CONDITIONAL PROBABILITIES OF OCCURRENCE OF WET AND DRY YEARS OVER A LARGE CONTINENTAL AREA

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ABSTRACT

The probability of occurrence of wet and dry years over an area for each year from 1931 to 1960 is investigated by a stochastic approach. To use this concept, years are classified into one of five categories: very dry year $[0 < F(x) \leq 0.15]$, dry year $[0.15 < F(x) \leq 0.35]$, normal year $[0.35 < F(x) \leq 0.65]$, wet year $[0.65 < F(x) \leq 0.85]$, and very wet year $[0.85 < F(x) \leq 1.00]$, with $F(x)$ the probability distribution function of an annual precipitation and annual runoff.

Four mathematical models are used to describe the probabilities of occurrence of wet and dry years over an area if the type of year at the central sub-area or central station is known. Model I, which advances the hypothesis that the occurrences of wet and dry years over an area are independent, is rejected. Model III, which has the linear dependence of the occurrences of a wet or a dry year at two sub-areas, or at two stations, is considered an acceptable model. Models II and IV, which have an exponential dependence and an hyperbolic dependence, respectively, are applicable in specific cases. An example of these specific cases occurs when the types of years in two sub-areas, or at two stations, are different but not opposite, such as very dry and dry, normal and either very dry or very wet.

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CHAPTER I

INTRODUCTION

1. Significance of this study. Planning of water resources development and operation of large-scale water resources projects include many types of problems. Planning problems involve engineering, economic and financial feasibility studies with the engineering feasibility being of primary concern. Efficient operation requires a good knowledge of hydrologic processes.

Both planning and operation require hydrologic data of which precipitation and river flow records are of a major importance. The engineer must have a thorough understanding of the occurrences and properties of these factors before applying them to planning and operation of any water resource project. The problems associated with these two important phases of the hydrologic cycle refer to their distributions both in time and in space.

A large number of studies have been made on time distribution of rainfall and runoff. Investigations probing this problem revealed conflicting conclusions between two groups of researchers. But in more recent studies, most of the researchers have used appropriate statistical methods in their investigations and have arrived at similar conclusions. The conclusion is that the sequences of annual precipitation and annual river flow can be considered as stochastic processes.

Problems involving areal distribution of precipitation and runoff depend upon the physiographic features of the area and the general circulation in the atmosphere. These large-scale phenomena are complex and are not completely understood theoretically. This study is an attempt to describe mathematically the simultaneous occurrences of wet and dry years over large continental areas.

2. Definition of wet and dry years. A wet or dry year can be indicated by several phenomena such as rainfall, stream flow, water levels in wells, moisture content in the air, or similar variables. Hydraulic engineers and agriculturists consider a wet year as being one where there are excesses in annual river flow or in annual soil moisture. Meteorologists consider a wet year as one with an excess of annual precipitation. The definitions of wet and dry years vary from one field of interest to another and from place to place. For instance, 15 inches of annual precipitation in Louisiana is a dry year, whereas 15 inches of annual precipitation in some parts of Arizona is a wet year. Definitions of wet and dry years in this study are based on the use of the probability concept in sorting annual values of precipitation or runoff in wet and dry years. They are defined as follows:

Let X be a random variable of annual precipitation or annual runoff with a distribution function $F(x)$ which is defined as:

$$F(x) = P[X \leq x]. \tag{1}$$

If $F(x)$ is known, x_1, x_2, x_3 and x_4 can be determined by:

$$F(x_1) = 0.15;$$

$$F(x_2) = 0.35;$$

$$F(x_3) = 0.65;$$

and

$$F(x_4) = 0.85.$$

In this study, a year is considered to be

	"very dry" if for that year,	$X \leq x_1$;
	"dry" if for that year,	$x_1 < X \leq x_2$;
	"normal" if for that year,	$x_2 < X \leq x_3$;
	"wet" if for that year,	$x_3 < X \leq x_4$;
and,	"very wet" if for that year,	$x_4 < X$.

Since $F(x)$ is not known, it is necessary to estimate the percentiles x_1, x_2, x_3 and x_4 from the sample of observed data. The random variable X, which is a standardized variable of the variable Q, is computed from a set of observed data such that

$$X = \frac{Q - \bar{Q}}{s} \tag{2}$$

where X is the standardized variable of the variable Q, $\bar{Q}$ is the mean of the variable Q, and s is the standard deviation of the variable Q.

The method of using plotting positions and smooth curve fittings to obtain the distribution $F(x)$ may be used to estimate the percentiles x_1, x_2, x_3 and x_4. This was found to be a time consuming method because investigators were working with a large number of stations. In this study a digital computer was used to derive these percentiles in the following manner:

The standardized variables of each station, and the record of each station for a 30-year period (1931-1960) were serialized according to their magnitudes so that

$$X_1 \leq X_2 \leq X_3 \leq \ldots . \leq X_{30}.$$

Therefore, a year is considered:

"very dry" if for that year,	$X \leq XX_1$;
"dry" if for that year,	$XX_1 < X \leq XX_2$;
"normal" if for that year,	$XX_2 < X \leq XX_3$;
"wet" if for that year,	$XX_3 < X \leq XX_4$;
"very wet" if for that year,	$XX_4 < X$

where

$$XX_1 = \tfrac{1}{2}(X_4 + X_5);$$

$$XX_2 = \tfrac{1}{2}(X_{10} + X_{11});$$

$$XX_3 = \tfrac{1}{2}(X_{19} + X_{20});$$

and

$$XX_4 = \tfrac{1}{2}(X_{25} + X_{26}).$$

3. Objective. The objective of this study is to investigate the simultaneous occurrence of wet and dry years over an area, for any given year. Dependence in sequence of precipitation and runoff is not investigated. This study is intended to increase the knowledge of basic hydrologic processes by attempting to answer the following two questions: (1) Are there systematic patterns in the distributions of wet and dry years over an area? (2) If these patterns are systematic, what kind of regularity do they follow?

4. Method of investigation. Hydrologic system investigation methods fall into two principal categories entitled "parametric hydrology" and "stochastic hydrology." The parametric hydrologic method is the search for deterministic relationsnips among physical factors involved in hydrologic processes. The stochastic hydrologic method consists of the use of statistical and probability methods in analyzing the variables which follow hydrologic stochastic processes. As the annual precipitation and the annual river flow are stochastic variables, the second method of investigation was used exclusively in this study. First, distribution characteristics of observed data were studied. Then, a sample technique, such as the concept of conditional probability of wet and dry years was employed to analyze the problems investigated.

5. Two methods of computation and analysis of data. Two methods were used for computing the conditional probabilities. The first method included the computation of the synoptic maps of wet and dry years by dividing the total continental area into sub-areas in such a way that the types of wet or dry years of sub-areas are determined from all precipitation stations falling inside each sub-area. Then the conditional probabilities of wet and dry years at various sub-areas, given the type of year at a central sub-area, were computed from the synoptic maps. This first method and the results obtained are described in Chapter III.

The second method included the computation of conditional probabilities of wet and dry years of precipitation stations at a given distance k from the central station, given the type of year at the central station. This second method and the results obtained are described in Chapter IV.

The terms "first method" and "second method," which represent the methods described above, are used in the further text of this paper.

CHAPTER II

MATHEMATICAL MODELS

1. Selection of mathematical models. In the past the simultaneous occurrence of wet and dry years over an area has been studied and described on a qualitative basis only. This limitation is the result of the complexity of the problem and the lack of knowledge of how hydrologic variables, such as precipitation and river flow, are related over large regions. This study employs the conditional probability concept to describe this phenomenon. In Chapter I, a year was classified into one of five categories. Since each category has one point in common with all other categories there are twenty-five possible combinations of conditional probability. The conditional probability of each combination has its own characteristic. Therefore, several mathematical models are required to describe each of these twenty-five combinations.

This chapter proposes and investigates mathematical models for the simultaneous occurrence of wet and dry sub-areas or stations for a given year. The first model, entitled the independent model, may be considered a null hypothesis. This hypothesis assumes that wet and dry sub-areas or stations for a given year are distributed on a chance basis. All remaining models are based on the following assumption: the conditional probability of a sub-area or station having wet or dry years similar to its adjacent sub-area or station, depends only on the distance between the two sub-areas or stations. Different models assume different functional forms for this dependence. Thus, the second model assumes an exponential function, the third model a linear function and the fourth model a hyperbolic function. The following criteria was used as a guide to develop mathematical models other than the independent model.

(a) The mathematical function should be simple and not involve more than two parameters;

(b) $p_{ij}(0) = 0$, for $i \neq j$ and $p_{jj}(0) = 1$;

(c) $\lim_{k \to \infty} p_{ij}(k) = p_i$ for all i, j

where

$$p_{ij}(k) = P\,[X_i(k)/X_j(0)] \qquad (3)$$

and it is the conditional probability of X_i at the distance k, given X_j at the distance zero. Several other models were also considered, for example:

$$p_{ij}(k) = p_i - \frac{p_i}{1 + \omega_{ij}k^2} \qquad \text{for } i \neq j$$

$$= p_j + \frac{1 - p_j}{1 + \omega_{jj}k^2} \qquad \text{for } i = j \qquad (4)$$

$$p_{ij}(k) = p_i - \frac{p_i}{(1 + \omega_{ij}k)^2} \qquad \text{for } i \neq j$$

$$= p_j + \frac{1 - p_j}{(1 + \omega_{jj}k)^2} \qquad \text{for } i = j \qquad (5)$$

$$p_{ij}(k) = p_i - \frac{p_i}{1 + \omega_{ij}\sqrt{k}} \qquad \text{for } i \neq j$$

$$= p_j + \frac{1 - p_j}{1 + \omega_{jj}\sqrt{k}} \qquad \text{for } i = j \qquad (6)$$

However, a graphical comparison indicated that these functions are not very well suited for the observed data in comparison with the four functions.

2. Model I: Independence. Throughout this study wet or dry years are considered to occur independently from year to year. The hypothesis of this model is that sub-areas or stations having a wet or a dry year occur randomly over the total area. In other words, there is no regular pattern in the simultaneous occurrence of wet and dry years over the large-area for any particular year, provided that the time series of wet and dry years at any sub-area is independent.

From the hypothesis of the independent model and the definition of wet and dry years, the conditional probability of X_i at the distance k, given X_j at the distance zero is:

$$p_{ij}(k) = p_i \quad \text{for } k \neq 0 \text{ and } i,\ j = 1, 2, \ldots, 5$$

$$= 1 \quad \text{for } k = 0 \text{ and } i = j$$

$$= 0 \quad \text{for } k = 0 \text{ and } i \neq j. \qquad (7)$$

The conditional probability of wet and dry years as a function of distance described by Model I is shown in fig. 1a.

3. Model II: Exponential dependence. The hypothesis of the exponential model is that the probability of occurrence of a wet or a dry year in one sub-area or station depends upon the occurrence of a wet or a dry year in the adjacent sub-areas or stations. The degree of dependence decreases as the distance between the sub-areas or stations increases and is asymptotic to the value of zero. These conditional probabilities of wet and dry years as a function of distance are described by the exponential function.

$$p_{ij}(k) = p_i\,(1 - e^{-\lambda_{ij}k}) \quad \text{for } i \neq j$$

$$= p_j + (1 - p_j)\,e^{-\lambda_{jj}k} \quad \text{for } i = j \qquad (8)$$

where $i, j = 1, 2, \ldots, 5$; λ_{ij} is the parameter which is dependent on i and j. The value of λ_{ij} measures the degree of dependence between sub-areas or stations. The greater λ_{ij}, the smaller the dependence. The conditional probability $p_{ij}(k)$ as a function of distance k described by Model II is shown in fig. 1 b.

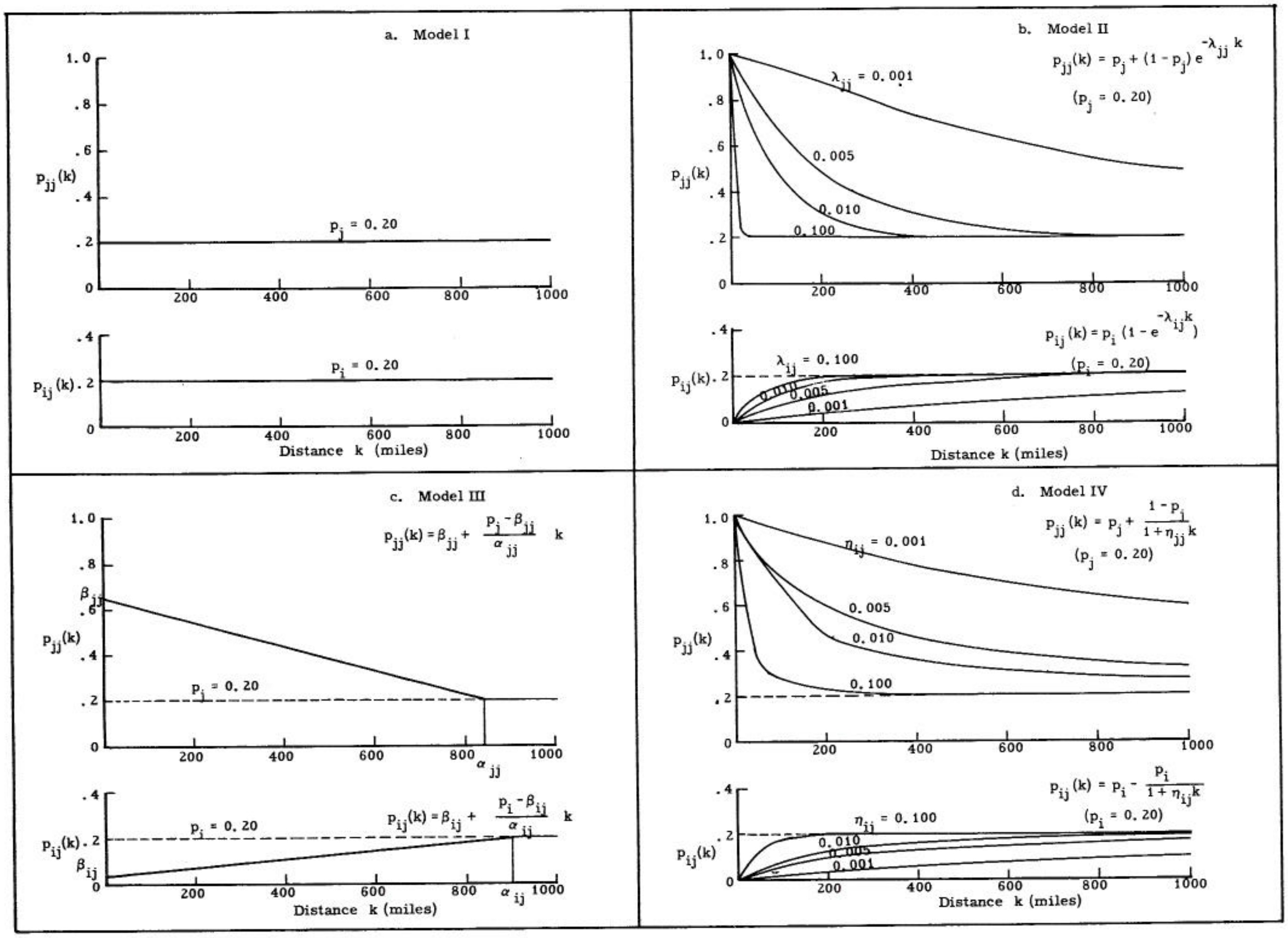

Fig. 1 Conditional probabilities of wet and dry years as a function of distance described by mathematical Models I, II, III and IV

Curve fitting, using the least-squares method was used to estimate the parameter λ_{ij}. Let S be the average squared deviation of the observations about the estimated regression function

$$S = \frac{1}{n} \sum_{k=1}^{n} \left[p_{ij}(k) - p_i (1 - e^{-\hat{\lambda}_{ij} k}) \right]^2 \tag{9}$$

where $\hat{\lambda}_{ij}$ is the estimated parameter of λ_{ij}.

The minimum value of S is obtained by taking the derivative of eq. (9) with respect to $\hat{\lambda}_{ij}$ and equating to zero.

$$\frac{dS}{d\hat{\lambda}_{ij}} = \sum_{k=1}^{n} \left[p_{ij}(k) - p_i (1 - e^{-\hat{\lambda}_{ij} k}) \right] \left[p_i e^{-\hat{\lambda}_{ij} k} k \right] = 0 \tag{10}$$

4. Model III: Linear dependence. The hypothesis of this model is that the probability of occurrence of a wet or a dry year in one sub-area or station depends linearly upon the occurrence of a wet or a dry year in the adjacent sub-areas or stations. The conditional probability of X_i at the distance k, given X_j at the distance zero is:

$$\begin{aligned} p_{ij}(k) &= \beta_{ij} + \frac{p_i - \beta_{ij}}{\alpha_{ij}} k && \text{for } 0 < k < \alpha_{ij} \\ &= p_i && \text{for } k > \alpha_{ij} \\ &= 1 && \text{for } k = 0 \text{ and } i = j \\ &= 0 && \text{for } k = 0 \text{ and } i \neq j \end{aligned} \tag{11}$$

where i, j = 1, 2, . . . , 5; β_{ij} and α_{ij} are parameters of the model. Let

$$\nu_{ij} = \frac{p_i - \beta_{ij}}{\alpha_{ij}} \tag{12}$$

Then, ν_{ij} is the slope of eq. (11) for $0 \le k \le \alpha_{ij}$ and the measure of degree of dependence between sub-areas or stations. The greater ν_{ij} the larger the dependence. Figure 1 c shows the conditional probability $p_{ij}(k)$ as a function of the distance k, given the values of β_{ij} and α_{ij}.

The least-squares method for curve fitting was used to estimate the parameters α_{ij} and β_{ij}. A straight forward method of finding the least-squares estimate of α_{ij} and β_{ij} is as follows:

$$\hat{\beta}_{ij} = \frac{\sum_{k=1}^{m} p_{ij}(k) \sum_{k=1}^{m} k^2 - \sum_{k=1}^{m} k \sum_{k=1}^{m} k \; p_{ij}(k)}{m \sum_{k=1}^{m} k^2 - \left[\sum_{k=1}^{m} k \right]^2} \tag{13}$$

and

$$\hat{\alpha}_{ij} = (p_i - \hat{\beta}_{ij}) \frac{m \sum_{k=1}^{m} k^2 - \left[\sum_{k=1}^{m} k \right]^2}{m \sum_{k=1}^{m} k \, p_{ij}(k) - \sum_{k=1}^{m} k \sum_{k=1}^{m} p_{ij}(k)} \tag{14}$$

where $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ are the estimates of the parameters α_{ij} and β_{ij}, respectively.

5. Model IV: Hyperbolic dependence. This mathematical function was proposed by a graphical comparison between the shape of the function and the observed data. The conditional probability of X_i at the distance k, given X_j at the distance 0 is:

$$\begin{aligned} p_{ij}(k) &= p_i - \frac{p_i}{1 + \eta_{ij} k} && \text{for } i \neq j \\ &= p_j + \frac{1 - p_j}{1 + \eta_{jj} k} && \text{for } i = j \end{aligned} \tag{15}$$

Figure 1d shows the conditional probability $p_{ij}(k)$ as a function of the distance k for the different values of η_{ij}.

The method of least-squares was used to estimate the parameter η_{ij}. Let S be the average squared deviation of the observations about the estimated regression function

$$S = \frac{1}{n} \sum_{k=1}^{n} \left[p_{ij}(k) - (p_i - \frac{p_i}{1 + \hat{\eta}_{ij} k}) \right]^2 \tag{16}$$

where $\hat{\eta}_{ij}$ is the estimated parameter of η_{ij}.

A minimum value of S is obtained by taking the derivitive of eq. (16) with respect to $\hat{\eta}_{ij}$ and equating to zero.

$$\frac{dS}{d\hat{\eta}_{ij}} = \sum_{k=1}^{n} \left[p_{ij}(k) - (p_i - \frac{p_i}{1 + \hat{\eta}_{ij} k}) \right] \left[\frac{p_i k}{(1 + \hat{\eta}_{ij} k)^2} \right] = 0 \tag{17}$$

6. Measurement of goodness of fit of mathematical models. The goodness of fit of mathematical models to the observed data is measured by the standard errors of estimate which is denoted by S_{ij} and is defined by

$$S_{ij} = \sqrt{\frac{\sum_{k=1}^{N} \left[p_{ij}(k) - \hat{p}_{ij}(k) \right]^2}{N}} \tag{18}$$

where $\hat{p}_{ij}(k)$ is the estimated conditional probability.

CHAPTER III

INVESTIGATION OF GOODNESS OF FIT OF MATHEMATICAL MODELS

BY THE FIRST METHOD

This chapter describes the method used to obtain synoptic maps of wet and dry years at sub-areas in the Western United States for each year from 1931 to 1960, the computation of conditional probabilities of wet and dry years of sub-areas as a function of distance, and the investigation of goodness of fit of mathematical models.

1. Data used. The accuracy of results in this study depends on the density of rainfall and runoff stations over the continental area used for the analysis. The amount of rainfall station data exceeded that from runoff stations, thus investigator decided to use only the annual precipitation data. This data taken from 1141 stations during the period from 1931 to 1960, is from 21 Western States and the provinces of British Columbia, Alberta and Saskatchewan in Southwestern Canada. Primary source of data was the published records of the United States Weather Bureau and the Canadian Department of Commerce, Meteorological Branch. Distribution of the precipitation stations over this large area is shown in fig. 2 (stations cover different climatic and physiographic features).

This identical data was used by Yevdjevich (1963), Caffey (1965), Markovic (1965) and Pinkayan (1965). Characteristic details of the stations and the accuracies of the records can be found in these references.

Fig. 2 Areal distribution of precipitation stations

2. Synoptic maps of wet and dry years in the Western United States. The Western United States was superimposed by a grid system, as shown in fig. 3, consisting of grid sub-areas, with each sub-area being one-degree quadrangle of latitude and longitude. Each sub-area is assumed to have approximately a hydrological and meteorological homogeneity, or that the correlation coefficient of annual precipitation at any two stations inside a sub-area is very high (say, greater than 0.70). This assumption is supported by several results, particularly by Caffey's (1965). The assumption further implies that every point in a sub-area has approximately the same precipitation characteristics concerning the classification of wet and dry years. Precipitation data recorded at one point in a sub-area was used to represent the precipitation of the total sub-area. If more than one precipitation station was available in a sub-area, then the average annual precipitation of all stations was considered as the annual value for that sub-area.

The occurrence of a very dry, dry, normal, wet, or very wet year for each sub-area and for each year from 1931 to 1960, was determined from the standardized variables of annual precipitation by using definitions of wet and dry years given in Chapter I. The number of stations in each sub-area ranged from nine to zero. In the event there was no precipitation station in a sub-area, the type of year for the sub-area was specified as the same type of year as the adjacent or nearest sub-area which contained precipitation stations.

Synoptic maps of wet and dry years for each year from 1931 to 1960 were computed on a CDC 3600 high-speed computer, and are presented in figs. 4 through 8. These thirty individual maps, presented in groups of six maps per figure, represent the basic research material for the first method of analysis given later in this chapter.

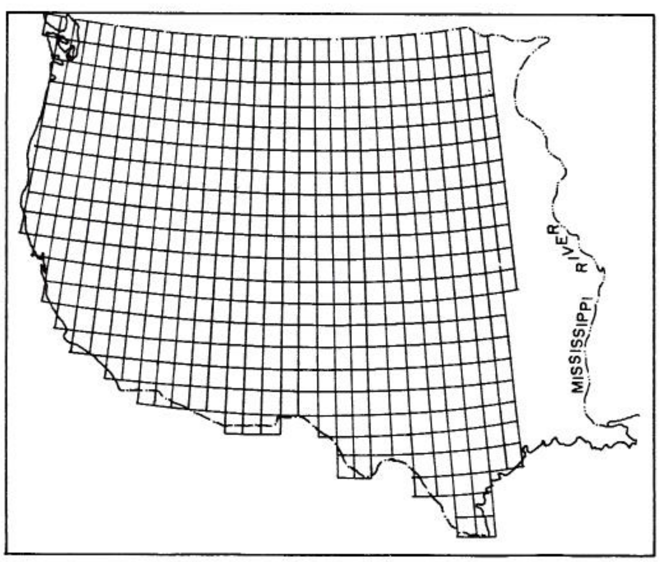

Fig. 3 Grid system for determining synoptic maps of wet and dry years in the Western United States

3. Sequence of the percent of areas having wet and dry years. The convergence of the longitudinal lines towards the North Pole does not allow all the sub-areas in the grid system to be equal. Therefore, the surface of a sub-area depends upon where it is located. The percentages of the sub-areas having very dry, dry, normal, wet and very wet years were computed for each year from 1931 to 1960, by using the thirty synoptic maps of wet and dry years, and the results are presented in fig. 9.

The computed mean values of the percent of sub-areas having very dry, dry, normal, wet and very wet years are 13.40 percent, 20.12 percent, 29.88 percent, 19.44 percent, and 16.61 percent. These values deviate from the expected means of 15.00 percent, 20.00 percent, 30.00 percent, 20.00 percent and 15.00 percent, respectively. The two main reasons for these deviations are:

(1) The number of wet and dry years on record could not be evenly divided into five categories using standard classifying procedures. In the thirty year period covered in this study, there must be 4.5 6, 9, 6, 4.5 years being very dry, dry, normal, wet and very wet in each sub-area, respectively.

(2) Errors were introduced by assigning a type of year to sub-areas which do not contain a precipitation station. However, the deviations from the expected means are not large.

The correlation coefficients taken between two variables at a time, between the percent of sub-areas in the Western United States having very dry, dry, normal, wet and very wet years, were computed as shown in Table 1. Correlation coefficients between the two adjacent time series in fig. 9; or for the sub-areas having very dry, and dry years, sub-areas of dry and normal years, sub-areas of normal and wet years, and sub-areas of wet and very wet years; are positive. They range from 0.2142 to 0.6108. The correlation coefficients between time series for the sub-areas which are not adjacent in fig. 9; or the sub-areas having very dry years and either normal, wet or very wet years, and sub-areas of normal years and very wet years or very dry years; are negative. They range from -0.2864 to -0.8523. These patterns in correlation coefficients should be expected whenever there is a conditional probability of distribution of wet and dry years, and where sub-areas of five categories are restricted by the total area being 100 percent. The probability that the sub-areas surrounding a sub-area having a very wet year will also have a wet year which is greater than the probability in the case of an independent distribution of wet and dry years across the sub-areas. This is due to the fact that sub-areas that are close together face a greater probability of being subjected to the same hydrologic conditions than in the case of independent distribution of wet and dry years. This is the primary reason for a positive correlation coefficient. The correlation coefficients of a very wet year versus normal, dry or very dry years are negative because the conditional probabilities are smaller than the probabilities in the case of independent joint distribution of wet and dry years.

