THESIS

INDISPENSABLE INFINITIES: AN ASPECT-PLURALIST APPROACH TO MATHEMATICAL IDEALIZATION AND ONTOLOGY

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ABSTRACT

INDISPENSABLE INFINITIES: AN ASPECT-PLURALIST APPROACH TO MATHEMATICAL IDEALIZATION AND ONTOLOGY

I investigate some epistemic and metaontological implications of infinite idealizations in mathematical physics. I situate the discussion within the framework of *indispensability arguments* for mathematical Platonism, which typically assume that mathematics is indispensable to science in virtue of a single distinctive *explanatory* role. In chapter two, through case studies of limit-taking techniques and infinitely idealized models in statistical and quantum mechanics, I argue that scientific practice reveals instead a plurality of genuinely epistemic (as opposed to merely pragmatic) reasons for their prevalence. These reasons correspond to a plurality of productive epistemic *roles* mathematics might play in fundamental physics. To make sense of this plurality, I develop a neo-Wittgensteinian contextualist framework, scientific *aspect-realism*. On this view, mathematical models remain truth-apt without requiring de-idealization or strict structural correspondence as necessary conditions on interpretation.

Building on these conclusions, chapter three defends *indispensability pluralism*, the claim that mathematics is genuinely indispensable to science in multiple unique, irreducible ways. Finally, I argue that this observation neither straightforwardly supports traditional Platonist realism nor undermines indispensability arguments altogether. Instead, it motivates a novel form of *ontological pluralism* as applied to the mathematical. According to the resulting view, *aspect-pluralism*, the ontological commitments warranted by scientific practice depend directly on the epistemic roles mathematical structures perform, thereby licensing commitment to multiple *kinds* of mathematical entities while staving off metaphysical overinflation.

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DEDICATION

To the mountains

“How wearily familiar we have become with that ‘nothing but space, time, matter and motion,’ that ‘nothing but sex,’ that ‘nothing but economics!’ And the no less intolerant ‘nothing but spirit,’ ‘nothing but psychology’—how boring and tiresome they also are! ‘Nothing but’ is mean as well as stupid. It lacks generosity. Enough of ‘nothing but.’ It is time to say again, with primitive common sense (but *for better reasons*), ‘not only, but *also*.’”

-Aldous Huxley, *Meditation on The Moon* (1931)

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Chapter 1: Indispensability, Idealization, and Ontology

1.1 Introduction to the Introduction

In what follows I will attempt to address two critical questions in the philosophy of science and the philosophy of mathematics. First, I will consider the question of the epistemic *justification* for the use of explicitly idealized mathematical models in fundamental physics.[1] The second question concerns if and how the striking applicability of these kinds of models warrants our ontological commitment to the mathematical objects and structures they reference—roughly, whether (and in what way) we ought to take mathematical entities to be included in our best description of what exists in the mind-independent world. As I will argue, the answers we adopt concerning the former heavily constrain and influence the answers we ought to give for the latter.

The nature, and order, of the questions considered is due to what is perhaps the most (in)famous argument for Platonist realism in the philosophy of mathematics: *the indispensability argument*. These arguments use epistemic premises—about how mathematics functions in our best scientific theories, models, or explanations—to motivate ampliative, *metaphysical conclusions* about the existence or status of mathematical objects/structures. On the traditional formulation on the argument, found in Quine (1977) and Putnam (1979), roughly, we are ontologically committed to all and only those entities indispensable to our best *theories*.[2]

Thereby, in the same way that a *scientific* realist takes the concrete unobservable entities quantified over in our best theories like electrons, quarks, and wave-functions as really *existing*, we are warranted in taking a realistic attitude to the mathematical entities (objects and structures) involved in our theories.

[1] This framing of the question follows Shech (2023, 34-45)
[2] See Colyvan (2001) for an overview of indispensability arguments. Empirical arguments of this sort can be traced all the way back to Mill, at least (1865).

A more recent formulation of the indispensability argument, Alan Baker's (2009) *Enhanced* Indispensability Argument (EIA) concentrates on *explanatory indispensability,* and goes as follows:

(1) We ought to rationally believe in the existence of any entity that plays an indispensable explanatory role in our best scientific theories.
(2) Mathematical objects play an indispensable explanatory role in science.
(3) Hence, we ought rationally to believe in the existence of mathematical objects [and structures] (Baker, 2009, 613).

Analyzing each of these premises more closely, I think they ought to be understood as claims of the following type: (1) is a *methodological* or metaphilosophical premise, (2) an *epistemic* one, and (3) *metaphysical*. This connection between epistemic role and ontological commitment is the fulcrum on which my argument turns. If epistemic or theoretical indispensability considerations in (2) are to do any metaphysical work in (3), we must first say what it means for some piece of mathematics to *be epistemically indispensable*!

This requires, as might seem trivial, scrutinizing more closely the natures of the epistemically relevant *roles* mathematics plays and how scientists understand and justify these applications in practice. What we'll come to see is that the traditional formulations take indispensability to be univocal –either "theoretical" or "explanatory" in nature—but that this monism undersells and underexplains the striking and multifaceted roles that mathematical entities and models play in our most fundamental theories of nature.

Only after getting the epistemology right can we responsibly proceed to using the kinds of abductive arguments that seem most fitting here to argue for what are, *prima facie*, rather weighty metaphysical claims about the nature of mathematical reality. This insight motivates the structure of what follows: first focusing on epistemic roles by examining a unique, limiting case (infinitely idealized models) and next turning to the status of our ontological commitments concerning the entities involved in doing that epistemic work.

To put my cards on the table early, I defend a view that I will call *Aspect-Pluralism* according to which both epistemic justification and ontological commitment are context-sensitive, role dependent, and irreducibly plural. The central, and perhaps more substantive, claim is that this pluralism does not undermine *realism* concerning either the truth of scientific models and theories or the content of mathematical concepts. That is, even in limiting cases when scientific models rely on explicit and sometimes radical idealizations, they can still be *true* and capable of capturing objective features of the natural world this is the version of scientific realism (*aspect-realism*) established in chapter two.[3]

Likewise, pluralism about fruitful epistemic roles does not force the mathematical realist to capitulate. Instead, it lends support to a principled, local, commitment to the existence of a diverse range of kinds of mathematical entities and properties. Taken together, the two theses comprise *Aspect-Pluralism* as pushing towards a theory of the natural world, mathematics, and how they are fitted together.[4]

My account therefore aims to preserve what motivates realist interpretations of scientific modelling and mathematical discourse—namely, their apparent sensitivity to mind-independent

[3] Chakravarrtty (2017), Linneno (2023).
[4] At least some motivation for this project is, predictably, to be found in Wigner's famous (1960) paper on the "unreasonable" effectiveness of mathematics in natural science.

structure—while resisting both monistic accounts of indispensability and the threat of relativistic collapse. The chapters that follow develop and defend this position in detail. [5]

It will be helpful before delving into these arguments to construct a rough space of positions available in the literature. I'll begin, in section 2, with some methodological points about theory choice, metaphysics, and metaphilosophy. In section 3 I explicate the notion of idealization and some potential justificatory accounts. Finally in 4, I will reconstruct some of the extant positions in the metaphysics of mathematics.

1.2 On the Method of Scientific & Metaphysical Theory Choice

Two methodological commitments guide my approach to EIA. The first is a broadly naturalistic orientation, which I take to be entailed by premise (1) insofar as it embodies what I will call a *science-first* approach. What does this involve?

At minimum, it requires making explicit several assumptions that need to be in place before EIA can begin to do substantive philosophical work. First, with respect to my *explananda,* the approach treats features of actual scientific practice as legitimate sources of philosophical data. This includes physicists' background beliefs, ontological commitments, explanatory intuitions, and experiences of achieving understanding of the natural world. Any adequate philosophical account of applied mathematics, I take it, must aim—so far as it is possible—to accommodate and illuminate these phenomena rather than abstract away from them. This is not

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[5] That is, I hope to provide what has often been called (Feyerabend, 1975; Shaw, 2018; Ludwig & Ruphy, 2021; Bueno, 2011) a "pluralism worth having" for both science and mathematics. This is, of course, an instance of a much larger metaphilosophical debate between absolutism and relativism – it is widely acknowledged as an open question of whether there are any kinds of pluralism that do not, ultimately, collapse into epistemic or ontological relativism (Kusch, 2020; Baghramian & Carter, 2025). I'll take this framing of the question in more detail in chapter 2 section 6, mostly as concerns inter-community knowledge transfer and the nature of the properties that constitute the representational content of models. The title "aspect realism" draws from Juliet Floyd's *Aspect-Realist* interpretation of Wittgenstein, and Ben Martin's (2025) characterization of "Aspect Model Pluralism" which draws on Morrison (2011) to apply the modelling literature to logic – I extend the approach beyond both of these views insofar I apply it to mathematical ontology and to justification as a distinct epistemic achievement.

to say, however, that it is *merely* descriptive. This brings us to the second operative sense of naturalism.

This is that the productive method of philosophy—herein philosophy of science and mathematics—is *abductive* in character. It approaches both epistemology and metaphysics primarily through *inference to the best explanation*. Consequently, my analysis of applied mathematics proceeds with relatively little reliance on deduction in the classical logico-mathematical sense. In this respect, I follow Timothy Williamson (2007; 2024) and L. A. Paul (2012), both of whom argue that philosophy—whether epistemological or metaphysical—does not trade in certainty, but instead advances explanatory hypotheses that must be evaluated comparatively in light of certain theoretical virtues. As Williamson puts the point, and as I will assume throughout, "what is now called a broadly *abductive* method [is] the only viable approach to philosophical theorizing" (Williamson, 2024, X). [6]

From this perspective, metaphysics itself is best understood as a form of modelling—an attempt to construct systematic explanatory frameworks that capture the relevant explananda (Paul, 2012, 3). [7] EIA, or something closely resembling it, therefore emerges as a natural methodological tool for investigating the roles and nature of mathematical entities within scientific practice.[8]

A second methodological commitment distinct from naturalism—while not directly entailed by the structure of EIA itself—is suggested by a growing strand of recent literature

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[6] Douven (2025).

[7] Note a similar view posited by Swoyer (1999) which holds that the best hope for metaphysics is to adopt an inductive-explanatory approach to theory choice. See Allzen (2023) for a rejection of the cogency of this approach to metaphysics.

[8] For a full overview of the specific varieties of metaphysical naturalism and non-naturalism see Morganti (2024). The conclusion of this latest assessment is that the relevant "open-question" concerns that of *theory choice* and the role of so-called "non-empirical" factors in naturalistic metaphysics. I hope that the following discussion contributes to the broader debate, as I discuss the role of theoretical virtues in chapter 3 section 4. The following "extreme case" heuristic also, I think, contributes a significant desideratum for theory choice.

(Batterman, 2002; Shech 2019; 2023; McKenna, 2022). This commitment involves what I will call an *extreme-case heuristic*. Following Alan Hájek (2018), I take it to be philosophically illuminating to examine limiting or extreme cases when assessing the adequacy of a theory or when constructing viable alternatives. Extreme cases often reveal structural features that remain obscured in more ordinary contexts (Hájek, 2018, 352).

Guided by this heuristic, I approach both epistemically justified application and ontology through what I take to be paradigmatic extreme cases: scientific idealizations involving infinities. These include models that play indispensable roles in our best explanations of physical phenomena while representing systems in which magnitudes are taken to infinite or infinitesimal limits, as well as nomologically impossible configuration spaces treated as possible mathematical entities.[9]

Such cases stand, at least *prima facie,* apart from more familiar instances of scientific representation and idealization. As we will come to see, on the epistemic side they resist straightforward interpretation as so-called "mere approximations". Ontologically, they differ significantly from more intuitively tractable mathematical objects, such as canonical geometric forms, which often serve as standard examples in discussions of mathematical realism. It is precisely *because* these limiting cases appear especially demanding that they provide a useful testing ground. My working hypothesis is that an account capable of accommodating them will thereby possess the generality required to handle more ordinary forms of idealization and mathematical representation as well.

[9] The methodological claim isn't that extreme cases are representative, but that any adequate general account must be able to accommodate them without distortion. I won't directly defend this intuition here.

The methodological significance of this strategy becomes clear once we consider the ambitions of EIA itself. If any version of EIA is to carry genuine philosophical force, we must understand the precise nature of the epistemic work performed by mathematics in these extreme contexts as well. In particular, we must determine what renders the mathematics "indispensable" in such cases and whether this indispensability admits of a single functional basis—for example, *explanatory* indispensability—or instead exhibits a more pluralistic structure.

1.3 Mathematical Idealization and its Epistemic Roles

It will be helpful, before jumping into understanding their roles in scientific theorizing, to briefly get a handle on what exactly an idealization *is*. Intuitively, the term calls to mind the frictionless planes of high school physics and academic jokes about spherical cows.[10] On a closer look however, idealizations are not mere pedagogical tools, but instead important features of our most fundamental theories in almost all disciplines. While there is a great deal of debate on how exactly to characterize the broad term (see different definitions in e.g., Morrison (2014), Potochnik (2017), Frigg & Hartmann (2025), Pincock (2020), Rice (2021)), I take the concepts to represent a family resemblance class of models, representations, and scientific practices best operationalized as*: those in which some feature or magnitude is explicitly or intentionally distorted or falsified*. In short, cases where it appears that something like "representational accuracy" is traded in for another output that we might value.

Getting much more specific than this, and delineating between the closely related notions of idealization, approximation, and abstraction requires our recognizing that "generally, whether a representation is an idealization or not depends on how we take the world to actually *be* and this in turn depends on the background theory with which we are working" (Shech, 2023, 15;

[10] Harte, 1988, xxii.

Thompson-Jones, 2012). That is, what is idealized as opposed to approximate or abstract will depend both on which background scientific metaphysics we adopt and the theoretical landscape of the specific discipline one is concerned with.

At this general level, I think it is sufficient to, in a Wittgensteinian manner, be satisfied with something like a family resemblance class consisting of various existing definitions and practices (Wittgenstein, 1953). For simplicity, when needed, I'll refer back to my italicized operative definition. Insofar as we want to define "idealization" for all scientists and all models everywhere, this will be sufficient. Attending to individual cases at a local level will, I hope to show, prove much more fruitful than trying to capture *idealization* universally or trying to taxonomize them in any entirely general fashion.[11]

Focusing on "mathematical" idealizations helps, again, to prevent some of the potential permeation of the present conclusions into these other areas.[12] Christopher Pincock gives a useful definition of our current target: "an idealization will be *mathematical* just in case these assumptions, or the resulting representation, involve mathematics in some crucial way. Here we find a technique for arriving at representations that is used across the sciences but seems largely

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[11] Note here a couple of different classification projects that might be of interest. First, Wiesberg (2007) classifies idealizations into three types based on *why* they are implemented: "*Galilean Idealizations*: "simplifications" motivated by our rendering some system computationally tractable., *Minimalist Idealizations*: models which isolate key "difference-making" factors of a given phenomenon, *Multiple-Model Idealizations*: related but mutually incompatible models of the same target with different representational aims." Nowak (2000) takes an historical approach, based on the *theoretical* role of idealizations: "*Neo-Duhemian paradigm*: Idealization is a method of transforming raw data, such as when systematic errors are corrected, into data that can be used in the scientific enterprise. *Neo-Weberian paradigm*: Idealization is a method of constructing scientific notions. *Neo-Leibnizian paradigm*: Idealization is a deliberate falsification. *Neo-Millian paradigm*: Idealization is taken to be a consequence of the discrepancy between mathematical representation and the physical world. *Neo-Hegelian paradigm*: Idealization is a process that focuses only on essential and relevant features of some phenomenon." While useful, I take it that such taxonomies will inevitably face recalcitrance from some case study in one of the many disparate contexts of applied mathematics, and thereby are useful perhaps, at best, as heuristics.

[12] Concrete scale models of infrastructure, ecosystems, and molecules are often thought to count as idealizations (see e.g., Pincock, 2024; Rohwer, 2025) along with computer simulations (Laymon, 1990), and certain economic models (Grune-Yanoff & Wirling, 2024). The list goes on to include ethical theories (Rawls 1972) and formal-epistemic frameworks like Bayesianism and possible-worlds semantics (Greco, 2023). I note this merely to acknowledge that demarcating the *scientific* (not just the mathematical) is itself a difficult question that I will, regretfully, ignore.

absent in non-scientific contexts" (Pincock, 2007, 957). Here, I'll concentrate on a number of instances hailing from fundamental mathematical physics. Each will be explained more fully in what follows, but a cursory exposition of the main characters is appropriate.

First, *thermodynamic limit-taking* (TDL) in statistical mechanics provides a canonical example of an infinite idealization doing apparent explanatory work. By taking the number of constituents or the system size to infinity, singularities and non-analytic behavior appear in thermodynamic quantities—features that correspond to sharp phase transitions in real systems. The philosophical significance of TDL lies in the fact in many influential treatments, the explanation of phase transitions and critical behavior is most naturally formulated only in the infinite limit (Batterman, 2002; Kadanoff, 2000; Batterman & Rice, 2014). As a result, TDL represents a case in which an explicitly "false" assumption is *prima facie* tightly bound with explanation and understanding.[13]

Second, *renormalization-group* (RG) methods sharpen this challenge by introducing a potential second "infinity" into the explanation of phase transitions. In RG explanations, systems are repeatedly "coarse-grained" and rescaled, ideally without bound, and their behavior is characterized in terms of fixed points and flows in a space of possible Hamiltonians (Kadanoff, 2000). This framework seems to explain why microscopically diverse systems display the same critical behavior: many micro-differences become irrelevant under renormalization (Batterman, 2002; 2010). The infinite iteration of the RG operation and the ideal *minimal model* it generates are thus explanatorily central to understanding this striking property of phase transitions.

Finally, the *Aharonov–Bohm* effect introduces a distinct but equally philosophically rich form of idealization in quantum mechanics (Shech, 2019; Aharonov & Bohm, 1959). Here, the

[13] See Baron (2019, 1159) for an explanation of how taking this practice clashes with general-relativistic background assumptions about Schwarzschild radii.

epistemic value of the math is tied not to limit procedures but to the structure of the theory's *configuration space*—specifically, the global properties of electromagnetic potentials and their associated topology (Shech, 2019). The effect shows that physically observable phenomena can depend on features that are not localized in the classical causal field variables, challenging simple causal-mechanical reconstructions. In the even more abstract version I return to in Chapter 3, the configuration space itself becomes increasingly idealized—taking on *infinite length*.

1.3.2 Monist Accounts of Epistemic Justification

Given that baseline idea of what idealizations are and which ones I'll be focusing on, we can turn to a few proposals for what they *do* for us. While I will provide a more detailed account in chapter two, it will be useful to briefly survey the relevant approaches. What to notice here is that as I characterize them, each of these accounts posits one distinct role for an idealization, I call them for this reason "monist" accounts of justification.

Prominent epistemic *realist* accounts of applied mathematics—e.g., Pincock's (2004; 2007; 2011; 2012) mapping account—seek out justification by establishing objective structure-preserving relations (isomorphisms, homomorphisms) between a de-idealized "matching model" and the target, and then showing the idealized model is an acceptable mathematical transformation of that match. The hope is to render the "infinite" bits schematic or contentless, preserving the "underlying truth" of inferences drawn and revealing which parts of the model don't *really* make a difference (Pincock, 2020). However, as we will see in chapter two, the construction of the requisite "matching model" faces principled, not merely practical, difficulties.

On the other hand, distinctly *anti-realist* approaches— like Mark Povich's (2024) *Fully Inferentialist Theory* (FIT) and Parker's *adequacy-for-purpose* (AFP) pragmatism—center conventions, aims, and inferential permissions as justificatory grounds.[14] On these views, idealizations are largely unproblematic – they ought to be expected insofar as we are concerned with the "assertability" or "usefulness" of a given bit of mathematics as its success conditions – accuracy is not a necessary condition when there are many things we want to do (Povich, 2024, 214; Parker, 2020).

We might idealize for many a reason on these views—see Weisberg's taxonomy in *fn 9*. These capture the holism and context-sensitivity of practice but risk making justification too cheap, struggling to answer what I'll go on to call our basic *No Miracles Constraint* (what makes the success non-accidental and truth-apt).[15] To my mind, it is clear that in at least some, if not many, cases—and in particular cases in the most fundamental of the sciences like mechanics—accuracy does matter, one must have something satisfying to say about *truth*!

The central task, therefore, is to articulate an account that acknowledges context and plurality without losing truth-aptness and objective tethers where they exist. Of course, other attempts have been made to sail between these two traditional extremes: take for example Angela Potochnik's (2017) *causal patterns* account and Collin Rice's (2021) *understanding holism*. Potochnik's account characterizes science as the search for *causal patterns* in an overwhelmingly complex world. On her view, idealization is a central representational strategy that allows limited agents to isolate "stable" causal regularities in the face of complexity.

[14] Povich (2024) offers "anti-realist" accounts of mathematical content (glossed above as *normativism*) and of model content (here FIT). I will treat them separately, as they concern different topics, but note that this project can be read as an overall response to Povich's reading of both model content and metaonotlogy.

[15] See Chakravartty (2017) for a history of "no miracles" arguments. This schematic version, meant to constrain the current account to a minimally "realist" scientific metaphysics (as far as the *representational* content of mathematical models goes), is drawn from Shech (2023). It is my belief that adopting something like this constraint will be instrumental in constructing a plausible version of the indispensability argument.

The guiding idea is that scientific representations typically do not aim to mirror the full causal nexus of their targets; instead opting to depict key causal patterns that admit of exceptions, and they do so precisely by *simplifying, distorting, and abstracting away* from many interacting influences (Potochnik, 2017, 120). This picture is explicitly aim-sensitive: what counts as an acceptable idealization depends on the cognitive and practical ends of inquiry.

