THESIS

CLASSIFYING SIMPLICIAL DISSECTIONS OF CONVEX POLYHEDRA WITH SYMMETRY

Submitted by

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ABSTRACT

CLASSIFYING SIMPLICIAL DISSECTIONS OF CONVEX POLYHEDRA WITH SYMMETRY

A convex polyhedron is the convex hull of a finite set of points in $\Re^3$. A triangulation of a convex polyhedron is a decomposition into a finite number of 3-simplices such that any two intersect in a common face or are disjoint. A simplicial dissection is a decomposition into a finite number of 3-simplices such that no two share an interior point. We present an algorithm to classify the simplicial dissections of a regular polyhedron under the symmetry group of the polyhedron.

ACKNOWLEDGEMENTS

I would like to thank my advisor Dr.Anton Betten for giving me an oppurtunity to explore Combinatorics and Group theory. I also want to thank him for being patient with me and allowing me to work at my own pace. I want to thank Jesse Drendel for being available to answer questions at any time of the day. Finally I want to thank my parents for their unconditional love and support.

DEDICATION

To MOM and DAD.

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Chapter 1

Introduction

In combinatorics the problem of classifying combinatorial objects is important. Classification means to determine the distinct ways in which an object can exist subject to an equivalence class. Equivalence relation arises naturally from the space in which the combinatorial object lives in. We show that there are upto equivalence exactly 10 dissections of a cube under the action of its symmetric group. Six of these dissections are triangulation's and have been described by other authors [1]. Our main goal is to show that this list is exhaustive. For this a technique called POSET classification will be used.

In this thesis we are interested in subdivisions of platonic solids into non-overlapping tetrahedra. Analogous to the 2-D case we call these subdivisions as dissections. Two dissections are equivalent if there is a symmetry of the underlying platonic solid that takes one tetrahedron to another.

Classification problem for dissections is about finding a complete set of pairwise inequivalent dissections. In this work we use techniques from combinatorics and group theory to classify the dissections of cube.

Chapter 2

Dissections

Dissections appear in many areas of mathematics, in particular applied mathematics, computational geometry, algebraic geometry and integer programming. In this thesis we consider the dissections of polyhedra. In particular we are interested in problem of classification of dissections. This means that we want to determine distinct dissections after considering the action of the symmetry group of the polyhedra. In this chapter we will introduce terminology to formally define dissections of polyhedra. We will also introduce the notation fo identifying the dissections.

2.1 Basic Definitions

Definition 2.1.1. Point configuration A point configuration is a finite collection of points $A = \{a_1, ..., a_n\}$ in Euclidean space $\Re^d$

Definition 2.1.2. Convex set A subset of a Euclidean space is convex if, given any two points X and Y in that set, the line XY joining them lies entirely within that set.

Let us consider an example in $\Re^2$. In Figure 2.1 the points in the circle forms a convex set because a straight line drawn between any two points in the circle will lie inside the circle. The points inside the star do not form a convex set because as shown in the picture the exists a line joining two points in star that dose not lie inside the star.

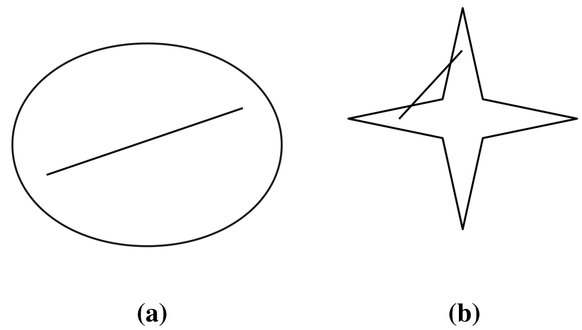


Figure 2.1: (a) represents a Convex set (b) represents a Non Convex set.

Definition 2.1.3. Convex hull: A convex hull of point configuration A is the intersection of all convex sets containing the points in A.

A convex hull of A is the smallest convex set containing A. We denote this by $conv(A)$. The convex hull of a bounded subset of a plane can be visualized as a rubber band stretched around the subset. For example in Figure 2.2 below the solid dots are points of A and the area enclosed by the black band is the convex hull of A.

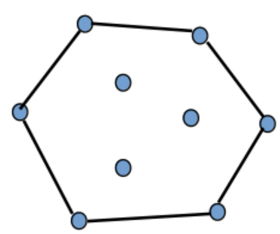


Figure 2.2: Convex hull of solid dots is the area enclosed by rubber band.

We say that k+1 points $u_0, ...u_k$ in $\Re^d$ are affinely independent if $u_1 - u_0, ...u_k - u_0$ are linearly independent.

Definition 2.1.4. Polyhedra: A polyhedra is a convex hull of point configuration.

Definition 2.1.5. k-simplex: A k-simplex is the convex hull of $k + 1$ affinely independent points in $\Re^d$, where $d \geq k$.

A 0-simplex is a point, 1-simplex is a line segment, 2-simplex is a triangle, 3-simplex is a tetrahedron.

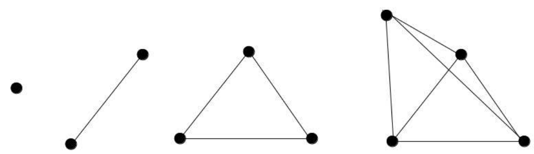


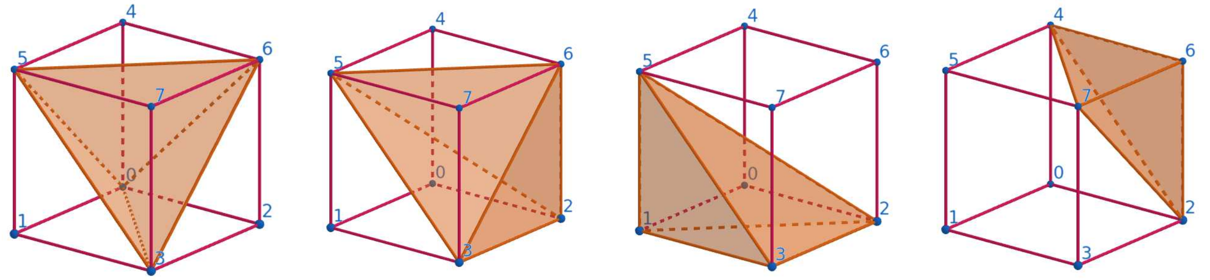
Figure 2.3: From left to right: 0-simplex, 1-simplex, 2-simplex, 3-simplex.

Definition 2.1.6. Dissection: A dissection of polyhedra in a collection of d-simplices all of whoes vertices are points in A such that union of all these simplicies equals a polyhedra.

Definition 2.1.7. Triangulation: A triangulation of polyhedra in a collection of d-simplices all of whoes vertices are points in A that satisfies following two properties:

1. Union property: The union of all these simplicies equals a polyhedra.
2. Intersection property: Any pair of these simplices intersect in a common face. (possibly empty)

In this project we are interested in dissections of cube in $\Re^3$. Dissections of a cube in $\Re^3$ contains subdivisions of cube into non overlapping tetrahedra formed from vertices of cube. Upto symmetry of the cube, there are only four types of tetrahedra shown in Figure 2.4. Any dissection of the cube is composed of tetrahedra equivalent to these four types of tetrahedra.



(a) Core **(b)** Slanted **(c)** Staircase **(d)** Corner

Figure 2.4

In this paper we describe tetrahedra using their rank. To get rank of tetrahedron we list 4-subsets of the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$ in lexicographic order. Then the position of the subset containing the vetrices of the tetrahedron in the list is its rank. Ranking starts from 0. Tetrahedron with rank 0 has vertices $(0, 1, 2, 3)$. A list of ranks of tetrahedron and corresponding vertices in given at the end of this document.

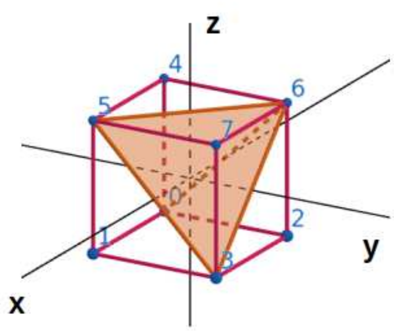
Example: Consider following dissection of cube: $\{1, 35, 45, 57, 63, 66\}$

This dissection consists of tetrahedra with ranks $1, 35, 45, 57, 63, 66$, that is it has tetrahedra with vertices $(0, 1, 2, 4), (1, 2, 3, 4), (1, 3, 4, 5), (2, 3, 4, 7), (2, 4, 6, 7), (3, 4, 5, 7)$

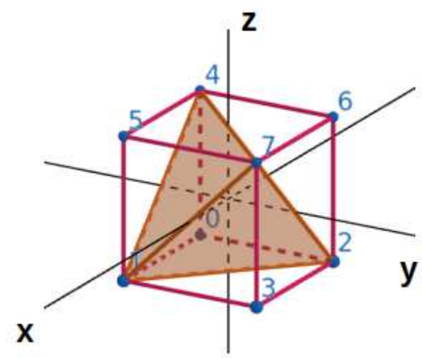
Chapter 3

Classification

In group theory two geometric objects are said to be equivalent upto symmetry if one can be transformed into other using rotations or flips along fixed axis. Figure 3.1a represents a core tetrahedron in a cube. When we rotate this cube in clockwise direction by 90 degrees we get the core tetrahedron show in Figure 3.1b. Hence two tetrahedra in Figure 3.1 are equivalent.



(a)



(b)

Figure 3.1: Two equivalent core tetrahedra.

Collection of all equivalent objects is called an equivalent class. All equivalent objects possess same properties up to symmetry. Hence when working with collection of geometric objects, to avoid redundancy, we only have to consider non-equivalent geometric objects. This can be achieved by picking one object from each equivalent class. The process of picking representatives from each class is called classification. The goal of this chapter is to develop the group theory to formally define classification.