4. Distributions of the percent of areas having wet and dry years. Frequency distributions of the percent of sub-areas having very dry, dry, normal, wet and very wet years were computed from data presented in fig. 10. It was found that at each year of the 1931 to 1960 period the percent of the sub-areas in the Western United States having very dry

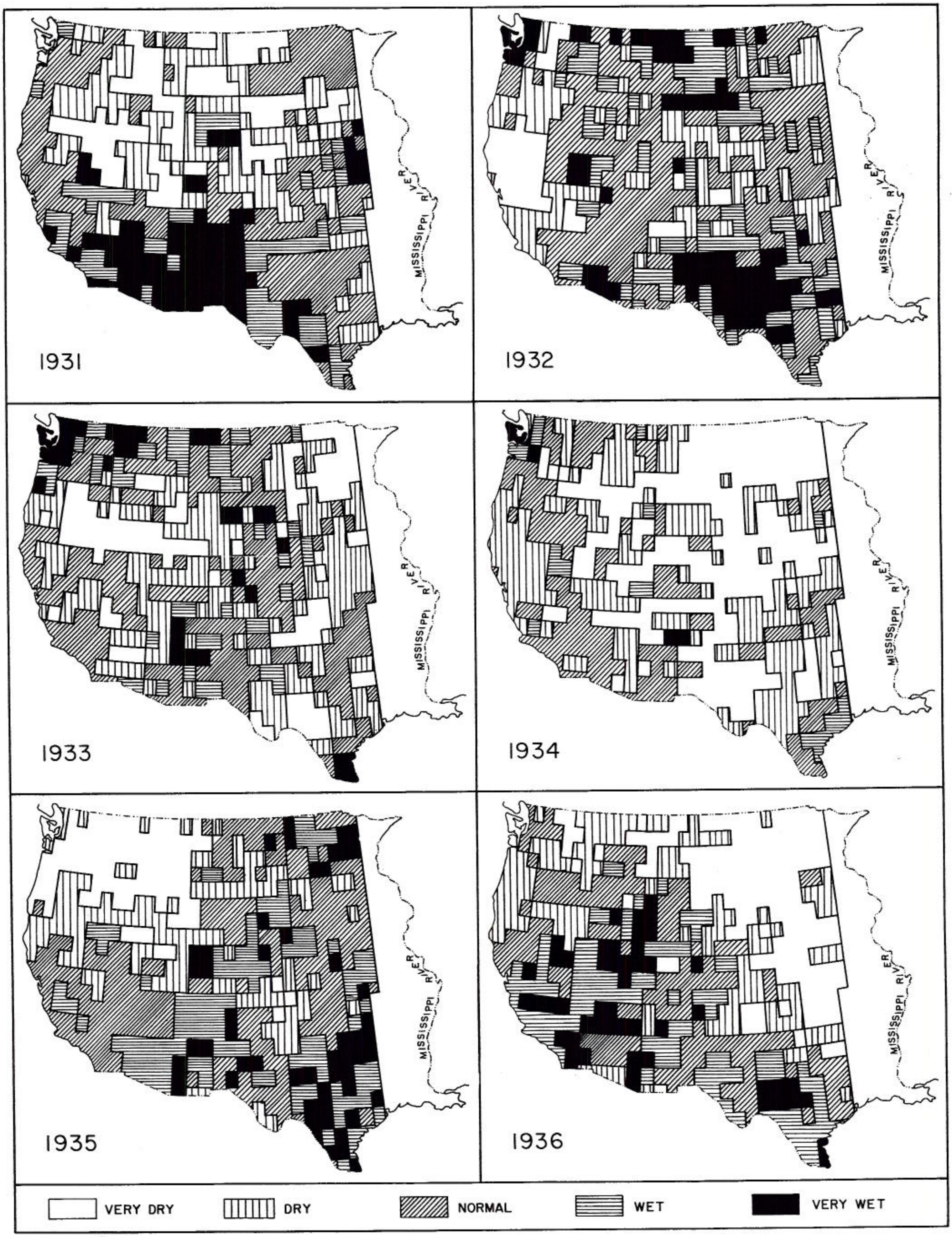

Fig. 4 Synoptic maps of wet and dry years in the Western United States for individual years during the period 1931 to 1960

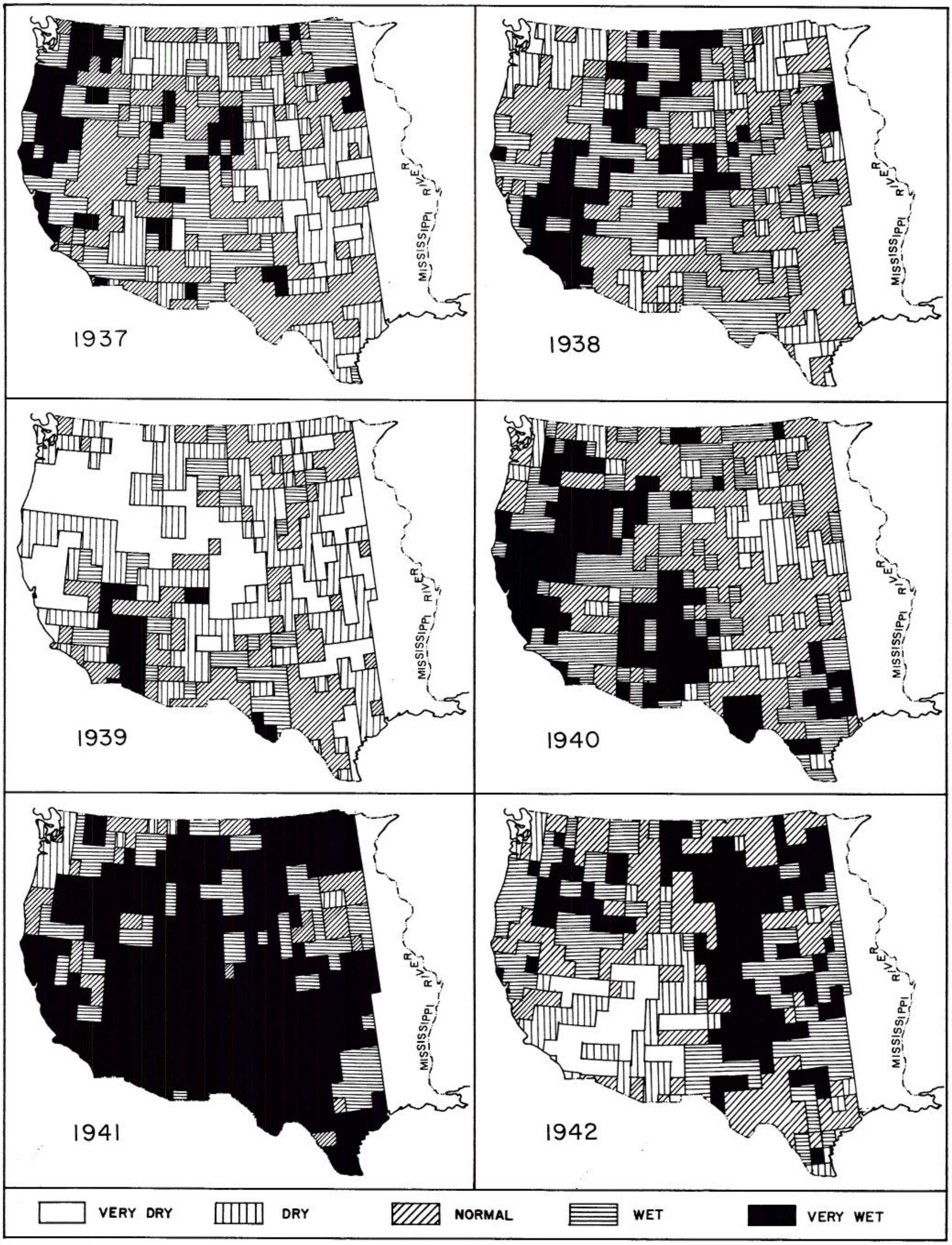

Fig. 5 Synoptic maps of wet and dry years in the Western United States for individual years during the period 1937 to 1942

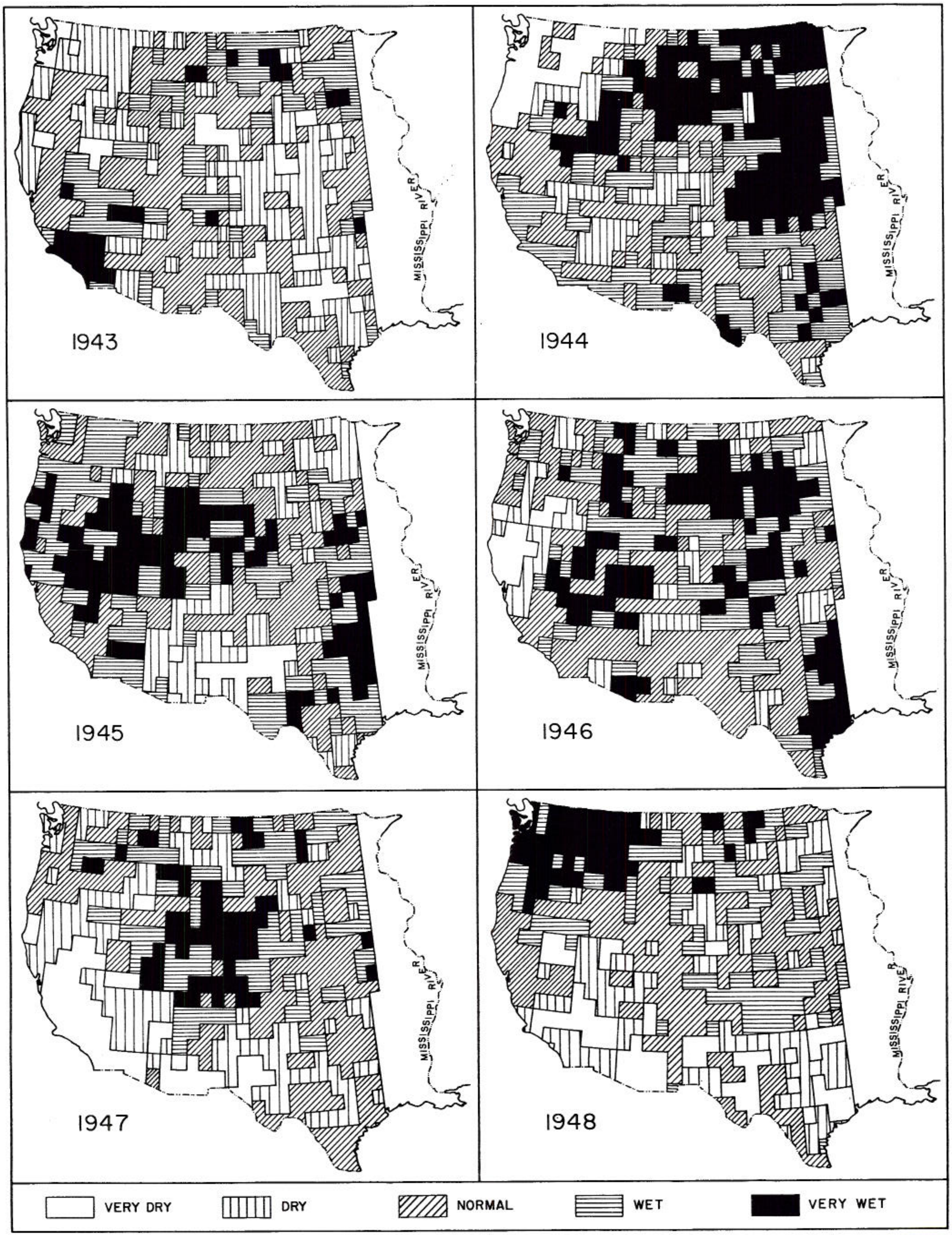

Fig. 6 Synoptic maps of wet and dry years in the Western United States for individual years during the period 1943 to 1948

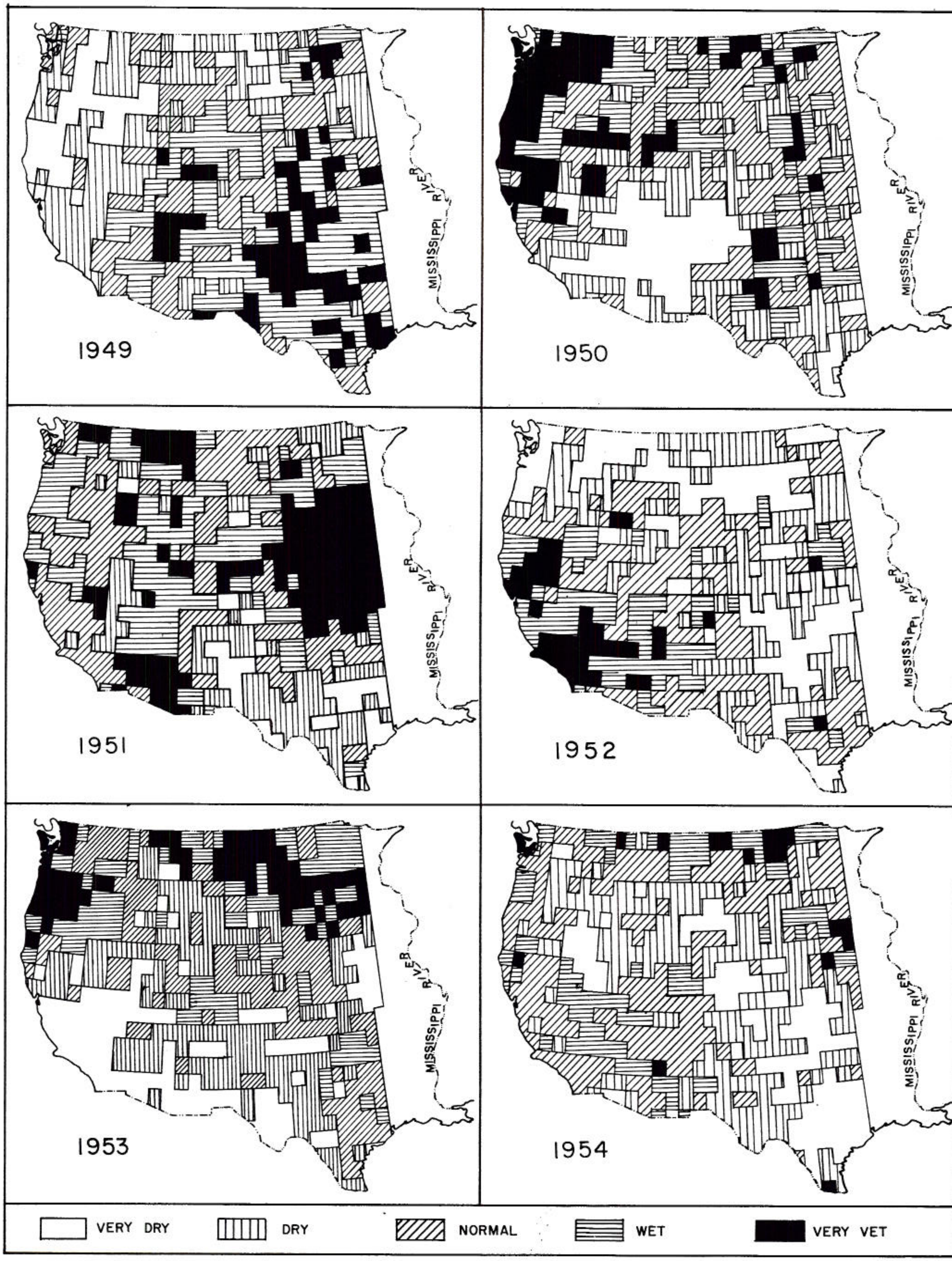

Fig. 7 Synoptic maps of wet and dry years in the Western United States for individual years during the period 1949 to 1954

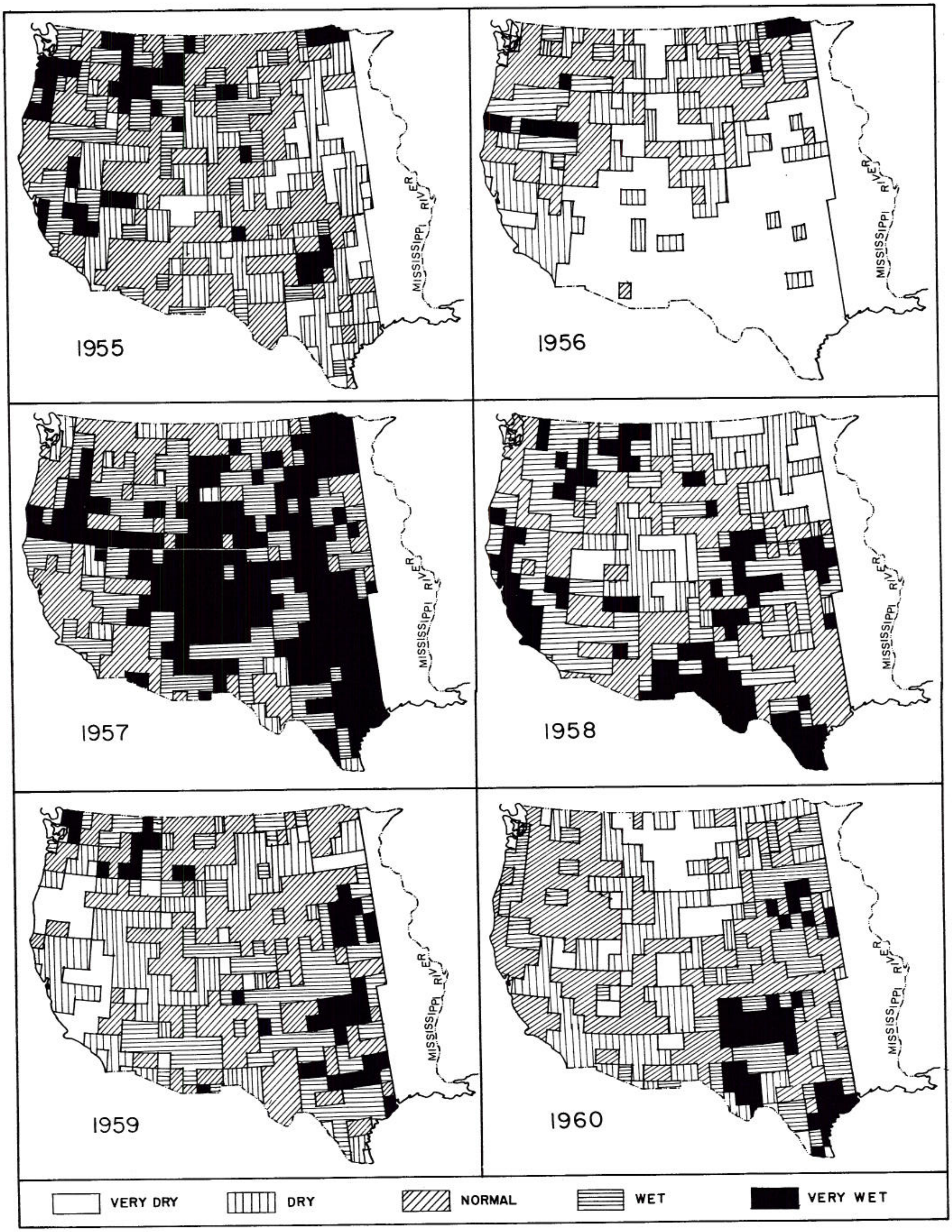

Fig. 8 Synoptic maps of wet and dry years in the Western United States for individual years during the period 1955 to 1960

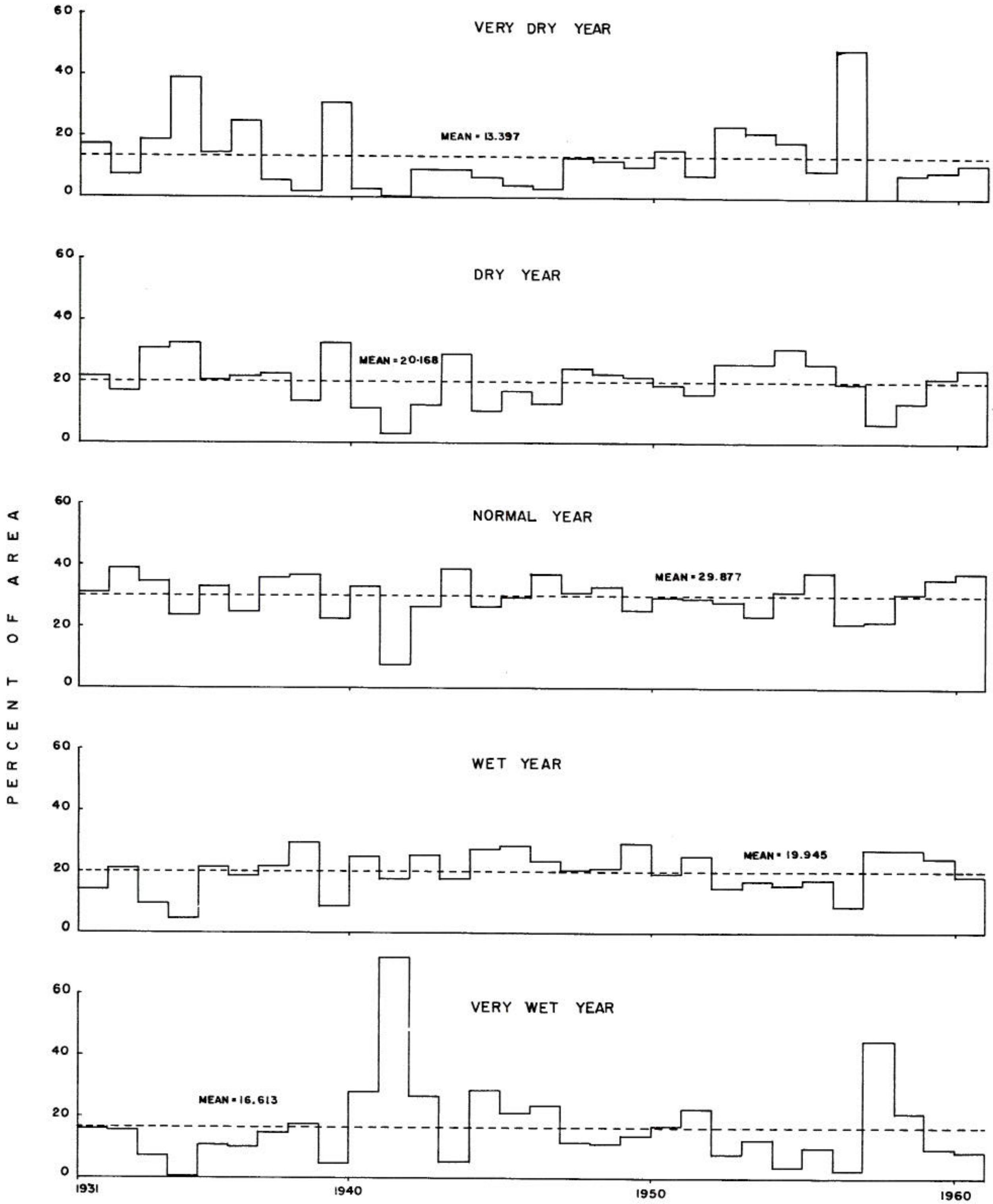

Fig. 9 Time series of the percent of areas in the Western United States having very dry, dry, normal, wet and very wet years

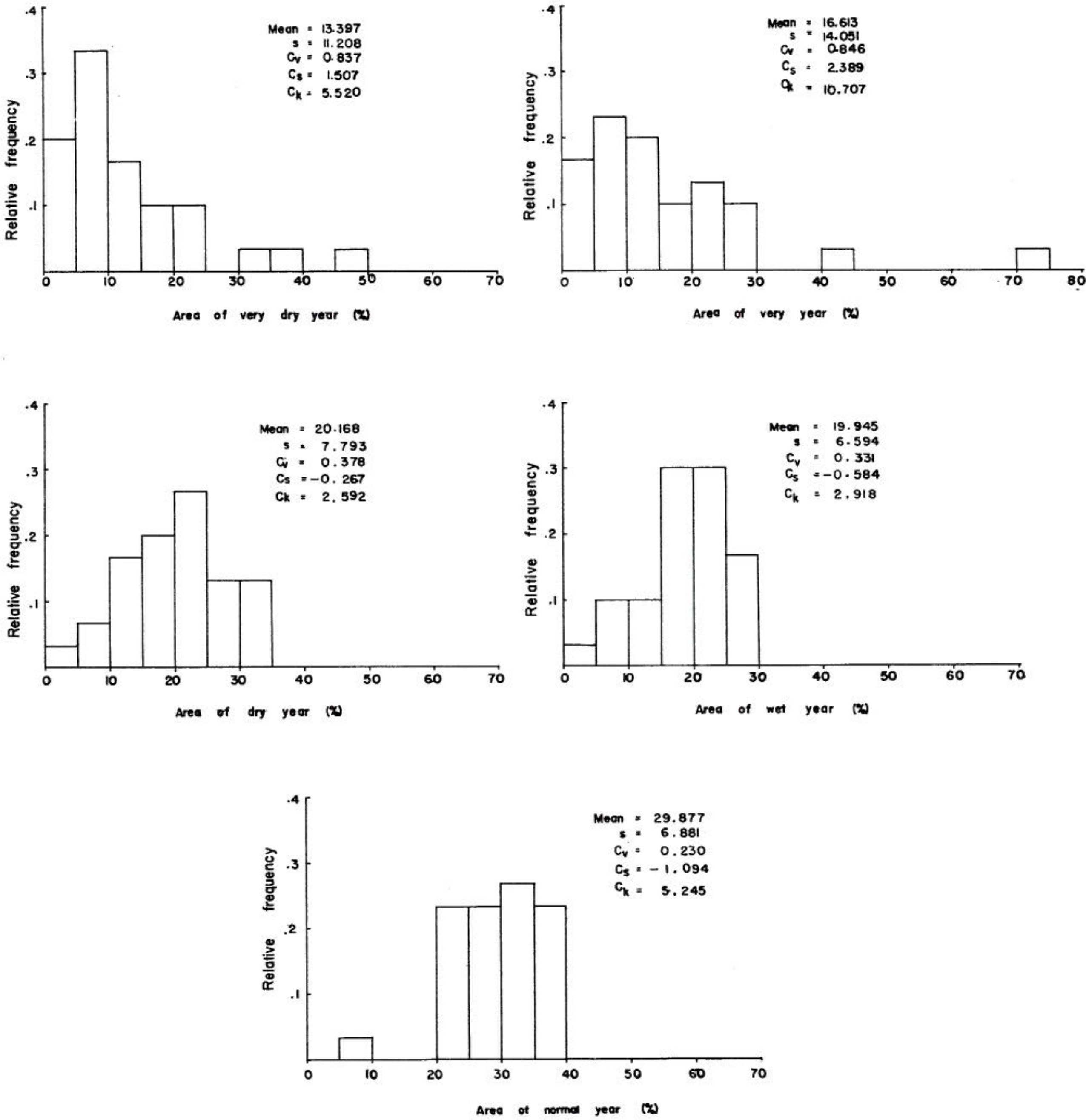

Fig. 10 Frequency distributions of percent of areas in the Western United States having very dry, dry, normal, wet and very wet years

TABLE 1

Correlation coefficients of the percent of areas having X_1, X_2, X_3, X_4 and X_5

	X_1	X_2	X_3	X_4	X_5
X_1	-	0.6108	-0.2864	-0.8036	-0.6118
X_2		-	0.2737	-0.6644	-0.8523
X_3			-	0.2142	-0.5103
X_4				-	0.4275
X_5					-

with X_1 = very dry year; X_2 = dry year; X_3 = normal year; and X_4 = wet year; and X_5 = very wet year.

and very wet years deviated somewhat from their expected values (15.00), whereas the percent of areas having dry, normal and wet years deviated very little from their expected values (20.00, 30.00, 20.00, respectively). Positive skewnesses were found in distributions of the sub-areas having very dry and very wet years, and negative skewnesses were found in dry, normal and wet years. The coefficient of variation and the skewness coefficient change with the types of wet and dry years as shown in fig. 11. Both increased toward the extremes.

5. Conditional probability of wet and dry years. By using the grid system, as shown in fig. 40, and described in Appendix A, the conditional probability X_i at the sub-areas at the distance k from the central sub-area, given X_j at the central sub-area, is

$$p_{ij}(k) = \frac{N_{ki}}{N_{kj}} \qquad (19)$$

where N_{ki} is the number of the sub-areas having X_i at the distance k from the central sub-area, given X_j at the central sub-area, and N_{kj} is the total number of sub-areas at the distance k, given X_j at the central sub-area.

Fifty-four central sub-areas, with various types of climates and different physiographic features, were selected throughout the Western United States for studying the goodness of fit of mathematical models to the observed data. The number of the central sub-areas were selected to minimize the use of the computer. Locations of central sub-areas centers are shown in fig. 12. The conditional probabilities of wet and dry years as a function of distance were computed on a CDC 3600 computer. Examples of the results of these computations are presented in figs. 13 through 16.

6. Model I: Independence. The null hypothesis of the mathematical model states that wet and dry sub-areas for a given year are distributed on a chance basis. Therefore, the joint areal distribution of very dry, dry, normal, wet and very wet years is a multinomial distribution of five variables. The chi-square test was used for testing the goodness of fit. For the jth year, with $j = 1, 2, \ldots, 30$, the statistic χ^2 is computed by

$$\chi^2 = \sum_{i=1}^{5} \frac{(x_i - m\,p_i)^2}{m\,p_i} \qquad (20)$$

where m is the sample size, p_i is the probability of X_i, and x_i is the observed value of X_i. From the property of the multinomial distribution, the statistic χ^2 is distributed as chi-square with 4 degrees of freedom. The 10 percent rejection limit of significance being the 0.90 percentile of the chi-square distribution with 4 degrees of freedom was used here, and it is 7.78. Computed chi-squres with 4 degrees of freedom for each year are presented in fig. 17.