A different, more holistic, middle path is developed by Collin Rice in his *understanding realism* (2021). The central claim is that idealized models of phenomena are piecemeal in such a way that their epistemic payoff emerges only at the level of scientific understanding as a *whole*—especially in contexts involving universality (Rice, 2021a; 2021b). In those cases, distortions are not merely tolerable because they target the "irrelevant" or non-difference making (à la *the mapping account*). Instead, the deliberate misrepresentation of difference-making features can be precisely what enables scientists to identify stable patterns *across* heterogeneous systems, to track why micro-details wash out, and thereby to secure a kind of factive (or truth-sensitive) holistic *understanding*.[16]

My proposal is sympathetic to the conciliatory motivations behind these newer views and will draw in large part on Rice's (2021;2025a) treatment of *universality classes* and focus on the *mathematical* flavor of these kinds of idealizations. However, neither framework to my mind captures the particular pressure exerted by the extreme cases that organize the current approach, nor extend their conclusions adequately to discuss what the models in question might actually *be* (the so-called *question of ontolog*) (Frigg & Hartmann, 2022).

[16] Here we see the important epistemic distinction between *understanding*, *explanation*, *knowledge*, and *representation*. Rice focuses on understanding, while other accounts are concerned with mere knowledge (Pincock, 2020). The cutting up of these exact achievements is a tricky and rich literature to parse—and for the most part will be left undisturbed. The understanding/explanation distinction will, however, be dealt with in chapter 3 section 2.2.2.

Potochnik's causal-pattern picture can make idealization look too closely tied to pragmatic tractability and to the selective depiction of patterns for human ends, threatening collapse into a kind of deflationism or relativism. As she notes—on a view like hers, science no longer can be considered as "aiming at truth" (Potochnik, 2017, ch 4).[17] Additionally, we might note—it is not a given that science is restricted to the causal, particularly in mathematics. Rice's holism, by contrast and in picking up on these points, powerfully resists decompositional strategies, but it can invite the impression that what ultimately matters is the holistic production of "understanding", with truth-aptness relegated to a *downstream or global verdict* about the overall scientific output.[18]

Aspect-Pluralism is thus motivated by the thought that we need an account that simultaneously (i) says something epistemically substantive about why idealization is so pervasive, one which makes their success *non-miraculous* and—and thereby (ii) fine-grained enough to evaluate the truth-conditions and inferential licenses of *particular* mathematical moves in particular contexts, rather than only the holistic epistemic product they furnish.

1.4 Mathematical Ontology: Realisms and Anti-Realisms

Given that brief exposition of the *roles* that idealized models might be playing that are epistemically beneficial, such that a handle on premise (2) might be found—e.g., signaling irrelevance, isolating out causal patterns, and expressing aims and inferential conventions—it makes sense to turn to (3) and ask what *kinds* of things might be able to help play those roles. As with most metaphysical debates, the one concerning the correct approach to mathematical ontology divides roughly along the *realist/anti-realist* distinction. The mathematical realist, like

[17] Like Shech (2023, 35), it is unclear to me how exactly this squares with the causal representation account and take its outright rejection of truth tracking as a *prima facie* count against the view as an account of what scientists think they are up to.
[18] I note these differences here given that I do not return to these views while establishing my own.

the scientific realist when entertaining the unobservable, maintains that our mathematical discourse and concepts successfully refer and as such can be sufficiently truth-apt. While there are many extant formulations of realism, I will briefly present three that will become of interest to us in chapter three: Platonism, Aristotelianism, and Pythagoreanism.

1.4.1 Mathematical Realisms

Platonism, according to Linnebo (2023,1) consists in three claims: "*Existence*: There are mathematical objects; *Abstractness*: Mathematical objects are abstract; and *Independence*: Mathematical objects are independent of intelligent agents and their language, thought, and practices." Interpretations of exactly how one ought to interpret these claims abound. Prominent versions range from "lightweight" versions, like Balaguer's (1998) *full-blooded Platonism* on which "any mathematical objects that could exist actually do exist," to much stronger views like Shapiro's (1997) *ante-rem* and Resnik's (1997) structuralisms which hold that mathematical concepts are about (relational) features of mathematical *structures* which themselves are best characterized as unique "abstract, platonic entities" (Horsten, 2022, 1).[19]

Alternatively, and of increasing interest in the literature, *Aristotelianism*, defended most prominently by James Franklin (2017) and Marc Lange (2013, 2021), seeks to bring Platonism back down to the physical world without giving up on the realist intuition of successful reference. On this view, mathematical concepts refer to real features and structures instantiated in the physical world. Franklin's approach emphasizes that mathematics describes patterns, ratios, and quantitative structures that exist in nature (e.g., symmetries, proportions, and spatial relations) making mathematics a *science of the real world.* This approach promises to explain

[19] The word "lightweight" here may seem counterintuitive, but it is the correct one for the following reason: Baluguer makes clear that while it is true that all possible mathematical entities exist, this existence is "shallow" in the sense that it cannot interreact in *any* meaningful way with the world (Linnebo, 2023, 1).

both the objectivity and applicability of mathematics with relative ontological economy: its truths hold because they are grounded in the structural fabric of reality itself (Franklin, 2017, 1).

Lange's (2013, 2021) approach builds on this picture by focusing on the *modal* and *explanatory* roles of mathematics in science. In multiple places, he has argued that mathematics often figures in necessary truths that underwrite scientific explanations, especially in cases where phenomena exhibit a kind of necessity that goes *beyond* the robustness afforded by nomological (or causal-law based) necessity. For Lange, these necessities arise from the essential modal properties of physical systems, reinforcing the Aristotelian idea that mathematics finds itself so usefully applied insofar as it articulates the modal profile of the natural world—it tells us how things *must, cannot, or may possibly be*.[20]

Finally, in a recent attempt to reconcile these views, Sam Baron (2024) articulates what he calls a *partial Pythagorean* view which holds that the best way to *be* a Platonist (and to accommodate applicability concerns) is to *also* insist that the Platonic and the natural realms are intimately connected. So intimately, in fact, that mathematical and physical objects and structures can be said to "*share properties*" (Baron, 2024, 2).

The view then, assumes Platonism of a rather vague sort, but helps to solve some issues in applicability that the Aristotelians have historically fared far better with. I mention it here only to note the most recent developments in the literature on mathematical ontology and because I will, briefly in chapter three, return to it as a candidate description of what is going on in one mathematical explanation involving the surface areas of honeycombs.

[20] Another relevant account in this vein is *Ontic Structural Realism* (OSR) as most fully articulated in Ladyman & Ross (2007). I will return to a characterization of OSR in section 5 of chapter 2 as something like a foil to my own view. For now, a slogan summary of the view will suffice on OSR, one takes the view that "

1.4.2 Mathematical Anti-Realisms

Now on to the anti-realists. I'll offer two potential broad positions that I find most plausible and well-argued on this end, but by no means claim to be exhaustive of the available stances. These are *nominalism* (of two flavors) and *modal structuralism*.

Nominalism, like Platonism, is best conceived of as an umbrella theory that encompasses at least several distinct approaches. The *locus classicus* of nominalism is Hartry Field's (1980) monograph *Science Without Numbers* in which he argues, in an attempt to finish work begun by David Hilbert (1935), that science can be done *without* our supposing anything in the neighborhood of the *truth* of mathematical theories and discourse. All it requires is that those theories be *conservative* over natural science. By this, Field means that "whenever a statement of an empirical theory can be derived using mathematics, it can in principle also be derived without using any mathematical theories" (Field, 1980, 13; Horsten, 2022, 1).[21] The Fieldian project to establish this, as the reader might expect, is in direct response to the original formulations of the indispensability argument, and is meant to undercut the inductive jump from premise (2) to premise (3). Mathematics is (insofar as it is conservative) isn't "indispensable" at all!

In the rest of the book, he goes on to provide a valiant attempt to "nominalize" some fundamental parts of Newtonian mechanics – a crucial step in arguing in a certain manner against *indispensability* arguments for the kinds of realisms we just went through.

Field's project has spawned a number of interesting complementary positive ontological theses, which take the guise of stances like *fictionalism,* defended by Field himself and Mary

[21] Note also that for Field the theory must be *consistent* where consistency is (roughly) the property of a mathematical theory that it does not include logically contradictory claims in its axioms or conclusions. However, functionally these two requirements are closely related. As Field puts it: "the gap between the claim of consistency and the full claim of conservativeness is, in the case of mathematics, a very tiny one" (Field, 1980, 13).

Leng (2010) on which mathematical entities are "useful fictions".[22] Doing mathematics, on ficitonalist views, amounts to being "invited" to participate in game of make believe with our fellow inquirers (Horsten, 2022, 1).

Another way a nominalist might go here is to adopt the related doctrine of *normativism*, as has most recently and extensively been defended by Mark Povich (2024).[23] The normativist asserts that, at the end of the day, all mathematical concepts track is the "assertability conditions" – within a given conventional rule-set—and as such serve to *express* rules of discourse and the "transformation of empirical descriptions", mathematical entities then are to be understood as conventional-constructions (Povich, 2024, 1).

Finally, and somewhat alone in its character (though generally being taken to be an anti-realist approach) is *modal structuralism*. First articulated by Geoffrey Hellman (1989), this view holds that mathematical discourse is concerned with *possible* structures. That is, all mathematical discourse can be translated into modal (counterfactual) discourse. Consider the following example from Reck & Schiemer (2025): "For Hellman, an arithmetic sentence such as "2+3=5" is to be analyzed as follows":

> Necessarily, for all relational systems *M*, if *M* is a model of the Dedekind-Peano axioms, then $2_M+3_M=5_M$.

This is, as Hellman puts it himself, an attempt at "structuralism without structures" (Hellman, 1996). However, as the reader may anticipate, this leaves the modal structuralist with a great deal to say about what exactly *modality* itself is—there is other commitments that need to

[22] Note that Leng's approach differs from Field's in not requiring a full axiomatization project. Call this version "easy road" nominalism. Yablo (2014) also offers a sophisticated defense of a fictionalist program.
[23] Povich's view is, in many senses, similar to Jarred Warren's (2020) *conventionalist* position – but Povich does substantial work to differentiate the two (pp. 226-246). It is, in essence, an attempt to revive a *Carnapian* approach to metaontological questions.

be made. In chapter three I will return to modal structuralism as an enticing approach to understanding idealized mathematical situations that deal not only with *possible* structures but seemingly *impossible* ones. There I will say a bit more about what kind of metaphysics of modality is warranted. For now, it is sufficient to recognize modal structuralism as a viable alternative for understanding the legitimacy of applied mathematics and autonomy of mathematical discourse more broadly. Its status as genuine nominalist, however, is unclear– and its relationship with EIA (as Bueno (2013, 1) notes) is underexplored.

Thus, in general, it seems that we have many live and compelling realist and anti-realist options, each with a story about the mathematical content of mathematical concepts and representations, and how they are "hooked up" to the world. The question of *how* we ought to go about choosing between then is, as I take it to have been shown in my exposition of my naturalistic methodology, one responsive only to an abductive method– settled by something like *EIA*.[24] Its failure traditionally has entailed nominalism of some form (see e.g., Povich, 2024, 217), and its success Platonism. I seek to show however, that if we accept the pluralist claim about roles – and the entailing claim about the plurality of indispensability—*monism* as concerns the nature of mathematical reality becomes untenable and a more defensible kind of realism emerges.

1.5 Thesis Cartography

The rest of the thesis will proceed as follows. Chapter 2 will approach the epistemic premise via the justification of the use of infinite idealizations. I give a more in-depth survey of

[24] Many philosophers have found the question of mathematical metaphysics either an uninteresting one or an impossible one (issues stem from Benaceraff's (1965) infamous critique of Platonism on the grounds of its mysterious epistemology). EIA gives us one plausible route of defusing these worries and one that purports to be continuous (abductively) with the generally more palatable scientific realist stance. If a defensible contemporary version of EIA can be maintained, mathematical ontology sits in epistemically respectable company.

three of the strongest contemporary accounts of applied mathematics: Pincock's (2004, 2007, 2012) structural mapping account, Povich's (2024) *fully inferentialist theory*, and Wendy Parker's (2020) pragmatism, before attempting to offering my account that avoids apparent problems the case study of phase transitions presents.

To do this I draw on neo-Wittgenstein approaches to applied mathematics, particularly that given by Juliet Floyd (2021) and *perspectival* approaches to scientific representation (Massimi, 2022). I articulate, before closing, what kind of "realism" the *aspect-realist* component of *aspect-pluralism* purports to be by rejecting appeals to *intrinsic* properties and focusing on the purely relational as the content of models (Massimi, 2022; Reuger, 2005).

With this conception of the roles and justification of applied mathematics that hooks up to a story about how those models might possibly be *true* about the world, I turn in chapter three to premise (3) and the question of ontology. Here I introduce the Abstract Aharonov-Bohm effect along with some more regular examples to illustrate the usefulness of the *extreme* case heuristic. My discussion leads to the conclusion that *indispensability* is achieved differently in the different distinct role's mathematics plays.

This in hand, I argue that the most charitable reading of EIA thus demands a turn to *meta*-ontology, and further that we be, once again, a certain kind of *ontological pluralist* (that is, accepting the legitimacy of multiple ontological commitments regarding mathematical truth-makers).[25] Interpretation here is again as sensitive as possible to the contextual features that help reveal what exactly it is we need from mathematics for certain problems given the different justificatory and indispensable roles played. Motivating my construction is the possibility that

[25] Section four of chapter three is devoted to distinguishing the relatively novel kind of meta-ontological pluralism that I am advancing here on the basis of an extended indispensability argument. Others who entertain the metaontological turn are Plebani (2020), Jonas (2024), and Povich (2024). Kinds of "pluralism" are relatively diverse but offered in Hamkins (2012), Priest (2024), and surveyed in Zalta (2024).

the mathematical universe, insofar as it is anything like the natural world, is extremely complex—its inhabitants diverse and bearing many properties.

In recognition of the broad scope of these topics I'll simply note in closing the introduction that my aim is emphatically not, by any means, to *settle* any debates about mathematical application or ontology once and for all. I am, instead, merely out to show that once we attend carefully to infinite mathematical idealizations in physics, sensitivity to pluralist and contextualist concerns becomes unavoidable, though it need not undermine our (sometimes seemingly tragic) optimistic intuitions that science is still in the business of truth with a capital T.

Chapter 2: Phase Transitions and the Roles of Mathematics in Natural Science

2.1 Introduction

In order to explore the epistemic premise, (2), of EIA, which tells us that math plays an "explanatory role" in natural science, I think it apt to look to some of the most perplexing, but most useful instances of mathematics at the most fundamental scientific levels: idealizations. Philosophers of science have, in recent years, grappled more explicitly with the tension between scientists' aims to represent and explain real systems accurately, and the fact that their most successful theories often rely on highly idealized models. [26] This tension becomes especially acute in the philosophy of applied mathematics where abstract models and operations frequently involve distortions which appear indispensable for representation, explanation, and prediction. [27] Among the most striking cases are *infinite idealizations*—instances where a physical magnitude or iterative operation is taken to an infinite or infinitesimal limit (Shech, 2018,1).[28]

While this tension brings into relief a multitude of thorny problems, I will approach it on first pass here as an epistemic challenge about the *justificatory status* of the achievements produced by such models. [29] That is, understanding *why* we use idealized models ought to reveal something about their roles, and in turn the ways in which they might be crucial and ineliminable features of our theories.

It is best understood as follows, *Justification*: in virtue of what can explanations and inferences derived from infinitely idealized mathematical models be justified? [30] That is, how is

[26] Frigg, 2002; Weisberg, 2007, 2013; Potochnik, 2017; Rice, 2021; Shech, 2023.
[27] Wigner, 1960; Pincock, 2007; Morrison, 2015; Frigg & Nguyen, 2020; Mancosu et al. 2023; Sher, 2023
[28] See Easwaran et. al (2024) for an overview of the concept of mathematical infinity.
[29] Fine distinctions between some of these epistemic achievements are discussed at length in Grimm (2024) and Strevens, (2013).
[30] Worries about justification and its nature in this epistemic sense go back to Plato's *Meno* where he suggests that knowledge is "tied down" such that it is stable. One implication of my argument, one which can also be read into Plato, is that there are *many ways* of tying down some cognitive state to the world (see e.g., Fine, 2004).

it that we ought to go about negotiating the "epistemic-logical" debt that we take on by employing (infinite) mathematical idealizations (Shech, 2023, 32)?

I argue that a reasonable desideratum for an account of the role of mathematical idealization in natural science is that it responds to a variant of the *No Miracles Argument*:

> (*NMI*): the predictive and explanatory success of highly idealized models would be miraculous or underexplained unless some *truth-apt* relation holds between the model and its target.[31]

NMI will help ensure that whatever epistemic role is being played in a given case, it will indeed be genuinely *epistemic*, and not merely pragmatic.[32] To explore how this justificatory question unfolds in practice, I focus on limiting cases of infinite idealization from mathematical physics: the *thermodynamic limit-taking* (TDL) and *renormalization group* (RG) techniques used to explain critical phase transitions and their universality across microstructurally diverse systems.[33] These techniques appear indispensable in our best explanations of why fluids exhibit singular behavior at critical points and why such behavior is stable across systems as different as boiling kettles and ferromagnets. Yet both rely on infinite idealizations—assuming infinite particle density or iterating RG transformations to infinity—which seem to defy straightforward truth-functional, or realist, interpretation.[34]

[31] For discussion of the no miracles argument and kinds of justification see Lipton, 1994; Chakravartty, 2017; Carter & Sosa, 2022.
[32] I take it that this is an important condition to observe when turning to back to EIA. If the roles are not truth sensitive, then it becomes (to my mind) less persuasive that those roles would ground metaphysical conclusions. Shech (2023, 43) distinguishes *justifying* from *motivating* reasons here.
[33] Batterman, (2002); Batterman & Rice, (2014); Shech, (2018), (2023); McKenna, (2022); Palacios, (2019).
[34] While the literature on realism and its many flavors is vast, three main approaches are recognized (Chakravartty, 2017): (1) the epistemic view that the best theories are true or approximately true, (2) the successful reference-view that scientific terms successfully refer to both observable and unobservable objects, and (3) the ontological view that the metaphysical furniture posited by our best theories exists mind-independently. Largely, I am concerned here

I argue that traditional approaches to justification, whether realist accounts that appeal to structural mappings and *de-idealization*, or anti-realist accounts that reduce justification to inferential or pragmatic adequacy, either struggle to capture the full epistemic value of these techniques or cannot respond to NMI. In response, I develop a pluralist-contextualist account of applied mathematics, *Aspect-Realism (AR)*, which combines contextualism about truth conditions with a pluralism about model-world relations.[35] On this view, idealized models can be justified by a diverse set of truth-apt "tethers," including structural, quasi-structural, and non-representational explanatory relations, situated within the evolving practice of mathematical physics.[36] This approach preserves the realist intuition that is consistent with NMI and scientific practice, while accommodating the indispensability of infinite idealizations and several conceptual failures in *de-idealization*. I argue further that AR has resources to be generalized as an account of applied mathematics.

This chapter proceeds as follows. §2 introduces the case study of phase transitions and universality; In §3 I evaluate three dominant monist approaches—Christopher Pincock's (2004, 2011) mapping account, Mark Povich's (2024) inferentialism, and Wendy Parker's (2020) adequacy-for-a-purpose pragmatism; §4 develops the structure of *Aspect-Realism* by drawing on Juliet Floyd's (2021) neo-Wittgensteinian philosophy of mathematics and contextual approaches to naturalized metaphysics as they apply to the case studies. §5 concludes with implications for the general philosophy of applied mathematics and the realism debate.

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with (1) and (2) concerning individual models in particular (both with reference to theory). Though (3) will come into play, I will remain largely agnostic about the precise ontological nature of the content of models—thus, AR ought to be construed as a kind of limited or modest realism. Note also that this paper is not concerned with mathematical or model ontology (in the sense of aiming at the "kinds of things" that the mathematical models and techniques herein considered might be), but rather only with the kinds of relations they can stand in and to what (this agnosticism is consistent with much of the literature on applied mathematics e.g., Priest, 2022).

[35] Floyd, 2021; Davey, 2011; Liu, 2019; Massimi, 2022

[36] Shech & McGivern, 2021; Ladyman & Ross, 2007; Kasirzadeh, 2021; Rice, 2022; McKenna, 2022.

2.2 Phase Transitions and Universality

To bring out this tension, it is helpful to reconstruct a case study that has become of interest to philosophers of physics in recent years given its ubiquity and nuance: phase transitions.[37] Phase transitions (PT) are those instances of *critical behavior* in fluid systems wherein the fluid changes phase (e.g., from liquid to gaseous) at a given "critical point" without progressing through a hybrid liquid-gaseous "regime"—are one of the primary phenomena of interest in statistical mechanics and thermodynamics (Batterman, 2002; Frigg & Werndl, 2024).

There are, importantly, at least two distinct explanatory targets involved in understanding the entirety of relevant processes and phenomena—and hence two distinct infinite idealizations at play (Batterman, 2002; McKenna, 2021). The first concerns what Batterman (2002) calls type (i) questions, which ask why one given instance of critical behavior obtains, and the second are type (ii) questions, which ask why the *pattern* of critical behavior, does and can be expected to obtain across microstructurally diverse systems from boiling kettles to the ferro-magnetization of iron bars (its *universality*) (Batterman, 2002, 23). Universality—the robustness of some property or behavior across different microstructural systems in general—is a relatively familiar yet striking phenomenon that remains of significant interest as an explanandum in physics (Kadanoff, 2000).[38]

Understanding (i) requires TDL and supposing the idealized claim that a system has, in essence, "infinite density", but additionally understanding (ii), seems to require appeal to another potential infinite idealization in applying the renormalization group (RG) operation iteratively to

[37] Shech, (2023); Palacios, (2019); Morrison, (2014;2015).
[38] It is apt to characterize universality as closely related or equivalent to "multiple realizability" (MR) (Batterman, 2002), though I do not take it that this reduces its interest as an independent explanandum (contra. Povich, 2024, ch. 3). See McKenna (2021) for a similar defense of universality.

a topological space of possible systems (Fischer, 1998; Kadanoff, 2000, 2013; Batterman, 2002, 38; Baron, 2019).

