DISSERTATION

A QUANTUM $H^*(T)$-MODULE VIA QUASIMAP INVARIANTS

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ABSTRACT

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For X a smooth projective variety, the quantum cohomology ring $QH^*(X)$ is a deformation of the usual cohomology ring $H^*(X)$, where the product structure is modified to incorporate quantum corrections. These correction terms are defined using Gromov–Witten invariants. When X is toric with geometric quotient description $V//T$, the cohomology ring $H^*(V//T)$ also has the structure of a $H^*(T)$-module. In this paper, we introduce a new deformation of the cohomology of X using quasimap invariants with a light point. This defines a quantum $H^*(T)$-module structure on $H^*(X)$ through a modified version of the WDVV equations. We explicitly compute this structure for the Hirzebruch surface of type 2. We conjecture that this new quantum module structure is isomorphic to the natural module structure of the Batyrev ring for a semipositive toric variety.

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DEDICATION

Dedicate to my parents.

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Chapter 1

Introduction

Quantum cohomology is a central object of interests both in mathematics and physics, as it is related to string theory and mirror symmetry [1]. The idea of quantum cohomology first appeared in physics [2]. The first mathematical construction was given in terms of symplectic geometry for semi-positive symplectic manifolds [3, 4].

The quantum product on a quantum cohomology is a deformation of the product of the ordinary cohomology. It can be defined through Gromov–Witten invariants, which uses the moduli space of stable maps [5–7]. One notable application of the quantum product, especially its *associativity*, has been shown in [8] to count the number of rational curves of degree d on the projective plane $\mathbb{P}^2$.

It is not easy to compute quantum cohomology in general. When the space X is a smooth projective Fano toric variety, the quantum cohomology ring $QH^*(X)$ agrees with the Batyrev ring $\mathrm{Bat}^*(X)$, defined in [9] (see [5, Example 8.1.2.2 or Example 11.2.5.2] or an extension of [10]). In [11, §1.7], it is conjectured that Gromov–Witten invariants and quasimap invariants are the same for Fano index at least 2. This is proved in [12, Corollary 1.3.3] and [13] for smooth projective toric Fano varieties using different ways. Thus, there are no interesting phenomena observed in this case. On the other hand, the Hirzebruch surface of type 2, say $\mathbb{F}_2$, is not Fano, but *semipositive*, i.e., the anticanonical divisor is nef. The Gromov–Witten invariants and the quasimap invariants of $\mathbb{F}_2$ do not agree [14, §6.4]. Also, it shows a failure of the equality between $\mathrm{Bat}^*(\mathbb{F}_2)$ and $QH^*(\mathbb{F}_2)$ [5, Example 11.2.5.2]. Thus, one may expect that the deformation given by 3-pointed quasimap invariants defines a ring structure isomorphic to $\mathrm{Bat}^*(\mathbb{F}_2)$. In [15, §1.1.8], the author pointed out that this expectation is not true as well, due to the failure of the divisor equation in the case of quasimap invariants.

In this paper, we define a new quantum deformation using $2|1$-quasimap invariants. Let X_Σ be a smooth projective toric variety whose toric geometric quotient description (Theorem 2.1.1) is given by $V//T$, where V is a finite dimensional $\mathbb{C}$-vector space and T is a complex torus. Denote by $H^*(T)$ the T-equivariant cohomology of a point. Let $\{T_i\}$ be a basis for $H^*(X_\Sigma)$ and $\{T^i\}$ be the dual basis under the intersection pairing.

Definition 3.1.1. *For $\xi \in H^*(T)$ and $\phi \in H^*(X_\Sigma)$, define the* ***quantum*** $H^*(T)$***-action*** *on $H^*(X_\Sigma)$ via the $2|1$-pointed quasimap invariants*

$$\xi \star \phi := \sum_{\beta \in \mathit{Eff}(X_\Sigma)} q^\beta \sum_i \langle \phi, T_i \,| \xi \rangle_{0,2|1,\beta} T^i.$$

The moduli space of $m|k$-pointed quasimaps were introduced in [16] (see Definition 2.2.1). Here, $m|k$ represents the number of heavy and light markings, respectively, where heavy markings are the ordinary ones and light markings are infinitesimally weighted ones [17]. This operation defines not a product structure but a module structure that satisfies an analogue of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. The following is the main result which defines the (small) quantum $H^*(T)$-module structure on $H^*(X_\Sigma)$.

Theorem 3.1.2 (Quantum $H^*(T)$-module structure)**.** *For $\xi, \zeta \in H^*(T)$ and $\phi \in H^*(X_\Sigma)$,*

$$\xi \star (\zeta \star \phi) = (\xi \cdot \zeta) \star \phi.$$

We explicitly compute the quantum module structure of the Hirzebruch surface $\mathbb{F}_2$ using the Atiyah–Bott localization theorem. Suppose that $\mathrm{Pic}(\mathbb{F}_2) \simeq \mathbb{Z}D_2 \oplus \mathbb{Z}D_4$, where D_2 and D_4 are the torus-invariant divisors such that $D_2 \cdot D_2 = 0$ and $D_4 \cdot D_4 = -2pt$[1]. The geometric quotient construction of $\mathbb{F}_2$ is $\mathbb{C}^4//(\mathbb{C}^*)^2$. The $(\mathbb{C}^*)^2$-equivariant cohomology of $\mathbb{C}^4$ is given by $\mathbb{Q}[\sigma_2, \sigma_4]$. A full description of the quantum module structure of $\mathbb{F}_2$ is listed below.

---

[1] The divisor D_2 is the class of fiber and D_4 is the class of ∞-section.

Theorem 4.0.1. *The quantum* $H^*((\mathbb{C}^*)^2)$*-module structure for* $\mathbb{F}_2$ *is given by the following:*

$$\begin{aligned}
\sigma_2 \star 1 &= D_2 - \frac{1}{2} f(q_4) D_4 & \sigma_4 \star 1 &= (1 + f(q_4)) D_4 \\
\sigma_2 \star D_2 &= q_2 q_4 (1 + f(q_4)) - \frac{1}{2} f(q_4) pt & \sigma_4 \star D_2 &= -\frac{1}{2} q_2 f(q_4) + (1 + f(q_4)) pt \\
\sigma_2 \star D_4 &= -2 q_2 q_4 (1 + f(q_4)) + (1 + f(q_4)) pt & \sigma_4 \star D_4 &= q_2 (1 + f(q_4)) - 2(1 + f(q_4)) pt \\
\sigma_2 \star pt &= q_2 q_4 (1 + f(q_4)) D_4 & \sigma_4 \star pt &= q_2 D_2 - \frac{1}{2} q_2 (1 + f(q_4)) D_4,
\end{aligned}$$

where $f(z) = \sum_{d \geq 1} \binom{2d}{d} z^d = \frac{1}{\sqrt{1-4z}} - 1$.

An interesting observation from Theorem 4.0.1 is that the quantum module structure of $\mathbb{F}_2$ coincides with the Batyrev ring of $\mathbb{F}_2$ regarded as a module. In other words, we found a geometric interpretation of the Batyrev ring of $\mathbb{F}_2$ through $2|1$-quasimap invariants. We conjecture the following.

Conjecture 1.0.1. *For a smooth semipositive toric variety* $V//T$*, the quantum* $H^*(T)$*-module structure of* $V//T$ *coincides with a natural module structure of the Batyrev ring of* $V//T$.

To prove Theorem 3.1.2, which is an analogue of the WDVV equations, a type of a splitting axiom as in [18–20] is required. This involves with the virtual fundamental classes of the quasimap moduli spaces. Instead of using perfect obstruction theory to prove the splitting axiom as in [19], we use localized top Chern class from [21, ch14.1], which is a more elementary notion. In [14, 22], a construction of the moduli space of quasimaps with light points is given as a zero locus of a section of a vector bundle on a smooth Deligne–Mumford stack. This global model gives rise to the virtual fundamental class as the localized top Chern class.

In the localization computation to obtain the quantum module structure of $\mathbb{F}_2$ in Theorem 4.0.1, there are some key features that we would like to highlight:

1. The module structure in Theorem 3.1.2 allows us to assume that the degree of the insertion from the light point is one.

2. In general, the map forgetting a heavy point does not define a universal curve in the quasimap case. However, the map forgetting a light point gives rise to the universal curve [17]. This allows us to have the divisor equation as in [23]. Thus, the computation of the module structure boils down to the computation of all possible 2-pointed quasimap invariants.

3. Having two heavy markings gives rise to a chain of $\mathbb{P}^1$'s for the source curve of a quasimap.

4. When applying the localization theorem to a 2-pointed quasimap invariant, the contribution of each fixed locus can be expressed as a fraction whose numerator is a homogeneous polynomial in V_1 and W_1 and the denominator is W_1^N for some natural number N. Here, V_1 and W_1 are weights given in (4.7). Since the invariant is a rational number, we are allowed to consider particular types of fixed loci whose corresponding term in the localization formula has a numerator not divisible by V_1. In other words, the coefficient of W_1^N in the numerator is not zero for such a fixed locus. This observation reduces considerably the number of fixed loci that we have to consider in the computation of a single 2-pointed quasimap invariant. We call this the necessary fixed locus. See the paragraph above Corollary 5.0.2, and Figure 5.1, and Figure 6.2.

This paper is organized as follows. In Section 2, we give preliminaries on the quotient construction of a toric variety and recall the definition of the moduli space of stable toric quasimaps with light points. The moduli space of quasimaps will be constructed as a zero locus of a section of a vector bundle over a smooth Deligne–Mumford stack with a global embedding. The definition of the new quantum deformation using these moduli spaces is given in Section 3. We prove that this deformation defines a quantum module structure. Through Section 4-6, we elaborate the localization computation for the quantum module structure of the Hirzebruch surface $\mathbb{F}_2$. In Section 5, we briefly explain calculus on the Losev–Manin space and some combinatorics to simplify the localization formula obtained in Section 4. In Section 7, we verify that the quantum module structure of $\mathbb{F}_2$ agrees with the Batyrev ring realized as a module.

Chapter 2

Moduli spaces of stable toric $m|k$-pointed quasimaps to X_Σ

Throughout this paper, our base field is the complex numbers $\mathbb{C}$. Let M be a $\mathbb{Z}$-lattice, and N the dual lattice, and $\Sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ a smooth complete fan. Write X_Σ for the corresponding smooth projective toric variety with torus $N \otimes_{\mathbb{Z}} \mathbb{C}^*$. For $\sigma \in \Sigma$, denote the set of all 1 dimensional cones, say *rays*, of σ by $\sigma(1)$, and set $\Sigma(1) := \cup_{\sigma\in\Sigma}\sigma(1)$.

2.1 Geometric Quotients for Toric Varieties

The toric variety X_Σ can be expressed as a geometric quotient using the data of the fan Σ. A *primitive collection* is a subset P of rays in $\Sigma(1)$ such that

i) $P \nsubseteq \sigma(1)$ for all $\sigma \in \Sigma$;

ii) every proper subset of P is contained in $\sigma(1)$ for some $\sigma \in \Sigma$.

Define

$$Z_\Sigma := \bigcup_{P\text{: a primitive collection}} \mathbf{V}(x_\rho \mid \rho \in P) \subseteq \mathbb{C}^{\Sigma(1)},$$

where ρ is chosen to be the minimal generator of the ray. This is the *irrelevant subset* to the fan Σ.

Since the fan Σ is complete, we have an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \longrightarrow \mathrm{Cl}(X_\Sigma) \longrightarrow 0, \tag{2.1}$$

where $m \in M$ goes to $\sum_{\rho\in\Sigma(1)}\langle m, \rho\rangle$. Smoothness of X_Σ allows us to identify the class group $\mathrm{Cl}(X_\Sigma)$ with the Picard group $\mathrm{Pic}(X_\Sigma)$. Denote the matrix of the map $\mathbb{Z}^{\Sigma(1)} \to \mathrm{Pic}(X_\Sigma)$ by $(a_{i\rho})$ where $\rho = 1, 2, \dots, n$ and $i = 1, 2, \dots, r$ with $n := |\Sigma(1)|$ and $r := \operatorname{rank}\mathrm{Pic}(X_\Sigma)$. Applying

$\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$, we obtain the exact sequence

$$0 \longrightarrow (\mathbb{C}^*)^r \longrightarrow (\mathbb{C}^*)^{\Sigma(1)} \longrightarrow N \otimes_{\mathbb{Z}} \mathbb{C}^* \longrightarrow 0.$$

Thus, the torus $(\mathbb{C}^*)^r$ is acting on $\mathbb{C}^{\Sigma(1)}$ by componentwise multiplication, which can be represented by the transposition of the $r \times n$ matrix $(a_{i\rho})$. In this case, the geometric quotient associated to X_Σ is given as follows from [24, Theorem 5.1.11].

Theorem 2.1.1. *Given a smooth complete fan Σ, there is a natural isomorphism between the corresponding toric variety and the geometric quotient*

$$X_\Sigma \simeq \mathbb{C}^{\Sigma(1)} \backslash Z_\Sigma \,//\, (\mathbb{C}^*)^r.$$

2.2 The moduli space of stable toric $m|k$-quasimaps

We recall the definition of the moduli space of stable toric quasimaps with m heavy points and k light points to a smooth projective toric variety X_Σ. For details, we refer to [14] for a construction of the moduli space of stable toric quasimaps with heavy markings and [16] for the one with light markings.

Choose $\alpha_\rho \in \mathbb{Z}$ so that the line bundle $\mathcal{O}_{X_\Sigma}(1) := \otimes_{\rho \in \Sigma(1)} \mathcal{O}(D_\rho)^{\otimes \alpha_\rho}$ on X_Σ is ample, where D_ρ is the torus invariant divisor on X_Σ corresponding to ρ.

Definition 2.2.1. A **stable toric $m|k$-pointed quasimap** to X_Σ of genus g is the data

$$((C; x_1, \ldots, x_m; y_1, \ldots, y_k), \{L_\rho\}_{\rho \in \Sigma(1)}, \{s_\rho\}_{\rho \in \Sigma(1)}, \{\phi_m\}_{m \in M})$$

where

- C is a connected, at most nodal, projective curve of genus g,
- $\{x_1, \ldots, x_m, y_1, \ldots, y_k\}$ are nonsingular marked points,

- $\{x_1, \dots, x_m\}$ are distinct and disjoint from $\{y_1, \dots, y_k\}$,
- L_ρ are line bundles on C,
- $s_\rho \in \Gamma(C, L_\rho)$ are global sections,
- (compatibility) the trivializations $\phi_m : \otimes_\rho L_\rho^{\langle m,\rho\rangle} \to \mathcal{O}_C$ are isomorphisms satisfying $\phi_m \otimes \phi_{m'} = \phi_{m+m'}$ for all $m, m' \in M$,

satisfying

1. (nondegeneracy) there is a finite (possibly empty) set of smooth points $B \subset C$, disjoint from $\{x_1, \dots, x_m\} \subset C$, such that for every $z \in C \backslash B$, there exists a maximal cone $\sigma \in \Sigma_{max}$ such that $s_\rho(z) \neq 0, \ \forall \rho \not\subset \sigma$,
2. (stability) $\omega_C(x_1 + \cdots + x_m + \epsilon(y_1 + \cdots + y_k)) \otimes \mathcal{L}^\epsilon$ is ample for every rational number $\epsilon > 0$, where $\mathcal{L} := \otimes_{\rho \in \Sigma(1)} L_\rho^{\otimes \alpha_\rho}$.

Denote a stable quasimap by $(C; \underline{x}; \underline{y}, \underline{L}, \underline{s}, \underline{\phi})$. Two stable quasimaps $(C; \underline{x}; \underline{y}, \underline{L}, \underline{s}, \underline{\phi})$ and $(C'; \underline{x'}; \underline{y'}, \underline{L'}, \underline{s'}, \underline{\phi'})$ are *isomorphic* if there exists

$$(f : C \to C', \{\theta_\rho : L_\rho \to f^*(L'_\rho)\}_{\rho \in \Sigma(1)}),$$

where f and θ_ρ are isomorphism such that

$$f(x_i) = x'_i, \ f(y_i) = y'_i, \ \theta_\rho(s_\rho) = f^*(s'_\rho), \ \phi_m = f^*(\phi'_m) \circ (\otimes_\rho \theta_\rho^{\langle m,\rho\rangle}).$$

Definition 2.2.2. Given a stable toric quasimap $(C; \underline{x}; \underline{y}, \underline{L}, \underline{s}, \underline{\phi})$, the map $\mathrm{Pic}(X_\Sigma) \to \mathrm{Pic}(C)$ sending $\mathcal{O}_{X_\Sigma}(D_\rho) \mapsto L_\rho$ is a well-defined homomorphism because of the compatibility condition of the trivializations ϕ_m. Composing with the degree map to $\mathbb{Z}$, there is a $\mathbb{Z}$-linear homomorphism from $\mathrm{Pic}(X_\Sigma)$ to $\mathbb{Z}$. By Poincaré duality, the perfect pairing between $H_2(X_\Sigma, \mathbb{Z})$ and $H^2(X_\Sigma, \mathbb{Z}) \simeq$

$\mathrm{Pic}(X_\Sigma)$ gives rise to a unique curve class $\beta \in H_2(X_\Sigma, \mathbb{Z})$ determined by

$$\beta \cdot D_\rho = \deg L_\rho,$$

for all $\rho \in \Sigma(1)$. The class β is called the **degree** of the stable quasimap. For an irreducible component C' of the source curve of the given quasimap, we can similarly define $\beta_{C'}$ to be the degree of the quasimap restricted to C'. Then, it satisfies that $\beta = \sum_{C'} \beta_{C'}$, where C' runs over all irreducible components.

For $\beta \in H_2(X_\Sigma, \mathbb{Z})$, we say β is **(quasimap)-effective** if β can be realized as classes of stable toric quasimaps to X_Σ. Denote the set of all (quasimap)-effective classes as Eff(X_Σ). The set Eff(X_Σ) forms a semigroup with the property that if $\beta_1 + \beta_2 = 0$ for $\beta_i \in$Eff(X_Σ), then $\beta_1 = \beta_2 = 0$ [14, Lem 3.1.3].

Remark 2.2.3. The following observations are useful for our purposes.

1. We call x_i a *heavy marking* and y_j a *light marking*.

2. From the degree of the log-canonical bundle $\omega_C(\sum x_i + \epsilon \sum y_j) \otimes \mathcal{L}^\epsilon$, we obtain $2g-2+m \geq 0$.

3. On a rational component C' of C, if $\beta_{C'} > 0$, then C' must have at least two special points, i.e., a heavy marking or a node. When $\beta_{C'} = 0$, there are at least three special points, allowing at most one of them to be replaced by a light point.

4. On a genus one component, there is at least one special point or a light point. Otherwise, the line bundle $\mathcal{L}$ restricted to the component must be of positive degree.

5. The subset $B \subset C$ is the set of *base points* of a quasimap. Away from each base point, the sections of a quasimap defines a map to X_Σ.

6. Observe that the nondegeneracy condition is only related to heavy markings. Thus, light markings can collide with base points.

7. We do not have any *rational tails*, i.e., a component without any markings, since a quasimap with such a component is not stable.

8. The number of components of a quasimap is finite [14, Cor 3.1.5].

Remark 2.2.4. There is an equivalent description of a quasimap $(C; \underline{x}; \underline{y}, \underline{L}, \underline{s}, \underline{\phi})$ without the trivializations ϕ_m [14, Lem 3.1.8]. Choosing an integral basis for Pic(X_Σ), say $\{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$, the following data represent the same quasimap

$$(C; \underline{x}; \underline{y}, \underline{P}, \underline{s}),$$

where P_i are line bundles on C with $\deg P_i = \int_\beta c_1(\mathcal{P}_i)$, and $L_\rho = \bigotimes_{i=1}^r P_i^{\otimes a_{i\rho}}$. In Section 2.6, we will use this equivalent description for quasimaps.

Definition 2.2.5. Fix $g, m, k \geq 0$, $\beta \in H_2(X_\Sigma)$. The **moduli space of degree β stable toric quasimaps** to X_Σ is the moduli stack parametrizing isomorphism classes of families of stable toric $m|k$-quasimaps of degree β. Denote it by $Q_{g,m|k}(X_\Sigma, \beta)$.

The moduli space of quasimaps was constructed in [14, 16].

2.3 Stack quotients

Due to the presence of base points, a quasimap does not always define a map to the target toric variety. When a toric variety X_Σ has the toric quotient description $V \backslash Z_\Sigma // T$ from Theorem 2.1.1, where $V := \mathbb{C}^{\Sigma(1)}$ and $T := (\mathbb{C}^*)^r$, the natural target of a quasimap is the stack quotient $[V/T]$. Note that X_Σ is an open substack of $[V/T]$.

The cohomology of the stack quotient is given by

$$H^*([V/T]) = H^*([\text{pt}/T]) = H^*_T(\text{pt}) = H^*(T) = \mathbb{Q}[\sigma_1, \ldots, \sigma_r],$$

where $H^*(T)$ is the group cohomology of T. The variables σ_j have a geometric interpretation. Suppose that we have an integral basis $\{\mathcal{P}_j\}_{j=1}^r$ of Pic(X_Σ), where $\mathcal{P}_j$ correspond to $\mathcal{O}_{X_\Sigma}(D_j)$.

The jth column of the $r \times n$ action matrix $(a_{i\rho})$ from the toric quotient defines a T-equivariant line bundle $\mathbb{C} \to$pt. Then, the equivariant Euler class of this bundle gives rise to $\sigma_i \in H_T^*(\mathrm{pt})$.