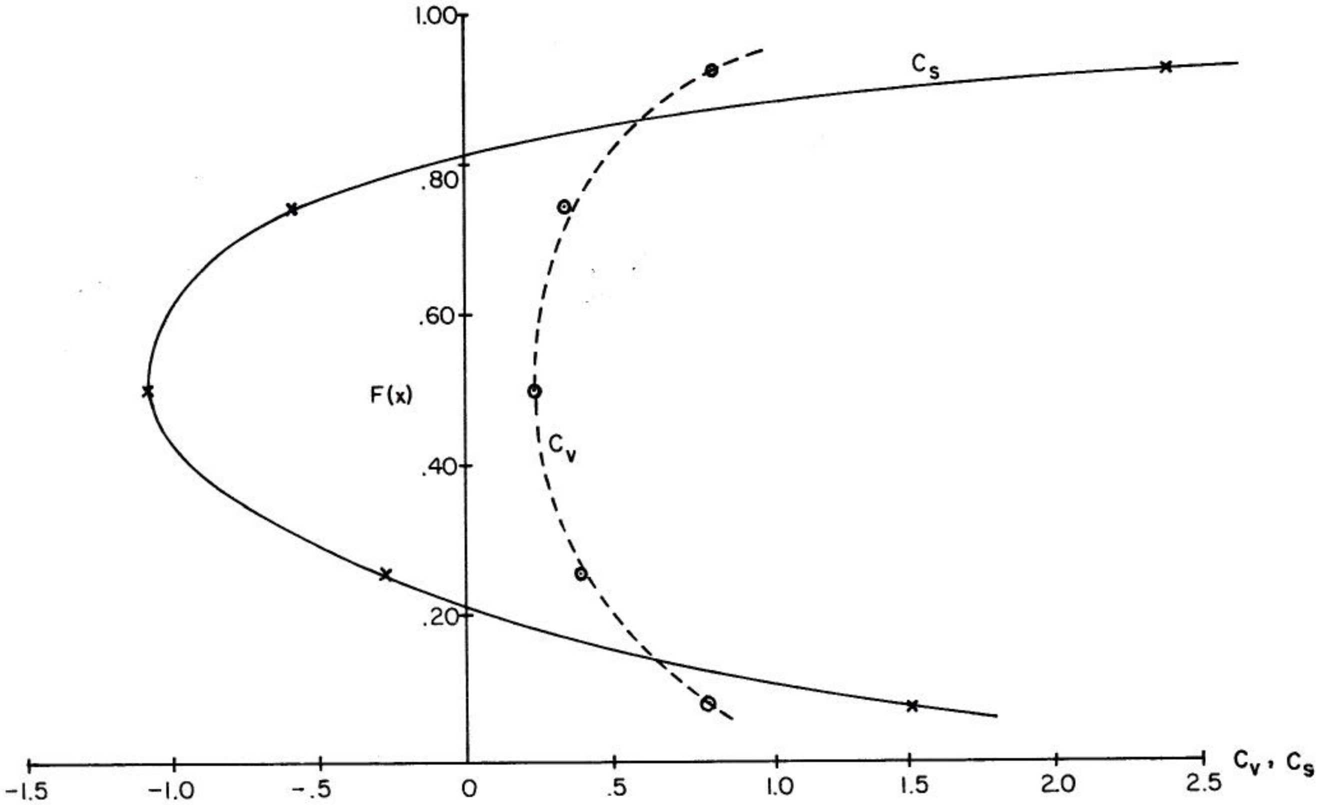

Fig. 11 Relationship among coefficients of variation, skewness coefficients and types of wet and dry years

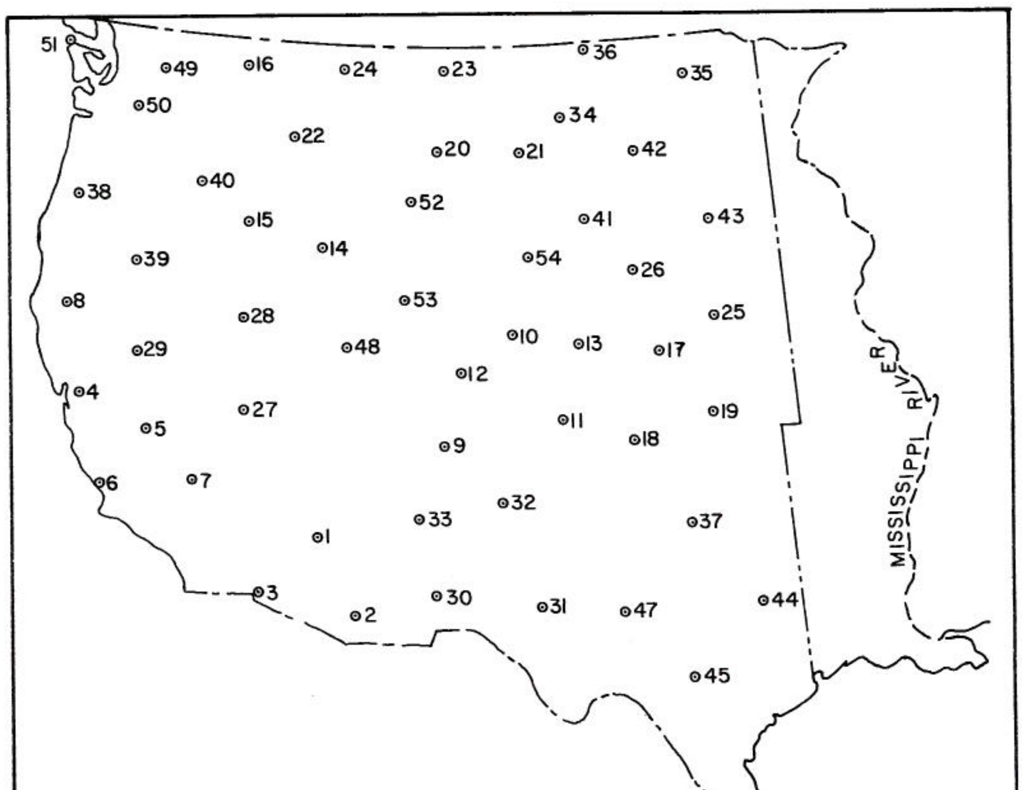

Fig. 12 Locations of central sub-area centers in the Western United States

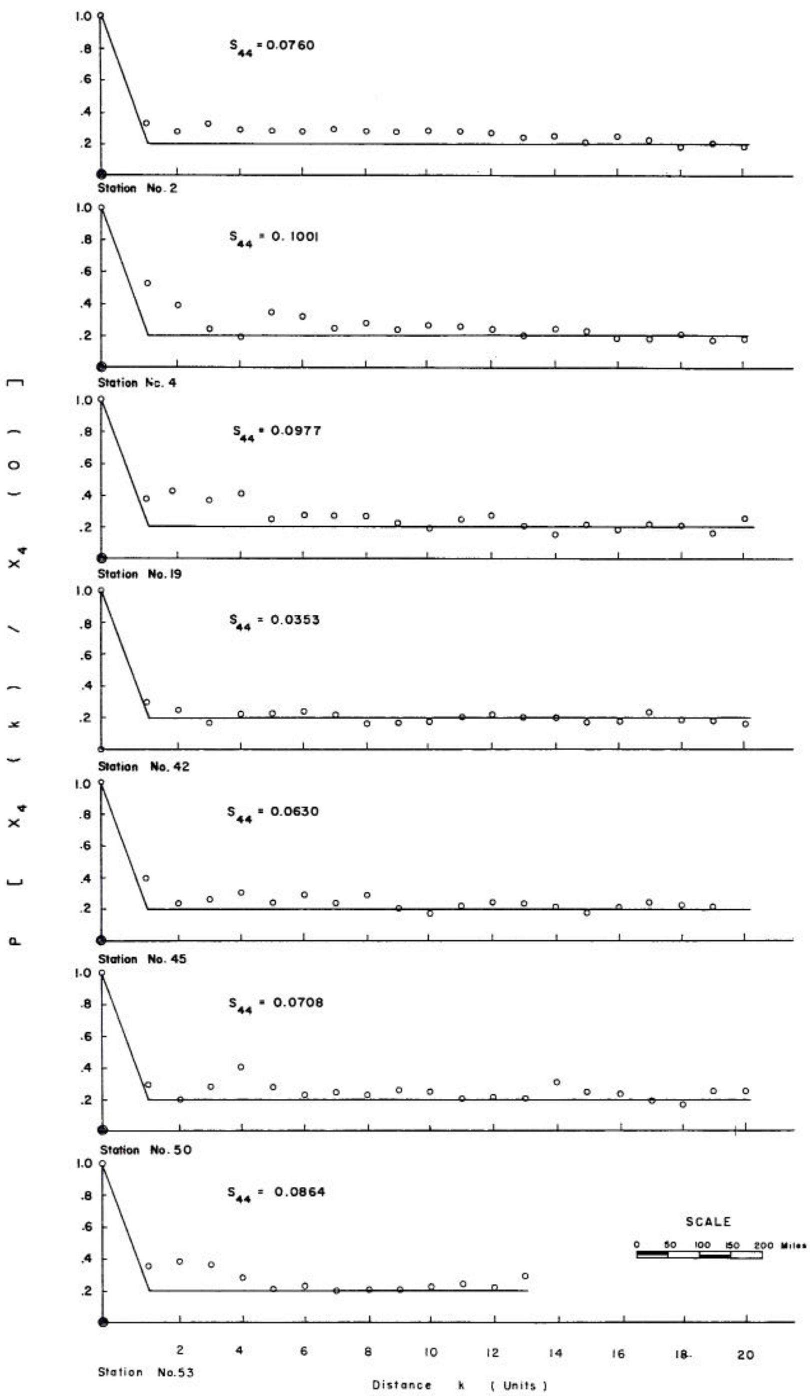

Fig. 13 Comparison of conditional probabilities of having a wet year at the sub-areas at the distance k from the central sub-area, given a wet year at the central sub-area, with those of Model I. Points are the computed conditional probabilities: the solid line represents Model I

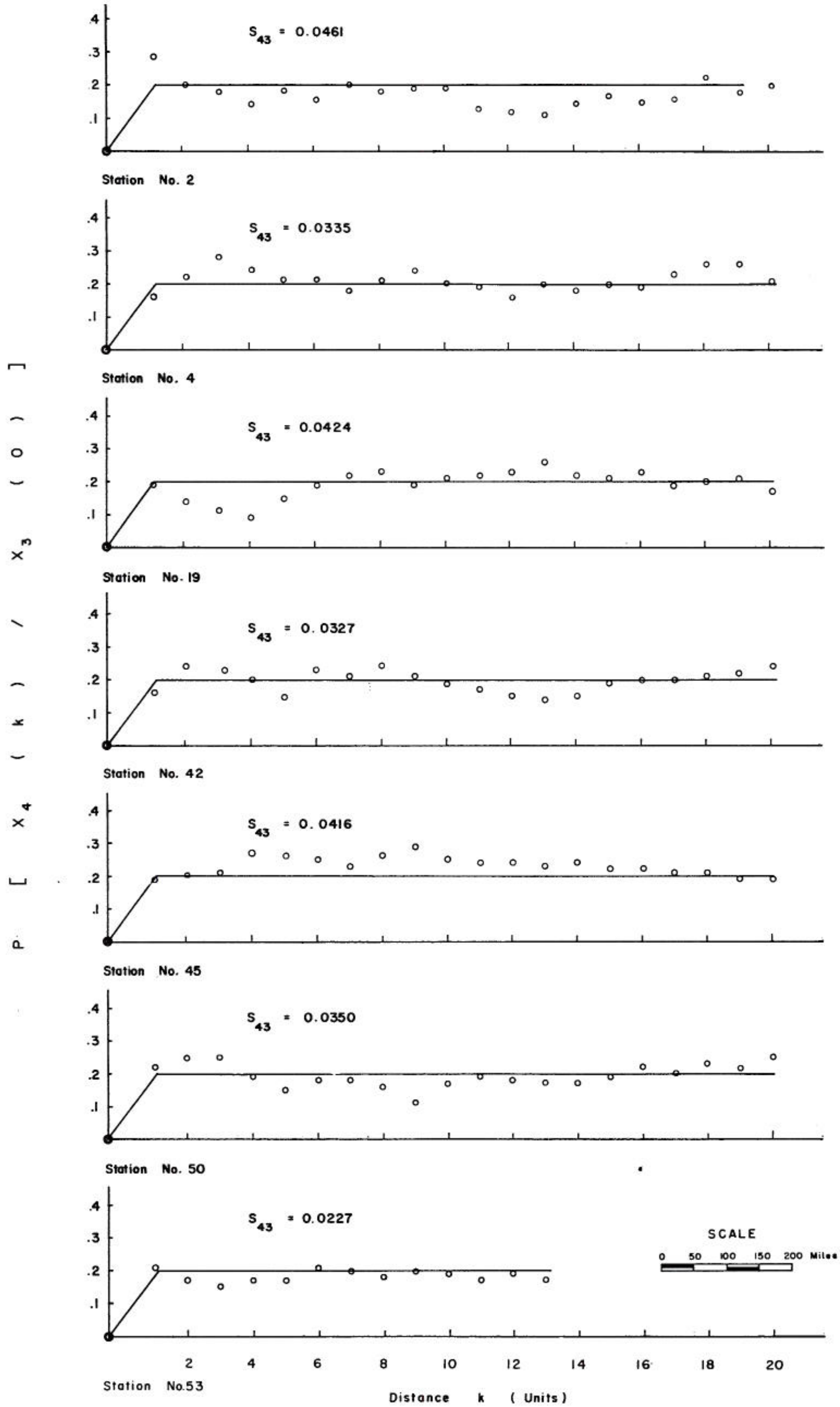

Fig. 14 Comparison of conditional probabilities of having a wet year at the sub-areas at the distance k from the central sub-area, given a normal year at the central sub-area, with those of Model I. Points are the computed conditional probabilities: the solid line represents Model I.

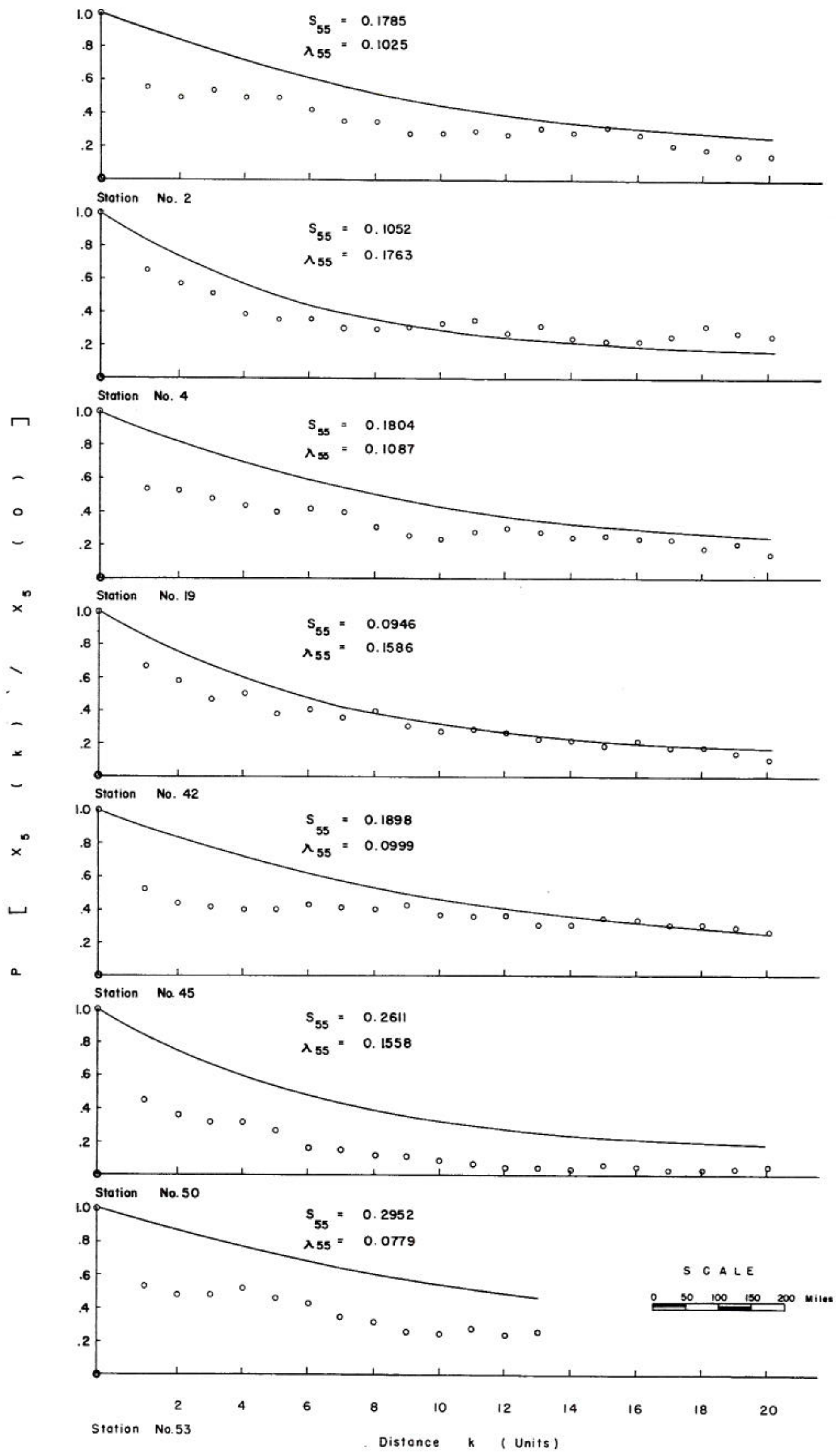

Fig. 15 Comparison of conditional probabilities of having a very wet year at the sub-areas at the distance k from the central sub-area, given a very wet year at the central sub-area, with those of Model II. Points are the computed conditional probabilities: the solid line represents Model II.

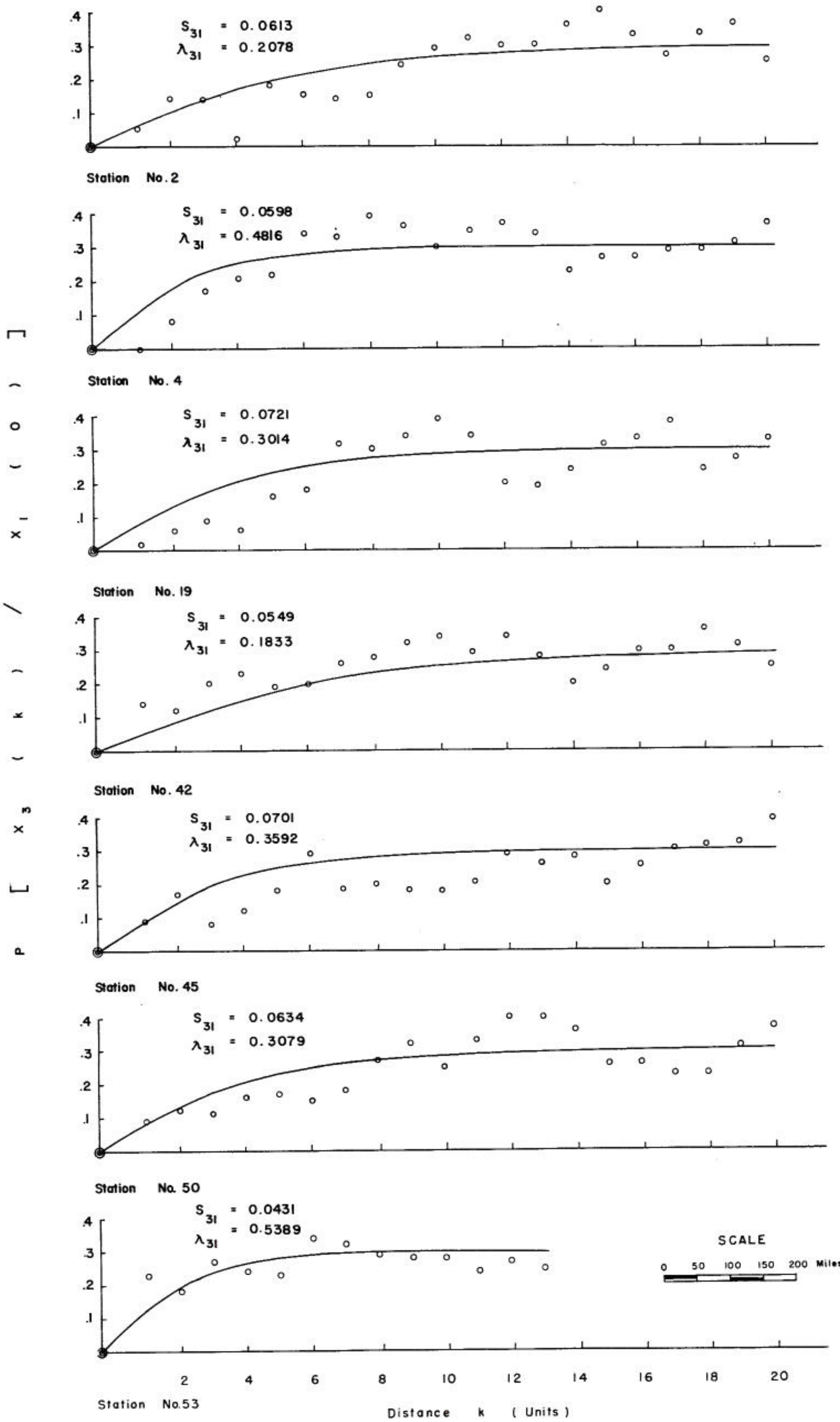

Fig. 16 Comparison of conditional probabilities of having a normal year at the sub-areas at the distance k from the central sub-area, given a very dry year at the central sub-area, with those of Model II. Points are the computed conditional probabilities: the solid line represents Model II.

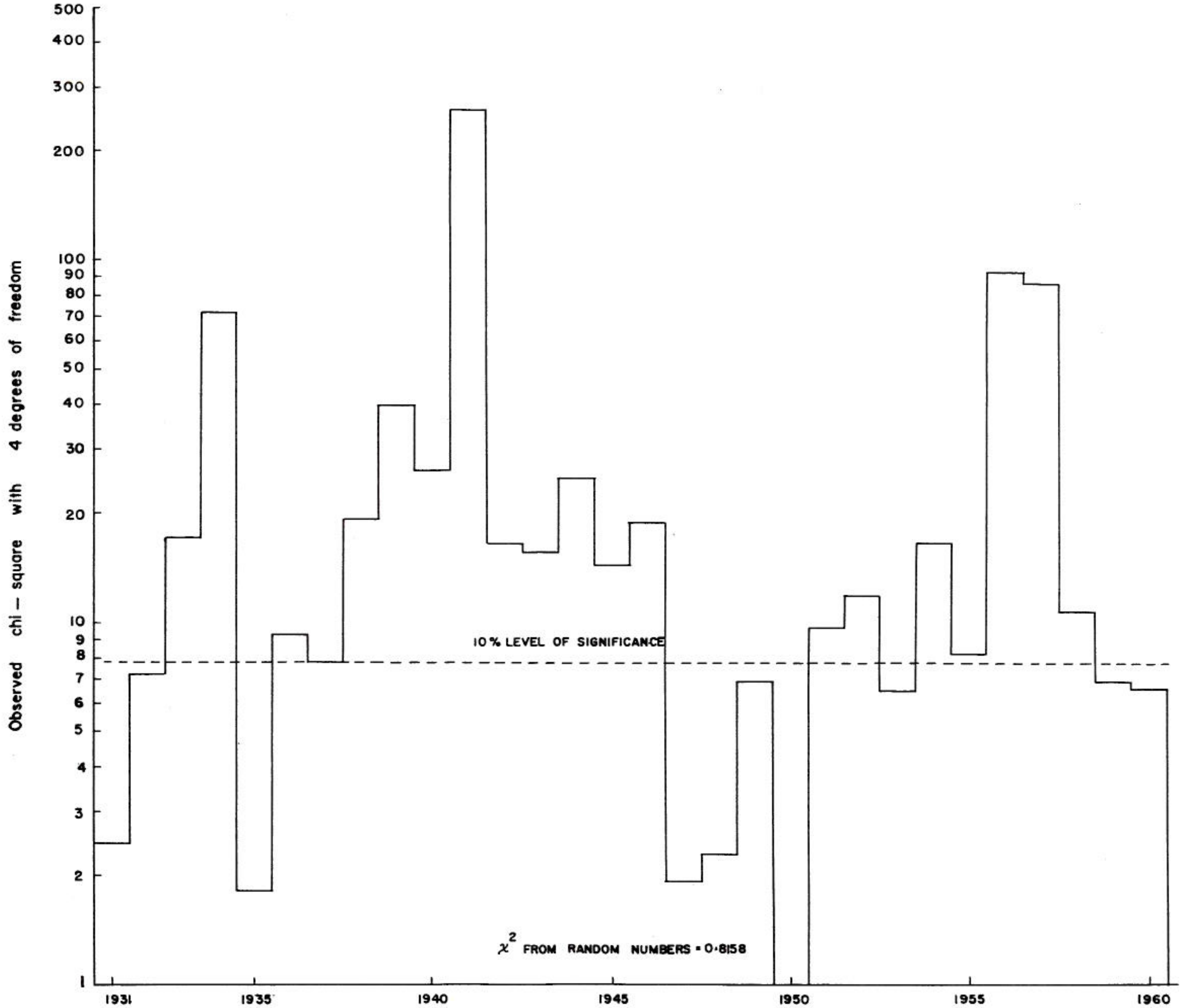

Fig. 17 Chi-squares for testing for each year of the independence of wet and dry years (Model I).

For 66.67 percent of years on record (20 years out of 30) the chi-squares are greater than 7.78. Those 10 years (33.33 percent) which have chi-squres less than 7.78 were found to be the years for which the very dry, dry, normal, wet and very wet areas are approximately 15.00 percent, 20.00 percent, 30.00 percent, 20.00 percent and 15.00 percent, respectively. In other words, the distribution across the area follows approximately the definitions of wet and dry years at any sub-area. Therefore, the null hypothesis of joint areal multinomial distribution of five variables is rejected at a 10 percent rejection limit of significance.

In comparison of observed data with random numbers, let a table of independent random numbers as large as the number of sub-areas represent an area under consideration for a given year in the period from 1931 to 1960. Each number in the table represents the annual precipitation in a sub-area for that year. Values from the table of random independent numbers were assigned into the grid system of the Western United States to allow one number per sub-area. By using the definitions of wet and dry years, the type of year for each sub-area was determined from the independent number in the sub-area, whose numbers ranged from 00 to 99. The assigned independent numbers are interpreted as follows:

Number	Type of Year
00 - 14	very dry
15 - 34	dry
35 - 64	normal
65 - 84	wet
85 - 99	very wet

The synoptic map derived from the table of independent numbers is presented in fig. 18. Statistic χ^2 computed by eq. (20) is 0.8158. Since this value is less

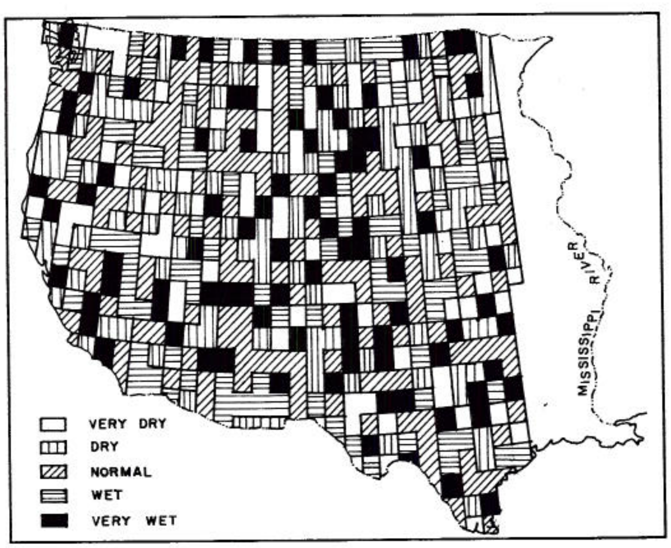

Fig. 18 Synoptic map of wet and dry years, as assigned to sub-areas of the Western United States, for a given year, obtained from the table of independent numbers.

than 7.78 (10 percent rejection limit of significance being the 0.90 percentile of the chi-square distribution with 4 degrees of freedom) the independent numbers used are considered to be distributed by the multinomial distribution of five variables. This shows a clear distinction between the observed distributions of wet and dry years and the multinomial distribution. Visual comparison of fig. 18 with figs. 4 through 8 also show this difference clearly.

Conditional probabilities of a variable X_i at the distance k from the central sub-area, given a variable X_j at the central sub-area were tested against the binomial distribution of the two independent variables X_i and X_j. These independent variables were obtained from the table of independent numbers and are presented in fig. 19. The conditional probabilities are defined by eq. (7). The number of possible combinations of the five variables, considering two variables at a time, is twenty-five. Two examples of observed data for seven central sub-areas, with different hydrologic and physiographic characteristics fitted by Model I, are presented in figs. 13 and 14. Mathematical Model I is suited for the observed data.