Taking the type (i) question in more detail, we can understand (Fig.1) as demonstrating the critical point C. This point "indicates the existence of a *qualitative* change in the behavior of the system. Below Tc the distinction between liquid and vapor makes sense; above Tc, it apparently does not." (Batterman, 2010, my emphasis). In thermodynamic terms, the phenomenal shift at C is "represented" by a mathematical singularity.

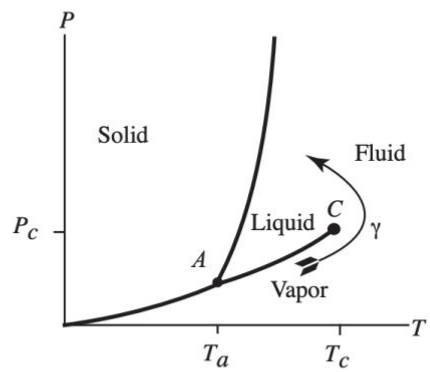

Figure 1: Temperature-Pressure Phase Diagram for a "Fluid" (Batterman, 2010, 6).

In essence, this means that crossing one of the bold lines represents either a first-order or higher-order PT and correspond to "non-analyticities" (or singularities) in either first order or higher-order derivatives wherein the point is no longer infinitely differentiable. Statistical Mechanics follows suit here in representing PT with non-analyticities in the free energies—i.e., a partition function that conveys information about the microstates of a system and their probabilities in the following form: $Z = \sum_s e^{-\beta H(s)}$where H(s) is the Hamiltonian (or energy) associated with microstate s.

What is important here is that the partition function is given by a sum of analytic functions, but confusingly no finite sum can get us the non-analyticity we need (Shech, 2023, 56). To mitigate this, physicists take the thermodynamic limit (TDL) in which the number of particles N assumed in the target system diverges to infinity. Taking $N \to \infty$ allows the partition function to tend towards a sum of $\frac{1}{1-x}$ and achieve the non-analyticity at 1.

In essence, what this illustrates for many physicists and philosophers is that "the existence of phase transitions requires an infinite system. No phase transitions occur in systems with finite degrees of freedom (Kadanoff, 2000, 238). TDL appears to be indispensable—every time a kettle boils, it only does so insofar as we assume that it to have an infinite number of particles. This is the first infinite idealization and should immediately strike us as problematic. As Baron (2019) puts it: "the only way for infinite mass to reside in a finite region is for that region to possess infinite density. But, according to the general theory of relativity, any occurrence of infinite density induces a singularity—a black hole" (Baron, 2019, 1951).

Recall that it is not only the case that these limit taking techniques are required for understanding instances of PT—but also that the same critical exponent apparently describes the experimental behavior of various microstructurally diverse systems experimentally plotted as order parameters with coexistence curves (the difference between temperature and density of coexisting phases), scaling to the same power law t^{β} (fig.2). In order to explain this *universality*, that is, to answer the type (ii) question, physicists often appeal to *renormalization group* arguments (RG).

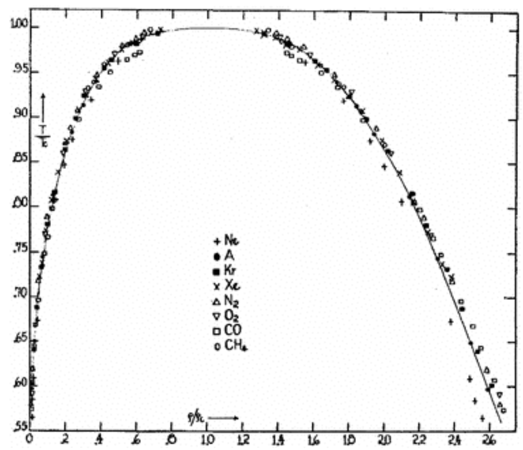

Figure 2: Universality of critical phenomena (Guggenheim, 1945, 256)

In applying RG, we consider each relevant thermodynamic system in SM as represented by its Hamiltonian – which characterizes the systems "degrees of freedom" from influence by external fields etc. (Frigg & Werndl, 2023, 7). As these systems approach critical points, the correlation lengths, the distance range demarcating the mutual causal efficacy of the parts of a system, of the Hamiltonians themselves diverge to infinity as we saw above, despite the fact that the actual observed interactions remain localized (have short correlation lengths). For fluids with extremely high numbers of particles interacting (as is the case in real-world systems of interest), this is a "highly singular, intractable mathematical problem" (Batterman, 2002, 40). That is, the infinite system in which the correlation length diverges displays fundamentally different (or unique) behaviors in the limit from those that occur in finite systems.

RG arguments apparently make the problem tractable by approaching the systems in the abstract mathematical phase space (a topological structure) of possible Hamiltonians. Repeatedly applying a particular renormalization technique to Hamiltonians in the space allows physicists to transform physical systems into abstract possible Hamiltonian systems (minimal models) with fewer degrees of freedom. That is, when we take the number of RG transformations to infinity in the case of PT, we observe that the abstract space reaches a fixed point to which the different physical systems that exhibit the same critical behavior flow to as we iterate RG to the limit (fig. 3).

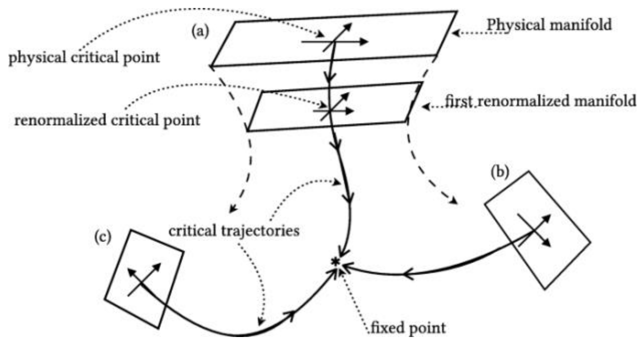

Figure 3: Fixed-Point and Universality Class (Sullivan, 2019, 1).

In such cases, the fixed point (the minimal model Hamiltonian) possesses certain universal properties (e.g., dimension and symmetry). Physicists use these universal features that remain in the minimal model to understand additionally which microstructural features of the systems in the universality class are irrelevant—thereby demarcating and explaining the "universality class" all the phase-transition systems participate in (Batterman, 2010,7; Rice, 2020).

2.3 Mapping Structure, Affording Inference, and Pragmatic Adequacy

I'll now turn, in more detail, to three approaches in the literature on how exactly mathematical models and techniques are applied to the natural world and see what resources they can draw on to answer *Justification* in these singular cases. I begin with Pincock's (2004,2011) *mapping* account, as it has been considered a standard approach in the field. More generally, *mapping* can be understood as a subspecies of "referentialist" accounts which understand epistemic justification generally to derive from the demonstration of some objective relation (specific to each view) between the model and target (Povich, 2024, 214).[39]

Though AR will largely draw on these accounts, we will come to see how it is structurally distinct in its incorporation of social and intentional considerations and ecumenical inclusion of objective relations. For now, the focus on Pincock's *mapping* alone will help shed light on how this typically realist view plays out from a distinctly mathematical perspective where the relevant relations are various "morphisms" and the relata are *structures* both physical and mathematical.

2.3.1 Mapping Accounts and De-idealization

The paradigm case of structural mapping is an *isomorphism*—"such that two structures $S= \langle D, R_1, R_2, \ldots \rangle$ and $S^*=\langle D^*, R_1^*, R_2^*, \ldots \rangle$ are isomorphic IFF there exists a function f from the domain D of S to the domain D* of S* such that the matching is "one-to-one" over ($x_1, \ldots. x_n$)" (Pincock, 2012, 34). This is the kind of relation that mapping accounts rely on in canonical "non-idealized" cases in the literature on mathematical representation and explanation like Pincock's own *Bridges of Königsberg* case wherein we come to understand why a set of bridge crossings is physically impossible by seeing it as isomorphic to a non-Eulerian graph-theoretic structure in which such a progression is mathematically impossible (Pincock, 2007, fig. 4).

[39] Elgin, (2017); Weisberg, (2013); Frigg & Nguyen, (2021); Pincock, (2007, 2012).

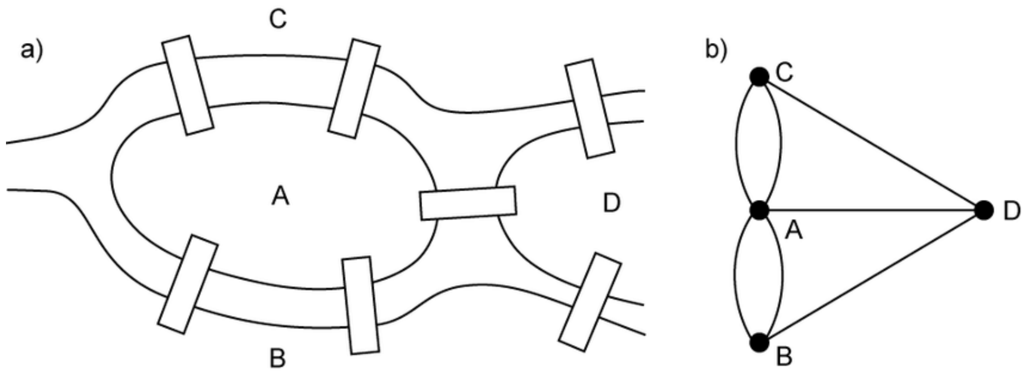

Figure 4: The Bridges of Königsberg (Pincock, 2007)

In this simple case, the existence and representation of such objective and difference-making *physical instantiations* of structures and the iso/homeomorphisms between them and mathematical models seems to supply the best answer to something like *no miracles* about the predictive success of such mathematical explanations for all of the cases of bridge-crossing to come. This is an intuitively clean example and does not cry out for a great deal of interpretation—this changes however when the examples introduce more drastic distortions.

Pincock here appeals to what I will call the method of stepwise interpretation. The steps are twofold, (i) identifying a structural isomorphism relation between a de-idealized "matching model" and the physical system and then (ii) identifying an "acceptable" mathematical transformation between the matching model and the idealized model equation or system where "acceptable" is determined by the aims and goals of scientists in a particular context (Pincock, 2007, 612).

This two-step interpretive process is what results in our being *justified* in construing the idealized claim, here the infinite degrees of freedom, as *schematic* – as having no physical content, therefore blocking our imputing the infinite depth to the physical system and preserving the veridicality, what Pincock more recently calls the "*underlying truth"* of the inferences drawn from the distorted model (Pincock, 2012, 32-34; Pincock, 2020).

To see how this works consider another more tractable example of an infinite idealization from example from Pincock (2007), *wave dispersion*:

> "When a rock is dropped in an otherwise calm ocean, it produces an irregular disturbance in the surface. In some cases, as the disturbance propagates outwards, it becomes more regular as the waves in the original superposition with a longer wavelength move more quickly than the waves with a shorter wave- length… To explain it, we start with a suitable version of the Navier–Stokes equations from fluid mechanics. This is our first model A and its content can be assigned using the mapping account. Then we consider the limit where the ratio of the depth of the ocean to the wavelength goes to infinity. This leads to the second model B. The second model is crucial to our explanation because it allows one to derive the following equation for the velocity c of a wave with wavelength"

Clearly, there is no physical analogue to this magnitude, but Pincock argues that we can, in this case mark the infinity as schematic in virtue of the mathematical link. The nature of this link and the content of model A shows "that the specific depth of the ocean is irrelevant as long as it exceeds a certain threshold" (Pincock, 2011, 215). Now, what is going on in the fixing of the content of A, the *matching model construction*, is where the philosophical interest lies. This is because construction of such a model is meant to capture *all* parts of the physical system (Pincock, 2007, 963).

As I read it, this is an instance of a much broader concept of *de-idealization*, which reveals itself as the justificatory ground of the idealization. Here idealized parameters are interpreted as irrelevant, mere-approximations, or "non-difference making" features given that we can reasonably expect them to disappear from successor theories that more completely and accurately captures the "full" mechanical picture of the physical system (Shech, 2023, 44). This kind of principle is captured in the literature on theories as opposed to models by Ruetsche (2011, 336) in her *Sounder Principle*: "No effect predicted by a non-final theory can be counted as a genuine physical effect if it disappears and stays disappeared from that theory's successors."

On this kind of procedure, optimism in total structural fidelity (is/homeomorphism) to the physical system provides the foundation upon which contextual aims can then operate, justifying the partial morphism between the idealized and de-idealized models and allowing us to rank their fidelity. These aims do not alter the objective relations but guide which transformations count as 'acceptable,' thereby completing the two-step interpretive process.

The important point for us is that these aims and interests that allow us to fix the schematic nature of idealized claims in (ii) are *ancillary* to, and separate from, the representational content of the model – the partial objective relations come first, then contextual interpretation follows. Interests and goals in no way determine the kinds of relations and transformations available, but merely how idealization via abstracting in might be is interpreted, *post hoc*, in light of them (Povich, 2024, 214).[40]

As may be predicted, the infinite idealizations in PT and RG do *not* behave in the same manner as the kind of infinite idealization in wave dispersion—the latter is, in Batterman's sense regular, while our infinities of interest are *singular* (Batterman, 2002; 2010). I argue that this is for two distinct reasons—both of which are supported by a conceptual point about *de-*idealization.

First, as regards the possibility of "matching model" construction, we noted that according to prominent physicists and philosophers, it is currently unclear whether *singular* cases of TDL and limit taking operations in RG are in principle or in practice dispensable (e.g., they feature "stubbornly" in our current best fundamental theories and explanations), a fact

[40] We might also note that aside from the method of stepwise interpretation, Pincock is relatively uncommitted as to which contextual considerations will be more or less relevant in which contexts. AR is meant to remedy what I take to be this problem of "unshared" standards. If there is no agreement on these features across contexts, then what Pincock calls a "limited realism" (2011, 221) is no realism at all.

which would prevent the construction of the matching model altogether, stopping mapping's process of justification at step one.[41]

Turning from practice, we can follow Davey (2011), in observing that there is, in fact, a three-pronged conceptual tension underlying the concept of *de-idealization*.[42] First, that in constructing or putting justificatory stock in a de-idealized matching model, we will necessarily employ further (though perhaps less "drastic") idealization (Davey, 2011, 21). In order to get from some physical system P_1 to some mathematical model M_1, at least *some* idealization must occur—particularly in our viewing P_1 as apt for a structural description in the first place.

From this follows the second and third points that fundamental theories are, in a sense, also essentially idealized and further that science itself is, in practice, *incomplete* (Davey, 2011, 33).[43] Whichever route to refuting de-idealization we choose—from essential idealizations or from more global pessimism—it seems that any account that requires the matching model for model evaluation will, in principle, make truth-sensitive justification too expensive and potentially unattainable in a large swath of important circumstances where the concept itself begins to unravel.

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[41] While there have been attempts to show that either PT or RG or both are, in fact, "dispensable," I will not discuss them at length here, and follow those authors (and physicists) who hold otherwise (see e.g., Sullivan, 2019; Wu, 2021 for critical views & Kadanoff, 2000; Batterman, 2002, 2010; Ruetsche, 2011; Morrison, 2015; and Palacios, 2019 for *indispensablist* views). We will tackle the notion of *indispensability* again in chapter three.

[42] See also Knuuttila & Morgan (2019) for a similar argument for the "irreversibility" of idealization.

[43] This observation of the ubiquity of idealization seems to thwart attitudes that often undergird "difference-making" + structuralist accounts of justification like Baron's (2019) and Strevens (2019) in which we can dispense of infinite idealizations given that they are only *in practice* indispensable and not "in-prinicple" (Baron, 2019). The attitude in question is that, based on a historical induction that science progressively approaches a more *complete* and *coherent* picture of the world, we ought to expect infinite idealizations to ultimately be idealized. One problem with this attitude and refuting them both leaves the basic logical premise of *no miracles* and the limited realism I argue for untouched. The first is the cogency of historical inductions in general (Shech, 2019). A line of argument I see being fruitful for approaching questions about de-idealization is to appeal to a direct implication between idealization and metaphilosophy (both epistemic and ontological), wherein it can be shown that no non-ideal theory is possible: human experience and inquiry is idealized all the way down. See D. Greco's (2023) *Idealization in Epistemology* for one interesting argument in this vein which I cannot pursue here.

The second major tension appears in Pincock's second step where the acceptable "transformation" between the matching model and the idealized (equation model) must be made in respect to RG specifically. Largely, it concerns the fact that these ancillary considerations will involve those of explanatory relevance and virtue. Batterman (2010) puts it as follows:

> "There are no structures (properties or entities) that are involved in the limiting mathematical operations. That is, limiting mathematical operations typically do not yield anything like the abstract non-Eulerian structure of the bridge system in Pincock's example. If the limits are not regular [de-idealizable], then they yield various types of divergences and singularities for which there are no physical analogues. Nevertheless… these singularities are essential for genuine explanation… In sum, if nontraditional idealizations do play an explanatory role, then mapping accounts simply cannot be the whole story about the *explanatory applicability* of applied mathematics in natural science." (Batterman, 2010, 19).

That is, even if we could lean on de-idealization and stepwise interpretation, it seems that on the strict view of structure and mathematical transformation traditional mapping accounts employ, they cannot account for the complete epistemic value of such essential idealizations, particularly the limiting *operations* (in RG specifically) invoked in the case study on critical phenomena wherein no kind of clear causal-representational role applies (McKenna, 2022). If we maintain, as I suggested earlier we should, that explanations be justified in a way that gives a robust response to NMI, then more must be said about how mathematics interacts with the world in a truth-functional sense without necessary appeal to structures and partial-isomorphisms in these cases.

In short, I take these observations to reveal that structural mapping accounts of applied mathematics make good on, in many existing cases in the literature like *Konigsberg* and potentially *wave-dispersion*, the demand for robust justificatory conditions that satisfy the reasons/justification distinction underlying NMI. They supplement an account of representational content of individual models with an account of exactly how their mathematical relationships supply *accurate* information about a system (via a given morphism).

However, the failure of de-idealization in the case of singular limits reveals (i) that de-idealization *cannot* be a necessary condition for justified idealization, and (ii) that limit taking operations, on the formal mapping view of physical and mathematical structure, are explanatorily valuable in a way that does not involve structure-preserving mappings in the strict mathematical sense (Frigg, 2002; Batterman, 2010). Mapping, then, cannot be the whole justificatory story about the epistemic role of mathematics.

2.3.2 Fully Inferentialist Theory and Pragmatism

These problems with mapping accounts have pushed many authors to reject the referentialist condition for the existence of objective relations in combination with de-idealization all together. One route is pursued by the inferentialist. Though there are many flavors of inferentialism dating back to Sellars' and Brandom's work in the philosophy of language, I will focus on Mark Povich's (2024) *Fully Inferentialist Theory* (FIT) as a recent and sophisticated example also particularly concerned with applied mathematics.[44]

The central idea behind FIT is that "the content of a model is *determined* by the inferences that are made from it, and the inferences to be made from it are determined by the form of the model and the denotational conventions surrounding it" (Povich, 2024, 191, my emphasis). When we invoke and interpret structures involved in applying concepts – there is emphatically not any implication of "metaphysical dependence" between the structural-

[44] Sellars, (1968); Brandom, (2000); Khalifa et. al, (2022); Suarez, (2024); Warren, 2020.

mathematical description and any worldly truth-maker (presumably a physical structure or property) (Povich, 2024, 173).[45]

Mathematical representation, in this way on FIT, is deflated to an entirely intentional and rule-governed procedure. In light of the apparent awkwardness of the incorporation of explanatory aims and pragmatic goals of science on traditional mapping accounts when faced with the problem of essential idealization, FIT simply reverses the fit direction of fit. The structural morphic and non-morphic relations that are relevant are determined solely by the interests and conventions of scientists applying mathematical models insofar as they become the primary grounds for the acceptability of the inferences gleaned from applied mathematical content (Povich, 2024, 171).

This being the case, the above problems posed by Davey and Batterman concerning the existence of "non-traditional" idealizations which are nonetheless warranted seems trivial. On FIT, "we ought to expect that there should be significant, ineluctable, and often unspecified differences between the form of representation [the model] and the structure of the target" (Povich, 2024, 215). So long as taking the TDL affords us with the kinds of empirically adequate inferences about and of universal critical behavior that are consistent with the conventions surrounding limit taking operations in mathematical physics, FIT can happily account for the inferential and explanatory power of such operations without requiring a clear-cut accuracy condition for justification.

[45] Bueno & Colyvan, (2011) & Bueno & French, (2018)'s "inferentialist-mapping" accounts seem to present a viable middle-way, but Batterman's second critique applies to them equally in that they think that operations are set-theoretic structures all the way down (thanks to Travis Mckenna for pointing out this alternative weakness of such accounts). I add additionally that they will (insofar as they retain their inferentialist bent) fail to satisfy NMI as well. Thus, these accounts in fact take on the *problems* of both mapping and FIT (a lack of sufficient tethering *and* an inability to account for non-structural applications).

A similar, yet even more liberal take on model success is proposed by Wendy Parker (2020) in her adequacy-for-purpose (AFP) view.[46] Again, the basic thrust is here is that, eschewing the mapping request for accuracy conditions based on a distinct morphism or transformation, "model quality is to be assessed relative to a purpose; model evaluation seeks to learn whether a model is adequate or fit for a particular purpose" (Parker, 2020, 458).