In [16], the moduli space of quasimaps with light points $Q_{g,m|k}(X_\Sigma, \beta)$ was identified with the following moduli space of quasimaps to a stack quotient without any light points

$$Q_{g,m|0}([\mathbb{C}^n/(\mathbb{C}^*)^r] \times [\mathbb{C}/\mathbb{C}^*]^k, (\beta, 1, \ldots, 1)).$$

In [22], the authors showed that such a moduli space has a perfect obstruction theory which allows them to define the virtual fundamental class.

2.4 Evaluation maps

For the ith heavy marking, the evaluation map

$$ev_i : Q_{g,m|k}(X_\Sigma, \beta) \to X_\Sigma$$

is well-defined since heavy markings are distinct from base points and light markings, so that sections of a quasimap defines a map to X_Σ. In contrast, since light markings can collide with base points, sections of a quasimap might not define a map to X_Σ to define an evaluation map. In this case, the stack quotient $[V/T]$ can be used as the target of an evaluation map at each light marking. Thus, the evaluation map at the jth light marking is given as follows:

$$\hat{ev}_j : Q_{g,m|k}(X_\Sigma, \beta) \to [V/T].$$

For a fixed genus g and a degree β, we define the $m|k$-pointed quasimap invariants.

Definition 2.4.1. For $\phi_1, \ldots, \phi_m \in H^*(X_\Sigma)$ and $\xi_1, \ldots, \xi_k \in H^*([V/T])$, an $m|k$*-pointed quasimap invariant* is defined by

$$\langle \phi_1, \ldots, \phi_m \,|\xi_1, \ldots, \xi_k \rangle_{g,m|k,\beta} := \int_{[Q_{g,m|k}(X_\Sigma,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{m} ev_i^*(\phi_i) \prod_{j=1}^{k} \hat{ev}_j^*(\xi_j).$$

The virtual fundamental class $[Q_{g,m|k}(X_\Sigma, \beta)]^{\mathrm{vir}}$ will be defined in Section 2.6 using the notion of localized Euler class.

2.5 Localized Euler classes

We briefly review the localized Euler class in [21, §14.1]. For a fiber square of schemes

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

with i a regular embedding of codimension d, there exists an induced homomorphism between the Chows groups

$$i^! : A_* Y' \to A_* X',$$

which is called the *Gysin homomorphism* [21, §6.2].

Let E be a rank e vector bundle over a pure n-dimensional scheme X, and s is a section of E. Denote the zero scheme by $Z(s)$. Consider the fiber square

$$\begin{tikzcd} Z(s) \arrow[r, "i"] \arrow[d, "i"'] & X \arrow[d, "s"] \\ X \arrow[r, "0"] & E \end{tikzcd}$$

where 0 is the zero section. The *localized Euler class* is defined in [21, §14.1] as the class

$$e(E, s) := 0^! [X] \in A_{n-e}(Z(s)). \tag{2.2}$$

It satisfies

$$i_*(e(E, s)) = c_e(E) \cap [X] \in A_{n-e} X.$$

For later use, we give the following proposition.

Proposition 2.5.1 (Multiplicativity of localized Euler classes, [21, Example 17.4.8])**.** Let E_i be a rank e_i vector bundle over a pure n-dimensional scheme X with a section s_i, where $i = 1, 2$, and $E = E_1 \oplus E_2,\ s = s_1 \oplus s_2$. Then,

$$e(E, s) = e(E_1, s_1) \cup e(E_2, s_2) \in A_{n-(e_1+e_2)}(Z(s)).$$

2.6 Global construction of $Q_{g,m|k}(X_\Sigma, \beta)$

For a smooth project toric variety X_Σ with fan Σ, the quasimap moduli spaces $Q_{g,m|k}(X_\Sigma, \beta)$ can be embedded into a smooth Deligne–Mumford stack over which there is a vector bundle with a section that cuts out the quasimap moduli space. In this way, one may construct the virtual fundamental class of the quasimap moduli space using the localized Euler class. A detailed construction is given in [14].

Let $\mathfrak{M}_{g,m|k}$ be the moduli stack of prestable curves with m heavy points and k light and $\mathfrak{Pic}_{g,m|k}$ the Picard stack. Denote $r := \operatorname{rank} \mathrm{Pic}(X_\Sigma)$. Write $\mathfrak{Pic}^r_{g,m|k}$ for the r-fold fibered product of $\mathfrak{Pic}_{g,m|k}$ over $\mathfrak{M}_{g,m|k}$. There are natural forgetful maps $\mathrm{fgt_s} : Q_{g,m|k}(X_\Sigma, \beta) \to \mathfrak{Pic}^r_{g,m|k}$ by forgetting sections, and $\mathrm{fgt_{s,l}} : Q_{g,m|k}(X_\Sigma, \beta) \to \mathfrak{M}_{g,m|k}$ by forgetting both sections and line bundles. The following diagram commutes:

$$\begin{tikzcd}
Q_{g,m|k}(X_\Sigma, \beta) \arrow[r, "\mathrm{fgt_s}"] \arrow[dr, "\mathrm{fgt_{s,l}}"'] & \mathfrak{Pic}^r_{g,m|k} \arrow[d, "\mathrm{fgt_l}"] \\
& \mathfrak{M}_{g,m|k}
\end{tikzcd}$$

From [14, proof of Thm 3.2.1], there is an open and closed substack of finite type $\mathfrak{M}^\circ$ such that the map $\mathrm{fgt_{s,l}}$ factors through. This substack parametrizes all possible prestable curves coming from the source curves of quasimaps in $Q_{g,m|k}(X_\Sigma, \beta)$. These prestable curves in $\mathfrak{M}^\circ$ have no rational tails. Choose an integral basis $\{\mathcal{P}_i\}$ for $\mathrm{Pic}(X_\Sigma)$. Denote by $\mathfrak{Pic}^r_\beta$ the substack of $\mathfrak{Pic}^r_{g,m|k}$ over $\mathfrak{M}^\circ$ by imposing the conditions

1. the degree of P_i is equal to $\int_\beta c_1(\mathcal{P}_i)$

2. the stability condition holds,

$$\omega_C(\sum_{l=1}^{m} x_l + \sum_{l'=1}^{k} \epsilon y_{l'}) \otimes \big(\otimes_{\rho \in \Sigma(1)} L_\rho^{\otimes \alpha_\rho} \big)^\epsilon$$

is ample for every rational number $\epsilon > 0$, where $L_\rho := \otimes_i P_i^{\otimes a_{i\rho}}$.

These conditions are open conditions on the base, so that $\mathfrak{Pic}^r_\beta$ is an open substack.

Let $\mathcal{C} \to \mathfrak{M}^\circ$ be the universal curve. Write $\mathcal{C}\mathfrak{Pic}^r_\beta$ for the fiber product of $\mathcal{C}$ and $\mathfrak{Pic}^r_\beta$ over $\mathfrak{M}^\circ$ with the two projections $\pi_1 : \mathcal{C}\mathfrak{Pic}^r_\beta \to \mathcal{C}$ and $\pi_2 : \mathcal{C}\mathfrak{Pic}^r_\beta \to \mathfrak{Pic}^r_\beta$.

$$\begin{tikzcd}
\mathcal{C}\mathfrak{Pic}^r_\beta \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathfrak{Pic}^r_\beta \arrow[d] \\
\mathcal{C} \arrow[r] & \mathfrak{M}^\circ
\end{tikzcd}$$

By the condition (2), one can take a π_2-relative ample bundle $\mathcal{O}(1)$ over $\mathcal{C}\mathfrak{Pic}^r_\beta$, whose fiber at $((C, \underline{x}, \underline{y}), (P_i)_{i=1}^r) \in \mathfrak{Pic}^r_\beta$ is given by

$$\omega_C(\sum_{l=1}^{m} x_l + \sum_{l'=1}^{k} \epsilon y_{l'}) \otimes \big(\otimes_{\rho \in \Sigma(1)} L_\rho^{\otimes \alpha_\rho} \big). \tag{2.3}$$

For a family of quasimaps $C \to S$, there exists a positive integer $M = M(g, \beta, m, k)$ such that for all geometric points $s \in S$, $H^1(C_s, L_\rho(M)) = 0$ for all ρ [14, Cor 3.1.11]. We obtain the substack $\mathfrak{Pic}^{r,\circ}_\beta$ in $\mathfrak{Pic}^r_\beta$ by imposing the following conditions:

3. $H^1(C, L_\rho(M)) = 0$ for all $\rho \in \Sigma(1)$.

This is an open condition, so the substack $\mathfrak{Pic}^{r,\circ}_\beta$ is open. Denote by $\mathcal{C}\mathfrak{Pic}^{r,\circ}_\beta$ the restriction of $\mathcal{C}\mathfrak{Pic}^r_\beta$ to $\mathfrak{Pic}^{r,\circ}_\beta$.

There are universal line bundles $\mathcal{L}_i$ over $\mathcal{C}\mathfrak{Pic}^{r,\circ}_\beta$ with $i = 1, \ldots, r$, whose fiber over a point $((C, \underline{x}, \underline{y}), (P_i)_{i=1}^r) \in \mathfrak{Pic}^{r,\circ}_\beta$ is P_i. Write $\mathcal{L}_\rho := \otimes_i (\mathcal{L}_i)^{\otimes(a_{i\rho})}$ and set $\mathcal{V} := \oplus_\rho \mathcal{L}_\rho$.

As in the proof of [25, Proposition 5] or [26, Lemma 2.5], we take

$$\mathcal{B} := (\pi^*(\pi_*(\mathcal{V}^\vee(M))))^\vee(M), \tag{2.4}$$

where $\pi := \pi_2$ and $\mathcal{V}(M) := \mathcal{V} \otimes \mathcal{O}(M)$. This gives an embedding

$$\mathcal{V} \hookrightarrow \mathcal{B} \twoheadrightarrow \mathcal{E} \tag{2.5}$$

with $\mathbf{R}^1\pi_*(\mathcal{B}) = 0$, where $\mathcal{E}$ is the cokernel of the embedding. Then, we also have $\mathbf{R}^1\pi_*(\mathcal{E}) = 0$. Thus, $[\mathbf{R}\pi_*(\mathcal{B}) \to \mathbf{R}\pi_*(\mathcal{E})]$ forms a two-term resolution of $\mathbf{R}\pi_*(\mathcal{V})$ by vector bundles.

Define the total space of sections of $\pi_*(\mathcal{B})$ by

$$\mathrm{tot}(\pi_*(\mathcal{B})) := \mathrm{Spec}(\mathrm{Sym}(\mathbf{R}^1\pi_*(\omega_\pi \otimes \mathcal{B}^\vee)), \tag{2.6}$$

where ω_π is the relative dualizing sheaf for π and denote the corresponding map by

$$p : \mathrm{tot}(\pi_*(\mathcal{B})) \to \mathfrak{Pic}^{r,\circ}_\beta.$$

The fiber of p at $((C, \underline{x}, \underline{y}), (P_i)_{i=1}^r)$ is $\oplus_\rho H^0(C, L_\rho)$. Impose the generic nondegeneracy condition appearing in Definition 2.2.1, and the condition that there are no base points at nodes or heavy markings. Then, we obtain an open substack in $\mathrm{tot}(\pi_*(\mathcal{B}))$ which is smooth and Deligne–Mumford, say B. By pulling back $\pi_*(\mathcal{E})$ via p and restricting to B, we obtain a vector bundle $E := p^*\pi_*(\mathcal{E})|_B$, and a tautological section s induced from the map

$$H^0(\mathcal{C}, \mathcal{B}) \longrightarrow H^0(\mathcal{C}, \mathcal{E}).$$

The zero locus $Z(s)$ of the section s is exactly the quasimap moduli space $Q_{g,m|k}(X_\Sigma, \beta)$. Thus, we obtain a global model for a construction of the moduli space of quasimaps

$$\begin{tikzcd} & E \arrow[d] \\ Z(s) \arrow[r] & B \arrow[u, bend right, "s"'] \end{tikzcd} . \tag{2.7}$$

Definition 2.6.1. In the above situation, the virtual fundamental class of the moduli space of quasimaps $Q_{g,m|k}(X_\Sigma, \beta)$ is defined by taking the localized Euler class in the diagram (2.7):

$$[Q_{g,m|k}(X_\Sigma, \beta)]^{\mathrm{vir}} := e(E, s).$$

This construction was introduced in [14, Thm 3.2.1]. The virtual fundamental class defined above does not depend on the choice of embeddings and vector bundles. Also, it agrees with the Behrend–Fantechi version of virtual fundamental class in [25] defined using relative perfect obstruction $\mathbf{R}\pi_*\mathcal{B}^\vee$ (see [26, Prop 2.14]).

Chapter 3

Quantum $H^*(T)$-module structure on $H^*(X_\Sigma)$

3.1 Quantum $H^*(T)$-module structure on $H^*(X_\Sigma)$

To give a quantum deformation of $H^*(T)$-module, we introduce a coefficient ring. The *Novikov ring* is defined by

$$\Lambda := \Big\{ \sum_{\beta \in \mathrm{Eff}(X_\Sigma)} a_\beta q^\beta : a_\beta \in \mathbb{Q} \Big\}.$$

Fix $g = 0,\ m = 2$ and $k = 1$. Let $\{T_i\}$ be a basis for $H^*(X_\Sigma)$ and $\{T^j\}$ the dual basis under the intersection pairing on X_Σ.

Definition 3.1.1. For $\xi \in H^*(T)$ and $\phi \in H^*(X_\Sigma)$, define the **quantum** $H^*(T)$**-action** on $H^*(X_\Sigma)$ via the $2|1$-pointed quasimap invariants

$$\xi \star \phi := \sum_\beta q^\beta \sum_i \langle \phi, T_i\ |\xi\rangle_{0,2|1,\beta} T^i \in H^*(X_\Sigma, \Lambda),$$

where β runs over $\mathrm{Eff}(X_\Sigma)$.

The following theorem defines a quantum $H^*(T)$-module structure on $H^*(X_\Sigma)$.

Theorem 3.1.2 (Quantum $H^*(T)$-module structure)**.** *For $\xi, \zeta \in H^*(T)$ and $\phi \in H^*(X_\Sigma)$,*

$$\xi \star (\zeta \star \phi) = (\xi \cdot \zeta) \star \phi. \tag{3.1}$$

This can be viewed as a modified version of WDVV equations via 3-pointed Gromov–Witten invariants. We will prove Theorem 3.1.2 following a similar argument to the one used in Gromov–Witten theory. The key difference is that we use an equivalence of divisors in $H^*(\overline{M}_{0,2|2})$ instead of $H^*(\overline{M}_{0,4})$. Here, $\overline{M}_{0,2|d}$ is the Losev–Manin space defined in [27]. Introduce the following notations:

- $D(13|24)$: the class in $H^*(\overline{M}_{0,2|2})$ given by the locus of nodal curves that the one component has a heavy point marked by 1 and a light point marked by 3, and the other component has a heavy point marked by 2 and a light point marked by 4,
- $D(3=4)$: the class in $H^*(\overline{M}_{0,2|2})$ given by the locus of an irreducible component where the 3rd and 4th light markings are colliding.

Since $\overline{M}_{0,2|2} \simeq \mathbb{P}^1$, we obtain the following equivalence

$$D(13|24) = D(3=4).$$

3.2 Splitting axiom

We prove the splitting axiom in quasimap theory with a light point for the left-hand side of (3.1). Our proof uses elementary intersection theory, instead of obstruction theory.

Write $Q := Q_{0,2|2}(X_\Sigma, \beta)$ and $Q_{\beta_i} := Q_{0,2|1}(X_\Sigma, \beta_i),\ i = 1, 2$. There is a gluing morphism

$$\begin{aligned} \mathrm{gl} : \mathfrak{M}^\circ_{0,2|1} \times \mathfrak{M}^\circ_{0,2|1} &\to \mathfrak{M}^\circ_{0,2|2} \\ (C_1, (x_1, n_1; y_1)) \times (C_2, (x_2, n_2; y_2)) &\mapsto (C_1 \sqcup C_2/n_1 \sim n_2, (x_1, x_2; y_1, y_2)), \end{aligned}$$

where $C_1 \sqcup C_2/n_1 \sim n_2$ is the nodal curve given by C_1 and C_2 gluing n_1 and n_2. As in [18, Prop 5.2.2], there is a fiber square,

$$\begin{tikzcd} \bigsqcup_{\beta_1+\beta_2=\beta} Q_{\beta_1} \times_{X_\Sigma} Q_{\beta_2} \arrow[r, "l"] \arrow[d] & Q \arrow[d] \\ \mathfrak{M}^\circ_{0,2|1} \times \mathfrak{M}^\circ_{0,2|1} \arrow[r, "\mathrm{gl}"] & \mathfrak{M}^\circ_{0,2|2} \end{tikzcd}$$

For β_1 and β_2 with $\beta_1+\beta_2 = \beta$, write the natural inclusion as $i_{\beta_1|\beta_2} : Q_{\beta_1} \times_{X_\Sigma} Q_{\beta_2} \to \bigsqcup_{\beta_1+\beta_2=\beta} Q_{\beta_1} \times_{X_\Sigma} Q_{\beta_2}$. Let $\Delta : X_\Sigma \to X_\Sigma \times X_\Sigma$ be the diagonal embedding for a smooth projective toric variety

X_Σ. It is regular of codimension equal to $\dim X_\Sigma$. For a basis $\{T_i\}$ and the dual basis $\{T^j\}$,

$$[\Delta(X_\Sigma)] = \sum_i T_i \otimes T^i \in H^*(X_\Sigma) \otimes H^*(X_\Sigma) \simeq H^*(X_\Sigma \times X_\Sigma),$$

where i in T_i and T^i means dimension and codimension, respectively. The splitting axiom in quasimap theory with a light point is as follows:

Proposition 3.2.1 (Splitting axiom)**.** The following holds

$$i^*_{\beta_1|\beta_2}\mathrm{gl}^![Q]^{\mathrm{vir}} = \Delta^!([Q_{\beta_1}]^{\mathrm{vir}} \times [Q_{\beta_2}]^{\mathrm{vir}}).$$

Hence,

$$\mathrm{gl}^![Q]^{\mathrm{vir}} = \sum_{\beta_1+\beta_2=\beta} \Delta^!([Q_{\beta_1}]^{\mathrm{vir}} \times [Q_{\beta_2}]^{\mathrm{vir}}).$$

We will prove this proposition after giving some preliminary setup and lemmas.

Define $\mathcal{C}_i$ from the following fibered square:

$$\begin{tikzcd}
\mathcal{C}_i \arrow[r, "\overline{p}_i"] \arrow[d] & \mathcal{C}\mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_i} \arrow[d] \\
\mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_1} \times \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_2} \arrow[r, "p_i"] & \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_i}
\end{tikzcd}$$

where p_i are the projections. There is a universal curve

$$\mathcal{C}_1 \bigsqcup \mathcal{C}_2 \to \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_1} \times \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_2},$$

with the inclusions $c_i : \mathcal{C}_i \to \mathcal{C}_1 \bigsqcup \mathcal{C}_2$.

There is a natural inclusion $\mathrm{gl}^*\mathfrak{Pic}^{r,\circ}_{0,2|2,\beta_1,\beta_2} \to \mathrm{gl}^*\mathfrak{Pic}^{r,\circ}_{0,2|2,\beta}$, where $\mathrm{gl}^*\mathfrak{Pic}^{r,\circ}_{0,2|2,\beta_1|\beta_2}$ is the substack parametrizing r-uple of line bundles over a nodal curve whose restriction on one curve has degree β_1 and degree β_2 on the other curve. Consider the restriction morphism

$$r : \mathrm{gl}^*\mathfrak{Pic}^{r,\circ}_{0,2|2,\beta_1|\beta_2} \longrightarrow \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_1} \times \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_2},$$

given in the following way: over $(C_i, (x_i, n_i; y_i))_{i=1}^2 \in \mathfrak{M}^\circ_{0,2|1} \times \mathfrak{M}^\circ_{0,2|1}$,

$$(C_1 \sqcup C_2/n_1 \sim n_2, \{L_j\}_{j=1}^r) \longmapsto ((C_1, \{L_j|_{C_1}\}_{j=1}^r), (C_2, \{L_j|_{C_2}\}_{j=1}^r)).$$

Denote $\mathrm{gl}^*\mathcal{C}\mathfrak{Pic}^{r,\circ}_{0,2|2,\beta_1|\beta_2}$ by $\mathcal{C}$ and $\pi : \mathcal{C} \to \mathrm{gl}^*\mathfrak{Pic}^{r,\circ}_{0,2|2,\beta_1|\beta_2}$. There is a morphism $\overline{r} : r^*(\mathcal{C}_1 \bigsqcup \mathcal{C}_2) \to \mathcal{C}_1 \bigsqcup \mathcal{C}_2$ induced by the restriction morphism r. Also, there is a natural morphism

$$\nu : r^*\big(\mathcal{C}_1 \bigsqcup \mathcal{C}_2\big) \longrightarrow \mathcal{C},$$

given by normalizing a nodal curve at the node. By abuse of notation, we write

$$\nu_*\mathcal{V}_i := \nu_*\overline{r}^* c_{i*}\overline{p}_i^*\mathcal{V}_i, \quad \nu_*\mathcal{B}_i := \nu_*\overline{r}^* c_{i*}\overline{p}_i^*\mathcal{B}_i.$$

Write $\mathrm{gl}^*\mathcal{V}$ as the pullback of $\mathcal{V}$ over $\mathcal{C}$. Observe that we have the normalization exact sequence

$$\mathrm{gl}^*\mathcal{V} \hookrightarrow \bigoplus_{i=1}^2 \nu_*\mathcal{V}_i \xrightarrow{d} \mathrm{gl}^*\mathcal{V}|_n,$$

where the map d is given by the difference.

