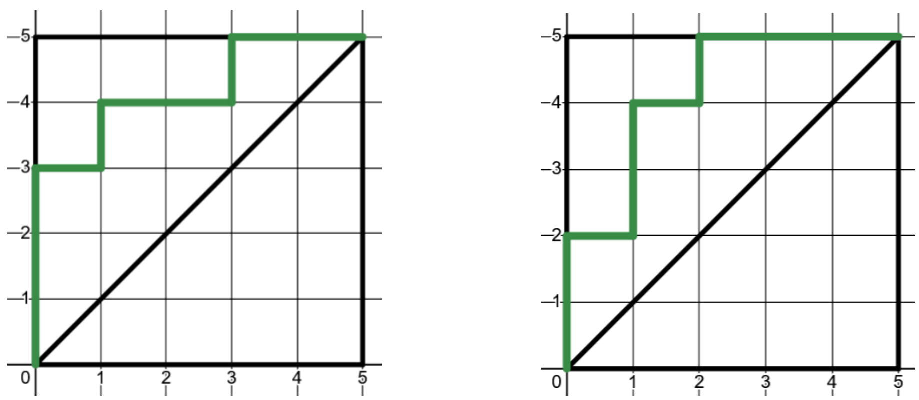


Figure 2.3: The Hessenberg function $h = (3, 4, 4, 5, 5)$ and its transpose $h' = (2, 4, 5, 5, 5)$.

The following observation is thanks to John Shareshian [26]:

Proposition 2.1.2. The space $\mathrm{Hess}(X, h)$ is isomorphic to $\mathrm{Hess}(X^T, h')$.

Proof. Let W be the permutation matrix for the permutation $\pi(i) = n + 1 - i$ for all $i \in [n]$, and so $W = W^{-1} = W^T$. Notice that for any $n \times n$ matrix X, we have that WXW is the matrix whose entries are the same as X, but rotated 180°, which we call X^R. Let $g \in \mathrm{GL}_n(\mathbb{C})$. We will show that if $gB \in \mathrm{Hess}(X, h)$, then $((gW)^{-1})^T B \in \mathrm{Hess}(X^T, h')$:

$$\begin{aligned}(gW)^T X^T ((gW)^{-1})^T &= W g^T X^T (g^{-1})^T W \\ &= W (g^{-1} X g)^T W \\ &= ((g^{-1} X g)^T)^R\end{aligned}$$

Since $gB \in \mathrm{Hess}(X,h)$, we know $g^{-1}Xg \in H(h)$, so $(g^{-1}Xg)_{i,j} = 0$ whenever $i > h(j)$. Taking the transpose, we get that $(g^{-1}Xg)^T_{i,j} = 0$ whenever $j > h(i)$. Finally, rotating by 180°, we have $((g^{-1}Xg)^T)^R_{i,j} = 0$ when $i > h'(j)$. Hence $((gW)^{-1})^T B \in \mathrm{Hess}(X^T, h')$. Further note that since B is the space of upper-triangular matrices, $(WBW)^T = B$.

Since the map $g \mapsto ((gW)^{-1})^T$ is an involution, and each entry in the resulting matrix is a polynomial in the entries of g, this is an invertible algebraic morphism $\mathrm{Hess}(X,h) \to \mathrm{Hess}(X^T, h')$, and so the two varieties are isomorphic. □

In this paper, we focus on Hessenberg varieties $\mathrm{Hess}(S,h)$ where S is a regular semisimple matrix, meaning that S is diagonalizable and has distinct eigenvalues. In particular, these Hessenberg varieties are smooth [2], and for a fixed Hessenberg function h and any two regular semisimple matrices S and T, $\mathrm{Hess}(S,h) \cong \mathrm{Hess}(T,h)$, per [1]. So, we will not need to specify the underlying matrices for regular semisimple Hessenberg varieties. In fact, regular semisimple Hessenberg varieties also satisfy the following nice property.

Corollary 2.1.3. *When S is a semisimple matrix,* $\mathrm{Hess}(S,h)$ *is isomorphic to* $\mathrm{Hess}(S,h')$.

Proof. Since S is semisimple over $\mathbb{C}$, it is diagonalizable, so there exists some $m \in \mathrm{GL}_n(\mathbb{C})$ such that $m^{-1}Sm$ is diagonal. By Equation 2.1.1, we get that $\mathrm{Hess}(S,h) \cong \mathrm{Hess}(m^{-1}Sm,h)$. By Proposition 2.1.2, we have that $\mathrm{Hess}(m^{-1}Sm,h) \cong \mathrm{Hess}((m^{-1}Sm)^T,h') = \mathrm{Hess}(m^{-1}Sm,h')$. Finally, again by Equation 2.1.1, we have that $\mathrm{Hess}(m^{-1}Sm,h') \cong \mathrm{Hess}(S,h')$.

Therefore $\mathrm{Hess}(S,h) \cong \mathrm{Hess}(S,h')$, as desired. □

2.2 Symmetric functions

In this section, we define the necessary background on symmetric and quasisymmetric functions - a standard reference is [27].

We start by defining a **weak composition** α of n to be a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of non-negative integers such that $\sum_{i=1}^{\infty} \alpha_i = n$. Note that this condition requires that a finite number of entries of α are nonzero. If λ is a weak composition such that $\lambda_i \geq \lambda_{i+1}$ for all i, we call λ a **partition** of n, and use the notation $\lambda \vdash n$.

Definition 2.2.1. A **homogeneous symmetric function** of degree n over a commutative ring R is a formal power series $f(\mathbf{x})$ with coefficients in R such that for any weak composition $\alpha = (\alpha_1, \ldots, \alpha_j)$ of n, the coefficient of $x_1^{\alpha_1} \cdots x_j^{\alpha_j}$ is the same as the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_j}^{\alpha_j}$ for any set of mutually-distinct values $i_1, i_2, \ldots, i_j$. Intuitively, $f(\mathbf{x})$ is fixed under any permutation of the indeterminates.

We will denote the ring of homogeneous symmetric functions of degree n over R by Sym_R^n. If we take the direct sum of these rings for all n, we obtain the graded algebra of all symmetric functions over R, which we denote by Sym_R, or simply Sym when the underlying ring is evident (usually in this paper, we will use $R = \mathbb{Q}$). Further, a **symmetric polynomial** is a symmetric function restricted to a finite set of indeterminates. Equivalently, a symmetric polynomial is a polynomial $f(x_1, \ldots, x_k) \in R[x_1, \ldots, x_k]$ such that for any permutation $\pi \in \mathfrak{S}_k$, the polynomials $f(x_1, \ldots, x_k)$ and $f(x_{\pi(1)}, \ldots, x_{\pi(k)})$ are equal. We now define some important families of symmetric functions.

Definition 2.2.2. Given a partition λ of n, the **monomial symmetric function** $m_\lambda(\mathbf{x})$ is

$$m_\lambda(\mathbf{x}) = \sum_\alpha x^\alpha,$$

where the index α ranges over all distinct permutations of the partition $\lambda = (\lambda_1, \lambda_2, \ldots)$.

For example, we have

$$m_{(3,1)}(\mathbf{x}) = \sum_{i \neq j} x_i^3 x_j.$$

It is not hard to show that any symmetric function $f \in \mathrm{Sym}$ can be written as a sum of different monomial symmetric functions, and so the monomial symmetric functions span the algebra Sym. In fact, they satisfy a notion of linear independence, and so they form a basis of Sym. Another basis of symmetric functions is as follows.

Definition 2.2.3. Given a positive integer k, we define the **elementary symmetric function**

$$e_k(\mathbf{x}) = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Further, for a partition λ of n, define $e_\lambda = e_{\lambda_1}\, e_{\lambda_2}\, \cdots\, e_{\lambda_{\ell(\lambda)}}$.

For example, we have

$$\begin{aligned} e_{(3,2)}(\mathbf{x}) &= \left(\sum_{i_1 < i_2 < i_3} x_{i_1} x_{i_2} x_{i_3} \right) \left(\sum_{i_1 < i_2} x_{i_1} x_{i_2} \right) \\ &= (x_1x_2x_3 + x_1x_2x_4 + \ldots)(x_1x_2 + x_1x_3 + \ldots) \\ &= 10m_{(1,1,1,1,1)}(\mathbf{x}) + 3m_{(2,1,1,1)}(\mathbf{x}) + m_{(2,2,1)}(\mathbf{x}). \end{aligned}$$

To define our third basis, we need to introduce the notion of tableaux for partitions.

Given a partition $\lambda \vdash n$, the **Young diagram** of λ is an array of n unit boxes in the plane, left-aligned, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. We will use the convention where the first row is on the bottom of the diagram. The **transpose** or **conjugate** partition λ' is obtained by flipping λ across the line $y = x$. Notably, given a Hessenberg function $h : [m] \to [m]$, one can construct a partition λ of $n = \sum_i (m - h(i))$ by setting λ_i to be the number of boxes above the Dyck path for h in row $m + 1 - i$. Then, the transpose partition λ' is exactly the partition formed by the boxes above the Dyck path for the transpose Hessenberg function h'.

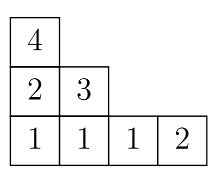


Figure 2.4: A semistandard Young tableau of shape $(4, 2, 1)$

Given the Young diagram of λ, a **Young tableau** T of shape λ is a filling of the Young diagram for λ with positive integers. We say T is **standard** if the entries strictly increase up columns and left-to-right across rows. We say T is **semistandard** if the entries strictly increase up columns, and *weakly* increase left-to-right across rows. See Figure 2.4 for an example of a semistandard Young tableau. We write $\mathrm{SSYT}(\lambda)$ to be the set of all semistandard Young tableaux of shape λ. We can now define the Schur function for a partition λ.

Definition 2.2.4. Given a partition λ of n, the **Schur function** $s_\lambda(\mathbf{x})$ is:

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} x_1^{\mathrm{wt}_1(T)} x_2^{\mathrm{wt}_2(T)} \cdots x_k^{\mathrm{wt}_k(T)},$$

where $\mathrm{wt}_i(T)$ counts the number of instances of i in the tableau T, and k is the largest integer present in T.

For example, consider the following semistandard Young tableaux of shape $\lambda = (3, 1)$:

2		
1	1	1

2		
1	1	2

2		
1	2	2

3		
1	1	2

2		
1	1	3

From these tableaux, the Schur function $s_{(3,1)}$ includes the following:

$$s_{(3,1)}(\mathbf{x}) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + 2x_1^2 x_2 x_3 + \cdots$$

In fact, we can write this Schur function in the monomial basis as follows:

$$s_{(3,1)}(\mathbf{x}) = 3m_{(1,1,1,1)}(\mathbf{x}) + 2m_{(2,1,1)}(\mathbf{x}) + m_{(2,2)}(\mathbf{x}) + m_{(3,1)}(\mathbf{x}).$$

In combinatorics, we are often interested in the expansions of symmetric functions into other bases. We say that a symmetric function $f(\mathbf{x})$ is m**-positive** (resp. **Schur-positive** and e**-positive**) if $f(\mathbf{x})$ expands with nonnegative coefficients in the monomial basis (resp. the Schur basis and the elementary basis). It is known [27] that the elementary symmetric functions are Schur-positive, and that both the elementary and Schur functions are m-positive. In general, understanding when other symmetric functions satisfy certain positivity conditions can tell us about geometric or algebraic properties of related objects.

2.3 Chromatic quasisymmetric functions

We begin by defining quasisymmetric functions.

Definition 2.3.1. A **quasisymmetric function** of degree n over a commutative ring R is a formal power series $f(\mathbf{x})$ with coefficients in R such that for any weak composition $\alpha = (\alpha_1, \ldots, \alpha_j)$ of n, the coefficient of $x_1^{\alpha_1} \cdots x_j^{\alpha_j}$ is the same as the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_j}^{\alpha_j}$ for any sequence $1 \leq i_1 < i_2 < \cdots < i_j$.

Notably, all symmetric functions are quasisymmetric, but not all quasisymmetric functions are symmetric. We denote the graded algebra of quasisymmetric functions by QSym. As with symmetric functions, a **quasisymmetric polynomial** is a quasisymmetric function restricted to a finite set of indeterminates.

There are several natural bases for QSym, and the most relevant basis for our purposes are the monomial quasisymmetric functions.

Definition 2.3.2. Given a weak composition α of n, the **monomial quasisymmetric function** $M_\alpha(\mathbf{x})$ is

$$M_\alpha(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_j} x_{i_1}^{\alpha_1} \cdots x_{i_j}^{\alpha_j}$$

Given a partition λ, if we take the sum over all rearrangements α of λ, then the monomial quasisymmetric functions yield the monomial symmetric function: $m_\lambda(\mathbf{x}) = \sum_\alpha M_\alpha(\mathbf{x})$. So, to show that a given quasisymmetric function $f(\mathbf{x})$ is not symmetric, it suffices to show that there are two compositions α and β which are rearrangements of each other, such that in the monomial quasisymmetric expansion of $f(\mathbf{x})$, the coefficients of $M_\alpha(\mathbf{x})$ and $M_\beta(\mathbf{x})$ are different. This argument will be seen in Chapter 8.

Our main object of study is as follows. In [9], Stanley defined the **chromatic symmetric function** of a finite graph G, which is a certain generalization of the chromatic polynomial of a graph. This definition was later generalized by Shareshian and Wachs in [13] to form a quasisymmetric function.

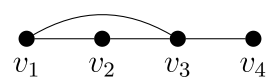
Definition 2.3.3. Let G be a graph with an (ordered) vertex set $V = \{v_1, \ldots, v_d\}$. Then the **chromatic quasisymmetric function** of G is

$$X_G(\mathbf{x}; q) = \sum_{\kappa : V \to \mathbb{N}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)} \, q^{\operatorname{asc}(\kappa)}$$

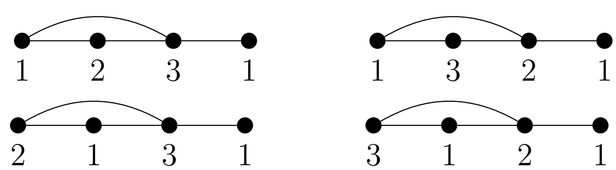
where the sum is over all proper colorings $\kappa : V \to \mathbb{N}$ of the vertices of G, and where $\operatorname{asc}(\kappa)$ is the number of vertices $v_i < v_j$ such that $\kappa(v_i) < \kappa(v_j)$.

Each term in the sum has the form $x_1^{i_1} \cdots x_n^{i_n} \, q^{\operatorname{asc}(\kappa)}$, where i_k is the number of times k was used in the coloring κ, and n is the largest color used. Stanley's chromatic symmetric function is attained from $X_G(\mathbf{x}; q)$ by setting $q = 1$.

Example 2.3.4. Consider the following graph G, with its vertices labeled:



There are four colorings of G with colors $1, 1, 2, 3$:



These colorings have 3, 2, 2, and 1 ascents respectively, so they contribute $(q + 2q^2 + q^3)x_1^2 x_2 x_3$ to $X_G(\mathbf{x}; q)$. With a little more analysis, we can write $X_G(\mathbf{x}; q)$ in the monomial symmetric basis as

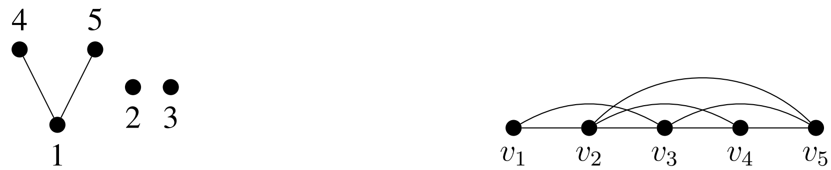
$$X_G(\mathbf{x}; q) = (q + 2q^2 + q^3) m_{(2,1,1)} + (1 + 6q + 10q^2 + 6q^3 + q^4) m_{(1,1,1,1)}.$$

Notice that this is a symmetric function with coefficients in $\mathbb{Z}[q]$.

One of the main conjectures of study for chromatic symmetric functions comes from Stanley and Stembridge in [11], and this conjecture also naturally extends to the quasisymmetric case.

Definition 2.3.5. Let h be a Hessenberg function. Define the poset P_h on the set $[n]$ as follows: For $i, j \in \mathbb{N}$, we say $i <_{P_h} j$ if and only if $h(i) < j$.

We will usually refer to the incomparability graph of P_h as G_h. See Figure 2.5 for an example of such a poset.



(a) The poset P_h **(b)** The incomparability graph G_h

Figure 2.5: The poset and incomparability graph for $h = (3, 5, 5, 5, 5)$.

These posets are $(2+2)$- and $(3+1)$- free, so their incompatibility graphs relate to the original Stanley-Stembridge conjecture [28].The following generalization of the conjecture is equivalent to the one given in [13] by Shareshian and Wachs.

Conjecture 2.3.6 ([13], Conjecture 1.3)**.** *If G_h is the incomparability graph of the poset P_h for some Hessenberg function h, then $X_{G_h}(\mathbf{x}; q)$ is e-positive and e-unimodal.*

Shareshian and Wachs showed that, for the incomparability graphs $G_h = \text{inc}(P_h)$, the chromatic quasisymmetric function is in fact symmetric, although this is not true for every graph (for instance, any labeled claw graph has non-symmetric chromatic quasisymmetric function).

In [10], Gasharov proved a weaker condition – that the chromatic symmetric function for this class of graphs is Schur-positive. This result was also extended by Shareshian and Wachs in [13] as follows.

Definition 2.3.7. Let P be a poset on the set $[n]$, and let λ be a partition of n. A P**-tableau** T of shape λ is a filling of the Young diagram of λ with entries of P such that the following hold:

- Each entry is used exactly once;
- If $j \in P$ appears directly to the right of $i \in P$, then $j >_P i$;
- If $j \in P$ appears directly above $i \in P$, then $j \not<_P i$.

We say a P-**inversion** in a P-tableau is a pair of entries i, j such that $i < j$ as integers, i appears in a higher row than j, and i and j are incomparable in P. Define $\mathrm{Inv}_P(T)$ to be the set of P-inversions of T, and $\mathrm{inv}_P(T) := |\mathrm{Inv}_P(T)|$ to be the number of P-inversions.

Proposition 2.3.8 ([13], Theorem 6.3)**.** Let G be the incomparability graph of a $(3+1)$-free poset P. Then

$$X_G(\mathbf{x}; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{PT}(\lambda)} q^{\mathrm{inv}_P(T)} \right) s_\lambda(\mathbf{x})$$

where $\mathrm{PT}(\lambda)$ is the set of P-tableaux of shape λ and s_λ is the Schur function for the partition λ.

Example 2.3.9. Consider the poset P on $\{1, 2, 3, 4, 5\}$ given by the single relation $1 <_P 5$. The graph $G = \mathrm{inc}(P)$ is the complete graph on $\{1, 2, 3, 4, 5\}$ without the edge from 1 to 5. There are six P-tableaux of shape $(2, 1, 1, 1)$, given below, with $3, 4, 4, 5, 5$, and 6 inversions, respectively. These P-tableaux contribute $(q^3 + 2q^4 + 2q^5 + q^6)s_{(2,1,1,1)}$ to the Schur expansion of $X_G(\mathbf{x}; q)$.

$$\begin{array}{|c|c}\hline 4 \\ \cline{1-1} 3 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 4 \\ \cline{1-1} 2 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 2 \\ \hline 1 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 3 \\ \cline{1-1} 2 \\ \cline{1-1} 4 \\ \hline 1 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 2 \\ \cline{1-1} 4 \\ \cline{1-1} 3 \\ \hline 1 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \begin{array}{|c|c}\hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} 4 \\ \hline 1 & \multicolumn{1}{c|}{5} \\ \hline \end{array}$$

We will now examine the main connection between chromatic quasisymmetric functions and Hessenberg varieties.

2.4 GKM graphs

Suppose S is a regular semisimple matrix, so S is diagonalizable with distinct eigenvalues. We saw in Chapter 1 that $\mathrm{Hess}(S, h)$ is smooth, and all Hessenberg varieties of this form are isomorphic to each other for a fixed h. In [8], Tymoczko exhibited an action of the symmetric group $\mathfrak{S}_n$ on the cohomology ring $H^*(\mathrm{Hess}(S, h))$ using GKM theory, which realizes the equivariant cohomology of Hessenberg varieties as follows:

Proposition 2.4.1 ([8], Proposition 5.4)**.** The equivariant cohomology ring $H_T^*(\mathrm{Hess}(S,h))$ is isomorphic (as rings) to

$$\left\{\alpha \in \bigoplus_{w\in\mathfrak{S}_n} \mathbb{C}[t_1,\dots,t_n] \,\middle|\, t_{w(i)} - t_{w(j)} \text{ divides } \alpha(w)-\alpha(w') \text{ if } w' = w(ji) \text{ for some } j < i \leq h(j)\right\}$$

where $\alpha(w)$ is the w-component of α and $(ji) \in \mathfrak{S}_n$ is the transposition of j and i. We call the division criterion for the polynomials in two parts of a tuple the **GKM condition**.