What exactly it means to be adequate for a purpose is a complex and context-sensitive matter (in much the way that "adequacy" (ii) is determined on mapping in the face of idealization), but "in general terms what is required is that the model stand in a suitable relationship with a target, a (type of) user, (type of) methodology, (type of) circumstances, and purpose jointly" (Parker, 2020, 475).

Both of these views, I believe, even in the schematic presentation given here, face the same major worries. In leaving the standards of assertability and adequacy deeply ambiguous, they explicitly eschew the reasons/justifications distinction, and as such cannot give our desired kind of reply to NMI. That is, they make justification 'too cheap', in part by leaving what are supposed to be "shared" standards largely *unshared*, or idiosyncratic.

While there is a sense in which this is likely to be seen by these authors as a feature rather than a bug of their accounts, they both still voice concerns about this rather unfortunate lacuna. Take Povich: "the form of representation and its representational conventions determine which inferences we are allowed to make. Note that this does not imply that the conclusions of those inferences—the claims about the target—are justified, true, or accurate… denotation is prior to and meant to constrain inference. It is then a substantive question whether those inferences are

[46] One might understand this approach to adequacy and explanation as a descendent of Van Fraassen's (1980) *erotetic model* of explanation and approach more generally. See also Fagan (2025) for a "particularist" account of model/explanatory success which is based on a minimalist account of *scientific understanding*.

justified or not" (Povich, 2024, 196). And Parker: "in some cases, of course, evaluators may remain highly uncertain whether a model is adequate for any purposes that are currently of significant interest. But this is not a shortcoming of an adequacy-for-purpose view; it is just an *unfortunate* epistemic reality" (Parker, 2020, 473; my emphasis).

Povich perhaps makes a more conscious effort than Parker here by appealing briefly to some forward-looking options the inferentialist has for justification and truth. Of note as options for him, deflationary notions of "trueing" wherein the model and the world are brought "together" relative to some purpose, or else a "models-as-tools" view wherein some undefined "cooperative" reliable relation between a model-artifact and the world delineates set of correct applications in virtue of practical constraints (that come both from the world and our limited nature) (Andersen, 2023; Knuuttila, 2011).

To my mind however the more fundamental worry about anti-realist views in relation to the kinds of applications the above case studies represent, is that they don't currently have adequate resources to respect what actual physicists take to be one fundamental constraint on interpreting statistical mechanics and thermodynamics. Namely, that a good explanation of "how and why processes in nature take place… surely has something to do with how… the systems [*really*] behave and not with what we happen (or fail) to know about them" (Frigg & Werndl, 2023, 63).

While AR will indeed help itself to *some* social and intentional elements as at play in the justifying of idealizations, it remains consistent with this kind of practice-focused realism that respects NMI to insist that at least some of these purposes—and hence standards—will need to have something substantial to say about accuracy conditions in order to be, in fact, adequate or assertable (Rice, 2025).

2.3.3 Taking Stock

At this point, the justificatory dilemma presents itself more clearly: structural mapping accounts of applied mathematical models and modelling techniques can give an easy answer to the question of justified inference in many regular cases (e.g., isomorphisms in *Königsberg*) but struggle to successfully justify in the cases of singular limits where de-idealization fails and partial-structural relations are insufficient characterizations of mathematics' epistemic role.

Reversing the direction of justificatory fit on deflationary FIT and AFP pragmatism reveals conversely that some level of contextual, intentional, and non-structural (in the traditional morphic-sense) explanatory considerations are required for a universalizable account of applied mathematics and idealization. However, the resources available for satisfying NMI are either unsatisfactory or non-existent.

On both FIT and AFP, there is an almost trivial sense in which representation and explanation will be a holistic, "open-ended, and idiosyncratic practice" (Povich, 2024, 218). And while I agree that both holism and open-endedness are features of applied mathematical practice that any account should, in fact seek to capture rather than mitigate, such an account need not imply idiosyncrasy. Rather, even in the limiting cases of infinite mathematical idealizations, this is compatible with our aiming at (and hence being justified by) the truth of our models. Let's see how this might be done.

2.4 General Mathematical Aspect Realism

The shortcomings of these approaches reveal a deeper issue: neither can fully explain how mathematical techniques like limit-taking remain epistemically valuable when they lack access to de-idealization or epistemic optimism is unwarranted. To adequately account for this, an approach is needed that respects both the realist intuition behind NMI and the holism/contextual sensitivity of scientific practice. This is where a certain neo-Wittgensteinian philosophy of mathematics becomes compelling as a framework for understanding truth and justification as embedded in modelling perspectives, or *forms of life*.[47] By situating mathematical techniques in the evolving conventions, explanatory aims, and multi-scale phenomena of science, we can preserve their *truth-aptness* without demanding a structural correspondence of a "god's eye" variety.

The approach I favor here is Juliet Floyd's (2021) *aspect-realism* (AR). The basic idea is that we take very seriously Wittgenstein's injunction that mathematics is made up of a "motley" of techniques, which are "invented", and aspects are "discovered" (RFM, pp. 88):

> "The later Wittgenstein emphasizes the importance of projectability and plasticity; the work of fitting concepts to reality, including the reality of mathematical (and other) experiences. This form of realism glosses the wide and multifarious kinds of applicability of mathematics—to empirical situations, to mathematics itself, to experiences, and to concepts." (Floyd, 2021, 2).

To begin explicating aspect, technique, and the kinds of justificatory relation in which they stand, it is necessary to start with the most salient feature of AR: *contextualism* about truth-functional justification. As Wittgenstein himself puts the guiding intuition—particularly in the context of interpreting limit taking techniques: "I must not say, not: … 'we must not express ourselves like this'… but 'test the justification in this way'. You cannot survey the justification of an expression unless you survey its employment" (RFM, II, 18).

AR thus captures the thrust of Davey's (2011) and Liu's (2019) approaches which hold that we can understand the idealizations in PT as holistically "true" and as such justified relative

[47] *Perspectivalist* accounts like Giere (2006), Massimi (2022), represent promising avenues of exploration here, though I do not have the space to engage directly with these views here paper. These views are largely neo-Kantian in nature, and as such will naturally have some subtle contrasts with AR over the nature of the properties and models involved. I return to this question in the final section of this chapter.

to a set of objective standards for the attribution of properties and structure (Davey, 2011, 3; Liu, 2019, 1885). On AR, as Floyd puts it, statements and model-inferences "may be said to express truth-conditions *sharply*…but individuating which among the several possible truth-conditions expressible with a [model]-form is expressed on a particular occasion is a matter of being able to embed it in specific situations meaningfully" (Floyd, 2021, 51). These "situations", I'll argue, are local scientific *forms of life* that anchor the "shared standards" for interpreting both s*tructure-based* representation and non-structural explanatory relevance as they are expressed in practice.

Shapiro (1997) puts the insight as follows: "there is no reason why the present program of … realism must presuppose this God's-eye view of reality," rather, it would be a serious philosophical misstep to deny Putnam's (1999) claim that any plausible "realism is… internal to our 'form of life'. It is hard to imagine a perspective external to that, but this, I suppose, is Putnam's point" (Shapiro, 1997, 65).[48]

In the case of empirical application, I take this now-platitude as a rejection of de-idealization optimism as a necessary condition on justification for infinite idealizations in cases like PT and RG, and an endorsement that despite it, such constrained mathematical perspectives and singularities can still help us "see" through "windows to reality" and not merely clarify our ossified rules of practice (Floyd, 2021, 5; Massimi, 2022).

In an effort to avoid overdoing the exegesis and remain in conversation with the above contemporary views I will make explicit what I take to be constitutive of the *form of life*. Broadly, the idea is that the "shared standards" for interpreting potential *aspect-morphic*

[48] An enlightening discussion of the reasons why such an anti- "view from nowhere" realism is possible in light of the problem of idealization (and of overlapping representations), despite criticisms (see, e.g., Chakravartty, 2017; Morrison, 2015) is given in Massimi (2022, pp. 57-72) and Ladyman & Ross (2007). The main thrust of the argument is that we need not assume realism to imply realism about *essential* or deep properties – but restrict ourselves to what Massimi calls "modal phenomenal" properties, or what I here call *aspects.*

relations which justify the inferences, explanations, and understandings derived from idealized mathematical models are neither entirely rule-governed (conventional), nor entirely "brute" stipulations given by "self-standing" formal structure of a *naively* mind-independent world populated by those structures (Floyd, 2021, 57; PI §88; Ladyman & Ross, 2007; Maddy, 2007, 343).

Instead, one comes to see that the kind of "agreement" which constitutes the justificatory bedrock for truth-functional relations undergirding applied mathematical arguments involves a "naturalistic, evolving web of contingencies on which ride the necessities of…mathematics" (Floyd, 2021, 57). More specifically, I take the web to be a conjunction of three factors.[49] The first is empirical information related phenomenal "scales."[50]For example, in the case of TDL in PT, statistical mechanical and thermodynamical representations (discrete v. continuous) may be said to differ in *scale* (Shech & McGivern, 2021, 1414).[51]

Second and third are social-conventional facts and intentional content. Here I draw on Povich's (2024) inclusion of these factors into the importation-set that constrains denotational inferences on FIT but take them to have a very different function. As he puts it idealized models and mathematical techniques more generally are "public artifacts, with surrounding accepted practices of use and shared [theoretical] knowledge about targets. But representations are often employed and reasoned with by individuals who have their own…purposes and knowledge" – I agree that these observations are trivial and will even add to this content individual physicist's

[49] Note that this construction of the form of life is similar to Maddy's (2014) "trio of our interests, nature, and the world's regularities". See Bangu (*forthcoming*) for further discussion on Wittgenstein on mathematical concept formation that builds on Maddy's naturalism in this vein.
[50] By scales here I mean observational and experimental content differentiating between "energy levels and regimes, as well as different lengths and time scales" that are relevant to our interpreting and understanding the ontological content physical systems (Shech & McGivern, 2021, 1413).
[51] This is a very schematic definition. See Batterman (2013) for further discussion.

experiences of "seeing" physical systems and models from a particular stance or perspective (Batterman, 2002, 24; Povich, 2024, 200; Baker, 2022).

Recall however that on FIT social and intentional content—here related to both "general and individual aims in explanation and understanding"—was construed as merely constraining conventional assertability of inferences (Povich, 2024, 201). I wish to include this set of content more constitutively and holistically as follows. On AR, in conjunction with *scale*, the domain of relational and modal features of the world (*aspects*) that might be accurately represented, explained, or discovered by a given technique *constrained* and *individuated*, but not thereby constructed, by these social-epistemic and intentional features.[52]

The second central feature of AR, which I take it falls out rather readily from (though is not entailed by) contextualism: *pluralism*. This is simply the view that there are many different *ways* in which a model might be tethered to features of the world (qua form of life) such that truth-functional justification is supported, and NMI is answered. This is a relatively simple thesis (though not, of course, uncontroversial), and as such it will be best to define by ostension to PT and RG.[53]

2.4.1 AR Applied

With this basic view about justification and truth in hand, lets return to our cases. There, this holistic but world-sensitive determination factors in a plurality of important ways. Understanding this plurality helps us to avoid monist missteps in accounting for NMI and explanatory power. First, I'll consider how AR interprets our determining the "epistemically

[52] Part of the motivation for presenting a neo-Wittgensteinian alternative to neo-Carnapian FIT comes from the frequent inclusion of Wittgenstein among the progenitors of conventionalism and inferentialism (Povich, 2024,1; Zalta, 2024). This is based on the (at least at one point) orthodox reading that derives largely from Kripke (1982). The heterodox readings I take as my starting point, are more closely related to the tradition of Putnam (1999), and Floyd (2021). For a similar holism— "foundational holism" that AR is largely sympathetic to see Sher (2023).
[53] An interesting connection here would be an application of alethic pluralism (in particular Gila Sher's (2023) *correspondence pluralism*) but I leave this for further research.

fundamental" ontology of the system or *real pattern* relevant to a representation, explanation, or inference (Ladyman & Ross, 2006; Liu, 2019; Wallace, forthcoming). Second, I argue that these models capture truth-functional *modal information* about universality classes and *exemplify* universality qua explananda in itself (Rice, 2022; Morrison, 2014).

By way of this I'll also argue that in more regular (non-essentially idealized) cases where clear structural dependence relations do hold like *Konigsberg* (as I'll argue is the case in TDL for PT but not RG), AR-like contextualism is required for coming to see systems as instantiating morphism-apt structures in the first place (Baker, 2022). Finally, I will gesture towards how AR makes room for some newer, less straightforward relations which appear important for scientific practice.

Let's take them in turn. Specifically in the case of the first singular infinite idealization in PT, consider how Shech & McGivern (2021) describe what it might mean for an idealized model to capture a property of the target system that is *epistemically fundamental*:

> "In taking the thermodynamic limit, it is suggested that we are in essence transitioning from the (ontologically fundamental) microscopic scale and realm of finite number of degrees of freedom, i.e., the regime of statistical mechanics proper, to the (perhaps epistemologically fundamental) macroscopic scale and realm of an infinite number of degrees of freedom, i.e., the regime of thermodynamics. Thus, while statistical mechanics remains fundamental in the ontological sense—real systems are finite!— there is a sense in which it is *explanatorily inadequate* without the thermodynamic limit. In this sense, the realm of thermodynamics is supposed to be epistemological fundamental." (Shech & McGivern, 2021, 1415).

On AR, the information about phenomenal differences between statistical and thermodynamic scales, social-epistemic content related to explanatory virtues and relevance (e.g., that continuous phenomena are relevant targets of representation and explanation), and individual background beliefs and commitments like Kadanoff's (2000, 235) and Batterman's (2002) insistence on the indispensability of TDL for the "existence" of critical phenomena intertwine to modulate our interpretation of what exactly the techniques and models are up to when we employ them.

Taken together, they allow us to understand what at first appeared an intractable idealization as, from this situated perspective, a descriptively *accurate* modelling technique that captures the relational, epistemically fundamental, property that the finite system in question is qualitatively continuous, and not discrete (Reuger, 2005; Liu, 2019).[54] In such cases de-idealization, in the traditional sense of a matching model wherein all model magnitudes have correlate physical magnitudes, even if it were possible, would seem not to be desirable—the higher level perspective of description is what we need to *get the ontology right*.

I find it helpful here, as a brief interlude, to turn back to the more regular case of applied mathematics in the *bridges*, because it appears that a similar sort of embedding may also be required. It seems clear that the second kind of intertwining is at play. As Baker (2022) notes, one problem with structural morphism-based explanations *in general* is that in order to get off the ground, they require us to treat physical phenomenon *as if* they instantiate a particular physical *structure* (Baker, 2022, 53).[55] In order to take this stance and isolate a single relevant structure (of the many possible ones), he argues, certain intentional and explanatory factors must be appealed to.

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[54] One might argue that this kind of induction violates NMI. However, this kind of explanatory indispensability (or *explanatory autonomy*) arguments are strongly motivated elsewhere in in the philosophy of physics, mathematics, and metaethics (see Colyvan, 2010; Baker, 2009; Enoch, 2011; Rice, 2022). The reifications of continuous systems should be less contentious than that of mathematical or normative facts given that they are much more easily construed as causally efficacious. The naturalistic construal of AR is amenable to such arguments, though in this paper I remain relatively abstinent on what "reification" amounts to for *epistemically fundamental* aspects.

[55] This "assumed structure problem" or "problem of structural overdetermination" is one perennially acknowledged, but rarely dealt with by philosophers of mathematics – see Mac Lane (1996) for a discussion of these difficulties. Essentially the problem is that physical systems are complex to a degree that they may inevitably approximately instantiate a great deal of possible structures (which exactly of these are more "instantiable" than the others is a different and difficult question—though again, I think a pluralism will be required). See also Maddy (2007, pp. 318-328).

For the *bridges,* “the structural stance is adopted, in which the physical phenomenon is treated as if it consists solely of certain objects (in this case, landmasses) and relations between these objects (in this case, being connected by a bridge). The resulting structure is then identified, whose amenability to mathematical analysis makes this stance not only structural but also mathematical.” (Baker, 2022, 11).

What I want to say here is that the taking of the stance required to vault this “assumed structure problem” represents a similar, contextually embedded appeal to something like epistemic fundamentality in applying mathematics as in the thermodynamic interpretation above. That is, even in more regular cases of successful applied-mathematical explanations, appeal to the *intentional and social features* of the form of life are required for us to accurately interpret the complex mind-independent world *as* structural to “get the ontology right” in a context. This reveals to my mind that more idealized cases have a similar interpretive structure to more regular ones. Exploring justified idealization via applied mathematics then has important implications for applied mathematical explanation in general.

That said, let’s return to the infinite idealizations again to expand on the set of model-world tethers we might appeal to satisfy NMI. Here AR can, I argue, respect at least two additional possible justificatory relations in the more complicated TDL cases: those between model-systems and real *universality-classes*, and between RG and *universality* simpliciter, understood as an instance of a “real pattern.” In the first case, we can follow Rice (2020) in observing that:

> “Discovering universality classes [by way of appealing to the idealized model given by TDL] can demonstrate that there *is* a class of systems, which includes an idealized model system and its target system(s), that will exhibit the same stable patterns of behavior. It is precisely this *stability* of these universal behaviors across perturbations of the system’s

> physical features that can enable scientists to justifiably use models that drastically distort difference-making causes" (Rice, 2020, 832-33).[56]

That is, even in cases where no story about "difference-making" per se (which we have seen generally requires optimism in a de-idealization narrative and some matching-model construction), infinite idealization can be justified by an objective relationship, here one of mutual *participation* in a modally stable set, between model system M_1 and target system P_1.

The case of universality *simpliciter* is perhaps more interesting and highlights additional justificatory conditions. My intuition, drawing from Batterman (2002; 2010) and McKenna (2021), is that indispensable mathematical techniques like RG that help us explain and understand type (ii) questions about universality allow us to justify, again without de-idealizing, "*seeing*" disparate systems as real instances of a common pattern that itself exists in the world and not merely as cognitive coincidence. The kind of relation I think is at work here—and one which fits well with the broader framework of AR—is one of *exemplification*, or "representation-as" (Elgin, 2017; Frigg & Nguyen, 2021).

Consider Batterman's remarking that "the asymptotic investigation of the behavior of systems at or near these singularities lets us '*observe*' properties as they issue from concealment and obscurity", or Ladyman & Ross: "the universalities on which [Batterman] focuses would be *invisible* to inquirers who confined their attention to the scale on which micro-properties (relative to them) are measured" (Batterman, 2002, 125; Ladyman & Ross, 2007, 204, my emphasis). Though we need not agree univocally with Ladyman and Ross that the fundamental properties we take to be described accurately are *structural* in nature, on AR they will be nonetheless

[56] This appeal is separate from the renormalization group explanation which can be considered a different kind of justifying story—though I disagree here with Shech (2023) that RG serves as a kind of *de-idealizing* story broadly conceived given that it too involves TDL (to my mind this is in line with Davey's observations above). Thanks to Collin Rice for correcting this interpretation of his own view.

largely relational (and modal), as opposed to the traditional *essential* properties proposed by the realist (Massimi, 2022, 52).

In *exemplification*, we understand some model X as exemplifying a property of a system Y given that it affords "unique epistemic access" to that property (Elgin, 2017; Frigg & Nguyen, 2021).[57] What I'm suggesting is that we ought to construe *universality itself* (not only universal common features like dimension and symmetry) as higher-level property of things in the world that can be *exemplified* by RG explanations in the same way that "a wave on the beach is a real pattern to a surfer, or a lifeguard, because it is taken as the basis for prediction and explanation" (Ladyman & Ross, 2006, 103-104; McKenna, 2021, 731). Asymptotic mathematical techniques not only represent and help pick out real classes of systems, but actually allow us to *see new physics* in these cases.

Thus, type (ii) explanations that appeal to RG do indeed pick out real features of the world, though not ones restricted to *individual* target systems and physical structures (contra. Ladyman & Ross's ontic structural realism), even given the fact that individual target systems will retain the "universal" properties of symmetry and dimension. Following Morrison (2014) we can understand this "structurally stable behavior" as the justificatory grounds for understanding the "robustness of predictions" that RG operations afford (Morrison, 2014, 1155).

[57] It has been raised as a problem for *"representation-as" accounts* that they do not answer the question of the applicability of mathematics and the distinctiveness of scientific representation more broadly. In response, I'll follow Povich (2024, 168) and Priest (2022) in taking a "disjunctive" or descriptive approach to demarcating mathematical and non-mathematical representation. I take it that AR is compatible with both kinds of concept application and that it is a strength of the view. The same kind of roughly functionalist approach to demarcation also applies to how AR will respond to demands that an account of representation be able to tell us what is "distinctly" scientific about some representations or explanations as opposed to others (linguistic or aesthetic), and further what is distinctly "mathematical" about mathematical models. Though one characteristic is being "structural" broadly construed, on AR this is a sufficient but not necessary condition for being mathematical (hence the ability to include RG).

Recalling the alethically-basic status of facts about scale, conventional, and intentional content here we can see that this is—given contextualism—a legitimate way situating model relationships to relational properties very generally. Our surrogative reasoning in all such cases then is to be *justified* in terms of an objective relationship to the properties imputed to the target with no recourse to stepwise interpretation *or* de-idealization: the idealization's truth-value is taken holistically (Frigg & Nguyen, 2020,1).

Since the "shared-standards" for the interpretation and translation of the instantiation relation are grounded in *both* worldly and social/intentional features of the form of life, we can see how these objective relations become *epistemically* valuable beyond "mere" representation. That is, appeal to this relation is interpreted in light of certain *explanatory* and understanding-oriented interests like unification and simplicity, thereby helping individuate and bring out the structurally stable behavior as the relevant explanandum for RG in particular (Batterman, 2002, 34).