To have evaluation maps, using Lemma 3.4.1 in [28], we replace $\bigoplus_{i=1}^2 \mathcal{B}_i$ by $\bigoplus_{i=1}^2 \tilde{\mathcal{B}}_i$ with a surjective morphism δ commuting the following diagram:

$$\begin{tikzcd}
\bigoplus_{i=1}^2 \nu_*\mathcal{V}_i \arrow[r, hook] \arrow[d, two heads, "d"'] & \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{B}}_i \arrow[dl, two heads, "\delta"] \\
\mathrm{gl}^*\mathcal{V}|_n &
\end{tikzcd}$$

and $\mathbf{R}^1\pi_*\tilde{\mathcal{B}}_i = 0$, where $\pi_i : \mathcal{C}\mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_i} \to \mathfrak{Pic}^{r,\circ}_{0,2|1,\beta_i}$. Define $\tilde{\mathcal{B}}_i$ to be the equalizer of the following two morphisms:

$$\mathcal{B}_i \oplus (\mathcal{V}_i)|_{n_i} \xrightarrow{\text{proj}_1} \mathcal{B}_i \xrightarrow{\text{restriction}} (\mathcal{B}_i)|_{n_i}$$

$$\mathcal{B}_i \oplus (\mathcal{V}_i)|_{n_i} \xrightarrow{\text{proj}_2} (\mathcal{V}_i)|_{n_i} \xrightarrow{\text{inclusion}} (\mathcal{B}_i)|_{n_i}$$

Then, we obtain maps

$$e_i : \tilde{\mathcal{B}}_i \to (\mathcal{V}_i)|_{n_i}. \tag{3.2}$$

Taking the difference of e_1 and e_2 defines the desired morphism δ. Therefore, over $\mathcal{C}$, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{gl}^*\mathcal{V} \arrow[r, hook] \arrow[d, hook] & \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{B}}_i \arrow[r, two heads, "s"] \arrow[d, equal] & \tilde{\mathcal{E}} \arrow[d, two heads] \\
\bigoplus_{i=1}^2 \nu_*\mathcal{V}_i \arrow[r, hook] \arrow[d, two heads, "d"'] & \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{B}}_i \arrow[r, two heads, "s_1\oplus s_2"] \arrow[dl, two heads, "\delta"] & \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{E}}_i \\
\mathrm{gl}^*\mathcal{V}|_n & &
\end{tikzcd} \tag{3.3}$$

where $\tilde{\mathcal{E}}$ and $\bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{E}}_i$ are defined as the cokernels.

The exact sequence in the second row defines $e(E_1 \oplus E_2, s_1 \oplus s_2) = [Q_{\beta_1}]^{\mathrm{vir}} \times [Q_{\beta_2}]^{\mathrm{vir}}$, where $E_1 \oplus E_2$ is the pullback of $\pi_* \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{E}}_i$ along $p : \mathrm{tot}(\pi_* \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{B}}_i) \to \mathcal{C}$ as in the global construction method in Section 2.6. The exact sequence in the first row in (3.3) gives rises to $e(E, s) = i^*_{\beta_1|\beta_2}\mathrm{gl}^![Q]^{\mathrm{vir}}$, where $E := p^*\pi_*\tilde{\mathcal{E}}$. The exact sequence

$$\ker(\delta) \hookrightarrow \pi_* \bigoplus_{i=1}^2 \nu_*\tilde{\mathcal{B}}_i \twoheadrightarrow \mathrm{gl}^*\mathcal{V} \tag{3.4}$$

defines $e(V_n, \delta)$ where $V_n := p^*\pi_*\mathrm{gl}^*\mathcal{V}|_n$. We give a lemma to relate these three localized Euler classes $e(E, s),\ e(E_1 \oplus E_2, s_1 \oplus s_2)$, and $e(V_n, \delta)$.

Lemma 3.2.2. *Given a commutative diagram*

$$\begin{tikzcd}
& & & P \\
A \arrow[rr, hook, "\iota"] \arrow[dr, hook, "\rho"] & & M \arrow[ur, two heads, "q_P"] \arrow[r, two heads, "q"] \arrow[dr, two heads, "\nu_M"] & B \\
& N \arrow[ur, hook, "\iota_N"] \arrow[rr, two heads, "\nu_N"] & & P'
\end{tikzcd}$$

with exact sequences $A \hookrightarrow M \twoheadrightarrow B$, $N \hookrightarrow M \twoheadrightarrow P$ and $A \hookrightarrow N \twoheadrightarrow P'$, we can construct the following commutative diagram

$$\begin{tikzcd}
& K' \arrow[dr, hook, "\rho_M"] & & P & \\
A \arrow[ur, hook] \arrow[r, hook, two heads, "\sim", "\varphi"'] \arrow[dr, hook] & K \arrow[u, hook, "i_{K'}"] \arrow[d, hook, "i_N"'] \arrow[r, hook, "\overline{\iota}"'] & M \arrow[ur, two heads] \arrow[r, two heads] \arrow[dr, two heads] & B \arrow[u, two heads, "\tilde{q}_P"] \arrow[r, hook, two heads, "\psi", "\sim"'] \arrow[d, two heads, "\nu_B"] & P \oplus P' \arrow[ul, two heads, "\pi_P"'] \arrow[dl, two heads, "\pi_{P'}"] \\
& N \arrow[ur, hook] \arrow[rr, two heads] & & P' &
\end{tikzcd}$$

with exact sequences $K \hookrightarrow M \twoheadrightarrow P \oplus P'$ and $K' \hookrightarrow M \twoheadrightarrow P'$. In particular, B is canonically isomorphic to $P \oplus P'$.

Proof. (Construction of $\tilde{q}_P$ and its surjectivity) For $b \in B$, take $m \in q^{-1}(b)$, and define $\tilde{q}_P(b) := q_P(m)$. It is well-defined; for some $m' \in q^{-1}(b)$, $q(m - m') = 0$, thus $\iota(a) = m - m'$ for some $a \in A$. Since $\iota(a) = \iota_N(\rho(a))$, By the exactness of $N \hookrightarrow M \twoheadrightarrow P$, $q_P(m - m') = 0$. By the construction, $q_P = \tilde{q}_P \circ q$ holds and $\tilde{q}_P$ is surjective.

(Existence of an exact sequence $K \hookrightarrow M \to P \oplus P'$) By the universal property of the product, there exists a unique morphism $\overline{q} : M \to P \oplus P'$. Take $K := \ker \overline{q}$ and denote $\overline{\iota} : K \to M$.

(Construction of $\psi : B \to P \oplus P'$ and $\nu_B : B \to P'$) Let $a \in A$. Then, $q(\iota(a)) = 0$. Consider $q_P(\iota(a)) = \tilde{q}_P(q(\iota(a))) = 0$. Also, $\iota = \iota_N \circ \rho$ and the exactness of $A \hookrightarrow N \twoheadrightarrow P'$ imply $\nu_M(\iota(a)) = 0$. Thus, $\overline{q}(\iota(a)) = 0$. Since $B = \operatorname{coker} \iota$, by the universal property of the cokernel, there exists a unique morphism $\psi : B \to P \oplus P'$ satisfying $\overline{q} = \psi \circ q$. Define $\nu_B := \pi_{P'} \circ \psi$.

(Surjectivity of ν_B and $\tilde{q}_P$) The surjectivity of ν_B and $\tilde{q}_P$ follows from $\nu_M = \nu_B \circ q$ and $q_P = \tilde{q}_B \circ q$.

(Construction of K', i_N and $i_{K'}$, and injectivity) Define $K' := \ker \nu_M$ and set $\rho_M : K' \to M$. For $k \in K$, then $q_P(\bar{\iota}(k)) = 0$ and $\nu_M(\bar{\iota}(k)) = 0$ by projecting via π_P and $\pi_{P'}$. By the universal property of the kernels N and K', there exist unique morphisms $i_N : K \to N$ and $i_{K'} : K \to K'$, respectively. Since $\bar{\iota}$ is injective, one can show that i_N and $i_{K'}$ are injective.

(Construction of an isomorphism φ) Note that $\nu_M(\iota(a)) = 0$ because of $\iota = \iota_N \circ \rho$ and $\nu_N = \nu_M \circ \iota_N$, and the exactness of $A \hookrightarrow N \twoheadrightarrow P'$. Also, one can see that $q_P(\iota(a)) = 0$. Thus, $\overline{q}(\iota(a)) = 0$. By the universal property of the kernel K, there exist a unique map $\varphi : A \to K$ satisfying $\iota = \bar{\iota} \circ \varphi$. On the other hand, consider that the composition $K \hookrightarrow M \twoheadrightarrow P'$ is the zero morphism since $K' \hookrightarrow M \twoheadrightarrow P'$ is exact and $\bar{\iota} = \rho_M \circ i_{K'}$. Thus, from $\iota_N \circ i_N = \bar{\iota}$ and $\nu_N = \nu_M \circ \iota_N$, the composition $K \hookrightarrow N \twoheadrightarrow P'$ is also the zero morphism. Therefore, By the universal property of the kernel A being the kernel of ν_N, there exists a unique morphism $\tilde{\varphi} : K \to A$ such that $i_N = \rho \circ \tilde{\varphi}$. Last, we show that φ is an isomorphism. Consider that the injectivity of ρ implies the injectivity of φ. To show that φ is surjective, for $k \in K$, set $a := \tilde{\varphi}(k)$. Observe that

$$i_N(\varphi(a) - k) = \rho(a) - i_N(k) = \rho(\tilde{\varphi}(k)) - i_N(k) = i_N(k) - i_N(k) = 0.$$

Since i_N is injective, $\varphi(a) = k$, which implies that φ is surjective.

(Claim: ψ is an isomorphism) Since $A \simeq K$, we have $B \simeq \mathrm{Im}(\overline{q})$. Thus, it is enough to show that ψ is surjective. Let $(p, p') \in P \oplus P'$. Using the projections π_P and $\pi_{P'}$, by the surjectivity of q_P and ν_N, there exist $m \in M$ and $n \in N$ such that $q_P(m) = p$ and $\nu_N(n) = p'$. Set $x := \nu_M(m)$ and $y := q_P(\iota_N(n))$. Then, $\overline{q}(m) = (p, x)$ and $\overline{q}(\iota_N(n)) = (y, p')$. Note that the exactness of $N \hookrightarrow M \twoheadrightarrow P$ implies that $y = 0$. For $x \in P'$, there exists $n' \in N$ such that $\nu_N(n') = x$. Then, $\overline{q}(\iota_N(n')) = (0, x)$. Observe that

$$\overline{q}(m + \iota_N(n) - \iota_N(n')) = (p, x) + (0, p') - (0, x) = (p, p').$$

Hence, ψ is an isomorphism. $\square$

We apply Lemma 3.2.2 to the diagram (3.3) to relate the three localized Euler classes.

Lemma 3.2.3. *The three localized Euler classes* $e(E, s)$, $e(E_1 \oplus E_2, s_1 \oplus s_2)$, *and* $e(V_n, \delta)$ *are related*

$$e(E, s) = e(E_1 \oplus E_2, s_1 \oplus s_2) \cup e(V_n, \delta) \in A_*(Z(s)).$$

Proof of Lemma 3.2.3. Take

$$\begin{array}{lll} A := \mathrm{gl}^*\mathcal{V}, & M := \bigoplus_{i=1}^{2} \nu_*\tilde{\mathcal{B}}_i, & B := \tilde{\mathcal{E}} \\ N := \bigoplus_{i=1}^{2} \nu_*\mathcal{V}_i, & P := \bigoplus_{i=1}^{2} \nu_*\tilde{\mathcal{E}}_i, & P' := \mathrm{gl}^*\mathcal{V}|_n \end{array}$$

By Lemma 3.2.2,

$$\tilde{\mathcal{E}} \simeq \left(\bigoplus_{i=1}^{2} \nu_*\tilde{\mathcal{E}}_i \right) \bigoplus \mathrm{gl}^*\mathcal{V}|_n.$$

Applying Proposition 2.5.1 gives us the relation among three classes. $\square$

Recall that $e(E, s) = i^*_{\beta_1|\beta_2}\mathrm{gl}^![Q]^{\mathrm{vir}}$, $e(E_1 \oplus E_2, s_1 \oplus s_2) = [Q_{\beta_1}]^{\mathrm{vir}} \times [Q_{\beta_2}]^{\mathrm{vir}}$. We prove Proposition 3.2.1.

Proof of Proposition 3.2.1. Let $Y := \mathrm{tot}\big(\bigoplus_{i=1}^{2} \nu_*\tilde{\mathcal{B}}_i\big)$. Note that Y is smooth.

(Step 1: $\Delta^![Y] = e(V_n, \delta)$) Consider the following diagram of fiber squares:

$$\begin{tikzcd}
Z(\delta) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & Z(\delta) \arrow[r] \arrow[d] & Y \arrow[d, "0_{V_n}"] \\
Z(\delta) \arrow[r] \arrow[d] & Y \arrow[r, "\delta"] \arrow[d] & V_n \\
X_\Sigma \arrow[r, "\Delta"] & X_\Sigma \times X_\Sigma &
\end{tikzcd}$$

The evaluation map $Y \to [V/T] \times [V/T]$ at n_1 and n_2 is induced from e_1 and e_2 in (3.2). The evaluation map $Z(\delta) \to [V/T]$ is well-defined since $Z(\delta)$ is defined where the evaluation maps induced from e_1 and e_2 coincide. Since n_1 and n_2 do not collide with any base points and light markings, those evaluation maps factor through $X_\Sigma \times X_\Sigma$ and X_Σ, respectively. Then, from [21, Theorem 6.4],

$$0^!_{V_n}\Delta^![Y] = \Delta^! 0^!_{V_n}[Y] \tag{3.5}$$

Using the definition of the localized top Chern class (2.2) and [21, Remark 6.2.1], the equation (3.5) becomes

$$\begin{aligned} \text{id}^*\Delta^![Y] &= \text{id}^* e(V_n, \delta) \\ \Delta^![Y] &= e(V_n, \delta) \end{aligned}$$

(Step 2: $\Delta^! e(E_1 \oplus E_2, s_1 \oplus s_2) = e(E_1 \oplus E_2, s_1 \oplus s_2) \cup e(V_n, \delta)$) Consider the following diagram of fiber squares:

$$\begin{tikzcd}
Z(s) \arrow[r] \arrow[d] & Z(s_1 \oplus s_2) \arrow[r] \arrow[d] & Y \arrow[d, "0_{E_1 \oplus E_2}"] \\
Z(\delta) \arrow[r] \arrow[d] & Y \arrow[r, "s_1 \oplus s_2"] \arrow[d] & E_1 \oplus E_2 \\
X_\Sigma \arrow[r, "\Delta"] & X_\Sigma \times X_\Sigma &
\end{tikzcd}$$

We apply [21, Theorem 6.4], again.

$$0^!_{E_1 \oplus E_2}\Delta^![Y] = \Delta^! 0^!_{E_1 \oplus E_2}[Y] = \Delta^! e(E_1 \oplus E_2, s_1 \oplus s_2) \tag{3.6}$$

By Step 1, the left-hand side of (3.6) becomes $0^!_{E_1 \oplus E_2} e(V_n, \delta)$. Note that surjectivity of $\delta : \oplus \tilde{\mathcal{B}}_i \to \text{gl}^*\mathcal{V}|_n$ implies that $Z(\delta)$ is smooth. Thus,

$$0^!_{E_1 \oplus E_2} e(V_n, \delta) = e(E_1 \oplus E_2, s_1 \oplus s_2) \cup e(V_n, \delta).$$

Last, applying Lemma 3.2.3, we conclude with $\Delta^! e(E_1 \oplus E_2, s_1 \oplus s_2) = e(E, s)$. $\square$

3.3 Proof of Theorem 3.1.2

We expand both sides of the equation (3.1) in Theorem 3.1.2. The left-hand side gives us

$$\begin{aligned}\xi \star (\zeta \star \phi) &= \xi \star \big(\sum_{\beta} q^{\beta} \sum_{i} \langle \phi, T_i \,|\zeta\rangle_{0,2|1,\beta} T^i\big) \\ &= \sum_{\beta} q^{\beta} \sum_{\substack{\beta_1,\beta_2 \\ \beta_1+\beta_2=\beta}} \sum_{i,j} \langle \phi, T_i \,|\zeta\rangle_{0,2|1,\beta_1} \langle T^i, T_j \,|\xi\rangle_{0,2|1,\beta_2} T^j.\end{aligned} \tag{3.7}$$

The expansion of the right-hand side is

$$(\xi \cdot \zeta) \star \phi = \sum_{\beta} q^{\beta} \sum_{j} \langle \phi, T_j \,|\xi \cdot \zeta\rangle_{0,2|1,\beta} T^j. \tag{3.8}$$

To show the equality between (3.7) and (3.8), it is enough to verify that the coefficients of T^j in the degree β part are the same, i.e.,

$$\sum_{\substack{\beta_1,\beta_2 \\ \beta_1+\beta_2=\beta}} \sum_{i} \langle \phi, T_i \,|\zeta\rangle_{0,2|1,\beta_1} \langle T^i, T_j \,|\xi\rangle_{0,2|1,\beta_2} = \langle \phi, T_j \,|\xi \cdot \zeta\rangle_{0,2|1,\beta}. \tag{3.9}$$

Consider that there is a forgetful morphism

$$\mathrm{ft} : Q_{0,2|2}(X_{\Sigma}, \beta) \to \overline{M}_{0,2|2}.$$

This morphism is given by the map $\mathrm{fgt}_{s,l}$ forgetting line bundles and sections, followed by the stabilizing map $\mathfrak{M}^{\circ}_{0,2|2} \to \overline{M}_{0,2|2}$. The equivalence $D(13|24) = D(3 = 4)$ implies

$$\mathrm{ft}^*(D(13|24)) = \mathrm{ft}^*(D(3 = 4)). \tag{3.10}$$

For the pullback $\mathrm{ft}^*(D(13|24))$, as in [18, Prop 6.2.2], the following holds:

$$\mathrm{ft}^*(D(13|24)) \cap [Q]^{\mathrm{vir}} = l_* \mathrm{gl}^! [Q]^{\mathrm{vir}}. \tag{3.11}$$

Thus, by the projection formula,

$$\begin{aligned}&\int_{[Q]^{\mathrm{vir}}} ev_1^*(\phi)ev_2^*(T_j)\hat{ev}_3^*(\xi)\hat{ev}_4^*(\zeta)\cdot \mathrm{ft}^*(D(13|24)) \\ &= \int_{\mathrm{gl}^![Q]^{\mathrm{vir}}} l^*(ev_1^*(\phi)ev_2^*(T_j)\hat{ev}_3^*(\xi)\hat{ev}_4^*(\zeta)).\end{aligned} \tag{3.12}$$

For the pullback $\mathrm{ft}^*(D(3=4))$, similar to (3.11), one can see that

$$\mathrm{ft}^*(D(3=4))\cap[Q]^{\mathrm{vir}} = l'_*\mathrm{inc}^![Q]^{\mathrm{vir}}.$$

Using the projection formula,

$$\begin{aligned}&\int_{[Q]^{\mathrm{vir}}} ev_1^*(\phi)ev_2^*(T_j)\hat{ev}_3^*(\xi)\hat{ev}_4^*(\zeta)\cdot \mathrm{ft}^*(D(3=4)) \\ &= \int_{\mathrm{inc}^![Q]^{\mathrm{vir}}} l'^*(ev_1^*(\phi)ev_2^*(T_j)\hat{ev}_3^*(\xi)\hat{ev}_4^*(\zeta)).\end{aligned} \tag{3.13}$$

The equation (3.10) implies that (3.12) and (3.13) are equal. Thus, to conclude Theorem 3.1.2, it suffices to show that (3.12) and (3.13) are the same as the left-hand side and the right-hand side of (3.9), respectively.

Set $X := X_\Sigma$. Write e_1, e_2, and $\hat{e}_3$ for the evaluation maps on $Q_1 := Q_{\beta_1}$, and $\overline{e}_1$, $\overline{e}_2$, and $\overline{\hat{e}}_3$ on $Q_2 := Q_{\beta_2}$ to distinguish from evaluation maps on Q. One can write down

$$e_{2*}(e_1^*(\phi)\hat{e}_3^*(\xi)\cap[Q_1]^{\mathrm{vir}}) = \sum_j c_j T^j.$$

Applying the projection formula gives us

$$\begin{aligned}\langle \phi, T_i \mid \xi\rangle_{\beta_1} &= e_{2*}(e_1^*(\phi)e_2^*(T_i)\hat{e}_3^*(\xi)\cap[Q_1]^{\mathrm{vir}})\cap[X] \\ &= T_i e_{2*}(e_1^*(\phi)\hat{e}_3^*(\xi)\cap[Q_1]^{\mathrm{vir}})\cap[X] = c_i.\end{aligned}$$

Write $\overline{ev}$ for the evaluations on Q_2. Then, it is easy to see that

$$\begin{aligned}\sum_i \langle \phi, T_i \mid \xi\rangle_{\beta_1}\langle \psi, T^i \mid \zeta\rangle_{\beta_2} &= \langle \psi, \sum_i \langle \phi, T_i \mid \xi\rangle_{\beta_1} T^i \mid \zeta\rangle_{\beta_2} \\ &= \overline{e}_1^*(\psi)\overline{\hat{e}}_3^*(\zeta)\overline{e}_2^*(e_{2*}(e_1^*(\phi)\hat{e}_3^*(\xi)\cap[Q_1]^{\mathrm{vir}}))\cap[Q_2]^{\mathrm{vir}}. \qquad (3.14)\end{aligned}$$

Consider the following diagram:

$$\begin{tikzcd} Q_1 \times_X Q_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & Q_2 \arrow[d, "\overline{e}_2"] \\ Q_1 \arrow[r, "e_2"] & X \end{tikzcd}$$

We prove the following lemma, which is Lemma 6.2.7 in [18] modified to our case.