Notice that, for each $i = 1,\dots,n$, the tuple $t_i := (t_i)_{w\in\mathfrak{S}_n}$ is an element of the above set. From the theory of torus-equivariant cohomology (see [23] for details), we get a presentation of the cohomology ring of $\mathrm{Hess}(S,h)$:

$$H^*(\mathrm{Hess}(S,h)) \cong H_T^*(\mathrm{Hess}(S,h))/\langle t_1,\dots,t_n\rangle$$

The $\mathfrak{S}_n$ action described by Tymoczko is as follows: For $v \in \mathfrak{S}_n$, and $\alpha = (\alpha(w))_{w\in\mathfrak{S}_n} \in \bigoplus_{w\in\mathfrak{S}_n} \mathbb{C}[t_1,\dots,t_n]$ define $v\cdot\alpha$ by $(v\cdot\alpha)(w) = v\cdot\alpha(v^{-1}w)$ for all $w \in \mathfrak{S}_n$, where the action on the right permutes the order of the tuple by the permutation v.

Rather than thinking of elements of this ring as tuples of polynomials, it is often useful to think of them as labeled graphs with vertex set $\mathfrak{S}_n$, and edges whenever $w' = w(ij)$ with $j < i \leq h(j)$, where the vertex label at w is $\alpha(w)$.

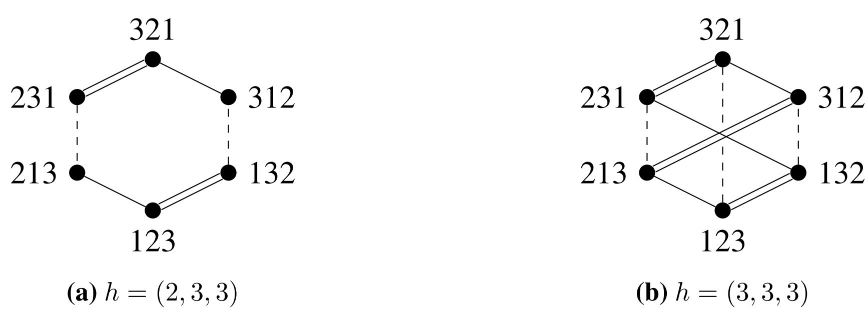


Figure 2.6: The (unlabeled) GKM graphs for two Hessenberg functions. Dashed lines represent the permutation (13), solid lines represent (12), and doubled lines represent (23).

The following theorem ties together the geometry of Hessenberg varieties with the algebraic structure of chromatic symmetric functions. It was originally conjectured in [13] by Shareshian and Wachs, and was proven by Brosnan and Chow in [4], and separately by Guay-Paquet in [5].

Proposition 2.4.2 ([4], [5]). Let h be a Hessenberg vector, P_h be the associated poset with incomparability graph $G = (V, E)$. Let $\text{Hess}(S, h)$ be the regular semisimple Hessenberg variety for h. Then

$$\omega X_G(\mathbf{x}; q) = \sum_{j=0}^{|E|} \text{Frob}(H^{2j}(\text{Hess}(S, h)))q^j,$$

where $\text{Frob}(H^{2j}(\text{Hess}(S, h)))$ is the Frobenius characteristic of the representation of $\mathfrak{S}_n$ on the $(2j)^{th}$ cohomology group of $\text{Hess}(S, h)$ and ω is the involution on symmetric functions that sends the Schur function s_λ to $s_{\lambda'}$, where λ' is the transpose partition to λ.

Combining Propositions 2.3.8 and 2.4.2, we can connect the graded representation of the cohomology ring for Hessenberg varieties with P-tableaux as follows:

$$\sum_{j=0}^{|E|} \text{Frob}(H^{2j}(\text{Hess}(S, h)))q^j = \omega X_G(\mathbf{x}; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\text{inv}_h(T)} \right) s_{\lambda'}(\mathbf{x}). \tag{2.4.1}$$

From this, polynomial presentations of the cohomology ring $H^*(\text{Hess}(S, h))$ can be connected explicitly to sets of P_h-tableaux, by understanding the action of $\mathfrak{S}_n$ and its decomposition into irreducible representations. For example, in Section 5, we give explicit bijections between monomials in the cohomology ring for $h = (h(1), n, \ldots, n)$ with P_h-tableaux of shapes (1^n) and $(2, 1^{n-2})$.

2.5 Symmetric group representations

Irreducible representations of $\mathfrak{S}_n$ are indexed by partitions of n, and we write them as V_λ. The trivial representation is indexed by $\lambda = (n)$, and the standard representation is indexed by $\lambda = (n-1, 1)$. For more details on the following facts, and a general overview of $\mathfrak{S}_n$ representation theory, see [29].

Given a standard tableau T of shape λ, define the **Specht polynomial** F_T as

$$F_T = \prod_{C} \prod_{\{i<j\} \in C} (x_j - x_i)$$

where the outer product is over all columns of T. For example, given the standard tableau T below, we can compute the corresponding Specht polynomial:

$$T = \begin{array}{|c|c|c|} \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-2} 2 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 6 \\ \hline \end{array} \qquad F_T = (x_4 - x_2)(x_4 - x_1)(x_2 - x_1)(x_5 - x_3)$$

Given a fixed partition λ of n, the subspace of $\mathbb{C}[x_1, \ldots, x_n]$ generated by $\{F_T\}_{T \in \mathrm{SYT}(\lambda)}$, where $\mathrm{SYT}(\lambda)$ is the set of standard tableaux T of shape λ, is isomorphic to the irreducible $\mathfrak{S}_n$-module V_λ. These submodules are called the **Specht modules**. $\mathfrak{S}_n$ acts on a Specht module by permuting the indices: For $w \in \mathfrak{S}_n$, $w \cdot (x_j - x_i) = (x_{w(j)} - x_{w(i)})$.

Definition 2.5.1 ([30], Definition 1.5)**.** If R is an $\mathfrak{S}_n$-module which decomposes into irreducible $\mathfrak{S}_n$-modules as

$$R = \bigoplus_{\lambda} c_\lambda V_\lambda \,,$$

then a **higher Specht basis of** R is a set of elements $\mathcal{B}$ with a decomposition $\mathcal{B} = \bigcup_\lambda \bigcup_{i=1}^{c_\lambda} \mathcal{B}_{i,\lambda}$ such that the elements of $\mathcal{B}_{i,\lambda}$ are a basis of the i-th copy of V_λ in the decomposition of R.

Higher Specht bases for $\mathfrak{S}_n$-modules have been constructed in the context of the coinvariant ring [31] and the generalized coinvariant ring [30]. These bases are valuable since they give a natural grouping of basis elements into irreducible modules.

A representation V of $\mathfrak{S}_n$ is a **permutation representation** if there exists a set A acted on by $\mathfrak{S}_n$ such that $V \cong \mathbb{C}A$, the vector space of $\mathbb{C}$-linear combinations of elements of A. In other words, elements of $\mathfrak{S}_n$ act via permutation matrices on $\mathbb{C}A$.

If the trivial representation of $\mathfrak{S}_n$ is $V_{(n)}$, then the **natural permutation representation** of $\mathfrak{S}_n$ is the induced representation $V_{(n)} \uparrow_{\mathfrak{S}_{n-1} \times \mathfrak{S}_1}^{\mathfrak{S}_n}$. Here, $\mathfrak{S}_n$ acts on a set of n disjoint cosets of the subgroup $\mathfrak{S}_{n-1} \times \mathfrak{S}_1$, so this representation is isomorphic to the $\mathfrak{S}_n$-module $\mathbb{C}^n$, where $\mathfrak{S}_n$ acts by permuting the standard unit vectors $\vec{e_1}, \ldots, \vec{e_n}$. The natural permutation representation V decomposes into irreducibles as $V \cong V_{(n)} \oplus V_{(n-1,1)}$. Then we have:

$$\mathrm{Frob}(V) = \mathrm{Frob}(V_{(n)}) + \mathrm{Frob}(V_{(n-1,1)}) = s_{(n)} + s_{(n-1,1)} = h_{(n-1,1)}.$$

Notice that $\omega(h_{(n-1,1)}) = e_{(n-1,1)}$, so if an $\mathfrak{S}_n$-module V is a direct sum of natural permutation representations (and trivial representations, which have Frobenius character $s_{(n)} = h_{(n)}$), then we get the h-positivity of the character of this representation. In general, permutation representations yield h-positive symmetric functions under the Frobenius map. When the $\mathfrak{S}_n$-module is the cohomology ring $H^*(\mathrm{Hess}(S, h))$, we can apply ω to get the e-positivity of the associated chromatic symmetric function.

Chapter 3

Basis elements when $h' = ((n-1)^{n-m}, n^m)$

In this section, we introduce the results of Abe, Horiguchi, and Masuda in [23], including their realization of $H^*(\mathrm{Hess}(S,h))$ as a polynomial quotient ring, in the case that $h = (h(1), n, \ldots, n)$. We then define our basis for the transpose Hessenberg function $h' = ((n-1)^{n-m}, n^m)$.

Abe, Horiguchi, and Masuda defined the following elements of $H^*(\mathrm{Hess}(S,h))$ in terms of GKM graphs: Let $h = (h(1), n, \ldots, n)$. For $1 \le k \le n$, and $w \in \mathfrak{S}_n$:

$$x_k(w) := t_{w(k)}$$

$$y_k(w) := \begin{cases} \prod_{\ell=2}^{h(1)} (t_k - t_{w(\ell)}) & \text{if } w(1) = k \\ 0 & \text{if } w(1) \neq k \end{cases}$$

(a) The GKM class x_2. **(b)** The GKM class y_2.

Figure 3.1: Two GKM classes for $h = (2,3,3)$, as defined in [23].

The authors showed that for each k, the tuples $x_k := (x_k(w))_{w \in \mathfrak{S}_n}$ and $y_k := (y_k(w))_{w \in \mathfrak{S}_n}$ satisfy the GKM condition. Further, they show that the x_k and y_k classes satisfy the following:

Lemma 3.0.1. *([23], Lemma 4.1) Suppose* $h = (h(1), n, \ldots, n)$. *The following hold:*

1. $y_k \, y_{k'} = 0$ *for* $k \neq k'$.

2. $x_1 \, y_k = t_k \, y_k$ *for all* k.

3. $y_k \prod_{\ell=h(1)+1}^{n}(t_k - x_\ell) = \prod_{\ell=2}^{n}(t_k - x_\ell)$ *for all* k.

4. $\sum_{k=1}^{n} y_k = \prod_{\ell=2}^{h(1)}(x_1 - x_\ell)$.

We use the convention $\prod_{\ell=n+1}^{n}(t_k - x_\ell) = 1$ *in equation* (3) *and* $\prod_{\ell=2}^{1}(x_1 - x_\ell) = 1$ *in equation* (4).

Using these relations, the authors show that the classes t_k, x_k, and y_k generate $H_T^*(\mathrm{Hess}(S, h))$. In particular, via the homomorphism $H_T^*(\mathrm{Hess}(S, h)) \to H^*(\mathrm{Hess}(S, h))$ which takes the quotient by the torus fixed points $\langle t_1, \dots, t_n \rangle$, they show that $H^*(\mathrm{Hess}(S, h))$ has the following presentation:

Proposition 3.0.2 ([23], Theorem 4.3)**.** If $h = (h(1), n, \dots, n)$, then the cohomology ring of the regular semisimple Hessenberg variety $\mathrm{Hess}(S, h)$ is given by

$$H^*(\mathrm{Hess}(S, h)) \cong \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]/I$$

where $\deg(x_i) = 2$, $\deg(y_k) = 2(h(1) - 1)$, and I is the homogeneous ideal generated by the following types of elements:

1. $y_k y_{k'}$ $(1 \leq k \neq k' \leq n)$

2. $x_1 y_k$ $(1 \leq k \leq n)$

3. $(\prod_{\ell=h(1)+1}^{n}(-x_\ell))y_k - \prod_{\ell=2}^{n}(-x_\ell)$ $(1 \leq k \leq n)$

4. $\sum_{k=1}^{n} y_k - \prod_{\ell=2}^{h(1)}(x_1 - x_\ell)$

5. The $i-$th elementary symmetric polynomial $e_i(x_1, \dots, x_n)$ $(1 \leq i \leq n)$

Notice that some of the relations in the ideal above are the result of setting each $t_i = 0$ in the previous lemma, as a result of taking the quotient by $\langle t_1, \dots, t_n \rangle$. Additionally, the authors use the relations in I to identify a basis of this space:

Proposition 3.0.3 ([23], Remark 4.5)**.** From the proof of Proposition 3.0.2, the following two types of monomials form a $\mathbb{Z}$-basis of $H^*(\mathrm{Hess}(S,h))$ when $h = (h(1), n, \ldots, n)$:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{which does not contain the factor } \prod_{\ell=1}^{h(1)} x_\ell \tag{3.0.1}$$

$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k \quad \text{which does not contain the factor } \prod_{\ell=h(1)+1}^{n} x_\ell \tag{3.0.2}$$

where $0 \leq i_j \leq n-j$ in the first equation, and $0 \leq \ell_j \leq n-1-j$, and $1 \leq k \leq n-1$ in the second equation.

3.1 Basis for the transpose Hessenberg variety

Recall from Proposition 2.1.2 that if S is a regular semisimple matrix, then $\mathrm{Hess}(S,h) \cong \mathrm{Hess}(S,h')$, where h' is the transpose Hessenberg function to h. When $h = (h(1), n, \ldots, n)$, the transpose Hessenberg function is $h' = ((n-1)^{n-h(1)}, n^{h(1)})$. So, these Hessenberg functions will yield isomorphic Hessenberg varieties, and thus isomorphic cohomology rings. However, the basis elements of $H^*(\mathrm{Hess}(S,h))$ will not necessarily form a basis of $H^*(\mathrm{Hess}(S,h'))$ - the defining properties of the x_i and y_i variables will differ, and so the basis elements will exist in different quotient rings. Therefore, we are motivated to find a similar basis for $H^*(\mathrm{Hess}(S,h'))$ as Abe, Horiguchi, and Masuda did for $H^*(\mathrm{Hess}(S,h))$.

Building from the GKM graphs defined in [23], we define the following when the Hessenberg function is $h' = ((n-1)^{n-m},\, n^m)$.

Definition 3.1.1. For $1 \leq k \leq n$, and $w \in \mathfrak{S}_n$, define:

$$x_k(w) := t_{w(k)}$$

$$y_k(w) := \begin{cases} \prod_{\ell=n-m+1}^{n-1} (t_k - t_{w(\ell)}) & \text{if } w(n) = k \\ 0 & \text{if } w(n) \neq k \end{cases}$$

An example of these GKM graphs is given in Figure 3.2.

(a) The class x_2. **(b)** The class y_2.

Figure 3.2: Two GKM classes for $h = (2, 3, 3)$, as given in Definition 3.1.1.

As above, define $x_k := (x_k(w))_{w\in\mathfrak{S}_n}$ and $y_k := (y_k(w))_{w\in\mathfrak{S}_n}$ First, we show that these tuples are elements of $H^*(\mathrm{Hess}(S, h'))$ by showing that they satisfy the GKM condition.

Lemma 3.1.2. *Given $w \in \mathfrak{S}_n$, if there exists $i < j \leq h'(i)$ such that $w' = w(ij)$, then $x_k(w) - x_k(w')$ and $y_k(w) - y_k(w')$ are divisible by $t_{w(i)} - t_{w(j)}$.*

Proof. Suppose that we have $w \in \mathfrak{S}_n$, and $i < j$ such that $j \leq h'(i)$ and $w' = w(ij)$.

First, we show that the GKM condition holds for all x_k. If $k = i$, then $x_i(w) - x_i(w') = t_{w(i)} - t_{w'(i)} = t_{w(i)} - t_{w(j)}$. Similarly, if $k = j$, then $x_j(w) - x_j(w') = t_{w(j)} - t_{w(i)}$. If $k \neq i$ and $k \neq j$, then $x_k(w) - x_k(w') = t_{w(k)} - t_{w'(k)} = t_{w(k)} - t_{w(k)} = 0$.

Now, we show that the GKM condition holds for all y_k. For the first case, suppose that $j < n$. If $w(n) \neq k$, then we have $w'(n) \neq k$, so $y_k(w) - y_k(w') = 0 - 0 = 0$. If $w(n) = k$, then we have $w'(n) = k$. If $n-m+1 \leq i$ or $j < n-m+1$, then $\prod_{\ell=n-m+1}^{n-1}(t_k - t_{w(\ell)}) = \prod_{\ell=n-m+1}^{n-1}(t_k - t_{w'(\ell)})$, since the index ℓ includes both or neither of i and j. Hence $y_k(w) = y_k(w')$, so $y_k(w) - y_k(w') = 0$. If $i < n - m + 1 \leq j$, then $y_k(w)$ contains the factor $(t_k - t_{w(j)})$, $y_k(w')$ contains the factor $(t_k - t_{w'(j)}) = (t_k - t_{w(i)})$, and all other factors are shared. Hence $(t_k - t_{w(j)}) - (t_k - t_{w(i)}) = t_{w(i)} - t_{w(j)}$ divides $y_k(w) - y_k(w')$.

For the second case, suppose that $j = n$, so $w(j) = w(n)$. Since $j \leq h'(i)$, we have that $h'(i) = n$, so $n - m + 1 \leq i$. If $w(n) \neq k$ and $w'(n) \neq k$, then $y_k(w) - y_k(w') = 0$. If $w(n) = k$, then $w' = w \cdot (in)$, so $w'(n) \neq k$. Thus $y_k(w)$ contains the factor $(t_k - t_{w(i)}) = (t_{w(j)} - t_{w(i)})$, and $y_k(w') = 0$. Similarly, if $w'(n) = k$ and $w(n) \neq k$, then $y_k(w) = 0$, and $y_k(w')$ contains the factor $(t_k - t_{w'(i)}) = (t_{w'(j)} - t_{w'(i)}) = (t_{w(i)} - t_{w(j)})$. Hence, in all cases, $t_{w(i)} - t_{w(j)}$ divides $y_k(w) - y_k(w')$. $\square$

Since the x_k and y_k satisfy the GKM condition, they are well-defined elements of $H^*(\mathrm{Hess}(S, h'))$. We next find a set of similar relations to Lemma 3.0.1.

Lemma 3.1.3. *The following hold:*

1. $y_k\, y_{k'} = 0$ *for all* $k \neq k'$.

2. $x_n\, y_k = t_k\, y_k$ *for all* k.

3. $y_k \prod_{\ell=1}^{n-m}(t_k - x_\ell) = \prod_{\ell=1}^{n-1}(t_k - x_\ell)$ *for all* k.

4. $\sum_{k=1}^{n} y_k = \prod_{\ell=n-m+1}^{n-1}(x_n - x_\ell)$.

We have the convention $\prod_{\ell=1}^{0}(t_k - x_\ell) = 1$ *in equation* (3)*, and* $\prod_{\ell=n}^{n-1}(x_n - x_\ell) = 1$ *in equation* (4).

Proof. Let $w \in \mathfrak{S}_n$.

(1) If $k \neq k'$, then $w(n) = k$ or $w(n) = k'$ (or neither), but not both. So we have $y_k(w)y_{k'}(w) = 0$.

(2) If $w(n) = k$, then $x_n(w) = t_{w(n)} = t_k$. If $w(n) \neq k$, then $y_k(w) = 0$. In either case, $x_n(w)y_k(w) = t_k\, y_k(w)$.

(3) If $w(n) = k$, then we have

$$y_k(w) \cdot \prod_{\ell=1}^{n-m}(t_k - x_\ell(w)) = \prod_{\ell=n-m+1}^{n-1}(t_k - t_{w(\ell)}) \cdot \prod_{\ell=1}^{n-m}(t_k - t_{w(\ell)}) = \prod_{\ell=1}^{n-1}(t_k - x_\ell(w))$$

If $w(n) \neq k$, then $y_k(w) = 0$. Further, we must have that $w(i) = k$ for some $i < n$, and so $\prod_{\ell=1}^{n-1}(t_k - t_{w(\ell)})$ contains the factor $(t_k - t_k)$, and is thus 0.

(4) Suppose $w(n) = k'$. Then for any $k \neq k'$, we have $y_k(w) = 0$. We have

$$\sum_{k=1}^{n} y_k(w) = \prod_{\ell=n-m+1}^{n-1} (t_{k'} - t_{w(\ell)}) = \prod_{\ell=n-m+1}^{n-1} (t_{w(n)} - t_{w(\ell)}) = \prod_{\ell=n-m+1}^{n-1} (x_n(w) - x_\ell(w))$$

□

Notice that the relations in Lemma 3.1.3 are the same as those in Lemma 3.0.1, with each x_i swapped with x_{n+1-i}, and supposing that $m = h(1)$. Under this isomorphism of polynomial rings, we obtain the following results.