What to notice here is that in broadening the scope of the relations that might hold between an idealized mathematical model or a given limit taking technique and the world, we've moved in a sense away from full and partial structural isomorphism and ended up at something closer to exemplification of both modal and higher level structural properties—though showing that even in cases where structure *is* in play, the modelling *context* plays equally anchoring roles in evaluation and interpretation. There are many *good* reasons for using idealizations, and they play different roles.

Yet, despite their plurality and contextual determination, all these "tethers" are sufficiently truth-apt such that the inferential move between P_1 and idealized-M_1 need not be translated in a *stepwise* fashion, rather, it is both holistic and naturalized. This recognition has also, in a generalized form, helped us make progress on circumventing the *assumed structure problem* without accepting deflationary inferentialism or pragmatism.

This set of relations exhibited by PT and RG fails to exhaust justificatory roles countenanced by AR. That is, even if there is a way of construing RG as somehow standing in a quasi-representational relation (as I have above) Batterman's second critique—of the failure of structure—still holds in that there will be less clearly representational sources of justification that are sensitive to features of the form of life, thus setting the account even further apart from more "flexible" versions of mapping (e.g., Pincock, 2020).

Though a contentious issue, it is now widely accepted that there are many *kinds* of mathematical explanations (Mancosu et al. 2023; Pincock, 2023). Not all of them involve representation of target systems—certainly, as we've seen, not *difference-making* factors, whether they are of the causal or non-causal sort. This reflects a generalization of Batterman's (2010) and Rice's (2022) earlier points that some mathematical model and operation applications stand in more complex relationships to the explananda than is allowed on structural mapping accounts. One can expect that more relevant roles and relations will be uncovered as our modelling technologies and procedures become more complex and localized.

Mckenna's (2022) and Kasirzadeh's (2021) accounts represent such possible extensions into much more complex justificatory relationships that pervade applied mathematics, and which cannot be captured in a strict notion of structural mapping. This is particularly problematic, McKenna claims, in cases of *multi-scale modeling* where different pieces of mathematics are related not only by means of formal transformations, but on the basis of "empirical information" (McKenna, 2022, 1).

Kasirzadeh similarly proposes a "bridging" relationship that is equally complex that holds between models, interpreters with distinct aims, and empirical targets. Her proposal "illustrates how mathematics acts as a reliable connecting scheme in our explanatory reasoning about different representations of an empirical phenomenon," (Kasirzadeh, 2021, 704). I'd like to suggest that something like these mathematical bridges, insofar as they are modally and communicably robust across models and context, could certainly constitute another *justificatory* reason (as opposed to a purely pragmatic reason) that one might use to provide a minimal response to NMI.[58]

AR's pluralist-contextualist approach is meant to be ecumenical over these emerging roles with its explicit inclusion of scalar information and explanatory salience in the FOL, which allows us to understand how the formal and the empirical intertwine holistically across a variety of cases and contexts to individuate and constrain tethers between model and world, as such providing a promising framework for such multiscale modelling endeavors. AR thus respects the "open-texture" of scientific investigation and the role of mathematics, particularly as applied to emerging problems in the sciences, as *both* invention and discovery (Wittgenstein, 1956).

2.5 Objections and Replies: Relativism and Overdetermination

I have argued, primarily through illustrative cases, that the justificatory conditions for infinite idealization in applied mathematics are both plural and context sensitive. At the same time, I have maintained that aspects can still be represented and explained accurately, such that the *accuracy* of our best essentially idealized models and theories (like statistical mechanics) can

[58] Again, work here ought to be done to combine the literature on applied mathematics with *perspectivalist* approaches to scientific epistemology like Massimi's (2022) which emphasize the role of models as blueprints which are robust enough *across* perspectives to licenses minimal-realist conclusions about their content. Stemeroff (2022) is critical of such an approach for fundamental physics in the case of conservation laws, but more case studies should be considered.

be preserved. At this point, however, AR confronts a familiar tension: the apparent incompatibility between plurality, novelty, and objectivity (Ludwig & Ruphy, 2021).

The central challenge can be stated simply: on what grounds can AR plausibly claim to be a form of scientific realism? One initial answer appeals to intuitive continuity with traditional realist commitments, particularly the reification of unobservables and the idea of historical scientific progress (Rice, 2021, 262). Nevertheless, a more compelling response—especially for anti-realist critics—requires addressing two distinct but related worries. The first concerns a potential collapse into relativism or constructivism. The second concerns explanatory overdetermination and the causal closure of the physical.

. Because AR distances itself from ontic structural realism regarding the physical relata in aspect-morphic relations—specifically, by refusing to restrict the content of mathematical application to purely structural features and instead treating contextual features as partially constitutive—it must clarify what, if anything, is preserved across Forms of Life such that AR qualifies as realist in a substantive sense (Ladyman & Ross, 2007; Morrison, 2015, 169; Chakravartty, 2017; Shech, 2023).

The core of this objection concerns the relationship between contextualism as an epistemic thesis and contextualism as an ontological thesis (Massimi, 2022, 58). The epistemic claim holds that scientific knowledge is always obtained from within particular modelling contexts. The stronger ontological inference asserts that, because knowledge is context-bound, the metaphysical structure of the world itself becomes unknowable or indeterminate. The relevant question, in the present case, is whether contextually fundamental descriptions—such as thermodynamic descriptions in phase transitions or universality relations in renormalization group (RG) explanations—can be regarded as real features of the world.

The most promising response available to AR is to clarify that its conception of properties, grounded in aspects, treats many scientifically relevant properties as *relational* and *modal*. This position leaves AR largely agnostic about intrinsic or essential properties (Massimi, 2022, 74). When combined with the claim that truth-conditions can be plural and contextually constrained, AR can follow Rueger (2005, 580) in maintaining that apparently inconsistent models can nevertheless be true descriptions without undermining commitment to a mind-independent reality:

> "The important point for the realist here is that the two different spatial scales or perspectives are different assumptions with respect to a relational property of the system rather than with respect to one of its intrinsic properties… what look like incompatible intrinsic properties in our models are only ascriptions of intrinsic properties from different perspectives" (Rueger, 2005, 586).

On this view, AR endorses a modest inductive commitment to complementary relational properties plural and idealized models successfully describe. These properties are treated as consistent with an underlying, though only partially conceptualized, independent reality. This approach blocks the worry that successful models fail to make contact with the world. Instead, models capture relational features of the world while remaining agnostic about deeper intrinsic structure. [59]

This position aligns naturally with AR's inheritance from Quinean and Putnamian naturalistic and "realistic" realism. Although knowledge is irreducibly situated, the domain of the "unthinkable" or non-truth-apt remains historically open and revisable (Floyd, 2021, 52). Forms of Life thus function as local and partial modelling contexts that remain responsive to empirical discovery and to the open-textured character of scientific concepts (Putnam, 1962).

[59] Massimi (2022, ch 9.2) argues that the root of the traditional realist's problem with contextual or perspectival existence claims lies in their assumption that realism must be a realism about "deeply essential" properties. This kind of anti-essentialist realism is, I take it, a direct consequence of the more conceptual worries with de-idealization explicated above.

AR therefore incorporates both passive and active dimensions, what I take to be an apt characterization of scientific practice (Floyd, 2021, 2). While remaining non-committal about essential features of the world—particularly concerning theoretical reduction and intrinsic properties—its contextualism maintains, following Liu (2019), that context-relative property ascriptions still successfully describe the world's furniture in relevant respects, perhaps the only scientifically relevant respects. As Liu observes, "without certain anchoring assumptions that ensure a context, cats and dogs will be no different from heaps and molecules, and phase transitions will appear no sharper than any changes of the configurations of heaps of molecules" (Liu, 2019, 1912).

If we accept the existence of ordinary objects and properties—and plausibly a richly heterogeneous "rainforest" ontology that includes universality classes and real patterns—then contextual sensitivity becomes a condition for realism as opposed to deflation. Similar, though often more formalism-dependent, considerations apply to scientific practice generally (Putnam, 1999; Massimi, 2018).

This conclusion modestly extends Massimi's (2022) own realist defense of perspectivalism, which appeals to "inferential stability" regarding phenomena. On her account, perspectival models generate a "garden of forking inferential paths" through which scientists converge on inferentially stable characterizations of modally robust phenomena (Massimi, 2022, 74). AR, however, implies a somewhat richer ontology: phenomena may be characterized not only modally but also in terms of relational properties more broadly.[60]

Moreover, AR assigns a stronger stabilizing role to mathematics than Massimi's primarily inferential conception allows (Massimi, 2022, 144). Mathematical modelling

[60] See Marshall & Weatherson (2023) for an overview of the properties literature.

techniques are not merely blueprint-like public artifacts (a la Massimi) or even tools—a view associated with deflationary approaches such as Knuuttila (2011)—but possess a deeper naturalness. On AR's liberal naturalism, mathematical application is emphatically not anarchism or pragmatism. The novelty experienced when mathematics is applied to physical systems reflects a "reorientation of our way of seeing that comes when we encounter mathematical features, forms, and structures anew" (Floyd, 2021, 5).

In applying mathematics, this plurality of constraints show us not only what feels right, but what *is true*: success in either case is no miracle. Accordingly, the formal structures that underwrite applied mathematical modelling belong to the relevant Form of Life—here, mathematical physics—and possess more than merely instrumental significance (Stemeroff, 2022, 517).

A related concern arises from AR's pluralism about *explanation*. Critics may argue that allowing causal, non-causal, modal, unificatory, exemplificatory, and other explanatory relations to function as contextually legitimate forms of explanation risks overdetermining the explanandum in a manner incompatible with realism (Mancosu et al., 2023, 1; Pincock, 2023, 45). This objection parallels Kim's (2005) exclusion argument, which appeals to the causal closure of the physical domain to challenge the explanatory relevance of higher-level properties or structures. Comparable concerns are often advanced by explanatory monists, such as Woodward (2021), although the dominance of monistic approaches has declined in recent literature.

AR responds by rejecting the assumption that explanatory pluralism entails incompatibility. Drawing on Batterman (2021) and Baker's (2022) mathematical stance, AR instead holds that different explanatory modes correspond to different legitimate questions about complex worldly systems. The multiplicity of explanations reflects the multiplicity of explanatorily relevant aspects—modal, structural, phenomenological, and otherwise—that modelling techniques can reveal. Insisting that these explanatory modes are mutually exclusive would obscure *a central feature of scientific practice*: the joint natural and conceptual complexity of the phenomena under investigation (Psillos, 2007).

By adopting structural perspectives differently across modelling contexts, scientists can isolate particular aspects or structures for explanatory purposes. In some cases, these perspectives yield explanations that outperform fully causal accounts. While AR remains neutral regarding reduction, it can nevertheless support the weaker claim that higher-level explanations are explanatorily superior to lower-level explanations in certain respects. This "aspectual" character of these explanations illustrates the plurality of ways in which modelling techniques remain *tethered* to reality.

2.6 Conclusion

In this chapter hope to have demonstrated what a pluralistic and context-sensitive set of epistemically justified relations available to a philosophical account of applied mathematics might look like as applied to one case of mathematical idealization in scientific practice. In doing so, I've argued that explanatory relevance and other local considerations can do a great deal of heavy lifting in helping us interpret the content of idealized models realistically in cases where de-idealization is either unavailable or, perhaps more often, undesirable.

Mathematical techniques, herein graph-theoretic procedures and asymptotics, help to explain and make epistemically available "aspects" – the broad set of modal-relational "necessities, facts, and characteristics"—of the world as approached, as it must be, from inside the situated form of life (Floyd, 2021, 51). AR thereby localizes (domesticates), rather than deflates, the notion of accurate representation and epistemic justification such that NMI might still be given something like an answer grounded in truth across multiple distinctive roles.

While this analysis is to an extent schematic, as our brief consideration of the *assumed structure problem* indicated I believe it points to the broader conclusion that appeal to accounts like AR helps reframe how we think about the distinctive roles of mathematics in science along with the nature of scientific realism and explanation. If the limiting cases of singular infinite idealizations can be justified and explanatory without de-idealization optimism, then the links between truth, representation, and epistemic value are far more diverse and context-sensitive than traditional realist or anti-realist accounts allow. This suggests that mathematical models are not useful tools or "mere" representations but active epistemic mediators capable of revealing unique modal and relational features of reality that would be otherwise unintelligible. In short, the success of applied mathematical models, even ones that involve singular limits, is no miracle, but is grounded in the complex interplay of human formal techniques and heterogeneous aspects of a rich natural world.

Chapter 3: Towards a Pluralist Metaontology of Mathematics

3.1 Introduction

As we've just established, mathematical entities, structures, models, and operations are important and manifold features of our best physical models (Mancosu et al, 2023, 1). In other words, mathematics seems to be successfully applicable to the natural world in a *variety* of interesting and justified ways. This was in the conclusion of the previous chapter. Moreover, just a cursory glance at the development of mathematics and the physical sciences suggests a deep coevolution, with some going as far as to claim that "even the purest mathematics has its roots in physical sources" (Maddy, 2007, 342). The natural question that follows from these observations concerns how we ought to interpret the kinds of things mathematics purports to be "about" and to what extent that interpretation should be continuous with how we understand the more "concrete" content of those same theories. This amounts to our turning to the move in EIA from premise (2) to premise (3).

As we established, scientific realists have committed themselves to the existence of many things they cannot observe—subatomic particles, fields, continuous fluids, centers of gravity etc., —but which are nonetheless quantified over by our best theories and explanations (Chakravartty, 2017; Baker, 2022).[61] Generally, these kinds of theoretical commitments are rationally supported by Inferences to the Best Explanation (hereafter IBE). In applying IBE, as opposed to giving deductively valid arguments for a given explanatory entity, we infer "what would, if true, provide the best explanation of evidence" (Lipton, 2004, 1). The relative ubiquity of this species

[61] Note that given the conclusions of AR in 2, I am in this very position (hence my finding these kinds of empirical arguments for realism persuasive).

of reasoning illuminates the limits and contours of the logic of theory and model evaluation in science and philosophy alike.[62]

In our specific case of the philosophy of mathematics, originating explicitly in the work of Quine (1976) and Putnam (1979)[63], Platonists have utilized this same kind of reasoning, in the form of *indispensability arguments*, to support a continuous belief in the existence of mind-independent non-spatiotemporal mathematical entities and structures (Colyvan, 2001; Baker, 2005, 2009; Brown, 2024). I will restate the version adapted to focus on *explanatory* indispensability, EIA, as it was established in chapter one:

(1) We ought to rationally believe in the existence of any entity [or structure] that plays an indispensable explanatory role in our best scientific theories.

(2) Mathematical objects [and structures] play an indispensable explanatory role in science.

(3) Hence, we ought rationally to believe in the existence of mathematical objects [and structures] (Baker, 2009, 613).

This argument will be the main character in the final narrative I'll construct about how to approach naturalistic mathematical ontology given the conclusions of last chapter. This narrative will occur in two acts corresponding to two questions, both taking place against chapter two's *plurality in application*—our understanding of applied mathematics as a "MOTLEY" of epistemic techniques and aspects of reality which are nonetheless *accurate* in a wide range of cases (Wittgenstein, 1956, 88; Mancosu et al, 2023; Priest, 2024, 66).

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[62] Harman (1965); Swoyer, (1999); Lipton (2004); Williamson (2017); Paul (2012); Greco (2023); Douven (2025).
[63] Note that these species of so called "empirical" arguments for mathematical and logical realism originate in Mill (Mill, 1865).

The first question, addressed in §2, is one of scope. Here I'll examine which kinds of things *would* count as candidates for reification according to the EIA. In particular, I'll ask whether some of the *infinite mathematical idealizations*, like those we examined in the last chapter, play a similar-enough epistemic role to structures and entities in models and explanations to warrant EIA-like ontological commitment (at this point I'll remain agnostic about what this commitment might amount to).[64] To do this I'll present, in §2.1, two more traditional *explanatory* roles that appeal to clear dependence relation between explanandum and a mathematical structure, and one trickier case from quantum mechanics, the Aharonov-Bohm effect, that requires a more careful treatment and a potential diversion from explanation to *understanding* (Shech, 2018; 2023).

These new additions to our mathematical menagerie in hand, I'll argue that attempts to break the inductive similarity ("symmetry") between types of cases fail on both conceptual and empirical grounds. Further, I will explore the intuition that understanding what philosophers and physicists have to say about how *idealizations* are "indispensable" or "essential" (e.g., to *particular* models and explanations in qualified contexts) can shed a great deal of light on how the notion operates in applied mathematics more generally (Shech, 2019;2023; Sher, 2023). What I propose is that if one accepts MEP—and a certain modest attitude towards scientific progress—one should also accept *indispensability pluralism* (hereafter IP).

In §3 I turn to the second and more ontologically substantive question: the cogency of EIA given IP, and in turn, what exactly we would be committed to in the cases presented. I'll

[64] Pinning down an exact definition of "idealization" and differentiating it in a principled way from other similar concepts like "abstraction" and "approximation" is a tricky business. This ambiguity is, I think, more telling of how a new approach ought to be constructed than it is problematic. The account presented here will not cede to any universal "symmetry-breaking" argument such that the distinctions become relevant beyond the particular explanatory context in which they are invoked, plurality and context sensitivity of concept application is a key feature, not a bug (Pincock, 2007; Morrison, 2015; Norton, 2012; Leng, 2012; Shech, 2023, 54). See 2.2.1.

argue, following Povich (2024) that EIA as it stands is an enthymemic and requires reconstruction *(EIA*)*, and a turn to corresponding *metaontology.* However, against Povich's deflationary normativist approach, and Jonas (2024) and Plebani's (2020) pessimism towards EIA's success given IP, I'll contend that EIA* instead suggests a more flexible, ontological-pluralist approach to understanding mathematical reality: one which adopts a *localized* approach to standards of interpretation and reification of mathematical and modal structures. Before concluding, I will draw out in §4 some broader implications of my interpretation for epistemic constraints on IBE and the method of naturalistic metaphysics more generally (Longino, 1995; Pincock, 2023; Paul, 2012; Ladyman & Ross, 2007; Wimsatt, 2007).

3.2 Scope and Symmetry: Idealization and Indispensability

The natural place to begin this investigation given the focus of the current literature is with an exploration of mathematical *explanation* and explanatory indispensability.[65] One common way of defining the latter insofar as a piece of applied mathematics is concerned is by somehow showing that there is a distinctive kind of mathematical explanation (DME)—one that is entirely autonomous from any alternative "lower-level" explanation of a given phenomenon or regularity that might be given in non-mathematical (typically causal-mechanical) terms (Putnam, 1975; Pincock, 2023).

To get a taste for what exactly I mean, consider this primer example. A question we might want to ask is, "why are hive-bee honeycombs hexagonal?" (Lyon & Colyvan, 2008). What needs explaining here *prima facie* is precisely why honeycomb is a hexagonal lattice, and not triangular or squared etc.? According to Lyon and Colyvan there are two distinct parts to the answer; the biological fact that bees who minimize the amount of wax they use to build their

[65] See Mancosu et al (2023) for an overview of mathematical explanation and Colyvan (1998) for an overview of indispensability arguments.

hives are selected for evolutionarily and the mathematical fact, long surmised but finally proved in Hales (2001), that "a hexagonal grid represents the best way to divide a surface into regions of equal area with the least total perimeter" (Lyon & Colyvan, 2008, 3). While there are two parts here, the relevant intuition is that the mathematical fact is *required* to get us the answer to the contrastive why-question originally posed by the curious comb observer.

Prominent attempts to capture relevant intuitions about such cases find very different reasons for why this intuition might be accurate. For example, Pincock (2015) largely defends the view that mathematical explanations demonstrate a *sui generis* metaphysical dependence relation between the mathematical and physical facts, while Lange (2013) argues that mathematics is unique in its ability to explain via properties of systems that impose modal constraints *stronger* than causal laws (Lange, 2013, 485). However, of late, the literature – as we saw in the last chapter (section 5) has taken an explicit turn towards *pluralism* (Batterman, 2002; McKenna, 2022; Pincock, 2018, 2023).

One main line of support for this derives from Batterman's (2002;2010) work on *infinite idealizations* in physics we explored last chapter, taken up perhaps most prominently in Morrison (2015), Shech (2018;2023), and McKenna (2022).[66] According to these authors, there are *genuinely explanatory* instances of limit taking operations in science that do not fit into entity/structure facing accounts like Pincock's and Lange's, yet still seem to be indispensable or ineliminable in the sense that that they are unlikely to give way to unconditionally better lower-level models or explanations in the future.

[66] Note that this the discussion of "essential" idealizations goes beyond infinite idealization. Within the sphere of mathematical explanation however infinite idealizations are particularly salient and so I will take them here as both a paradigm and limiting case (no pun intended).

This expansion appears to pose a threat to our being capable of cordoning off a *distinct* indispensable role for mathematics such that EIA can be meaningfully applied: the Platonist wants to commit herself to the existence of sets, numbers, topological structures, and structural relations but extending the argument to infinite and infinitesimal systems and spaces required for Batterman-type explanations over-inflates her ontology to include systems that are intuitively more impossible even than the existence of objects like the Platonic solids (Knowles, 2020; Shech, 2023, 54). Attempts to save EIA-defenders from overinflated ontological commitments in the face of MEP have taken the form of "breaking the symmetry" between the kind of indispensability displayed in Batterman-type cases and that at play in traditional mathematical explanations (Baron, 2019).

I'm going to argue that such attempts to treat such infinites and infinite idealizations differently from other kinds of mathematical entities face serious difficulties. Just as explanation is a pluralistic concept, so is explanatory indispensability. Understanding both notions, requires recruiting contextual considerations and further attending to the relations between explanation and other important epistemic values like *understanding* and *exploration.* If we want to use EIA at all to guide our ontological interpretations, its scope will necessarily be quite broad (Shech, 2023, 54).