In this thesis we will work with unit cube shown in figure Figure 3.2. Coordinates of unit cube are given in Table 3.1.

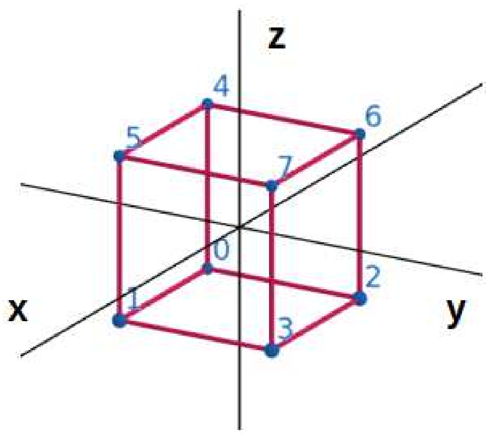


Figure 3.2: Unit cube

Table 3.1: Coordinates of vertices of unit cube

Vertex	Coordinate
0	(-1,-1,-1)
1	(1,-1,-1)
2	(-1,1,-1)
3	(1,1,-1)
4	(-1,-1,1)
5	(1,-1,1)
6	(-1,1,1)
7	(1,1,1)

Assume that the cube is glued to the coordinate axis. If we rotate the cube along y-axis by 90°in clockwise direction the spatial orientation of cube is preserved. This operation is same as sending x axis to z axis and z axis to negative x axis. Similar to this we want to find all the permutations of coordinate axis that preserve the spatial orientation of cube.

There are $3!$ ways to organize the x, y, z coordinate axis. Each of the axis can take either positive or negative sign. Hence there are $3! * 2^3 = 48$ ways to organize the axis that will preserve the spatial orientation of cube. We can describe each of these transformations using matrix notation.

Matrix in Figure 3.3 is a transformation that preserves the spatial orientation of the unit cube. First, second and third column of the matrix correspond to x-axis, y-axis and z-axis respectively. First, second and third rows correspond to x,y,z-axis respectively. 1 in first column indicates that x-axis goes to x-axis. -1 in column 2 indicates that y-axis goes to negative z-axis and 1 in column 3 indicates that z-axis in sent to positive y-axis. Other set of transformation that preserve the symmetry of cube are given in section 3.1.

$$\begin{array}{c} \\ \mathsf{x} \\ \mathsf{y} \\ \mathsf{z} \end{array} \begin{array}{c} \begin{array}{ccc} \mathsf{x} & \mathsf{y} & \mathsf{z} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{array}$$

Figure 3.3: matrix representation of transformation $x, y, z - axis \rightarrow x, -z, y - axis$

3.1 Basic definitions of group theory

Definition 3.1.1. Group: Let G be a set together with a binary operation that assigns to each ordered pair G of elements of G an element in G denoted by ab. We say G is a group under this operation if the following properties are satisfied.

1. Associativity: The operation is associative if *(ab)c = a(bc)*

2. Identity: There is an element *e* in *G* such that *ae* = *ea* = *a* for all *a* in *G*

3. Inverses: For each element a in G , there is an element b in G (called an inverse of a) such that $ab = ba = e$

Set of all 48 transformations given in section 3.1 forms a group. This group is called the symmetric group of cube. Identity element is the unit matrix corresponding to element 0. Inverse of every transformation is also a transformation. Finally matrix multiplication is associative. Since the product of two transformations is also a transformation the set of all 48 transformations is closed under matrix multiplication.

Definition 3.1.2. A **group action** of a group G on a set A is a map from $G \times A$ to A (written as $g \cdot a$ for all $g \in G$ and $a \in A$) satisfying following properties:

1. $(a \cdot g_1) \cdot g_2 = a \cdot (g_1 \cdot g_2)$ for all $g_1, g_2 \in G$, $a \in A$

2. $1 \cdot a = a$ for all $a \in A$ and 1 is identity of G

If group action of G on set A exists we say that G is acting on A.

The symmetric group G of a cube is acting on the set of vertices (A) of the cube.

$$A = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

Each element of the symmetry group is a mapping from $A \rightarrow A$. For example consider the transformation

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Transformation M can be visualized as rotating the cube along x-axis in clockwise direction. New coordinates for $a \in A$ under the transformation M is equal to $M \cdot (\bar{a})^T$, where $\bar{a}$ is the coordinate points of vertex a and $(\bar{a})^T$ is the column vector of $\bar{a}$. For example vertex 1 has coordinates $\bar{a} = (1, -1, -1)$.

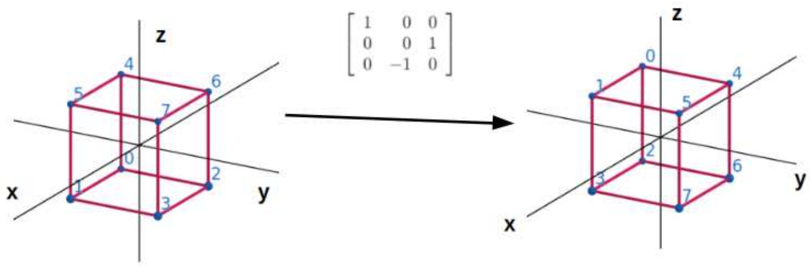


Figure 3.4: Transformation M rotates the cube along x-axis in clockwise direction

$$M \cdot \bar{a} = M \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$(1, -1, 1)$ is the new coordinate of vertex 1 under the transformation M. Since $(1, -1, 1)$ was the coordinate of vertex 5 before transformation, we say that M moves the vertex 1 to vertex 5. Similarly M moves vertices $5 \to 7$, $7 \to 3$ and $3 \to 1$. It also moves $0 \to 4$, $4 \to 6$, $6 \to 2$ and $2 \to 0$. Hence as shown in Figure 3.5 M is a map from $A \to A$.

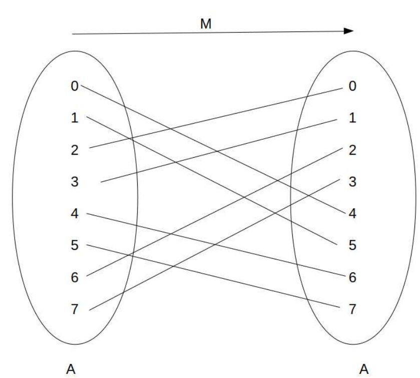


Figure 3.5: Transformation M is a map from $A \to A$

For $a \in A$ and $g_1, g_2 \in G$ we have $g_1 \cdot (g_2 \cdot a) = (g_1 \cdot g_2) \cdot a$. Remember that a is the column vector corresponding to vertex of a cube. Also for $a \in A$ and $1 \in G$ we have $1 \cdot a = a$ (1 is the identity matrix). Hence the symmetric group (G) of cube is acting on the vertices of cube.

Definition 3.1.3. Let G be a symmetric group of a set A. For each a in A, let

$$Stab_G(a) = \{g \in G | g \cdot a = a\}$$

$Stab_G(a)$ is called the **stabilizer** of a. Stabilizer of a point a forms a group.

Example 3.1.1. Consider the transformation

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Coordinates of vertex 7 are $(1, 1, 1)$.

$$M \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The transformation M keeps the vertex 7 fixed. Hence $M \in Stab_G(7)$

Coordinates of vertex 0 are $(-1, -1, -1)$.

$$M \cdot \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

The transformation M keeps the vertex 0 fixed as well. Hence $M \in Stab_G(0)$

Definition 3.1.4. Let G be a symmetry group of set A. For each a in A, let $Orb_G(a) = \{g \cdot a | \forall g \in G\}$. The set $Orb_G(a)$ is a subset of A. It is called **orbit** of a under G.

Alternatively we say that $a_1, a_2 \in A$ belong to the same orbit if $\exists g \in G$ such that $g \cdot a_1 = a_2$. Under a given group action, a set can have many orbits. Each orbit can be represented by an arbitrary element from that orbit. Such an element is called **orbit representative** of respective orbit.

Definition 3.1.5. **Order** of a set A represents number of elements in A. It is denoted as $|A|$.

Lemma 1. Orbit-Stabilizer Theorem: Let G be a symmetry group of a set A. Then for any $a \in A$, $|G| = |orb_G(a)| \times |stab_G(a)|$.

Definition 3.1.6. To **classify** a set under the action of its permutation group means to find orbit representatives of orbits under this action.

In this thesis we want to classify the dissections of cube under the action of symmetry group of the cube. That is we want to find the orbit representatives of set of dissections of cube under the action of symmetry group of cube. In next chapter we will discuss the classification algorithm that will allow us to classify the dissections without generating all dissections of cube.

Chapter 4

POSET and Classification algorithm

Classical approach of classifying a set under a group action involved enumerating all the elements of the set and then determining the group action on every element. Such an approach is computationally expensive as the size of the set increases. Hence we use an alternative approach to classify the dissections of cube. Dissections discussed in chapter 1 has an underlying POSET structure. We can take advantage of this structure to classify our dissections with out enumerating all all of them. In this chapter we formally define POSET and then discuss classification algorithm that can be used on sets having POSET structure to classify them under respective symmetry groups

4.1 POSET and Lattice

Definition 4.1.1. A **partial ordering** on a finite set P is a relation " $\preceq$ " on P which satisfies following properties:

1. Reflexive property: $x \preceq x$
2. Antisymmetric property: $x \preceq y$ and $y \preceq x$ implies $x = y$
3. transitive property: $x \preceq y$ and $y \preceq z$ implies $x \preceq z$

where $x, y, x \in P$. The set P along with partial ordering $\preceq$ $(P, \preceq)$ is called a **partially ordered set** which is abbreviated as POSET.

Finite POSETs are represented using Hasse diagram. Hasse diagram is a graph with vertices representing elements of POSET. Vertices are arranged such that if $x \preceq y$ then x is placed above y. We place an edge between two elements x, y of POSET if:

1. $x \preceq y$ and

2. There dose not exist a z such that $x \preceq z$ and $z \preceq y$.

Example 4.1.1. Let $P = \mathcal{P}(\{1, 2, 3\})$, the power set of $\{1, 2, 3\}$. P is a POSET whoes ordering is the subset relation $\subseteq$. Following diagram displays Hasse diagram of POSET P.