Proposition 3.1.4. If $h' = ((n-1)^{n-m}, n^m)$, then the classes x_k, y_k, t_k (for $1 \leq k \leq n$) generate $H_T^*(\mathrm{Hess}(S, h'))$ as a $\mathbb{Z}$-algebra.

We now give the quotient ring presentation of $H^*(\mathrm{Hess}(S, h'))$.

Theorem 3.1.5. *If $h' = ((n-1)^{n-m}, n^m)$, then the cohomology ring of $H^*(\mathrm{Hess}(S, h'))$ is given by*

$$H^*(\mathrm{Hess}(S, h')) \cong \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I$$

where $\deg(x_k) = 2$, $\deg(y_k) = 2(m-1)$, and I is the homogeneous ideal generated by the following types of elements:

1. *$y_k \, y_{k'}$ for all $k \neq k'$.*
2. *$x_n \, y_k$ for all k.*
3. *$y_k \prod_{\ell=1}^{n-m}(-x_\ell) - \prod_{\ell=1}^{n-1}(-x_\ell)$ for all k.*
4. *$\sum_{k=1}^{n} y_k - \prod_{\ell=n-m+1}^{n-1}(x_n - x_\ell)$.*
5. *The i-th elementary symmetric polynomial $e_i(x_1, \ldots, x_n)$ for $1 \leq i \leq n$.*

As a consequence of this quotient realization, we can identify the basis elements of $H^*(\mathrm{Hess}(S, h'))$:

Remark 3.1.6. If $h' = ((n-1)^{n-m}, n^m)$, then following two types of monomials form a $\mathbb{Z}$-basis of $H^*(\mathrm{Hess}(S, h'))$:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{which does not contain the factor } \prod_{\ell=n-m+1}^{n} x_\ell \tag{3.1.1}$$

$$x_{n-1}^{\ell_1} x_{n-2}^{\ell_2} \cdots x_1^{\ell_{n-1}}\, y_k \quad \text{which does not contain the factor } \prod_{\ell=1}^{n-m} x_\ell \tag{3.1.2}$$

where $0 \leq i_j \leq j-1$ in the first equation, and $0 \leq \ell_j \leq j-1$ and $1 \leq k \leq n-1$ in the second equation.

Chapter 4

New bases for $H^*(\mathrm{Hess}(S,h))$

Now that we understand the basis elements of $H^*(\mathrm{Hess}(S,h))$ for $h = (h(1), n, \ldots, n)$, and for the transpose Hessenberg function h', our goal is to better understand the action of the symmetric group $\mathfrak{S}_n$ on these monomials.

4.1 Higher Specht bases

In this section, we give a new basis for $H^*(\mathrm{Hess}(S,h))$ when S is regular semisimple and $h = (h(1), n, \ldots, n)$, and show that it is a higher Specht basis. Recall the set of monomial basis elements of $H^*(\mathrm{Hess}(S,h))$ defined by the authors in [23] for $h = (h(1), n, \ldots, n)$ in the previous section:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{which does not contain the factor} \prod_{\ell=1}^{h(1)} x_\ell \tag{4.1.1}$$

$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k \quad \text{which does not contain the factor} \prod_{\ell=h(1)+1}^{n} x_\ell \tag{4.1.2}$$

running over all $0 \le i_j \le n-j$ in the first equation, and over all $0 \le \ell_j \le n-1-j$, and $1 \le k \le n-1$ in the second equation.

The action of $\mathfrak{S}_n$ on the above polynomials is defined by fixing each of the x_i, and permuting the y_i, that is, for $w \in \mathfrak{S}_n$, we have $w \cdot x_i = x_i$ and $w \cdot y_i = y_{w(i)}$. This group action gives a representation of $\mathfrak{S}_n$, and it is known that this representation decomposes into the direct sum of trivial representations and standard representations. However, it is not necessarily easy to see how certain permutations of basis elements decompose in the basis.

Define B_1 to be the set of monomials in equation (4.1.1) above, and B_2 to be the set of monomials in equation (4.1.2) above. We construct an alternate basis of $H^*(\mathrm{Hess}(S,h))$ motivated by the basis elements of Specht modules. Let B_3 be the set of monomials of the form:

$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}}(y_{k+1}-y_1) \quad \text{which does not contain the factor} \prod_{\ell=h(1)+1}^{n} x_\ell \tag{4.1.3}$$

running over all $0 \leq \ell_j \leq n-1-j$ and $1 \leq k \leq n-1$.

From Proposition 3.0.3, the set $B_1 \cup B_2$ forms a basis of $H^*(\mathrm{Hess}(S,h))$ when the Hessenberg function is $h = (h(1), n, \ldots, n)$. We claim that $B_1 \cup B_3$ also forms a basis.

Lemma 4.1.1. *Let $f(x_1, \ldots, x_n)$ be homogeneous in this presentation of $H^*(\mathrm{Hess}(S,h))$. Then $f(x_1, \ldots, x_n)$ can be expressed only in terms of the elements of B_1.*

Proof. Since $B_1 \cup B_2$ forms a basis of $H^*(\mathrm{Hess}(S,h))$, we can write

$$f(x_1, \ldots, x_n) = \sum_{B_1} b_{\underline{i}} x_1^{i_1} \cdots x_n^{i_n} + \sum_{B_2} c_{\underline{\ell},k} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$$

for some constants $b_{\underline{i}}$ and $c_{\underline{\ell},k}$ ranging over the lists of exponents $\underline{i}$ and $\underline{\ell}$ of elements of $B_1 \cup B_2$, and with $1 \leq k \leq n-1$. From the dot action of $\mathfrak{S}_n$ on $H^*(\mathrm{Hess}(S,h))$, we know that for $w \in \mathfrak{S}_n$, $w \cdot x_i = x_i$ and $w \cdot y_i = y_{w(i)}$ for all i. So, since $f(x_1, \ldots, x_n)$ is fixed by the action of $\mathfrak{S}_n$, we get that $c_{\underline{\ell},k} = c_{\underline{\ell},k'}$ for all $k \neq k'$. Define $c_{\underline{\ell}} := c_{\underline{\ell},k}$.

If $c_{\underline{\ell}} \neq 0$, then for $w \in \mathfrak{S}_n$, w fixes $f(x_1, \ldots x_n)$, and fixes $\sum_{B_1} b_{\underline{i}} x_1^{i_1} \cdots x_n^{i_n}$, w must also fix $\sum_{B_2} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}$. Hence we can use the transposition $(i,j) \in \mathfrak{S}_n$ to find:

$$\sum_{k \neq i} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k = \sum_{k \neq j} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$$

$$\sum_{k \neq i} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k - \sum_{k \neq i,j} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k = \sum_{k \neq j} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k - \sum_{k \neq i,j} c_{\underline{\ell}} x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$$

$$x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_j = x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_i$$

However, this is a contradiction, since these are distinct basis elements in $B_1 \cup B_2$, so they are linearly independent. Hence $c_{\underline{\ell}} = 0$, so $f(x_1, \ldots, x_n)$ can be expressed solely using basis elements from B_1. □

We now show that B_1 and B_3 together form a basis for $H^*(\text{Hess}(S,h))$.

Theorem 1.0.1. The following sets form a basis of $H^*(\text{Hess}(S,h))$ when $h = (h(1), n, \ldots, n)$:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{which does not contain the factor } \prod_{\ell=1}^{h(1)} x_\ell$$
$$x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_{k+1} - y_1) \quad \text{which does not contain the factor } \prod_{\ell=h(1)+1}^{n} x_\ell$$

running over all $0 \leq i_j \leq n - j$ in the first equation, and over all $0 \leq \ell_j \leq n - 1 - j$, and $1 \leq k \leq n - 1$ in the second equation.

Proof. Note that $B_1 \cup B_2$ form the basis of $H^*(\text{Hess}(S,h))$ given in Proposition 3.0.3, and our desired basis is $B_1 \cup B_3$. Consider the degree-lexicographic ordering on the monomials given by $x_1 < x_2 < \ldots < x_n < y_2 < \ldots < y_n < y_1$, and order the elements of B_1, B_2, and B_3 correspondingly, with elements of B_3 ordered by their initial monomial. We will form the transition matrix M from $B_1 \cup B_2$ to $B_1 \cup B_3$, and show that it is invertible. Note that since $H^*(\text{Hess}(S,h))$ is graded with $\deg(x_i) = 2$ and $\deg(y_i) = 2(h(1) - 1)$ for all i, the transition matrix is block diagonal, with blocks M_d according to degree d. The columns of M_d correspond to ways of writing one element of $B_1 \cup B_3$ of degree d in terms of the basis elements from $B_1 \cup B_2$.

For M_0, there are two cases. If $h(1) > 1$, then $M_0 = [\ 1\ ]$, which is invertible. If $h(1) = 1$, then B_2 contains elements $y_1, \ldots, y_{n-1}$ of degree 0, and B_3 contains elements $y_2 - y_1, \ldots y_n - y_1$ of degree 0. For the ordering given above, the transition matrix is given in Figure 4.1. This matrix is invertible, since we can use row operations to add rows 2 through $n - 1$ to the last row to get an upper-triangular matrix with nonzero entries on the diagonal.

When $d > 0$, consider the transition matrix M_d. Columns corresponding to an element of B_1 will have a 1 on the diagonal, and 0's elsewhere, since these basis elements exist in both sets. For $k < n$, we can write $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) = x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k - x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_1$, so columns corresponding to elements in B_3 of this form will have a 1 on the diagonal, corresponding to the

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & -1 & -1 & \cdots & -1 & -2 \end{bmatrix}$$

Figure 4.1: The transition matrix between $B_1 \cup B_2$ and $B_1 \cup B_3$ for elements of degree 0 when $h(1) = 1$.

row for $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$, a (-1) below the diagonal, corresponding to the row for $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_1$, and 0's elsewhere.

For elements in B_3 of the form $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_n - y_1)$, we use relation (4) from Proposition 3.0.2 to rewrite:

$$x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_n - y_1)$$
$$= x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} (-2y_1 - y_2 - \cdots - y_{n-1}) + x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} \left(\prod_{m=2}^{h(1)} (x_1 - x_m) \right)$$

The first term of this expansion is clearly a sum of basis elements from B_2. The second term is a polynomial containing only the variables $x_1, \ldots, x_n$, so by Lemma 4.1.1, we know that it can be written as a sum of basis elements from B_1. So we have

$$\begin{aligned} &x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_n - y_1) \\ &= x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} (-2y_1 - y_2 - \cdots - y_{n-1}) + x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} \left(\prod_{m=2}^{h(1)} (x_1 - x_m) \right) \\ &= x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} (-2y_1 - y_2 - \cdots - y_{n-1}) + \sum_{B_1} b_{\underline{i}} x_1^{i_1} \cdots x_n^{i_n} \end{aligned}$$

Thus the column of M_d corresponding to $x_n^{\ell_1} \cdots x_n^{\ell_{n-1}}(y_n - y_1)$ has a $b_{\underline{i}}$ in rows corresponding to elements from B_1, a (-1) in the rows corresponding to $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$ for $k < n$ and a (-2) in the row for $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_1$. So the matrix has the form shown in Figure 4.2. Then, using row operations, adding the row for the basis element $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$ to the row for the basis element

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & b_{\underline{i}^{(1)}} & \cdots & 0 & \cdots & b_{\underline{j}^{(1)}} \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & b_{\underline{i}^{(2)}} & \cdots & 0 & \cdots & b_{\underline{j}^{(2)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & b_{\underline{i}^{(k)}} & \cdots & 0 & \cdots & b_{\underline{j}^{(k)}} \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & -1 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & -1 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & -1 & -1 & \cdots & -2 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & -1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & \cdots & -2 \end{bmatrix}$$

Figure 4.2: The general form of the transition matrix between $B_1 \cup B_2$ and $B_1 \cup B_3$.

$x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_1$ clears the entries below the diagonal, and maintains that the diagonal is nonzero, because of the formula found above for expressing $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_n - y_1)$ in terms of basis elements from $B_1 \cup B_2$, namely that the coefficient for $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} y_k$ is -1 when $2 \leq k \leq n-1$.

Therefore we can use elementary row operations to make M_d an upper-triangular matrix with nonzero entries on the diagonal. Hence each M_d is invertible, so the transition matrix from $B_1 \cup B_2$ to $B_1 \cup B_3$ is invertible, and so $B_1 \cup B_3$ forms a basis of $H^*(\mathrm{Hess}(S, h))$. $\square$

In other words, we have succeeded in finding a higher Specht basis for the cohomology ring of $\mathrm{Hess}(S, h)$, where there is a natural projection from $H^*(\mathrm{Hess}(S, h))$ to $V_{(n)}$ and $V_{(n-1,1)}$. This mapping is given by $x_1^{i_1} \cdots x_n^{i_n} \mapsto 1$ and $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) \mapsto y_k - y_1$.

By counting the monomials in each set, we can use these basis elements to understand the multiplicity of irreducible representations in the decomposition of $H^*(\mathrm{Hess}(S, h))$.

Proposition 4.1.2. When $h = (h(1), n, \ldots, n)$, the dot action of $\mathfrak{S}_n$ on $H^*(\mathrm{Hess}(S, h))$ decomposes into $h(1)(n-1)!$ copies of the trivial representation $V_{(n)}$ and $(n - h(1))(n-2)!$ copies of the standard representation $V_{(n-1,1)}$.

Proof. The dot action acts on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/I \cong H^*(\mathrm{Hess}(S, h))$ as follows: For $w \in \mathfrak{S}_n$, we have $w \cdot x_i = x_i$ and $w \cdot y_i = y_{w(i)}$. Since the set $B_1 \cup B_3$ forms a basis, we can examine the dot action on these basis elements. By counting the possible strings of exponents on each type of monomial, we find that there are $h(1)(n-1)!$ basis elements in B_1 and $(n-h(1))(n-1)!$ basis elements from B_3. Each of the basis elements from B_1 is fixed by the dot action, so this representation includes $h(1)(n-1)!$ copies of the trivial representation.

Let $V_{(n-1,1)}$ be the Specht module for the standard representation, so $V_{(n-1,1)}$ has a basis of the form $\{y_k - y_1\}_{2 \le k \le n}$. Consider a map $\varphi : B_3 \to V_{(n-1,1)}$ that maps a given polynomial as follows: $\varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)) = y_k - y_1$. We note that φ preserves the action of $\mathfrak{S}_n$, since for any $w \in \mathfrak{S}_n$,

$$\begin{aligned}
\varphi(w \cdot (x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)) &= \varphi((x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_{w(k)} - y_{w(1)})) \\
&= y_{w(k)} - y_{w(1)} \\
&= w \cdot (y_k - y_1) \\
&= w \cdot \varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1))
\end{aligned}$$

For a fixed list of exponents $\underline{\ell}$, the restriction of φ to the elements of B_3 with a common factor $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}$ is a bijection between these polynomials and the basis $\{y_k - y_1\}_{2 \le k \le n}$ of $V_{(n-1,1)}$. From above, we know that there are $(n-h(1))(n-1)!$ basis elements in B_3, so there are $\frac{(n-h(1))(n-1)!}{n-1} = (n-h(1))(n-2)!$ elements for each fixed exponent vector ℓ. For each such exponent vector, the monomials $\{x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)\}_{2 \le k \le n}$ span a submodule isomorphic to the Specht module $V_{(n-1)}$ for the standard representation. The monomials from B_1 each span a trivial representation, and there are $h(1)(n-1)!$ choices for exponent vector on each of these monomials. Hence $H^*(\mathrm{Hess}(S, h))$ decomposes into irreducibles as desired. □

By a similar argument to Theorem 1.0.1, we get a higher Specht basis for the cohomology ring of $\mathrm{Hess}(S, h')$, using the isomorphism which sends x_i to x_{n-i+1}:

Theorem 1.0.5. The following sets form a basis of $H^*(\mathrm{Hess}(S,h))$ when $h = ((n-1)^{n-m}, n^m)$:

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \quad \text{which does not contain the factor } \prod_{\ell=n-m+1}^{n} x_\ell$$

$$x_{n-1}^{\ell_1} x_{n-2}^{\ell_2} \cdots x_1^{\ell_{n-1}} (y_{k+1} - y_1) \quad \text{which does not contain the factor } \prod_{\ell=1}^{n-m} x_\ell$$

running over all $0 \leq i_j \leq j-1$ in the first equation, and over all $0 \leq \ell_j \leq j-1$ and $1 \leq k \leq n-1$ in the second equation.

4.2 Permutation representations

In the previous section, we saw that for $h = (h(1), n, \ldots, n)$, the sets B_1 and B_3 formed a higher Specht basis of $H^*(\mathrm{Hess}(S,h))$. Each element of B_1 spans a trivial representation isomorphic to $V_{(n)}$, and each set of $n-1$ elements of B_3 with the same degree on each x_i span a standard representation isomorphic to $V_{(n-1,1)}$. We now turn towards permutation representations, since these connect to the question of e-positivity for the chromatic symmetric function.

Proposition 4.2.1. Suppose $x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}}$ does not contain the factor $\prod_{\ell=h(1)+1}^{n} x_\ell$, and for all j, $0 \leq \ell_j \leq n-1-j$. Then the set $\{x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k\}_{1 \leq k \leq n}$ forms a linearly independent set of elements of $H^*(\mathrm{Hess}(S,h))$.

Proof. If $h(1) = n$, then there are no such elements defined above, so we assume that $h(1) < n$. Notice that it is equivalent to show that the set

$$\{x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} y_k\}_{1 \leq k \leq n-1} \cup \{x_n^{\ell_1} x_{n-1}^{\ell_2} \cdots x_2^{\ell_{n-1}} (y_1 + \cdots + y_n)\}$$

is linearly independent, since the transition matrix between the given set and this set is invertible. Since elements of B_2 form distinct basis elements, we have that the set of $n-1$ elements on the left is linearly independent. Using relations (1) and (2) from Proposition 3.0.2, $H^*(\mathrm{Hess}(S,h))$ does not contain elements divisible by $y_k y_\ell$ for $k \neq \ell$, or elements divisible by $x_1 y_k$ for any k. Under relation (3), we have that $0 = \prod_{\ell=2}^{n}(-x_\ell) - y_k(\prod_{\ell=h(1)+1}^{n}(-x_\ell))$. Notice that $\prod_{\ell=h(1)+1}^{n}(-x_\ell)$

divides this term, but does not divide $x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}$ by definition, so this relation does not factor into our calculation of linear independence.

Using relation (4), we have the following equivalence in $H^*(\mathrm{Hess}(S,h))$:

$$x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}(y_1+\cdots+y_n) = x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}\left(\prod_{\ell=2}^{h(1)}(x_1-x_\ell)\right). \tag{4.2.1}$$

Since relations $(1)-(3)$ each contain some y_k, it suffices to show that this term is nonzero modulo relation (5).

Recall the coinvariant ring $H^*(\mathrm{Fl}(\mathbb{C}^n)) \cong R = \mathbb{C}[x_1,\ldots,x_n]/I$ where I is the ideal generated by the elementary symmetric functions $e_1,\ldots,e_n$. If the right-hand side of Equation 4.2.1 is nonzero in R, then it is nonzero in $H^*(\mathrm{Hess}(S,h))$. The **Artin basis** of R consists of monomials $\{x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n} \,|\, 0\le i_j \le j-1\}$. Any permutation of the Artin basis $\{x_{\pi(1)}^{i_1}x_{\pi(2)}^{i_2}\cdots x_{\pi(n)}^{i_n} \,|\, 0 \le i_j \le j-1\}_{\pi\in\mathfrak{S}_n}$ also forms a basis of R.

Suppose that

$$x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}\left(\prod_{\ell=2}^{h(1)}(x_1-x_\ell)\right) = \sum c_k\, x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n}$$

Let m be the largest index such that the exponent on x_m in $x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}$ is 0, which exists since it does not contain a factor of $\prod_{\ell=h(1)+1}^{n} x_\ell$. In particular, $m \ge h(1)+1$. We show that each term of the sum above is an element of the permuted Artin basis for the permutation $\pi = (1m)$.

First, since x_m does not divide $x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}$, and $m \ge h(1)+1$, we have that $k_m = 0$. Next, for each x_j with $2 \le j \le h(1)$, the exponent ℓ_{n-j+1} satisfies $0 \le \ell_{n-j+1} \le j-2$ by definition. Hence, the exponent k_j satisfies $0 \le k_j \le j-1$. For all x_j with $h(1)+1 \le j \le n$, other than x_m, we have $k_j = \ell_{n-j-1} \le j-2$. Lastly, we have that the exponent on x_1 is $k_1 = h(1)-1 \le m-2$. Thus each term of the sum is an element of this permuted Artin basis.