Lemma 3.3.1. *For $z_i \in A^*(Q_i)$, the following equality holds:*

$$p_{2*}(p_1^* z_1 p_2^* z_2 \cap \Delta^!([Q_1]^{vir} \times [Q_2]^{vir})) = (\overline{e}_2^* e_{2*}(z_1 \cap [Q_1]^{vir}))z_2 \cap [Q_2]^{vir}. \qquad (3.15)$$

Proof. For convenience, write $e := e_2$ and $\overline{e} := \overline{e}_2$. Consider the following diagram:

$$\begin{tikzcd} Q_1 \times_X Q_2 \arrow[r, "p_2"] \arrow[d] & Q_2 \arrow[r, "\overline{e}"] \arrow[d, "\Gamma_{\overline{e}}"] & X \arrow[d, "\Delta"] \\ Q_1 \times Q_2 \arrow[r, "e\times id"] \arrow[d, "\pi_1"'] & X \times Q_2 \arrow[r, "id\times\overline{e}"] \arrow[d, "\overline{\pi}_1"] & X \times X \\ Q_1 \arrow[r, "e"] & X & \end{tikzcd}$$

where $\Gamma_{\overline{e}}$ is the graph of $\overline{e}$. Denote the projections on the second factor by $\pi_2 : Q_1 \times Q_2 \to Q_2$ and $\overline{\pi}_2 : X \times Q_2 \to Q_2$. Then, $\overline{\pi}_2 \circ \Gamma_{\overline{e}} = \text{id}$ and $\overline{\pi}_1 \circ \Gamma_{\overline{e}} = \overline{e}$. Since X is smooth, $\Gamma_{\overline{e}}$ is regular. Thus, one can apply [21, Thm 6.2(c), Thm 6.2(a) and Remark 6.2.1] and the projection formula to the left-hand side of (3.15). Then,

$$\begin{aligned} p_{2*}(p_1^*z_1p_2^*z_2 \cap \Delta^!([Q_1]^{\text{vir}} \times [Q_2]^{\text{vir}})) &= p_{2*}\Delta^!(\pi_1^*z_1\pi_2^*z_2 \cap ([Q_1]^{\text{vir}} \times [Q_2]^{\text{vir}})) \\ &= p_{2*}\Gamma_{\overline{e}}^!(\pi_1^*z_1\pi_2^*z_2 \cap ([Q_1]^{\text{vir}} \times [Q_2]^{\text{vir}})) \\ &= \Gamma_{\overline{e}}^!(e \times id)_*(\pi_1^*z_1\pi_2^*z_2 \cap ([Q_1]^{\text{vir}} \times [Q_2]^{\text{vir}})) \\ &= \Gamma_{\overline{e}}^*(e \times id)_*(\pi_1^*z_1\pi_2^*z_2 \cap ([Q_1]^{\text{vir}} \times [Q_2]^{\text{vir}})) \\ &= \Gamma_{\overline{e}}^*(e \times id)_*(\pi_1^*(z_1 \cap [Q_1]^{\text{vir}})\pi_2^*(z_2 \cap [Q_2]^{\text{vir}})) \\ &= \Gamma_{\overline{e}}^*(\overline{\pi}_1^*e_*(z_1 \cap [Q_1]^{\text{vir}})\overline{\pi}_2^*(z_2 \cap [Q_2]^{\text{vir}})) \\ &= \Gamma_{\overline{e}}^*\overline{\pi}_1^*e_*(z_1 \cap [Q_1]^{\text{vir}})\Gamma_{\overline{e}}^*\overline{\pi}_2^*(z_2 \cap [Q_2]^{\text{vir}})) \\ &= (\overline{e}^*e_*(z_1 \cap [Q_1]^{\text{vir}}))z_2 \cap [Q_2]^{\text{vir}}) \end{aligned}$$

□

The following corollary proves that (3.12) is the same as the left-hand side (3.9).

Corollary 3.3.2. *For* $\beta_1, \beta_2 \in \text{Eff}(X_\Sigma)$ *with* $\beta_1 + \beta_2 = \beta$, $\phi, \psi \in H^*(X_\Sigma)$, *and* $\xi, \zeta \in H^*([V/T])$,

$$\int_{i^*_{\beta_1|\beta_2}\text{gl}^![Q]^{vir}} i^*_{\beta_1|\beta_2} l^*(ev_1^*(\phi)ev_2^*(\psi)\hat{ev}_3^*(\xi)\hat{ev}_4^*(\zeta)) = \sum_i \langle \phi, T_i \mid \xi\rangle_{0,2|1,\beta_1} \langle \psi, T^i \mid \zeta\rangle_{0,2|1,\beta_2} \tag{3.16}$$

Proof. Take $z_1 := e_1^*(\phi)\hat{e}_3^*(\xi)$ and $z_2 := \overline{e}_1^*(\psi)\overline{\hat{e}}_3^*(\zeta)$ and apply Lemma 3.3.1. Then, (3.14) becomes

$$p_{2*}(p_1^*(e_1^*(\phi)\hat{e}_3^*(\xi))p_2^*(\overline{e}_1^*(\psi)\overline{\hat{e}}_3^*(\zeta)) \cap \Delta^!([Q_1]^{\text{vir}} \times [Q_2]^{\text{vir}})). \tag{3.17}$$

Applying Proposition 3.2.1 to (3.17) gives us

$$p_{2*}(p_1^*(e_1^*(\phi)\hat{e}_3^*(\xi))p_2^*(\overline{e}_1^*(\psi)\overline{\hat{e}}_3^*(\zeta)) \cap i^*_{\beta_1|\beta_2}\text{gl}^![Q]^{\text{vir}}). \tag{3.18}$$

Pushing forward (3.18) to the Chow ring of a point, we obtain (3.16). □

Consider that there is a natural inclusion

$$\mathrm{inc} : \mathfrak{M}^{\circ}_{0,2|1} \hookrightarrow \mathfrak{M}^{\circ}_{0,2|2},$$

by sending $(C; x_1, x_2, y_1)$ to $(C; x_1, x_2, y_1, y_1)$. Denote $Q' := Q_{0,2|1}(X_\Sigma, \beta)$. There is a fibered square

$$\begin{tikzcd}
Q' \arrow[r, "l'"] \arrow[d] & Q \arrow[d] \\
\mathfrak{M}^{\circ}_{0,2|1} \arrow[r, "\mathrm{inc}"] & \mathfrak{M}^{\circ}_{0,2|2}
\end{tikzcd}$$

Successively pulling back along the inclusion, we obtain the following comutative diagram:

$$\begin{tikzcd}
\mathrm{inc}^*\mathcal{V} \arrow[r] \arrow[d] & \mathcal{V} \arrow[d] \\
\mathrm{inc}^*\mathcal{C}\mathfrak{Pic}^{r,\circ}_{0,2|2} \arrow[r] \arrow[d, "\pi'"'] & \mathcal{C}\mathfrak{Pic}^{r,\circ}_{0,2|2} \arrow[d, "\pi"'] \\
\mathrm{inc}^*\mathfrak{Pic}^{r,\circ}_{0,2|2} \arrow[r] \arrow[d] & \mathfrak{Pic}^{r,\circ}_{0,2|2} \arrow[d] \\
\mathfrak{M}^{\circ}_{0,2|1} \arrow[r, "\mathrm{inc}"] & \mathfrak{M}^{\circ}_{0,2|2}
\end{tikzcd}$$

One can observe that the embedding $\mathcal{V} \to \mathcal{B}$ pulled back over $\mathfrak{M}^{\circ}_{0,2|1}$ via the inclusion defines the virtual fundamental classes $[Q']^{\mathrm{vir}}$ using the method in Subsection 2.6. By the definition of the total space of sections (2.6), we have an induced map

$$Y' := \mathrm{tot}(\pi'_*\mathcal{B}') \to Y := \mathrm{tot}(\pi_*\mathcal{B}),$$

where $\mathcal{B}' := \mathrm{inc}^*\mathcal{B}$. Denote $\mathcal{E}' := \mathrm{inc}^*\mathcal{E}$ and $E' := p'^*\pi'_*(\mathcal{E}')$, where $p' : \mathrm{tot}(\pi^*_*\mathcal{B}') \to \mathrm{inc}^*\mathfrak{Pic}^{r,\circ}_{0,2|1}$. There are induced maps $Q' \to Q$ and $E' \to E$. We relate the virtual fundamental classes $[Q]^{\mathrm{vir}}$ and $[Q']^{\mathrm{vir}}$.

Lemma 3.3.3. *The following holds*

$$\mathrm{inc}^![Q]^{vir} = [Q']^{vir}.$$

Proof. From [21, Thm 6.4], the fibered diagram

$$\begin{tikzcd}
Q' \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & Y \arrow[d, "0"] \\
Y' \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & E \\
\mathfrak{M}^{\circ}_{0,2|1} \arrow[r, "\mathrm{inc}"] & \mathfrak{M}^{\circ}_{0,2|2} &
\end{tikzcd}$$

implies

$$\mathrm{inc}^![Q]^{\mathrm{vir}} = \mathrm{inc}^! 0^! [Y] = 0^! \mathrm{inc}^! [Y].$$

Then, since the zero sections 0 and $0'$ are regular embeddings, by [21, Thm 6.2(c)], the following fibered squares

$$\begin{tikzcd}
Q' \arrow[r] \arrow[d] & Y' \arrow[d] \\
Y' \arrow[r, "0'"] \arrow[d] & E' \arrow[d] \\
Y \arrow[r, "0"] & E
\end{tikzcd}$$

gives rise to

$$0^! \mathrm{inc}^! [Y] = 0'^! \mathrm{inc}^! [Y] = 0'^! [Y'] = [Q']^{\mathrm{vir}}.$$

$\square$

We show that (3.13) is the same as the right-hand side of (3.9). With commutativity of the following diagram

$$\begin{tikzcd}
Q \arrow[r, "ev_1 \times ev_2 \times \hat{ev}_3 \times \hat{ev}_4"] & X_\Sigma \times X_\Sigma \times [V/T] \times [V/T] \\
Q' \arrow[u, "l'"] \arrow[r, "ev'_1 \times ev'_2 \times \hat{ev}'_3 \times \hat{ev}'_3"] & X_\Sigma \times X_\Sigma \times [V/T] \times [V/T] \arrow[u, "id"']
\end{tikzcd}$$

applying Lemma 3.3.3 to (3.13) gives us

$$\begin{aligned}
&\int_{[Q']^{\mathrm{vir}}} ev'^*_1(\phi) ev'^*_2(T_j) \hat{ev}'^*_3(\xi) \hat{ev}'^*_3(\zeta) \\
&= \int_{[Q']^{\mathrm{vir}}} ev'^*_1(\phi) ev'^*_2(\psi) \hat{ev}'^*_3(\xi \cdot \zeta) = \langle \phi, T_j \mid \xi \cdot \zeta \rangle_{0,2|1,\beta}.
\end{aligned} \tag{3.19}$$

Therefore, we proved Theorem 3.1.2. □

Chapter 4

Localization formula for $\mathbb{F}_2$

In this section, we derive a localization formula for the quantum $H^*((\mathbb{C}^*)^2)$-module structure for the Hirzebruch surface $\mathbb{F}_2$.

The Hirzebruch surface $\mathbb{F}_2$ is a toric variety with the fan Σ depicted in Figure 4.1, where $\rho_1 = (-1, 2),\ \rho_2 = (1, 0),\ \rho_3 = (0, 1),\ \rho_4 = (0, -1)$ are primitive ray generators. The corresponding toric diagram is depicted in Figure 4.2. Note that $|\Sigma(1)| = 4$ and $\operatorname{rank}\operatorname{Pic}(\mathbb{F}_2) = 2$. Thus, there is

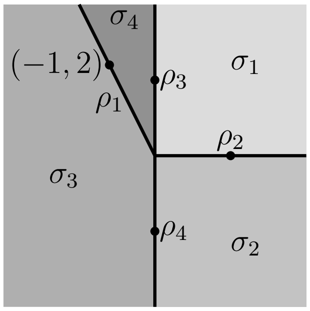

Figure 4.1: The fan for $\mathbb{F}_2$

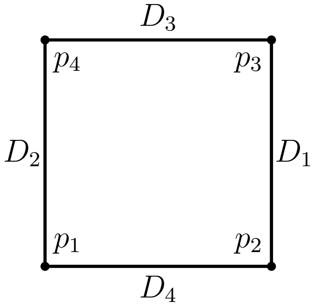

Figure 4.2: A toric diagram of $\mathbb{F}_2$

a $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4$ given by a 4×2 matrix; in this case, the action-matrix is

$$\begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{4.1}$$

The geometric quotient description for $\mathbb{F}_2$ from Section 2 is

$$\mathbb{F}_2 \simeq \mathbb{C}^4 /\!/ (\mathbb{C}^*)^2.$$

Each column of the matrix (4.1) gives rise to a line bundle $\mathcal{O}_{\mathbb{F}_2}(D_i)$, where D_i is the prime torus invariant divisor on $\mathbb{F}_2$ defined as the zero locus of the ith coordinate function on $\mathbb{C}^4$. We omit the subscript $\mathbb{F}_2$ in the notation of such a line bundle when the context is clear.

From the fan Σ of $\mathbb{F}_2$, the cohomology ring of $\mathbb{F}_2$ can be written as follows:

$$H^*(\mathbb{F}_2) \simeq \mathbb{Q}[D_2, D_4]/\langle D_2^2, D_4^2 + 2D_2D_4\rangle.$$

The reason why we chose the divisors D_2 and D_4 as generators will be discussed in Subsection 7.0.1. The cohomology ring has a graded structure as a $\mathbb{C}$-vector space:

$$H^*(\mathbb{F}_2) = H^0(\mathbb{F}_2) \oplus H^2(\mathbb{F}_2) \oplus H^4(\mathbb{F}_2) = \mathbb{C} \cdot [\mathbb{F}_2] \oplus (\mathbb{C} \cdot D_2 \oplus \mathbb{C} \cdot D_4) \oplus \mathbb{C} \cdot [pt],$$

where $D_2D_4 = [pt]$. The cohomology of the corresponding stack quotient is given as follows:

$$H^*([\mathbb{C}^4/(\mathbb{C}^*)^2]) \simeq H^*((\mathbb{C}^*)^2) \simeq \mathbb{Q}[\sigma_2, \sigma_4],$$

where σ_2 and σ_4 are the classes from the line bundles determined by the second column $(1, 0)^T$ and fourth column $(0, 1)^T$ of the action matrix (4.1).

The following Theorem gives an explicit description of the quantum $H^*((\mathbb{C}^*)^2)$-module structure on $H^*(\mathbb{F}_2)$.

Theorem 4.0.1. *The quantum $H^*((\mathbb{C}^*)^2)$-module structure for $\mathbb{F}_2$ is given by the following:*

$$\begin{aligned}
\sigma_2 \star 1 &= D_2 - \frac{1}{2} f(q_4) D_4 & \sigma_4 \star 1 &= (1+f(q_4))D_4 \\
\sigma_2 \star D_2 &= q_2q_4(1+f(q_4)) - \frac{1}{2} f(q_4)pt & \sigma_4 \star D_2 &= -\frac{1}{2} q_2 f(q_4) + (1+f(q_4))pt \\
\sigma_2 \star D_4 &= -2q_2q_4(1+f(q_4)) + (1+f(q_4))pt & \sigma_4 \star D_4 &= q_2(1+f(q_4)) - 2(1+f(q_4))pt \\
\sigma_2 \star pt &= q_2q_4(1+f(q_4))D_4 & \sigma_4 \star pt &= q_2D_2 - \frac{1}{2} q_2 f(q_4) D_4,
\end{aligned}$$

where $f(z) = \sum_{d\geq 1} \binom{2d}{d} z^d = \frac{1}{\sqrt{1-4z}} - 1$.

For the complete description of the $H^*((\mathbb{C}^*)^2)$-module structure on $H^*(\mathbb{F}_2)$, it is required to compute all $2|1$-pointed quasimap invariants for $\mathbb{F}_2$. We verify Theorem 4.0.1 through a series of reductions, and applying the Atiyah–Bott localization formula in [6, §4.3].

4.1 Reduction to particular 2-pointed invariants

By linearity, we only consider monomials in $\mathbb{Q}[\sigma_2, \sigma_4]$ for the insertion coming from the light point. From the module structure in Theorem 3.1.2, one can successively reduce the degree of the monomial-insertion. Then, it suffices to compute

$$\langle T_i, T_j \mid \sigma_k \rangle_{0,2|1,\beta},$$

where $T_0 := [\mathbb{F}_2],\ T_1 := [D_2],\ T_2 := [D_4],\ T_3 := [pt]$ form a basis for $H^*(\mathbb{F}_2)$.

In general, there does not exist a forgetful map $Q_{0,m+1|k}(X,\beta) \to Q_{0,m|k}(X,\beta)$ in the case of the moduli space of quasimaps. On the other hand, forgetting a light marking $Q_{0,m|k+1}(X,\beta) \to Q_{0,m|k}(X,\beta)$ is well-defined. In fact, this map coincides with the universal curve of $Q_{0,m|k}(X,\beta)$ [17, §2.2]. Then, following the argument in Gromov–Witten theory, the divisor equation holds for the insertion from the one light marking, i.e.,

$$\langle T_i, T_j | \sigma_k \rangle_{0,2|1,\beta} = (D_k \cdot \beta) \langle T_i, T_j \rangle_{0,2,\beta}, \tag{4.2}$$

where we took D_k the image of σ_k under the map $H^*([\mathbb{C}^4/(\mathbb{C}^*)^2]) \twoheadrightarrow H^*(\mathbb{F}_2)$. Therefore, Theorem 4.0.1 reduces to the computation of 2-pointed quasimap invariants of $\mathbb{F}_2$ with all possible insertions for all $\beta \in \operatorname{Eff}(\mathbb{F}_2)$. In this case, one can verify that $\operatorname{Eff}(\mathbb{F}_2) = \mathbb{Z}_{\geq 0}D_2 \oplus \mathbb{Z}_{\geq 0}D_4$.

Write $\beta = aD_2 + bD_4 \in \operatorname{Eff}(\mathbb{F}_2)$. Then, the virtual dimension of $Q_{0,2}(\mathbb{F}_2, \beta)$ is

$$\operatorname{v.dim} Q_{0,2}(\mathbb{F}_2, \beta) = 1 + 2a.$$

Observe that one insertion from $H^*(\mathbb{F}_2)$ can have codimension up to 2. To obtain a nonzero integral over $Q_{0,2}(\mathbb{F}_2, \beta)$, a must be 0 or 1. The only possible pairs of insertions that we need to consider are

$$\begin{aligned} &\langle D_i, 1\rangle_{0,2,\beta} \quad \text{when } \beta = dD_4, \\ &\langle D_i, pt\rangle_{0,2,\beta} \quad \text{when } \beta = D_2 + dD_4, \end{aligned}$$

where $i = 2, 4$ and $d \geq 0$. We will focus on the computation of $\langle D_i, 1\rangle_{0,2,dD_4}$ in detail, since computing $\langle D_i, pt\rangle_{0,2,D_2+dD_4}$ is similar.

For a component C' of a quasimap of degree dD_4, there exists a positive number d', such that $\beta_{C'} = (d', d', 0, -2d')$, i.e.,

$$L_1|_{C'} \simeq \mathcal{O}_{C'}(d'),\ L_2|_{C'} \simeq \mathcal{O}_{C'}(d'),\ L_3|_{C'} \simeq \mathcal{O}_{C'},\ L_4|_{C'} \simeq \mathcal{O}_{C'}(-2d').$$

One key observation is that, on a component C', the negative degree of the fourth line bundle forces the corresponding section to be the zero section along C'. Therefore, every component maps to D_4 in $\mathbb{F}_2$. In this case, $N_{D_4/\mathbb{F}_2} \simeq O_{\mathbb{P}^1}(-2)$ with $H^0(\mathbb{P}^1, O_{\mathbb{P}^1}(-2)) = 0$. This property is called *rigidity* in [29]. Thus, the fact that D_3 and D_4 are disjoint implies

$$\langle D_3, 1\rangle_{0,2,dD_4} = 0. \tag{4.3}$$

Since $D_3 = 2D_2 + D_4$ and $D_1 = D_2$, there are relations

$$\langle D_4, 1\rangle_{0,2,dD_4} = -2\langle D_2, 1\rangle_{0,2,dD_4}, \quad \langle D_1, 1\rangle_{0,2,dD_4} = \langle D_2, 1\rangle_{0,2,dD_4}. \tag{4.4}$$

Therefore, the computation of $\langle D_i, 1\rangle_{0,2,dD_4}$ for all i is reduced to the computation of

$$\langle D_2, 1\rangle_{0,2,dD_4}.$$

4.2 Linearization

Following [30, §2], the $\mathbb{C}^*$-equivariant cohomology of a point, denoted by $H^*(\mathbb{C}^*) := H^*_{\mathbb{C}^*}(\mathrm{pt})$, is the polynomial ring $\mathbb{Q}[\alpha]$, where α represents the Chern class of the hyperplane line bundle $\mathcal{O}_{\mathbb{P}^\infty}(1)$ (this convention is from [6, §27.1]). Consider the complex line bundle over a point $L_k \to \mathrm{pt}$ with a $\mathbb{C}^*$-action on the fiber given by $t.v = t^k v$. With the choice of the convention α, the $\mathbb{C}^*$-equivariant top Chern class of L_k, denoted by $e^{\mathbb{C}^*}(L_k)$, is $-k\alpha$. We call this the *weight* of L_k.