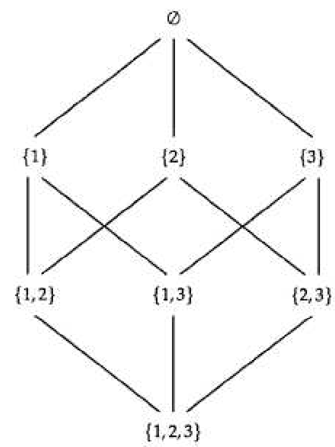


Figure 4.1: Hasse Diagram of P

We say that a group G acts on POSET P if:

$$x \preceq y \implies g \cdot x \preceq g \cdot y$$

for all $x, y \in P$ and for all $g \in G$.

Let Q be a subset of the POSET $(p, \preceq)$. An upper bond of Q is an element $w \in P$ such that $q \preceq w$ for all $q \preceq Q$. The least upper bound of Q also known as supremum of Q is its upper bound w such that $w \preceq w'$ for all other upper bounds w' of Q. If the supremum of set exists it is unique.

Similarly a lower bound of Q is an element $l \in P$ such that $l \preceq q$ for all $q \in Q$. The greatest lower bound of Q also known as the infinimum of Q is the lower bound l of Q such that $l' \preceq l$ for all other lower bounds l' of Q. If the infimum of a set exists then it is unique.

Definition 4.1.2. A finite set L is called a **lattice** if it satisfies the following properties:

1. $(l, \preceq)$ is a poset

2. Any two elements $x, y \in L$ have an infimum and a supremum.

A indexing function of lattice $(L, \preceq)$ is a mapping $rk; L \rightarrow \mathbb{N}$, satisfying

$$x \preceq y \implies idx(x) \preceq idx(y)$$

for all $x, y \in L$. A lattice with such an indexing function is called a indexed lattice. The indexing function induces levels on lattice L. The ith level contains elements of index i.

Example 4.1.2. The POSET of power function $\mathcal{P}(\{1,2,3\})$ previously discussed is an indexed lattice. The indexing function if defined as follows:

$$idx : \mathcal{P} \rightarrow \mathbb{N}, x \rightarrow |x|$$

where $|x|$ is the cardinality of subset x of $\{1,2,3\}$. Following table defines elements at each level.

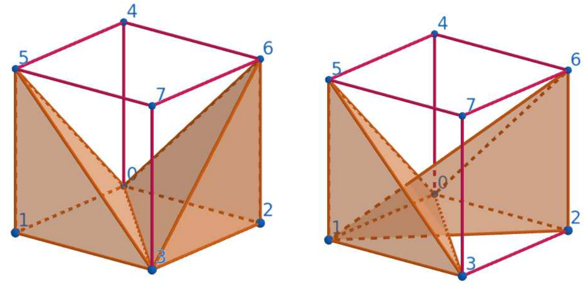
Level	elements at level
0	$\emptyset$
1	$\{1\}$, $\{2\}$, $\{3\}$
2	$\{1,2\}$,$\{1,3\}$,$\{2,3\}$
3	$\{1,2,3\}$

We will now discuss the POSET structure underlying the dissections of cube discussed in chapter 1.

Consider the cube with vertices $0, 1, 2, 3, 4, 5, 6, 7$.
Let $\mathcal{H}$ be set of non-coplanar tetrahedra formed from vertices of the cube.
Let $\mathcal{P}(\mathcal{H})$ be power set of $\mathcal{H}$. $\mathcal{P}(\mathcal{H})$ has all combinations of tetrahedrons formed from vertices of cube.

A partial dissection is a set of disjoint tetrahedra.

Let $\mathcal{T} \subseteq \mathcal{P}(\mathcal{H})$ be set of all partial dissections such that none of the tetrahedra intersect. For example partial dissection $a \in \mathcal{T}$ but $b \notin \mathcal{T}$.



(a) **(b)**

Figure 4.2: $a \in \mathcal{T}$ but $b \notin \mathcal{T}$

$\mathcal{T}$ is a partially ordered set. The ordering on the set is inclusion. That is $a \preceq b$ if $a \subseteq b.(a, b \in \mathcal{T})$.

We can represent the POSET $\mathcal{T}$ using a lattice. We can define the indexing function as following.

$$idx : \mathcal{T} \to \mathbb{N}, x \to |x|$$

Where $|x|$ is cardinality of x that is number of tetrahedra in partial dissection x.Nodes in level i represent partial dissections with i tetrahedra in it.

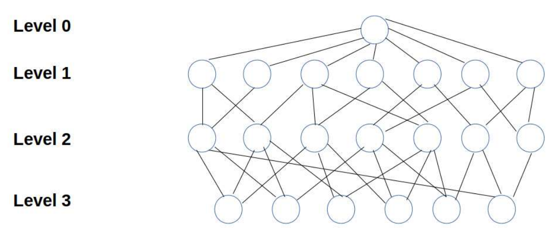


Figure 4.3: Lattice Diagram of $\mathcal{T}$

As discussed in chapter one, there are maximum six tetrahedra in dissections of cube. Hence the lattice of dissections has six levels.

4.2 General theory of Classification

Let G be a group that acts on finite sets A and B.

Let I be a G-invariant relation between A and B:

$$(a,b) \in I \iff (ag,bg) \in I$$

For $a \in A$ let,

$$Up(a) = \{(a,y) \in I | \forall y \in B\}$$

For $b \in B$, let

$$Down(b) = \{(x,b) \in I | \forall x \in A\}$$

Let

$$P_1, ..., P_m$$

be the orbit representatives corresponding to action of G on A.

Let

$$Q_1, ..., Q_n$$

be the orbit representatives corresponding to action of G on B.

Let $\mathcal{T}_{i,r}$ be the orbits of $Stab_G(P_i)$ on $Up(P_i)$ and let $t_{i,r} = (P_i, b_{ir})$ be the orbit representative of $\mathcal{T}_{i,r}$; so

$$Up(P_i) = \bigcup_{r=1,..,r_i} \mathcal{T}_{i,r}$$

Where $r = 1, ..., r_i$ and $\mathcal{T}_{i,r} = Orb_{Stab_{(G)}(P_i)}(t_{i,r})$

Let $\mathcal{S}_{j,l}$ be the orbits of $Stab_G(Q_j)$ on $Down(Q_j)$ and let $s_{j,l}$ be the orbit representatives of $\mathcal{S}_{j,l}$ so

$$Down(Q_j) = \bigcup_{t=1,..,l_j} \mathcal{S}_{j,l}$$

Where $l = 1, ..., l_j$ and $\mathcal{S}_{j,l} = Orb_{Stab_{(G)}(Q_j)}(s_{j,l})$

Let

$$\mathcal{I}_k \ k = 1, ..., \delta$$

be the orbit representatives under the action of G on I.

Lemma 2. (1)There is a canonical bijection between sets

$$\{\mathcal{T}_{i,r} \mid i = 1, .., m \ \ r = 1, ..., r_i\}$$

and

$$\{\mathcal{S}_{j,l} \mid j = 1, .., n \ \ r = 1, ..., l_j\}$$

(2) $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} l_j$

Proof: We will show that for each $\mathcal{I}_k$, there exists exactly one $\mathcal{T}_{i,r}$ such that $\mathcal{T}_{i,r} \subset \mathcal{I}_k$. For $n = 1, 2$, let Π_n be the projection on to n^{th} coordinate.
For contradiction assume $\mathcal{T}_{i,r} \subset \mathcal{I}_k$ and $\mathcal{T}_{u,v} \subset \mathcal{I}_k$ where $(i, r) \neq (u, v)$. Then there exists $g \in G$ such that $t_{i,r} \cdot g = t_{u,v}$. This implies

$$P_i \cdot g = \Pi_1(t_{i,r} \cdot g) = \Pi_1(t_{u,v}) = P_u$$

This implies P_i and P_u belong to same orbit under the action of G on A. Hence $i = u$ and $g \in Stab_G(P_i)$.

Since $g \in Stab_G(P_i)$, we have

$$b_{i,r} \cdot g = \Pi_2(t_{i,r} \cdot g) = \Pi_2(t_{u,v}) = b_{i,v}$$

This implies that $b_{i,r}$ and $b_{i,v}$ belong to same orbit under the action of $stab_G(P_i)$. Hence $b_{i,r} = b_{i,v}$, that is $r = v$. This is a contradiction since $(i, r) \neq (u, v)$.

Similarly one can show that for a given $\mathcal{R}_k$ there is a unique $\mathcal{S}_{j,l}$ such that $\mathcal{S}_{j,l} \subset \mathcal{R}_k$.