Therefore $x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}\left(\prod_{\ell=2}^{h(1)}(x_1-x_\ell)\right)$ is nonzero in R, and thus is also nonzero in the cohomology ring $H^*(\mathrm{Hess}(S,h))$. So, modulo the relations in the ideal I, the given set of elements of $H^*(\mathrm{Hess}(S,h))$ are linearly independent. Consequently, the set $\{x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}\,y_k\}_{1\leq k\leq n}$ is linearly independent. □

This new set behaves nicely with regard to the $\mathfrak{S}_n$ action - since $\mathfrak{S}_n$ fixes each of the x_k variables, and permutes the y_k variables in the natural way, this basis decomposes into a number of copies of the natural permutation representation of $\mathfrak{S}_n$.

Corollary 4.2.2. *The $\mathfrak{S}_n$-module $H^*(\mathrm{Hess}(S,h))$ decomposes into $(n-h(1))(n-2)!$ copies of the natural permutation representation, and $n(h(1)-1)(n-2)!$ copies of the trivial representation.*

Proof. Since $\{x_n^{\ell_1}x_{n-1}^{\ell_2}\cdots x_2^{\ell_{n-1}}\,y_k\}_{1\leq k\leq n}$ forms a linearly independent set of polynomial elements of $H^*(\mathrm{Hess}(S,h))$, for a fixed list of exponents $(\ell_1,\ldots,\ell_{n-1})$, the submodule generated by these monomials yield a copy of the natural permutation representation. The number of such vectors $(\ell_1,\ldots,\ell_{n-1})$ such that $\ell_i=0$ for some $1\leq i\leq n-h(1)$, and $0\leq \ell_j\leq n-j-1$ is $(n-h(1))(n-2)!$. The other basis elements of $H^*(\mathrm{Hess}(S,h))$ are fixed, and there are $n!-n((n-h(1))(n-2)!)=n(h(1)-1)(n-2)!$ of these basis elements. □

As discussed in Section 2, the natural permutation representation is sent by the Frobenius character map to $h_{(n-1,1)}$, and the trivial representation is sent to $h_{(n)}$. So, applying the involution ω, the corresponding chromatic symmetric function decomposes as $X_{G_h}(\mathbf{x})=(n-h(1))(n-2)!\,e_{(n-1,1)}+n(h(1)-1)(n-2)!\,e_{(n)}$. Hence, the corresponding chromatic symmetric function for these graphs is e-positive.

Chapter 5

Tableaux bijections

In this section, we examine bijections between the monomial basis elements from the previous sections, and sets of tableaux. In [10], Gasharov showed that we can expand the the chromatic symmetric function $X_G(\mathbf{x})$ in the Schur basis using P-tableaux: If h is a Hessenberg function, P_h is the corresponding poset, and $G_h = \mathrm{inc}(P_h)$, then

$$X_{G_h}(\mathbf{x}) = \sum_{\lambda \vdash n} |\mathrm{PT}(\lambda)| \; s_\lambda$$

where $\mathrm{PT}(\lambda)$ is the set of P_h-tableaux of shape λ. As mentioned in Proposition 2.3.8, Shareshian and Wachs generalized this result to chromatic quasisymmetric functions, using the inversion statistic for P-tableaux:

$$X_{G_h}(\mathbf{x}; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{PT}(\lambda)} q^{\mathrm{inv}(T)} \right) s_\lambda .$$

From Chapter 4, we know how the $\mathfrak{S}_n$-module $H^*(\mathrm{Hess}(S,h))$ decomposes into irreducible modules when $h = (h(1), n, \ldots, n)$, and equivalently, when h' is the transpose Hessenberg function. Further recall that if V_λ is the Specht module indexed by the partition λ, then we know that $\mathrm{Frob}(V_\lambda) = s_\lambda$. We restate Equation 2.4.1 here:

$$\sum_{j=0}^{|E|} \mathrm{Frob}(H^{2j}(\mathrm{Hess}(S,h)))q^j = \omega X_G(\mathbf{x}; q) = \sum_{\lambda \vdash n} \left(\sum_{T \in PT(\lambda)} q^{\mathrm{inv}_h(T)} \right) s_{\lambda'}(\mathbf{x}).$$

In other words, for each partition shape λ, there are the same number of basis elements generating instances of V_λ as there are P_h-tableaux of shape λ'. In this section, we will work towards finding explicit, combinatorial bijections between the basis elements of $H^*(\mathrm{Hess}(S,h))$ and sets of P_h-tableaux for the case $h = (h(1), n, \ldots, n)$. We hope that these bijections will inform the construction of basis elements for other cases.

5.1 Regular nilpotent Hessenberg varieties

We begin this section by looking at regular nilpotent Hessenberg varieties, rather than regular semisimple Hessenberg varieties, since the cohomology rings in this case are understood for all Hessenberg functions. In this section, N is a regular nilpotent matrix, so in Jordan canonical form, N has a single Jordan block with eigenvalue 0.

Proposition 5.1.1 ([6], Corollary 7.3)**.** Let N be a regular nilpotent matrix and let $h : [n] \to [n]$ be any Hessenberg function. Then the following set of monomials form an additive basis for $H^*(\mathrm{Hess}(N, h))$:

$$\mathcal{N}_h := \left\{ x_1^{i_1} \cdots x_n^{i_n} \,|\, 0 \leq i_k \leq h(k) - k \text{ for } 1 \leq k \leq n \right\}$$

Given a Hessenberg function h, we visualize the basis elements as follows: For each k, let b_k be the number of full boxes below the corresponding Dyck path and above the main diagonal, in the k-th column. Then $b_k = h(k) - k$, so b_k is the largest power allowed on the x_k in a basis element. See Figure 5.1 for an example of this construction.

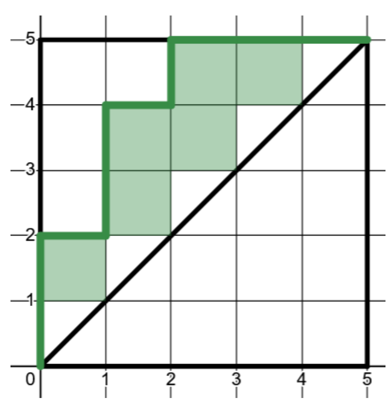


Figure 5.1: The Dyck path D_h for $h = (2, 4, 5, 5, 5)$. The highest degree basis element in $\mathcal{N}_h$ is $x_1 x_2^2 x_3^2 x_4$, and any monomial that divides this element is a basis element.

Regular nilpotent Hessenberg varieties are useful in the study of regular semisimple Hessenberg varieties in the following way. When S is regular semisimple, $\mathfrak{S}_n$ acts on $H^*(\mathrm{Hess}(S, h))$ by Tymoczko's dot action, and the fixed points of this action return $H^*(\mathrm{Hess}(N, h))$.

Proposition 5.1.2 ([3], Theorem B)**.** Let N be a regular nilpotent matrix, S be a regular semisimple matrix, and let h be any Hessenberg function of length n. Then there exists an isomorphism of graded algebras

$$\mathcal{A} : H^*(\mathrm{Hess}(N, h)) \to H^*(\mathrm{Hess}(S, h))^{\mathfrak{S}_n}$$

where $H^*(\mathrm{Hess}(S, h))^{\mathfrak{S}_n}$ is the set of $\mathfrak{S}_n$-fixed points of $H^*(\mathrm{Hess}(S, h))$.

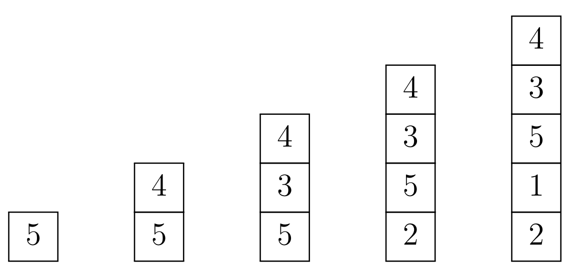
We can translate the action of $\mathfrak{S}_n$ on $H^*(\mathrm{Hess}(S, h))$ to a trivial action on $H^*(\mathrm{Hess}(N, h))$. Since the action is trivial, the corresponding representation of $\mathfrak{S}_n$ decomposes into the direct sum of one-dimensional trivial representations. Under the graded Frobenius map, these map to $q^d s_{(n)}$, for d the degree of the grading on each subspace. From Equation (2.4.1), we should be able to find a weight-preserving bijection between the set of monomial basis elements of $\mathcal{N}_h$ to the set of P_h-tableaux of shape $(n)' = (1^n)$.

Definition 5.1.3. Define $\mathrm{PT}(h, \lambda)$ to be the set of P_h-tableaux of shape λ.

Let $\lambda = (1^n)$. We define a map $\varphi : \mathcal{N}_h \to \mathrm{PT}(h, \lambda)$ as follows.

Definition 5.1.4. Let $x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{N}_h$. Begin with a P_h-tableau T of a single box whose entry is n. For each $k = n-1, \ldots 1$, if $i_k = 0$, insert k into a new box at the bottom of T, so k occurs directly below some $\ell > k$. If $i_k > 0$, then since $i_k \leq h(k) - k$, we have $k < h(k)$. List the entries $k+1, \ldots, h(k)$, which already exist in T, in order from the lowest to highest row position in T. Insert k in a new box directly above the i_k-th lowest entry of this list. Define $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ to be the resulting tableau from this process.

Example 5.1.5. Let $h = (2, 3, 5, 5, 5)$, and consider the monomial $x_1 x_3 x_4 \in \mathcal{N}_h$. We construct $\varphi(x_1 x_3 x_4)$ as follows.



We start with a single box containing a 5. Then, to insert 4 with $i_4 = 1$ P_h-inversion, we insert the 4 above the 5. To insert 3 with $i_3 = 1$ P_h-inversion, we insert the 3 above the 5 but below the 4. Similarly insert a 2 with no P_h-inversions, and a 1 with one P_h-inversion. Notice that at each step, the number of elements in P_h greater than k that are incomparable to k is $h(k) - k$, which is also the largest possible power i_k.

Theorem 1.0.4. The map φ is a well-defined, weight-preserving bijection.

Proof. First, at each insertion step, if $i_k = 0$, then k is inserted at the bottom of the tableau, so the entry immediately above k is larger, so adjacent entries are still P_h-non-decreasing. If $i_k > 0$, by the definition of P_h, k is incomparable to any $k < \ell \leq h(k)$, so adding k above one of these ℓ maintains that the column of T is P_h-non-decreasing. Hence after each insertion, T is still a P_h-tableau, and so φ is a well-defined function.

Now we show that φ is injective: Consider a function $\psi : PT(h, \lambda) \to \mathcal{N}_h$ defined as follows: Given a P_h-tableau T of shape (1^n), let $\psi(T) = x_1^{i_1} \cdots x_n^{i_n}$ where i_k is the number of inversions of T with k as the smaller entry. For each $k = 1, \ldots, n$, note that k can form an inversion as the smaller entry with at most $h(k) - k$ entries: $k + 1, \ldots, h(k)$. Hence each i_k is between 0 and $h(k) - k$, so $\psi(T)$ is in $\mathcal{N}_h$, and thus ψ is well-defined. Further, given a monomial $x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{N}_h$, $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ is constructed so that each k is inserted to form i_k inversions as the smaller entry, since each k is inserted above i_k incomparable elements greater than it. Thus for any $T \in PT(h, \lambda)$, we have $\varphi(\psi(T)) = T$.

Next, given $x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{N}_h$, we show by induction on n that $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ is the unique P_h-tableau of shape (1^n) that maps to $x_1^{i_1} \cdots x_n^{i_n}$ under ψ. When $n = 1$, the only monomial in $\mathcal{N}_h$ is 1, and the only P_h-tableau consists of a single box containing a 1. Given $x_1^{i_1} \cdots x_n^{i_n} \in \mathcal{N}_h$, by the induction hypothesis, there is a unique T with entries $2, \ldots, n$ such that $\psi(T) = x_2^{i_2} \cdots x_n^{i_n}$. Now consider the choices for where the 1 entry can be inserted. If the 1 is inserted in the bottom row, then it creates no new inversions and $\psi(T) = x_1^0 x_2^{i_2} \cdots x_n^{i_n}$. If the 1 is inserted above any entry $\ell > h(1)$, then 1 and ℓ are comparable in P_h, and so T is no longer a valid P_h-tableau. If the 1 is inserted above any entry ℓ with $2 \leq \ell \leq h(1)$, then there the same number of entries from the list

$2, \ldots, h(1)$ below the 1 in T as there are inversions with 1 as the smaller entry. Hence, for any i_1, there is exactly one way of inserting 1 into T so that $\psi(T) = x_1^{i_1} \cdots x_n^{i_n}$. Thus $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ is the unique tableau with $\psi(\varphi(x_1^{i_1} \cdots x_n^{i_n})) = x_1^{i_1} \cdots x_n^{i_n}$, and therefore φ is a bijection.

Lastly, we note that for any $x_1^{i_1} \cdots x_n^{i_n}$, by construction, $T = \varphi(x_1^{i_1} \cdots x_n^{i_n})$ has i_k inversions with k as the smaller entry for all k, so $\deg(x_1^{i_1} \cdots x_n^{i_n}) = \operatorname{inv}(T) = i_1 + \cdots + i_n$. $\square$

5.2 Regular semisimple Hessenberg varieties

Now we turn our attention to regular semisimple Hessenberg varieties. Recall the partial set of basis elements B_1 of the cohomology ring $H^*(\operatorname{Hess}(S, h))$ when $h = (h(1), n, \ldots, n)$:

$$B_1 = \left\{ x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \text{ without a factor of } \prod_{\ell=1}^{h(1)} x_\ell \,\middle|\, 0 \le i_j \le n - j \right\}$$

Let $\lambda = (1^n)$. We will define a map φ between B_1 and $\operatorname{PT}(h, \lambda)$, and the corresponding inverse map ψ.

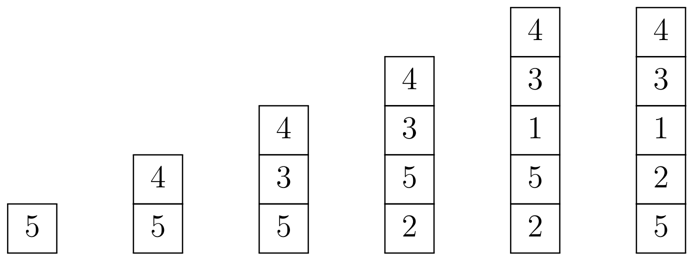
Definition 5.2.1. Let $x_1^{i_1} \cdots x_n^{i_n} \in B_1$. Begin with a P_h-tableau T of a single box whose entry is n. For each $k = n-1, \ldots, 1$, insert k into T above exactly i_k of the existing entries.

Let k' be the smallest index in $1, \ldots, h(1)$ such that $i_{k'} = 0$, which exists by the definition of B_1. By the above construction, after inserting n through 1, entry k' will be on the bottom of T. If $k' = 1$, then set $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ to be T. If $1 < k' \le h(1)$, then slide the entry k' up until it is directly below the 1, and define $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ to be T after this slide.

We now define the map ψ from $\operatorname{PT}(h, \lambda)$ to B_1. Let P_N be the poset corresponding to $h = (n, \ldots, n)$, in which all elements are incomparable.

Definition 5.2.2. Let $T \in \operatorname{PT}(h, \lambda)$. If 1 is in the bottom row or the second row, let $\hat{T} = T$. Otherwise, construct $\hat{T}$ by shifting the entry directly below the 1 to be in the bottom row of T. Define $\psi(T)$ to be $x_1^{i_1} \cdots x_n^{i_n}$ where i_k is the number of P_N-inversions in $\hat{T}$ with k as the smaller entry.

Example 5.2.3. Let $h = (3, 5, 5, 5, 5)$, and consider the monomial $x_1^2 x_3 x_4 \in B_1$. We construct $\varphi(x_1^2 x_3 x_4)$ as follows.



We start with a single box containing a 5. Then (using the degrees on each variable) we insert the 4 above one existing entry, the 3 above one existing entry, the 2 above no existing entries, and the 1 above two existing entries. Since $1 <_{P_h} 5$ are comparable, the resulting tableau is not a P_h-tableau, so we shift the 2 (which is incomparable to the 1) to be directly below the 1. In the second-to-last tableau, the number of P_N-inversions with each k as the smaller entry is exactly i_k, so reading these inversions returns the monomial $x_1^2 x_3 x_4$.

Theorem 1.0.2. The map φ is a well-defined bijection.

Proof. Let $x_1^{i_1} \cdots x_n^{i_n} \in B_1$. First we show that $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ is a well-defined P_h-tableau. Since $h = (h(1), n, \ldots, n)$, the entries $2, \ldots, n$ are incomparable in P_h. So for $k = n-1, \ldots, 2$, inserting k directly above i_k existing entries in T creates a P_h-tableau with exactly i_k P_h-inversions where k is the smaller entry. Then, inserting a 1 above i_1 entries creates a P_N-tableau (but not necessarily a P_h-tableau) T where the 1 forms i_1 P_N-inversions. If $i_1 = 0$, then $\varphi(x_1^{i_1} \cdots x_n^{i_n}) = T$ is a P_h-tableau, since 1 is not directly above anything comparable to it.

If $i_1 \neq 0$, then since $x_1^{i_1} \cdots x_n^{i_n}$ is in B_1, there exists some smallest index $k' \in \{2, \ldots, h(1)\}$ such that $i_{k'} = 0$. By construction, k' will be in the bottom row of T, so k' forms no P_h-inversions as the smaller entry. Then, shifting k' to be directly below the 1 in T creates a P_h-tableau, since 1 is incomparable in P_h to k', and k' is incomparable in P_h to all other entries. Hence $\varphi(x_1^{i_1} \cdots x_n^{i_n})$ is a P_h-tableau, so φ is well-defined.

Now, let $T \in \mathrm{PT}(h, \lambda)$. If 1 is in the bottom row, then it forms no P_N-inversions as the smaller entry, so the exponent on x_1 in $\psi(T)$ is 0. When 1 is in a higher row, we construct T by shifting the entry k' that is below the 1 to the bottom of the tableau. So k' forms no P_N-inversions as the smaller entry, and in $\psi(T)$, the exponent on $x_{k'}$ is zero. Since T is a P_h-tableau, we have that $1 \le k' \le h(1)$, so the monomial $\prod_{\ell=1}^{h(1)} x_\ell$ does not divide $\psi(T)$. Then, after the shift, T is a P_N tableau, so for all $j \in [n]$, j can form a P_N-inversion as the smaller entry with at most $n-j$ other entries. Thus the exponent i_j on x_j satisfies $0 \le i_j \le n-j$. Therefore $\psi(T)$ is in B_1, so ψ is well-defined.

Next, given $x_1^{i_1} \cdots x_n^{i_n} \in B_1$, we show that $\psi(\varphi(x_1^{i_1} \cdots x_n^{i_n})) = x_1^{i_1} \cdots x_n^{i_n}$. In the construction of $\varphi(x_1^{i_1} \cdots x_n^{i_n})$, we constructed the intermediate P_N-tableau T such that each k in $1, \ldots, n$ has exactly i_k P_N-inversions, before shifting the entry k' from the bottom row to directly below the 1. The map ψ shifts the k' back to the bottom row, and then counts the P_N-inversions with each k as the smaller entry, which is how T is originally constructed. Hence $\psi(\varphi(x_1^{i_1} \cdots x_n^{i_n})) = x_1^{i_1} \cdots x_n^{i_n}$.

Further, for all $j = 1, \ldots, k'-1$, since k' is below j in T, we have that (j, k') is an inversion with j as the smaller entry, so the exponent on x_j in $\psi(T)$ is nonzero. So $x_{k'}$ is the variable with smallest index that has an exponent of 0. Therefore, in the construction of the intermediate tableau for $\varphi(\psi(T))$, k' is in the bottom row, and so k' is the entry that is shifted below the 1. Further, by construction, the intermediate tableau has i_k inversions with k as the smaller entry, which is also true of T. Therefore $\varphi(\psi(T)) = T$, and so we have that φ is a bijection. □

As with the nilpotent case, we have a bijection between the $\mathfrak{S}_n$-fixed points of $H^*(\mathrm{Hess}(S,h))$ and P_h-tableaux of shape (1^n) when $h = (h(1), n, \ldots, n)$. We now construct a bijection for the other monomials in $H^*(\mathrm{Hess}(S,h))$, given by the set B_3:

$$B_3 = \left\{ x_n^{\ell_1} \cdots x_2^{\ell_{n-1}} (y_{k+1} - y_1) \text{ with no factor of } \prod_{\ell=h(1)+1}^{n} x_\ell \,\middle|\, 0 \le \ell_j \le n-j-1,\ 1 \le k \le n-1 \right\}$$

Definition 5.2.4. Define $\mathrm{PSPT}(h, \lambda)$ to be the set of pairs (S, T), where S is a standard Young tableau of shape λ and T is a P_h-tableau of shape λ.