The standard errors of estimate for the mathematical model, as defined by eq. (18), was used in this study to measure the goodness of fit of mathematical Model I to the observed data. In this case, the estimated probabilities are: $\hat{p}_{ij}(k) = p_i$, $\hat{p}_{ij}(0) = 0$, and $\hat{p}_{jj}(0) = 1$. Two examples of areal distribution and frequency distribution of standard error of estimate, obtained by using the fifty-four selected central sub-areas, are presented in figs. 20 and 21. The areal distributions are very erratic forcing the isolines of standard error of estimate to be drawn schematically as an approximation. However, the coefficients of variation of standard errors of estimate, which range from 0.2789 to 0.2987, can be considered as being in tolerance. Therefore, the mean value of standard error of estimate is considered a sufficient statistic for measuring the applicability of Model I. The twenty-five possible combinations of the five variables, considering two variables at a time, can be represented by a 5 x 5 matrix. However, since wet and dry years are classified into five arbitrary categories (very dry, dry, normal, wet and very wet years) this matrix was transformed into a continuous graphical representation in the following way: Let $F(x)$ be a distribution function of annual precipitation of any station, as defined by eq. (1). If the annual precipitation at a station is between x_1 and x_2 so that $F(x_1) > 0.35$ and $F(x_2) \leq 0.65$, then that year is classified as normal. The distribution function $F(x)$ is $F(x_o)$ for any central sub-area, and $F(x)$ for any sub-area which is at the distance k from the central sub-area. The mean standard error of estimate obtained from fifty-four selected central sub-areas is denoted by $\overline{S}_{ij}$ where i and j refer to the types of years at the two sub-areas. Only twenty-five possible combinations of wet and dry years of $\overline{S}_{ij}$ are available. These twenty-five have the average values $F(x_o)$ or $F(x)$ of 0.075, 0.25, 0.50, 0.75 and 0.925 which are the central probabilities of very dry, dry, normal, wet and very wet years. In this way, a continuous representation of the mean standard error of estimate

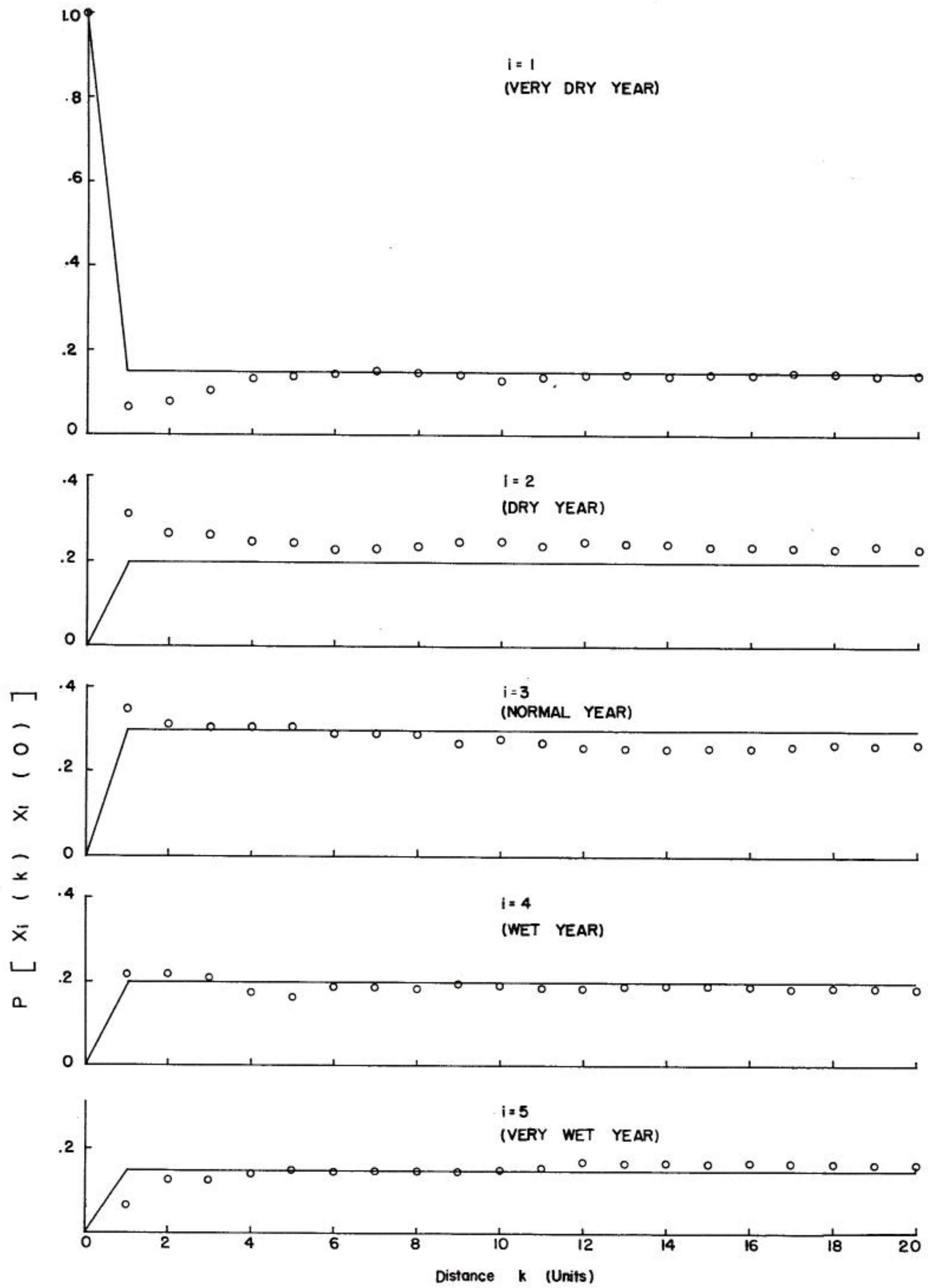

Fig. 19 Conditional probabilities of wet and dry years as a function of distance computed from the table of independent numbers

was obtained and presented in fig. 22, a through d.

A functional relationship between $F(x_o)$, $F(x)$ and $\bar{S}_{ij}$ could be fitted to the twenty-five points in fig. 22 a. However, the patterns of isolines in fig. 22 a are only qualitatively described here as follows: (1) Given a normal year, the probabilities of having any category of wet and dry years can be well described by Model I with the mean standard error of estimate $\bar{S}_{ij} \leq 0.06$; (2) Given a dry year, the probabilities of having any category of wet and dry years can be described sufficiently by Model I with $0.06 < \bar{S}_{ij} \leq 0.08$; (3) Given a wet year, the probabilities of having a very dry, dry, normal or wet year can be sufficiently described by Model I with $0.06 < \bar{S}_{ij} \leq 0.08$; (4) Given a very dry year, the probabilities of having a normal or a wet year can be described by Model I with $0.08 < \bar{S}_{ij} \leq 0.10$; (5) Given a very dry year, the probability of having a very dry year can be described by Model I only with $\bar{S}_{ij} > 0.10$; (6) Given a very wet year, the probabilities of having a very dry year, dry, normal

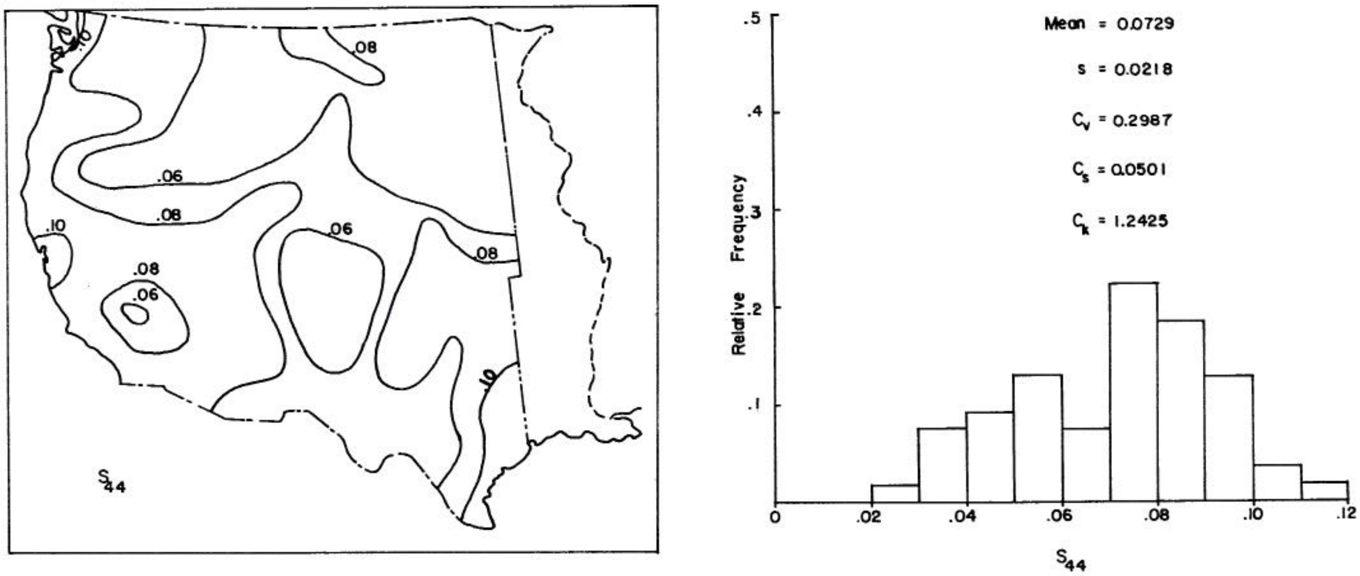

Fig. 20 Areal and frequency distributions of standard error of estimate S_{44} for Model I.

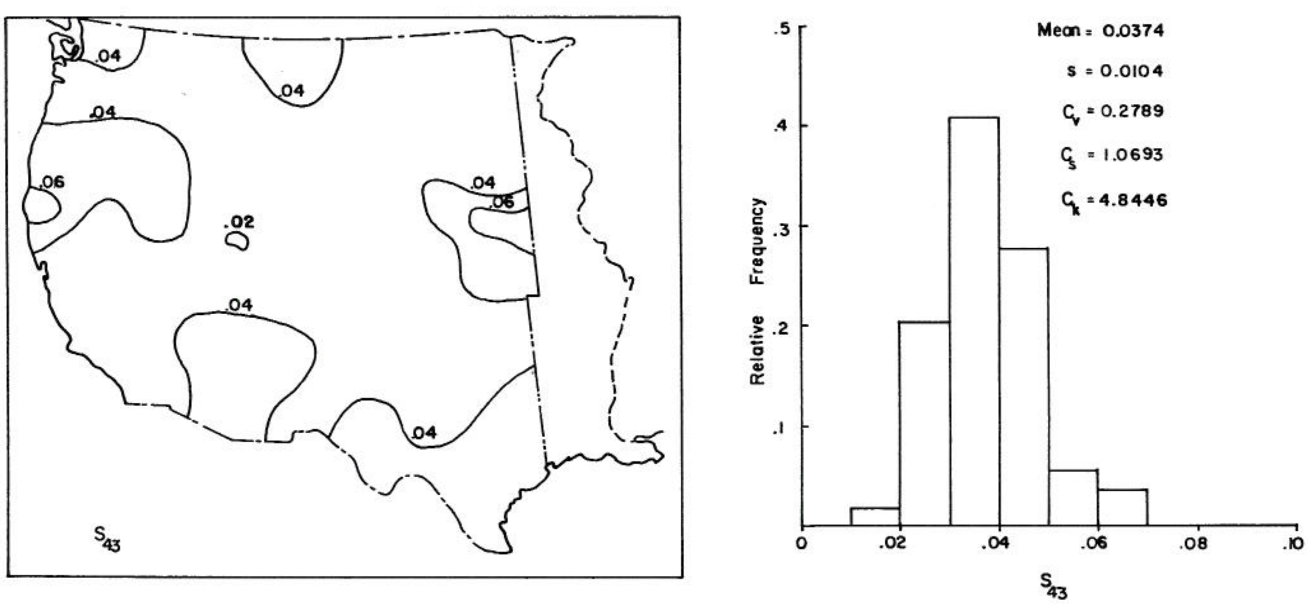

Fig. 21 Areal and frequency distributions of standard error of estimate S_{43} for Model I.

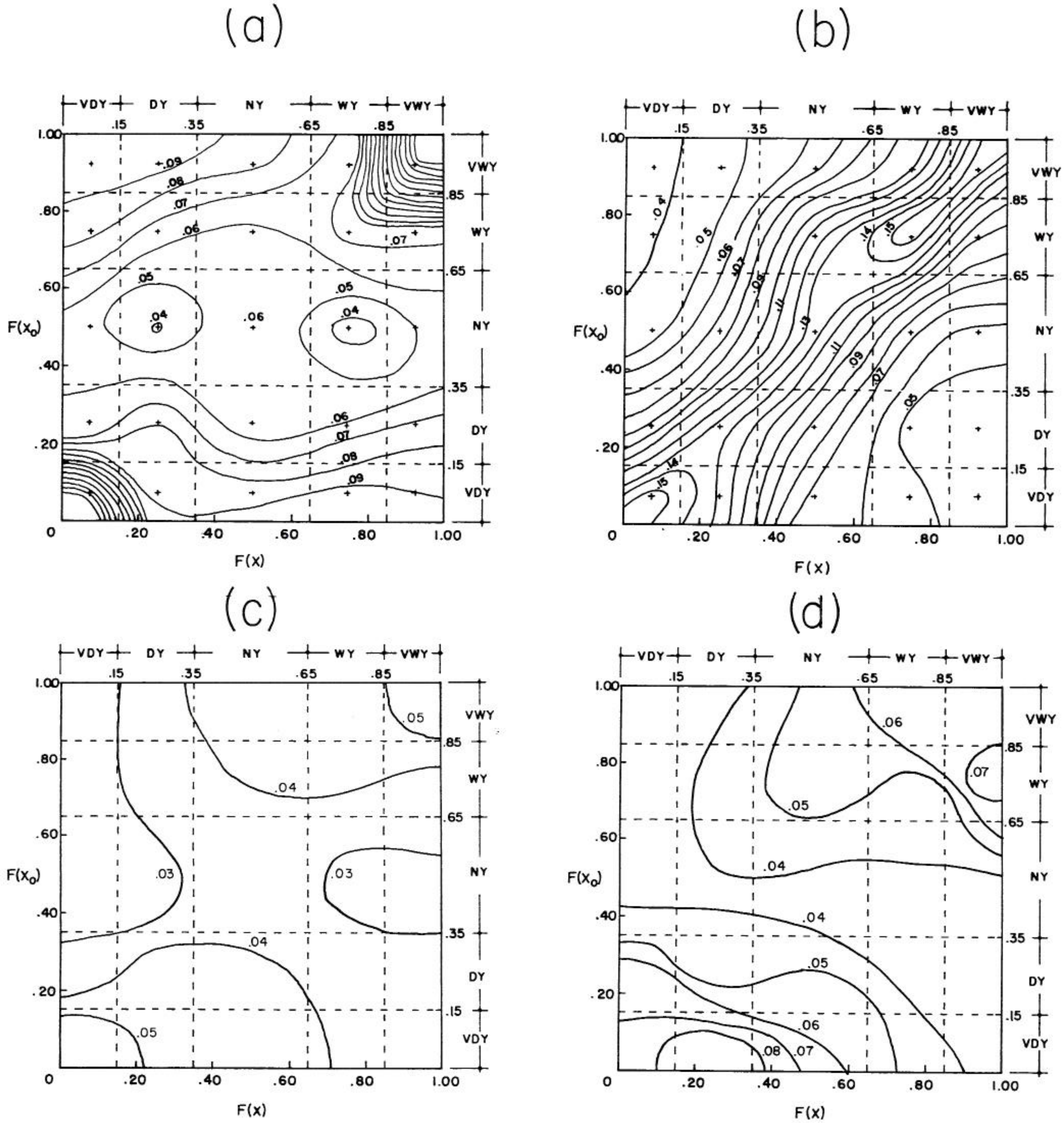

Fig. 22 Mean standard error of estimate, $\overline{S}_{ij}$, as a function of F (x_o) and F(x): (a) Model I; (b) Model II; (c) Model III; and (d) Model IV

or wet year can be described by Model I with $0.08 < \overline{S}_{ij} \leq 0.10$; and (7) Given a very wet year, the probability of having a very wet year can be described only with $\overline{S}_{ij} > 0.10$. The relationship between the coefficient of variation of S_{ij}, $F(x_o)$ and $F(x)$ has no regular pattern. It is presented in fig. 23a.

7. Model II: Exponential dependence. The mathematical Model II, as given by eq. (8) was used to fit the conditional probabilities of the five variables, considering two variables at a time, as a function of distance from the central sub-area to the sub-areas under consideration. The number of possible combinations of five variables, considering two variables at a time, is twenty-five. The goodness of fit of this model to the observed conditional probabilities was measured by the standard error of estimate.

Two examples of observed data fitted by Model II are presented in figs. 15 and 16. The standard errors of estimate were computed by eq. (18). Two examples of areal distribution and frequency distribution of standard error of estimate for fifty-four selected central sub-areas are presented in figs. 24 and 25.

The same notations for the mean standard errors of estimate which were used in the Model I were also used here. It was found from fig. 22b that the relationship between $\overline{S}_{ij}$, $F(x_o)$ and $F(x)$ have some patterns. They are not simple enough to be expressed by a fitted mathematical function. The main characteristics of these patterns are as follows: (1) Given a very dry or dry year, the conditional probabilities of having a wet or very wet year can be described by Model II with $\overline{S}_{ij} \leq 0.06$; (2) Given a normal year, the probabilities of having a very dry, dry, wet or very wet year can be described by Model II with $\overline{S}_{ij} \leq 0.08$; (3) Given a wet or very wet year, the probabilities of having a very dry or dry year can be described by Model II with $\overline{S}_{ij} \leq 0.06$; and (4) For the other combinations, besides those mentioned under (1), (2) and (3), the conditional probabilities can be described by this model only if $0.08 < \overline{S}_{ij} \leq 0.16$. It was also found that there is no apparent regular patterns in the relationship between the coefficients of variations, C_v, of standard error of estimate, S_{ij}, $F(x_o)$ and $F(x)$ as is shown in fig. 23b.

The parameter λ_{ij} of mathematical Model II was estimated by eq. (10). Two examples of the areal distribution and the frequency distribution of estimated parameters for the fifty-four selected central sub-areas are presented in figs. 24 and 25. The mean of estimated parameters $\hat{\lambda}_{ij}$ of the fifty-four selected central sub-areas is denoted by $\overline{\lambda}_{ij}$. The relationship between $\overline{\lambda}_{ij}$, $F(x_o)$ and $F(x)$ is presented in fig. 26a. No attempt has been made to express the relationship existing in fig. 26a by a simple fitted mathematical expression, though some regular patterns seem to exist. These patterns can be described qualitatively as follows: Parameters have high values in cases when a normal year occurs at a sub-area and a normal year occurs in the adjacent sub-areas, while the values decrease towards the corners. The higher $\overline{\lambda}_{ij}$, the more independent the two variables become.

8. Model III: Linear dependence. Mathematical Model III, as given by eq. (11), was used to fit the observed conditional probabilities of wet and dry years as a function of distance from the central sub-area to the sub-areas under consideration. The goodness of fit was measured by the standard error of estimate which is defined in eq. (18).

Two examples of the observed data being fitted by Model III are presented in figs. 27 and 28. It was found that the linear dependence fitted very well the observed data. However, the fitted function given by eq. (11) does not satisfy the boundary condition when $k > \alpha_{ij}$, and $p_{ij}(k) = p_i$. The ranges between $k > \alpha_{ij}$ and the maximum k were found to be very small when compared with the ranges between $k = 0$ and $k \leq \alpha_{ij}$. The condition $k > \alpha_{ij}$ (for distances much greater than 800 miles) is not of much interest in this study.

From the studies of Models I and II it was found that the areal distributions of the standard error of estimate and the estimated parameters are erratic, and their coefficients of variation were small. Therefore, the mean values are considered to be the most significant statistic. The mean standard errors of estimate from fifty-four selected central sub-areas for twenty-five combinations of wet and dry years, are presented graphically in fig. 22c. The same notations used in Models I and II were used here where $\overline{S}_{ij}$ is the mean of standard error of estimate S_{ij}. It was found that $\overline{S}_{ij}$ approximately ranges from 0.0300 to 0.0400 except for the cases of a very dry or very wet year. To describe the same category of year for this condition, the value of $\overline{S}_{ij}$ is approximately 0.500. However, this model is considered to be better fitted than Models I and II. The coefficients of variation of S_{ij} presented in fig. 23c, range from 0.30 to 0.40. These values are considered to be very small and do not seem to have a simple relationship with $F(x_o)$ and $F(x)$.

Parameters β_{ij} and α_{ij} of mathematical Model III were estimated by eqs. (13) and (14), respectively. The term ν_{ij}, which is defined by eq. (12), represents the slope of the straight line given by eq. (11). Instead of using the mean of ν_{ij}, the median was used in this study to eliminate the effects of extreme values of ν_{ij}. The median of ν_{ij} is a measure of the degree of dependence between sub-areas and is presented in fig. 26b. If ν_{ij} is small (0.0000 to $\pm$ 0.0050), then Model III becomes Model I as shown in fig. 26b for the following conditions: (1) Given a normal year, the conditional probability functions of having very dry, dry, wet or very wet year have the medians of ν_{ij} from -0.0002 to +0.0027; (2) Given a dry or wet year, the conditional probability of having a normal year have the medians of ν_{ij} from -0.0005 to -0.0029. It proves that under these two conditions Model III is identical to Model I. These results confirm the conclusions made with the investigation of Model I.

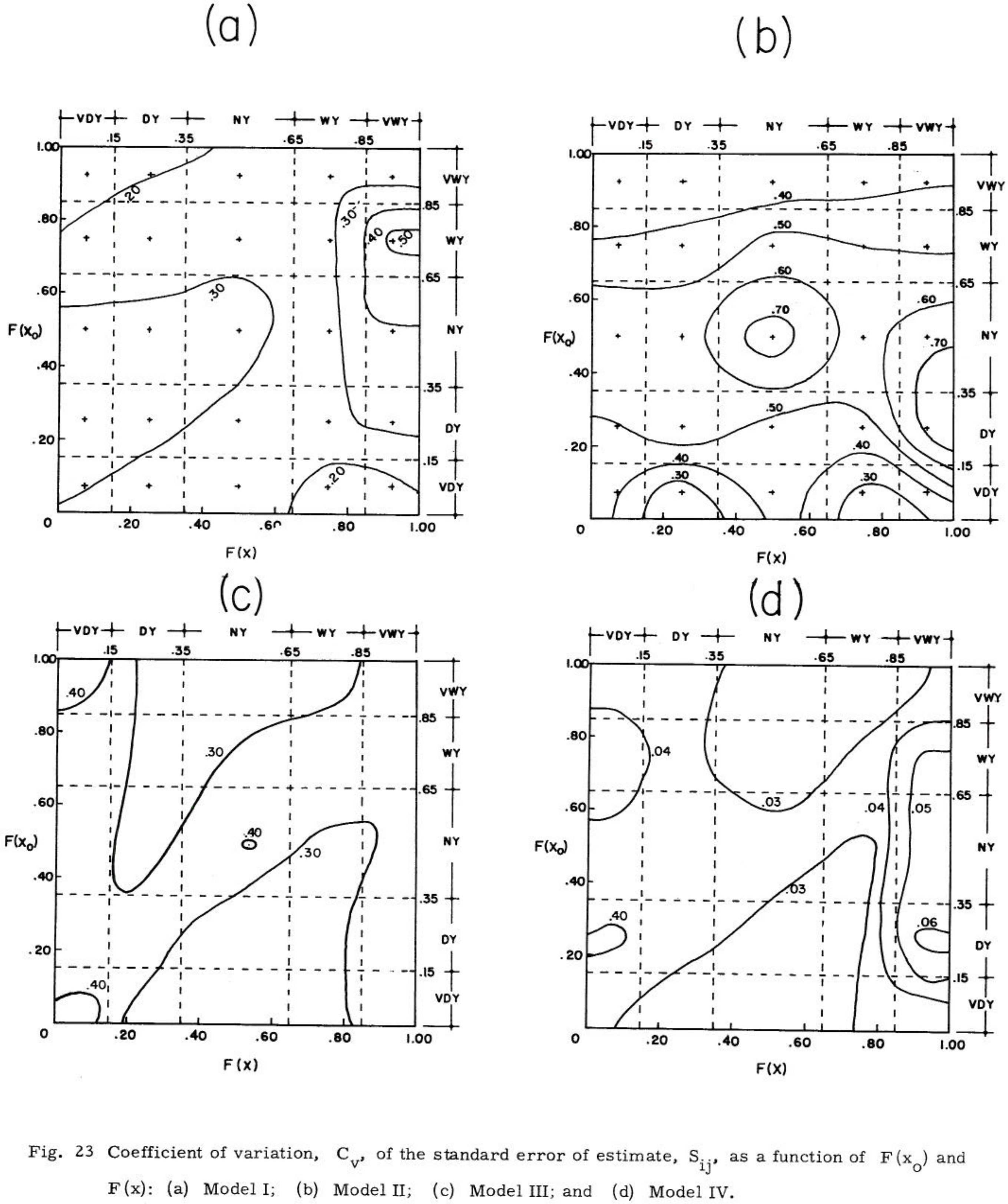

Fig. 23 Coefficient of variation, C_v, of the standard error of estimate, S_{ij}, as a function of $F(x_o)$ and $F(x)$: (a) Model I; (b) Model II; (c) Model III; and (d) Model IV.

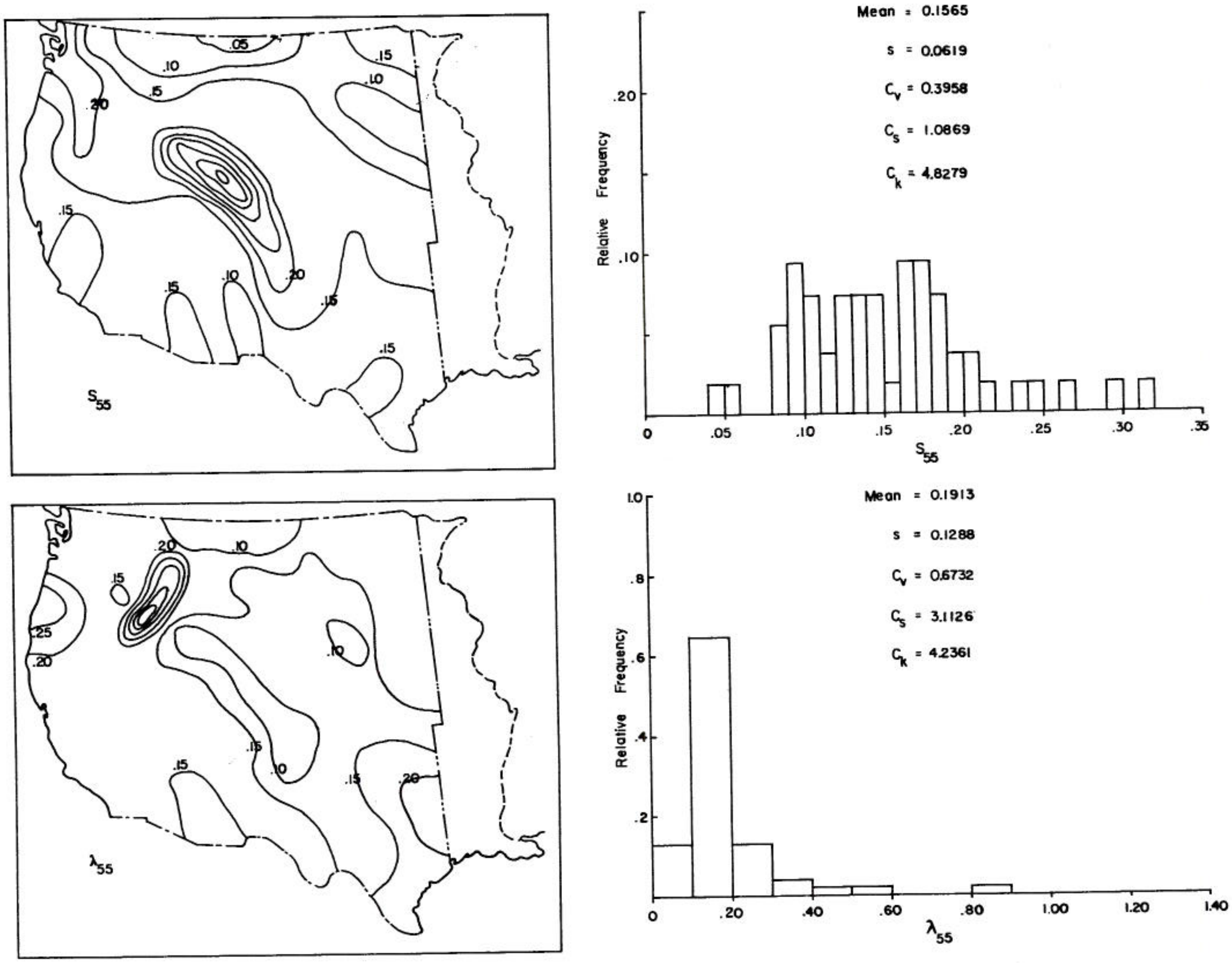

Fig. 24 Areal and frequency distributions of standard error of estimate, S_{55}, and of parameter λ_{55} for Model II.