This proves that there is a bjection between the sets $\mathcal{T}_{i,r}$ and $\mathcal{S}_{j,k}$

Lemma 3. For a given k there exists only one $\mathcal{T}_{i,r}$ and $\mathcal{S}_{j,l}$ such that $\mathcal{T}_{i,r} \subseteq \mathcal{I}_k$ and $\mathcal{S}_{j,l} \subseteq \mathcal{I}_k$

That is there exists a $g \in G$ such that

$$(P_i, b_{ir}) \cdot g = (a_{jl}, Q_j)$$

That is

$$b_{ir} \cdot g = Q_j$$

This means b_{ir} and Q_j belongs to same orbit under the action of G on B. So we can use b_{ir} as orbit representative instead of Q_j

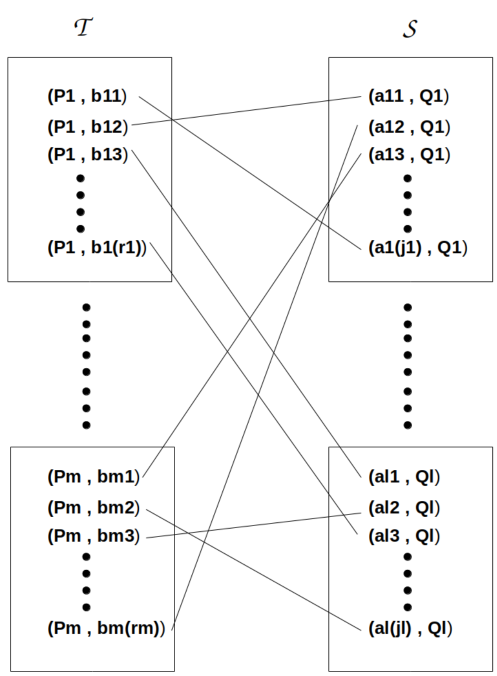


Figure 4.4: Relation between $\mathcal{T}$ and $\mathcal{S}$

In above figure an edge between elements of $\mathcal{T}$ and $\mathcal{S}$ implies that they belong to same flag orbit. For example the elements $(P1, b12) \in \mathcal{T}$ and $(a11, Q1) \in \mathcal{S}$ belong to the same flag orbit. Hence there exists an element $g \in G$ such that $g \cdot (P1, b12) = (a11, Q1)$. Since the group acts component wise we have $g \cdot b12 = Q1$. Which mean $b12, Q1$ belong to the same orbit under the action of G. This means one can use second components of $\mathcal{T}$ as representatives of set B.

This is very convenient way of determining the representatives of B if representatives of A are knows. One has to be careful about avoiding redundant representatives. For example $(P1, b12) \in \mathcal{T}$ and $(a11, Q1) \in \mathcal{S}$ belong to the same flag orbit and $(Pm, bm1) \in \mathcal{T}$ and $(a12, Q1) \in \mathcal{S}$ also belong to same flag orbit. Which means $b12, bm1$ and $Q1$ all belong to same orbit. So we only need to pick one of $b12, bm1$ as the representative of orbit containing $Q1$. To remove such redundant orbits we use technique called isomorphic rejection discussed in next section.

4.3 Classification algorithm applied to POSET

Consider a partially ordered set

$$(\mathcal{P}, \preceq)$$

. The ordering on this set ($\preceq$) is inclusion. i.e

$$x < y \iff x \subseteq y \quad x, y \in \mathcal{P}$$

The group G acts on POSET $\mathcal{P}$, as following:

$$g \cdot x \preceq g \cdot y \iff x \preceq y$$

which is a G invariant relation.

We want to determine orbit representatives of every level of $\mathcal{P}$ under the action of G.

POSET classification algorithm uses lemma 2 to determine orbit representatives of level $i + 1$ from orbit representatives of level i.

Let A be objects at level i

Let B be the objects at level $i+1$

Let $\mathcal{I}$ be relation between A and B given by POSET structure. That is

$$(x, y) \in \mathcal{I} \iff x \preceq y$$

Assume $P_1, ...P_m$, the orbit representatives corresponding to action of G on A are known.

Assume $Stab_G(P_i)$, for $i = 1, .., m$ are also known.

For $i = 1, ..., m$ calculate the orbit representatives

$$t_{i,1}, t_{i,2}, ..., t_{i,r_i}$$

under the action of $Stab_G(P_i)$ on $Up(P_i)$

Let

$$\mathcal{T} = \bigcup t_{i,r}$$

for $i = 1, 2, ...m$ and $r = 1, 2, ...r_i$

Let Π_i denote projection on to i^{th} component.

Let $\mathcal{C} = \{\Pi_2(t) \mid t \in \mathcal{T}\}$

From Lemma 2, we know that $\mathcal{C}$ contains orbit representatives of level $i+1$. $\mathcal{C}$ can contain elements that belong to same orbit. So we want to prune this list to eliminate any redundant representatives. We use isomorphic rejection for this.

Steps in isomorphic rejection:

1. For every $t_{i,r} \in \mathcal{T}$

2. Let $Q_j = \Pi_2(t_{i,r})$. Where $\Pi_2(a, b) = b$

3. Let $\mathcal{D} = Down(Q_j) = \{(x, y) | \forall x \in A, x \preceq y\}$

4. For every element d in $\mathcal{D}$ do the following:

(a) Let a = $\Pi_1(d)$. Find $\alpha \in G$ such that $a\alpha == P_i'$ for some i'. (P_i is representatives of A)

(b) Find β in stabilizer of $P_{i'}$ such that $d\alpha\beta = t_{i',r'}$

5. If $i' = i$ and $r' = r$ move to next element in $\mathcal{T}$. Else remove the element $t_{i',r'}$ from $\mathcal{T}$ before moving on to next element in $\mathcal{T}$

Upon finishing isomorphic rejection, the elements left in $\mathcal{T}$ are orbit representatives of level $i+1$.

Example 4.3.1. We apply the above POSET classification algorithm to the POSET of dissections of cube. Here we will discuss the generation of orbit representatives of level one and two.

1. POSET algorithm starts at level-0 which just consists of the empty set.

2. Next $Up(\emptyset)$ is calculated. $Up(\emptyset)$ will contain 58 elements which are cartesian products of the $\emptyset$ and all non-planar tetrahedra in level-1.

3. The orbit representatives of $Up(\emptyset)$ is calculated under the action of stabilizer of $\emptyset$,which is the entire symmetry group of cube. $\mathcal{T}1$ is the collection of these orbit representatives. $\mathcal{T}1 = \{\{(\emptyset, 1)\}, \{(\emptyset, 2)\}, \{(\emptyset, 4)\}, \{(\emptyset, 28)\}\}$. $(1, 2, 4, 28)$ are rank numbers of tetrahedra.

4. Next we calculate $\Pi_2(\mathcal{T}1)$ which is equal to $\{1, 2, 4, 28\}$

5. We perform isomorphic rejection on the set $\Pi_2(\mathcal{T}1)$. Since all elements in $\{1, 2, 4, 28\}$ are non-isomorphic, isomorphic rejection dose not eliminate any elements from this list. Hence $\{1, 2, 4, 28\}$ are orbits of level 1.

6. $\mathcal{S}1$ layer in figure below represents the orbit representatives of $Stab_G(1)$ on $Down(1)$, $Stab_G(2)$ on $Down(2)$, $Stab_G(4)$ on $Down(4)$ and $Stab_G(28)$ on $Down(28)$. The POSET program dose not compute these representatives. We only include it in the graph to emphasize that there is a one-one bijection between $\mathcal{T}1$ and $\mathcal{S}1$

7. POSET algorithm uses these representatives of level-1 to determine the representatives of level-2. The sets $Up(1), Up(2), Up(4), Up(28)$ are calculated.

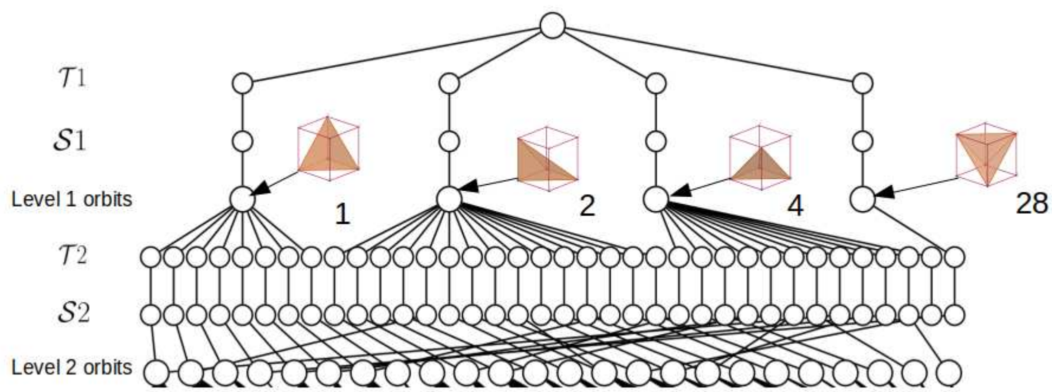
8. The representatives from action of $Stab_G(1)$ on $Up(1)$, $Stab_G(2)$ on $Up(2)$, $Stab_G(4)$ on $Up(4)$ and $Stab_G(28)$ on $Up(28)$ are calculated. These representatives form the set $\mathcal{T}2$ in the picture below.

9. The set $\Pi_2(\mathcal{T}2)$ is computed. It has 36 elements, some of which are isomorphic.

10. After performing the isomorphic rejection on $\Pi_2(\mathcal{T}2)$ we get the orbit representatives of level-2 which are 24 in count. In the graph below two elements in $\Pi_2(\mathcal{T}2)$ connect to same node in Level-2 representatives when they are isomorphic.

11. This process is continued upto level-6.

The complete list of representatives for all six levels of POSET of dissections is given in last chapter.



$\mathcal{T}1$
$\mathcal{S}1$
Level 1 orbits
1
2
4
28
$\mathcal{T}2$
$\mathcal{S}2$
Level 2 orbits

Figure 4.5

Chapter 5

Intersection Matrix

POSET classification algorithm was implemented as part of open source software called orbiter [2, 3]. We use this software to classify the dissections of cube.

Remember from previous section that POSET algorithm uses orbit representatives of level i to determine representatives of level $i+1$. This algorithm requires all partial dissections in level $i+1$ that are super sets of representatives of level i. These partial dissections were supplied to POSET method in orbiter using the test function called $tetrahedronTestFunction$.

Partial dissections have non-intersecting tetrahedra. So $tetrahedronTestFunction$ must eliminate any sets that have intersecting tetrahedra.

There are 70 different tetrahedra that can fit into a cube. Of these 70, 12 tetrahedra are planar and hence can be discarded. For the remaining 58 tetrahedra, we determine if two tetrahedra intersect based on following conditions.

Two tetraherdons intersect if they have common face and opposite vertices corresponding to this face of both tetrahedra lie on the same side of the face. An opposite vertex to a face of the tetrahedron is the fourth vertex that is not part of the face.