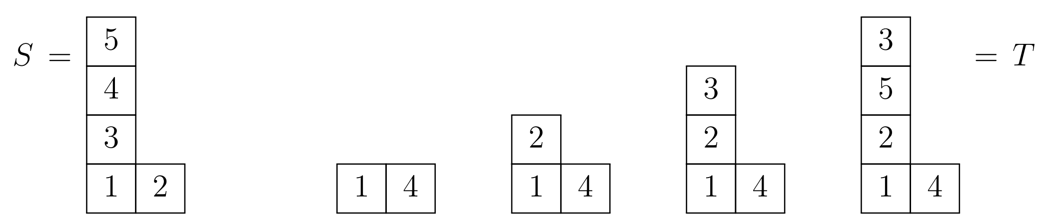
Let $\mu = (2, 1^{n-2})$ and $h = (h(1), n \ldots, n)$. We define a map $\varphi : B_3 \to \mathrm{PSPT}(h, \mu)$ as follows.

Definition 5.2.5. Let $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) \in B_3$. First, let S be the unique standard tableau of shape μ with entries 1 and k in the bottom row. Next, let j be the largest entry among $h(1)+1, \ldots, n$ such that the exponent ℓ_{n-j+1} on x_j is zero. Begin with a tableau T with a single row of two boxes, containing a 1 and j. Then for each $i \geq 2$, other than j, insert i into the left column so that it is under exactly $(i-2) - \ell_{n-i+1}$ of the current entries in the left column. Define $\varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1))$ to be the pair (S, T) resulting from this construction.

We now define the inverse map $\psi : \mathrm{PSPT}(h, \mu) \to B_3$.

Definition 5.2.6. Let I_h be the set of potential P_h-inversions in a P_h-tableau T, excluding those containing a 1, so $I_h = \{(i,j) \,|\, 1 < i < j \text{ and } j \leq h(i)\}$. When $h = (h(1), n, \ldots, n)$, we get $I_h = \{(i,j) \,|\, 1 < i < j\}$. Let $(S,T) \in \mathrm{PSPT}(h, \mu)$. Let 1 and k be the entries in the bottom row of S. Let ℓ_{n-j+1} be the number of P_h-inversions in $I_h \setminus \mathrm{Inv}_h(T)$ with j as the larger entry. Define $\psi(S,T)$ to be the monomial $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)$ with k and ℓ_{n-j+1} defined above.

Example 5.2.7. Let $h = (3,5,5,5,5)$, and consider the monomial $x_5^2 x_3(y_2 - y_1) \in B_3$. Note that $j = 4$ is the largest index where x_j has an exponent of zero. We construct $\varphi(x_5^2 x_3(y_2 - y_1))$ as follows.



S is defined to be the unique standard Young tableaux of shape $(2, 1^{n-2})$ with a 1 and 2 in the bottom row. Then, to form T, we start with a single row containing a 1 and a 4. We then insert a 2 in the left column underneath $(2-2) - 0 = 0$ entries, a 3 in the left column underneath $(3-2) - 1 = 0$ entries, and a 5 in the left column underneath $(5-2) - 2 = 1$ entry.

To find the image under ψ of this pair, the exponent ℓ_{n-j+1} on x_j is the number of potential P_h-inversions (i,j) with $i \neq 1$ that do not appear in T. The 2 has no possible P_h-inversions as the larger entry, so $\ell_4 = 0$, but the 3 is missing the inversion with 2, so $\ell_3 = 1$. Since the 4 is in the bottom row, it forms all possible P_h-inversions as the larger entry, so $\ell_2 = 0$. Then the 5 forms an inversion with 3, but not 2 or 4, so $\ell_1 = 2$. Hence $\psi(S,T) = x_5^2 x_3(y_2 - y_1)$.

Theorem 1.0.3. The map φ is a well-defined bijection.

Proof. Let $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) \in B_3$ and suppose that $\varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)) = (S,T)$. By construction, S is a standard tableau. Since $x_{h(1)+1} \cdots x_n$ does not divide the given monomial, there is some largest j among $h(1)+1, \ldots, n$ such that the exponent ℓ_{n-j+1} on x_j is zero. In the construction of T, the initial tableau has a row containing a 1 and j. Since $j \geq h(1)+1$, we have that $1 \to j$ forms a chain in P_h, so this row forms a valid P_h-tableau. Note that, by the definition of B_3, for each i, $0 \leq \ell_{n-i+1} \leq i-2$. Then, at each insertion of $i \geq 2$, i is inserted underneath at most $i-2$ entries in the column. If $i < j$, then i will not be inserted beneath the 1 in the bottom row. If $i > j$, then by the definition of j, $\ell_{n-j+1} \geq 1$, so i is inserted underneath at most $i-3$ entries in the left column, and again will not be inserted under the 1 in the bottom row. Since all of $2, \ldots, n$ are incomparable in P_h, after each i is inserted, T will remain a P_h tableau. Hence the map φ is well-defined.

Now suppose T is a P_h tableau of shape μ. For each x_j, the number of P_h-inversions (i,j) of T with $1 < i < j$ is at most $j-2$, so the exponent ℓ_j on x_{n-j+1} is at most $(n-j+1)-2 = n-j-1$. Next, by the definition of P_h-tableau, the bottom row of T is a chain in P_h, so it has entries 1 and m for some $m > h(1)$. Then all P_h-inversions (i,m) with $1 < i < m$ are present in T, so the exponent ℓ_{n-m+1} on x_m is 0. Hence $\prod_{\ell=h(1)+1}^{n} x_\ell$ does not divide $\psi(T,S)$. Finally, if S is a standard tableau, then the bottom row must contain entries 1 and k for some $2 \leq k \leq n$. Hence $\psi(T,S) \in B_3$, and so ψ is well-defined.

We now show that φ and ψ are inverses. For $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1) \in B_3$, note that the pair $\varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)) = (S,T)$ is constructed so that each entry $i = 2, \ldots, n$ in T (excluding $j > 1$ in the bottom row) forms $i - 2 - \ell_{n-i+1}$ P_h-inversions as the larger value. So there are ℓ_{n-i+1} P_h-inversions in $I_h \setminus \mathrm{Inv}(T)$ with i as the larger entry. Further, S is constructed so that the bottom row of S corresponds to the factor $(y_k - y_1)$. Thus $\psi(\varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1))) = x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)$.

Next, note that, at each step of the construction of T from $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)$, since $2, \ldots, n$ are all incomparable in P_h, inserting an entry $i > 1$ into the left column below k existing entries gives k P_h-inversions with i as the larger entry. Hence there is a unique place to insert i so that i forms $(i-2)-\ell_{n-i+1}$ P_h-inversions as the larger entry. Thus $\varphi(x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1))$ is the unique pair of tableaux such that ψ maps the pair to $x_n^{\ell_1} \cdots x_2^{\ell_{n-1}}(y_k - y_1)$. Therefore ψ is a left-inverse of φ, and so φ is a bijection. $\square$

Chapter 6

New proof of the Poincaré polynomial

Recall that, given a graded vector space V over a field k, if $V = \bigoplus_{i\in\mathbb{N}} V_i$ with each subspace V_i consisting of vectors of degree i being finite dimensional, then the Poincaré polynomial of V is

$$\mathrm{Poin}(V,q) = \sum_{i\in\mathbb{N}} \dim_k(V_i) q^i$$

In Equation 2.4.1, we saw that we could express the image of regular semisimple Hessenberg varieties under the graded Frobenius map in terms of Schur functions in the following way, due to the chromatic quasisymmetric function defined by Shareshian and Wachs:

$$\sum_{j=0}^{|E|} \mathrm{Frob}(H^{2j}(\mathrm{Hess}(S,h)))q^j = \omega X_G(\mathbf{x};q) = \sum_{\lambda\vdash n}\left(\sum_{T\in PT(\lambda)} q^{\mathrm{inv}_h(T)}\right) s_{\lambda'} \tag{6.0.1}$$

So, we can write the Poincaré polynomial of a regular semisimple Hessenberg variety as follows:

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = \sum_{\lambda\vdash n}\left(\sum_{T\in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)}\right)\#\mathrm{SYT}(\lambda)$$

since the dimension of the irreducible representation V_λ is the number of standard Young tableaux of shape λ. We use this formula to provide an alternate proof of the formula of the Poincaré polynomial of the regular semisimple Hessenberg variety $\mathrm{Hess}(S,h)$ when $h = (h(1), n, \dots, n)$, originally proven by Abe, Horiguchi, and Masuda in [23].

Theorem 6.0.1 ([23], Lemma 3.2). *If $h = (h(1), n, \dots, n)$, then the Poincaré polynomial of $\mathrm{Hess}(S,h)$ is given by*

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = \frac{1-q^{h(1)}}{1-q}\prod_{j=1}^{n-1}\frac{1-q^j}{1-q} + (n-1)q^{h(1)-1}\frac{1-q^{n-h(1)}}{1-q}\prod_{j=1}^{n-2}\frac{1-q^j}{1-q}$$

Writing this in terms of q-analogues, we get the following expression:

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = h(1)_q(n-1)_q! + (n-1)q^{h(1)-1}(n-h(1))_q(n-2)_q!$$

Proof. From above, we know that

$$\mathrm{Poin}(\mathrm{Hess}(S,h),q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} \right) \#\mathrm{SYT}(\lambda).$$

Let $h = (h(1), n, \ldots, n)$. All chains in P_h have length two and include the element 1. Since distinct rows in a P_h tableaux need to contain entries from distinct chains in P_h, the only shapes λ with a nonzero number of P_h-tableaux are $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$. Further, we have that $\#\mathrm{SYT}(\lambda) = 1$ and $\#\mathrm{SYT}(\mu) = n-1$.

For $\lambda = (1^n)$, we need to count the P_h-inversions in the P_h tableaux of this shape. Since the element 1 is incomparable to each of $2, \ldots, h(1)$, it can form between 0 and $h(1)-1$ inversions as the smaller entry. For each $i = 2, \ldots, n$, the entry i can form up to $n-i$ inversions as the smaller entry. Hence, we get that

$$\sum_{T \in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} = (1 + q + \cdots + q^{h(1)-1})(1 + q + \cdots + q^{n-2})! = h(1)_q(n-1)_q!.$$

For $\mu = (2, 1^{n-2})$, the bottom row of any P_h-tableaux of shape μ must be filled with entries from a chain in P_h, so it contains a 1 and an i for some $i = h(1)+1, \ldots, n$. Then, since $i > 1$, it is incomparable with all other $j \neq 1$, so the entry i in the bottom row forms inversions as the larger entry with the entries $2, \ldots, i-1$, of which there are $i-2$. So this entry contributes between $h(1)-1$ and $n-2$ inversions to the P_h-tableaux as the larger entry. Then, for the column entries of $j = 2, \ldots, n$ and $j \neq i$, if $j < i$, then j forms an inversion with i where j is the smaller entry (which was already counted), and can form an inversion as the smaller entry with the other $n-j-1$ entries larger than j. If $j > i$, then j does not form an inversion with i, and can form an inversion as the smaller entry with any of the $n-j$ entries larger than j. In each case, there is a unique placement for j giving each set of inversions. Hence we have

$$\sum_{T\in \mathrm{PT}(h,\mu)} q^{\mathrm{inv}_h(T)} = (q^{h(1)-1}+q^{h(1)}+\cdots+q^{n-1})(1+q+\cdots+q^{n-3})! = q^{h(1)-1}(n-h(1))_q(n-2)_q!$$

Therefore, for $\lambda = (1^n)$ and $\mu = (2, 1^{n-2})$, we have the Poincaré polynomial of $\mathrm{Hess}(S, h)$ as follows:

$$\begin{aligned}\mathrm{Poin}(\mathrm{Hess}(S,h),q) &= \sum_{T\in \mathrm{PT}(h,\lambda)} q^{\mathrm{inv}_h(T)} + (n-1)\sum_{T\in \mathrm{PT}(h,\mu)} q^{\mathrm{inv}_h(T)} \\ &= h(1)_q(n-1)_q! + (n-1)q^{h(1)-1}(n-h(1))_q(n-2)_q!\end{aligned}$$

This completes the proof. □

Chapter 7

Products of chromatic quasisymmetric functions

In this chapter, we will begin to investigate the question of when the chromatic quasisymmetric function is symmetric. The results here, and those in Chapters 8 and 9, were originally posted in [25]. To approach the given question, it would be handy to restrict the problem to the set of connected graphs. An immediate consequence of the definition of the chromatic quasisymmetric function is the following:

Proposition 7.0.1 ([13], Proposition 2.2)**.** If G and H are finite graphs, and $G+H$ is the disjoint union of G and H, then $X_{G+H}(\mathbf{x};q) = X_G(\mathbf{x};q)\,X_H(\mathbf{x};q)$.

We aim to show that the property of a quasisymmetric function being symmetric follows a similar structure. That is, if f and g are quasisymmetric functions, and $f \cdot g$ is symmetric, then f and g are both symmetric. First, we introduce some notation and facts from the theory of quasisymmetric functions.

Recall that if R is a ring, QSym_R is the ring of quasisymmetric functions over R, and $\{M_\alpha\}_\alpha$ is the monomial quasisymmetric basis of QSym_R, indexed by compositions α. We will use the notation $\mathrm{QSym}_R[x_1,\ldots,x_n]$ for the ring of quasisymmetric polynomials (i.e. quasisymmetric functions restricted to a finite number of indeterminates). Since $\mathrm{QSym}_R[x_1,\ldots,x_n]$ is a subring of $R[x_1,\ldots,x_n]$, the following result is of use:

Lemma 7.0.2. *Let R be any ring. If $h \in R[x_1,\ldots,x_n]$ is irreducible, then it is also irreducible in $R[x_1,\ldots,x_n,x_{n+1},\ldots,x_{n+m}]$.*

Proof. By induction, it suffices to prove the statement for $m=1$.

Suppose for contradiction that a polynomial $h \in R[x_1,\ldots,x_n]$ is irreducible, but $h = f \cdot g$ in $R[x_1,\ldots,x_n,x_{n+1}]$ where f and g both have degree at least 1. Set $x_{n+1}=0$ on both sides of the equation. Since h is a polynomial in $x_1,\ldots,x_n$, the left hand side is still simply h. On the right hand side, if neither f nor g become a constant, then h factors in $R[x_1, \ldots, x_n]$ and we have a contradiction.

On the other hand, if f or g become a constant at $x_{n+1} = 0$, say $g(x_1, \ldots, x_n, 0) = c \in R$, then $g = c + x_{n+1}r$ for some nonzero $r \in R[x_1, \ldots, x_{n+1}]$. Consider both f and g as polynomials in x_{n+1} with coefficients in $R[x_1, \ldots, x_n]$, and consider the highest power of x_{n+1} in each. The product of these terms gives the highest power of x_{n+1} in $h = f \cdot g$, which is nonvanishing. This is a contradiction since h does not depend on x_{n+1}. □

We now use a theorem of [32] that establishes generators for $\mathrm{QSym}_{\mathbb{Z}}$ using λ-ring theory. A **Lyndon word** is a word $\alpha = \alpha_1, \ldots, \alpha_n$ of positive integers where α is strictly lexicographically smaller than any cycling $\alpha_i, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{i-1}$. For example, 14615 is a Lyndon word, since the cyclings 46151, 61514, 15146, and 51461 are all lexicographically larger. In [32], Hazewinkle defines quasisymmetric functions $\lambda_n(M_\alpha)$ for each Lyndon word α and positive integer n, of degree $n \cdot \sum_i \alpha_i$.

Theorem 7.0.3 ([32], Theorem 3.1)**.** *We have* $\mathrm{QSym}_{\mathbb{Z}} = \mathbb{Z}[\lambda_n(M_\alpha)]$ *where* α *ranges over all Lyndon words with* $\gcd(\alpha_i) = 1$. *Further,* $\lambda_n(M_1) = e_n$ *for all* n.

Corollary 7.0.4. *We have* $\mathrm{QSym}_{\mathbb{Z}} = \mathbb{Z}[e_1, e_2, \ldots, f_1, f_2, \ldots]$ *for some functions* f_i *such that there are finitely many* f*'s of each degree.*

Proof. The generators $\lambda_n(M_\alpha)$ give the elementary symmetric functions for $\alpha = (1)$, and there are a finite number of generators $\lambda_n(M_\alpha)$ of a given degree K. □

Following this corollary, we have our main theorem of this section.

Theorem 7.0.5. *Suppose* f *and* g *are quasisymmetric functions,* h *is a symmetric function, and* $f \cdot g = h$. *Then* f *and* g *are in fact symmetric.*

Proof. We induct on the degree of h. If h has degree 0, then f and g are also constants and are therefore symmetric.

Now, suppose h has degree d, and assume for strong induction that the statement is true for all degrees less than d. We consider two cases.

Case 1. Suppose h is irreducible in the ring of symmetric functions $\mathrm{Sym} = \mathbb{Z}[e_1, e_2, \ldots]$. Then it is irreducible over $\mathbb{Z}[e_1, \ldots, e_d]$ because h has degree d. Thus, by Lemma 7.0.2, h is irreducible in $\mathbb{Z}[e_1, \ldots, e_d, f_1, \ldots, f_N]$ where $f_1, \ldots, f_N$ are the generators $\lambda_n(M_\alpha)$ other than the elementary symmetric functions that have degree $\leq d$ (see Corollary 7.0.4). It follows from Theorem 7.0.3 that h is irreducible in QSym, and so the factorization does not occur.

Case 2. Suppose h factors in Sym as $h = u \cdot v$ where $u, v \in \mathrm{Sym}$ both have lower degree than d. Then by Theorem 7.0.3, we know QSym is a unique factorization domain (since there are a finite number of generators of each degree, so any given polynomial lies in a finitely generated ring that has unique factorization). Thus we can consider the unique factorizations

$$u = a_1 a_2 \cdots a_r \qquad\qquad v = b_1 b_2 \cdots b_s$$

in QSym. By the induction hypothesis, all of $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ are symmetric. Therefore, any factorization of $h = a_1 \cdots a_r b_1 \cdots b_s$ into a product of two factors f, g must have that f, g are products of disjoint subsets of the a and b factors up to scaling by a unit. Thus f and g are both symmetric as well, as desired. □

Interestingly, while the above result holds for symmetric and quasisymmetric functions in infinitely many indeterminates, it does not hold for symmetric and quasisymmetric polynomials:

Remark 7.0.6. Theorem 7.0.5 does not hold when restricted to finitely many variables. For example, the monomials $x_1^n x_2^{n-1} \ldots x_n^1$ and $x_1^1 x_2^2 \ldots x_n^n$ are both elements of $\mathrm{QSym}[x_1, \ldots, x_n]$ and not of $\mathrm{Sym}[x_1, \ldots, x_n]$. However, their product $(x_1 x_2 \ldots x_n)^{n+1}$ is an element of $\mathrm{Sym}[x_1, \ldots, x_n]$.

As a result of Theorem 7.0.5, when studying the chromatic quasisymmetric function of a graph, we can restrict to the case of connected graphs.

Corollary 7.0.7. *Suppose G is a graph with multiple connected components $G_1, \ldots, G_k$. Then $X_G(\mathbf{x}; q)$ is symmetric if and only if each $X_{G_i}(x; q)$ is symmetric.*

Chapter 8

Chromatic quasisymmetric functions for trees

In this section, we will prove Theorem 1.0.7, describing the chromatic quasisymmetric function for connected, directed acyclic graphs. This result further yields an exact classification of when the chromatic quasisymmetric function for a tree is symmetric. In this chapter, let $G = (V, E)$ be a finite graph with n vertices. First, we start with some general results on necessary conditions for symmetry.

8.1 General results for directed acyclic graphs

Every labeling of a graph G induces a directed acyclic orientation on G by directing every edge from a smaller label to a larger one. Moreover, any directed acyclic graph can be given a labeling that respects the orientation of the edges. Thus, it equivalent to define the chromatic quasisymmetric function on a directed acyclic graph G where given a proper coloring κ of G, a directed edge (u, v) is said to have an ascent if $\kappa(u) < \kappa(v)$. Given a directed acyclic graph G, let G^{rev} denote the acyclic graph obtained from G by reversing the orientation of all of the edges.

Lemma 8.1.1. *Let G be a directed acyclic graph. Then $X_G(\mathbf{x}; q)$ is symmetric if and only if $X_{G^{\text{rev}}}(x; q)$ is symmetric.*

Proof. As the underlying undirected graph does not change, any proper coloring of G is also a proper coloring of G^{rev}. Any proper coloring κ of G with $\operatorname{asc}(\kappa)$ ascents will have $|E| - \operatorname{asc}(\kappa)$ ascents in G^{rev}. Thus, $X_{G^{\text{rev}}}(x; q) = q^{|E|} X_G(x; q^{-1})$ which implies the desired result. □

Any vertex of a directed acyclic graph that only has edges exiting the vertex is called a **source**. Likewise, any vertex with only incoming edges is said to be a **sink**. The following observation is due to Liu [33].