3.2.1 Soap Bubbles and the Aharonov-Bohm Effect

In order to compare and contrast idealization-based explanations and the canonical mathematical explanations used to support EIA it will be helpful to first clarify in more detail another example of the latter. While many have been offered in a vast swath of sciences—from the span of cicada lifecycles (Baker, 2005) to the impossibility of evenly-spaced golf ball dimples (Brown, 2024)—I'll focus for interpretive clarity on the treatment of soap bubbles that satisfy *Plateau's laws* as discussed in Lyon (2012), Pincock (2015), and Knowles (2021).

These laws—developed from the nineteenth century Belgian physicist Joseph Plateau—aim to capture three striking patterns of behavior in soap-bubble formation (Fig. 1):

(1) Soap formations consist of finite flat or smoothly curved surfaces smoothly joined together.

(2) Within a soap formation there are three possible meetings of surfaces: (i) no surfaces meet; (ii) exactly three surfaces meet along a smooth curve; (iii) exactly six surfaces (together with four curves) meet at a vertex.

(3) When three surfaces meet along a curve, they do so at angles of 120°; when four curves meet at a point, they do so at angles of ≈109°. (Knowles, 2021, 624).

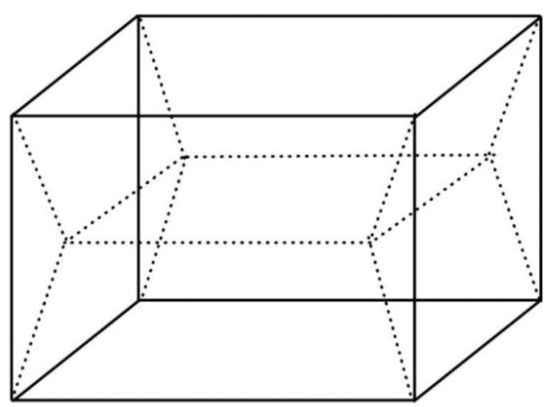

Figure 5: Soap formation on a rectangular frame (Lyon, 2012)

A novel explanation of these laws was given by mathematician Jean Taylor and can be broken down into three rough steps (Almgren & Taylor, 1976; Pincock, 2015, 861). First, one defines a mathematical analog of the soap films by approximating bubble formations as a two-dimensional surface in Euclidean R^3 space. A given set is soap-like if it is *minimal*, that is, "their total area cannot be decreased by certain small deformations that leave their frame or enclosed volume fixed"—such a configuration instantiates a (**M,** ε, δ)- minimal set (Knowles, 2021, 626). Given this mathematical definition, we come to see that "any such configuration of surfaces must with mathematical necessity conform exactly to the three geometric principles stated in the beginning"—*Plateau's laws* (Almgren & Taylor, 1976, 86; Pincock, 2015, 861).

Second, one goes on to prove that those configurations that *do* satisfy (1) also, by necessity, satisfy (2) and (3). The final, and most interesting, step consists in showing that indeed, "every soap bubble-like configuration is composed of smooth surfaces whose total area is finite" – that is, that almost-minimal sets *exist* and satisfy (1) (Pincock, 2015, 817).[67]

With this proof in hand, according to Pincock, Taylor has, by employing novel mathematics to a longstanding application problem, given a special kind of explanation of *why* soap formations satisfy Plateau's laws. This explanation is distinct from a causal one (wherein some piece of mathematics merely represents causal dependence-relations between microstructural facts), but rather trades on a *sui generis* "abstract" dependence relation between physical soap-bubble systems and the mathematical fact that (**M,** ε, δ)- minimal sets have the property of being minimal (Pincock, 2015, 866). This is a paradigmatic case of an applied mathematical explanation: it occurs when we have a "(1) classification of a system using (2) a more abstract entity [minimal sets] that is (3) appropriately linked to the phenomenon being explained" (Pincock, 2015, 867).

What is doing the real work in Taylor's explanation, however, is not merely the presence of a mathematical structure, but the *contextual stabilization* of a particular representational aspect under specific explanatory aims. The appeal to (M,ε,δ)-minimal sets does not purport to track the full microphysical constitution of soap films, nor to approximate it asymptotically.

[67] Taylor draws on geometric measure theory here—specifically the geometric properties of sets, to define the soap-like configurations as existent 'solution measures' that are 'well-behaved' (Almgren & Taylor, 1976, 92).

Rather, within the context of explaining Plateau's laws (invariant across wide ranges of material and dynamical variation) the formalism functions as a constraint on which deformations, perturbations, and counterfactual variations are explanatorily admissible. In this sense, the mathematical structure fixes an *aspect* of the physical system.

Contextual factors here – again, like information about scales and mathematicians background theories—help to determine which mathematical properties count as explanatory and which idealizations are licensed as truth-preserving with respect to the explanandum. Seen this way, Taylor's explanation exemplifies how mathematical indispensability arises not from a global metaphysical relation between abstracta and concreta, but from the local alignment of representational aspects and explanatory aims.

Now for a more complex case. Here we'll consider the example reconstructed by Shech (2018;2023) and Earman (2019) of a quantum electro-magnetic phenomenon, the Aharonov-Bohm effect. As we'll come to see, the explanation of this particular pattern of theoretical and experimental behavior involves the application of an *infinite idealization* – one of a similar ilk to those employed in what we've seen Batterman (2002;2010) refer to as *asymptotic explanations*.

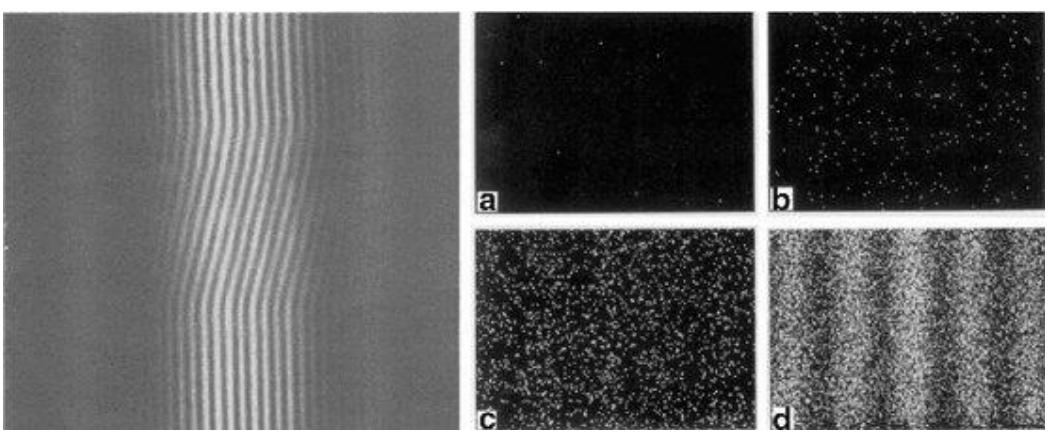

Figure 6: (left) example of double-slit interference (Möllenstedt & Bayh, 1962). (right) single-electron buildup of (biprism) interference pattern (from Tonomura, 1999; Shech, 2018).

When a beam of electrons is shot at a double-slit screen and made to rejoin at the detector screen, a certain interference pattern emerges (Fig 2). When we introduce into this experimental set up a "completely shielded" (impenetrable) magnetic field, one which is produced by an *infinitely long solenoid*, the interference pattern shifts by $\Delta x = \frac{l\lambda e}{2\pi d\hbar}\Phi_\infty$, where λ is the wave length of the electron, *e* is its charge, $\hbar$ the reduced plank constant, *d* the distance between the slits, and Φ_∞ the magnetic flux through S_∞ when the solenoid is "on" (Fig. 3; Shech, 2023, 6). This shift is the magnetic (abstract) Arahonov-Bohm effect (AB) (Aharanov & Bohm, 1959; Earman, 2019).[68] In this idealized model, we represent the quantum state of the system as the electron wave function Ψ, which *cannot* enter the region S_{in} inhabited by the magnetic field but must remain in S_{out} (Fig 3).

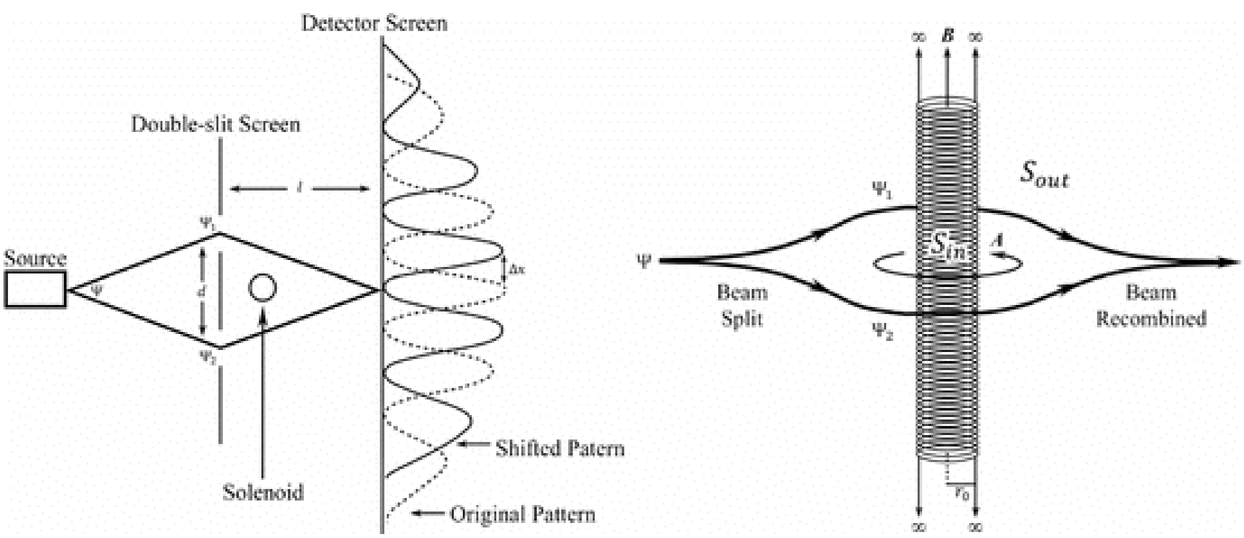

Figure 7: An illustration of the magnetic AB effect (from Shech, 2018)

[68] Note that this has been experimentally confirmed *without* the use of infinite solenoids (Tonomura et al. 1982). Shech refers to this experimental replication as a different phenomenon entirely – the *concrete AB* effect. That these are different will become important for how the infinite idealization used here is epistemically valuable—potentially in an indispensable sense. In this reconstruction I'll focus for simplicity on A&B's original higher-order derivation as the example of mathematical explanation.

One thing that is interesting about this model is that it potentially demonstrates a novel kind of *quantum non-locality* wherein the magnetic field affects the beam represented by the wave function in the absence of any "local" interaction between the two. Thus, the magnetic AB effect demonstrates a species of "action-at-a-distance" (Shech, 2018, 4; Maudlin, 1998; Earman, 2019). Additionally, this kind of model targets an odd semiclassical scale in which appeals are made to both external, unquantized electromagnetic fields *and* an electron quantized in non-relativistic quantum mechanics (Earman, 2019, 2014-2016). This latter detail will become an important plot point when we return to capturing the relevant notion of indispensability.

The *mathematics* of the explanation can be characterized as follows. We say that the configuration space on which the electron wave function is defined here is "non-simply" or "multiply" connected, where this configuration space roughly represents all the possible configurations of electrons. For a space to be multiply connected is (without going into too much detail) to say that any loops in that space *can't* be shrunk down to a single point without encountering an obstruction (See figures 4 and 5 for examples of simple and non-simple spaces).

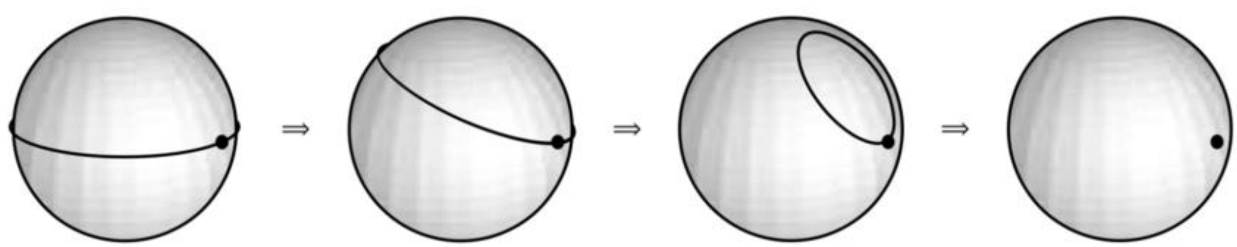

Figure 8: simply connected space on which loops can be continuously deformed (Shech, 2019).

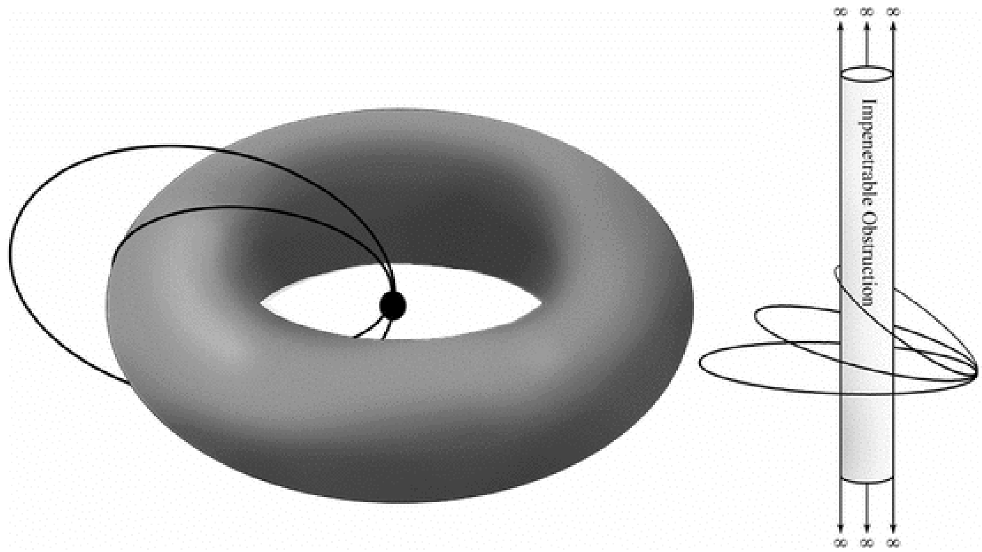

Figure 9: Non-simply connected space with a non-trivial fundamental group in which some loops cannot be continuously shrunk to a point in the context of an impenetrable torus (left) and an impenetrable infinite cylinder (right) (Shech, 2019).

The line among physicists then is that the multiply connected space is *indispensable* for the explanation of the AB effect (Aharonov & Bohm 1959; Ryder, 1996; Lyre, 2001). As we've seen however, we do not get this without appealing to the infinite idealization: the cylindrical solenoid *must be* infinitely long and infinitely impenetrable (Shech, 2023, 52). If not, the solenoid is not completely shielded, and the effect as theoretically understood in the original paper (concerning vector potentials) simply does not arise. In this case then we get something similar to the infinite idealizations in thermodynamic models of phase transitions in which we must consider the system as infinitely dense as it approaches the critical point in order to capture the qualitative change in the system (Batterman, 2002; Reutsche, 2011; Shech, 2023).

However, the solenoid in this model of AB seems to be slightly different from the (**M,** ε, δ)- minimal sets in the above soap-bubble example. Specifically, beyond explaining single experimental instances, they additionally give us unique *exploratory* and *understanding-based* access to the structures of classical and quantum mechanics (Shech, 2018, 12; 2023, 72; Earman, 2019). That is, they still seem to be largely indispensable in the sense that they provide a *unique* *kind of epistemic value* that would not be achieved by a microstructural explanation, but in very different ways depending on the specifics of the application. I'll now argue that understanding such disparate cases both clarifies and complicates the structure of EIA.

3.2.2 Indispensability and Contextual Induction

Even canvasing these two examples suggests that there is a much finer structure to the notion of epistemic indispensability (and the kinds of relations and virtues it trades on) than we might have thought. This potential scope expansion is no unqualified challenge to EIA however, as some (e.g, Knowles, 2020; Pincock, 2023) have argued it is. Rather, to my mind it provides the opportunity to *clarify* the notion of scientifically indispensable mathematics in terms of two distinct questions that we might ask about a given piece of mathematics: (1) indispensable *for what*? And (2) indispensable in *what way*? Different answers to these are exactly what motivate many of the ontological positions sketched in 1.4.

That is, to what domain or context ought we index a given mathematical entity or idealization's indispensability, and further, whether it is necessary that this indispensability be of the *explanatory* sort at all. (3.2.1) will tackle the first question in reference to explanation alone, and I'll argue that *explanatory* pluralism translates into a plural and contextual notion of *indispensability* (IP) by way of clarifying the structure of IBE more broadly. (3.2.2) tackles the second – proposing tentatively that there are in fact *other* relevant epistemic states beyond explanation that mathematics can be indispensable for furnishing, here understanding and modal exploration.

Before engaging directly with these, it is necessary to dispel one intuitive objection. This is the fact that "idealizations" like our infinite solenoid and mathematical entities *simply aren't the same kinds of things* and as such should not be inductively comparable to begin with.

Idealizations, conceived of traditionally, are "cleaned up model systems" while mathematical objects are more like fictional characters (Norton, 2012; Shech, 2023, 54). This is misleading however – this is a very *narrow* construal of idealizations that in fact fails to capture the epistemic dynamics of the abstract AB effect as sketched above along with other intuitive notions of idealization like the infinite systems in Batterman's *phase transitions* examples (Batterman, 2002).

Further, the distinction seems even more difficult to maintain given the sociological fact that definitions of idealization – and how they stand in relation to other concepts in its neighborhood like approximation and abstraction—is a matter of very little agreement. For example, Norton (2012) delineates an approximation– which is still factually plausible, from an idealization which is deliberately false and physically inconsistent. While Morrison (2009) holds that "abstraction is a process whereby we describe phenomena in ways that can't possibly be realized… (for example, infinite populations)…idealization, on the other hand, typically involves a process of approximation whereby the system can become *less* idealized by adding correcting factors) (Morrison, 2009, 125). What is most relevant for EIA-aptness then is not going to be a particular *character* of the relevant abstracta involved, but rather their *epistemic roles*. That is what we now investigate.

3.2.2.1 Explanation and Reasons to Believe

Beginning with the notion of explanation alone, I hope to show that accepting a legitimate pluralism leads to a minimal commitment to indispensability pluralism. In order to understand how exactly EIA works as a form of IBE, it is important to see *how exactly* positing the existence of mathematical entities might be the "best" explanation, given that the first premise of EIA already assumes this to be the case.

Putnam (1975) provides an intuitive framework for understanding this notion on the basis of "explanatory autonomy" (Pincock, 2023, 59). A given mathematical explanation may be autonomous in the *strong* sense if the mathematics in an explanation targets a structural or higher-level phenomenon that more "fundamental" sciences like physics *cannot* explain. By contrast, an explanation might be *weakly* autonomous in that although there may be a possible micro-structural explanation of a given phenomenon, the mathematics explains it *better* in some respects—that is, the mathematical version of the explanation has more *explanatory power* on some grounds (Lyon & Colyvan, 2008).

At least in our Honeycomb and Soap Bubbles cases, *strong* autonomy will seem a difficult bar to reach.[69] That is—it will require an odd sort of historical attitude towards the ability of fundamental physics to provide more stable reductive explanations as science progresses. In an illustration of this kind of (teleological) thinking, Baron (2019) distinguishes between two kinds of explanatory indispensability: *substantive* and *constructive*. In cases of the former, in addition to being explanatorily indispensable *now*, we also have good reason that it will always be—that is, we cannot expect science to dispose of it given sufficient time (Baron, 2019, 1946). The latter are merely those features of explanations that are currently indispensable, but we have good reason to believe will be "de-idealized" out of our model and explanations down the line.[70]

Such historical inductions are suspicious, and I argue largely unhelpful in these scenarios. For one, the original notion of indispensability in EIA that rides on the principles of IBE does not

[69] I'll argue with Pincock (2023) that this generally extrapolates over cases of mathematical explanation – even under a "brute pluralist" conception.

[70] Note here that this distinction between kinds of indispensability mirrors in large part the discussion of de-idealization (as a justificatory condition for idealization). Here we note that the kinds of mathematical idealizations that have been deemed "explanatory" may not be amenable to "de-idealization" to a physical / causal explanation. We dealt precisely with the concept of de-idealization in chapter two, showing it to be largely tenuous – either in principle impossible in some cases, or *undesirable* in others.

require the above notion of strong autonomy, but only the *weak* version: it seems not to matter whether the mathematical entity will feature in the ideal, coherent scientific picture of the universe but only whether it features in a *conditionally better* manner of expressing the explanation or getting at the phenomena. Take Colyvan's (2001) definition of what it is to be *dispensable* – such that if either (1) or (2) fail, an entity is indispensable to theory T:

(1) There exists a modification of *T*, *T** with exactly the same observational consequences as *T*, in which the entity in question is neither mentioned or predicted.

(2) *T** must be preferable to T. (Colyvan, 2001, 77).

This second condition is where the action is. That is, whatever kind of indispensability that motivates EIA is based on a version of IBE that invokes the "preferability" or uniqueness of a *particular* mathematical explanation over an alternative physical explanation—it rides on the *weak* autonomy of mathematics. Pincock (2023, 61) considers the roles that mathematics plays as supporting weak autonomy to be "generality" and "robustness" (i.e., invariance across microstructural perturbations). This is, indeed, often true of the "unique" role that a piece of mathematics might play in an explanation, but as we saw in chapter two, this is *not* the only role that it plays! And further, given our consideration of NMI, not the only role that is unique in the kind of *accurate* epistemic access that it affords to certain kinds of properties of systems: affording true modal information, capturing epistemically fundamental structural features, discovering real classes of things, bridging, etc.