To determine if two points lie on same side of a plane, we use the hesse normal form of that plane. Consider a plane with equation $ax + by + cz + d = 0$. The hesse normal form of this plane is given by $\hat{\mathbf{n}} \cdot \mathbf{x} = -\mathbf{p}$, where $\hat{\mathbf{n}} = (n_x, n_y, n_z)$

$$n_x = \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$n_y = \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

$$n_z = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$\mathbf{p} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

and

$$\mathbf{x} = (x, y, z)$$

Distance $\mathbf{D}$ of a point $\mathbf{x_0}$ from the plane $\hat{\mathbf{n}} \cdot \mathbf{x} = -\mathbf{p}$ is given by $\hat{\mathbf{n}} \cdot \mathbf{x_0} + \mathbf{p} = \mathbf{D}$. If $\mathbf{D} > 0$ then x_0 lies on the positive side of plane. If $\mathbf{D} < 0$ then x_0 lies on negative side of plane. If $\mathbf{D} = 0$ the x_0 lies on the plane.

Two points x_0, x_1 lie on same side of the plane if they both lie on positive side of the plane or negative side of plane or on the plane. If points dont lie on same side of plane we say that they lie on different sides of plane.

Example 5.0.1. In Figure 5.1, 3 and 6 are opposite vertices to the face 410 and both of them lie on same side of the face. Hence we know that these tetrahedra intersect.

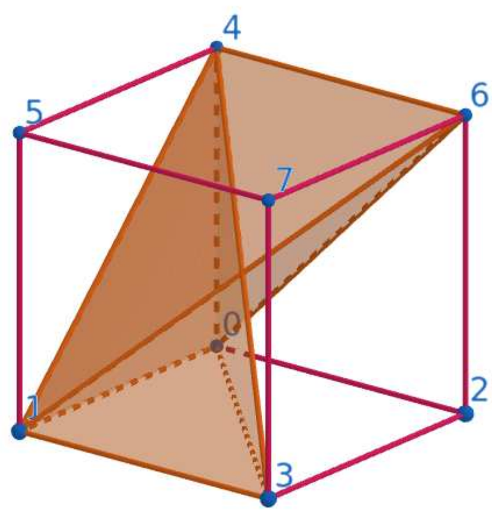


Figure 5.1: Tetrahedra intersect because opposite vertices are on opposite sides of common face

In Figure 5.2, 1 and 2 are opposite vertices of each tetrahedron corresponding to the common face 037. Both these vertices lie on different side of the common face and hence we know that they dont intersect.

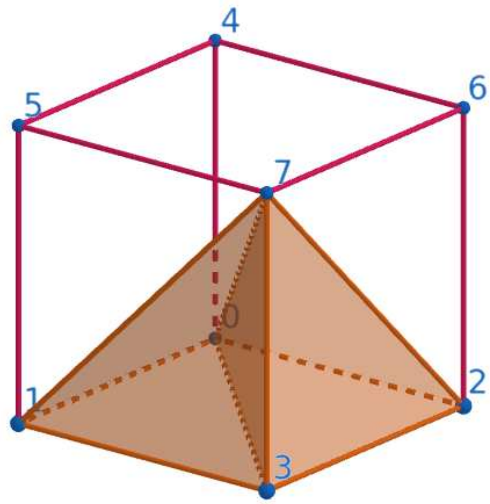


Figure 5.2: Tetrahedra dont intersect because opposite vertices are on same sides of common face

When tetrahedra dont have common faces we use following approach to determine their intersection.

1. pick 1st of two tetrahedra and intersect it with a face f_k of second tetrahedron ($1 \leq k \leq 4$). Clearly these intersection points will lie on f_k
2. Calculate the local coordinates of intersection points with respect to face f_k
3. Pick the point of intersections from above and calculate their center of gravity
4. If center of gravity lies inside the face f_k then both tetrahedra intersect. If center of gravity is outside the face f_k then repeat above steps for remaining faces of second tetrahedron until an intersection is found. If no intersection is found for all for faces this means tetrahedra do not intersect.

Example 5.0.2. Lets consider the intersection of tetrahedra with rank numbers 1 and 6. Tetrahedron 1 has vertices $(0, 1, 2, 4)$ and tetrahedron 6 has vertices $(0, 1, 3, 5)$. Intersecting the tetrahedron $(0, 1, 2, 4)$ with the face $(0, 3, 5)$ of second tetrahedron we get intersecting points $A, B, 0$. Center of gravity of these three points clearly lies inside the face $(0, 3, 5)$ and hence we know that these two tetrahedra intersect.

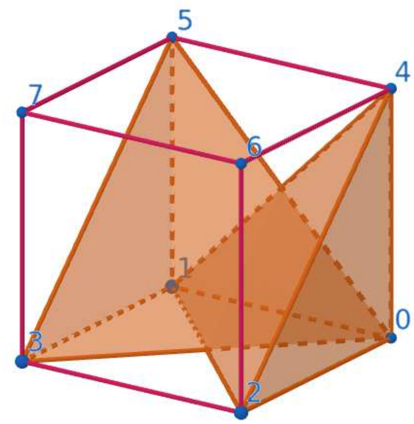


Figure 5.3: Intersection of tetrahedra with ranks 1 and 6

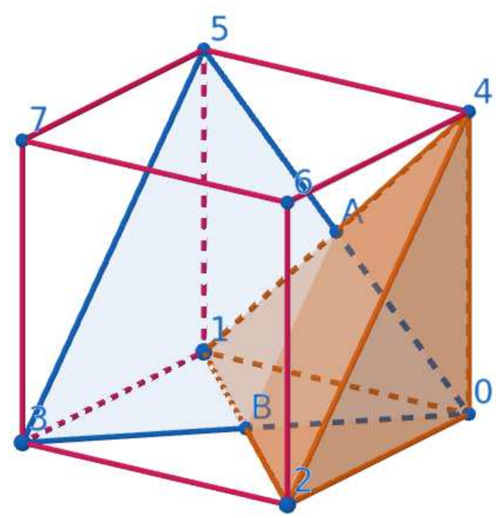


Figure 5.4: Intersection of face $(0, 3, 5)$ of tetrahedron 6 with tetrahedron 1

Intersection of all pairs of 70 tetrahedra is determined and populated in to $70x70$ array called $tetraIntersection$. This intersection matrix is given in Figure 5.5. The column index and row index correspond to rank of the tetrahedron. For example $(4, 7)$ element in the intersection matrix gives the intersection of tetrahedra with ranks 4 and 7. The white in matrix picture indicates that the tetrahedron is planar and hence can be ignored. The diagonal of the matrix represents the intersection of tetrahedron with itself and hence can be ignored as well. Black in intersection matrix indicates that 2 tetrahedra do not intersect and red indicates that two tetrahedra intersect.

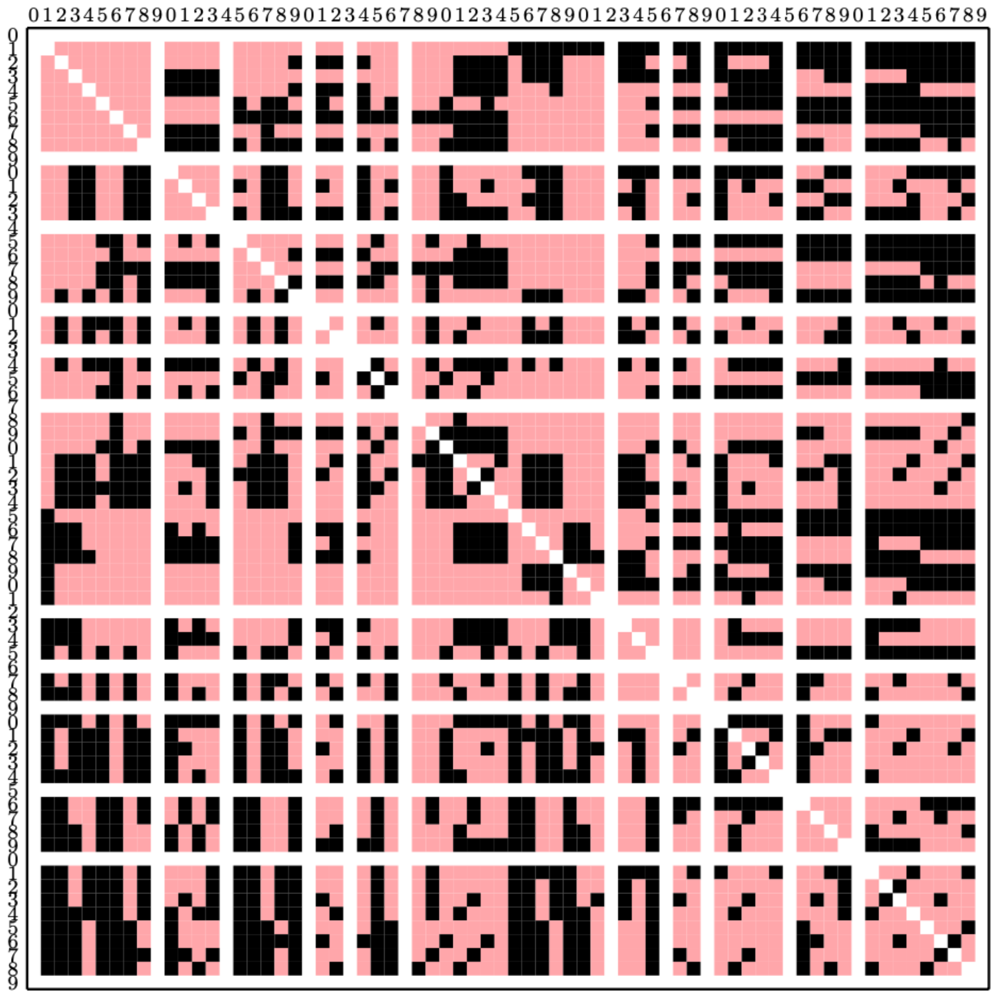


Figure 5.5: Intersection matrix of tetrahedra

Chapter 6

Results

6.1 Orbit representatives of dissections of cube

POSET algorithm ran for 6 levels and following 10 orbit representatives of dissections were found. Table 6.1 lists these ten orbit representatives. Table 6.2 gives the type of tetrahedra in each representative. For example representative R_1 has one tetrahedron of type core, with rank 41 and four tetrahedra of type corner with ranks $1, 38, 52, 63$. Figure 6.1 shows the tetrahedra in each representative R_i where $1 \leq i \leq 10$. For example the row corresponding to R_1 shows one core tetrahedron and four corner tetrahedra that make R_1.