Lemma 8.1.2. *If G has a different number of sources and sinks, then $X_G(\mathbf{x}; q)$ is not symmetric.*

Proof. Let a be the number of sources and b be the number of sinks in G. As G^{rev} has b sources and a sinks, it suffices to assume that $a < b$ by Lemma 8.1.1. We will show that in the monomial quasisymmetric function expansion of $X_G(\mathbf{x}; q)$, the coefficients of $M_{(a,1^{n-a-b},b)}$ and $M_{(b,1^{n-a-b},a)}$ are different. First, consider a coloring of G such that each source has color 1, each sink has color $n-a-b+1$, and the intermediate vertices are colored in such a way to achieve the maximum ascent statistic. Such a coloring exists because of the equivalence between directed acyclic graphs and labeled graphs. Since every intermediate vertex has a color between 1 and $n-a-b+1$, this coloring κ has $\operatorname{asc}(\kappa) = |E|$, so $q^{|E|}$ appears with a nonzero coefficient on $M_{(a,1^{n-a-b},b)}$ in the expansion of $X_G(\mathbf{x}; q)$.

Next, consider a coloring of the vertices of G with content $(b, 1^{n-a-b}, a)$. As $a < b$, there are more sinks in G than sources, and so the coloring must contain a non-source vertex colored with a 1. Any edge directed towards this vertex does not count towards the ascent number of κ, and such an edge exists because the vertex is not a source. Thus, no such coloring contains $|E|$ ascents, and so $q^{|E|}$ does not appear in the coefficient of $M_{(b,1^{n-a-b},a)}$ in the expansion of $X_G(\mathbf{x}; q)$. Hence $X_G(\mathbf{x}; q)$ is not symmetric. □

The set of sources (or sinks) of a graph G forms an induced subgraph containing no edges - this is an example of the broader notion of antichains:

Definition 8.1.3. An **antichain** in a directed acyclic graph is a set of vertices $\{v_1, \ldots, v_k\}$ such that there is not a directed path from any v_i to v_j.

Note that this matches the definition of an antichain on a poset when we consider the directed acyclic graph as a poset such that $v_i < v_j$ if and only if there is a directed path from v_i to v_j.

Lemma 8.1.4. *Let G be a directed acyclic graph. If G has an antichain whose size is larger than the number of sinks (or the number of sources), then $X_G(\mathbf{x}; q)$ is not symmetric.*

Proof. Let V be the set of vertices of G and let a be the number of sources. By assumption, there exists an antichain S in G with cardinality $|S| > a$. Consider the subset T of V consisting of all vertices that are not in S and have a directed path to a vertex in S. (Viewing G as a poset, T consists all elements that are strictly less than at least one element of S.)

We construct a coloring with maximum ascent statistic $|E|$ and content $(1^{|T|}, |S|, 1^{n-|S|-|T|})$. Color all of the vertices in S with the color $|T|+1$ and the vertices in T with the colors 1 through $|T|$ such that every edge between vertices in T contributes an ascent. Likewise, color the vertices in $V-(S \cup T)$ with the remaining colors such that every edge between these vertices contributes an ascent. As S is an antichain, this construction gives a proper coloring of G. Observe that any directed edge crossing between the sets S, T, and $V-(S\cup T)$ must go from T to S, T to $V-(S\cup T)$, or S to $V-(S\cup T)$. The constructed coloring respects this relationship, and therefore has maximum ascent statistic $|E|$. Thus, $q^{|E|}$ appears in the coefficient of $M_{(1^{|T|},|S|,1^{n-|S|-|T|})}$

Now consider a coloring of G with content $(|S|, 1^{n-|S|})$. Since $|S| > a$, there exists a nonsource vertex v colored 1. Any incoming edge to this vertex will not contribute an ascent, and so there is no proper coloring of G with content $(|S|, 1^{n-|S|})$ and maximum ascent statistic $|E|$. Hence $q^{|E|}$ does not appear in the coefficient of $M_{(|S|,1^{n-|S|})}$, and thus, $X_G(\mathbf{x};q)$ is not symmetric. □

8.2 Directed acyclic graphs with at least 2 sources

Consider a connected, directed acyclic graph G with at least two sources. We will show that the chromatic quasisymmetric function of G is not symmetric. As before, we will view G as a poset where a vertex w is less than v if and only if there is a directed path from w to v. Suppose that G has a sources, for $a \geq 2$, and by Lemma 8.1.2, assume that G has a sinks as well. We define $S(G)$ to be the set of all vertices v of G having at least two sources that are smaller than v.

Lemma 8.2.1. *The set $S(G)$ is nonempty.*

Proof. Assume that every sink has exactly one source that is smaller than it. As G has an equal number of sources and sinks, every sink is paired with a unique source. Since G has at least two sinks, this would imply that G is not connected. Thus, at least one sink of G has at least two sources smaller than it, and so $S(G)$ is nonempty. □

Given $v \in S(G)$, we define $\mathrm{stat}(v)$ to be the number of nonsource vertices smaller than v. We set $k := 1 + \min_{v \in S(G)} \mathrm{stat}(v)$. Denote by $K_\alpha^{|E|}$ the set of proper colorings of G having weight α and $|E|$ ascents. We will construct a strictly injective map from $K^{|E|}_{(1^k,a,1^{n-k-a})}$ to $K^{|E|}_{(a,1^{n-a})}$. Before defining this map, we recall Dilworth's Theorem on posets.

Theorem 8.2.2. *(Dilworth's Theorem, [34]) Suppose that P is a finite poset. Then, the size of the largest antichain of P is equal to the minimum number of disjoint chains needed to cover the vertices of P.*

By Lemma 8.1.4, we may assume that the largest antichain of G has size a. Thus, G can be covered by a disjoint chains, each of which contains exactly one source and sink. Fix such a minimal chain decomposition R of the graph G. We are now ready to define the map of interest.

Definition 8.2.3. Let $\varphi_{G,R}\colon K^{|E|}_{(1^k,a,1^{n-k-a})} \to K^{|E|}_{(a,1^{n-a})}$ be the map where $\varphi_{G,R}(\kappa)$ is the coloring obtained by iterating over every chain in R as follows:

- If the chain contains a vertex colored 1, do nothing.
- Otherwise, recolor the vertex colored $k+1$ with the color 1. Then sort the colors in the chain such that they are increasing from the chain's source to the chain's sink.

Before showing that $\varphi_{G,R}$ is well-defined, we first prove the following lemma.

Lemma 8.2.4. *If $\kappa \in K^{|E|}_{(1^k,a,1^{n-k-a})}$, then κ assigns every vertex in $S(G)$ a color strictly larger than $k+1$.*

Proof. Given a vertex v in $S(G)$, the number of nonsource vertices smaller than v is at least $k-1$, by the definition of k. Moreover, as v is greater than at least two sources, there are at least $k+1$ vertices smaller than v. Thus, in order for a proper coloring κ of weight $(1^k, a, 1^{n-k-a})$ to have the maximum number $|E|$ of ascents, we must have $\kappa(v) \geq k+2$ which proves the lemma. □

Lemma 8.2.5. *The map $\varphi_{G,R}$ is well-defined.*

Proof. Let $\kappa \in K^{|E|}_{(1^k,a,1^{n-k-a})}$. By construction, $\varphi_{G,R}(\kappa)$ has the appropriate weight $(a, 1^{n-a})$. It remains to show that the coloring is proper and has maximum ascent statistic $|E|$. Consider two adjacent vertices v and w in G where v is larger than w. We show that the coloring of v is strictly larger than that of w in $\varphi_{G,R}(\kappa)$. If v and w lie within the same chain in the minimal chain decomposition R, then the color of v is larger than w by construction. Thus, we may assume that v and w lie in different chains in R.

As the color of every vertex in $\varphi_{G,R}(\kappa)$ is weakly smaller than their respective color in κ, it suffices to show that v is painted the same color in both κ and $\varphi_{G,R}(\kappa)$. Moreover, since $\varphi_{G,R}$ does not recolor any vertex having color larger than $k+1$ in κ, it suffices to show that the color of v in κ is larger than $k+1$. Since v and w lie in different chains, v must be greater than at least two different sources of G: the source in the chain of the decomposition that covers v, and the source in the chain that covers w. This implies that $v \in S(G)$, and v has color larger than $k+1$ in κ by Lemma 8.2.4. Therefore, the map $\varphi_{G,R}$ is well-defined. □

Lemma 8.2.6. *The map $\varphi_{G,R}$ is injective, but not surjective.*

Proof. A coloring $\kappa \in K^{|E|}_{(1^k,a,1^{n-k-a})}$ can be recovered from its image $\varphi_{G,R}(\kappa)$ as follows. In κ, every chain contains a vertex colored $k+1$, and the chains that change are precisely those in $\varphi_{G,R}(\kappa)$ that no longer contain a vertex colored $k+1$. Moreover, in $\varphi_{G,R}(\kappa)$, every chain in R has its source colored 1 by the definition of $\varphi_{G,R}$. Thus we can recover κ by, for each chain of R that does not contain a vertex colored $k+1$, recoloring its source with the color $k+1$ and sorting the colors in the chain such that they are increasing from the chain's source to the chain's sink. Hence, the map $\varphi_{G,R}$ is injective.

To show that $\varphi_{G,R}$ is not surjective, we find a coloring $\tilde{\kappa} \in K^{|E|}_{(a,1^{n-a})}$ that is not in the image of $\varphi_{G,R}$. Let v be a vertex in $S(G)$ such that $k = \mathrm{stat}(v)+1$. As $\mathrm{stat}(v) = k-1$, there are $k-1$ nonsource vertices smaller than v. Construct the coloring $\tilde{\kappa}$ by coloring all sources 1, coloring the nonsource vertices smaller than v with the colors $2, \ldots, k$ such that every edge has an ascent, coloring v with $k+1$, and coloring the rest of the vertices with the remaining colors so as to obtain the maximum ascent statistic $|E|$. By Lemma 8.2.4, the vertex v must have color larger than $k+1$

in every coloring in $K^{|E|}_{(1^k,a,1^{n-k-a})}$. Since $\varphi_{G,R}$ does not change the color of any vertex having color larger than $k+1$, $\tilde{\kappa}$ cannot be in the image of $\varphi_{G,R}$. Thus, the map $\varphi_{G,R}$ is not surjective. □

Lemma 8.2.6 gives us the desired result on directed acyclic graphs.

Theorem 1.0.7. Let G be a connected, directed acyclic graph. If G has at least two sources, then $X_G(\mathbf{x};q)$ is not symmetric.

Proof. The fact that $\varphi_{G,R}$ is injective, but not surjective, tells us that in $X_G(\mathbf{x};q)$, the coefficient of $M_{(1^k,a,1^{n-k-a})}$ is strictly smaller than the coefficient of $M_{(a,1^{n-a})}$. Hence the chromatic quasisymmetric function $X_G(\mathbf{x};q)$ is not symmetric. □

Combining Dilworth's Theorem [34] with Lemma 8.1.4 and Theorem 1.0.7, we also obtain the following.

Corollary 8.2.7. *Let G be a connected, directed acyclic graph such that $X_G(\mathbf{x};q)$ is symmetric. Then G must have a directed path from its source to sink that includes every vertex of G.*

As an additional corollary, we obtain the following characterization of all directed trees with symmetric chromatic quasisymmetric functions thereby settling an open question in [24].

Corollary 8.2.8. *Let T be a directed tree. $X_T(x;q)$ is symmetric if and only if T is a directed path.*

Proof. As directed paths are natural unit interval graphs, the symmetry of the chromatic quasisymmetric functions for directed paths follows from Shareshian and Wachs [13].

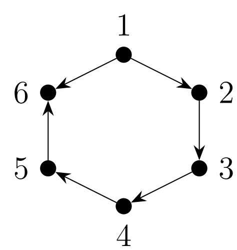
Let T be a directed tree with a symmetric quasisymmetric function. Theorem 1.0.7 implies that T has exactly one source and sink, and Corollary 8.2.7 implies that T must be a directed path from its source to sink. □

We can also characterize which directed acyclic orientations on cycles have symmetric chromatic quasisymmetric functions. A directed acyclic orientation on a cycle is naturally oriented if the cycle has exactly one source and sink with an edge going from its source to sink.

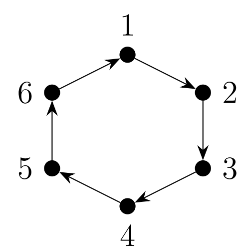
Corollary 8.2.9. *Let C be a directed acyclic cycle graph. Then $X_C(q;t)$ is symmetric if and only if C is naturally oriented.*

Proof. The symmetry of naturally ordered cycles was proven by Ellzey and Wachs [35]. Let C be a directed acyclic cycle such that $X_C(q;t)$ is symmetric. By Corollary 8.2.7, C has a directed path from its source to its sink that includes every vertex of C. In order for C to be a cycle, its source must share edge with its sink implying that C is naturally oriented. □

Remark 8.2.10. Ellzey [36] and Alexandersson and Panova [37] studied a generalization of the chromatic quasisymmetric function that allows for directed cycles. In their papers, they show that directed cycles have a symmetric chromatic quasisymmetric function. Thus, with respect to this more general setting, we have that an oriented cycle has a symmetric chromatic quasisymmetric function if and only if it is a naturally oriented cycle or a directed cycle:



(a) A naturally oriented cycle.



(b) A directed cycle.

Chapter 9

Chromatic quasisymmetric functions for mixed mountain graphs

We begin with the case of ordinary mountain graphs in which only k-cliques are used.

9.1 Mountain graphs

We define a mountain graph as follows.

Definition 9.1.1. The (p, k)**-mountain graph** $M_{p,k}$ is the graph formed by replacing all but one edge in a cycle of length $p + 1$ with a k-clique (we require $p \geq 2$ and $k \geq 2$). The edge that was not replaced is called the **bottom edge**. We call each of the k-cliques the **mountains** of $M_{p,k}$.

Notice that $|V(M_{p,k})| = p(k - 1) + 1$. In this section, we will consider $G = M_{p,k}$.

We assign an ordering to the vertices by drawing the mountain graph in the plane with the bottom edge connecting the leftmost vertex to the rightmost vertex, and the mountains connected left to right in between, with the vertices in each clique that are not along the $p + 1$-cycle all lying strictly between the two endpoints of the clique in left to right order (see Figure 9.1). We then name the vertices $v_1, \ldots, v_n$ left to right in this order. A vertex along the induced $p + 1$-cycle is called a **lower vertex**, and any other vertex is called an **upper vertex**.

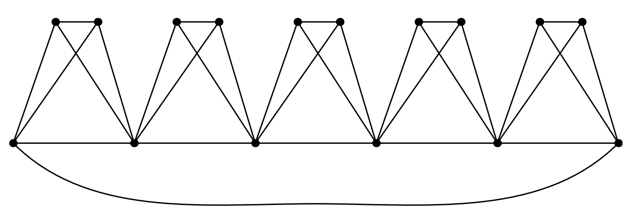


Figure 9.1: The $(5, 4)$-mountain graph

Notably, these graphs are not natural unit interval graphs unless $p = 2$ and $k = 2$ (or a re-labeling of a natural unit interval graph when $p = 2$ and $k \geq 3$). We will show that if G is a

(p,k)-mountain graph, then $X_G(\mathbf{x};q)$ is symmetric. In the case of $(p,2)$-mountain graphs, our work recovers the result that the chromatic quasisymmetric functions of naturally labeled cycles are symmetric [35]. Given a proper coloring $\kappa : V(G) \to \mathbb{N}$, define the $(a, a+1)$**-colored subgraph** to be the induced subgraph on the vertex set $\kappa^{-1}(\{a, a+1\})$.

Lemma 9.1.2. *Let G be a (p,k)-mountain graph, and let $\kappa : V(G) \to \mathbb{N}$ be a proper coloring. Then for all a, the $(a, a+1)$-colored subgraph is:*

- *A cycle of length $p+1$, or;*
- *A (possibly empty) disjoint union of paths, each of the form $v_{i_1} - v_{i_2} - \cdots - v_{i_j}$, with*
  - *$i_1 < i_2 < \cdots < i_j$ or*
  - *$i_1 < i_2 < \cdots < i_k = n$ and $1 = i_{k+1} < i_{k+2} < \cdots < i_j$ for some $2 \le k \le j-1$.*

Proof. If the $(a, a+1)$-colored subgraph contains all of the lower vertices (which is only possible when p is odd), then no other vertex in any k-clique can be colored a or $a+1$. Thus the subgraph is a cycle of length $p+1$.

Define G' to be the graph G without the bottom edge, so G' is a natural unit interval graph. If vertices v_1 and v_n are not both colored at least one of a or $a+1$, then the $(a, a+1)$-colored subgraph is contained in G'. By [13] (Lemma 4.4), the $(a, a+1)$-colored subgraph is a disjoint union of paths, each of the form $v_{i_1} - v_{i_2} - \cdots - v_{i_j}$, with $i_1 < i_2 < \cdots < i_j$.

Now suppose that v_1 and v_n are colored a or $a+1$, and not every lower vertex is colored a or $a+1$. Then in G', vertices v_1 and v_n are in different connected components of the $(a, a+1)$-colored subgraph. Again, since G' is a natural unit interval graph, these connected components form paths $v_1 = v_{i_{k+1}} - \cdots - v_{i_j}$ and $v_{i_1} - \cdots - v_{i_k} = v_n$ such that $i_{k+1} < \cdots < i_j$ and $i_1 < \cdots < i_k$. In G, this yields a path of the form $v_{i_1} - v_{i_2} - \cdots - v_{i_j}$, which includes the bottom edge. □

Notice that for any coloring of a (p,k)-mountain graph, if the $(a, a+1)$-colored subgraph contains a path, then the middle vertices of the path cannot be upper vertices of the mountain graph.

Let G be a (p,k)-mountain graph, and define $K_{a,a+1}(G)$ to be the set of proper colorings of G such that the $(a,a+1)$-colored subgraph does not include the bottom edge.

Proposition 9.1.3. *There is an ascent-preserving involution on $K_{a,a+1}(G)$ which swaps the number of occurrences of the colors a and $a+1$.*

Proof. As above, define G' to be the graph G without the bottom edge. Since G' is a natural unit interval graph, the above statement follows from the proof of [13], Theorem 4.5. We summarize this argument here for clarity.

From Lemma 9.1.2, for $\kappa \in K_{a,a+1}(G)$, each connected component of the $(a,a+1)$-colored subgraph is a path of the form $v_{i_1} - v_{i_2} - \cdots - v_{i_j}$, with $i_1 < i_2 < \cdots < i_j$. Let $\psi_a(\kappa)$ be the coloring of G obtained by swapping a and $a+1$ exactly when the number of vertices in the connected path is odd (if this number is even, leave the coloring of the component unchanged). Notice that $\psi_a(\kappa)$ is still a proper coloring.

Since each path with an even number of vertices contains the same number of vertices colored a and $a+1$, it follows that $\psi_a(\kappa)$ swaps the number of instances of colors a and $a+1$. The paths with an odd number of vertices have an even number of edges, so swapping a and $a+1$ preserves the number of ascents in these paths. Further, since a and $a+1$ are adjacent colors, ascents between a vertex in one of these paths and a vertex colored b with $b \neq a$ and $b \neq a+1$ are preserved. $\square$

Now define $L_{a,a+1}(G)$ to be the set of proper colorings of G such that the $(a,a+1)$-colored subgraph includes the bottom edge.

Proposition 9.1.4. *There is an ascent-preserving automorphism on $L_{a,a+1}(G)$ which swaps the number of occurrences of the colors a and $a+1$.*

To prove this, we first establish several lemmas about operations that we will put together to show the symmetry.

Definition 9.1.5. Let $a > 1$, and let $\kappa \in L_{a,a+1}(G)$ be a coloring with maximal color c. We define a new coloring $\mathrm{cycle}(\kappa) \in L_{a-1,a}(G)$ as follows:

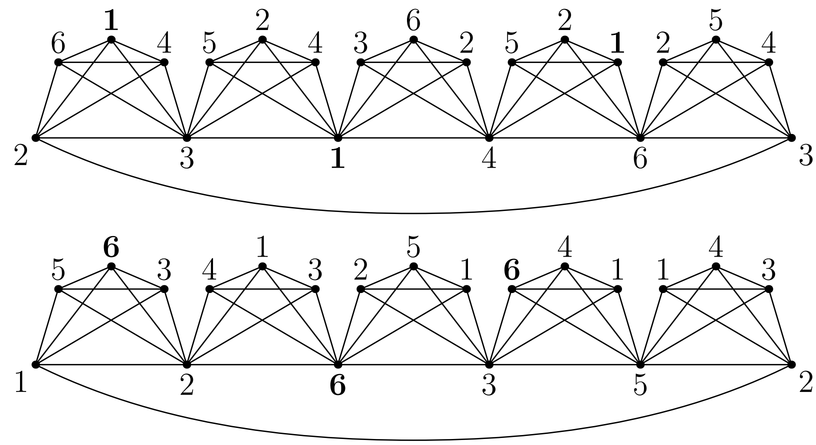


Figure 9.2: Above, an element $\kappa \in L_{2,3}(G)$, and below, the output $\mathrm{cycle}(\kappa) \in L_{1,2}(G)$.