Thus, it seems that a kind of *weak* autonomy of mathematical explanations over the physical is sufficient to show that "indispensability" is something indexed to the particular *roles* that pieces of mathematics play in different explanations. Further, that attempts to break the explanatory symmetry between *kinds* of autonomy or indispensability (e.g., weak vs strong) at a wholesale level rely on too speculative an inductive foundation.[71]

Re-indexing the notion of the indispensability of some mathematical fact or object to the *particular role* it plays in a given explanation for specific aims highlights a main feature of IBE more generally, one emphasized by Psillos (2007): *its contextual nature*. All forms of ampliative inference in science that rely on explanatory considerations he contends must be "sensitive to the context, background information, cognitive aims and values etc." (Psillos, 2007, 442).[72] That is, regardless the (ontological) fruits of our use of IBE (i.e., its validity regarding causally efficacious un-observables like electrons *and* mathematical structures like minimal sets) the determination of *best explanation* is a deeply contextual matter that exhibits a distinctively "fine structure" (Psillos, 2007, 43; Norton, 2021; McKay, 2023).

It seems clear to me then that even at the level of *explanation*, given that mathematics might, in different contexts, play different roles that fail to reduce to one single distinct form (either a monist conception of "distinctively mathematical explanation" or as having long-run *strong autonomy*), there will be multiple ways of being *indispensable* (IP).[73]

This is all to say that minimally if we, (1) construe EIA as turning on *only* explanatory concerns, (2) understand the relative autonomy/virtuosity of some given explanation via IBE as a *context-sensitive matter*, and (3) reject Baron's precarious historical inductions that allow us to

---

[71] See Shech (2019), Mizrahi (2015), Lange (2002) for arguments against historical inductions like this. Largely, this is a "conceptual" point against Baron's move to distinguish between kinds of indispensability in a way that generates *hierarchy*. There certainly may be different ways of being indispensable, but they cannot be *ranked* in a way that would *necessarily* give us more inductive warrant for one over another. [Bob—Middle Way]

[72] This kind of point has been brought out most aggressively by Norton's (2021) *material theory of induction* which categorically denies the existence of any uniform principle on inductive practice. Without belaboring the details of his account, what is important here is that this heterogeneity of inferential contexts does *not* preclude some of those inferences from being either justified or correct.

[73] See Delarivière et al. (2017) for another explicitly contextualist approach to mathematical explanation.

break symmetry between entity-indispensability and idealization-indispensability we get two conclusions:

(A) That *indispensability* is also a pluralist notion (what I will now refer to as indispensability pluralism, or **IP**);

(B) that the EIA, if effective, will have a broad range of applications.

3.2.2.2. Beyond Explanation

While I think the two final conclusions of the above section have serious implications for EIA I'd like to extend (A) and (B) beyond the realm of "role in an explanation" in light of the AB case. That is, we can take our question (2) posed about the nature of indispensability: "indispensable *in what way,"* further. I argue that if we grant an expansion of domain for indispensability arguments based on *explanatory* pluralism, we ought also expand it to other unique epistemic achievements to which a piece of mathematics might be indispensable for attaining, in particular modal *understanding*.

Understanding as a distinctive and desirable epistemic value in science has regained popularity in recent years, particularly in relation to the role of idealizations (Kvanvig, 2003; Elgin, 2017; Khalifa, 2017; Rice, 2021, Shech, 2022). The debates as to the nature of understanding are ongoing, but for the current purpose I'll characterize it in a certain way before demonstrating how exactly the mathematical idealization in AB furnishes it for us in an *indispensable way*.

What is important here is that we can, following in the footsteps of authors like Lipton (2019), Le Bihan (2017), and Rice (2021), Wilkenfeld (2024), accept that scientific *understanding*, in some cases, comes cleanly apart from *explanation* even when we try to construe explanation as a pluralistic concept itself.[74] Lipton and Wilkenfeld investigate the famous Galilean thought experiment which, by logical *reductio* alone, gives us insight into the physics of falling bodies (Lipton, 2009, xx; Wilkenfeld, 2024, xxx). Consider another elegant example presented in Rice (2021) that I find more apt for this discussion given that it explicitly involves *mathematical idealization* (2021, 240-242).

Here, astrophysicists Daniel, Heggie, & Varri (2017) use a mathematical model of "possible star orbits to demonstrate that *even if* a star has an energy that exceeds the energy required to escape from a [star] cluster, the star may be unable to escape" (Rice, 2021, 240). While it is clear from observation that these "potentially escaping" stars play a crucial role in understanding the kinematics on the outskirts of clusters, they note that "theoretical understanding of the phase space" is insufficient to deal with potential escapers (Daniel, Heggie, & Varri, 2017, 2). To rectify the theoretical lacuna, they constructed the following idealized model (Fig. 7).

[74] Others, such as Strevens (2013) and Khalifa (2017) reject such a distinction, arguing, in the end that all "understanding-why" reduces to explanation. This is, however, not the place to argue along these particular lines and is sufficient to acknowledge that *even in* "explanation-first" presentations such as Strevens, there are more fundamental epistemic achievements that are understanding-based (e.g., his notion of theoretical "understanding-with").

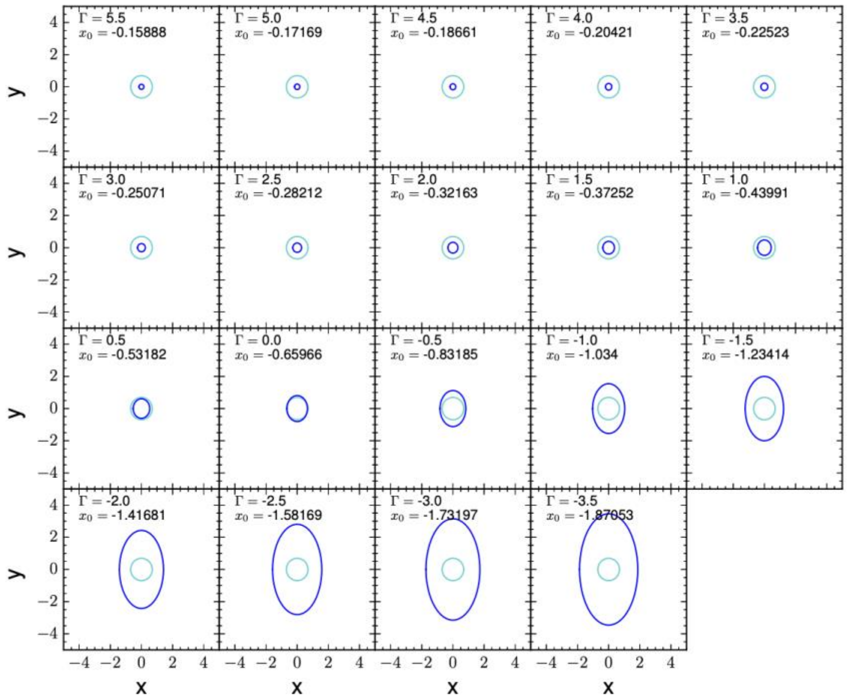

Figure 10: examples of circular and elliptical f-orbits. Each star's orbit is shown by the darker line, while tidal radii are shown by lighter lines. The latter is where the cluster's gravity is dominated by that of the galaxy it orbits (Daniel, Heggie & Varri, 5).

This basic model allowed our astrophysicists to gain insight into features of the phase space of possible orbits, and the conclusion that contained in this space is the possibility of a sufficiently energetic star to stay put nonetheless (Rice, 2021, 242). As Rice puts it, rather than give us accurate representation of causal features of systems, this kind of model produces "understanding of stellar systems by providing modal information concerning which orbits are possible," (Rice, 2021, 242).

Largely, I think we can further specify the understanding in our AB case in much a similar way to be the grasping of true modal information about some given phenomenon or theory which is distinct from explanation in its being to some degree *objectual* (Le Bihan, 2017;

Earman, 2019; Shech, 2023).[75] That is, we can understand infinite idealization here as providing a distinct kind of non-explanatory "understanding [of] a thing or a domain of things, broadly construed, so that one can grasp the relations among events, objects, domains, and so forth, in an appropriate manner" (Shech, 2023, 66).

Targets here might be as broad as something like a "subject matter" (Elgin, 2017) or as narrow as a single phenomenon or thing (Kelp, 2015)[76]—in our case what is at issue is both understanding (1) the *modal structure of quantum mechanics and classical mechanics and their relations* (Maudlin, 1998), and (2) understanding (what Aharonov and Bohm themselves took to be) the physical instantiation of the abstract AB effect: the concrete AB effect (Shech, 2023, 69). The mathematical (infinite) idealizations in both cases appear to be *essential* (indispensable). Let me quickly demonstrate how.

For (1), we can cache this out, by observing that "it is by exploring the modal structure of quantum mechanics, particularly what it says about possible worlds that are most naturally construed as idealizations (vis-à-vis the abstract AB effect)," that we facilitate the understanding of the *foundations* of those theories as such and their *intertheoretic relations* (Shech, 2023, 69). For (2) we might think that there is in fact a "physical counterpart" of the abstract effect, the concrete effect (Aharonov & Bohm, 1959). Without belaboring the technical details, it turns out that if we do away with the idealizing assumptions that keep the configuration space partitioned (in the solenoid and out), we need not even appeal to quantum-mechanics, and the phenomenon can be "explained away classically" (Shech, 2023, 70). We entirely lose a handle on the

[75] See the recent edited collection *Modeling the Possible* (Knuuttila, Grune-Yanoff, Kosiken, Wirling eds) (2025)
[76] Note that oftentimes understanding at more particular levels invariably involves understanding at more systematic levels of theory – this is not to say that observation and understanding itself is entirely theory-ladden in a naïve way (Hanson, 1958), but merely an acknowledgement of the difficulty we face in disambiguating between different epistemic roles and levels of generality more broadly.

phenomenon we were trying to understand to begin with – that is, the idealizations in abstract AB are *necessary* for understanding what concrete AB might even *"be in the first place"* (Shech, 2023, 66). [77]

Accordingly, I think we can add two more conclusions to (A) and (B) above: (C) that IP, if we want it to correctly carve the realities of scientific practice, must include non-explanatory epistemic roles, one central example being *modal understanding* furnished by infinite idealization (Brown, 2024, 496). I'll cache out what exactly I mean by this in the next section. With these four conclusions cemented, we can now return to EIA as an argument for Platonism and see how it might fair with a new notion of indispensability.

3.3 From Indispensability Pluralism to Ontological Pluralism

What happens to EIA given IP and claims A-C? I contend that the result is that the character of our ontological commitments regarding non-causal explanatory entities (mathematical properties, minimal sets, and infinite solenoids) ought to be *varied* and that the correct description of their character will depend primarily on the epistemic *roles* they play in specific contexts. The question is, therefore, no longer whether mathematics is indispensable *tout court*, but what kind of existence is licensed by which indispensable epistemic role.

Introducing and accepting IP gives us good reasons to be *ontological pluralists* about mathematical objects it licenses and *contextualists* about the appropriateness of our commitments to them. This pushes the discussion of mathematical metaphysics, following Plebani (2020) and

[77] There are other features of mathematical understanding that seem to make it distinct. One such property of note is its *communicative stability* across disciplines. Mathematical models and explanations are typically robust across contexts insofar as they are understood across them *and* provide methods for facilitating further collaboration and exploration of discovery space. See Ch.2 Sec. 6.

Povich (2024), in the direction of *metaontology* and away from the first order global dispute between Platonism and nominalism.[78]

In order to motivate this empirically grounded ontological liberalism, I take it that EIA needs to be slightly reformulated from Baker's original (2009) formulation. Following Povich, I'll add in the intermediary notion of a mathematical concept with a premise (2.5):

> (P1*) We ought rationally to believe in the existence of any entity or structure referred to by a concept that plays an indispensable…role in our best scientific theories.
>
> (P2*) Mathematical concepts play a variety of indispensable…roles in science.
>
> (P**2.5*)** Mathematical concepts refer to mathematical entities and structures.
>
> (C*) Hence, we ought to believe in the existence of mathematical objects (Povich, 2024, 23).

Introducing 2.5 gives us some flexibility in determining the correct description of the truth-makers for the mathematical concepts that were invoked above and thereby making the existential quantifier in (C*) *plural* over kinds of mathematical entities(abstracta). Additionally, notice that the argument no longer turns on explanation alone—thus allowing us to formulate versions of EIA based on understanding (i.e., an *understanding indispensability argument*).[79] To

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[78] It seems to me that moving in this direction is the best way to approach this debate which appears to be rather stagnant (save Baker, 2022) since the original positing of the two sides in, e.g., Field (1980) and Colyvan (2001). Most progress in mathematical ontology of late has required serious assumptions to get off the ground. For example, Baron (2024)'s most recent *Pythagorean* approach to Platonism, while ingenious and deeply philosophical interesting, needs to assume a univocal Platonism commitment. With Povich (2024) and Plebani (2020) I think that "going meta" is what is needed to move things forward productively.

[79] Notice that given IP I have removed the "explanatory" preceding "indispensable" in the formulation, this should come as no shock at this point given my commitment to MEP, but the reader is reminded that there are accounts on which "explanation" can be taken pluralistically such that it is *defined* by its furnishing of understanding and so my separation is not a necessary one (Rice & Rohwer, 2020). I separate MEP from explanatory pluralism to exemplify the MOTLEY. Povich introduces 2.5 to show the *failure* of EIA and bolster his own neo-Carnapian normativist metaonology for mathematics – a deflationary anti-realist approach to mathematical content. There may indeed be,

get a flavor of the proposal, consider the applications of mathematics given above and what we might reasonably be committed to in each context. Ontological commitment to the entities will be indexed to the epistemic roles those entities play in a given model, and different roles license commitment to different *kinds* of mathematical structure.

Honeycombs

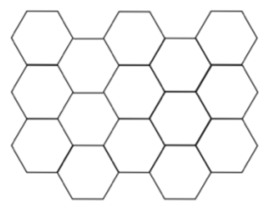

Figure 11: Hales, 2001, 1.

In the case of the Honeycombs, we can note that the role is (1) explanatory, and (2) *mixed*. Let me explain what I mean by the latter. That is, the role this mathematical fact (the honeycomb conjecture) plays in the explanation of the fact that hive bee honeycombs are hexagonal, is supplanted by information the nature of selection pressures on hive bees. As Lange (2013) puts it – "this explanation works by describing relevant features felt by honeybees, so it is an ordinary, causal explanation, not distinctively mathematical" (Lange, 2013, 500; 2021).[80] However, just because the explanation requires our supplanting it with more causally efficacious or empirical content, we need not conclude that mathematics is playing no interesting role that licenses some ontological commitment.

on my view, contexts in which normativism is the most apt commitment to make, I therefore reject the global nature of Povich's normativism.

[80] Note here that Lange *does* think that there is a related explanandum in the area that may in fact count as what he calls "distinctively mathematical" – but given our assumption of MEP, there is likely to be no single unified "distinctive" role of mathematics. To my mind, this voids Lange's criticism of the "ordinariness" of the causal explanation when citing the mathematical facts involved.

A better view of these kinds of cases, I think, is something closer to the idea that physical systems can, in a certain sense, actually possess mathematical properties, but in a certain sense. Here approaches like Lange's (2021) Aristotelian Realism and Ladyman & Ross (2007) and Wallace's (forthcoming) *ontic structural realism* (OSR) stand out as apt descriptions of the relevant mathematical properties. Both views countenance the explanatory role at play in the case, given that real concrete honeycomb systems will literally instantiate either brute-mathematical (Lange) or structural-relational (OSR) properties that can play causally efficacious roles.

Without belaboring the details of such an account, I suggest that a hybrid Aristotelian-structuralist view may be best suited for this role wherein we are committed to mathematical properties as "property-like" aspects of systems that are, essentially relational. That is, systems can instantiate concrete structural features like hexagonality (as a relation between parts of the system) that mathematical concepts in explanations then cite as genuine causal difference makers *in* the world (Lyon, 2012). Some mathematical properties have this character.

Soap Bubbles

In the case of Plateau's laws, the role is similar, but I argue relevantly distinct such that EIA* licenses commitment to a different kind of mathematical object. The main motivating difference, brought out in Pincock's (2015) original treatment of the case, is its determinately non-causal character. That is, there is a distinction to be drawn between the honeycomb case—where the mathematical content is tied to physically instantiated structural properties—and the soap-bubble case, where the explanatory work is carried by a mathematical structure whose physical realizations are multiply heterogeneous and only partially constrained by shared microphysical constitution. The minimal set is therefore more abstract than any individual soap-film configuration that realizes it.

In this latter scenario, it is plausible to posit a distinctive, sui generis relation of ontological dependence between physical soap-bubble systems and the mathematical fact that (M,ε,δ)-minimal sets necessarily satisfy Plateau's laws (Pincock, 2015, 880). However, the precise character of the relata in this dependence relation requires further specification. I suggest that what Taylor's proof licenses commitment to is best understood as a substantive abstract structural entity individuated by its explanatory role.[81]

More specifically, and as opposed to instantiated properties or simple abstract objects, they function as *structural* universals that delineate equivalence classes of physically possible film configurations under admissible deformation. What the mathematical theory contributes is a principled partitioning of the space of physically realizable surface configurations into those that satisfy Plateau's laws and those that do not. The explanatory force of the theory derives from the fact that any configuration that falls within the structural class determined by minimal-set theory must, with mathematical necessity, exhibit the observed geometric regularities.

Under this interpretation, the dependence relation between mathematics and physics is neither causal nor constitutive in the Aristotelian sense operative in the honeycomb case. Instead, it is mediated by structural determination: physical soap films *realize* Plateau-compliant configurations because they occupy positions within a deformation-invariant structural pattern captured by minimal-set theory. The relevant mathematical entity is therefore not the individual

[81] See Reck & Schiemer (2025) for a survey of the various *non-eliminative* interpretations of structuralism. The point here is less to characterize the exact metaphysical conditions, but to show that the explanatory role of the case seems to generate good reasons to commit to the existence of such instantiable structures.

minimal surface but the abstract structure that organizes the invariance relations governing admissible deformations of such surfaces.

The way we have been understanding context helps clarify why this structural interpretation is appropriate. As Knowles (2020) emphasizes, the choice of algebraic or geometric formalism in applied modeling is rarely dictated by a unique underlying ontology but rather by the transformations, equivalences, and invariances that must be tracked for specific explanatory purposes (Knowles, 2020, 13). In the soap-bubble case, minimal-set theory is explanatorily apt not because it mirrors the microphysical constitution of films, but because its structural content renders salient precisely those deformation relations under which Plateau's laws remain invariant.

Context therefore fixes not only which mathematical objects are invoked, but which structural relations among them are treated as representationally significant. From the perspective of indispensability pluralism, this suggests that the ontological commitment licensed by EIA* in this case is best understood as a form of local structural realism in which we are committed to the existence of abstract structural entities whose identity conditions are determined by their role in capturing stable features of classes of physical systems.

This form of commitment remains broadly Platonist in spirit, insofar as it treats the relevant structures as mind-independent and non-spatiotemporal. However, it departs from traditional object-based Platonism by individuating mathematical entities primarily through their position within networks of structural relations rather than through intrinsic objecthood. Minimal-set structures exist as abstract deformation-invariant patterns that multiple heterogeneous physical systems can realize without sharing any deeper microphysical identity.

AB Effect: Configuration Spaces as Possible Worlds

Finally, to our more exotic example – the *non-simply connected* configuration space (the infinite solenoid) utilized in the Abstract AB effect. Here is where we depart most clearly from the other cases and where ontological pluralism becomes a crucial ally. While there is little work done on the ontology of configuration spaces here are two claims from the literature to situate the discussion:

> "At the end of the day one might simply insist that the configuration space's role is only as a convenience in the representation of systems. However, we would like to ask just as so many have asked about mathematics generally, why does this representation work so well, and why can its virtues not provide some pressure towards taking it with ontological seriousness" (Ney & Phillips, 2013).
>
> "Finally, it is by exploring (1) the modal structure of quantum mechanics, particularly what it says about possible worlds that are most naturally construed as idealizations (vis-à-vis the abstract AB effects), that we facilitate (2) understanding theory's foundations and (3) understanding its intertheoretical relations." (Shech, 2023, 68).

Here I think it is best that we do, in fact, respect the plausibility of running indispensability type arguments for the existence of things like phase spaces (Lyon & Colyvan, 2008) and configuration spaces, insofar as they can be construed as possible (and impossible) topological structures (Shech, 2019, 17). Largely, this is because such systems as the infinite solenoid are not physically actualizable and as such *cannot* be construed as some "mere approximation" of a physical system in any sense. Such infinite mathematical idealizations play a role that is *literally* indispensable insofar as the epistemic benefits that we receive from understanding the idealized configuration space (in the ways discussed above) could not be furnished by any physical system (Shech, 2019, 17).[82]

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[82] This response is developed in Shech (2019) as a response to "easy road" approaches to nominalism. That is, those which seek to show how mathematical entities are non-essential/ dispensable without providing a full Field-type nominalization program for all of the mathematical sciences. I take it that such an appeal applies to most anti-realist accounts more generally, including Povich's (2024) metaontological normativism.