Table 6.1: Orbit representatives of dissection of cube

Representative	Dissections
R_1	$\{1, 38, 41, 52, 63\}$
R_2	$\{1, 35, 45, 56, 65, 68\}$
R_3	$\{1, 35, 45, 56, 66, 67\}$
R_4	$\{1, 35, 45, 57, 63, 66\}$
R_5	$\{1, 35, 45, 58, 61, 68\}$
R_6	$\{1, 35, 45, 59, 61, 64\}$
R_7	$\{1, 35, 45, 59, 62, 63\}$
R_8	$\{1, 35, 47, 52, 57, 63\}$
R_9	$\{2, 19, 36, 56, 66, 67\}$
R_{10}	$\{2, 19, 36, 59, 61, 64\}$

Table 6.2: Orbit representatives of dissections of cube

Representative	Core	Corner	Staircase	Slanted
R_1	41	1,38,52,63		
R_2		1,68	45,56	35,65
R_3		1	45, 56, 66, 67	35
R_4		1, 63	45, 66	35, 57
R_5		1, 68	45, 61	35, 58
R_6		1	45, 59, 61, 64	35
R_7		1, 63	45, 59	35, 62
R_8		1, 52, 63		35, 47, 57
R_9		2, 19, 36, 56, 66, 67		
R_{10}		2, 19, 36, 59, 61, 64		

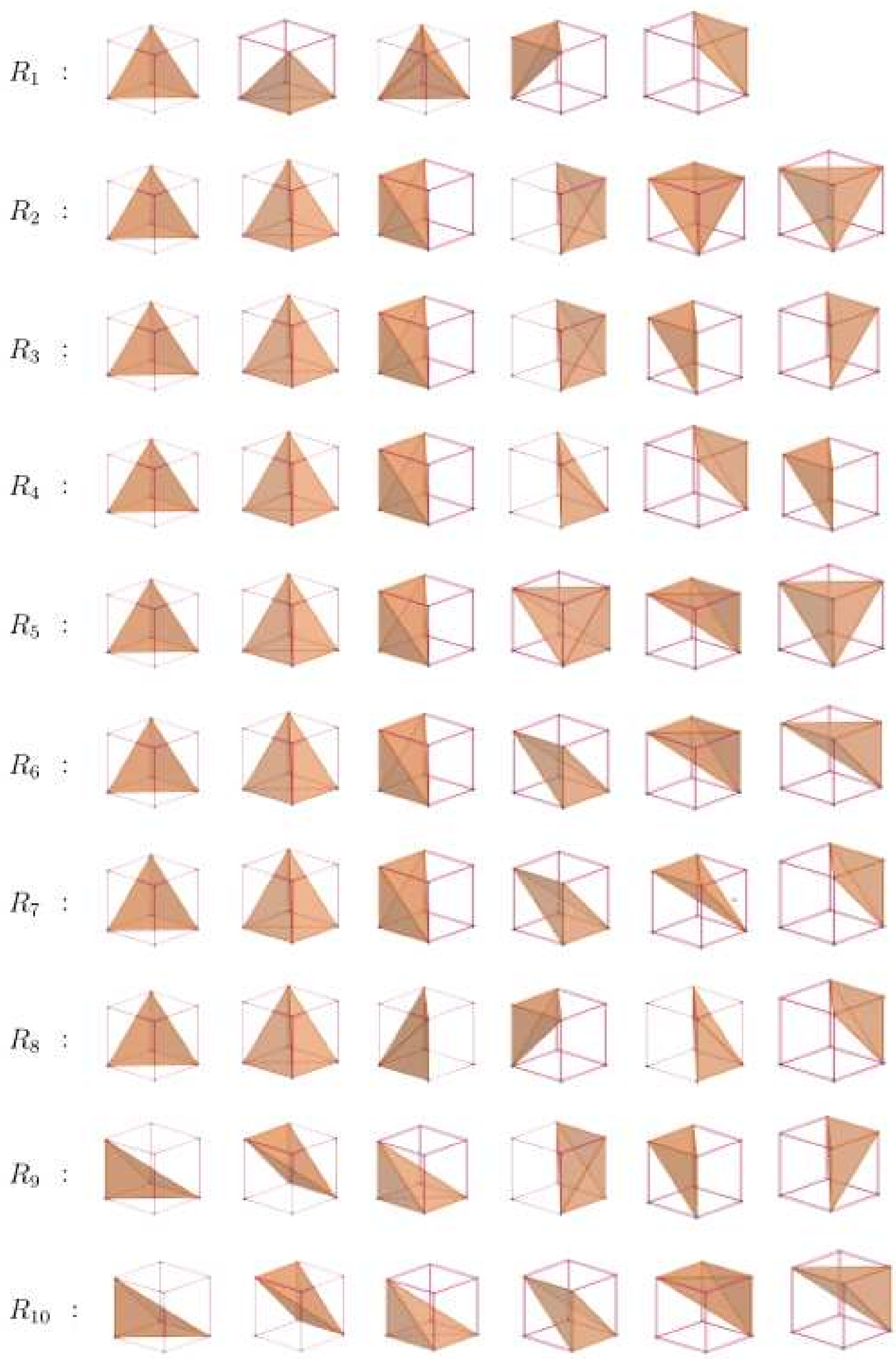


Figure 6.1: Tetrahedra in each orbit representative

6.2 POSET of orbit representatives

Table 6.3 gives the orbit representatives at each level of POSET of dissections. Figure 6.2 represents the POSET of orbit representatives at each level. The rank of tetrahedra in each partial-dissection or dissections can be read from Table 6.4. For a given level, the top most element in the table for that level corresponds to left most node in that level of POSET graph. For example the left most node in level-2 is equal to the partial-dissection $\{1, 35\}$. As we move from left to right in a level of POSET graph, we have to move from top to bottom in the table to get relevant information. So the right most node in level-2 corresponds to the partial-dissection $\{4, 53\}$.

In Table 6.4, column "SO" gives the order of stabilizer of respective representative and length of orbit containing that respective. Notice that the produce of elements in "SO" column for a representative is always 48. This is due to Orbit-Stabilizer theorem discussed in chapter 2.

In Figure 6.2 note that if a node in level-j ($j < 6$) dose not have an edge to level-(j+1) then it is considered a bad node. Bad node occur due to partial-dissection with hole in it that can be filled. We ignore such nodes.

Table 6.3: Total number of orbit representatives at each level of POSET.

Level	Number of orbit representatives
0	1
1	4
2	24
3	59
4	72
5	32
6	9

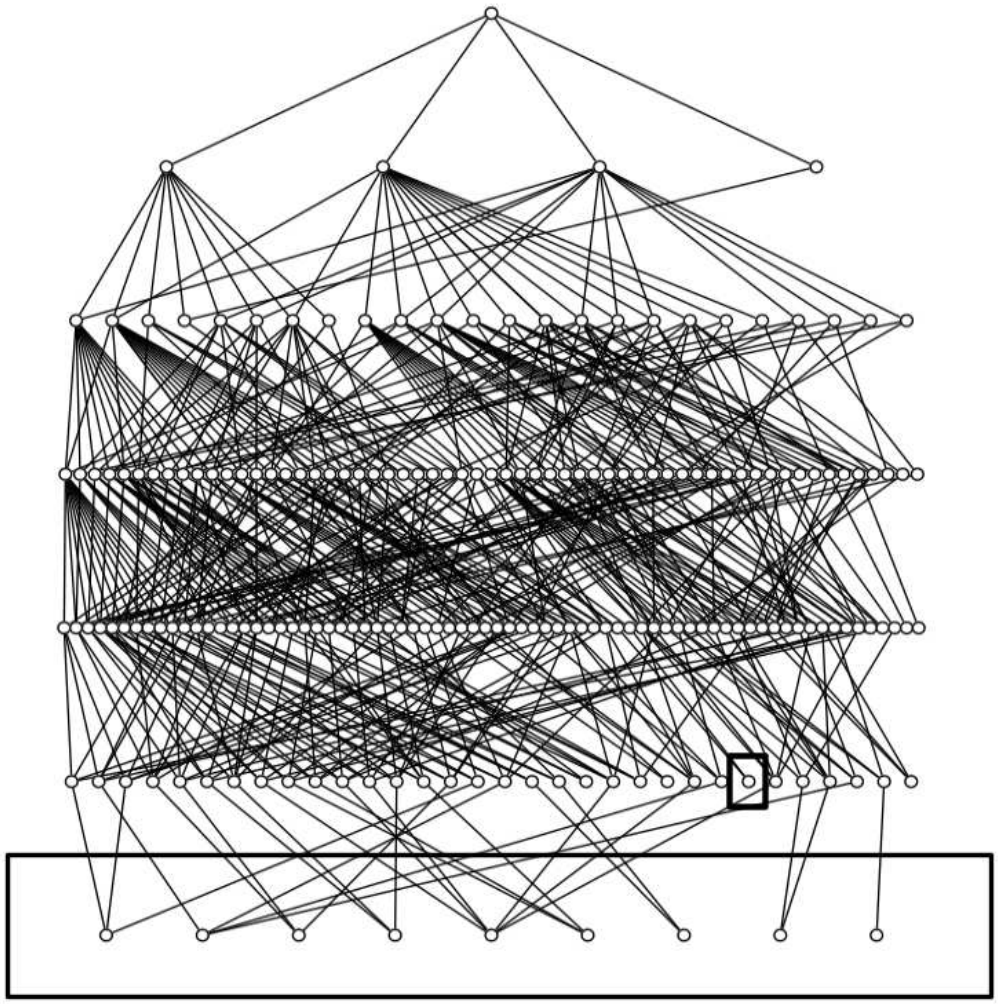


Figure 6.2: POSET of Orbits

N = node

D = depth or level

O = orbit with a level

Rep = orbit representative

SO = (order of stabilizer, orbit length)

F = number of flags

Gen = number of generators for the stabilizer of the orbit rep.