Consider a $\mathcal{T} := (\mathbb{C}^*)^4$-action on $\mathbb{F}_2 \simeq \mathbb{C}^4//(\mathbb{C}^*)^2$ given by

$$(t_1, t_2, t_3, t_4).(z_1, z_2, z_3, z_4) = (t_1 z_1, t_2 z_2, t_3 z_3, t_4 z_4).$$

The corresponding fixed points are

$$p_1 = (1:0:1:0),\ p_2 = (0:1:1:0),\ p_3 = (0:1:0:1),\ p_4 = (1:0:0:1).$$

One can linearize the line bundles $\mathcal{O}(D_i)$ using the weights of the tangent spaces at each $\mathcal{T}$-fixed point with respect to the $\mathcal{T}$-action. For example, at the fixed point $p_1 = (1:0:1:0) = D_2 \cap D_4$, we have $(1 : \frac{t_2}{t_1} : 1 : \frac{t_1^2 t_4}{t_3})$. It follows that

$$e^{\mathcal{T}}(\mathcal{T}_{p_1}\mathbb{F}_2) = (\alpha_1 - \alpha_2) \cdot (\alpha_3 - \alpha_4 - 2\alpha_1),$$

where $\alpha_1 - \alpha_2$ and $\alpha_3 - \alpha_4 - 2\alpha_1$ are the weights at p_1 along D_4-direction and D_2-direction, respectively. From [31, §4.3], we have

$$e^{\mathcal{T}}(T_{p_1}D_4) = e^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2)|_{p_1}), \quad e^{\mathcal{T}}(T_{p_1}D_2) = e^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_4)|_{p_1}).$$

Thus, $e^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2)|_{p_1}) = (\alpha_1 - \alpha_2)$ and $e^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_4)|_{p_1}) = (\alpha_3 - \alpha_4 - 2\alpha_1)$. In the same fashion, the four line bundles $\mathcal{O}(D_i)$ are canonically linearized from the $\mathcal{T}$-action. Table 4.1 shows all the weights of $\mathcal{O}^{\mathcal{T}}(D_i)|_{p_k}$.

Table 4.1: The weights of line bundles at each fixed points

	$\mathcal{O}^{\mathcal{T}}(D_1)$	$\mathcal{O}^{\mathcal{T}}(D_2)$	$\mathcal{O}^{\mathcal{T}}(D_3)$	$\mathcal{O}^{\mathcal{T}}(D_4)$
$p_1 = (1:0:1:0)$	0	$\alpha_1 - \alpha_2$	0	$\alpha_3 - \alpha_4 - 2\alpha_1$
$p_2 = (0:1:1:0)$	$\alpha_2 - \alpha_1$	0	0	$\alpha_3 - \alpha_4 - 2\alpha_2$
$p_3 = (0:1:0:1)$	$\alpha_2 - \alpha_1$	0	$2\alpha_2 - \alpha_3 + \alpha_4$	0
$p_4 = (1:0:0:1)$	0	$\alpha_1 - \alpha_2$	$2\alpha_1 - \alpha_3 + \alpha_4$	0

4.3 Fixed loci for $\langle D_2, 1\rangle_{0,2,dD_4}$

The $\mathcal{T}$-action on $\mathbb{F}_2$ induces an action on $Q_{0,2}(\mathbb{F}_2, \beta)$, so we can apply Atiyah–Bott localization to compute $\langle D_2, 1\rangle_{0,2,dD_4}$.

Depending on existence of a base point, there are two types of components of the source curve for a quasimap in a fixed locus F. For convenience, we give the following definitions to distinguish them.

Definition 4.3.1. Let F be a $\mathcal{T}$-fixed locus in $Q_{0,2}(\mathbb{F}_2, \beta)$ under the $\mathcal{T}$-action, and $(C; \underline{p}; \underline{q}, \underline{L}, \underline{\phi})$ a quasimap in F. We say that a component $C' \subseteq C$ is of **base-type** if the quasimap has a base point in C'. Otherwise, we say C' is of **map-type**. When all the components of C are of map-type, we say F is a fixed locus of **map-type**. Otherwise, F is called a fixed locus of **base-type**.

One can represent a fixed locus in $Q_{0,2}(\mathbb{F}_2, dD_4)$ as a decorated chain graph. The fact that the moduli spaces of quasimaps we consider have two marked points forces quasimaps to have a chain of $\mathbb{P}^1$'s for their source curve. Thus, a fixed locus can be represented by to a chain graph as follows.

- An edge represents one map-type component of a quasimap.
- A vertex represents a heavy marking, one node, or two nodes.

We decorate our chain graph to encode the rest of the information as follows.

- A dashed half-edge at a vertex corresponds to one base-type component.
- A blue labelling at a vertex stands for the fixed point in the image of a quasimap where the vertex goes.
- A red labelling at one end stands for a heavy marking.
- A positive number e_i on an edge is the degree of the map-type component represented by the edge.
- A positive number b_j attached to a dashed half-edge is the degree of the base-type component represented by the half-edge.

We provide informative notes on a decorated chain graph for a fixed locus.

- When a dashed half-edge is attached to a vertex, the number of nodes represented by the vertex is equal to the number of edges attached to the vertex.
- Without loss of generality, we fix the order of the red labellings for heavy markings.
- We put only one blue labelling, since these will alternate along consecutive vertices by rigidity. In fact, it is redundant, but it reminds us where the marking goes.
- We omit the red and blue color when it is clear.

Figure 4.3 shows a decorated chain graph of a fixed locus, including all possible types of vertices:

1. a vertex at one end without any dashed half-edge;
2. an *interior vertex* (i.e., not at one end) without any dashed half-edge;
3. an interior vertex with a dashed half-edge;
4. a vertex at one end with a dashed half-edge.

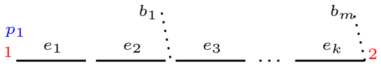

Figure 4.3: A decorated chain graph of a fixed locus for $\beta = dD_4$ with all possible vertex-types

We group the vertices as follows. For $k \in \{1, 2\}$, denote by

- I_k^m: the set of all interior vertices mapping to p_k and not carrying any dashed half-edges;
- I_k^b: the set of all interior vertices mapping to p_k and carrying a dashed half-edge;
- I_k^{end}: the set of all vertices at one end mapping to p_k and carrying a dashed half-edge.

To collect vertices going to either p_1 or p_2, denote

$$I^m := I_1^m \sqcup I_2^m, \qquad I^b := I_1^b \sqcup I_2^b, \qquad I^{end} := I_1^{end} \sqcup I_2^{end}, \quad I^B := I^b \sqcup I^{end}.$$

Last, introduce the following notations to count vertices in each sets

$$N_k^m := |I_k^m|, \qquad N_k^b := |I_k^b|, \qquad N_k^{end} := |I_k^{end}|,$$
$$N^m := |I^m|, \qquad N^b := |I^b|, \qquad N^{end} := |I^{end}|.$$

We assign to each vertex v the data $(i(v), b(v), e(v), e'(v), n(v), n'(v))$, where

- $i(v)$: the fixed point p_i to which the vertex v goes;

- $b(v)$: the degree of dashed half-edge attached to v; 0 if nothing is attached;
- $e(v)$: the degree of an edge attached to v (this is always positive);
- $e'(v)$: the degree of another line, if it exists; otherwise, it is 0;
- $n(v)$: one node that is represented by v;
- $n'(v)$: another node, if it exists.

We omit v and write i, b, e, e', n, n' when it is clear in the context. See Figure 4.4.

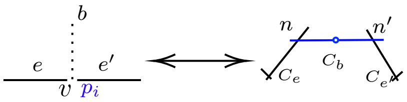

Figure 4.4: A picture of a local vertex

4.4 Virtual normal bundle

Applying the localization theorem, we obtain a formula for the invariant $\langle D_2, 1\rangle_{0,2,dD_4}$ in terms of equivariant classes

$$\langle D_2, 1\rangle_{0,2,dD_4} = \sum_{F:\text{ fixed loci}} \int_F \frac{ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_F}{e^{\mathcal{T}}(N_F^{\text{vir}})}, \tag{4.5}$$

where N_F^{vir} is the virtual normal bundle to F in the moduli space. One can view each fixed locus F as follow: for the decorated chain graph Γ_F,

$$F \simeq \prod_{\text{edge in }\Gamma_F} \overline{M}_{0,2} \times \prod_{\substack{\text{dashed half-edge in }\Gamma_F\\ \text{with degree } b_j}} \overline{M}_{0,2|b_j},$$

where $\overline{M}_{0,2}$ is interpreted as a point. For a detailed explanation, we refer readers to [5, §9.2], [32, §4], or [6, §27.4].

For a fixed locus F in $Q_{0,2}(\mathbb{F}_2, dD_4)$, we compute $e^{\mathcal{T}}(N_F^{\mathrm{vir}})$. Given a vector bundle E with a torus action, denote its nonzero weight part by E^{mov}. From the tangent-obstruction sequence as in [5, ch7 §1.4, ch9 §2.1] or [6, §24.4], the formula for the inverse of the equivariant Euler class of the virtual normal bundle to F is given by

$$\frac{1}{e^{\mathcal{T}}(N_F^{\mathrm{vir}})} = \frac{e^{\mathcal{T}}(\mathrm{Aut}(C, x_1, x_2)^{\mathrm{mov}})}{e^{\mathcal{T}}(\mathrm{Def}(C, x_1, x_2)^{\mathrm{mov}})\, e^{\mathcal{T}}(\mathrm{Def}(\underline{s})^{\mathrm{mov}})}, \tag{4.6}$$

where $(C; \underline{x}, \underline{P}, \underline{s})$ is a quasimap in F.

For convenience, set notations for weights

$$\begin{aligned} W_1 &:= e^{\mathcal{T}}(\mathcal{O}(D_2)|_{p_1}) = \alpha_1 - \alpha_2, \quad V_1 := e^{\mathcal{T}}(\mathcal{O}(D_4)|_{p_1}) = \alpha_3 - \alpha_4 - 2\alpha_1, \\ W_2 &:= e^{\mathcal{T}}(\mathcal{O}(D_2)|_{p_2}) = \alpha_2 - \alpha_1, \quad V_2 := e^{\mathcal{T}}(\mathcal{O}(D_4)|_{p_2}) = \alpha_3 - \alpha_4 - 2\alpha_2. \end{aligned}$$

Then,

$$W_1 = -W_2,\ V_2 = V_1 + 2W_1. \tag{4.7}$$

We compute each factor in (4.6). For a more detailed explanation, we refer [6, §27.4] to the reader.

1) Automorphisms of $(C; x_1, x_2)$: we do not have any contributions from automorphisms of a pointed source curve. The source curve of a quasimap in a fixed locus looks like a chain of $\mathbb{P}^1$'s. Each component has exactly two special points: a node or a heavy marking (recall that a red labelling in a decorated graph represents a heavy marking).

2) Deformation of $(C; x_1, x_2)$: a vertex $v \in I^m$ has valence 2 (dashed half-edges are not counted for valance). Since the two map-type components attached to this vertex are not contracting, the weight from the deformation of the source curve comes from the tensor product of the two tangent lines at the node. Thus, the contribution to $e^{\mathcal{T}}(N_F^{\mathrm{vir}})$ is

$$(-1)^{N_2^m} W_1^{N^m} \prod_{v \in I^m} \left(\frac{1}{e} + \frac{1}{e'} \right). \tag{4.8}$$

Consider that a vertex $v \in I^B$ can have one or two edges. Define

$$\epsilon(v) := \begin{cases} 0, & v \in I^{end} \\ 1, & v \in I^b. \end{cases}$$

For a vertex $v \in I^{end}$, there is one contracting component(dashed half-edge) and one non-contracting component(edge). On the other hand, a vertex $v \in I^b$ has one contracting component with two nodes to which two non-contracting components are attached. Figure 4.4 shows a local picture around a vertex in I^b, where the right figure is a picture of the graph in geometric side. The weight contribution from the deformation of the source curve is obtained as follows. At the nodes n(resp. n',) along C_e(resp. $C_{e'}$), we obtain the weight $\frac{W_i}{e}$(resp. $\frac{W_i}{e'}$). The base-type component C_b attached to this vertex $v \in I^B$ may be viewed as an element of the Losev–Manin space $\overline{M}_{2|b}$. The deformation of this contracting component gives rise to ψ-classes at the two nodes, see [6, §25.2]. Along C_b, we have $\psi_n := \psi_1(\overline{M}_{2|b})$ and $\psi_{n'} := \psi_2(\overline{M}_{2|b})$. Thus, the contribution to $e^{\mathcal{T}}(N_F^{\text{vir}})$ is

$$\prod_{v \in I^B} \left(\frac{W_i}{e} - \psi_n\right)\left(\frac{W_i}{e'} - \psi_{n'}\right)^{\epsilon(v)}. \tag{4.9}$$

3) Deformation of sections $\underline{s}$: There is an Euler sequence

$$0 \to \mathcal{O}_{\mathcal{C}}^{\oplus 2} \to \oplus_\rho \mathcal{L}_\rho \to \mathcal{F} \to 0 \tag{4.10}$$

over the universal curve $\pi : \mathcal{C} \to Q := Q_{0,2}(\mathbb{F}_2, \beta)$. In [14, §5.3], the relative obstruction theory over $\mathfrak{M}_{0,2}^{\circ}$ is given by

$$E^{\bullet}_{Q/\mathfrak{M}_{0,2}^{\circ}} = (\mathbf{R}^{\bullet}\pi_* \mathcal{F})^{\vee}. \tag{4.11}$$

We use this to compute the weight of Def$(\underline{s})^{\text{mov}}$. For the source curve C of a given quasimap in a fixed locus, one can break C into its components using normalization. The normalization exact sequence is given in the following way:

$$0 \to \mathcal{O}_C \to \bigoplus_{C_e:\text{map-type}} \mathcal{O}_{C_e} \oplus \bigoplus_{C_b:\text{base-type}} \mathcal{O}_{C_b} \to \bigoplus_{n:\text{node}} \mathcal{O}_n \to 0$$

Tensoring $\mathcal{F}$ and taking cohomology yields

$$\begin{aligned} 0 \to H^0(C,\mathcal{F}) \to \bigoplus_{C_e:\text{map-type}} H^0(C_e,\mathcal{F}) \oplus \bigoplus_{C_b:\text{base-type}} H^0(C_b,\mathcal{F}) \to \bigoplus_{n:\text{node}} T_{p_n}\mathbb{F}_2 \\ \to H^1(C,\mathcal{F}) \to \bigoplus_{C_e:\text{map-type}} H^1(C_e,\mathcal{F}) \oplus \bigoplus_{C_b:\text{base-type}} H^1(C_b,\mathcal{F}) \to 0, \end{aligned}$$

where p_n is the fixed point in $\mathbb{F}_2$ where the node n goes to.

Over a map-type component C_e, since sections define a map, it amounts to compute the weights of $H^i(C_e, (\underline{s}|_{C_e})^* T\mathbb{F}_2)$. Thus, the contribution to $1/e^{\mathcal{T}}(N_F^{\text{vir}})$ for one map-type component of degree e is

$$\frac{e^{2e}\prod_{j=0}^{2e-2}\left(V_1 + \frac{1+j}{e}W_1\right)}{e(e!)^2 W_1^{2e}(-1)^e}, \tag{4.12}$$

which agrees with the one in the case of the moduli space of stable maps.

For the nodes, since $T_{p_n}\mathbb{F}_2 \simeq (\mathcal{O}(D_i) \oplus \mathcal{O}(D_4))|_{p_n}$ where $i = 1, 2$ depending on p_n, one can easily take the equivariant Euler class for their weights. The contribution to $1/e^{\mathcal{T}}(N_F^{\text{vir}})$ is

$$\Big(\prod_{v\in I^m} W_i V_i\Big)\Big(\prod_{v\in I^{end}} W_i V_i\Big)\Big(\prod_{v\in I^b} W_i^2 V_i^2\Big). \tag{4.13}$$

For a vertex $v \in I^B$, there is a corresponding base-type component C_b which may be regarded as an element in the Losev–Manin space $\overline{M}_{2|b}$. In this case, since C_b has base points, the component is contracting to a torus fixed point

$$p_1 = D_2 \cap D_4 \text{ or } p_2 = D_1 \cap D_4.$$

Since we are considering the quotient $\mathcal{F}$ in the Euler sequence (4.10), the relative obstruction theory (4.11) that we need to consider for the computation of weights is the following:

$$\mathbf{R}^{\bullet}\pi_*(\mathcal{L}_2 \oplus \mathcal{L}_4)|_F \text{ or } \mathbf{R}^{\bullet}\pi_*(\mathcal{L}_1 \oplus \mathcal{L}_4)|_F, \tag{4.14}$$

respectively. Because of the base points on C_b, the equivariant line bundles $\mathcal{L}_i (i = 1, 2)$ and $\mathcal{L}_4$ in (4.14) becomes

$$\mathcal{O}^{\mathcal{T}}_{C_b}(q_1 + \cdots + q_b) \text{ and } \mathcal{O}^{\mathcal{T}}_{C_b}(-2q_1 - \cdots - 2q_b),$$

respectively, where q_i are the base points on C_b. For the first line bundle, we consider the following divisor sequences.

$$\begin{gathered}
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(\sum_{k=1}^{b-1} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(\sum_{k=1}^{b} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(\sum_{k=1}^{b} q_k)|_{q_b} \to 0,\\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(\sum_{k=1}^{b-2} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(\sum_{k=1}^{b-1} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(\sum_{k=1}^{b-1} q_k)|_{q_{b-1}} \to 0\\
\vdots\\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b} \to \mathcal{O}^{\mathcal{T}}_{C_b}(q_1) \to \mathcal{O}^{\mathcal{T}}_{C_b}(q_1)|_{q_1} \to 0,
\end{gathered}$$

For the second line bundle, similarly, we consider the following divisor sequences.

$$
\begin{gathered}
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k - q_b) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k - q_b)|_{q_b} \to 0 \\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k - q_b) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k)|_{q_b} \to 0 \\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-2} q_k - q_{b-1}) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-1} q_k - q_{b-1})|_{q_{b-1}} \to 0 \\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-2} q_k - q_{b-1}) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-2} q_k) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2\sum_{k=1}^{b-2} q_k)|_{q_{b-1}} \to 0 \\
\vdots \\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(-2q_1) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-q_1) \to \mathcal{O}^{\mathcal{T}}_{C_b}(-q_1)|_{q_1} \to 0 \\
0 \to \mathcal{O}^{\mathcal{T}}_{C_b}(-q_1) \to \mathcal{O}^{\mathcal{T}}_{C_b} \to \mathcal{O}^{\mathcal{T}}_{C_b}|_{q_1} \to 0.
\end{gathered}
$$

Take the associated long exact sequences and apply the equivariant Euler class. Then, work inductively to obtain the contribution of the deformation of sections. For $v \in I^B$, the contribution to $1/e^{\mathcal{T}}(N_F^{\text{vir}})$ is

$$
\begin{aligned}
e^{\mathcal{T}}(H^0(C, &\mathcal{O}^{\mathcal{T}}_C(q_1 + \cdots + q_b)))^{-1} \cdot e^{\mathcal{T}}(H^1(C, \mathcal{O}^{\mathcal{T}}_C(-2q_1 - \cdots - 2q_b))) \\
&= \frac{1}{b!} \frac{V_i}{W_i(W_i - \hat{\psi}_1)} \prod_{j=2}^{b} \frac{(V_i - 2\Delta_j)(V_i - 2\Delta_j + \hat{\psi}_j)}{W_i + \Delta_j},
\end{aligned}
\tag{4.15}
$$

where D_{ij} is the divisor on $\overline{M}_{2|b}$ parameterizing curves with q_i and q_j colliding, $\Delta_i := D_{1i} + D_{2i} + \cdots + D_{i-1,i}$, and $\hat{\psi}_j = -e^{\mathcal{T}}(\mathcal{O}(q_j)|_{q_j})$ is the ψ-class in $\overline{M}_{0,2|b}$ at the light point q_j. The factor $1/b!$ comes from permuting the base points. For more detailed explanation, we refer readers to [33].

Combining (4.8), (4.9), (4.13), (4.12), (4.15), we derive a formula for the inverse of the Euler class of the virtual normal bundl.