- Change all vertices colored 1 to color $c+1$.
- For each vertex colored $c+1$:
  - If the vertex is a lower vertex, do nothing.
  - If the vertex is an upper vertex, suppose that it is the i-th upper vertex from the left in its clique. Then reorder the upper vertices in the clique so that this vertex is the i-th upper vertex from the right, while the relative order of the other vertices is preserved.
- Reduce the value of all colors by 1.

Note that the cycle operation applies a cyclic permutation to the content of κ. An example of the construction $\mathrm{cycle}(\kappa)$ is given in Figure 9.2.

Lemma 9.1.6. *Let $a > 1$. Then $\mathrm{cycle}\colon L_{a,a+1}(G) \to L_{a-1,a}(G)$ is an ascent-preserving bijection.*

Proof. As each step of cycle is invertible, the map cycle is bijective. It remains to show that it is ascent preserving. Suppose that $\kappa \in L_{a,a+1}(G)$ has largest color c. We will show that the number of ascents within each clique is preserved, and that any ascent formed by the bottom edge is preserved.

First, suppose that C is a clique in κ with no vertex colored 1. Then the only change to C is that the color of every vertex is reduced by 1, which preserves the relative orders of the vertices incident to each edge. Thus $\mathrm{cycle}(\kappa)$ has the same number of ascents within this clique.

Next, suppose that C is a clique of κ containing at least one (and thus, exactly one) vertex colored 1. We have two cases. If v is a lower vertex, then v cannot be the leftmost vertex v_1 or rightmost vertex v_n, since $\kappa \in L_{a,a+1}$ implies $\kappa(v_1)$ and $\kappa(v_n)$ are a or $a+1$. Thus v is connected to two of the cliques of G. Since v is colored 1, it forms an ascent with each of the $k-1$ vertices in the clique to its right, and no ascents with the vertices in the clique to its left. When the color of v is changed to $c+1$, since this color is greater than any present in κ, v forms an ascent with each vertex in the clique to its left, and none with the clique on its right. Since κ is a proper coloring, no other vertex in the cliques adjacent to v could have color 1, so no other vertices gain the color $c+1$. Reducing the value of each color by 1 again preserves the relative order of the vertices on each edge, so $\mathrm{cycle}(\kappa)$ has the same number of ascents within the cliques adjacent to v.

If v is an upper vertex, then since v has color 1, it forms an ascent with the vertices to its right within its clique - suppose there are ℓ such ascents. Recoloring v with $c+1$, and then moving v so that it is to the left of ℓ of the vertices in the clique preserves the number of ascents formed on an edge incident to v. Since the relative orders of the other vertices in the clique are preserved, the ascents between these vertices are preserved. Reducing the color of each vertex by 1 again preserves the number of ascents, so $\mathrm{cycle}(\kappa)$ has the same number of ascents within the clique containing v.

Lastly, for the bottom edge, the two vertices incident to the bottom edge are not labeled 1 because $a > 1$ and $\kappa \in L_{a,a+1}(G)$. Reducing the value of these colors by 1 does not change whether the bottom edge forms an ascent. Combining these cases, we get that $\mathrm{asc}(\kappa) = \mathrm{asc}(\mathrm{cycle}(\kappa))$, as desired. $\square$

Definition 9.1.7. Suppose $a = 1$, and let $\kappa \in L_{a,a+1}(G) = L_{1,2}(G)$ with largest color c. Define $\mathrm{reflect}(\kappa)$ to be the coloring obtained from κ as follows:

- Reflect κ horizontally.
- Recolor all 1's as 2's and vice versa.
- Recolor all 3's as c's, all 4's as $c-1$'s, and so on.

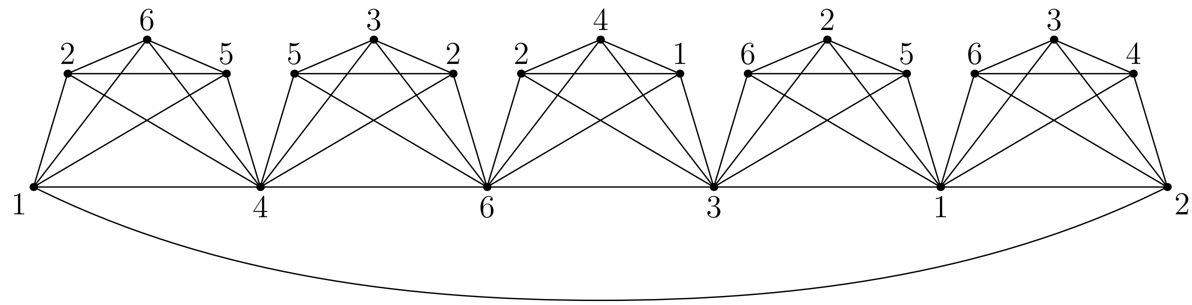


Figure 9.3: The output $\mathrm{reflect}(\mathrm{cycle}(\kappa))$ for κ as in Figure 9.2.

- Within each k-clique C of κ, consider the original positions of any 1 or 2 appearing in the upper entries (which can be no positions, a single index i, or two indices i and j from left to right). Then move any new 1's or 2's in the image of C under reflection to positions i or i, j in left to right order, keeping the relative order of the other upper entries the same.

See Figure 9.3 for an example of this construction.

Lemma 9.1.8. *The map* $\mathrm{reflect} : L_{1,2}(G) \to L_{1,2}(G)$ *is an ascent-preserving automorphism.*

Proof. As each step of $\mathrm{reflect}$ is invertible, it is bijective. It remains to show that it is ascent preserving. As in Definition 9.1.7, let $\kappa \in L_{1,2}(G)$ and let c denote the largest color in κ. Consider an edge e of G whose vertices are colored by κ with colors between 3 and c inclusive. In $\mathrm{reflect}(\kappa)$, these vertices are sent to a new edge $\tilde{e}$ of G. The horizontal reflection of the colors in κ followed by the swapping of all colors between 3 and c preserves the relative order of the colors of vertices incident to the edge e. Thus, e contributes to $\mathrm{asc}(G)$ if and only if $\tilde{e}$ contributes to $\mathrm{asc}(\mathrm{reflect}(\kappa))$.

Similarly, any edge of G whose vertices are colored with only the numbers 1 and 2 in κ contributes to $\mathrm{asc}(\kappa)$ if and only if its image in $\mathrm{reflect}(\kappa)$ contributes to $\mathrm{asc}(\mathrm{reflect}(\kappa))$. Note that this implies the bottom edge under κ contributes an ascent if and only if it contributes an ascent in $\mathrm{reflect}(\kappa)$.

We now examine the number of ascents in every k-clique under both colorings. By the previous paragraphs it suffices to only consider edges where exactly one vertex is colored 1 or 2. Let C be a k-clique of G under the coloring κ and let $\tilde{C}$ be its image under the coloring $\mathrm{reflect}(\kappa)$. We break into cases based on the colors of the lower vertices of C.

If the lower vertices of C are colored 1 and 2 in any order, then the edges where exactly one of the vertices is colored 1 or 2 are precisely the edges between the upper and lower vertices of C and $\tilde{C}$. As 1 and 2 are the smallest colors, the number of ascents for edges between upper vertices and lower vertices in both C and $\tilde{C}$ is $k-2$.

If the lower vertices of C are colored between 3 and c inclusive, then the positions of any vertices colored 1 or 2 in C are the same in $\tilde{C}$. This implies that the number of ascents for edges where exactly one of the vertices is colored 1 or 2 is the same in both C and $\tilde{C}$.

Now assume exactly one of the lower vertices of C is colored 1 or 2. We define a k-clique to be "left-colored" if the only lower vertex colored 1 or 2 in the k-clique is its left lower vertex. Similarly define the notion of being "right-colored". There exists a natural pairing of left-colored k-cliques to right-colored k-cliques by pairing a left-colored k-clique with the leftmost right-colored k-clique that sits to its right in G. Note that as v_1 and v_n are colored 1 and 2 (in some order), such a pairing must exist. Moreover, any k-clique falling between a paired left-colored and right-colored k-clique cannot have any lower vertices colored 1 or 2. We show that the total number of ascents within a pair of such k-cliques is equal to the total number of ascents within its image in $\text{reflect}(\kappa)$.

Without loss of generality, assume that C is left-colored and let D be its corresponding right-colored k-clique. Additionally, let $\tilde{C}$ and $\tilde{D}$ be the images of C and D under the coloring $\text{reflect}(\kappa)$ respectively. Recall that the position of any upper vertex colored 1 or 2 within a k-clique is the same as its position within the image of the k-clique in $\text{reflect}(\kappa)$. Thus, the number of ascents for edges between upper vertices where exactly one vertex is colored 1 or 2 is the same for C and $\tilde{C}$ and for D and $\tilde{D}$. Moreover, C and $\tilde{D}$ will each have an ascent between their lower vertices while $\tilde{C}$ and D do not.

We now consider edges between upper and lower vertices where exactly one vertex is colored 1 or 2. If C has no upper vertices colored 1 or 2, then there are $k-2$ ascents between the left lower vertex and upper vertices of C. If however, C has an upper vertex w colored 1 or 2, then there are $k-3$ ascents between the left lower vertex and upper vertices not colored 1 or 2. Note that the edge between w and the right lower vertex of C also forms an ascent. In either case, there are

$k-2$ ascents for edges between upper and lower vertices of C where exactly one vertex is colored 1 or 2. Similarly, $\tilde{D}$ has $k-2$ ascents for edges between upper and lower vertices where exactly one vertex is colored 1 or 2. A routine check tells us that there are no such ascents for the cliques D and $\tilde{C}$. Therefore, the total number of ascents in C and D is the same as the total number of ascents in $\tilde{C}$ and $\tilde{D}$.

Combining all of the above cases gives us the desired result $\mathrm{asc}(\mathrm{reflect}(\kappa)) = \mathrm{asc}(\kappa)$.

$\square$

We now have the tools to prove Proposition 9.1.4.

Proof. Let $\kappa \in L_{a,a+1}(G)$, and suppose its color content is

$$(c_1, \ldots, c_n).$$

We first apply cycle exactly $a-1$ times, so that $\kappa' := \mathrm{cycle}^{a-1}(\kappa)$ is in $L_{1,2}(G)$, with content

$$(c_a, c_{a+1}, c_{a+2}, \ldots, c_n, c_1, \ldots, c_{a-1}),$$

and has the same number of ascents as κ. Then we apply $\mathrm{reflect}$ to form $\kappa'' \in L_{1,2}(G)$ with content equal to

$$(c_{a+1}, c_a, c_{a-1}, \ldots, c_1, c_n, \ldots, c_{a+2}).$$

Finally, fix a reduced word for the permutation that reverses entries 3 through n, say,

$$(\sigma_3)(\sigma_4\sigma_3)(\sigma_5\sigma_4\sigma_3)\cdots(\sigma_{n-1}\cdots\sigma_3)$$

We then apply the automorphisms from Proposition 9.1.3 to apply these transpositions σ_i to the content in order, to obtain κ''' with content

$$(c_{a+1}, c_a, c_{a+2}, \ldots, c_n, c_1, \ldots, c_{a-1}).$$

Finally, we apply cycle^{-1} to κ''' exactly $a-1$ times to obtain a coloring κ'''' with content

$$(c_1, \ldots, c_{a-1}, c_{a+1}, c_a, c_{a+2}, \ldots, c_n).$$

The automorphism $\kappa \to \kappa''''$ is well-defined, bijective, and preserves ascents, because every step of the construction has these properties. This completes the proof. □

Combining Propositions 9.1.3 and 9.1.4, we obtain the following result.

Theorem 9.1.9. *Let $M_{p,k}$ be a (p,k)-mountain graph. Then $X_{M_{p,k}}(x;q)$ is symmetric.*

9.2 Bottomless Mountain graphs

From the family of (p,k)-mountain graphs, we construct another family of graphs whose chromatic quasisymmetric function is symmetric.

Definition 9.2.1. For $k \geq 3$, the (p,k)**-bottomless mountain graph** $B_{p,k}$ is the (p,k)-mountain graph $M_{p,k}$ where all edges between lower vertices are removed except for the bottom edge.

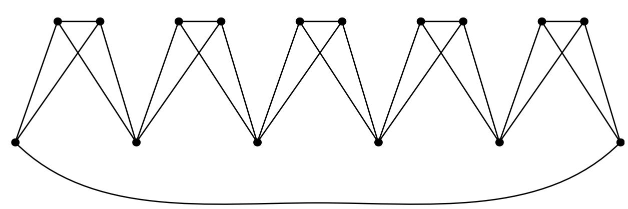


Figure 9.4: The $(5,4)$-bottomless mountain graph

We keep the same ordering on the vertices of $B_{p,k}$ as for $M_{p,k}$ as well as the definitions of the bottom edge, lower vertices, and upper vertices. To show the chromatic quasisymmetric function of $B_{p,k}$ is symmetric, it suffices to show that the ascent-preserving automorphisms presented in Section 9.1 for the colorings of $M_{p,k}$ hold true for $B_{p,k}$ as well.

Let $L_{a,a+1}(B_{p,k})$ (resp. $K_{a,a+1}(B_{p,k})$) be the set of proper colorings such that the $(a, a+1)$-colored subgraph does (resp. does not) include the bottom edge. As removing the bottom edge from any bottomless mountain results in a natural unit interval order, the involution given in Proposition 9.1.3 is also an ascent-preserving involution on $K_{a,a+1}(B_{p,k})$ that swaps the number of occurrences of the colors a and $a+1$.

Lemma 9.1.6 clearly holds for the graphs $B_{p,k}$, that is the cycle map gives an ascent-preserving map from $L_{a,a+1}(B_{p,k})$ to $L_{a-1,a}(B_{p,k})$ for all $a > 1$. In order for the proof of Proposition 9.1.4 to pass through for the graphs $B_{p,k}$, it remains to show that the $\mathrm{reflect}$ map is ascent-preserving automorphism on $L_{1,2}(B_{p,k})$.

Lemma 9.2.2. *Let* $\kappa \in L_{1,2}(B_{p,k})$*. Then* $\mathrm{asc}(\mathrm{reflect}(\kappa)) = \mathrm{asc}(\kappa)$.

Proof. As in Lemma 9.1.8 it suffices to consider edges where exactly one vertex is colored either 1 or 2 and the other vertex is not. To show that $\mathrm{asc}(\mathrm{reflect}(\kappa)) = \mathrm{asc}(\kappa)$, we examine the number of ascents coming from these edges in every bottomless k-clique. Let C be a bottomless k-clique of G under the coloring κ and let $\tilde{C}$ be its image under the coloring $\mathrm{reflect}(\kappa)$. We break into cases based on the colors of the lower vertices of C.

All cases follow directly from the proof of Lemma 9.1.8 with the exception of when the lower vertices are colored 1 or 2. In this case, it is possible for both lower vertices of C to have the same color and for an upper vertex to be colored with the opposite color in $\{1, 2\}$. The number of ascents in both C and $\tilde{C}$ coming from edges where exactly one vertex is colored 1 or 2 is then $k-3$. Thus, $\mathrm{asc}(\mathrm{reflect}(\kappa)) = \mathrm{asc}(\kappa)$ for all $\kappa \in L_{1,2}(B_{p,k})$. □

As the analogues of Lemmas 9.1.6 and 9.1.8 hold for the bottomless graphs $B_{p,k}$, the proof of Proposition 9.1.4 gives an ascent-preserving automorphism on $L_{a,a+1}(B_{p,k})$ that swaps the number of occurrences of the colors a and $a+1$. Pairing this ascent-preserving automorphism on $L_{a,a+1}(B_{p,k})$ with the ascent-preserving involution on $K_{a,a+1}(B_{p,k})$ gives an ascent-preserving automorphism on the set of all proper coloring of $B_{p,k}$ that interchanges the number of occurrences of the colors a and $a+1$. This proves the desired symmetry.

Theorem 9.2.3. *Let* $B_{p,k}$ *be a* (p,k)*-bottomless-mountain graph. Then* $X_{B_{p,k}}(x;q)$ *is symmetric.*

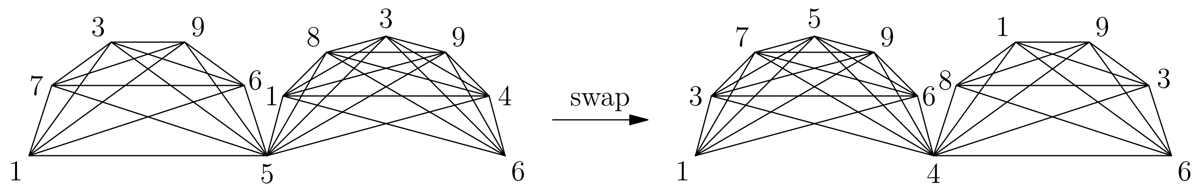


Figure 9.5: An example of the swap map, showing that we can swap a k-clique with an adjacent bottomless $k+1$-clique without changing the chromatic quasisymmetric function.

9.3 Mixed Mountain Graphs

A (p,k)-mixed mountain graph is a (p,k)-mountain graph where some number of the k-clique mountains are replaced with bottomless $k+1$-clique mountains.

We wish to define the swap map, which will swap a k-clique with a bottomless $k+1$-clique to its right in a way that preserves ascents. We first construct several auxiliary definitions.

Let G be a mixed mountain graph. Suppose $\mathcal{U}$ is a k-clique in G, and there is a bottomless $k+1$-clique $\mathcal{W}$ immediately to its right, with shared vertex $v \in \mathcal{U} \cap \mathcal{W}$. Then we name the remaining vertices of $\mathcal{U}$ by $u, u_1, \ldots, u_{k-2}$ from left to right, and the remaining vertices of $\mathcal{W}$ by $w_1, w_2, \ldots, w_{k-1}, w$.

Definition 9.3.1. Define the graph $\mathrm{swap}_v(G)$ to be the graph formed by replacing $\mathcal{U}$ with a bottomless $k+1$-clique, and replacing $\mathcal{W}$ with a k-clique.

Definition 9.3.2. Suppose κ is a proper coloring of G. Then we define the "smaller-colored" and "larger-colored" sets of upper vertices by

$$S_{\mathcal{U}} = \{u_i \in \mathcal{U} \mid \kappa(u_i) < \kappa(v)\} \qquad L_{\mathcal{U}} = \{u_i \in \mathcal{U} \mid \kappa(u_i) > \kappa(v)\}.$$

Define $S_{\mathcal{W}}$ and $L_{\mathcal{W}}$ similarly.

Example 9.3.3. In the graph in Figure 9.5 at left, we have $S_{\mathcal{U}} = \{u_2\}$, $L_{\mathcal{U}} = \{u_1, u_3, u_4\}$, $S_{\mathcal{W}} = \{w_1, w_3, w_5\}$, and $L_{\mathcal{W}} = \{w_2, w_4\}$. These correspond, respectively, to the color 3 on the left, the colors $7, 9, 6$ on the left, the colors $1, 3, 4$ on the right, and the colors $8, 9$ on the right.

Definition 9.3.4. The **special vertex** of a triple $(\mathcal{U}, \mathcal{W}, \kappa)$ is one of the w_i vertices, defined as follows.

- If $|S_{\mathcal{U}}| < |S_{\mathcal{W}}|$, then list the vertices of $S_{\mathcal{U}} \cup S_{\mathcal{W}}$ in decreasing order of vertex color, breaking ties by listing the elements of $S_{\mathcal{U}}$ first. Associate an open parenthesis to elements of $S_{\mathcal{U}}$, closed parenthesis to elements of $S_{\mathcal{W}}$, and pair them. Let w_i be the leftmost unpaired entry from $S_{\mathcal{W}}$ in this list.

- If $|L_{\mathcal{U}}| < |L_{\mathcal{W}}|$, then order the vertices of $L_{\mathcal{U}} \cup L_{\mathcal{W}}$ in increasing order of vertex color, breaking ties by listing the elements of $L_{\mathcal{U}}$ first. Associate an open parenthesis to elements of $L_{\mathcal{U}}$, a closed parenthesis to elements of $L_{\mathcal{W}}$, and pair them. Let w_i be the leftmost unpaired entry from $L_{\mathcal{W}}$ in this list.

Example 9.3.5. In the example above, we have $|S_{\mathcal{U}}| < |S_{\mathcal{W}}|$, so we list the vertices in decreasing order of vertex color. We list the colors, then the vertex labels, and then the parenthesization in the table below:

4	3	3	1
w_5	u_2	w_3	w_1
)	(	)	)

The leftmost unpaired entry is w_5, with color 4, so this is the special vertex.