Table 6.4: Orbit Representatives

N	D	O	Rep	SO	F	Gen
0	0	0	{ }	(48, 1)	4	3
1	1	0	{ 1 }	(6, 8)	8	2
2	1	1	{ 2 }	(2, 24)	15	1
3	1	2	{ 4 }	(2, 24)	12	1
4	1	3	{ 28 }	(24, 2)	1	3
5	2	0	{ 1, 35 }	(2, 24)	11	1
6	2	1	{ 1, 36 }	(1, 48)	15	0
7	2	2	{ 1, 38 }	(4, 12)	5	2
8	2	3	{ 1, 41 }	(6, 8)	1	2
9	2	4	{ 1, 43 }	(1, 48)	8	0
10	2	5	{ 1, 48 }	(2, 24)	5	1
11	2	6	{ 1, 50 }	(1, 48)	10	0
12	2	7	{ 1, 68 }	(12, 4)	2	3
13	2	8	{ 2, 19 }	(2, 24)	11	1
14	2	9	{ 2, 21 }	(1, 48)	7	0
15	2	10	{ 2, 22 }	(1, 48)	13	0

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
16	2	11	{ 2, 24 }	(2, 24)	6	1
17	2	12	{ 2, 32 }	(2, 24)	7	1
18	2	13	{ 2, 33 }	(1, 48)	10	0
19	2	14	{ 2, 34 }	(1, 48)	9	0
20	2	15	{ 2, 57 }	(1, 48)	5	0
21	2	16	{ 2, 58 }	(1, 48)	6	0
22	2	17	{ 2, 59 }	(2, 24)	8	1
23	2	18	{ 2, 64 }	(4, 12)	3	2
24	2	19	{ 2, 67 }	(4, 12)	2	2
25	2	20	{ 4, 11 }	(2, 24)	4	1
26	2	21	{ 4, 12 }	(2, 24)	3	1
27	2	22	{ 4, 34 }	(4, 12)	2	2
28	2	23	{ 4, 53 }	(2, 24)	2	1
29	3	0	{ 1, 35, 45 }	(1, 48)	12	0
30	3	1	{ 1, 35, 47 }	(1, 48)	5	0
31	3	2	{ 1, 35, 48 }	(1, 48)	4	0
32	3	3	{ 1, 35, 50 }	(1, 48)	6	0
33	3	4	{ 1, 35, 51 }	(1, 48)	8	0
34	3	5	{ 1, 35, 52 }	(1, 48)	7	0
35	3	6	{ 1, 35, 53 }	(1, 48)	3	0
36	3	7	{ 1, 35, 54 }	(1, 48)	4	0
37	3	8	{ 1, 35, 65 }	(2, 24)	2	1
38	3	9	{ 1, 35, 66 }	(1, 48)	5	0
39	3	10	{ 1, 35, 68 }	(2, 24)	4	1

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
40	3	11	{ 1, 36, 51 }	(1, 48)	6	0
41	3	12	{ 1, 36, 57 }	(1, 48)	3	0
42	3	13	{ 1, 36, 58 }	(1, 48)	5	0
43	3	14	{ 1, 36, 59 }	(1, 48)	7	0
44	3	15	{ 1, 36, 61 }	(2, 24)	4	1
45	3	16	{ 1, 36, 62 }	(1, 48)	3	0
46	3	17	{ 1, 36, 63 }	(1, 48)	5	0
47	3	18	{ 1, 36, 64 }	(1, 48)	5	0
48	3	19	{ 1, 36, 65 }	(1, 48)	4	0
49	3	20	{ 1, 36, 66 }	(1, 48)	6	0
50	3	21	{ 1, 36, 67 }	(1, 48)	4	0
51	3	22	{ 1, 36, 68 }	(2, 24)	4	1
52	3	23	{ 1, 38, 41 }	(4, 12)	1	2
53	3	24	{ 1, 38, 52 }	(6, 8)	3	3
54	3	25	{ 1, 38, 53 }	(2, 24)	2	1
55	3	26	{ 1, 43, 62 }	(2, 24)	2	1
56	3	27	{ 1, 43, 64 }	(1, 48)	5	0
57	3	28	{ 1, 50, 54 }	(2, 24)	4	1
58	3	29	{ 2, 19, 36 }	(2, 24)	7	1
59	3	30	{ 2, 19, 37 }	(1, 48)	8	0
60	3	31	{ 2, 19, 43 }	(1, 48)	5	0
61	3	32	{ 2, 19, 44 }	(1, 48)	3	0
62	3	33	{ 2, 19, 48 }	(1, 48)	4	0
63	3	34	{ 2, 19, 50 }	(1, 48)	4	0

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
64	3	35	{ 2, 19, 54 }	(1, 48)	6	0
65	3	36	{ 2, 19, 58 }	(2, 24)	2	1
66	3	37	{ 2, 19, 59 }	(1, 48)	5	0
67	3	38	{ 2, 21, 32 }	(1, 48)	5	0
68	3	39	{ 2, 21, 43 }	(1, 48)	3	0
69	3	40	{ 2, 21, 59 }	(1, 48)	3	0
70	3	41	{ 2, 21, 66 }	(1, 48)	2	0
71	3	42	{ 2, 22, 43 }	(2, 24)	2	1
72	3	43	{ 2, 22, 44 }	(1, 48)	3	0
73	3	44	{ 2, 22, 50 }	(1, 48)	3	0
74	3	45	{ 2, 22, 54 }	(1, 48)	5	0
75	3	46	{ 2, 22, 58 }	(1, 48)	3	0
76	3	47	{ 2, 22, 59 }	(1, 48)	3	0
77	3	48	{ 2, 22, 64 }	(2, 24)	4	1
78	3	49	{ 2, 24, 32 }	(1, 48)	5	0
79	3	50	{ 2, 24, 54 }	(6, 8)	1	2
80	3	51	{ 2, 24, 66 }	(2, 24)	1	1
81	3	52	{ 2, 32, 37 }	(2, 24)	2	1
82	3	53	{ 2, 32, 44 }	(1, 48)	2	0
83	3	54	{ 2, 32, 50 }	(1, 48)	3	0
84	3	55	{ 2, 33, 59 }	(2, 24)	2	1
85	3	56	{ 2, 34, 44 }	(2, 24)	1	1
86	3	57	{ 2, 59, 61 }	(6, 8)	1	2
87	3	58	{ 4, 11, 21 }	(6, 8)	1	2

Continued on next page

Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
88	4	0	{ 1, 35, 45, 56 }	(2, 24)	3	1
89	4	1	{ 1, 35, 45, 57 }	(1, 48)	2	0
90	4	2	{ 1, 35, 45, 58 }	(1, 48)	2	0
91	4	3	{ 1, 35, 45, 59 }	(1, 48)	4	0
92	4	4	{ 1, 35, 45, 61 }	(1, 48)	4	0
93	4	5	{ 1, 35, 45, 62 }	(1, 48)	2	0
94	4	6	{ 1, 35, 45, 63 }	(1, 48)	4	0
95	4	7	{ 1, 35, 45, 64 }	(1, 48)	2	0
96	4	8	{ 1, 35, 45, 65 }	(1, 48)	2	0
97	4	9	{ 1, 35, 45, 66 }	(1, 48)	4	0
98	4	10	{ 1, 35, 45, 67 }	(1, 48)	2	0
99	4	11	{ 1, 35, 45, 68 }	(1, 48)	4	0
100	4	12	{ 1, 35, 47, 52 }	(2, 24)	3	1
101	4	13	{ 1, 35, 47, 57 }	(2, 24)	1	1
102	4	14	{ 1, 35, 47, 63 }	(1, 48)	2	0
103	4	15	{ 1, 35, 47, 67 }	(1, 48)	2	0
104	4	16	{ 1, 35, 48, 51 }	(1, 48)	2	0
105	4	17	{ 1, 35, 48, 61 }	(1, 48)	1	0
106	4	18	{ 1, 35, 48, 68 }	(2, 24)	2	1
107	4	19	{ 1, 35, 50, 51 }	(1, 48)	2	0
108	4	20	{ 1, 35, 50, 52 }	(1, 48)	2	0
109	4	21	{ 1, 35, 50, 53 }	(1, 48)	2	0
110	4	22	{ 1, 35, 50, 54 }	(1, 48)	3	0
111	4	23	{ 1, 35, 50, 61 }	(1, 48)	1	0