Proposition 4.4.1. For a fixed locus F of $Q_{0,2}(\mathbb{F}_2, dD_4)$,

$$\int_F \frac{1}{e^{\mathcal{T}}(N_F^{\mathrm{vir}})} = \mathrm{Cont}(VC)\, \mathrm{Cont}_m(NS) \prod_{e\in \text{Edges}} \mathrm{Cont}_E(e) \prod_{v\in I^B} \mathrm{Cont}_B(v), \tag{4.16}$$

where

$$\begin{aligned}
\mathrm{Cont}(VC) &:= (-1)^{N_2^m+N_2^{end}+2N_2^b} V_1^{N_1^m+N_1^{end}+2N_1^b} V_2^{N_2^m+N_2^{end}+2N_2^b} W_1^{N^m+N^{end}+2N^b}\\
\mathrm{Cont}_m(NS) &:= \left((-1)^{N_2^m} W_1^{N^m} \prod_{v\in I^m} \left(\frac{1}{e_v}+\frac{1}{e'_v}\right)\right)^{-1}\\
\mathrm{Cont}_E(e) &:= \frac{1}{e}\frac{e^{2e}\prod_{j=0}^{2e-2}\left(V_1+\frac{1+j}{e}W_1\right)}{(e!)^2 W_1^{2e}(-1)^e}\\
\mathrm{Cont}_B(v) &:= \int_{\overline{M}_{0,2|b}} \frac{1}{b!}\frac{\frac{V_i}{W_i^2}\prod_{j=2}^{b}\frac{(V_i-2\Delta_j)^2}{W_i+\Delta_j}}{\left(\frac{W_i}{e}-\psi_n\right)\left(\frac{W_i}{e'}-\psi_{n'}\right)^{\epsilon(v)}}.
\end{aligned}$$

Remark 4.4.2. The contribution $\mathrm{Cont}(VC)$ is from vertex-counting in the decorated chain graph of F, $\mathrm{Cont}_m(NS)$ from node-smoothing at a map-type vertex, $\mathrm{Cont}_E(e)$ from an edge C_e of degree e, and $\mathrm{Cont}_B(v)$ from a base-type vertex. In this formula, we omitted all ψ-classes $\hat{\psi}_j$ at light points. The reason is that when an integral over $\overline{M}_{0,2|b}$ includes $\hat{\psi}_j$, it is zero. See the formula (5.4).

Chapter 5

Calculus on $\overline{M}_{0,2|b}$ and combinatorial simplifications

In this chapter, we start by reducing the localization formula (4.16) from key observations. Then, we introduce calculus on $\overline{M}_{0,2|b}$ and some combinatorics(unordered partitions and symmetric functions) to simplify further.

One can observe that the denominator in (4.16) is of the form W_1^N.

Lemma 5.0.1. *The denominator in* (4.16) *is of the form* W_1^N *for some nonnegative* N.

Proof. It is clear by applying geometric series to the term $\left(\frac{W_i}{e} - \psi_n\right)\left(\frac{W_i}{e'} - \psi_{n'}\right)$. $\square$

Lemma 5.0.1 allows us to consider particular fixed loci when $\beta = dD_4$ that contribute to the final answer in the computation of $\langle D_2, 1\rangle_{0,2,dD_4}$. As a result, we do not need to figure out the complete expansion of $\mathrm{Cont}_B(v)$ which requires a somewhat complicated combinatorics problem. Furthermore, this observation reduces considerably the number of fixed loci required to succeed the computation. We call such a fixed locus a *necessary fixed locus*. The following corollary classifies necessary fixed loci.

Corollary 5.0.2. *The decorated chain graphs of necessary fixed loci for* $\beta = dD_4$ *are those in Figure 5.1, where* $b + e + e' = d$*. Write it as* $F^b_{e,e'}$*, and we omit indices if they are 0.*

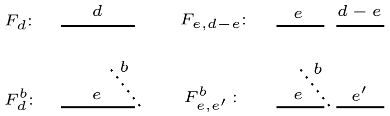

Figure 5.1: Necessary fixed loci for $\beta = dD_4$

Proof. For each fixed locus F, the insertion at the first/second marked point gives rise to the numerator

$$ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_F = W_1,$$

since the image of a quasimap in F lies on D_4. On the other hand, Lemma 5.0.1 tells us that for each fixed locus F, the denominator in (4.16) is W_1^N for some nonnegative N. Since the invariant $\langle D_2, 1\rangle_{0,2,dD_4}$ generates a rational number, the factor

$$V_1^{N_1^m+N_1^{end}+2N_1^b}$$

in (4.16) must vanish to contribute to the final answer. Hence, necessary fixed loci have $N_1^m = 0,\ N_1^{end} = 0,\ N_1^b = 0$. The four graphs in Figure 5.1 are all the decorated chain graphs of the necessary fixed loci satisfying this requirement. $\square$

Remark 5.0.3. This corollary tells us that a vertex with a dashed half-edge must contracts to p_2. Thus, we will set up $i = 2$ in V_i and W_i for such a vertex.

To reduce (4.16), it is necessary to manipulate further the denominator and the numerator of $\mathrm{Cont}_B(v)$ for a vertex $v \in I^B$. Recall that $\mathrm{Cont}_B(v)$ is given as an integration over the Losev–Manin space $\overline{M}_{0,2|b}$ whose $\dim$ is $b-1$.

i) (Denominator) The inverse of (4.9), when $\epsilon(v) = 1$, can be written as follows:

$$\begin{aligned}
&\frac{ee'}{W_i^2}\int_{\overline{M}_{0,2|b}} \sum_{s=0}^{b-1}\left(\frac{e\psi_n}{W_i}\right)^s \sum_{t=0}^{b-1}\left(\frac{e'\psi_{n'}}{W_i}\right)^t \\
&= \frac{ee'}{W_2^2}\int_{\overline{M}_{0,2|b}} \sum_{k=0}^{b-1}\frac{1}{W_2^k}\sum_{m=0}^{k}(e\psi_n)^{k-m}(e'\psi_{n'})^m \\
&= \frac{ee'}{W_1^2}\int_{\overline{M}_{0,2|b}} \sum_{k=0}^{b-1}\frac{(-1)^k}{W_1^k}\sum_{m=0}^{k}(e\psi_n)^{k-m}(e'\psi_{n'})^m.
\end{aligned}$$

ii) (Numerator) Recall the a vertex $v \in I^B$ must go to p_2, so set $i = 2$. We can write

$$\frac{V_i}{W_i^2}\prod_{j=2}^{b}\frac{(V_i - 2\Delta_j)^2}{W_i + \Delta_j} = \frac{V_2}{W_2^2}\prod_{j=2}^{b}\frac{(V_2 - 2\Delta_j)^2}{W_2 + \Delta_j} = \frac{V_1 + 2W_1}{W_1^2}\prod_{j=2}^{b}\frac{(V_1 + 2W_1 - 2\Delta_j)^2}{-W_1 + \Delta_j}. \tag{5.1}$$

Since a term in the numerator containing V_1 does not contribute to the final answer, we take $V_1 = 0$ in (5.1). Then, we obtain

$$\frac{(-1)^{b-1}2^{2b-2}2W_1}{W_1^2}\prod_{j=2}^{b}\frac{(W_1-\Delta_j)^2}{W_1-\Delta_j} = \frac{(-1)^{b-1}2^{2b-1}}{W_1}\prod_{j=2}^{b}(W_1-\Delta_j).$$

Thus, assuming $\epsilon(v) = 1$, i.e., $v \in I^b$, the contribution $\mathrm{Cont}_B(v)$ restricted to $V_1 = 0$ is

$$\begin{aligned}\mathrm{Cont}_B(v)|_{V_1=0} = {} & \frac{(-1)^{b-1}ee'2^{2b-1}W_1^{b-4}}{b!} \\ & \cdot \int_{\overline{M}_{0,2|b}}\prod_{j=2}^{b}(1-\frac{\Delta_j}{W_1})\sum_{k=0}^{b-1}\frac{1}{(-W_1)^k}\sum_{m=0}^{k}(e\psi_n)^{k-m}(e'\psi_{n'})^m\end{aligned} \tag{5.2}$$

5.1 Calculus on $\overline{M}_{0,2|b}$

To simplify (5.2), we give some facts about calculus on $\overline{M}_{0,2|b}$.

For the intersection of ψ-classes at heavy points, it is known from [34, Example 4.5] that

$$\int_{\overline{M}_{0,2|b}}\psi_n^{b-1-m}\psi_{n'}^{m} = \binom{b-1}{m}. \tag{5.3}$$

Unless $n+m = b-1$, then $\int_{\overline{M}_{0,2|b}}\psi_n^n\psi_{n'}^m = 0$. On the other hand, the intersection of ψ-classes at both heavy and light points is the following from [34, §4.6]

$$\int_{\overline{M}_{0,2|b}}\prod_{i=1}^{2}\psi_i^{n_i}\prod_{j=1}^{b}\hat{\psi}_j^{m_j} = 0 \tag{5.4}$$

if $m_j \neq 0$ for some j.

There are classes coming from collisions of light points.

Definition 5.1.1. Let b be a positive integer and $1 \leq l \leq b$. For a finite collection of mutually disjoint subsets $I_j \subseteq [b]$ where $j = 1, \ldots, l$, define

$$D_{I_1}D_{I_2}\cdots D_{I_l}$$

to be the cycle class in $H^*(\overline{M}_{0,2|b})$ of the closure of the locus, where for each $j = 1, \dots, l$, all light points marked by elements in I_j are colliding themselves. The degree of the class D_{I_j} is $|I_j| - 1$, so that the degree of the class $D_{I_1}D_{I_2}\cdots D_{I_l}$ is $\sum_{j=1}^{l}(|I_j| - 1)$. For a subset $I \subseteq [b]$ with $|I| = 1$, $D_I = [\overline{M}_{0,2|b}]$ denoted by 1.

Observe that the locus corresponding to $D_{\{i_1,\dots,i_l\}}$ is naturally isomorphic to as the Losev–Manin space $\overline{M}_{0,2|b-l+1}$. It follows that

$$\int_{\overline{M}_{0,2|b}} D_{\{i_1,\dots,i_{l+1}\}}\psi_n^{b-1-l-m}\psi_{n'}^{m} = \int_{\overline{M}_{0,2|b-l}} \psi_n^{b-1-l-m}\psi_{n'}^{m} \tag{5.5}$$

We provide a lemma for the product of two such cycle classes in $H^*(\overline{M}_{0,2|b})$.

Lemma 5.1.2. *For subsets $I, I' \subseteq [b]$,*

$$D_I \cdot D_{I'} = \begin{cases} D_I D_{I'}, & \text{if } |I \cap I'| = 0 \\ D_{I\cup I'}, & \text{if } |I \cap I'| = 1 \\ D_{I\cup I'}(-\hat{\psi}_{I\cap I'})^{|I\cap I'|-1}, & \text{if } |I \cap I'| > 1, \end{cases}$$

where $\hat{\psi}_J := \hat{\psi}_j|_{D_J}$ for every $j \in J$ (note that $\hat{\psi}_j|_{D_J} = \hat{\psi}_{j'}|_{D_J}$ for every $j, j' \in J$).

Proof. When $|I \cap I'| = 1$, $D_I \cdot D_{I'}$ is the locus where all light points in $I \cap I'$ are colliding without any repetition. Thus, $D_I \cdot D_{I'} = D_{I\cup I'}$.

Write $D_{12} := D_{\{1,2\}}$ and $-\hat{\psi}_{12} := -\hat{\psi}_{\{1,2\}}$. We claim

$$D_{12}^2 = (-\hat{\psi}_{12})D_{12}. \tag{5.6}$$

Recall the fact that $D_{12} \simeq \overline{M}_{0,2|1} \overset{i}{\hookrightarrow} \overline{M}_{0,2|2}$. Consider the exact sequence

$$T_{\overline{M}_{0,2|1}} \hookrightarrow i^*T_{\overline{M}_{0,2|2}} \twoheadrightarrow N_i.$$

Then, since $\overline{M}_{0,2|1} \simeq \mathrm{pt}$,

$$D_{12}^2 = D_{12} \cdot e(N_i) = -D_{12} \cdot e(T^*_{D_{12}}) = -D_{12} \cdot \hat{\psi}_{12}.$$

Observe that for $I = \{1, 2, \ldots, b\}$, using (5.6) gives

$$D_I^2 = (D_{12} \cdot D_{23} \cdot \cdots \cdot D_{b-1,b})^2 = (-\hat{\psi}_{12\cdots b})^{b-1}(D_{12} \cdot D_{23} \cdot \cdots \cdot D_{b-1,b}) = (-\hat{\psi}_I)^{b-1} D_I. \quad (5.7)$$

Thus, in general, for subsets $I, I' \subseteq [b]$ with $|I \cap I'| > 1$ and $i \in I \cap I'$, applying (5.7) implies

$$\begin{aligned} D_I \cdot D_{I'} &= D_{(I \setminus (I \cap I')) \cup \{i\}} \cdot D_{(I' \setminus (I \cap I')) \cup \{i\}} \cdot D^2_{I \cap I'} \\ &= D_{(I \setminus (I \cap I')) \cup \{i\}} \cdot D_{(I' \setminus (I \cap I')) \cup \{i\}} \cdot D_{I \cap I'} (-\hat{\psi}_{I \cap I'})^{|I \cap I'|-1} \\ &= D_{I \cup I'} (-\hat{\psi}_{I \cap I'})^{|I \cap I'|-1} \end{aligned}$$

When $|I \cap I'| = 0$, we claim that D_I and $D_{I'}$ are not equivalent in $H^*(\overline{M}_{0,2|b})$. Without loss of generality, assume that $|I| = |I'|$. Otherwise, codimensions of D_I and $D_{I'}$ are different. Let $b := |I| + |I'|$. Without loss of generality, we may assume $I = \{1, \ldots, k\},\ I' = \{k+1, \ldots, b\}$. Suppose that D_I and $D_{I'}$ are equivalent. Then, observe

$$D_I \cdot D_{I'} \cdot D_{k,k+1} = D_I^2 \cdot D_{k,k+1}.$$

Computing both sides using (5.7) gives us

$$D_{12\cdots b} = (-\hat{\psi}_I)^{k-1} D_{12\cdots k+1}.$$

From (5.4), taking the integration over $\overline{M}_{0,2|b}$ leads us to a contradiction

$$\int_{\overline{M}_{0,2|b}} D_{12\cdots b} = \int_{\overline{M}_{0,2|1}} 1 = 1 \neq 0 = \int_{\overline{M}_{0,2|b-k}} (-\hat{\psi}_I)^{k-1} = \int_{\overline{M}_{0,2|b}} (-\hat{\psi}_I)^{k-1} D_{12\cdots k+1}.$$

□

5.2 Unordered set partitions

We introduce basic combinatorics to reduce (5.2). Let b be a positive integer. Denote by $P(b)$ the set of all unordered set partitions of $[b]$. We always write an element $P \in P(b)$ as $P = (P_1, \dots, P_l)$ such that $p_1 := |P_1| \geq p_2 := |P_2| \geq \cdots \geq p_l := |P_l|$. Also, for $\lambda \vdash b$, denote $P(b, \lambda)$ by the subset of $P(b)$ satisfying $p_i = \lambda_i$ for all $i = 1, 2, \dots, l = l(\lambda)$, where $l(\lambda)$ is the length of λ. There is a decomposition of $P(b)$

$$P(b) = \bigsqcup_{\lambda \vdash b} P(b, \lambda).$$

For a partition $\lambda \vdash b$, the size $|P(b, \lambda)|$ is known as

$$|P(b, \lambda)| = \frac{b!}{\prod_{N=1}^{\infty} (N!)^{k_N} k_N!},$$

where $(k_N)_{N=1}^{\infty}$ is the multiplicity sequence for λ by letting k_N be the number of N's in λ.

We relate unordered set partitions and the class D_I with $I \subseteq [b]$. If $P = (P_1, \dots, P_l) \in P(b)$, write

$$D_P := D_{P_1} D_{P_2} \cdots D_{P_l}.$$

The codimension of D_P is $b - l$. For $\lambda \vdash b$, set

$$D_\lambda := \sum_{P \in P(b, \lambda)} D_P.$$

Using the combinatorial objects that we introduced, we simplifies the formula (5.2).

Lemma 5.2.1. *The following identities hold in* $H^*(\overline{M}_{0,2|b})$*:*

$$\prod_{j=2}^{b} (1 + \Delta_j) = \sum_{\lambda \vdash b} \left(\prod_{q=1}^{l(\lambda)} (\lambda_q - 1)! \right) D_\lambda = \sum_{l=1}^{b} \sum_{\lambda \vdash b : l(\lambda) = l} \left(\prod_{q=1}^{l(\lambda)} (\lambda_q - 1)! \right) D_\lambda.$$

Proof. We formally expand the left-hand side, apply Lemma 5.1.2, and then we count the number of D_I for a subset $I \subseteq [b]$. For $\lambda \vdash b$ and $P \in P(b, \lambda)$, there is a way to produce D_P by choosing 1 or D_{ij} in each factor $(1 + \Delta_j)$ and multiplying. Thus, one can write

$$\prod_{j=2}^{b}(1+\Delta_j) = \sum_{\lambda \vdash b} \sum_{P \in P(b,\lambda)} c_P D_P.$$

Since P_k are disjoint, the number of ways to form each D_{P_k} is independent of P_k's, say c_{P_k}. Thus, one can write

$$c_P D_P = \prod_{k=1}^{l(\lambda)} c_{P_k} D_{P_k}.$$

We count c_{P_k}. Without loss of generality, we may write $P_k = \{1, 2, \ldots, \lambda_k\}$. First, we choose $D_{i\lambda_k}$ in the factor $1 + \Delta_{\lambda_k}$, which is $|P_k \backslash \{\lambda_k\}|$ amount of choices. Next, we have $|P_k \backslash \{\lambda_k, \lambda_k - 1\}|$ amount of choices from $1 + \Delta_{\lambda_k - 1}$. This way amounts to

$$c_{P_k} = (\lambda_k - 1)!.$$

Therefore, it proves the first equality.

The second equality follows from expanding further with respect to codimensions. $\square$

Applying the second identity in Lemma 5.2.1 to the factor $\prod_{j=2}^{b}(1 - \frac{\Delta_j}{W_1})$, one can write (5.2) as follows:

$$\frac{(-1)^{b-1} e e' 2^{2b-1} W_1^{b-4}}{b!} \cdot \sum_{l=1}^{b} \sum_{\lambda \vdash b : l(\lambda) = l} \sum_{k=0}^{b-1} \sum_{m=0}^{k} \left(\prod_{q=1}^{l(\lambda)} (\lambda_q - 1)! \right) \frac{(-1)^{b-l+k} e^{k-m} e'^m}{W_1^{b-l+k}} \int_{\overline{M}_{0,2|b}} D_\lambda \psi_n^{k-m} \psi_{n'}^{m} \tag{5.8}$$

To obtain nonzero values for the integration, $\dim \overline{M}_{0,2|b} = b - 1$ must be the same as

$$\text{codim}\,(D_\lambda \psi_n^{k-m} \psi_{n'}^{m}) = (b-l) + (k-m) + m = b - l + k.$$

Thus, set k to be equal to $l-1$. Then, we use (5.3) and (5.5) to reduce (5.8), and obtain

$$\mathrm{Cont}_B(v) = \frac{ee'2^{2b-1}}{b!W_1^3} \cdot \sum_{l=1}^{b} (e+e')^{l-1} \sum_{\lambda \vdash b : l(\lambda)=l} \left(\prod_{q=1}^{l(\lambda)} (\lambda_q - 1)! \right) |P(b,\lambda)| \tag{5.9}$$

5.3 Symmetric functions theory

To reduce (5.9), we need symmetric function theory. We follow notations from [35]. Denote $p(b,\lambda)$ to be the subset of permutations in S_b whose cycle type is λ. Also, let

$$z_\lambda := \prod_{N=1}^{\infty} (N)^{k_N} k_N!,$$

where $(k_N)_{N=1}^{\infty}$ is the multiplicity sequence for λ as before. Then, one can easily see

$$\left(\prod_{q=1}^{l(\lambda)} (\lambda_q - 1)! \right) |P(b,\lambda)| = |p(b,\lambda)| = b! z_\lambda^{-1}. \tag{5.10}$$

For positive numbers b and e, introduce the multiset coefficient

$$\left(\!\!\binom{b}{e}\!\!\right) := \binom{b+e-1}{e},$$

which counts the number of monomials of degree e in b variables. One can have the following expression for a multiset coefficient.

Lemma 5.3.1. *For positive numbers b and e, the multiset coefficient $\left(\!\binom{b}{e}\!\right)$ can be written as follows:*

$$\left(\!\!\binom{b}{e}\!\!\right) = \sum_{l=1}^{b} \frac{e^{l-1}}{(b-1)!} \sum_{\lambda \vdash b : l(\lambda)=l} |p(b,\lambda)| \tag{5.11}$$

Recall the homogeneous and power sum symmetric functions:

$$h_b := \sum_{1\leq i_1\leq\cdots\leq i_b\leq b} x_{i_1}\cdots x_{i_b},$$
$$p_k := \sum_{i=1}^{b} x_i^k, \quad p_\lambda := p_{\lambda_1}\cdots p_{\lambda_l}.$$

Proof. From [35, Proposition 7.7.6], we have the following formula:

$$h_b = \sum_{\lambda\vdash b} z_\lambda^{-1} p_\lambda.$$

Evaluating $(1,\ldots,1)$ of length e to both h_b and p_λ above and using the second equality in (5.10) give us

$$\left(\!\!\binom{e}{b}\!\!\right) = h_b(1,\ldots,1) = \sum_{\lambda\vdash b} z_\lambda^{-1} p_\lambda(1,\ldots,1) = \sum_{l=1}^{b} \sum_{\lambda\vdash b: l(\lambda)=l} \frac{|p(b,\lambda)|}{b!} e^l.$$

Using the identity $\left(\!\!\binom{e}{b}\!\!\right)\frac{b}{e} = \left(\!\!\binom{b}{e}\!\!\right)$,

$$\left(\!\!\binom{b}{e}\!\!\right) = \sum_{l=1}^{b} \frac{e^{l-1}}{(b-1)!} \sum_{\lambda\vdash b: l(\lambda)=l} |p(b,\lambda)|.$$

□

We can obtain a concise simplification for (5.9).