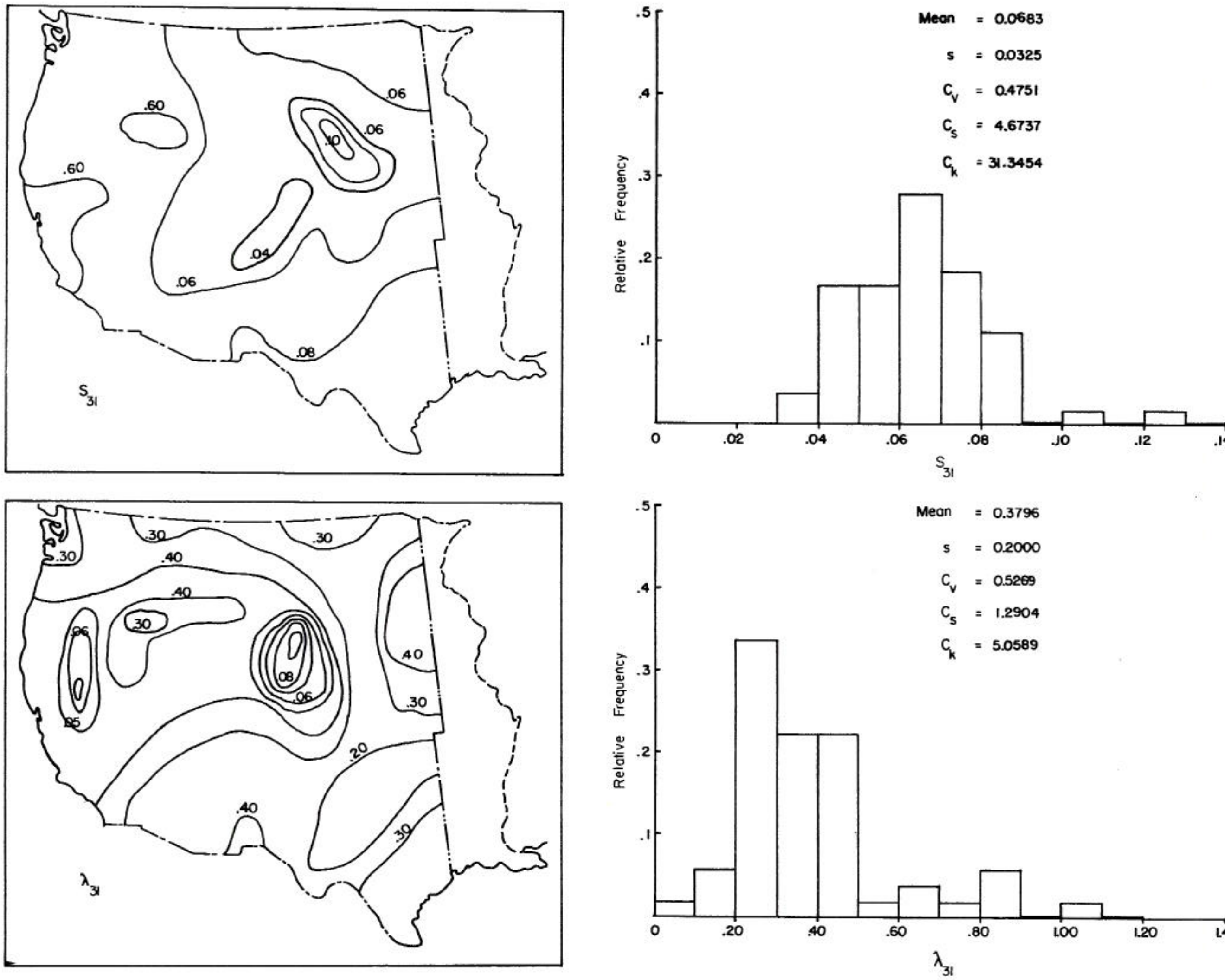

Fig. 25 Areal and frequency distributions of standard error of estimate, S_{31}, and of parameter λ_{31} for Model II.

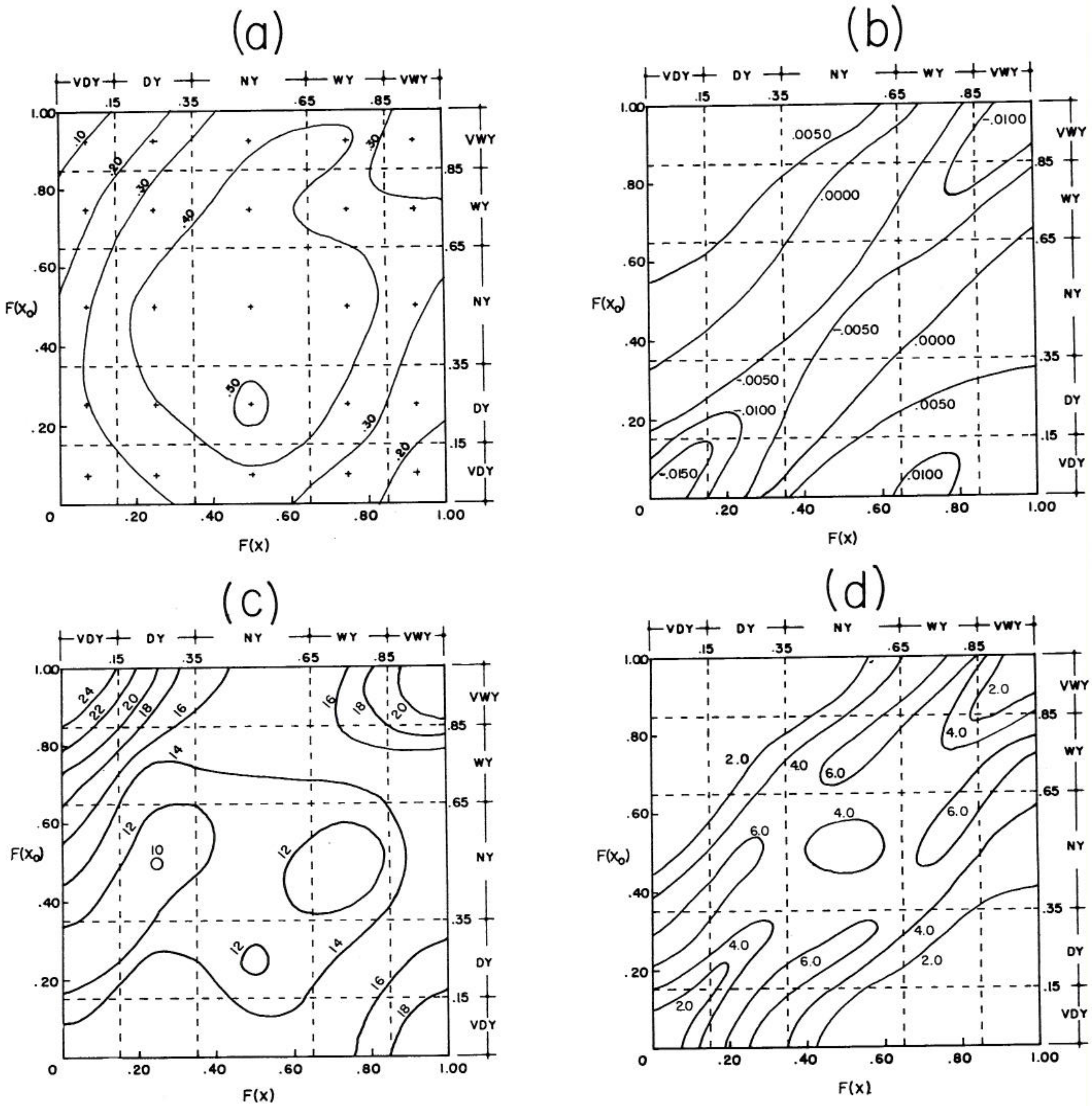

Fig. 26 Means or medians of parameters of mathematical models as functions of $F(x_o)$ and $F(x)$: (a) Model II, $\bar{\lambda}_{ij}$; (b) Model III, median of ν_{ij}; (c) Model III, median of α_{ij}; and (d) Model IV, $\bar{\eta}_{ij}$.

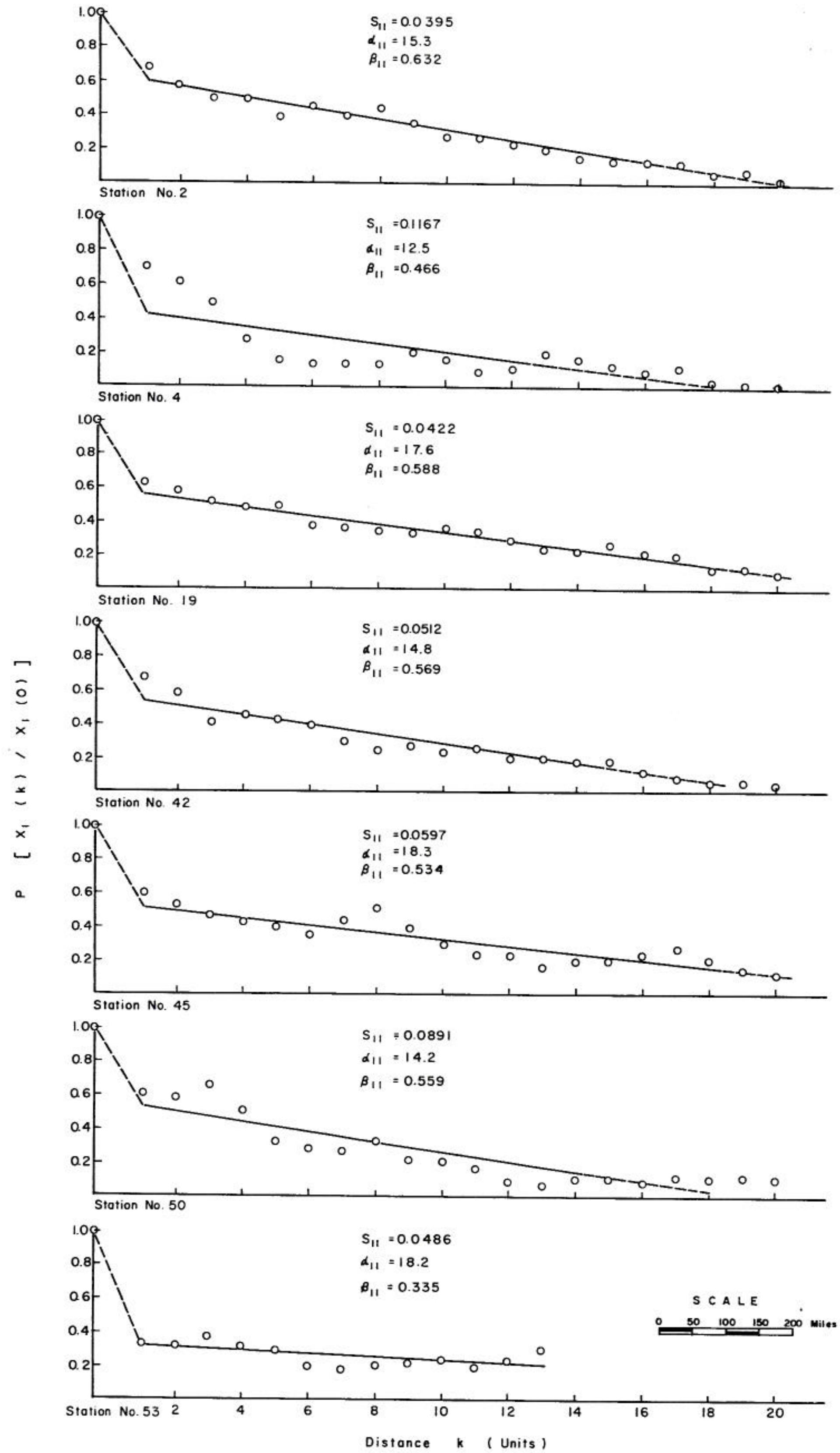

Fig. 27 Comparison of conditional probabilities of having a very dry year at the sub-areas at the distance k from the central sub-area, given a very dry year at the central sub-area, with those of Model III. Points are the computed conditional probabilities: the solid line represents Model III.

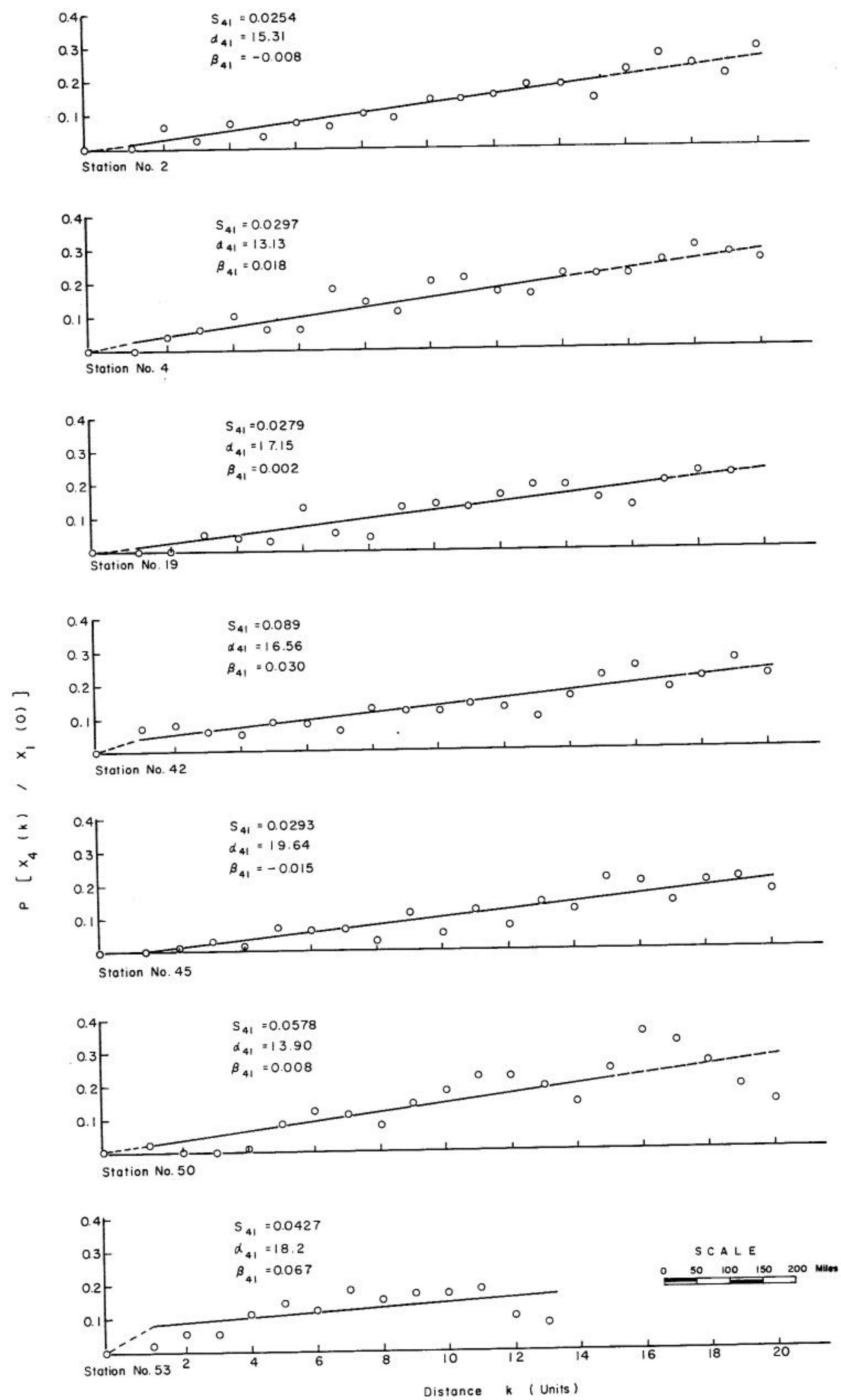

Fig. 28 Comparison of conditional probabilities of having a wet year at the sub-areas at the distance k from the central sub-area, given a very dry year at the central sub-area, with those of Model III. Points are the computed conditional probabilities: the solid line represents Model III.

Parameter α_{ij} is the distance from the central sub-area to the sub-areas in which their occurrence is independent of the occurrence of the central sub-area. The range of α_{ij} is very large. The median of α_{ij} was used in this study to eliminate the effect of extreme values and is presented in fig. 26 c as a function of $F(x_o)$ and $F(x)$. Values of α_{ij} range from 10 to 24 units or approximately 500 to 1200 miles. Larger values were found for the cases in which given a very wet year the occurrence of a very dry year is to be described or vice versa. It means that there is a high degree of dependence between a very dry year and a very wet year over the area. This statement was confirmed by the correlation coefficient which is -0.6118 between the percents of the sub-areas having very dry and very wet years.

9. Model IV: Hyperbolic dependence. Mathematical Model IV, as given by eq. (15), was used to fit the observed data and its goodness of fit was measured by the standard error of estimate defined by eq. (18). The parameter of the Model IV was estimated by a least square fitting. Since the normal equation of the least squares, eq. (17) is not linear, the estimated parameter η_{ij} was obtained by the graphical method with minimum standard error of estimate. A CDC 3600 computer was used to avoid the graphical method. Given a value of η_{ij}, the standard error of estimate was computed by eq. (17). One hundred and fifty values of η_{ij} ranging from 0.00 to 7.45 were used. In this way, the estimated parameter η_{ij} with minimum standard error of estimate was obtained.

Two examples of the observed data being fitted by Model IV are presented in figs. 29 and 30. The means of the standard error of estimate, $\overline{S}_{ij}$, range from 0.0320 to 0.0910 and are presented in fig. 22 d. It was found that this model fitted the observed data quite well except for the following cases: (1) Given a very dry year, then to describe a very dry or dry year, or vice versa; (2) Given a very wet year, then to describe a wet or very wet year, or vice versa. The coefficients of variation, C_v, of S_{ij} range from 0.251 to 0.612, and are presented in fig. 23 d. In most cases, the average coefficient of variation of S_{ij} is 0.300, except for the following cases: given a dry, normal or wet year, then to determine the probabilities of a very dry or a very wet year.

The mean parameter, $\overline{\eta}_{ij}$, of the mathematical model is a measure of the degree of dependence between sub-areas and is presented in fig. 26 d. If $\overline{\eta}_{ij}$ is larger (greater than 4.0) then Model IV becomes Model I (independence model). The cases which have $\overline{\eta}_{ij} > 4.0$ and $\overline{S}_{ij} < 0.06$ are: (1) Given a dry year, then to describe a very dry or normal year; (2) Given a normal year, then to describe a dry or wet year; (3) Given a wet year, then to describe a very wet or normal year. These results confirm the conclusions made from the studies of Model I.

10. Mean conditional probability of wet and dry years. The mean conditional probability denoted by $\overline{p}_{ij}(k)$ is defined as:

$$\overline{p}_{ij}(k) = \frac{p^1_{ij}(k) + p^2_{ij}(k) + \cdots + p^n_{ij}(k)}{n} \qquad (21)$$

where $p^1_{ij}(k)$, $p^2_{ij}(k)$, ..., $p^n_{ij}(k)$ are the conditional probabilities $p_{ij}(k)$ given X_j at the sub-area 1, at the sub-area 2, ..., at the sub-area n, respectively.

Means, standard deviations and coefficients of variation of $p_{ij}(k)$ for fifty-four central sub-areas, were computed and are presented in Appendix C, tables 2, 3 and 4, respectively. The maximum standard deviation of $p_{ij}(k)$ is 0.137. Standard deviations are excessive in the two following conditions: (1) When the distances k are small; for instance, 1 or 2 units (approximately 50 or 100 miles); (2) When the two categories of years under consideration are very different or opposite; for instance, given a very dry year then to describe a very wet year or vice versa. Most of the standard deviations range from 0.03 to 0.09 and are considered to be small. Therefore, the mean values are the best statistics to represent the conditional probability $p_{ij}(k)$ over the total area.

The four mathematical models were also fitted to the mean conditional probability, $\overline{p}_{ij}(k)$. Two examples of this fitting are presented in fig. 31. Mean values, $\overline{p}_{ij}(k)$, for fifty-four central sub-areas are much smoother than the values $p_{ij}(k)$ of individual sub-areas. This gave a better fit of the mathematical models to the observed data. However, the study of any individual central sub-area is still important because it presents information distributed over the whole area as well as its variations from one place to another. Comparisons between the mean values of the standard error of estimate and the parameters obtained from individual analysis of fifty-four central sub-areas and the values from the mean of fifty-four central sub-areas, are presented in Appendix C, tables 5 and 6, respectively.

11. Comparison of mathematical models. The comparison of the mathematical Models I, II, III and IV is based on the use of the mean standard errors of estimate, $\overline{S}_{ij} \leq 0.06$, and $\overline{S}_{ij} \leq 0.08$. The conditions dictating the use of these mathematical models depend upon the values of $F(x_o)$ and $F(x)$. The distributions $F(x_o)$ and $F(x)$ were introduced previously, with $F(x_o)$ being the probability of annual precipitation at any central sub-area, and $F(x)$ being the probability of annual precipitation at any sub-area at the distance k from the central sub-area. Conditions under which these four models are applicable were discussed earlier in this chapter. However, the general comparison can be made conveniently by the graphical presentations shown in fig. 32 a for $\overline{S}_{ij} \leq 0.06$ and fig. 32 b for $\overline{S}_{ij} \leq 0.08$. Each area in fig. 32 has numbers of those models among the four models, which fit well the computed

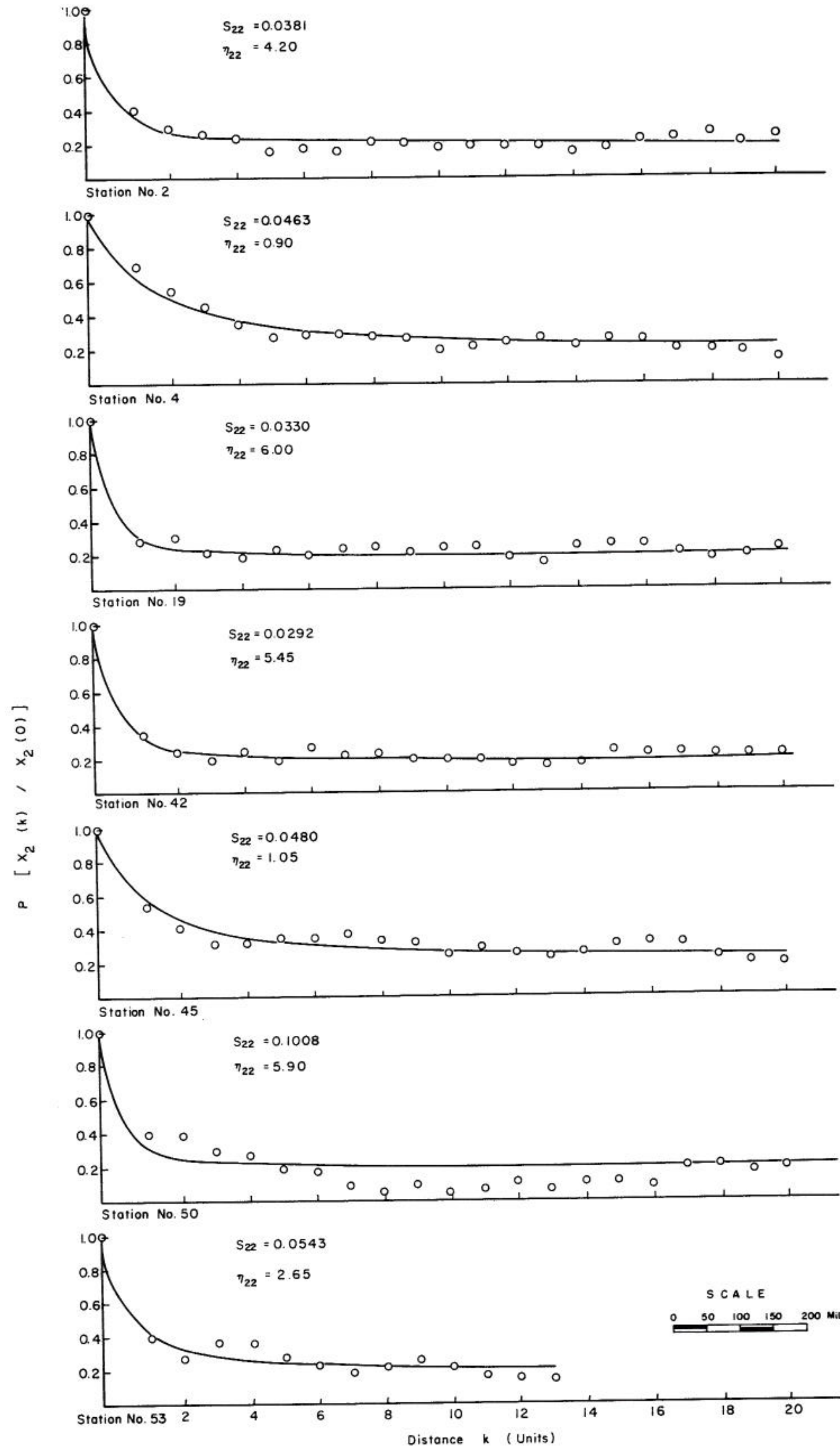

Fig. 29 Comparison of conditional probabilities of having a dry year at the sub-areas at the distance k from the central sub-area, given a dry year at the central sub-area, with those of Model IV. Points are the computed conditional probabilities: the solid line represents Model IV.

conditional probabilities under the criterion that $\overline{S}_{ij} \leq 0.06$ (fig. 32a) or $\overline{S}_{ij} \leq 0.08$ (fig. 32b). It is shown that Model III fits best the computed conditional probabilities in the occurrence either of very wet years or of very dry years on both sub-areas.

It is also shown that Model III fits well all cases, and the next best to this model is Model IV. Therefore, either a linear dependence or a hyperbolic dependence may be used in practically all cases, assuming that $\overline{S}_{ij}$ should be equal or less than 0.08.

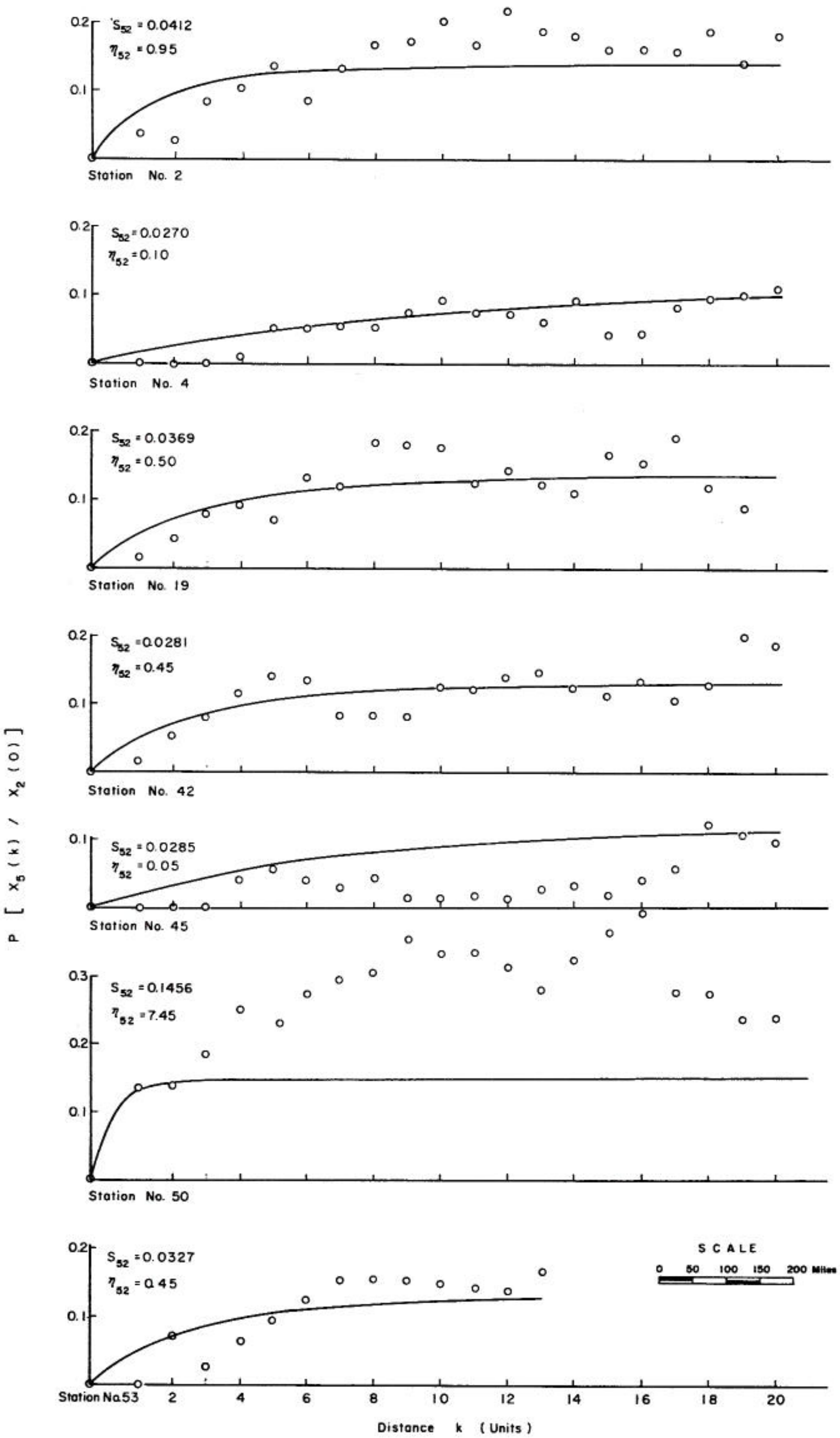

Fig. 30 Comparison of conditional probabilities of having a very wet year at the sub-areas at the distance k from the central sub-area, given a dry year at the central sub-area, with those of Model IV. Points are the computed conditional probabilities: the solid line represents Model IV.