Lemma 9.3.6. *The special vertex w_i for the triple $(\mathcal{U}, \mathcal{W}, \kappa)$ exists and is well-defined.*

Proof. Since κ is a proper coloring, the sets of colors $\kappa(\mathcal{U}) := \{\kappa(u_1), \ldots, \kappa(u_{k-2})\}$ and $\kappa(\mathcal{W}) := \{\kappa(w_1), \ldots, \kappa(w_{k-1})\}$ contain $k-2$ and $k-1$ elements, respectively. Further, $\mathcal{U} = S_{\mathcal{U}} \cup L_{\mathcal{U}}$ and $\mathcal{W} = S_{\mathcal{W}} \cup L_{\mathcal{W}}$ are unions of disjoint sets. Since $|\mathcal{U}| + 1 = |\mathcal{W}|$, we must have exactly one of $|S_{\mathcal{U}}| < |S_{\mathcal{W}}|$ or $|L_{\mathcal{U}}| < |L_{\mathcal{W}}|$. In either case, when associating parenthesis to the list of vertices, there are more vertices from $\mathcal{W}$ in the list, so there is some associated unpaired closed parenthesis. Hence the special vertex is well-defined. □

We now define the swap map for a coloring κ of G.

Definition 9.3.7. We define $\text{swap}(\kappa) : V(\text{swap}_v(G)) \to \mathbb{N}$ as follows. Color the vertices outside of $\mathcal{U} \cup \mathcal{W}$, as well as vertices u and w, with the same color as κ for G. Color the single vertex $v \in \mathcal{U} \cap \mathcal{W}$ with $\kappa(w_i)$, where w_i was the special vertex of $(\mathcal{U}, \mathcal{W}, \kappa)$. Then color the upper vertices of the new k-clique using the colors $\kappa(w_1), \ldots, \kappa(w_{i-1}), \kappa(w_{i+1}), \ldots, \kappa(w_{k-1})$ in the same relative order as $\kappa(u_1), \ldots, \kappa(u_{k-2})$, and the upper vertices of the new bottomless $k+1$-clique using the colors $\kappa(u_1), \ldots, \kappa(u_{k-2}), \kappa(v)$ in the same relative order as $\kappa(w_1), \ldots, \kappa(w_{k-1})$.

See Figure 9.5 for an example.

Lemma 9.3.8. *Suppose that G is a (p, k)-mixed mountain graph and κ is a proper coloring of G. Then* $\text{swap}(\kappa)$ *is a proper coloring, and κ and* $\text{swap}(\kappa)$ *have the same number of ascents.*

Proof. Let κ be a proper coloring of G, and let $\text{swap}(\kappa)$ be the associated coloring of $\text{swap}_v(G)$.

First, we show that $\text{swap}(\kappa)$ is a proper coloring. Let x and x' be adjacent vertices in $\text{swap}_v(G)$. If neither of x and x' are in either of the swapped cliques $\mathcal{U}$ and $\mathcal{W}$, then they are assigned the same colors in κ and $\text{swap}(\kappa)$, so $\text{swap}(\kappa)(x) \neq \text{swap}(\kappa)(x')$. If x and x' are both upper vertices in the new k-clique or bottomless $k+1$-clique, then $\text{swap}(\kappa)$ assigns them distinct colors from $\kappa(\mathcal{W} \setminus \{w_i\})$ or $\kappa(\mathcal{U} \cup \{v\})$, respectively. If x is an upper vertex and x' is not, then x' is one of the neighboring lower vertices of the new k-clique or bottomless $k+1$-clique. In either case, the colors of the vertices neighboring x in $\text{swap}_v(G)$ are a subset of the colors of the vertices neighboring the vertex with color $\text{swap}(\kappa)(x)$ in G. If x and x' are both lower vertices, then they must be the lower vertices of the new k-clique. In this case, $\text{swap}(\kappa)$ assigns x and x' colors from two adjacent vertices in $\mathcal{W}$. Hence x and x' are assigned different colors in $\text{swap}(\kappa)$.

Next, we show that κ and $\text{swap}(\kappa)$ have the same number of ascents. First, if e is an edge between any two vertices not in $\mathcal{U}$ or $\mathcal{W}$, then $\text{swap}(\kappa)$ assigns the same color to these vertices as κ, so it has an ascent exactly if κ has an ascent. If e is an edge between the upper vertices of the new k-clique, then these vertices were colored so as to maintain the relative order of colors of upper vertices of $\mathcal{U}$, so the edge e has an ascent if and only if the corresponding edge in the original k-clique $\mathcal{U}$ has an ascent. Likewise, if e is an edge between the upper vertices of the new bottomless $k+1$-clique, then since the ordering of the colors of the upper vertices is maintained, e has an ascent if and only if the corresponding edge in $\mathcal{W}$ has an ascent in κ.

Consider the set of edges between the left vertex u and upper vertices of the new bottomless $k+1$-clique. Since these upper vertices were assigned colors from $\kappa(\mathcal{U})$, which were the set of colors adjacent to u in G, these edges contain the same number of ascents as in the coloring κ. By the same argument, the edges between the upper vertices of the new k-clique and w contain the same number of ascents in κ and $\operatorname{swap}(\kappa)$.

Finally, consider the set of edges between the middle vertex v and an upper vertex in either clique. In κ, the number of ascents formed by these edges is $|S_{\mathcal{U}}| + |L_{\mathcal{W}}|$. If $|S_{\mathcal{U}}| < |S_{\mathcal{W}}|$, then consider the pairing defined in Definition 9.3.4, and let w_i be the special vertex with color $c = \kappa(w_i)$. Let $a = \kappa(v)$ be the color of v. Then $c = \operatorname{swap}(\kappa)(v)$, and since c was an unpaired color, each set of paired vertices are either both colored less or both colored greater than c. In particular, if u_j is paired with w_k, then the edges u_j — v — w_k contain exactly one ascent in both κ and $\operatorname{swap}(\kappa)$. By the definition of w_i, all unpaired vertices from $S_{\mathcal{U}}$ and $S_{\mathcal{W}}$ have color at most c, so the number of ascents in κ between these unpaired vertices and v is the same as the number of ascents in $\operatorname{swap}(\kappa)$ between these vertices. (Note that the edge between v and w_i corresponds to an edge between the new vertex labeled a in the k-clique and v, which will still be an ascent in $\operatorname{swap}(\kappa)$ if and only if v — w_i formed an ascent in κ.)

Since entries from $L_{\mathcal{U}}$ and $L_{\mathcal{W}}$ are given color greater than $\kappa(v)$ and $\operatorname{swap}(\kappa)(v)$, we get that the number of ascents formed by edges between $\mathcal{U}$, $\mathcal{W}$, and v in $\operatorname{swap}(\kappa)$ is also $|S_{\mathcal{U}}| + |L_{\mathcal{W}}|$.

An analogous argument shows that ascents are preserved in the case $|L_{\mathcal{U}}| < |L_{\mathcal{W}}|$. Therefore we have $\operatorname{asc}(\kappa) = \operatorname{asc}(\operatorname{swap}(\kappa))$. □

Lemma 9.3.9. *The map* $\kappa \mapsto \operatorname{swap}(\kappa)$ *is a bijection between colorings of* G *and* $\operatorname{swap}_v(G)$.

Proof. We first claim that it suffices to show that the map is injective. If it is, we can compose swap maps to form an injective map from colorings of the mixed mountain graph G_0 having exactly N k-cliques followed by M bottomless $k+1$-cliques, to the graph G_1 having M bottomless $k+1$-cliques followed by N k-cliques, which by symmetry have the same total number of proper colorings.

Thus this composed map is a bijection, so every intermediate swap map is a bijection as well (and every mixed mountain graph can be obtained via some sequence of swaps starting from G_0).

Suppose for contradiction that there exists colorings κ and ρ of G which both map to the coloring $\mathrm{swap}(\kappa) = \mathrm{swap}(\rho)$ on $\mathrm{swap}_v(G)$. For simplicity, we set $\kappa' := \mathrm{swap}(\kappa)$ and $\rho' = \mathrm{swap}(\rho)$. For a complete subgraph $H \subset G$, let $\kappa(H)$ be the set of colors on vertices of H. By construction of κ', since $\kappa' = \rho'$, we must have $\kappa(\mathcal{U}) = \rho(\mathcal{U})$ and $\kappa(\mathcal{W} \setminus \{v\}) = \rho(\mathcal{W} \setminus \{v\})$. If $\kappa(v) = \rho(v)$, then since swap preserves the relative orders of the upper vertices, we must have that $\kappa = \rho$.

Now assume that $\kappa(v) \neq \rho(v)$, and without loss of generality suppose that $\kappa(v) < \rho(v)$. We will show that $\kappa'(v) \neq \rho'(v)$. Let $S^\kappa_\mathcal{U}$ be the set of vertices of $\mathcal{U}$ colored smaller than $\kappa(v)$ and $S^\rho_\mathcal{U}$ be the set of vertices of $\mathcal{U}$ colored smaller than $\rho(v)$. Similarly define $L^\kappa_\mathcal{U}, L^\rho_\mathcal{U}$, $S^\kappa_\mathcal{W}$, $S^\rho_\mathcal{W}$, $L^\kappa_\mathcal{W}$, and $L^\rho_\mathcal{W}$. We have several cases:

Suppose that $|S^\kappa_\mathcal{U}| < |S^\kappa_\mathcal{W}|$ and $|L^\rho_\mathcal{U}| < |L^\rho_\mathcal{W}|$. Then in ρ', the color for v is chosen from the colors of $L^\rho_\mathcal{W}$, and in κ', the color for v is chosen from $S^\kappa_\mathcal{U}$. So we have $\rho'(v) > \rho(v) > \kappa(v) > \kappa'(v)$.

Suppose that $|S^\kappa_\mathcal{U}| < |S^\kappa_\mathcal{W}|$ and $|S^\rho_\mathcal{U}| < |S^\rho_\mathcal{W}|$. Let s^κ be the sequence consisting of vertices of $S^\kappa_\mathcal{U} \cup S^\kappa_\mathcal{W}$, ordered in decreasing order of vertex color, breaking ties by listing elements of $S^\kappa_\mathcal{U}$ first. Let s^ρ be the similarly defined word on $S^\rho_\mathcal{U} \cup S^\rho_\mathcal{W}$.

Since $\kappa(\mathcal{W} \setminus \{v\}) = \rho(\mathcal{W} \setminus \{v\})$ and the relative order for the colors of upper vertices in $\mathcal{W}$ must match the relative order of colors in κ' for the upper vertices of the $k+1$-bottomless clique , we have that κ and ρ color the upper vertices of $\mathcal{W}$ exactly the same. So, since $\kappa(v) < \rho(v)$, we have $S^\kappa_\mathcal{W} \subseteq S^\rho_\mathcal{W}$. Next, because $\kappa(\mathcal{U}) = \rho(\mathcal{U})$, and the order of upper vertices of $\mathcal{U}$ is the same in κ and ρ, we have that $S^\kappa_\mathcal{U} \subseteq S^\rho_\mathcal{U}$. Further, κ and ρ only differ in the positions of upper vertices u_j with $\kappa(v) \leq \kappa(u_j) \leq \rho(v)$. Therefore s^κ is a subword of s^ρ, and in fact since they are both decreasing we have $s^\rho = \tilde{s}s^\kappa$ for some word $\tilde{s}$.

Additionally, the vertex $x \in \mathcal{U}$ with color $\rho(x) = \kappa(v)$ is the first character to the left of s^κ in s^ρ, since it is colored smallest among $S^\rho_\mathcal{U} \setminus S^\kappa_\mathcal{U}$ and there can not be any colors tied with $\kappa(v)$. Since x is associated to an open parenthesis in s^ρ, it pairs with the leftmost unpaired closed parenthesis in s^κ. Therefore, the leftmost unpaired closed parenthesis in s^κ is different than in s^ρ, so $\kappa'(v) \neq \rho'(v)$, contradicting our assumption.

The two cases when $|L^\kappa_{\mathcal{U}}| < |L^\kappa_{\mathcal{W}}|$ follow similarly. Therefore, the map sending (G, κ) to $(\mathrm{swap}_v(G), \mathrm{swap}(\kappa))$ is injective. □

Since swap is a bijection, we can equivalently say that swap^{-1} is an ascent-preserving map on colorings for graphs where the swapped k-clique is to the right of the bottomless $k+1$-clique. Because of Lemmas 9.3.8 and 9.3.9, we conclude the following.

Theorem 9.3.10. *If G is a (p, k)-mixed mountain graph, and $\mathrm{swap}_v(G)$ is the (p, k)-mixed mountain graph obtained by swapping an adjacent k-clique and bottomless $k+1$-clique, then $X_G(\mathbf{x}; q) = X_{\mathrm{swap}_v(G)}(x; q)$.*

Remark 9.3.11. Note that the proof of the above theorem does not rely on the presence of the bottom edge in the mixed-mountain graph. Thus, this also implies a result on natural unit interval graphs. Let G be a natural unit interval graph obtained by deleting the bottom edge from a (p, k)-mixed mountain graph, and let H be the natural unit interval graph obtained by deleting the bottom edge from $\mathrm{swap}_v(G)$. Then $X_G(\mathbf{x}; q) = X_H(x; q)$.

As a consequence of Theorem 9.3.10, it suffices to show the chromatic quasisymmetric functions of (p, k)-mixed mountain graphs where every k-clique is to the left of a bottomless $(k+1)$-clique are symmetric. We denote by $M_{p,k,m}$ the (p, k)-mixed mountain graph with m k-cliques followed by $p - m$ bottomless $(k+1)$-cliques. Let $L_{a,a+1}(M_{p,k,m})$ denote the set of colorings whose $(a, a+1)$-colored subgraph includes the bottom edge, and $K_{a,a+1}(M_{p,k,m})$ denote the set of proper colorings of $M_{p,k,m}$ such that the $(a, a+1)$-colored subgraph does not include the bottom edge. As before, we demonstrate ascent-preserving automorphisms on the sets $L_{a,a+1}(M_{p,k,m})$ and $K_{a,a+1}(M_{p,k,m})$ which swap the number of occurrences of the colors a and $a+1$.

Removing the bottom edge of $M_{p,k,m}$ results in a unit interval graph. Thus, the proof of Proposition 9.1.3 immediately gives an ascent-preserving involution on $K_{a,a+1}(M_{p,k,m})$. In order to find an ascent-preserving automorphism on $L_{a,a+1}(M_{p,k,m})$, we make use of the swap, cycle and reflect maps. We first show that cycle and $\mathrm{reflect}$ are still ascent-preserving when generalized to the set $L_{a,a+1}(M_{p,k,m})$.

Lemma 9.3.12. *Suppose* $\kappa \in L_{a,a+1}(M_{p,k,m})$ *with* $a > 1$. *Then* $\mathrm{asc}(\mathrm{cycle}(\kappa)) = \mathrm{asc}(\kappa)$.

Proof. By the proof of Lemma 9.1.6 and the discussion in Section 9.2, it remains to consider the case where the lower vertex v shared between the adjacent k-clique and bottomless $(k+1)$-clique is colored 1. This vertex will have exactly $k-1$ ascents in κ coming from edges to upper vertices of the bottomless $(k+1)$-clique. In $\mathrm{cycle}(\kappa)$, this vertex will also have exactly $k-1$ ascents coming from all its edges with vertices in the k-clique. Thus, $\mathrm{asc}(\mathrm{cycle}(\kappa)) = \mathrm{asc}(\kappa)$. □

Unlike for mountain and bottomless mountain graphs, the $\mathrm{reflect}$ map on mixed mountain graphs is not an automorphism on the set $L_{1,2}(M_{p,k,m})$ but rather a map from $L_{1,2}(M_{p,k,m})$ to $L_{1,2}(M^{\mathrm{rev}}_{p,k,m})$. Nevertheless, this map still bijective and preserves ascents.

Lemma 9.3.13. *Let* $\kappa \in L_{1,2}(M_{p,k,m})$. *Then* $\mathrm{asc}(\mathrm{reflect}(\kappa)) = \mathrm{asc}(\kappa)$.

Proof. Let c denote the largest color in κ. By the proofs of Lemma 9.1.8 and 9.2.2, it remains to consider the case when a “left-colored" k-clique C is paired with a “right-colored" bottomless $(k+1)$-clique D. Let $\tilde{C}$ and $\tilde{D}$ be their respective images in the coloring $\mathrm{reflect}(\kappa)$. Given that the $\mathrm{reflect}$ map preserves ascents on edges whose vertices are colored 1 and 2 or whose vertices are colored with the numbers from the palette $\{3, \dots, c\}$, it suffices to consider edges where exactly one vertex is colored either 1 or 2. In addition, the relative position of any upper vertex colored 1 or 2 within C or D will remain the same in $\tilde{C}$ and $\tilde{D}$ respectively. This implies that the number of ascents for edges between upper vertices where exactly one vertex is colored 1 or 2 is the same for C and $\tilde{C}$ and the same for D and $\tilde{D}$.

We now examine edges in C, D, $\tilde{C}$, and $\tilde{D}$ where exactly one vertex is colored 1 or 2 and where at least one vertex is a lower vertex. As C is left-colored, its left lower vertex is colored 1 or 2 and there is an ascent along its edge to the right lower vertex. Moreover, by the same argument as in Lemma 9.1.8, there are $k-2$ ascents for edges between upper and lower vertices in C where exactly one vertex is colored 1 or 2. This implies that C has a total of $k-1$ ascents for edges where exactly one vertex is colored 1 or 2 and where at least one vertex is a lower vertex. The bottomless $(k+1)$-clique D and the k-clique $\tilde{C}$ have no such ascents as they are both right-colored. If an upper vertex of $\tilde{D}$ is colored 1 or 2, there are $k-2$ ascents from its left lower vertex to upper vertices and one ascent from the upper vertex colored 1 or 2 to the right lower vertex. If there are no upper vertices of $\tilde{D}$ colored 1 or 2, there are $k-1$ ascents from its left lower vertex to the upper vertices. In all cases $\tilde{D}$ has $k-1$ ascents for edges between upper and lower vertices where exactly one vertex is colored 1 or 2. As $\tilde{D}$ does not have an edge between its lower vertices, this is also the number of ascents of $\tilde{D}$ for edges where exactly one vertex is colored 1 or 2 and where at least one vertex is a lower vertex. Therefore, C and D have a total of $k-1$ ascents for such edges, and so do their images $\tilde{C}$ and $\tilde{D}$. Thus, $\mathrm{asc}(\mathrm{reflect}(\kappa)) = \mathrm{asc}(\kappa)$. □

Proposition 9.3.14. *There is an ascent-preserving automorphism on $L_{a,a+1}(M_{p,k,m})$ which swaps the number of occurrences of the colors a and $a+1$.*

Proof. Let $\kappa \in L_{a,a+1}(M_{p,k,m})$, and suppose its color content is

$$(c_1, \dots, c_n).$$

We start by applying cycle to κ exactly $a-1$ times, so that $\kappa' := \mathrm{cycle}^{a-1}(\kappa)$ is in $L_{1,2}(M_{p,k,m})$, with content

$$(c_a, c_{a+1}, c_{a+2}, \dots, c_n, c_1, \dots, c_{a-1}),$$

and has the same number of ascents as κ. By Lemmas 9.3.8 and 9.3.9, we now can apply swap $a(p-a)$ times to κ' to obtain κ'' in $L_{1,2}(M^{\mathrm{rev}}_{p,k,m})$ with the same content and number of ascents as κ'. Then we apply $\mathrm{reflect}^{-1}$ to form $\kappa''' \in L_{1,2}(M_{p,k,m})$ with content equal to

$$(c_{a+1}, c_a, c_{a-1}, \dots, c_1, c_n, \dots, c_{a+2}).$$

Finally, fix a reduced word for the permutation that reverses entries 3 through n, say,

$$(\sigma_3)(\sigma_4\sigma_3)(\sigma_5\sigma_4\sigma_3)\cdots(\sigma_{n-1}\cdots\sigma_3).$$

We then apply the automorphisms from Proposition 9.1.3 to apply these transpositions σ_i to the content in order, to obtain $\kappa^{(4)}$ with content

$$(c_{a+1}, c_a, c_{a+2}, \ldots, c_n, c_1, \ldots, c_{a-1}).$$

Finally, we apply cycle^{-1} to $\kappa^{(4)}$ exactly $a-1$ times to obtain a coloring $\kappa^{(5)}$ in $L_{a,a+1}(M_{p,k,m})$ with content

$$(c_1, \ldots, c_{a-1}, c_{a+1}, c_a, c_{a+2}, \ldots, c_n).$$

The automorphism $\kappa \to \kappa^{(5)}$ is well-defined, bijective, and preserves ascents, because every step of the construction has these properties. This completes the proof. □

Propositions 9.1.3 and 9.3.14 and Theorem 9.3.10 imply the desired symmetry of the chromatic quasisymmetric function of (p,k)-mixed mountain graphs.

Theorem 1.0.8. Let G be a (p,k)-mixed mountain graph. Then $X_G(\mathbf{x};q)$ is symmetric.

We finish this section with the following question, which could tie together Hessenberg varieties and chromatic quasisymmetric functions for mixed mountain graphs.

Question 9.3.15. Is there a modified Hessenberg variety X such that the Frobenius map of the cohomology ring is $X_G(\mathbf{x};q)$ for a mixed mountain graph G?

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