Corollary 5.3.2. *For a necessary fixed locus F in $Q_{0,2}(\mathbb{F}_2, dD_4)$, assuming $\epsilon(v)=1$, and a vertex $v\in I^B$,*

$$\mathrm{Cont}_B(v)|_{V_1=0} = \frac{ee'2^{2b-1}}{bW_1^3}\left(\!\!\binom{b}{e+e'}\!\!\right).$$

Proof. Use the first equality in (5.10) to (5.9), then apply Lemma 5.3.1. □

Chapter 6

Computation of the quantum module structure of $\mathbb{F}_2$

6.1 Computation of the invariants $\langle D_i, 1\rangle_{0,2,dD_4}$

We complete our computation of the invariants $\langle D_2, 1\rangle_{0,2,dD_4}$.

Proposition 6.1.1. Let F be a necessary fixed locus in $Q_{0,2}(\mathbb{F}_2, dD_4)$ and $b+e+e'=d$. Then, the contribution of F in (4.5) to the invariant $\langle D_2, 1\rangle_{0,2,dD_4}$ is the following:

$$\int_F \frac{ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_F}{e^{\mathcal{T}}(N_F^{\text{vir}})}\Bigg|_{V_1=0} = \begin{cases} \dfrac{(-1)^d}{2d}\dbinom{2d}{d}, & F = F_d; \\ \dfrac{(-1)^d}{2d}\dbinom{2e}{e}\dbinom{2e'}{e'}, & F = F_{e,e'}; \\ \dfrac{(-1)^{d-b}2^{2b}}{2b}\dbinom{d-1}{b-1}\dbinom{2e}{e}, & F = F_e^b; \\ \dfrac{(-1)^{d-b}2^{2b}}{2b}\dbinom{d-1}{b-1}\dbinom{2e}{e}\dbinom{2e'}{e'}, & F = F_{e,e'}^b. \end{cases} \tag{6.1}$$

Proof. As in the proof of Corollary 5.0.2, the numerator is $ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_F = W_1$. Once we derive the formulas in (6.1) when $F = F_{e,e'},\ F_{e,e'}^b$, the rest cases can be done in a similar fashion. Assume $F = F_{e,e'}$. Using Proposition 4.4.1 with Corollary 5.3.2 to F, one can derive

$$\begin{aligned} \frac{W_1}{e^{\mathcal{T}}(N_F^{\text{vir}})}\Bigg|_{V_1=0} &= W_1 \frac{e^{2e}\prod_{j=0}^{2e-2}\left(\frac{1+j}{e}W_1\right)}{e(e!)^2 W_1^{2e}(-1)^e} \cdot \frac{e'^{2e'}\prod_{j=0}^{2e'-2}\left(\frac{1+j}{e'}W_1\right)}{e'(e'!)^2 W_1^{2e'}(-1)^{e'}} \cdot \frac{(-1)(2W_1)W_1}{(-1)W_1\left(\frac{1}{e}+\frac{1}{e'}\right)} \\ &= W_1 \cdot \frac{(-1)^e}{2eW_1}\binom{2e}{e}\frac{(-1)^{e'}}{2e'W_1}\binom{2e'}{e'} \cdot \frac{2W_1^2 ee'}{W_1(e+e')} = \frac{(-1)^d}{2d}\binom{2e}{e}\binom{2e'}{e'}. \end{aligned}$$

Next, let $F = F^b_{e,e'}$. Similarly, we obtain

$$\begin{aligned}
\frac{W_1}{e^{\mathcal{T}}(N_F^{\mathrm{vir}})}\bigg|_{V_1=0} &= W_1 \frac{e^{2e}\prod_{j=0}^{2e-2}\left(\frac{1+j}{e}W_1\right)}{e(e!)^2 W_1^{2e}(-1)^e} \cdot \frac{e'^{2e'}\prod_{j=0}^{2e'-2}\left(\frac{1+j}{e'}W_1\right)}{e'(e'!)^2 W_1^{2e'}(-1)^{e'}} \\
&\quad \cdot \frac{(-1)^2(2W_1)^2W_1^2}{1} \cdot \frac{ee'2^{2b-1}}{bW_1^3}\left(\!\!\binom{b}{e+e'}\!\!\right) \\
&= W_1 \frac{(-1)^{e+e'}}{4ee'W_1^2}\binom{2e}{e}\binom{2e'}{e'} \cdot (-1)^2 4W_1^4 \cdot \frac{ee'2^{2b-1}}{bW_1^3}\left(\!\!\binom{b}{e+e'}\!\!\right) \\
&= \frac{(-1)^{d-b}2^{2b}}{2b}\binom{d-1}{b-1}\binom{2e}{e}\binom{2e'}{e'}.
\end{aligned}$$

□

Finally, we sum up all the cases in Proposition 6.1.1.

Proposition 6.1.2. All 2-pointed degree dD_4 quasimap invariants of $\mathbb{F}_2$ are given by

$$\begin{aligned}
&\langle D_1, 1\rangle_{0,2,dD_4} = \langle D_2, 1\rangle_{0,2,dD_4} = -\frac{1}{2d}\binom{2d}{d}, \\
&\langle D_3, 1\rangle_{0,2,dD_4} = 0, \quad \langle D_4, 1\rangle_{0,2,dD_4} = \frac{1}{d}\binom{2d}{d}.
\end{aligned}$$

Proof. By the argument below Lemma 5.0.1, the invariant $\langle D_2, 1\rangle_{0,2,dD_4}$ can be obtained from (4.5) by summing over only necessary fixed loci with the restriction $V_1 = 0$:

$$\begin{aligned}
\langle D_2, 1\rangle_{0,2,dD_4} &= \frac{ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_{F_d}}{e^{\mathcal{T}}(N^{\mathrm{vir}}_{F_d})}\bigg|_{V_1=0} + \sum_{e=1}^{d-1} \frac{ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_{F_{e,d-e}}}{e^{\mathcal{T}}(N^{\mathrm{vir}}_{F_{e,d-e}})}\bigg|_{V_1=0} \\
&\quad + \sum_{b=1}^{d-1} \frac{ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_{F^b_{d-b}}}{e^{\mathcal{T}}(N^{\mathrm{vir}}_{F^b_{d-b}})}\bigg|_{V_1=0} + \sum_{b=1}^{d-2}\sum_{e=1}^{d-b-1} \frac{ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2))|_{F^b_{e,d-b-e}}}{e^{\mathcal{T}}(N^{\mathrm{vir}}_{F^b_{e,d-b-e}})}\bigg|_{V_1=0}.
\end{aligned}$$

Apply Proposition 6.1.1, and observe that the first term is in fact a summand of the first summation as the case $e = 0$. Then,

$$\sum_{e=0}^{d-1} \frac{(-1)^d}{2d} \binom{2e}{e} \binom{2(d-e)}{d-e} + \sum_{b=1}^{d-1} \frac{(-1)^{d-b} 2^{2b}}{2b} \binom{d-1}{b-1} \binom{2(d-b)}{d-b}$$
$$+ \sum_{b=1}^{d-2} \sum_{e=1}^{d-b-1} \frac{(-1)^{d-b} 2^{2b}}{2b} \binom{d-1}{b-1} \binom{2e}{e} \binom{2(d-b-e)}{d-b-e}.$$

In the second summation, we extract the term with $b = d - 1$, which is -4^{d-1}. The rest of the second summation with $b = 1, \ldots d - 2$ can be put inside the last summation as the case $e = 0$, since $\frac{1}{b}\binom{d-1}{b-1} = \frac{1}{d}\binom{d}{b}$. Then,

$$\sum_{e=0}^{d-1} \frac{(-1)^d}{2d} \binom{2e}{e} \binom{2(d-e)}{d-e} - 4^{d-1} + \sum_{b=1}^{d-2} \frac{(-1)^{d-b} 2^{2b}}{2d} \binom{d}{b} \sum_{e=0}^{d-b-1} \binom{2e}{e} \binom{2(d-b-e)}{d-b-e}.$$

The first and second terms can collapse into the last summation as the case $b = 0$ and $b = d - 1$, respectively. Thus,

$$\sum_{b=0}^{d-1} \frac{(-1)^{d-b} 2^{2b}}{2d} \binom{d}{b} \sum_{e=0}^{d-b-1} \binom{2e}{e} \binom{2(d-b-e)}{d-b-e}. \tag{6.2}$$

Using the formula $\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n$, one can write the equation (6.2) as

$$\sum_{b=0}^{d-1} \frac{(-1)^{d-b} 2^{2b}}{2d} \binom{d}{b} \left(4^{d-b} - \binom{2(d-b)}{d-b} \right). \tag{6.3}$$

After distributing the summation, observe that the first summation is, in fact, the case $b = d$ of the next summation. Thus, the equation (6.3) becomes

$$\sum_{b=0}^{d} \frac{(-1)^{d-b-1} 4^b}{2d} \binom{d}{b} \binom{2(d-b)}{d-b}.$$

Apply the formula $\binom{2n}{n} = (-4)^n \binom{-1/2}{n}$ to have

$$-\frac{4^d}{2d} \sum_{b=0}^{d} \binom{d}{b} \binom{-1/2}{d-b}.$$

For the last step to obtain $-\frac{1}{2d}\binom{2d}{d}$, it is enough to show that the coefficient of x^d from the following is $4^{-d}\binom{2d}{d}$

$$(1+x)^{d-1/2} = \sum_{l=0}^{d} x^l \sum_{b=0}^{l} \binom{d}{b} \binom{-1/2}{d-b} + (\text{higher order terms}).$$

To achieve this, write

$$(1+x)^{d+1-1/2} = \cdots + a_{d+1}x^{d+1} + \cdots .$$

Observe that the induction hypothesis for d allows us to write the derivative of $(1+x)^{d+1-1/2}$ as

$$(d+1/2)(1+x)^{d-1/2} = (d+1/2)(\cdots + 4^{-d}\binom{2d}{d}x^d + \cdots).$$

Therefore, a_{d+1} is equal to

$$\frac{1}{d+1}(d+1/2)4^{-d}\binom{2d}{d} = 4^{-(d+1)}\binom{2(d+1)}{d+1}.$$

We computed $\langle D_3, 1\rangle_{0,2,dD_4} = 0$ from (4.3). From the relation (4.4), we can derive $\langle D_4, 1\rangle_{0,2,dD_4} = \frac{1}{d}\binom{2d}{d}$. □

We successfully computed all the invariants $\langle D_i, 1\rangle_{0,2,dD_4}$ for all $i = 1, 2, 3, 4$.

6.2 Computation of the invariants $\langle D_i, pt\rangle_{0,2,D_2+dD_4}$

To have a full description of the quantum module structure for $\mathbb{F}_2$, it still remains to compute 2-pointed quasimap invariants of degree $D_2 + dD_4$, with $d \geq 0$. This computation is very similar to the computation for degree dD_4 in the previous section. Thus, we omit details but point out all the features that are different from the previous computation.

In order to compute the invariant $\langle D_i, pt\rangle_{0,2,D_2+dD_4}$ by localization, choose an equivariant lift

$$[p_4]^{\mathcal{T}} = c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2)) \cdot c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_3))$$

for the insertion $[pt]$. Recall that $D_2 \cap D_3 = p_4$.

Lemma 6.2.1. *Let F a fixed locus in $Q_{0,2}(\mathbb{F}_2, D_2 + dD_4)$. Assume that we choose the above equivariant lift for the inserstion $[pt]$. Then, for the invariants $\langle D_i, pt\rangle_{0,2,D_2+dD_4}$, there exists a component C_0 of degree D_2 in the source curve C of a quasimap in F satisfying*

(i) C_0 must be at one end in C (here, C is regarded as a chain of $\mathbb{P}^1$'s);

(ii) the second marking must be on C_0;

(iii) the image of C_0 lies on the D_2-curve in $\mathbb{F}_2$;

(iv) C_0 is of map-type.

Hence,

$$\langle D_3, pt\rangle_{0,2,D_2+dD_4} = 0.$$

Proof. We are considering quasimaps is in the fixed locus F. Thus, there exists one and only one component of degree D_2, say C_0. The first assertion (i) is clear, since components except C_0 must land on D_4 in $\mathbb{F}_2$.

For (ii), due to our choice of the equivariant lift $[p_4]^{\mathcal{T}}$ for the second insertion $[pt]$, the second marking must be in C_0. Otherwise, there is no chance for the second marking to go to p_4.

Note that $(D_2 \cdot D_\rho)_{\rho=1}^4 = (0, 0, 1, 1)$ implies that the image of C_0 lies on either D_1 or D_2. By (ii), the image of C_0 must lie on the D_2-curve. Thus, (iii) is proved.

Claim that C_0 cannot have any base points for (iv). If there is a base point, C_0 contracts to either p_1 or p_4. It must be p_4, since the second marking on C_0 must map to p_4. However, it is also not true because the consecutive component of degree $d'D_4$ must land on the D_4-curve, contradiction occurs. Hence, C_0 must be of map-ype.

So far, the only component of degree D_2, which is of map-type and located at one end of the chain of $\mathbb{P}^1$'s, has the second marking that goes to p_4. To verify $\langle D_3, pt\rangle_{0,2,D_2+dD_4} = 0$, observe that the first marking is in a component of degree $d''D_4$ that is at the other end of the chain. This component maps to the D_4-curve. Recall that $D_3 \cap D_4 = \varnothing$. Thus, there is no such a quasimap in F. Therefore, $\langle D_3, pt\rangle_{0,2,D_2+dD_4} = 0$. $\square$

From the relations in $H^*(\mathbb{F}_2)$, it suffices to compute

$$\langle D_1, pt\rangle_{0,2,D_2+dD_4}.$$

In our decorated chain graph expression for a fixed locus, we depict the component of degree D_2 by a vertical line. Fix this vertical line on the left. The chain graph of a fixed locus is presented in Figure 6.1. The D_1-insertion forces that the number of horizontal edges in the decorated chain graph of a fixed locus must be odd, so that the first marking maps to D_1.

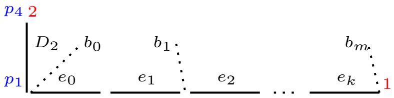

Figure 6.1: A decorated chain graph of a fixed locus for $\beta = D_2 + dD_4$ with all possible vertex-types, except the vertex with $b_0 = 0$

To have a formula similar to (4.16) in Proposition 4.4.1, we need to modify the notations for grouping vertices and introduce more. For $k \in \{1, 2\}$, denote by

- I_k^m: the set of all interior vertices mapping to p_k, not carrying any dashed half-edges, and not at the left end;
- I_k^b: the set of all interior vertices mapping to p_k, carrying a dashed half-edge, and not at the left end;

- I_2^{end}: the set of all vertices at the right end mapping to p_2, and carrying a dashed half-edge;
- $I_{D_2}^m$: the set of all interior vertices at the left end mapping to p_1, and not carrying any dashed half-edges;
- $I_{D_2}^b$: the set of all interior vertices at the left end mapping to p_1, and carrying a dashed half-edge.

Collect the vertices, regardless of where they go.

$$I^m := I_1^m \sqcup I_2^m, \qquad I^b := I_1^b \sqcup I_2^b, \qquad I^B := I^b \sqcup I_2^{end}.$$

Count them

$$\begin{aligned} &N_k^m := |I_k^m|, \qquad N_k^b := |I_k^b|, \qquad N_2^{end} := |I_2^{end}|, \\ &N^m := |I^m|, \qquad N^b := |I^b|, \\ &N_{D_2}^m := |I_{D_2}^m|, \qquad N_{D_2}^b := |I_{D_2}^b|. \end{aligned}$$

Note that $N_{D_2}^m,\ N_{D_2},\ N_2^{end} \in \{0,1\}$, and $N_{D_2}^m + N_{D_2}^b = 1$.

When $\beta = D_2 + dD_4$, a formula for the inverse of the Euler class of the virtual normal bundle of a fixed locus F is given through a similar fashion in Proposition 4.4.1. Let e_0 be the degree of the first horizontal edge on the left, and b_0 the degree of the dashed half-edge attached to the leftmost vertex, if such an half-edge exists, i.e., $N_{D_2}^b = 1$.

Proposition 6.2.2. For a fixed locus F in $Q_{0,2}(\mathbb{F}_2, D_2 + dD_4)$,

$$\begin{aligned} &\int_F \frac{\left(ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_1)) ev_2^*\left(c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2)) \cdot c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_3))\right)\right)\big|_F}{e^{\mathcal{T}}(N_F^{\text{vir}})} \\ &= -\frac{W_1^2 V_1}{W_1 V_1^2} \text{Cont}(VC) \text{Cont}_m(NS) \text{Cont}_{D_2}(NS) \\ &\cdot \prod_{\text{edges}} \text{Cont}_E(e) \prod_{v \in I^B} \text{Cont}_B(v) \cdot \text{Cont}_B(D_2), \end{aligned} \tag{6.4}$$

where

$$\text{Cont}(VC) := (-1)^{N_2^m + N_2^{end} + 2N_2^b} V_1^{N_1^m + 2N_1^b + N_{D_2}^m + 2N_{D_2}^b} V_2^{N_2^m + N_2^{end} + 2N_2^b} W_1^{N^m + N_2^{end} + N_{D_2}^m + 2N_{D_2}^b + 2N^b},$$

$$\text{Cont}_m(NS) := \left((-1)^{N_2^m} W_1^{N^m} \prod_{v \in I^m} \left(\frac{1}{e_v} + \frac{1}{e_v'}\right)\right)^{-1},$$

$$\text{Cont}_{D_2}(NS) := \left((V_1 + \frac{W_1}{e_0})^{N_{D_2}^m}\right)^{-1},$$

$$\text{Cont}_E(e) := \frac{e^{2e} \prod_{j=0}^{2e-2} \left(V_1 + \frac{1+j}{e} W_1\right)}{(e!)^2 W_1^{2e} (-1)^e},$$

$$\text{Cont}_B(v) := \frac{1}{b!} \int_{\overline{M}_{0,2|b}} \frac{\frac{V_i}{W_i^2} \prod_{j=2}^{b} \frac{(V_i - 2\Delta_j)^2}{W_i + \Delta_j}}{\left(\frac{W_i}{e} - \psi_n\right)\left(\frac{W_i}{e'} - \psi_{n'}\right)^{\epsilon(v)}},$$

$$\text{Cont}_B(D_2) := \left[\frac{1}{b_0!} \int_{\overline{M}_{0,2|b_0}} \frac{\frac{V_1}{W_1^2} \prod_{j=2}^{b_0} \frac{(V_1 - 2\Delta_j)^2}{W_1 + \Delta_j}}{\left(\frac{W_1}{e_0} - \psi_n\right)\left(V_1 - \psi_{n'}\right)}\right]^{N_{D_2}^b}.$$

Proof. Observe that the weight of the numerator coming from the insertions D_1 and pt is

$$\left(ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_1))\right) ev_2^*\left(c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2)) \cdot c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_3))\right)\big|_F = W_1^2 V_1.$$

The contribution from the deformation of the sections, as in (4.12), for the map-type component of degree D_2 is

$$-\frac{1}{W_1 V_1^2}.$$

To compute $\mathrm{Cont}_{D_2}(NS)$, we apply the same way used to derive (4.8). $\square$

Besides the numerator $W_1^2V_1$, the new factors appearing in the formula (6.4) are

$$-\frac{1}{W_1V_1^2},\ N_{D_2}^m,\ N_{D_2}^b, N_{D_2}^{end},\ (V_1+\frac{W_1}{e_0})^{N_{D_2}^m}.$$

All these are related to the component of degree D_2. We will pay attention to these factors in the formula (6.4).

Lemma 6.2.3. *The denominator of the formula* (6.4) *is of the form* W_1^N *for some nonnegative* N.

Proof. The factor $\frac{W_1^2V_1}{W_1V_1^2}$ in the formula (6.4) is $\frac{W_1}{V_1}$. We want to cancel the factor $\frac{1}{V_1}$. Note that we cannot just cancel this using the factor V_1 from either $\mathrm{Cont}_B(v)$ or $\mathrm{Cont}_B(D_2)$. The formal case is not possible because there is a map-type fixed locus that does not have this contribution term, or because there is a case where $v \in I^B$ might not map to p_1 so as to have V_1 factor. The latter case is not appropriate since the left corner vertex might not carry any dashed half-edges. Thus, we need to cancel the factor $\frac{1}{V_1}$ using a factor V_1 in some other places. Recall that $N_{D_2}^m + N_{D_2}^b = 1$ for all fixed locus. Therefore, we can cancel $\frac{1}{V_1}$ by $V_1^{N_{D_2}^m+N_{D_2}^b}$ in $\mathrm{Cont}(VC)$.

Consider the factor $(V_1+\frac{W_1}{e_0})^{N_{D_2}^m}$ in $\mathrm{Cont}_{D_2}(NS)$. Observe that the leftmost horizontal edge is the map-type component of degree e_0. There is a factor $(V_1+\frac{1+j}{e_0}W_1)$ in $\mathrm{Cont}_E(e)$ to cancel the factor $(V_1+\frac{W_1}{e_0})^{N_{D_2}^m}$ in $\mathrm{Cont}_{D_2}(NS)$ with $j=0$.