The exact character of the object reified here, I take it, ought to be—in line with the modal information that we saw it furnishing about phenomena and theories themselves—equally modal. Thus, for present purposes, it is best to adopt, and hence be committed to in this context, an actualist, ersatz ontology of impossible worlds sufficient to evaluate the counterpossible conditionals that arise in AB-style reasoning (Fine 1977; Nolan 1997; Sider, 2002).[83]

What this means is that we need not adopt the intuitively unappealing Lewisian modal realism wherein we take possible worlds as "sui generis entities of a kind with the concrete world we inhabit" (Sider, 2002, 279). Rather, the relevant impossible world may be construed as a structured *abstract object*—think: a maximal but inconsistent description of some state of affairs—at which certain mathematical impossibilities (for example, trivial holonomy with nonzero enclosed flux) are stipulated to obtain (Plantinga, 1974; Stalnaker 1984; Yablo 2014).[84] Truth-at such worlds functions as a semantic device for modeling the inferential content of *counterpossible* claims of the form "If (impossibly) … then …," which standard possible-worlds semantics trivializes (Berto 2017; Jago 2019).

This is sufficient to underwrite the physicists' counterfactual reasoning without unnecessarily inflating our metaphysics: mathematics here describes different possible (and impossible) configurations of a given space or structure (here the configuration space). This sets the stage for distinguishing the present account from pluralisms that either inflate ontology

[83] This interpretation will also, I argue, be apt for the case of the minimal model system in RG and the singular thermodynamic limit in TDL as were discussed in chapter 2.

[84] Note that the choice between modal metaphysical interpretations, which versions of modal structuralism are (I take it) eventually forced to confront insofar as they take modality to be fundamental, will be done via the method discussed in chapter 1. To my mind, abstractionism and actualism are superior explanatory posits that remain consistent with the other conclusions made here for abstract minimal sets in the soap bubbles case and shared properties in the honeycomb case and thereby are preferable to Lewisian concretism. The entire question of the metaphysics of modality, of course, cannot be given due time here. See Menzel (2016) for a full account of possible world interpretations.

indiscriminately, as in some contemporary Platonist views (e.g., Hamkins, 2012), or trivialize ontological commitment altogether, as in Carnapian frameworks (e.g., Povich (2024)).[85]

3.3.1. Pluralist Topography

There are a couple of disambiguations needed here between the kind of pluralism that I'm advocating for here and other extant approaches. First and foremost, it needs to be said that I do *not* directly take IP to motivate the view called *pluralist mathematical realism* as conceived of in a particularly inflationary sense (Jonas, 2024). This is the view held by some (e.g., Balaguer (1998) & Hamkins (2012)) that a robust "pluralist conception of mathematics can be squared [*univocally*] with a realist ontology," in such a way that is not entirely unintuitive (Jonas, 2024, 2899). That is, *all possible* mathematical (set-theoretic) universes exist mind-independently and non-spatiotemporally.

The reason why casting our Pluralism in these terms is unattractive is that, as Jonas articulates, empirical justifications for ontological commitment to an *entire* pluralistic (perhaps infinitely pluralistic) universe of real mind-independent mathematical things outstrips its own power (Jonas, 2024, 2911). That is, a great deal of higher-order pure mathematics does not find empirical application at all, and thus could not exist as parasitic on the parts that are operative in our case studies that justify reification (Jonas, 2024, 2913). Indispensability is not "transitive"

[85] Thanks to Collin Rice for raising the (important) question of what unifies the heterogeneous ontological commitments defended across the foregoing case studies as *mathematical*. A natural answer, defended in different ways by Marc Lange (2013), is that mathematical entities are unified by their role in articulating modal constraints that structure scientific explanation. While the present account is sympathetic to this insight, it departs from a *purely* modal unification strategy. Across the cases considered here, what marks an entity as mathematical is not solely its modal force, but its functional role within explanatory and representational practice. Mathematical entities, on this view, are abstract structures that organize relations of invariance, transformation, equivalence, or constraint that are indispensable to the construction and interpretation of scientific models. Although such roles often involve modal features—such as counterfactual robustness or the articulation of admissible physical possibilities—the modal character does not exhaust them.

from applied to unapplied concepts in a way that supports what we might call “full blooded” realist mathematical pluralism.

Thus, I am not committed to such a view—only to the specific epistemic roles and mathematical objects discussed. This is enough to establish pluralism concerning roles, indispensability, and kinds of objects and properties without having to endorse *all* possible mathematical universes and objects.

Additionally, the pluralism involved in my IP is not (as it is traditionally interpreted) the view called “heavy duty Platonism”. According to Knowles (2015), this view holds that “physical magnitudes, such as mass and temperature, are cases of physical objects being related to numbers” (Knowles, 2015, 1255). Roughly, the thrust here is that there is “no sharp distinction between concrete and abstract entities” to begin with (Knowles, 2015, 1255; Plebani, 2020, 35).

While I will hold that there ought to be no sharp *methodological* (i.e., inductive) distinction between the ways in which we interpret quantification over concrete causally-efficacious entities and various sorts of abstracta (mathematical structures and possible worlds), there will in fact be distinct *ways* of existing – this is the central tenet of what it means to be an *ontological* pluralist, the mathematical and the physical will not *always* be best understood as identical (Plebani, 2020, 12; Turner, 2021, 1).

Finally, it is important to note that what I'm after here however is *not* the rather trivial Carnapian version of ontological pluralism which says that there are competing ontological “visions”, none of which have “an objectively better claim to correctness” – thereby sucking all meaning out of debates about mathematical existence (Turner, 2021,1; Carnap, 1950; Povich, 2024).

Instead, I am interested in articulating the view that *in mathematics*—just as in naturalistic ontologies of science and in ordinary life—there are many *ways of being,* e.g., existential quantifiers are plural in reference (Turner, 2021, 1; Chakravartty, 2017). Further, and again as in scientific metaphysics, inconsistency between (and that contextual and value features constrain) these interpretations need not imply relativism or anti-realism about the targets of discourse. Mathematical reality, like the concrete reality it helps us explain and understand, is best understood neither as an ontological desert nor as an infinite heaven of crystalline structures but as a lush *rainforest* (Dupre, 1993; Wimsatt, 2007; Ladyman & Ross, 2007).

According to Zalta (2024), what I'm suggesting here represents a species of "metaphilosophical" pluralism, hereto "undiscussed in the literature", which holds roughly that "the principal philosophies of mathematics are all based on an insight or truth about the nature of mathematics that can be validated" (Zalta, 2024, 306). That is, traditional Platonism, modal structuralism, ontic structural realism, and even Fieldian fictionalism, get at true descriptions of properties of mathematical objects and structures *depending on* the relevant *indispensable* role that the entity, structure, or ideal system is playing in an explanation or model and the other epistemic features of the context (the relevant virtues of the model or explanation).[86]

To further motivate this approach, consider again the *assumed structure problem* from chapter two raised by MacLane (1996), Knowles (2020), and Baker (2022). This is fundamental issue for *any* account of applied mathematics that respects MEP: in applying mathematics to a target system or phenomena, we must *assume* that it is amenable to standing in some homomorphic relationship with a mathematical structure in the first place.

[86] Note that this kind of metaontological pluralism also helps absolve some of Jonas's (2024) worries about *transitivity*—if we can be committed to different regions of mathematics in different ways (analogously to how we are committed in a different sense to concrete vs. abstract entities) then those areas of mathematics which have not apparent chance of finding empirical support simply get a thinner correct description.

In light of this problem, he says that any successful account will have to deal with the fact that the truth or objectivity of mathematical concepts that we need for explanation does not *need* to entail existence. He then shows how applying Dennett's (1989) intentional stance towards "abstracta" (objects and structures) puts them somewhere "in the middle" between real and unreal (Baker, 2022, 53), particularly given the contextual features that constrain indispensability arguments. This is, he claims, parallel with the way in which scientists reify, via inference to the best explanation, *centers of gravity* (Baker, 2022, 54-55). He then poses two tentative options for moving forward: (1) mathematical structures are 'devices or artifacts, or we (2) adopt a form of nominalism about mathematical dialogue. The question of which remains open (Baker, 2022, 57).

This non-committal conclusion is unduly (albeit intentionally) unclear and indeed is *made* unclear by focus on the very first-order focus on the "Platonism/nominalism" debate at hand. Moving to the *metaontological* level allows us to recognize that there might plausibly be multiple categories of abstract existence and as such valid interpretations of the truth-makers of statements depending on what is *needed* for our application in a given situation of application (Bricker, 2014,1). To make explicit the underlying thrust of my argument: *ontological commitments* based on abductive, largely empirical reasoning (like EIA), will look locally for the conditions determining the apt description of some piece of mathematics' mode of existence.

Recall the methodological observations from chapter 1, which followed L.A. Paul (2012) in taking metaphysics as an endeavor of *modelling*, with an inferential practice structured in the same way as physical science, though simply turned to "different phenomena".[87] If we take this

[87] Paul (2012, 3) writes: "The questions metaphysicians address are different from those of scientists, but the methods employed to develop and select theories are often relevantly similar. And just as with natural and social-scientific theorizing, as long as we construct and evaluate our theories appropriately, we are justified in inferring conclusions using inference to the best explanation".

analogy seriously, and following Swoyer (1991) understand *models* as vehicles for "surrogative reasoning" about features of the target, we must recognize both the parallel "fine structure" and contextual nature of reification, and also accept the possibility of *inconsistent ontological models*.[88] While I do not have the space to discuss how exactly this plays out in the scientific literature, there is at least modest support for the idea that these seemingly ubiquitous features of natural inquiry do not preclude our taking them to be literal descriptions of aspects of physical and mathematical reality (Morrison, 2011; Paul, 2017; Massimi, 2022). That is, this inconsistency need not be incompatible with realism about explanatory entities and properties–mathematical or otherwise.

Taking this kind of approach voids the major objection Plebani (2020) has to the notion that indispensability arguments might license ontological pluralism about mathematics. As he puts it, proponents of the former ought not be in the business of positing a sharp divide between physical and abstract objects—just the kind of move the latter relies on (Plebani, 2020, 15). However, once we recognize that ontological pluralism is itself the proper description of the "scientific" domain *sans* the mathematical, then positing such an inferential continuity need not undercut plurality.

3.4 Novelty and Ontological Heterogeneity as Virtues

Before closing I think it is apt that I say a bit more about the character of the broader metaphysical methodology being proposed in this chapter. While I've already belabored the point about the so-called "fine-structure" or contextuality of "inference to the best explanation" and indispensability as such, there is something more general to say about exactly *which*

[88] For discussions of ontological pluralism in science—particularly in the special sciences (which indeed represent an interesting relationship with mathematics) see Ludwig & Ruphy (2021). I leave open the possibility that there may be in irreducible incompatibility in ontological descriptions even *of the same mathematical object or structure*.

theoretical virtues are at play in the *Aspect-Pluralist* approach. In particular, I take it that such an approach represents a renewed focus on the virtues of *novelty* and *ontological heterogeneity* insofar as they contrast with directly with Pincock's (2023) recent articulation of the virtues of *conservativeness* and *modesty* (Longino, 1996; King, 2025).

According to Pincock's most recent formulation, accepting an explanatory pluralism for mathematics and acknowledging that these roles are in a (weak) sense indispensable does *not* give us any good reasons to be Platonists (or realists at all for that matter) about the content of mathematics (Pincock, 2023, 69). The reason why is that EIA is sensitive to neither of the following considerations:

> ***Conservativeness* -** the requirement that our best explanation "deploy only entities that we have some prior and independent reason to believe in. X is a less conservative potential explanation than Y when X posits more kinds of new entities than Y does"
>
> ***Modesty* -** which insists that our new theory or explanation ought only deploy new entities "only when *all* the essential features of those new entities are fixed by the explanation of the phenomena in question" (Pincock, 2023, 71).

Pincock draws support for these virtues by pointing to an intuitive example, the beginning of the universe:

> "Consider someone who is agnostic about the existence of God, and yet already believes that the universe has been around for a finite amount of time. For this person, the God explanation is less conservative than the Big Bang explanation… they could then apply IBE and come to a justified belief in the Big Bang Theory. […] The key difference between the God explanation and the Big Bang explanation is that the addition of other phenomena besides the existence of the universe is sufficient to fix many of the essential features of the Big Bang, while not fixing the essential features that are traditionally ascribed to God…the proposed God explanation violates modesty to the extent that it assumes an all powerful God, over and above what is needed to explain the natural phenomena" (Pincock, 2023, 71-72).

The final upshot of this argument is that for Pincock, "the best defense of scientific realism will invoke a *highly restricted* form of IBE" the likes of which will never support rational commitment to the existence of abstracta, least of all a plurality of abstract kinds

(Pincock, 2023, 72, my emphasis). In closing, Pincock briefly alludes to something like a pessimistic meta-induction over the history of science to support this ontological skepticism (Psillos, 2022).

Intuitively, the view being proposed violates *both* conservativeness and modesty. This is a positive feature of ontological pluralism. The reason is that to my mind, both principles while, well-motivated, are neither exhaustive nor categorical. There are (at least) two other theoretical virtues in the literature on inference to the best explanation in science that I think deserve to be taken seriously in the philosophy of mathematics: *novelty* and *ontological heterogeneity*.

Perhaps championed most famously by authors in the feminist epistemology of science – particularly in Longino (1995; 1996)—I take it that both features have import not only on social/political grounds, those typically cited by the feminists, but are both (a) epistemically valuable in progressing science, and (b) compatible with a realist ontology on both the scientific and mathematical sides of the aisle. Take Longino's constructions as in contrast to Pincock's:

> ***Novelty-*** "By novelty, I understand theories that differ in significant ways from presently accepted theories, either by postulating different entities and processes, [or] adopting different principles of explanation,"
>
> ***Ontological Heterogeneity -*** "Treating individual differences as important and not to be elided in abstractions or idealizations which smooth out heterogeneity is valuing heterogeneity, taking it as a basic aspect, if not of the natural world, of one's theories of it" (Longino, 1996, 45-47).

Per the theme of the above discussion, the driving motivation for my siding with Longino is the truth of MEP and IP—which tell us that "best" *anything* will be a plural and contextual attribution. There is, it seems to be, a general tension between being a pluralist about indispensability and holding something like univocal *conservativeness* as far as a naturalistic approach to metaphysical theory choice is concerned.[89]

In particular, operating with a notion of IBE that draws on conservativeness and modesty seems to have trouble dealing with possible and impossible systems that accomplish a great deal of *exploratory* work and bring to our attention new phenomena in the world (e.g., in the AB effect case). Though of course Pincock is right that in general we prefer our theories to be relatively consistent with our background beliefs, what I want to say is that sometimes the content of those background ontological commitments are simply *inadequate* for making new discoveries and formulating novel models and theories that help move empirical inquiry beyond its current state.[90] Applied mathematics, I believe, is central to the kinds of radically novel explanations and understanding-furnishing models that help us achieve such progress.[91]

Given some minimal pluralist assumptions about scientific realism sketched above which posit something like a "rainforest" of levels of organization and species of being (e.g., Wimsatt, 2007; Ladyman & Ross, 2007), it makes sense that a naturalistic approach to mathematical ontology ought to countenance a similarly rich landscape.

What Pincock's line of reasoning presupposes is that conservativeness and modesty exhaust the epistemic virtues relevant to abductive inference. Yet once we acknowledge IP, this presupposition becomes untenable. Not only are conservativeness and modesty two among many

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[89] Note this is not a *direct* argument against Pincock because he does not make the stronger IP claim about strong autonomy, though I take it that the main thrust of the argument applies to his line of reasoning.

[90] It is interesting to note that while Pincock (2015) holds that novelty is indeed a desirable outcome of the Taylor case, his (2023) interestingly drops talk of such considerations.

[91] At this point, I'll note another point of departure of Aspect-pluralism from extant accounts. Consistent with Pincock's virtues, Rice (2025b) argues that mathematical modelers operate with an inclination to stick to the math they already know, as Rice puts it the "limited amount of modelling resources" plays a significant factor in model justification. I do not take this to be the case—mathematicians are engaged in a highly creative endeavor and though of course will often, by force of habit and ease, rely on existing methods but this is not categorical. I believe where *new* math can be used, it often is, and further *should be* in stagnant scientific areas (a la King's (2025) analysis of particle physics) (see Floyd, 2021).

possible virtues, but they also seem from this investigation to be *the wrong ones.* In contexts like the abstract AB effect, or renormalization-group explanations of universality (Batterman, 2002), *the background ontology is precisely what is under scrutiny.* Here, the introduction of novel modal or structural entities is far from philosophically "indulgent" – it is simply a sober-minded means of stabilizing new patterns, invariances, and counterfactual dependencies that were previously unknown. Ontological heterogeneity, then, is an epistemic *virtue* insofar as it enables the full articulation of new explanatory and understanding-furnishing roles.

Pincock's conservativeness constraint is therefore best understood as locally defeasible: appropriate in some situations, but ill-suited to contexts of theoretical and representational exploration. Once this is granted, it becomes clear that the pluralist reconstruction of EIA from a metaontological vantage does not *over generate* entities indiscriminately. Instead, it gives us a framework for determining differentiated ontological commitments precisely where they demonstrably enhance explanatory power, modal understanding, or theoretical unification.[92]

As Martin King writes in a recent article on the pursuitworthiness of *novel* and *"messy"* experiments in fundamental particle physics (given the stagnation of the standard model): "there is still a normative picture here, a rational story to tell about what is pursued, but it is open-ended and pluralistic" (King, 2025, 55). The same is true, I believe, of ontology. The more we learn about the complexity of the world around us, the less tenable ontological "desert landscapes" appear for describing it, I see no good reason why the same ought not hold for the mathematical.

[92] See Sober (1980)'s pioneering discussion which I take it supports the view against conservativeness that I am putting forward. The goal of IBE is to generate an explanation that explains *all* relevant phenomena. W.D. Ross, when considering the virtues of some moral theory, puts the pluralist intuition thus: "Loyalty to the facts is worth more than a symmetrical architectonic or a hastily reached simplicity" (Ross, 1930, 23).

3.5 Conclusion

When Alan Baker reformulated original Quine-Putnam-style indispensability arguments as indexed to particular explanations, he began to show how "holism" of theoretical commitment breaks down and how each particular explanation might be considered in its own right. Induction with EIA began to look like a more *local* affair indexed to *this* or *that* explanation. What I have tried to show here is that this localization of inductive inference can be taken even further—beyond particular explanations and towards more bespoke and diverse epistemic *roles* that are nonetheless indispensable to the practice of science in their own ways. Further, I've argued that rather than precluding a justified realist mathematical ontology altogether (e.g., as Povich (2024) argues), this move simply forces us to examine the metaontological commitments of science and mathematics more closely.

Even those who find current empirical arguments for realism uncompelling (e.g., Jonas (2024)) admit that with further work such assumptions *can*, in principle*,* get off the ground. This chapter offers one such move, and it involves attending carefully to both the scientific epistemology and the metaontological commitments which permeate our reasoning about ampliative inference in the most "extreme" cases of idealization.

This is, then, simply another attempt to articulate the original motivations of the indispensability arguments in the language of more up to date scientific epistemology and metaphysics. One important implication of this approach is a normative commitment to *novelty* and *heterogeneity* as epistemic, theoretic, and ontological virtues—and further that they are not incompatible with (but rather necessary for) truth and objectivity in science and mathematics alike.

3.5.1 The Overall Argument

This marks the end of my defense of mathematical epistemic pluralism, understood as grounded in scientific *aspect-realism*, and the mathematical metaontological pluralism that follows from it. These views together generate a unified picture of the application of mathematics to the natural world and the content of applied mathematical models themselves. I began by laying out the indispensability argument for Platonism which proceeds stepwise from a methodological premise (1) about how to do empirically informed metaphysics, to an epistemic premise (2) about the nature of the *role* of mathematics in science, and finally to a *metaphysical* conclusion (3) about what we ought to me committed to as concerns the profile of mathematical reality.

Traditionally, a monist picture has dominated: mathematics plays one role in which it is uniquely epistemically better than a physical analog *–representation* or *explanation*. Therefore, it is only when we have some irreducible and unequivocally *better* mathematical explanation that we are committed to the existence of the objects and structures it invokes. By probing a philosophically novel set of boundary cases involving the use of highly idealized models and techniques in statistical and quantum mechanics, I have shown this assumption to be false. Mathematics plays *many* roles that are justified and help to model, and furnish *true* understanding of, aspects of natural phenomena inaccessible by other means. This was the conclusion of chapter two.

Chapter three turned to the question of what it means for these roles to actually *be* indispensable, and there we found the concept of indispensability itself a fragile and underspecified one. Taking it *contextually* and being open to the idea that it may be the case that an irreducible plurality of epistemic roles implies an irreducible plurality of *indispensable* ones resulted in *indispensability pluralism* (IP). Understanding this to be the case, I argued that the best way to move forward to the metaphysical premise (3) of EIA was to abandon the first-order metaphysical debate between nominalism and Platonism, and approach commitment from the metaontological level.

This allowed us to accept that in different scenarios, when mathematics is epistemically indispensable in *different ways*, different *kinds* of commitments were warranted. In some cases, mathematical properties *are* in the world to be represented by concepts. In others, Platonic commitment to mind-independent *abstract* structures that can stand in objective dependence relations to systems is most explanatory. Finally, we saw that even when models get "weird"– as in the cases of infinite configuration spaces and systems with infinite correlation lengths—modal interpretations where mathematical structures are understood as *possible* and (non-vacuous) *impossible* structures are apt.

Understanding mathematics as a Wittgensteinian "MOTLEY" of techniques devised by historically situated human mathematicians and physicists, I have tried to show, does not – as many have supposed—undermine the requirement that we answer the questions of its striking applicability to the natural world or of its metaphysical content (Floyd, 2021). Just as we have seen in recent years in the general philosophy of science, plurality and incompatibility of possible descriptions do not *obscure* but instead reveal the even deeper truth that the phenomena and entities studied even by the most fundamental of sciences (here physics and mathematics) display a great deal of ontological complexity: *a plurality of aspects*.

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