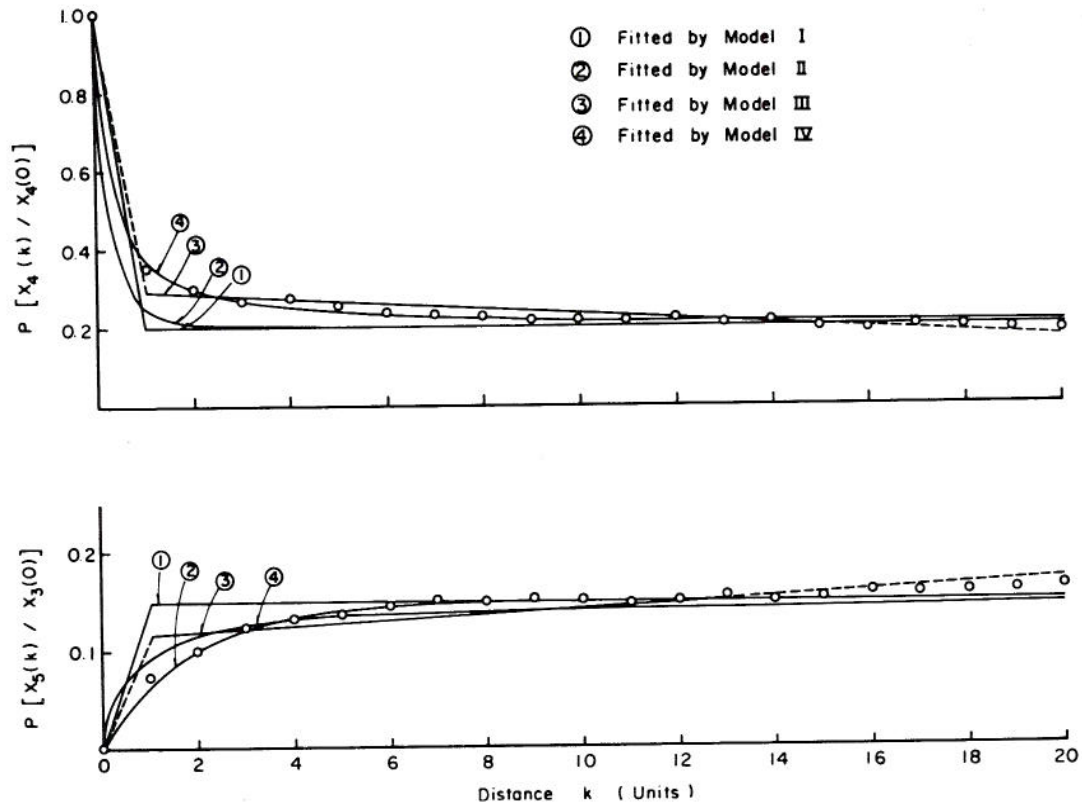

Fig. 31 Comparison of mean conditional probabilities of wet and dry years, $\overline{p}_{ij}(k)$, with those of Models I, II, III and IV. Points are the computed mean conditional probabilities: the solid lines represent Models I through IV.

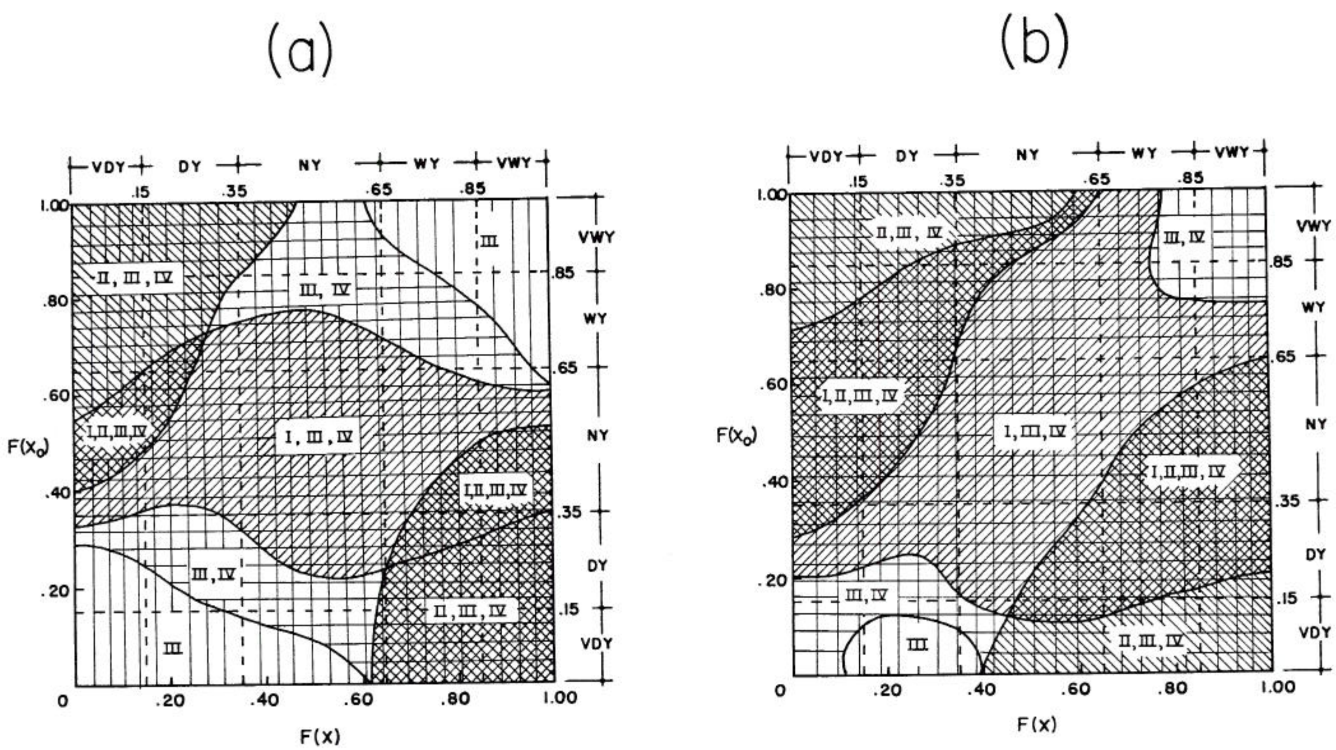

Fig. 32 Comparison of the goodness of fit of Models I, II, III and IV with various values of mean standard error of estimate being a function of $F(x_o)$ and $F(x)$: (a) $\overline{S}_{ij} \leq 0.06$; (b) $\overline{S}_{ij} \leq 0.08$.

CHAPTER IV

INVESTIGATION OF GOODNESS OF FIT OF MATHEMATICAL MODELS BY THE SECOND METHOD

This chapter describes the computation of conditional probabilities of having wet and dry years at stations or at points which are at a distance of k miles from the central station, given a wet or a dry year at the central station. The investigation of goodness of fit of mathematical models to computed conditional probabilities is the subject of this chapter.

1. Conditional probability of wet and dry years. By using the circular grid system, as shown in fig. 41 Appendix B, the conditional probability, $p_{ij}(k)$, of having X_i at the distance k from the central station, given X_j at the central station, is

$$p_{ij}(k) = \frac{\sum_{l=1}^{m} N_{ki}}{m \quad N_{kj}} \qquad (22)$$

where N_{ki} is the number of years of having X_i of a station at the distance k, m is the total number of stations being in the area at the distance k, and N_{kj} is the number of years at the central station having X_j.

All conditional probabilities for X_i occurring at the distance k from the central station, given X_j at the central station, were computed on a CDC 3600 computer. Four examples are shown in fig. 33, and each of them for two cases, p_{22} and p_{32}. The upper graph gives $p_{22}(k)$ as a function of distance k, with four individual examples (1) to (4), and the average value of $p_{22}(k)$ for all central stations (average of 1141 precipitation stations), line (5). The lower graph gives $p_{32}(k)$ as a function of distance k, with four individual stations (1) to (4), and the average value of $p_{32}(k)$ for all central stations (1141), line (5). Points on lines (1) to (4) for small values of k have a larger fluctuation than for large values of k. This is mainly caused by the small sample errors. For instance, in the case of a given very dry year at the central station, there are only four years out of a 30 year period that are very dry. At a distance of less than 100 miles the average number of stations used in computing the conditional probability is less than ten. This means that the sample size for this case of small k is less than 40. In

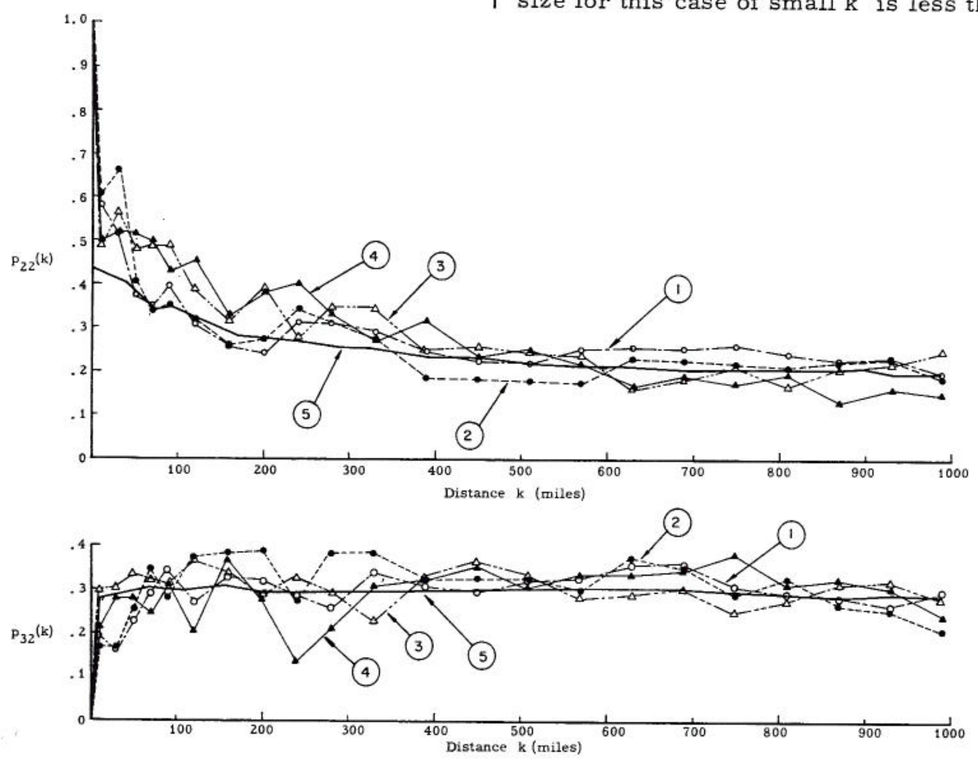

Fig. 33 Conditional probabilities of a dry year (upper graph) and a normal year (lower graph) at the distance k, given a dry year at a central station: (1) central station, Seward, Nebraska; (2) central station, Utica, Nebraska; (3) central station, Acton Escondido, California; (4) central station, Riverside Fire Station No. 3, California; and (5) Average conditional probabilities for all 1141 precipitation stations used in this study.

some cases, only one or two stations were present in the vicinity of the central stations. To avoid this large dispersion of points both for small values of k and large values of k, the mean conditional probability, $\overline{p}_{ij}(k)$, of all central stations (1141 in number) is used to represent the conditional probabilities of the continental area of Western North America.

The mean conditional probability, $\overline{p}_{ij}(k)$, of having X_i at the distance k from the central station, given X_j at the central station, is defined by

$$\overline{p}_{ij}(k) = \frac{p^1_{ij}(k) + p^2_{ij}(k) + \cdots + p^n_{ij}(k)}{n} \qquad (23)$$

where $p^1_{ij}(k)$, $p^2_{ij}(k)$, ... , $p^n_{ij}(k)$ are the conditional probabilities, $p_{ij}(k)$, of having X_i at the distance k, given X_j at the central station 1, at the central station 2, ... , at the central station n, respectively, with n the total number of stations (1141). The two lines (5) of fig. 33 give two examples of $\overline{p}_{ij}(k)$ as a function of distance.

From the definitions of wet and dry years, as given in Chapter I, a year was classified into one of five categories. Since each category at one station can be considered with any of the five categories occurring at another station, there are twenty-five combinations of the conditional probabilities of wet and dry years between the two stations with a distance k. These combinations can be represented by a 5 x 5 matrix. This matrix was transformed into a continuous form in such a way that F(x) is a probability distribution function of annual precipitation at any station, as defined by eq. (1). If the annual precipitation at a station is between x_1 and x_2 so that $F(x_1) > 0.35$ and $F(x_2) \leq 0.65$, then that year is classified as normal. The distribution function F(x) is considered as being $F(x_o)$ for any central station and F(x) for any other station which is at the distance k from the central station. The mean conditional probabilities of wet and dry years for various distances k were computed on a CDC 3600 computer. The results are presented in table 7, Appendix C. Their statistics, such as standard deviation and coefficient of variation, are presented in tables 8 and 9, Appendix C. Four examples of joint distribution of $F(x_o)$ and F(x) at various distances k, are given in fig. 34 which can be interpreted in the following way. If the distance between two stations is less than 100 miles, as shown in fig. 34 a, the conditional probabilities of the same category of years (such as wet year at central station and wet year at a station with the distance k from the central station, or other four same categories) range from 0.40 to 0.50. The larger the difference between $F(x_o)$ and F(x), - say as wet year at the central station and dry year at the station with the distance k - the smaller become the conditional probabilities. These joint probabilities of $F(x_o)$ and F(x) are nearly zero when the two categories are opposite, such as very dry and very wet years, or vice versa. Figure 34 shows the joint distribution of $F(x_o)$ and F(x) at the distance of 200 miles. The conditional probabilities of a normal year at stations at an approximate distance of 450 miles from the central station, given any of five categories of a year at the central station are approximately the probabilities of independent joint distributions. They are shown in fig. 34 c. At a distance of approximately k = 930 miles, the following conditions exist: (1) the joint probability which is represented by the 0.30 - isoline passes vertically through F(x) = 0.50, as shown in fig. 34 d; (2) the 0.20 - isoline passes vertically at approximately F(x) = 0.25 and F(x) = 0.75, as shown in fig. 34 d; and (3) the 0.15 - isoline passes vertically at approximately F(x) = 0.075 and F(x) = 0.925, as shown in the same figure. These results indicate that all twenty-five cases of conditional probabilities of wet and dry years equal the probabilities of independent joint distributions of $F(x_o)$ and F(x). In other words, the occurrences of wet and dry years at that distance are approximately independent of the occurrence of wet and dry years at the central station.

2. Fitting mathematical models to observed data. The four mathematical models, described in Chapter II, were fitted to the computed mean conditional probabilities as a function of distance. The goodness of fit of these models was measured by the standard error of estimate as defined by eq. (18). A presentation of the standard error of estimate of those fits for the twenty-five conditional probabilities, with a similar presentation as in fig. 34, is shown in fig. 35.

Model I. The goodness of fit of this model is shown in fig. 35 a. (1) Given a very dry year or a very wet year, the conditional probabilities of having any of the five categories of years can be described by Model I, with a standard error of estimate greater than 0.08; (2) Given a dry year or a wet year the probabilities of having any of the five categories of years can be described by Model I with standard errors of estimate between 0.04 and 0.08; (3) Given a normal year, the probabilities of having any of the five categories of years, except for a normal year, can be described by Model I with the standard error of estimate between 0.02 and 0.06.

Model II. The goodness of fit of this model is shown in fig. 35 b. (1) Given a very dry year or a very wet year, the conditional probabilities of having any of the five categories of years, except the same categories of given years, can be described by Model II with standard errors of estimate between 0.02 and 0.06; (2) Given a dry year or a wet year, the conditional probabilities of having any of the five categories of years can be described by Model II with standard errors of estimate between 0.02 and 0.06; (3) Given a normal year, the conditional probabilities of having any of the five categories of years, except for a normal year, can be described by Model II with a standard error of estimate of less than 0.02.

Model III. The goodness of fit of this model is shown in fig. 35 c. The standard errors of estimate of Model III range from 0.01 to 0.05 for all cases. This model may be applied to all cases in describing the occurrences of wet and dry years, regardless of the type of year.

Model IV. The goodness of fit of this model is shown in fig. 35 d. The standard errors of estimate of Model IV range from 0.01 to 0.05 for twenty-one cases. The other four cases (very dry and either very dry or dry; very wet and either very wet or wet) have standard errors of estimate between 0.06 to 0.08. This model is considered to be as accurate as Model III, except for those extreme cases.

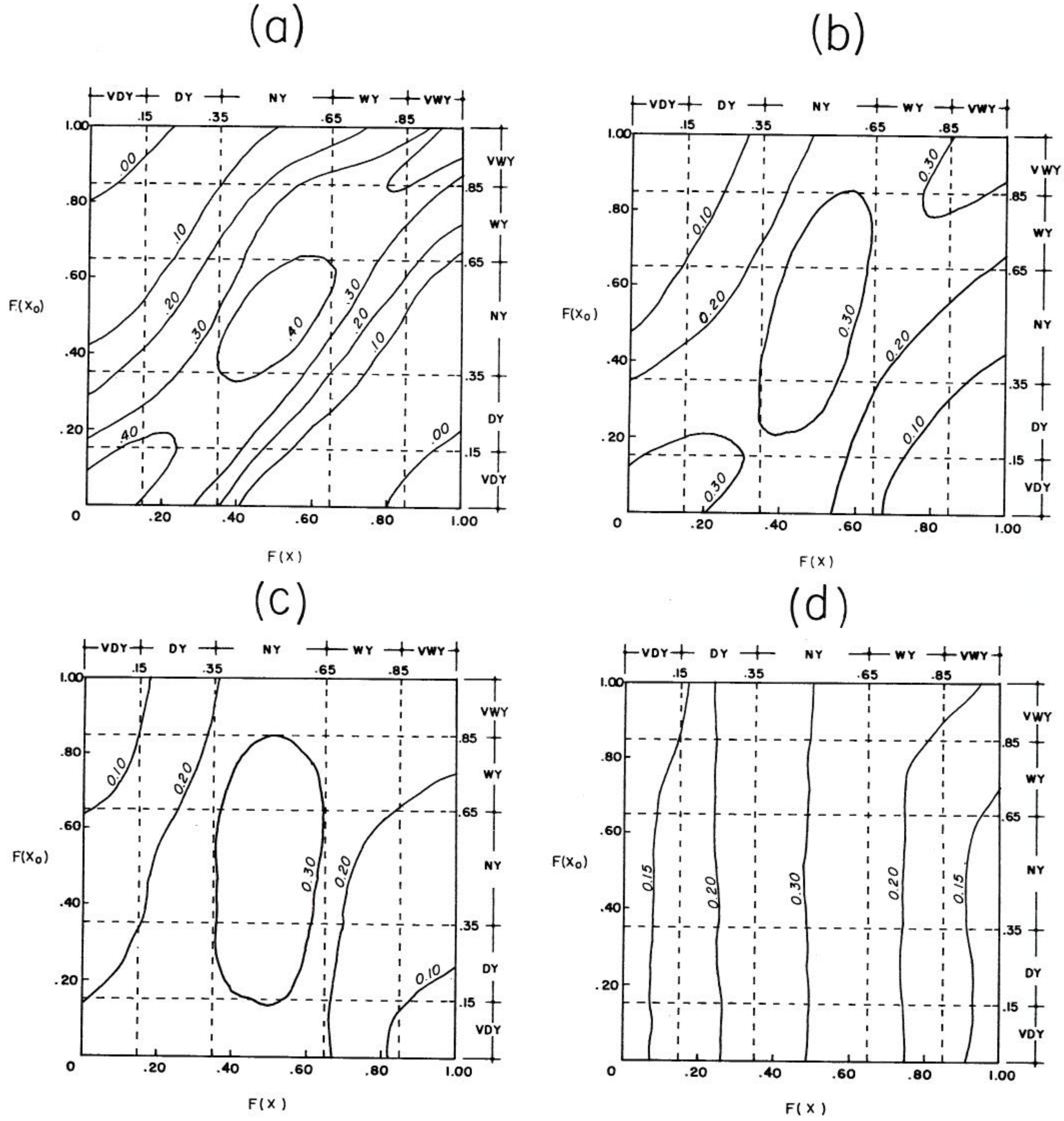

Fig. 34 Joint probability distribution function of probabilities $F(x_0)$ and $F(x)$ for various distances k: (a) k = 50 miles; (b) k = 200 miles; (c) k = 450 miles; and (d) k = 930 miles.

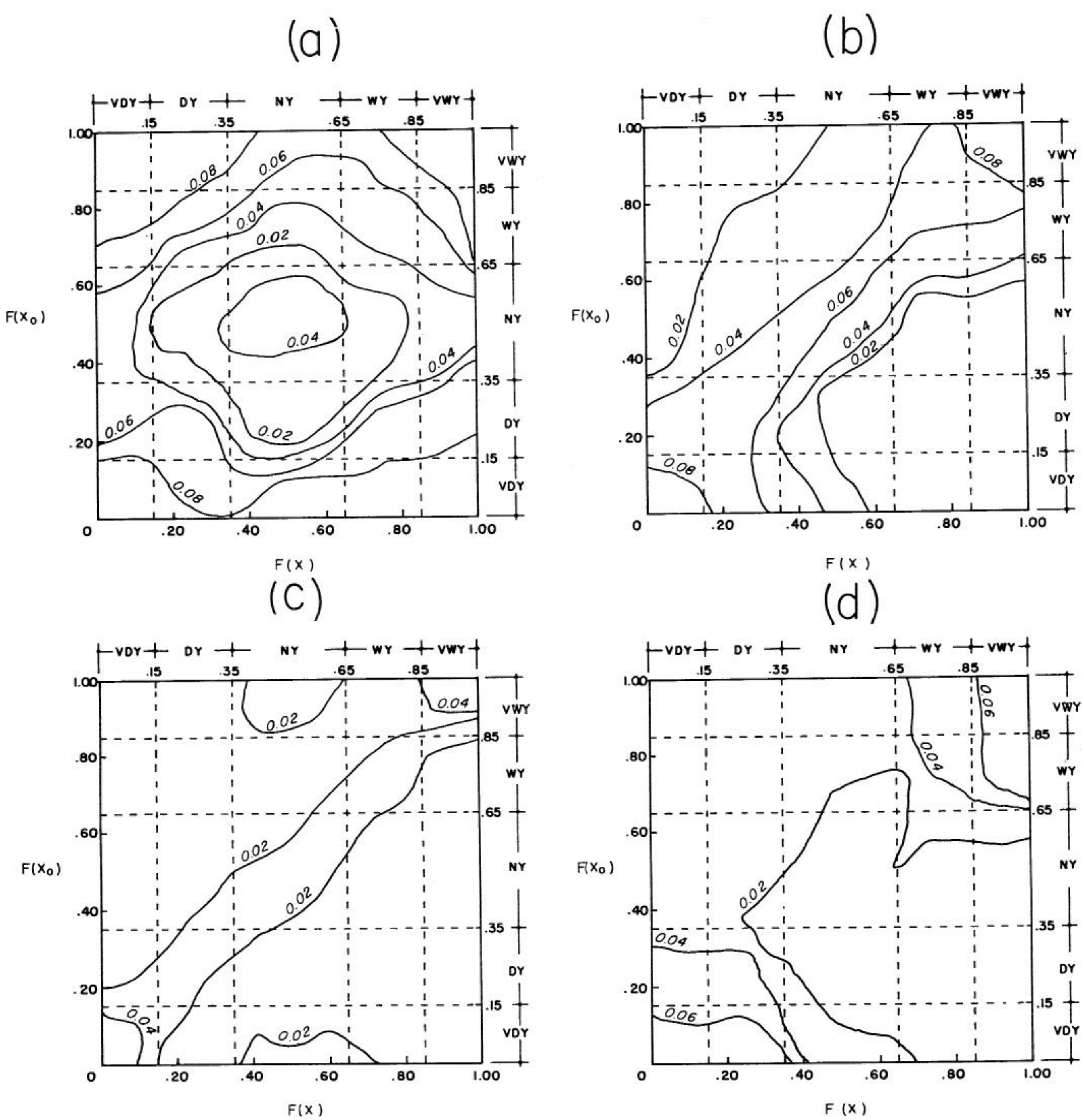

Fig. 35 Standard error of estimate of the four mathematical models as a function of $F(x_o)$ and $F(x)$: (a) Model I; (b) Model II; (c) Model III and (d) Model IV.

3. <u>Significance of parameters in mathematical models.</u> It was found that the number of parameters of models play an important role in the applicability of a model. Model III has two parameters and fits better the computed conditional probabilities than do Models I, II and IV. Model II, which has one parameter, fits better than Model I. Special cases of any model with two parameters are models with only one or no parameter. For instance, Model III becomes Model I when the parameter β_{ij} is equal to p_i; Model II becomes Model I when the parameters λ_{ij} is very large.

Parameters λ_{ij} of Model II has a high range of values. The range of λ_{ij} is shown in fig. 36 a. For a great difference of $F(x_o)$ and $F(x)$ the values are usually between 0.002 and 0.020. For small differences of $F(x_o)$ and $F(x)$ the range is large and in the neighborhood of 0.010 to 3.000. For $\lambda_{ij} \geq 0.010$ Model II becomes approximately the same as Model I.

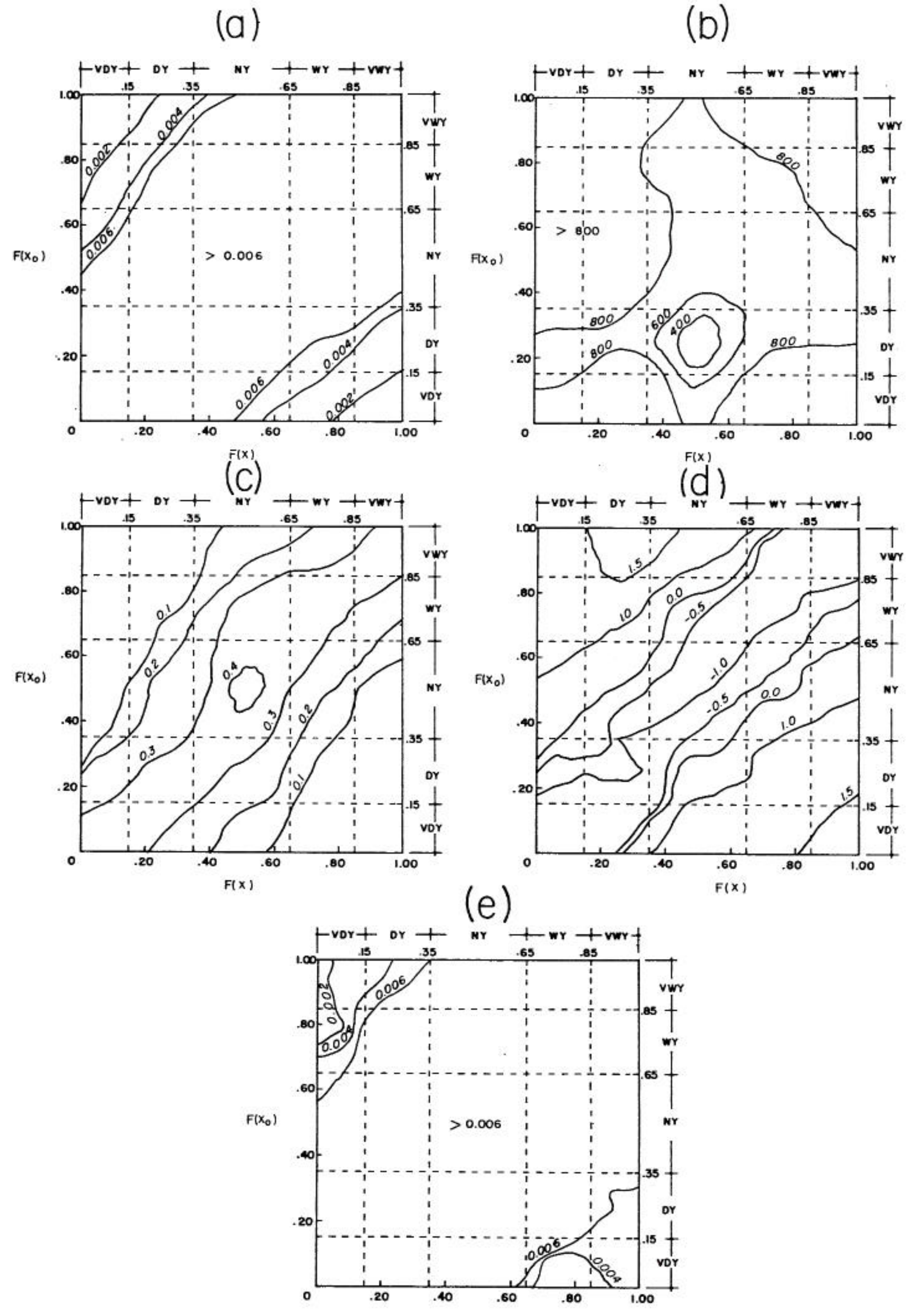

Fig. 36 Parameters of the four mathematical models as a function of $F(x_o)$ and $F(x)$: (a) Model II, λ_{ij}; (b) Model III, α_{ij}; (c) Model III, β_{ij}; (d) Model III, ν_{ij}; and (e) Model IV, η_{ij}.

Parameter α_{ij}, measured in miles, of Model III is shown in fig. 36 b; and is approximately 800 miles for most cases. This means that at a distance of approximately 800 miles the conditional probabilities of occurrences of wet and dry years between two stations are probabilities of independent joint distributions. Parameter β_{ij} of Model III is given in fig. 36 c. It shows a variation from 0.2 to 0.4. Parameter ν_{ij} of Model III, which is a combination of parameters α_{ij} and β_{ij} of this model, represents the slope of the linear mathematical model. The values of ν_{ij} are given in fig. 36 d. When ν_{ij} is small, from 0.000002 to 0.000026, Model III is equivalent to Model I. These results confirm the conclusions inferred from the investigation of Model I.