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
112	4	24	{ 1, 35, 51, 54 }	(1, 48)	2	0
113	4	25	{ 1, 35, 51, 64 }	(1, 48)	1	0
114	4	26	{ 1, 35, 51, 68 }	(1, 48)	3	0
115	4	27	{ 1, 35, 52, 53 }	(2, 24)	1	1
116	4	28	{ 1, 35, 52, 63 }	(2, 24)	1	1
117	4	29	{ 1, 35, 52, 67 }	(1, 48)	2	0
118	4	30	{ 1, 35, 65, 68 }	(4, 12)	1	2
119	4	31	{ 1, 35, 66, 67 }	(2, 24)	1	1
120	4	32	{ 1, 36, 51, 56 }	(3, 16)	1	1
121	4	33	{ 1, 36, 51, 58 }	(1, 48)	2	0
122	4	34	{ 1, 36, 51, 59 }	(1, 48)	2	0
123	4	35	{ 1, 36, 51, 64 }	(1, 48)	2	0
124	4	36	{ 1, 36, 51, 68 }	(2, 24)	2	1
125	4	37	{ 1, 36, 57, 66 }	(1, 48)	2	0
126	4	38	{ 1, 36, 58, 61 }	(2, 24)	2	1
127	4	39	{ 1, 36, 59, 61 }	(1, 48)	2	0
128	4	40	{ 1, 36, 59, 62 }	(1, 48)	2	0
129	4	41	{ 1, 36, 59, 63 }	(2, 24)	1	1
130	4	42	{ 1, 36, 59, 64 }	(1, 48)	4	0
131	4	43	{ 1, 36, 61, 68 }	(4, 12)	1	2
132	4	44	{ 1, 36, 63, 66 }	(2, 24)	1	1
133	4	45	{ 1, 36, 66, 67 }	(1, 48)	3	0
134	4	46	{ 1, 38, 41, 52 }	(6, 8)	1	3
135	4	47	{ 1, 38, 52, 63 }	(24, 2)	1	5

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
136	4	48	{ 2, 19, 36, 56 }	(1, 48)	4	0
137	4	49	{ 2, 19, 36, 57 }	(1, 48)	2	0
138	4	50	{ 2, 19, 36, 58 }	(1, 48)	2	0
139	4	51	{ 2, 19, 36, 59 }	(2, 24)	3	1
140	4	52	{ 2, 19, 36, 64 }	(2, 24)	1	1
141	4	53	{ 2, 19, 36, 67 }	(2, 24)	1	1
142	4	54	{ 2, 19, 37, 48 }	(1, 48)	2	0
143	4	55	{ 2, 19, 37, 50 }	(2, 24)	1	1
144	4	56	{ 2, 19, 37, 54 }	(1, 48)	2	0
145	4	57	{ 2, 19, 37, 65 }	(1, 48)	1	0
146	4	58	{ 2, 19, 37, 66 }	(1, 48)	1	0
147	4	59	{ 2, 19, 37, 67 }	(1, 48)	1	0
148	4	60	{ 2, 19, 43, 62 }	(2, 24)	1	1
149	4	61	{ 2, 19, 43, 64 }	(1, 48)	2	0
150	4	62	{ 2, 19, 44, 54 }	(1, 48)	2	0
151	4	63	{ 2, 19, 50, 54 }	(2, 24)	2	1
152	4	64	{ 2, 19, 59, 64 }	(4, 12)	1	2
153	4	65	{ 2, 21, 32, 43 }	(2, 24)	1	1
154	4	66	{ 2, 21, 32, 59 }	(1, 48)	2	0
155	4	67	{ 2, 22, 43, 64 }	(4, 12)	1	2
156	4	68	{ 2, 22, 44, 54 }	(2, 24)	1	1
157	4	69	{ 2, 24, 32, 59 }	(4, 12)	1	2
158	4	70	{ 2, 32, 37, 67 }	(8, 6)	0	2
159	4	71	{ 2, 32, 50, 56 }	(4, 12)	0	1

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
160	5	0	{ 1, 35, 45, 56, 65 }	(2, 24)	1	1
161	5	1	{ 1, 35, 45, 56, 66 }	(1, 48)	1	0
162	5	2	{ 1, 35, 45, 56, 68 }	(2, 24)	1	1
163	5	3	{ 1, 35, 45, 57, 63 }	(1, 48)	1	0
164	5	4	{ 1, 35, 45, 57, 66 }	(1, 48)	1	0
165	5	5	{ 1, 35, 45, 58, 61 }	(1, 48)	1	0
166	5	6	{ 1, 35, 45, 58, 68 }	(1, 48)	1	0
167	5	7	{ 1, 35, 45, 59, 61 }	(1, 48)	1	0
168	5	8	{ 1, 35, 45, 59, 62 }	(1, 48)	1	0
169	5	9	{ 1, 35, 45, 59, 63 }	(1, 48)	1	0
170	5	10	{ 1, 35, 45, 59, 64 }	(1, 48)	1	0
171	5	11	{ 1, 35, 45, 61, 64 }	(1, 48)	1	0
172	5	12	{ 1, 35, 45, 61, 68 }	(1, 48)	1	0
173	5	13	{ 1, 35, 45, 62, 63 }	(1, 48)	1	0
174	5	14	{ 1, 35, 45, 63, 66 }	(1, 48)	1	0
175	5	15	{ 1, 35, 45, 65, 68 }	(2, 24)	1	1
176	5	16	{ 1, 35, 45, 66, 67 }	(1, 48)	1	0
177	5	17	{ 1, 35, 47, 52, 57 }	(2, 24)	1	1
178	5	18	{ 1, 35, 47, 52, 63 }	(2, 24)	1	1
179	5	19	{ 1, 35, 48, 61, 68 }	(2, 24)	0	1
180	5	20	{ 1, 35, 50, 51, 54 }	(1, 48)	1	0
181	5	21	{ 1, 35, 50, 54, 61 }	(1, 48)	0	0
182	5	22	{ 1, 36, 51, 56, 68 }	(6, 8)	0	2
183	5	23	{ 1, 36, 51, 59, 64 }	(1, 48)	1	0

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Table 6.4 – continued from previous page

N	D	O	Rep	SO	F	Gen
184	5	24	{ 1, 36, 59, 61, 64 }	(2, 24)	1	1
185	5	25	{ 1, 38, 41, 52, 63 }	(24, 2)	0	5
186	5	26	{ 2, 19, 36, 56, 65 }	(1, 48)	1	0
187	5	27	{ 2, 19, 36, 56, 66 }	(2, 24)	1	1
188	5	28	{ 2, 19, 36, 56, 67 }	(1, 48)	1	0
189	5	29	{ 2, 19, 36, 58, 61 }	(2, 24)	1	1
190	5	30	{ 2, 19, 36, 59, 61 }	(2, 24)	1	1
191	5	31	{ 2, 19, 37, 66, 67 }	(2, 24)	0	1
192	6	0	{ 1, 35, 45, 56, 65, 68 }	(4, 12)		2
193	6	1	{ 1, 35, 45, 56, 66, 67 }	(2, 24)		1
194	6	2	{ 1, 35, 45, 57, 63, 66 }	(2, 24)		1
195	6	3	{ 1, 35, 45, 58, 61, 68 }	(2, 24)		1
196	6	4	{ 1, 35, 45, 59, 61, 64 }	(1, 48)		0
197	6	5	{ 1, 35, 45, 59, 62, 63 }	(2, 24)		1
198	6	6	{ 1, 35, 47, 52, 57, 63 }	(6, 8)		3
199	6	7	{ 2, 19, 36, 56, 66, 67 }	(4, 12)		2
200	6	8	{ 2, 19, 36, 59, 61, 64 }	(12, 4)		3

Appendix A

Rank of Tetrahedra

Rank of Tetrahedron	Vertices of Tetrahedron
0	(0, 1, 2, 3)
1	(0, 1, 2, 4)
2	(0, 1, 2, 5)
3	(0, 1, 2, 6)
4	(0, 1, 2, 7)
5	(0, 1, 3, 4)
6	(0, 1, 3, 5)
7	(0, 1, 3, 6)
8	(0, 1, 3, 7)
9	(0, 1, 4, 5)
10	(0, 1, 4, 6)
11	(0, 1, 4, 7)
12	(0, 1, 5, 6)
13	(0, 1, 5, 7)
14	(0, 1, 6, 7)
15	(0, 2, 3, 4)
16	(0, 2, 3, 5)
17	(0, 2, 3, 6)
18	(0, 2, 3, 7)
19	(0, 2, 4, 5)
20	(0, 2, 4, 6)
21	(0, 2, 4, 7)

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Table A.1 – continued from previous page

Rank of Tetrahedron	Vertices of Tetrahedron
22	(0, 2, 5, 6)
23	(0, 2, 5, 7)
24	(0, 2, 6, 7)
25	(0, 3, 4, 5)
26	(0, 3, 4, 6)
27	(0, 3, 4, 7)
28	(0, 3, 5, 6)
29	(0, 3, 5, 7)
30	(0, 3, 6, 7)
31	(0, 4, 5, 6)
32	(0, 4, 5, 7)
33	(0, 4, 6, 7)
34	(0, 5, 6, 7)
35	(1, 2, 3, 4)
36	(1, 2, 3, 5)
37	(1, 2, 3, 6)
38	(1, 2, 3, 7)
39	(1, 2, 4, 5)
40	(1, 2, 4, 6)
41	(1, 2, 4, 7)
42	(1, 2, 5, 6)
43	(1, 2, 5, 7)
44	(1, 2, 6, 7)
45	(1, 3, 4, 5)

Continued on next page

Table A.1 – continued from previous page

Rank of Tetrahedron	Vertices of Tetrahedron
46	(1, 3, 4, 6)
47	(1, 3, 4, 7)
48	(1, 3, 5, 6)
49	(1, 3, 5, 7)
50	(1, 3, 6, 7)
51	(1, 4, 5, 6)
52	(1, 4, 5, 7)
53	(1, 4, 6, 7)
54	(1, 5, 6, 7)
55	(2, 3, 4, 5)
56	(2, 3, 4, 6)
57	(2, 3, 4, 7)
58	(2, 3, 5, 6)
59	(2, 3, 5, 7)
60	(2, 3, 6, 7)
61	(2, 4, 5, 6)
62	(2, 4, 5, 7)
63	(2, 4, 6, 7)
64	(2, 5, 6, 7)
65	(3, 4, 5, 6)
66	(3, 4, 5, 7)
67	(3, 4, 6, 7)
68	(3, 5, 6, 7)

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Table A.1 – continued from previous page

Rank of Tetrahedron	Vertices of Tetrahedron
69	(4, 5, 6, 7)

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