Parameter η_{ij} of Model IV is given in fig. 36 e. It has a high range of values, from 0.010 to 700. The large values, 600 to 700, are in the center of the graph and are not presented in fig. 36 e. An exception to the foregoing statement occurs when two categories of years are the two extremes. In this situation the values of ν_{ij} range from 0.001 to 0.010. Model IV becomes Model I when ν_{ij} is greater than 0.010.

4. Comparison of mathematical models. A comparison of mathematical Models I, II, III and IV is based on the values of the standard error of estimate. For instance, at the standard error of estimate less or equal to 0.04 the conditional probabilities of having a normal year, given a dry year, can be described by either Model III or Model IV. The conditions dictating the use of these mathematical models depend upon the two categories of years which forms the conditional probability. This section discusses the conditions under which these four models may be applied. However, the general comparison of these models is illustrated in fig. 37 a for $S_{ij} \leq 0.04$ and fig. 37 b for $S_{ij} \leq 0.06$.

Figure 38 illustrates the mathematical model best suited to each of the twenty-five discrete cases of the combinations of $F(x_o)$ and $F(x)$. Model I, or $F(x)$ is independent of $F(x_o)$, and is not recommended for any case. Model II is well applicable for five cases. Model III is applicable for fourteen cases, and Model IV is applicable for six cases. A summary of recommended models, their goodness of fit, and their parameters are presented in fig. 38. Five examples of goodness of fit are given in fig. 39.

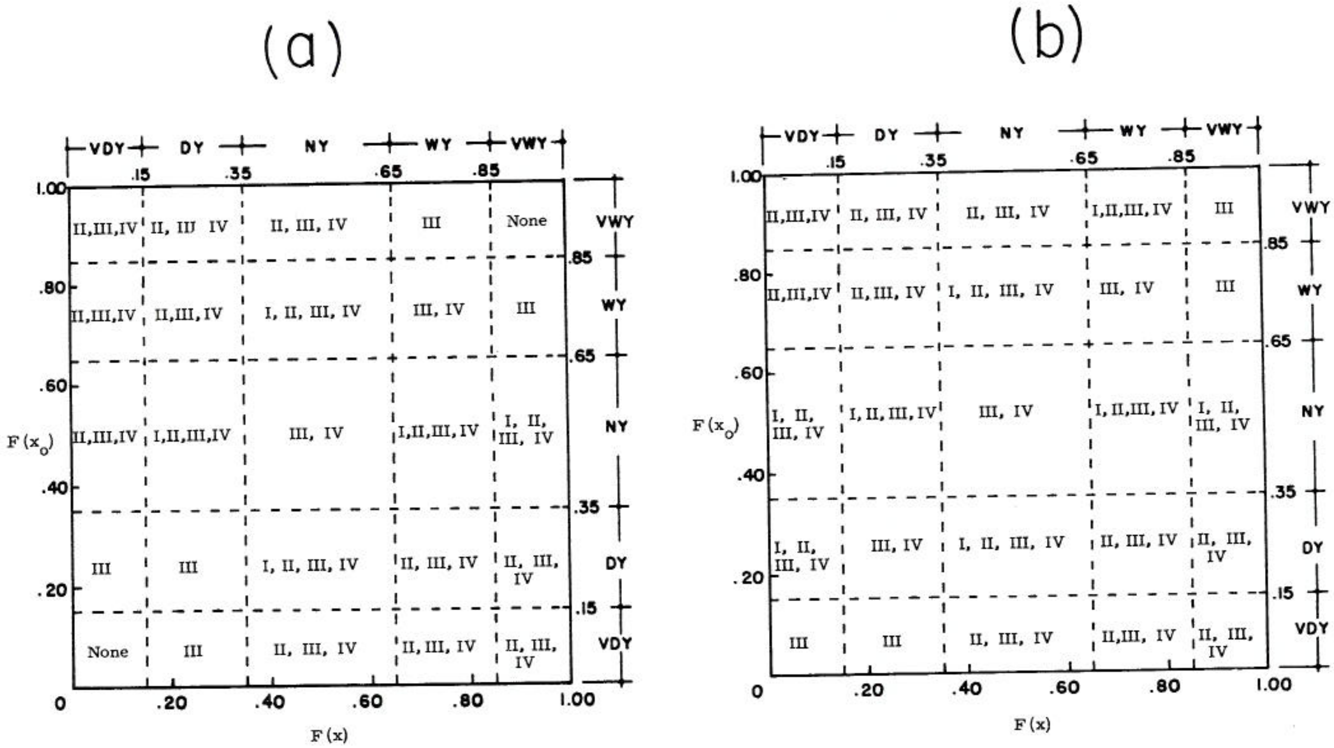

Fig. 37 Comparison of Models I, II, III and IV, with various values of standard error of estimate being functions of $F(x_o)$ and $F(x)$: (a) $S_{ij} \leq 0.04$; and (b) $S_{ij} \leq 0.06$.

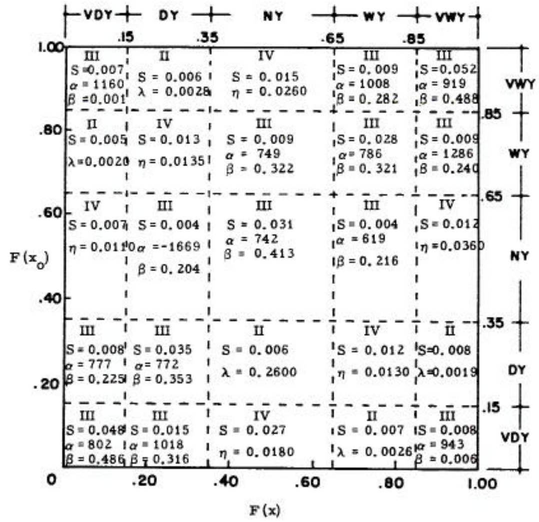

Fig. 38 Standard error of estimate and the parameters of recommended mathematical models for the studied twenty-five cases.

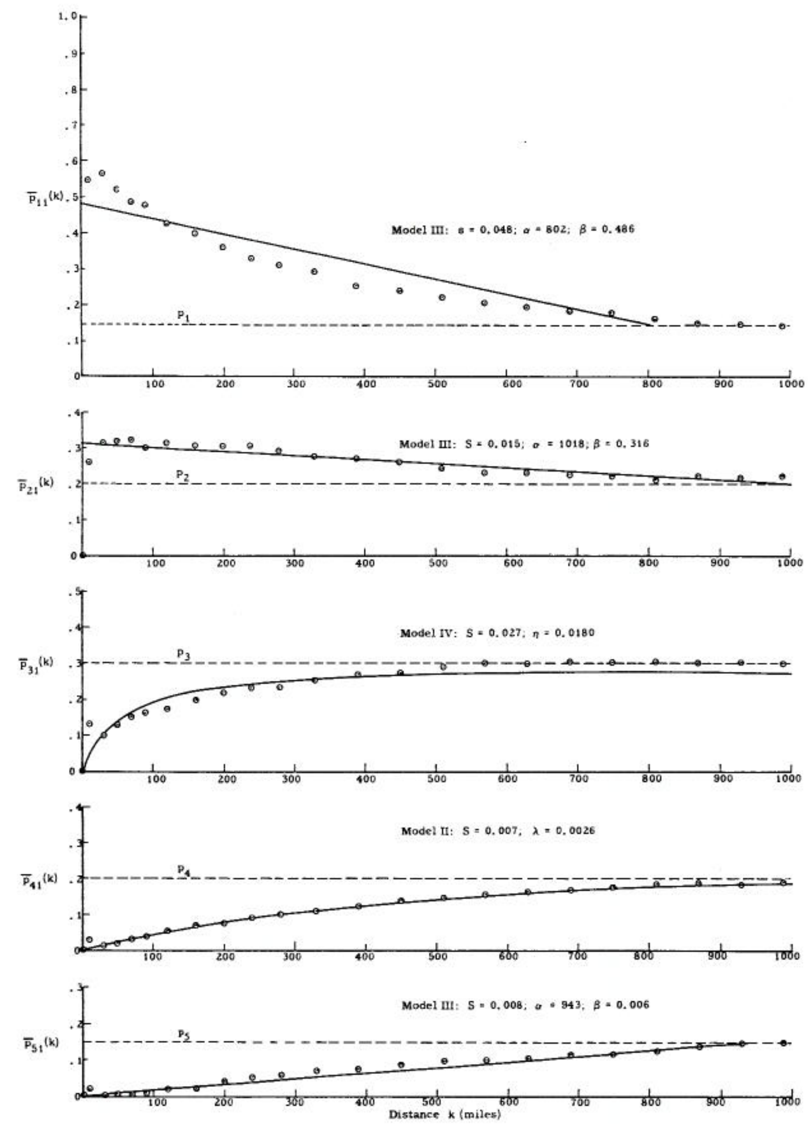

Fig. 39 Five cases of mean conditional probabilities, $\overline{p}_{ij}(k)$, fitted by the recommended mathematical models.

CHAPTER V

CONCLUSIONS

This study was conducted to find the conditional probability mathematical functions to describe the occurrence of wet and dry years over an area for any given year. Two different methods of investigation of mathematical models were used. First, average annual precipitation of the sub-areas were obtained, and these average or representative values of sub-areas were used in computing conditional probabilities of a given type of year occurring at a sub-area of k unit distance from a central sub-area given a type of year at this central sub-area. This investigation is designated as the first method. Second, the annual precipitation on the individual stations were used as the basic data in computing the conditional probabilities. The conditional probabilities for a given type of year, occurring at the stations which are at a distance k miles from a central station, given a type of year at the central station, were computed for a total of 1141 precipitation stations, and their conditional probabilities were averaged. This approach is designated as the second method of investigation.

The following conclusions were drawn from the results:

1. Mathematical Model I, being the independence model, with the hypothesis that the occurrences of wet and dry years over the sub-areas or stations are independent, is rejected for the application.

2. Mathematical Model II, being the exponential dependence model, is based on the hypothesis that the probability of occurrence of a wet or a dry year in one sub-area or at one station is dependent upon the occurrence of a wet or a dry year in the adjacent sub-areas or stations, respectively. The degree of dependence decreases as the distance between the sub-areas or stations increases. This dependence asymptotically converges to the independent case with the distance. The conditional probabilities of wet and dry years as a function of distance are described in this model by the exponential function. The model was found to be the best applied to five cases out of the twenty-five.

3. Mathematical Model III is the linear dependence for the occurrence of a wet or a dry year at two sub-areas or stations. Model III is considered to be the best mathematical model to describe the occurrence of wet and dry years over an area, but especially for fourteen cases out of the twenty-five.

4. Mathematical Model IV is a hyperbolic dependence model. This mathematical function was inferred by a graphical fit of the hyperbolic function to observed data. Model IV is recommended when the two categories of years of the two sub-areas or two stations are very different but not opposite, such as normal and either very dry or very wet, or vice versa. This model was found to be the best applied to six cases out of the twenty-five.

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APPENDIX A

GRID SYSTEM USED FOR COMPUTATION OF CONDITIONAL PROBABILITIES AS A FUNCTION OF DISTANCE BY USE OF THE FIRST METHOD

1. Grid system. Let Z_o and Y_o be respectively the longitude and latitude of the center of a central sub-area. A circular grid system is superimposed on the synoptic maps of wet and dry years. The center of the grid system is at the center of the central sub-area as shown in fig. 40. The circular area around the central sub-area with a radius r_o is called the "sub-area under consideration" having X_j. The area around this sub-area under consideration, which is bounded by the radius r_1, is called the "1 - unit - distance area." Similarly, the area bounded by the radii r_1 and r_2 is called the "2 - unit - distance area." The "k - unit - distance area" is the area bounded by the radii r_{k-1} and r_k. If r_{k-1} is given, r_k is determined so that the k - unit distance area is divided equally into N sub-areas A_o with the increment of the radial distance, denoted by Δr, being approximately constant.

2. Centroid of a sub-area. From the definition of the terms used in the grid system and the method of dividing the k - unit - distance area into equal sub-areas, the following equations are derived:

$$r_1 = r_o + \Delta r$$

$$r_k = r_o + k\Delta r$$

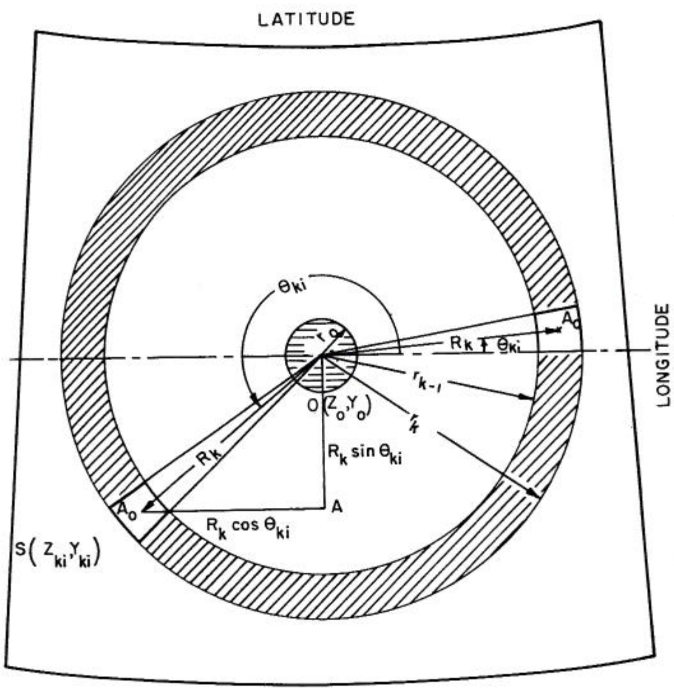

Fig. 40 Circular grid system used for computation of conditional probabilities of wet and dry years as a function of distance, by the use of the first method.

And

$$\pi (r_k^2 - r_{k-1}^2) = A_o N. \tag{24}$$

Substituting r_k and r_{k-1} in eq. (24),

$$N = \frac{\pi \Delta r}{A_o} (2 r_o + 2k \Delta r - \Delta r). \tag{25}$$

In finding the exact value of r_k, eq. (24) gives

$$r_k = \sqrt{\frac{A_o N}{\pi} - r_{k-1}^2} \tag{26}$$

where N is an integer computed from eq. (25).

The controid of the i-th area A_o is R_k and θ_{ki} as shown in fig. 40. R_k is the distance from the central sub-area to A_o and θ_{ki} is the angle measured counter-clockwise from the horizontal axis. By approximation R_k is

$$R_k = \frac{r_k + r_{k-1}}{2} \tag{27}$$

and

$$\theta_{ki} = i \frac{2\pi}{N} - \frac{\pi}{N} \tag{28}$$

where $i = 1, 2, \ldots, N$.

In this study, the central sub-area has a radius r_o of 50 miles, a sub-area is 2000 square miles, and the increments of the radial distance are approximately 50 miles.

The computation of longitude and latitude of the i-th area A_o is as follows: Let O be the center of a central sub-area with longitude Z_o and latitude Y_o. S is the center of a sub-area A_o at the distance R_k from O and the angle of direction θ_{ki}, as shown in fig. 40. For comparatively short distances, not greater than 1000 miles, the triangle SOA (in fig. 40) can be considered as a plane triangle. The degree of error introduced by this approximation is much less than the inaccuracies in the data used and the method of finding the areal distribution of wet and dry years. Kells and others (1942, p. 61) conceived a method for converting the distances along latitudinal lines to differences in longitude by the following expressions:

$$Y_{ki} = Y_o + \frac{R_k \sin\theta_{ki}}{60} \tag{29}$$

and

$$Z_{ki} = Z_o - \frac{1}{60} R_k \cos\theta_{ki} \sec\frac{\pi}{180} \cdot \frac{1}{2}(Y_o + Y_{ki}) \quad (30)$$

where Y_{ki} and Z_{ki} are respectively the latitude and longitude of the controid of sub-area at k-unit distance from the central sub-area.

APPENDIX B

GRID SYSTEM USED FOR COMPUTATION OF CONDITIONAL PROBABILITIES AS A FUNCTION OF DISTANCE BY USE OF THE SECOND METHOD

1. Grid system. Let Z_o and Y_o be the longitude and latitude of the central station, respectively. The method of computing the conditional probabilities of having a wet or a dry year at the distance k from the central station, given a wet or a dry year at this station is as follows: A circular grid system is superimposed on the area with its center located at the central station (as shown in fig. 41). The area of distance k is defined as the area of the ring between two circles which have a radius of R_1 and R_2, one of which encloses the other, where

$$k = \frac{R_1 + R_2}{2} \quad (31)$$

Consider the i-th station whose longitude and latitude are Z_i and Y_i, respectively. The distance between the central station and the i-th station was computed by the following expressions:

$$Y_i = Y_o + \frac{R_i \sin\theta_i}{60} \quad (32)$$

and

$$Z_i = Z_o - \frac{1}{60} R_i \cos\theta_i \sec\frac{\pi}{360}(Y_o + Y_i) \quad (33)$$

By solving eqs. (32) and (33)

$$\theta = \tan^{-1}\left[\frac{(Y_i - Y_o)\sec\frac{\pi}{360}(Y_o + Y_i)}{Z_o - Z_i}\right] \quad (34)$$

and

$$R_i = \frac{Y_i - Y_o) 60}{\sin\theta_i} \quad (35)$$

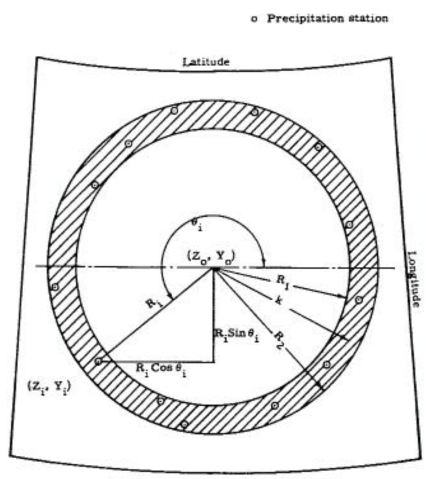

Fig. 41 Circular grid system used for computation of conditional probabilities of wet and dry years as a function of distance, by the use of the second method.

APPENDIX C

TABLES

(see following pages)

TABLE 2 - MEAN CONDITIONAL PROBABILITY $p_{ij}(k)$, COMPUTED BY THE FIRST METHOD

i	j	k (unit distance*) 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0.543	0.474	0.415	0.363	0.309	0.309	0.271	0.259	0.238	0.234	0.220	0.207	0.188	0.184	0.171	0.152	0.147	0.123	0.115	0.103
2	1	0.308	0.313	0.311	0.307	0.295	0.282	0.269	0.247	0.247	0.233	0.236	0.237	0.242	0.228	0.217	0.219	0.207	0.215	0.212	0.212
3	1	0.126	0.163	0.192	0.220	0.228	0.232	0.269	0.283	0.288	0.298	0.295	0.290	0.293	0.298	0.301	0.297	0.310	0.317	0.315	0.311
4	1	0.020	0.038	0.064	0.077	0.087	0.111	0.122	0.133	0.142	0.148	0.152	0.158	0.166	0.174	0.179	0.210	0.203	0.206	0.219	0.231
5	1	0.004	0.011	0.017	0.034	0.049	0.065	0.069	0.079	0.086	0.087	0.097	0.109	0.110	0.117	0.131	0.122	0.132	0.139	0.139	0.143
1	2	0.190	0.198	0.204	0.198	0.182	0.182	0.168	0.167	0.164	0.149	0.150	0.150	0.153	0.152	0.151	0.150	0.148	0.155	0.146	0.151
2	2	0.395	0.336	0.299	0.267	0.252	0.239	0.235	0.230	0.227	0.228	0.213	0.206	0.199	0.208	0.210	0.204	0.204	0.199	0.206	0.202
3	2	0.304	0.312	0.304	0.314	0.315	0.317	0.315	0.307	0.307	0.310	0.310	0.311	0.305	0.301	0.298	0.304	0.293	0.296	0.301	0.303
4	2	0.087	0.108	0.130	0.140	0.152	0.160	0.171	0.173	0.177	0.188	0.196	0.200	0.208	0.198	0.195	0.196	0.203	0.194	0.195	0.198
5	2	0.025	0.046	0.063	0.082	0.098	0.102	0.111	0.124	0.124	0.128	0.131	0.133	0.135	0.141	0.145	0.146	0.152	0.156	0.151	0.146
1	3	0.061	0.076	0.092	0.104	0.112	0.113	0.125	0.127	0.131	0.136	0.137	0.137	0.138	0.133	0.133	0.133	0.132	0.134	0.141	0.140
2	3	0.202	0.215	0.216	0.214	0.209	0.211	0.207	0.212	0.208	0.208	0.208	0.211	0.212	0.212	0.207	0.205	0.211	0.212	0.207	0.205
3	3	0.449	0.390	0.353	0.342	0.332	0.320	0.305	0.305	0.301	0.302	0.312	0.309	0.308	0.304	0.307	0.303	0.301	0.291	0.289	0.293
4	3	0.213	0.218	0.216	0.208	0.209	0.211	0.213	0.207	0.206	0.203	0.198	0.194	0.188	0.200	0.200	0.200	0.199	0.206	0.203	0.198
5	3	0.074	0.101	0.124	0.132	0.138	0.146	0.150	0.149	0.154	0.151	0.146	0.149	0.153	0.152	0.154	0.159	0.157	0.157	0.160	0.163
1	4	0.018	0.027	0.034	0.044	0.057	0.069	0.082	0.083	0.093	0.095	0.100	0.105	0.112	0.122	0.130	0.135	0.145	0.141	0.145	0.147
2	4	0.083	0.101	0.120	0.148	0.155	0.167	0.177	0.186	0.187	0.193	0.192	0.193	0.195	0.196	0.205	0.203	0.193	0.192	0.197	0.200
3	4	0.318	0.336	0.341	0.323	0.320	0.322	0.317	0.317	0.318	0.312	0.308	0.303	0.301	0.300	0.296	0.294	0.286	0.292	0.285	0.287
4	4	0.359	0.303	0.275	0.277	0.264	0.233	0.230	0.230	0.222	0.221	0.221	0.218	0.213	0.206	0.199	0.192	0.199	0.192	0.185	0.181
5	4	0.222	0.233	0.230	0.208	0.204	0.196	0.192	0.184	0.179	0.179	0.180	0.181	0.178	0.175	0.170	0.176	0.176	0.182	0.188	0.185
1	5	0.006	0.014	0.020	0.033	0.039	0.520	0.061	0.066	0.068	0.079	0.081	0.087	0.085	0.087	0.089	0.101	0.097	0.107	0.110	0.113
2	5	0.028	0.047	0.068	0.085	0.109	0.117	0.130	0.136	0.141	0.147	0.160	0.160	0.163	0.163	0.167	0.178	0.188	0.189	0.182	0.186
3	5	0.136	0.184	0.226	0.237	0.252	0.262	0.268	0.265	0.267	0.265	0.255	0.270	0.275	0.284	0.287	0.292	0.307	0.307	0.318	0.310
4	5	0.263	0.279	0.274	0.259	0.252	0.243	0.229	0.233	0.233	0.221	0.217	0.218	0.219	0.214	0.218	0.201	0.193	0.199	0.199	0.201
5	5	0.566	0.475	0.412	0.385	0.349	0.326	0.311	0.300	0.291	0.288	0.286	0.266	0.258	0.252	0.239	0.228	0.215	0.198	0.191	0.190

*1 unit distance ≃ 50 miles

TABLE 3 - STANDARD DEVIATION OF CONDITIONAL PROBABILITY $p_{ij}(k)$, COMPUTED BY THE FIRST METHOD

i	j	k (unit distance *) 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0.138	0.132	0.119	0.110	0.103	0.100	0.086	0.093	0.087	0.081	0.086	0.084	0.081	0.075	0.079	0.075	0.074	0.066	0.068	0.067
2	1	0.092	0.079	0.065	0.061	0.055	0.045	0.054	0.059	0.051	0.062	0.056	0.045	0.056	0.042	0.043	0.051	0.049	0.063	0.057	0.073
3	1	0.093	0.079	0.074	0.078	0.070	0.069	0.073	0.069	0.056	0.069	0.064	0.060	0.056	0.048	0.056	0.059	0.067	0.060	0.053	0.063
4	1	0.031	0.044	0.049	0.047	0.044	0.047	0.051	0.049	0.037	0.044	0.047	0.054	0.055	0.057	0.058	0.059	0.056	0.054	0.051	0.054
5	1	0.017	0.019	0.022	0.031	0.041	0.050	0.046	0.045	0.054	0.050	0.053	0.061	0.056	0.053	0.060	0.054	0.054	0.072	0.059	0.065
1	2	0.084	0.073	0.069	0.075	0.072	0.071	0.060	0.055	0.056	0.051	0.050	0.051	0.051	0.055	0.058	0.055	0.056	0.064	0.066	0.066
2	2	0.102	0.097	0.082	0.061	0.053	0.054	0.049	0.052	0.050	0.051	0.047	0.044	0.043	0.042	0.052	0.047	0.039	0.039	0.042	0.045
3	2	0.084	0.068	0.059	0.051	0.058	0.052	0.056	0.048	0.048	0.055	0.045	0.044	0.048	0.045	0.052	0.052	0.041	0.042	0.056	0.057
4	2	0.060	0.052	0.050	0.051	0.044	0.050	0.044	0.048	0.047	0.044	0.043	0.044	0.051	0.042	0.046	0.045	0.046	0.047	0.048	0.052
5	2	0.036	0.045	0.058	0.056	0.058	0.054	0.056	0.061	0.060	0.064	0.063	0.059	0.054	0.062	0.064	0.072	0.063	0.058	0.066	0.066
1	3	0.044	0.044	0.041	0.042	0.044	0.040	0.040	0.041	0.042	0.044	0.042	0.034	0.037	0.037	0.038	0.042	0.050	0.053	0.061	0.061
2	3	0.061	0.052	0.054	0.041	0.043	0.041	0.036	0.037	0.037	0.033	0.039	0.038	0.036	0.036	0.036	0.040	0.040	0.036	0.038	0.041
3	3	0.098	0.073	0.061	0.059	0.046	0.042	0.032	0.032	0.031	0.040	0.036	0.035	0.032	0.039	0.038	0.037	0.042	0.042	0.045	0.041
4	3	0.052	0.045	0.045	0.044	0.043	0.041	0.040	0.034	0.036	0.034	0.030	0.034	0.034	0.032	0.032	0.032	0.032	0.034	0.043	0.040
5	3	0.047	0.053	0.052	0.040	0.040	0.045	0.042	0.043	0.046	0.051	0.048	0.052	0.052	0.059	0.058	0.062	0.058	0.056	0.047	0.055
1	4	0.026	0.029	0.030	0.030	0.039	0.041	0.039	0.035	0.037	0.041	0.051	0.055	0.053	0.049	0.051	0.056	0.058	0.055	0.065	0.066
2	4	0.059	0.060	0.058	0.053	0.044	0.042	0.045	0.049	0.046	0.045	0.049	0.050	0.045	0.052	0.055	0.053	0.052	0.048	0.047	0.056
3	4	0.092	0.064	0.069	0.062	0.060	0.054	0.047	0.050	0.045	0.053	0.052	0.045	0.047	0.046	0.051	0.045	0.050	0.038	0.041	0.054
4	4	0.087	0.083	0.063	0.062	0.049	0.047	0.053	0.041	0.043	0.041	0.041	0.044	0.040	0.052	0.047	0.046	0.047	0.034	0.046	0.049
5	4	0.086	0.069	0.072	0.064	0.063	0.065	0.067	0.073	0.067	0.068	0.069	0.063	0.062	0.073	0.073	0.067	0.076	0.076	0.073	0.077
1	5	0.020	0.021	0.032	0.047	0.042	0.037	0.036	0.039	0.044	0.041	0.041	0.042	0.043	0.045	0.053	0.058	0.052	0.053	0.056	0.058
2	5	0.041	0.043	0.042	0.040	0.045	0.042	0.045	0.045	0.036	0.046	0.056	0.049	0.060	0.063	0.051	0.049	0.048	0.054	0.055	0.065
3	5	0.071	0.066	0.066	0.066	0.050	0.057	0.710	0.065	0.058	0.059	0.059	0.057	0.055	0.063	0.076	0.058	0.062	0.055	0.061	0.070
4	5	0.074	0.084	0.075	0.056	0.056	0.044	0.044	0.046	0.047	0.054	0.054	0.047	0.043	0.044	0.050	0.047	0.043	0.052	0.042	0.050
5	5	0.079	0.102	0.093	0.080	0.079	0.085	0.088	0.089	0.092	0.089	0.094	0.089	0.090	0.092	0.097	0.096	0.086	0.089	0.082	0.084

*1 unit distance ≃ 50 miles

TABLE 4 - COEFFICIENT OF VARIATION OF CONDITIONAL PROBABILITY $p_{ij}(k)$, COMPUTED BY THE FIRST METHOD

i	j	k (unit distance*)																			
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0.254	0.277	0.286	0.303	0.302	0.323	0.318	0.358	0.364	0.347	0.390	0.406	0.429	0.410	0.462	0.497	0.507	0.538	0.594	0.650
2	1	0.297	0.252	0.210	0.200	0.187	0.159	0.202	0.239	0.207	0.267	0.236	0.191	0.231	0.185	0.200	0.233	0.238	0.292	0.269	0.345
3	1	0.740	0.487	0.385	0.355	0.306	0.300	0.273	0.242	0.194	0.233	0.219	0.208	0.190	0.162	0.184	0.199	0.217	0.191	0.169	0.202
4	1	1.608	1.143	0.760	0.609	0.502	0.418	0.420	0.370	0.261	0.299	0.311	0.342	0.330	0.326	0.322	0.283	0.275	0.261	0.234	0.234
5	1	4.073	1.781	1.259	0.922	0.844	0.763	0.667	0.563	0.630	0.578	0.542	0.559	0.507	0.455	0.460	0.445	0.406	0.514	0.422	0.456
1	2	0.445	0.366	0.341	0.337	0.396	0.391	0.355	0.328	0.338	0.343	0.336	0.338	0.334	0.360	0.381	0.369	0.382	0.411	0.449	0.435
2	2	0.259	0.288	0.275	0.227	0.209	0.226	0.208	0.226	0.222	0.225	0.220	0.216	0.217	0.203	0.248	0.232	0.191	0.195	0.204	0.221
3	2	0.276	0.218	0.193	0.164	0.184	0.164	0.177	0.157	0.157	0.179	0.145	0.142	0.158	0.151	0.173	0.171	0.141	0.144	0.186	0.188
4	2	0.691	0.483	0.386	0.369	0.287	0.313	0.256	0.277	0.266	0.236	0.218	0.222	0.245	0.212	0.234	0.230	0.228	0.235	0.236	0.264
5	2	1.436	0.963	0.919	0.684	0.588	0.531	0.509	0.491	0.481	0.502	0.479	0.443	0.401	0.437	0.442	0.492	0.416	0.373	0.433	0.451
1	3	0.716	0.581	0.447	0.406	0.392	0.355	0.317	0.323	0.318	0.324	0.303	0.251	0.270	0.281	0.289	0.315	0.380	0.392	0.431	0.437
2	3	0.302	0.244	0.251	0.192	0.205	0.194	0.173	0.174	0.179	0.159	0.186	0.180	0.170	0.170	0.175	0.193	0.188	0.171	0.186	0.201
3	3	0.217	0.186	0.174	0.172	0.138	0.132	0.106	0.106	0.103	0.133	0.116	0.114	0.103	0.130	0.123	0.123	0.140	0.143	0.154	0.138
4	3	0.242	0.207	0.210	0.213	0.206	0.195	0.189	0.166	0.177	0.167	0.154	0.173	0.180	0.148	0.160	0.148	0.159	0.162	0.211	0.199
5	3	0.635	0.523	0.423	0.302	0.287	0.307	0.281	0.286	0.296	0.337	0.327	0.348	0.342	0.392	0.381	0.392	0.369	0.358	0.296	0.336
1	4	1.456	1.049	0.878	0.690	0.685	0.586	0.477	0.420	0.396	0.427	0.512	0.528	0.469	0.402	0.392	0.418	0.402	0.389	0.452	0.450
2	4	0.716	0.598	0.479	0.357	0.285	0.251	0.256	0.264	0.248	0.235	0.255	0.257	0.231	0.265	0.268	0.260	0.270	0.252	0.239	0.279
3	4	0.290	0.191	0.202	0.191	0.189	0.168	0.148	0.157	0.142	0.170	0.170	0.148	0.156	0.154	0.171	0.153	0.175	0.130	0.143	0.188
4	4	0.241	0.273	0.231	0.225	0.184	0.192	0.226	0.179	0.194	0.187	0.184	0.201	0.187	0.251	0.234	0.238	0.235	0.179	0.251	0.270
5	4	0.386	0.297	0.313	0.306	0.310	0.333	0.348	0.396	0.375	0.381	0.382	0.347	0.347	0.417	0.431	0.382	0.433	0.417	0.389	0.416
1	5	3.141	1.455	1.572	1.420	1.067	0.723	0.481	0.496	0.641	0.527	0.505	0.480	0.508	0.510	0.596	0.571	0.541	0.493	0.511	0.512
2	5	1.423	0.915	0.611	0.470	0.418	0.348	0.349	0.332	0.255	0.313	0.348	0.308	0.370	0.385	0.306	0.274	0.257	0.285	0.303	0.347
3	5	0.522	0.361	0.291	0.280	0.198	0.217	0.266	0.245	0.217	0.222	0.233	0.212	0.201	0.222	0.267	0.199	0.201	0.180	0.193	0.225
4	5	0.280	0.301	0.274	0.216	0.221	0.179	0.194	0.198	0.201	0.216	0.247	0.217	0.196	0.207	0.228	0.234	0.221	0.259	0.209	0.250
5	5	0.139	0.215	0.225	0.207	0.228	0.261	0.283	0.297	0.317	0.310	0.327	0.334	0.347	0.365	0.408	0.423	0.397	0.447	0.429	0.442

*1 unit distance ≃ 50 miles

TABLE 5 - COMPARISON OF STANDARD ERRORS OF ESTIMATE $\bar{S}_{ij}$ OBTAINED FROM $p_{ij}(k)$ AND S_{ij} OBTAINED FROM $\bar{p}_{ij}(k)$, BY THE FIRST METHOD

i	j	$\bar{S}_{ij}$ from $p_{ij}(k)$				S_{ij} from $\bar{p}_{ij}(k)$			
		Model I	Model II	Model III	Model IV	Model I	Model II	Model III	Model IV
1	1	.1783	.1592	.0605	.0763	.1571	.1558	.0367	.0431
2	1	.0865	.1300	.0485	.0914	.0635	.1506	.0114	.0695
3	1	.0889	.0683	.0549	.0598	.0630	.0204	.0225	.0229
4	1	.0977	.0521	.0377	.0485	.0829	.0208	.0114	.0289
5	1	.0920	.0447	.0332	.0413	.0768	.0117	.0092	.0190
1	2	.0626	.0909	.0334	.0618	.0245	.0877	.0092	.0289
2	2	.0809	.1367	.0463	.0445	.0628	.1394	.0292	.0113
3	2	.0525	.0899	.0419	.0533	.0088	.0406	.0047	.0169
4	2	.0620	.0462	.0362	.0403	.0425	.0245	.0168	.0120
5	2	.0733	.0442	.031E	.0407	.0490	.0069	.0149	.0140
1	2	.0533	.0443	.0246	.0322	.0358	.0304	.0125	.0043
2	3	.0398	.0734	.0274	.0404	.0102	.0351	.0034	.0154
3	3	.0604	.1371	.0404	.0394	.0431	.1229	.0248	.0119
4	3	.0374	.0710	.0281	.0390	.0088	.0236	.0052	.0146
5	3	.0501	.0516	.0279	.0384	.0222	.0057	.0131	.0112
1	4	.0827	.0382	.0251	.0353	.0690	.1241	.0069	.0168
2	4	.0641	.0514	.0375	.0432	.0430	.0198	.0191	.0126
3	4	.0560	.1165	.0410	.0590	.0181	.0382	.0052	.0263
4	4	.0729	.1586	.0421	.0457	.0540	.2021	.0182	.0149
5	4	.0726	.1074	.0383	.0747	.0448	.0870	.0115	.0492
1	5	.0975	.0366	.0264	.0324	.0863	.0232	.0074	.0076
2	5	.0932	.0437	.0359	.0395	.0782	.0156	.0160	.0140
3	5	.0835	.0636	.0488	.0537	.0560	.0249	.0205	.0140
4	5	.0640	.0987	.0411	.0673	.0380	.1301	.0080	.0441
5	5	.2011	.1565	.0554	.0675	.1786	.1384	.0367	.0276

TABLE 6 - COMPARISON OF PARAMETERS OF THE FOUR MATHEMATICAL MODELS OBTAINED FROM $p_{ij}(k)$ AND $\bar{p}_{ij}(k)$, BY THE FIRST METHOD

i j	Mean or median of parameters from p_{ij} (k)						Parameters from $\bar{p}_{ij}$ (k)					
	Model I	Model II	Model III			Model IV	Model I	Model II	Model III			Model IV
	-	$\bar{\lambda}_{ij}$	Median of α_{ij}	$\bar{\beta}_{ij}$	Median of ν_{ij}	$\bar{\eta}_{ij}$	-	λ_{ij}	α_{ij}	β_{ij}	ν_{ij}	η_{ij}
1 1	-	.2043	15.14	.4618	-.0211	1.316	-	.1608	15.75	.4586	-.0196	.95
2 1	-	.3066	15.76	.3175	-.0066	7.450	-	.1151	19.09	.3153	-.0060	7.45
3 1	-	.3796	14.52	.1771	.0088	1.792	-	.2667	14.65	.1810	.0081	.90
4 1	-	.2646	15.60	.0359	.0107	.304	-	.1804	16.25	.0361	.0101	.30
5 1	-	.1624	19.69	.0077	.0069	.208	-	.0889	18.98	.0090	.0074	.15
1 2	-	.2895	10.10	.1988	-.0041	6.008	-	.1186	15.82	.1959	-.0029	7.45
2 2	-	.3976	15.88	.2170	-.0079	3.710	-	.2900	15.87	.3121	-.0071	2.95
3 2	-	.5358	11.89	.3127	-.0009	6.405	-	.7270	18.87	.3138	-.0007	7.45
4 2	-	.3860	14.54	.1174	.0053	1.555	-	.1915	15.83	.1213	.0050	.80
5 2	-	.2331	16.38	.0540	.0061	1.042	-	.1707	16.26	.0569	.0057	.40
1 3	-	.2647	14.08	.0857	.0036	1.530	-	.1190	19.47	.0888	.0031	.60
2 3	-	.4506	9.66	.2114	.0006	6.292	-	.5826	65.43	.2114	-.0002	7.45
3 3	-	.4543	13.46	.3752	-.0052	4.624	-	.3215	14.71	.3725	-.0049	4.10
4 3	-	.4844	10.15	.2154	-.0015	6.884	-	1.0000	15.37	.2143	-.0009	7.45
5 3	-	.3042	14.23	.1123	.0033	2.754	-	.5828	12.81	.1130	.0029	1.70
1 4	-	.1587	19.93	.0205	.0067	.283	-	.0927	18.54	.0213	.0069	.20
2 4	-	.3606	13.06	.1165	.0058	1.530	-	.2134	15.73	.1222	.0049	.80
3 4	-	.4322	14.22	.3377	-.0026	6.718	-	1.0000	13.92	.3359	-.0026	7.45
4 4	-	.3613	14.46	.3059	-.0073	4.143	-	.2005	15.16	.3039	-.0068	3.65
5 4	-	.3102	14.13	.2192	-.0035	6.977	-	.2228	27.16	.2167	-.0024	7.45
1 5	-	.0938	24.60	.0144	.0052	.143	-	.0399	25.45	.0135	.0054	.10
2 5	-	.2103	18.26	.0564	.0075	.377	-	.1084	18.76	.0575	.0076	.25
3 5	-	.3966	14.17	.1951	.0065	1.626	-	.3042	16.22	.1960	.0064	.95
4 5	-	.4091	17.00	.2719	-.0051	7.112	-	.1025	17.27	.2721	-.0042	7.45
5 5	-	.1913	21.69	.4622	-.0161	.852	-	.1590	20.46	.4608	-.0152	.70

TABLE 7 - MEAN CONDITIONAL PROBABILITY $p_{ij}(k)$, COMPUTED BY THE SECOND METHOD

		k (miles)																					
i	j	10	30	50	70	90	120	160	200	240	280	330	390	450	510	570	630	690	750	810	870	930	990
1	1	0.548	0.569	0.521	0.490	0.485	0.435	0.400	0.360	0.331	0.314	0.293	0.257	0.242	0.225	0.206	0.198	0.184	0.181	0.166	0.151	0.148	0.142
2	1	0.263	0.318	0.322	0.323	0.303	0.317	0.307	0.307	0.305	0.293	0.278	0.276	0.260	0.241	0.238	0.234	0.226	0.222	0.216	0.222	0.219	0.222
3	1	0.136	0.101	0.133	0.154	0.163	0.178	0.201	0.220	0.230	0.238	0.253	0.270	0.275	0.290	0.303	0.301	0.306	0.304	0.307	0.303	0.306	0.303
4	1	0.029	0.010	0.022	0.028	0.041	0.054	0.068	0.075	0.087	0.099	0.109	0.122	0.136	0.149	0.154	0.164	0.172	0.177	0.185	0.186	0.182	0.188
5	1	0.023	0.002	0.003	0.005	0.008	0.016	0.024	0.038	0.047	0.056	0.067	0.076	0.087	0.095	0.099	0.104	0.112	0.117	0.125	0.138	0.145	0.144
1	2	0.202	0.216	0.221	0.222	0.217	0.221	0.217	0.217	0.216	0.206	0.194	0.188	0.178	0.167	0.160	0.153	0.150	0.151	0.147	0.148	0.139	0.136
2	2	0.432	0.416	0.383	0.347	0.349	0.319	0.288	0.280	0.266	0.258	0.256	0.238	0.229	0.223	0.218	0.211	0.210	0.213	0.209	0.210	0.199	0.194
3	2	0.277	0.290	0.298	0.306	0.300	0.300	0.310	0.297	0.298	0.300	0.301	0.301	0.304	0.310	0.311	0.315	0.312	0.303	0.296	0.295	0.297	0.306
4	2	0.059	0.062	0.077	0.092	0.096	0.116	0.121	0.138	0.142	0.148	0.153	0.165	0.175	0.181	0.183	0.188	0.186	0.189	0.196	0.193	0.207	0.206
5	2	0.031	0.015	0.021	0.034	0.038	0.045	0.065	0.068	0.078	0.089	0.096	0.108	0.114	0.119	0.128	0.133	0.142	0.143	0.152	0.155	0.158	0.158
1	3	0.041	0.043	0.057	0.065	0.072	0.081	0.090	0.100	0.104	0.107	0.111	0.119	0.123	0.130	0.135	0.135	0.135	0.132	0.137	0.137	0.143	0.144
2	3	0.195	0.200	0.202	0.215	0.212	0.204	0.205	0.204	0.200	0.205	0.205	0.206	0.208	0.208	0.207	0.209	0.207	0.206	0.206	0.203	0.206	0.206
3	3	0.485	0.478	0.441	0.407	0.399	0.381	0.364	0.350	0.347	0.340	0.332	0.323	0.318	0.310	0.309	0.304	0.305	0.308	0.306	0.303	0.296	0.290
4	3	0.210	0.214	0.219	0.220	0.213	0.213	0.217	0.214	0.212	0.210	0.208	0.200	0.200	0.198	0.196	0.196	0.199	0.196	0.194	0.198	0.195	0.194
5	3	0.069	0.065	0.081	0.093	0.104	0.122	0.125	0.133	0.137	0.138	0.144	0.151	0.152	0.154	0.154	0.156	0.154	0.158	0.156	0.159	0.161	0.166
1	4	0.023	0.013	0.011	0.019	0.017	0.030	0.038	0.046	0.056	0.066	0.076	0.086	0.089	0.096	0.103	0.111	0.117	0.119	0.120	0.125	0.122	0.128
2	4	0.078	0.058	0.078	0.091	0.100	0.117	0.135	0.137	0.147	0.152	0.157	0.163	0.173	0.180	0.184	0.185	0.188	0.187	0.189	0.191	0.196	0.200
3	4	0.307	0.323	0.330	0.339	0.328	0.322	0.314	0.323	0.312	0.312	0.305	0.302	0.300	0.298	0.297	0.294	0.294	0.296	0.297	0.303	0.313	0.302
4	4	0.390	0.371	0.341	0.312	0.316	0.293	0.276	0.264	0.255	0.247	0.239	0.236	0.226	0.222	0.220	0.214	0.210	0.209	0.207	0.207	0.201	0.199
5	4	0.202	0.234	0.239	0.239	0.239	0.238	0.237	0.230	0.230	0.223	0.223	0.214	0.211	0.202	0.196	0.197	0.190	0.189	0.186	0.175	0.168	0.170
1	5	0.020	0.001	0.003	0.005	0.005	0.010	0.019	0.023	0.027	0.034	0.045	0.056	0.070	0.076	0.081	0.088	0.096	0.099	0.105	0.109	0.116	0.115
2	5	0.033	0.016	0.025	0.030	0.040	0.059	0.079	0.088	0.103	0.108	0.116	0.131	0.139	0.151	0.159	0.166	0.173	0.176	0.182	0.184	0.188	0.184
3	5	0.120	0.128	0.159	0.170	0.199	0.229	0.237	0.253	0.260	0.267	0.275	0.280	0.281	0.279	0.273	0.284	0.280	0.286	0.293	0.294	0.292	0.306
4	5	0.248	0.274	0.275	0.296	0.284	0.275	0.276	0.266	0.268	0.262	0.261	0.255	0.244	0.235	0.234	0.226	0.223	0.219	0.212	0.209	0.207	0.209
5	5	0.579	0.580	0.538	0.499	0.472	0.426	0.390	0.370	0.343	0.328	0.302	0.278	0.265	0.259	0.252	0.236	0.229	0.221	0.208	0.204	0.197	0.187

TABLE 8 - STANDARD DEVIATION OF CONDITIONAL PROBABILITY $p_{ij}(k)$, COMPUTED BY THE SECOND METHOD

		k (miles)																					
i	j	10	30	50	70	90	120	160	200	240	280	330	390	450	510	570	630	690	750	810	870	930	990
1	1	0.188	0.172	0.133	0.148	0.137	0.109	0.091	0.106	0.099	0.102	0.087	0.076	0.084	0.090	0.083	0.078	0.076	0.074	0.071	0.073	0.072	0.075
2	1	0.129	0.145	0.106	0.106	0.106	0.078	0.077	0.079	0.069	0.059	0.051	0.059	0.056	0.055	0.055	0.059	0.055	0.056	0.055	0.063	0.066	0.063
3	1	0.133	0.106	0.105	0.110	0.096	0.068	0.063	0.064	0.064	0.072	0.066	0.067	0.057	0.056	0.058	0.057	0.057	0.049	0.050	0.058	0.058	0.062
4	1	0.054	0.031	0.044	0.043	0.057	0.048	0.049	0.043	0.044	0.043	0.041	0.042	0.045	0.049	0.050	0.051	0.050	0.051	0.051	0.051	0.048	0.052
5	1	0.065	0.012	0.025	0.016	0.024	0.022	0.031	0.044	0.047	0.045	0.041	0.046	0.057	0.062	0.055	0.054	0.063	0.056	0.063	0.076	0.076	0.065
1	2	0.102	0.105	0.093	0.095	0.088	0.073	0.070	0.072	0.067	0.069	0.059	0.056	0.060	0.058	0.053	0.054	0.054	0.058	0.056	0.051	0.054	0.054
2	2	0.161	0.130	0.106	0.103	0.105	0.071	0.061	0.055	0.055	0.059	0.051	0.045	0.041	0.042	0.045	0.045	0.037	0.040	0.041	0.043	0.045	0.046
3	2	0.110	0.123	0.103	0.093	0.087	0.059	0.060	0.058	0.059	0.059	0.048	0.045	0.044	0.044	0.044	0.044	0.043	0.042	0.042	0.036	0.043	0.048
4	2	0.070	0.070	0.066	0.076	0.069	0.055	0.045	0.052	0.041	0.041	0.040	0.043	0.039	0.042	0.041	0.042	0.038	0.039	0.037	0.038	0.045	0.043
5	2	0.052	0.035	0.038	0.043	0.044	0.035	0.047	0.045	0.045	0.051	0.052	0.052	0.052	0.054	0.051	0.049	0.055	0.055	0.054	0.054	0.058	0.064
1	3	0.047	0.053	0.045	0.046	0.046	0.039	0.036	0.039	0.040	0.037	0.035	0.036	0.036	0.040	0.041	0.037	0.041	0.041	0.038	0.041	0.042	0.040
2	3	0.086	0.082	0.070	0.061	0.065	0.039	0.043	0.044	0.041	0.040	0.036	0.037	0.033	0.034	0.033	0.032	0.032	0.028	0.029	0.031	0.035	0.031
3	3	0.122	0.120	0.085	0.077	0.080	0.061	0.053	0.051	0.044	0.042	0.039	0.036	0.035	0.037	0.036	0.033	0.035	0.033	0.031	0.035	0.032	0.036
4	3	0.080	0.071	0.063	0.055	0.054	0.036	0.034	0.042	0.032	0.032	0.030	0.029	0.028	0.032	0.031	0.030	0.028	0.028	0.029	0.028	0.029	0.029
5	3	0.061	0.063	0.066	0.061	0.067	0.052	0.043	0.047	0.045	0.042	0.047	0.047	0.049	0.046	0.045	0.044	0.042	0.041	0.040	0.040	0.044	0.047
1	4	0.050	0.035	0.028	0.036	0.031	0.031	0.032	0.037	0.041	0.047	0.045	0.045	0.043	0.042	0.046	0.046	0.047	0.052	0.043	0.053	0.053	0.059
2	4	0.076	0.068	0.079	0.080	0.071	0.055	0.056	0.051	0.046	0.047	0.045	0.046	0.050	0.052	0.044	0.038	0.045	0.038	0.039	0.037	0.043	0.039
3	4	0.123	0.110	0.096	0.095	0.095	0.062	0.068	0.050	0.050	0.052	0.043	0.046	0.047	0.050	0.047	0.043	0.041	0.037	0.040	0.045	0.047	0.045
4	4	0.163	0.137	0.091	0.090	0.089	0.058	0.059	0.054	0.047	0.047	0.042	0.042	0.040	0.041	0.039	0.035	0.039	0.038	0.033	0.038	0.040	0.044
5	4	0.103	0.106	0.096	0.079	0.076	0.072	0.065	0.066	0.064	0.063	0.065	0.060	0.069	0.066	0.058	0.057	0.055	0.057	0.055	0.057	0.058	0.058
1	5	0.044	0.010	0.022	0.016	0.012	0.021	0.023	0.024	0.025	0.028	0.032	0.036	0.041	0.046	0.053	0.061	0.058	0.053	0.049	0.056	0.058	0.057
2	5	0.059	0.037	0.048	0.044	0.048	0.047	0.050	0.053	0.056	0.050	0.047	0.048	0.053	0.052	0.051	0.052	0.053	0.054	0.050	0.058	0.056	0.057
3	5	0.105	0.121	0.103	0.091	0.105	0.080	0.076	0.075	0.066	0.054	0.054	0.050	0.046	0.054	0.051	0.050	0.042	0.050	0.055	0.056	0.059	0.052
4	5	0.118	0.119	0.101	0.088	0.090	0.062	0.052	0.060	0.053	0.044	0.048	0.043	0.043	0.044	0.047	0.049	0.047	0.044	0.044	0.049	0.050	0.053
5	5	0.183	0.153	0.144	0.130	0.138	0.110	0.103	0.098	0.095	0.085	0.081	0.080	0.083	0.085	0.084	0.080	0.080	0.076	0.073	0.077	0.085	0.078

TABLE 9 - COEFFICIENT OF VARIATION OF CONDITIONAL PROBABILITY $p_{ij}(k)$, COMPUTED BY THE SECOND METHOD

		k (miles)																					
i	j	10	30	50	70	90	120	160	200	240	280	330	390	450	510	570	630	690	750	810	870	930	990
1	1	0.343	0.302	0.255	0.302	0.281	0.252	0.226	0.295	0.299	0.323	0.298	0.295	0.345	0.400	0.405	0.395	0.414	0.411	0.429	0.481	0.486	0.527
2	1	0.490	0.458	0.328	0.329	0.349	0.247	0.250	0.257	0.226	0.201	0.185	0.214	0.215	0.229	0.231	0.252	0.245	0.250	0.255	0.285	0.303	0.285
3	1	0.972	1.047	0.796	0.716	0.589	0.382	0.315	0.291	0.278	0.302	0.262	0.250	0.208	0.194	0.193	0.191	0.187	0.162	0.163	0.191	0.191	0.206
4	1	1.842	3.145	2.030	1.538	1.401	0.880	0.729	0.570	0.500	0.434	0.380	0.342	0.329	0.332	0.327	0.313	0.289	0.288	0.276	0.274	0.262	0.274
5	1	2.854	5.201	8.456	3.291	3.204	1.319	1.296	1.157	1.002	0.797	0.609	0.605	0.651	0.654	0.559	0.522	0.561	0.481	0.504	0.552	0.524	0.454
1	2	0.505	0.485	0.420	0.427	0.403	0.330	0.323	0.330	0.312	0.333	0.306	0.299	0.336	0.348	0.334	0.356	0.361	0.383	0.379	0.342	0.390	0.396
2	2	0.374	0.313	0.276	0.296	0.301	0.224	0.213	0.196	0.207	0.228	0.199	0.188	0.181	0.187	0.208	0.213	0.177	0.188	0.196	0.207	0.226	0.236
3	2	0.395	0.425	0.346	0.303	0.289	0.195	0.193	0.195	0.196	0.198	0.160	0.149	0.145	0.142	0.141	0.138	0.138	0.139	0.140	0.121	0.143	0.157
4	2	1.200	1.131	0.858	0.831	0.717	0.472	0.374	0.375	0.287	0.277	0.264	0.258	0.222	0.232	0.225	0.223	0.202	0.208	0.189	0.195	0.215	0.208
5	2	1.683	2.326	1.874	1.280	1.185	0.777	0.735	0.659	0.578	0.578	0.541	0.480	0.453	0.456	0.401	0.368	0.389	0.384	0.355	0.351	0.366	0.405
1	3	1.155	1.242	0.795	0.708	0.634	0.479	0.398	0.395	0.380	0.350	0.313	0.298	0.292	0.309	0.304	0.278	0.306	0.309	0.279	0.297	0.296	0.277
2	3	0.438	0.410	0.345	0.283	0.305	0.192	0.212	0.214	0.203	0.193	0.175	0.180	0.159	0.164	0.160	0.156	0.154	0.136	0.139	0.151	0.169	0.151
3	3	0.251	0.251	0.192	0.190	0.201	0.160	0.146	0.145	0.127	0.125	0.119	0.112	0.109	0.118	0.117	0.108	0.114	0.106	0.101	0.115	0.107	0.125
4	3	0.379	0.331	0.285	0.250	0.253	0.168	0.155	0.196	0.153	0.150	0.145	0.147	0.141	0.160	0.156	0.152	0.143	0.145	0.147	0.144	0.148	0.148
5	3	0.886	0.982	0.813	0.660	0.647	0.431	0.349	0.357	0.328	0.306	0.326	0.309	0.322	0.300	0.295	0.279	0.276	0.259	0.255	0.255	0.272	0.280
1	4	2.150	2.600	2.426	1.928	1.803	1.052	0.858	0.816	0.740	0.711	0.590	0.524	0.480	0.439	0.449	0.419	0.398	0.437	0.357	0.421	0.436	0.460
2	4	0.984	1.169	1.003	0.880	0.709	0.471	0.414	0.370	0.313	0.310	0.287	0.281	0.289	0.288	0.240	0.207	0.237	0.206	0.206	0.193	0.217	0.196
3	4	0.401	0.339	0.289	0.279	0.291	0.192	0.216	0.156	0.159	0.168	0.142	0.153	0.156	0.168	0.158	0.147	0.139	0.127	0.135	0.148	0.151	0.150
4	4	0.419	0.370	0.266	0.289	0.282	0.198	0.213	0.206	0.185	0.190	0.178	0.180	0.177	0.185	0.175	0.165	0.186	0.179	0.161	0.182	0.198	0.223
5	4	0.511	0.453	0.402	0.330	0.319	0.302	0.275	0.287	0.277	0.284	0.291	0.281	0.328	0.328	0.298	0.291	0.289	0.303	0.294	0.328	0.343	0.341
1	5	2.164	10.00	6.308	3.272	2.625	1.985	1.224	1.040	0.961	0.827	0.703	0.638	0.587	0.604	0.654	0.685	0.600	0.530	0.464	0.509	0.499	0.500
2	5	1.776	2.344	1.876	1.484	1.214	0.795	0.630	0.610	0.544	0.465	0.405	0.366	0.379	0.346	0.317	0.312	0.306	0.306	0.275	0.314	0.299	0.310
3	5	0.877	0.947	0.651	0.536	0.527	0.350	0.319	0.294	0.253	0.202	0.196	0.177	0.166	0.193	0.186	0.177	0.152	0.175	0.186	0.190	0.201	0.170
4	5	0.478	0.435	0.366	0.298	0.317	0.225	0.190	0.226	0.200	0.168	0.186	0.170	0.178	0.187	0.202	0.216	0.212	0.201	0.207	0.234	0.240	0.254
5	5	0.316	0.264	0.267	0.260	0.293	0.258	0.263	0.264	0.276	0.260	0.269	0.287	0.314	0.329	0.334	0.341	0.349	0.345	0.351	0.379	0.431	0.416