Last, the factor $\frac{1}{V_1}$ from $\frac{1}{V_1-\psi_{n'}} = \frac{1}{V_1}\sum_l(\psi_{n'}/V_1)^l$ is cancelled by the factor V_1 coming from the numerator in $\mathrm{Cont}_B(D_2)$. To cancel the factor $\frac{1}{V_1^{b_0-1}}$ in $\sum_{l=0}^{b_0-1}(\psi_{n'}/V_1)^l$, we show that the expansion of $\prod_{j=2}^{b_0}(V_1-2\Delta_j)^2$ has the factor $V_1^{b_0-1}$. In the expression $V_1^{2b_0-2}\prod_{j=2}^{b_0}(1-2\frac{\Delta_j}{V_1})^2$, observe that the degree of each term in Δ_j is one, and so is the degree of V_1 in the denominator. Consider that the highest degree term in $\prod_{j=2}^{b_0}(1-2\Delta_j)^2$ is given by $D_{12\cdots b_0}$ with some coefficient, where $\deg D_{12\cdots b_0} = b_0-1$. Thus, $V_1^{b_0-1}$ in $V_1^{2b_0-2}$ cancels all factors of the form $\frac{1}{V_1^l}$ appearing in each term of the expansion of $\prod_{j=2}^{b_0}(1-2\frac{\Delta_j}{V_1})^2$. Thus, the factor $V_1^{b_0-1}$ remains to cancel out all the factors of the form $\frac{1}{V_1^l}$ in $\sum_{l=0}^{b_0-1}(\psi_{n'}/V_1)^l$.

Hence, the denominator of the formula (6.4) is W_1^N for some nonnegative N. $\square$

We determine necessary fixed loci whose contribution is nonzero.

Corollary 6.2.4. *The decorated chain graphs of necessary fixed loci for* $\beta = D_2 + dD_4$ *are in Figure 6.2, where* $b + e = d$.

Proof. In the proof of Lemma 6.2.3, we made

$$\frac{W_1^2 V_1}{W_1 V_1^2} V_1^{N_1^m + 2N_1^b + N_{D_2}^m + 2N_{D_2}^b} = W_1 V_1^{N_1^m + 2N_1^b + N_{D_2}^b}$$

in the formula (6.4) by applying $N_{D_2}^m + N_{D_2}^b = 1$ and cancellation. Because of Lemma 6.2.3, the exponent fixed $N_1^m + 2N_1^b + N_{D_2}^b$ must be 0 to contribute to the final answer. Hence, the only possible fixed loci for nonzero contribution will have

$$N_1^m = 0,\ 2N_1^b = 0,\ N_{D_2}^b = 0,\ N_{D_2}^m = 1.$$

$\square$

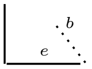

Figure 6.2: Necessary fixed loci for $\beta = D_2 + dD_4$

The formula (6.4) in Proposition 6.2.2 can be simplified with setting $V_1 = 0$.

Corollary 6.2.5. *Let* F *be a necessary fixed locus in* $Q_{0,2}(\mathbb{F}_2, D_2 + dD_4)$ *depicted in Figure 6.2, and* $b + e = d$ *with* $0 \leq b \leq d - 1$. *Then,*

$$\int_F \frac{\left(ev_1^* c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_1)) ev_2^*\left(c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_2)) \cdot c_1^{\mathcal{T}}(\mathcal{O}^{\mathcal{T}}(D_3))\right)\right)\big|_F}{e^{\mathcal{T}}(N_F^{vir})}\Bigg|_{V_1=0}$$
$$= \frac{(-1)^{d-b-1}4^b}{2}\binom{2(d-b)}{d-b}\binom{d-1}{b}.$$

Proof. Applying the contents in Sections 5.1, 5.2, and 5.3 with setting $V_1 = 0$, one can have the following simplification of the formula (6.4)

$$(-1)^{e-1}\frac{e(2e-1)!}{(e!)^2}\left(\frac{4^b e}{b}\binom{b+e-1}{b-1}\right)^{N_2^{end}}. \tag{6.5}$$

The factor $\frac{4^b e}{b}\binom{b+e-1}{b-1}$ comes from $\mathrm{Cont}_B(v)$ when $b > 0$. In this case, $N_2^{end} = 1$. The exponent N_2^{end} is for this reason. On the other hand, if $b = 0$, then $N_2^{end} = 0$. Replacing e by $d - b$, we derive the simplification. □

We compute the invariants $\langle D_1, pt\rangle_{0,2,D_2+dD_4}$.

Proposition 6.2.6. All 2-pointed degree $D_2 + dD_4$ quasimap invariants of $\mathbb{F}_2$ are given by

$$\langle D_1, pt\rangle_{0,2,D_2+dD_4} = \langle D_2, pt\rangle_{0,2,D_2+dD_4} = \frac{1}{2(2d-1)}\binom{2d}{d},$$
$$\langle D_3, pt\rangle_{0,2,D_2+dD_4} = 0, \quad \langle D_4, pt\rangle_{0,2,D_2+dD_4} = -\frac{1}{(2d-1)}\binom{2d}{d}.$$

Proof. Apply the Atiyah–Bott localization theorem to $\langle D_1, pt\rangle_{0,2,D_2+dD_4}$, and Corollary 6.2.5. Then, similar argument in the proof of Proposition 6.1.2 gives

$$\sum_{b=0}^{d-1}\frac{(-1)^{d-b-1}4^b}{2}\binom{2(d-b)}{d-b}\binom{d-1}{b} = -\frac{4^d}{2}\sum_{b=0}^{d-1}\binom{d-1}{d}\binom{-1/2}{d-b}.$$

Thus, it is enough to show

$$\sum_{b=0}^{d-1}\binom{d-1}{b}\binom{-1/2}{d-b} = -\frac{4^{-d}}{(2d-1)}\binom{2d}{d}.$$

The coefficient of x^d in $(1+x)^{d-1-1/2}$ is $\sum_{b=0}^{d-1}\binom{d-1}{b}\binom{-1/2}{d-b}$. We use induction. Let a_{d+1} be the coefficient of x^{d+1} in $(1+x)^{d-1/2}$. Then, applying $\frac{d}{dx}$ and the induction hypothesis give us

$$a_{d+1} = -\frac{4^{-d}}{2d-1}\frac{d-1/2}{d+1}\binom{2d}{d} = -\frac{4^{-(d+1)}(2d+2)(2d+1)}{(2d+1)(d+1)^2}\binom{2d}{d} = -\frac{4^{-(d+1)}}{(2d+1)}\binom{2(d+1)}{d+1}.$$

Hence, $\langle D_1, pt\rangle_{0,2,D_2+dD_4} = \frac{1}{2(2d-1)}\binom{2d}{d}$.

Lemma 6.2.1 computes $\langle D_3, pt\rangle_{0,2,D_2+dD_4} = 0$. The relation $D_3 = 2D_1 + D_4$ and $D_1 = D_2$ allow us to compute $\langle D_2, pt\rangle_{0,2,D_2+dD_4}$, and

$$\langle D_4, pt\rangle_{0,2,D_2+dD_4} = -\frac{1}{2d-1}\binom{2d}{d}.$$

□

Using Propositions 6.1.2 and 6.2.6, we derive the complete description of the quantum module structure of $\mathbb{F}_2$ written in Theorem 4.0.1.

Proof of Theorem 4.0.1. We show the computation of $\sigma_2 \star 1$, and the rest can be computed in a very similar way. Let $q_2 := q^{D_2}$ and $q_4 := q^{D_4}$ be the Novikov variables, and $f(z) := \sum_{d\geq 1}\binom{2d}{d}z^d = \frac{1}{\sqrt{1-4z}} - 1$. Applying Proposition 6.1.2, the divisor equation (4.2), and the Point mapping axiom and the Degree axiom in [5, §7.3] give us

$$\begin{aligned}
\sigma_2 \star 1 &:= \sum_i \sum_\beta q^\beta \langle 1, T_i \mid \sigma_2\rangle_{0,2|1,\beta} T^i \\
&= \sum_{d\geq 0} q_4^d \langle 1, D_2 \mid \sigma_2\rangle_{0,2|1,dD_4}(2D_2 + D_4) + \sum_{d\geq 0} q_4^d \langle 1, D_4 \mid \sigma_2\rangle_{0,2|1,dD_4} D_2 \\
&= (D_2 \cdot D_2)(2D_2 + D_4) + \sum_{d\geq 1} q_4^d (D_2 \cdot dD_4)\frac{-1}{2d}\binom{2d}{d}(2D_2 + D_4) \\
&\quad + (D_2 \cdot D_4)D_2 + \sum_{d\geq 1} q_4^d (D_2 \cdot dD_4)\frac{1}{d}\binom{2d}{d} D_2 \\
&= D_2 - \frac{1}{2} f(q_4) D_4.
\end{aligned}$$

□

Chapter 7

Comparison with the Batyrev ring of $\mathbb{F}_2$

For a smooth projective toric variety X_Σ, Batyrev defined in [9] a ring from the data of the fan Σ. In this section, we show that the quantum module structure of $\mathbb{F}_2$ agrees with the Batyrev ring of $\mathbb{F}_2$ realized as a natural module over the ring $\mathbb{Q}[[q_2, q_4]]$.

7.0.1 The Batyrev ring of X_Σ

Let $v_1, \ldots, v_s \in N \cap \Sigma(1)$ be primitive integral generators for the rays. There are two ideals in $\mathbb{Q}[x_1, \ldots, x_s]$. The first ideal is given by

$$P_\Sigma := \Big\langle \sum_{i=1}^{s} \langle m, v_i \rangle x_i \mid m \in M \Big\rangle.$$

For a primitive collection $P = \{v_{i_1}, \ldots, v_{i_k}\}$, we have the relation

$$v_{i_1} + \cdots + v_{i_k} = c_1 v_{j_1} + \cdots + c_l v_{j_l}, \tag{7.1}$$

where $v_{j_1}, \ldots, v_{j_l}$ are the generators of $\sigma \in \Sigma$ such that $v_{i_1} + \cdots + v_{i_k} \in \sigma$ and $c_1, \ldots, c_l \geq 0$. Using the dual of the exact sequence (2.1), the relation (7.1) gives rise to a class β_P in $H_2(X, \mathbb{Z})$. This class β_P is effective [36, Thm 2.15], [5, Example 8.1.2.2]. Then, the second ideal is defined by

$$SP_\Sigma := \langle x_{i_1} \cdots x_{i_k} - q^{\beta_P} x_{j_1}^{c_1} \cdots x_{j_l}^{c_l} \mid P : \text{ primitive collection} \rangle$$

This ideal is called the *quantum Stanley Reisner ideal*. Using these ideals, Batyrev defined the following ring

$$\text{Bat}H^*(X) := \mathbb{Q}[x_1, \ldots, x_s]/(P_\Sigma + SP_\Sigma).$$

When X_Σ is Fano, the quantum cohomology ring of X_Σ coincides with the Batyrev ring; however, this is false when X_Σ is not Fano, but *semipositive*, i.e., the anticanonical divisor is nef. The Hirzebruch surface $\mathbb{F}_2$ exactly shows the failure; see [5, Example 11.2.5.2]. The cone of effective curves(or the Mori cone) of $\mathbb{F}_2$ is generated by D_2 and D_4. Thus, the class β_P is given by the nonnegative linear combination of D_2 and D_4. The Batyrev ring of $\mathbb{F}_2$ can be written as follows:

$$\mathrm{Bat}H^*(\mathbb{F}_2) = \mathbb{Q}[x_2, x_4]/\langle x_2^2 - q_4 x_4^2, (2x_2 + x_4)x_4 - q_2\rangle.$$

The quantum $H^*((\mathbb{C}^*)^2)$-module of $\mathbb{F}_2$ has the following relations.

Lemma 7.0.1. *The following relations hold in the quantum $H^*((\mathbb{C}^*)^2)$-module of $\mathbb{F}_2$:*

$$(2\sigma_2 + \sigma_4) \star (\sigma_4 \star 1) = q_2, \quad \sigma_2 \star (\sigma_2 \star 1) = q_4 \sigma_4 \star (\sigma_4 \star 1).$$

Proof. Let $f(z) := \sum_{d \geq 1} \binom{2d}{d} z^d = \frac{1}{\sqrt{1-4z}} - 1$. Then, one can verify

$$(1 + f(z))^2 (1 - 4z) = 1, \tag{7.2}$$

$$4z(1 + f(z))^2 = f(z)(2 + f(z)). \tag{7.3}$$

We write f for $f(q_4)$. Observe the following:

$$\begin{aligned}(2\sigma_2 + \sigma_4) \star (\sigma_4 \star 1) &= (1 + f)(2\sigma_2 + \sigma_4) \star D_4 \\ &= q_2(1 + f)(-4q_4(1 + f) + (1 + f)) \\ &= q_2(1 + f)^2(-4q_4 + 1) \overset{(7.2)}{=} q_2,\end{aligned}$$

and

$$\begin{aligned}
\sigma_2 \star (\sigma_2 \star 1) &= \sigma_2 \star (D_2 - \frac{1}{2} f D_4) \\
&= \left(q_2 q_4 (1+f) - \frac{1}{2} f p t \right) - \frac{1}{2} f \left(-2 q_2 q_4 (1+f) + (1+f) p t \right) \\
&= q_2 q_4 (1+f)^2 - \frac{1}{2} p t f (2+f) \overset{(7.3)}{=} q_2 q_4 (1+f)^2 - p t 2 q_4 (1+f)^2 \\
&= q_4 (1+f)^2 (q_2 - 2pt), \\
\sigma_4 \star (\sigma_4 \star 1) &= (1+f) \sigma_4 \star D_4 \\
&= (1+f) \left(q_2 (1+f) - 2(1+f) p t \right) \\
&= (1+f)^2 (q_2 - 2pt).
\end{aligned}$$

Thus, we have

$$\sigma_2 \star (\sigma_2 \star 1) = q_4 \sigma_4 \star (\sigma_4 \star 1).$$

□

One can see that the two relations in Lemma 7.0.1 agree with the relations in Bat$H^*(\mathbb{F}_2)$ after identifying $\sigma_2 \star 1$ and $\sigma_4 \star 1$ with x_2 and x_4, respectively.

Lemma 7.0.2. *The Batyrev ring* Bat$H^*(\mathbb{F}_2)$ *is generated by* $1, x_2, x_4$ *and* $x_2 x_4$ *over* $\mathbb{Q}[[q_2, q_4]]$.

Proof. Applying Nakayama's lemma to the local ring $\mathbb{Q}[[q_2, q_4]]$ with the unique maximal ideal (q_2, q_4), one can show that the set $\{1, x_2, x_4, x_2 x_4\}$ generates Bat$H^*(\mathbb{F}_2)$ over $\mathbb{Q}[[q_2, q_4]]$. □

Remark 7.0.3. The proof of Lemma 7.0.2 does not give us a concrete way to write down an element in Bat$H^*(\mathbb{F}_2)$ with respect to the generating set $\{1, x_2, x_4, x_2 x_4\}$. It is worthwhile to observe, for instance, how $x_2^2 x_4$ can be written in terms of $1, x_2, x_4$, and $x_2 x_4$. Because it requires the equation (7.2) and the series $f(z) = \sum_{d \geq 1} \binom{2d}{d} z^d$ that contains some 2-pointed quasimap invariants of $\mathbb{F}_2$. Using the relations in $\mathrm{Bat}H^*(\mathbb{F}_2)$,

$$\begin{aligned} x_2^2x_4 &= q_4x_4^3 = x_4q_4(q_2 - 2x_2x_4) = q_2q_4x_4 - 2q_4x_2x_4^2 \\ &= q_2q_4x_4 - 2q_4x_2(q_2 - 2x_2x_4) = q_2q_4x_4 - 2q_2q_4x_2 + 4q_4x_2^2x_4. \end{aligned}$$

Thus, we have

$$(1 - 4q_4)x_2^2x_4 = q_2q_4(x_4 - 2x_2).$$

Applying the equation (7.2) with $z = q_4$, we obtain an expression of $x_2^2x_4$ as follows:

$$x_2^2x_4 = q_2q_4(1 + f(q_4))^2(x_4 - 2x_2).$$

This shows that finding generating sets over $\mathbb{Q}[[q_2, q_4]]$ using the relations in a Batyrev ring involves generating series whose coefficients may be related to 2-pointed quasimap invariants.

There is a natural $H^*((\mathbb{C}^*)^2) = \mathbb{Q}[\sigma_2, \sigma_4]$-module structure on $\mathrm{Bat}H^*(\mathbb{F}_2)$. This is given by

$$\sigma_2 \cdot x_2^ax_4^b = x_2^{a+1}x_4^b, \quad \sigma_4 \cdot x_2^ax_4^b = x_2^ax_4^{b+1}.$$

For convenience, we call this the *Batyrev module*.

Proposition 7.0.4. The Batyrev module of $\mathbb{F}_2$ is isomorphic to the quantum module of $\mathbb{F}_2$ as $H^*((\mathbb{C}^*)^2)$-modules.

Proof. Denote the generating sets by $\alpha := \{1, x_2, x_4, x_2x_4\}$ and $\mu := \{1, D_2, D_4, pt\}$ for the Batyrev module and the quantum module, respectively. Using Lemma 7.0.2, one can represent the action of σ_i as matrices using α and μ, respectively, say $[\sigma_i]_\alpha^\alpha$ and $[\sigma_i]_\mu^\mu$.

Define a linear function ϕ in the following way:

$$1 \mapsto 1, \quad x_2 \mapsto \sigma_2 \star 1, \quad x_4 \mapsto \sigma_4 \star 1, \quad x_2x_4 \mapsto (\sigma_2\sigma_4) \star 1.$$

This map extends linearly over $\mathbb{Q}[[q_2, q_4]]$. Then, the matrix presentation of ϕ over $\mathbb{Q}[[q_2, q_4]]$ is the following:

$$[\phi]^{\alpha}_{\mu} := \begin{pmatrix} 1 & 0 & 0 & -2(1+f)^2 q_2 q_4 \\ 0 & 1 & 0 & 0 \\ 0 & (-1/2)f & 1+f & 0 \\ 0 & 0 & 0 & (1+f)^2 \end{pmatrix},$$

where f is denoted for $f(q_4)$. One can check

$$[\phi]^{\alpha}_{\mu}[\sigma_i]^{\alpha}_{\alpha} = [\sigma_i]^{\mu}_{\mu}[\phi]^{\alpha}_{\mu}.$$

This shows that ϕ is $\mathbb{Q}[\sigma_2, \sigma_4]$-linear over $\mathbb{Q}[[q_2, q_4]]$. From Lemma 7.0.1, there exists an induced map from the Batyrev module to the quantum module. The determinant of $[\phi]^{\alpha}_{\mu}$ is $(1+f)^3$. By the equation (7.2) with $z = q_4$, the determinant $(1+f)^3$ is an invertible element in $\mathbb{Q}[[q_2, q_4]]$. Therefore, ϕ is an $\mathbb{Q}[\sigma_2, \sigma_4]$-isomorphism over $\mathbb{Q}[[q_2, q_4]]$. $\square$

Bibliography

[1] D. R. Morrison. Mathematical aspects of mirror symmetry. *IAS/Park City Mathematics Series*, pages 265–340, 1997.

[2] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner. Chiral rings in $n = 2$ superconformal theories. *Nuclear Physics B*, 324(2):427–474, 1989.

[3] D. McDuff and D. Salamon. *J-holomorphic Curves and Quantum Cohomology*. University lecture series. American Mathematical Society, 1994.

[4] Yongbin Ruan and Gang Tian. A mathematical theory of quantum cohomology. *Journal of Differential Geometry*, 42(2):259–367, 1995.

[5] D.A. Cox and S. Katz. *Mirror Symmetry and Algebraic Geometry*. American Mathematical Society, 1999.

[6] K. Hori. *Mirror Symmetry*. Clay mathematics monographs. American Mathematical Society, 2003.

[7] Maxim Kontsevich. *Enumeration of Rational Curves Via Torus Actions*. Birkhäuser Boston, Boston, MA, 1995.

[8] M Kontsevich and Yu Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Communications in Mathematical Physics*, 164(3):525–562, August 1994.

[9] Victor V Batyrev. Quantum cohomology rings of toric manifolds. *Journées de géométrie algébrique d'Orsay - Juillet*, (218):9–34, 1992.

[10] Zhenbo Qin and Yongbin Ruan. Quantum cohomology of projective bundles over $\mathbb{P}^n$. *Transactions of the American Mathematical Society*, 350, 08 1996.

[11] Ionuţ Ciocan-Fontanine and Bumsig Kim. Wall-crossing in genus zero quasimap theory and mirror maps. *Algebr. Geom.*, 1(4):400–448, 2014.

[12] Ionuţ Ciocan-Fontanine and Bumsig Kim. Higher genus quasimap wall-crossing for semi-positive targets. *J. Eur. Math. Soc. (JEMS)*, 19(7):2051–2102, 2017.

[13] Tom Coates and Cristina Manolache. Stable maps and quasimaps to toric Fano varieties, December 2013. hal-00923921.

[14] Ionuţ Ciocan-Fontanine and Bumsig Kim. Moduli stacks of stable toric quasimaps. *Advances in Mathematics*, 225(6):3022–3051, 2010.

[15] Luca Battistella. *Alternative compactifications in low genus Gromov-Witten theory*. PhD thesis, Imperial College London, 2018.

[16] Ionuţ Ciocan-Fontanine and Bumsig Kim. Big I-functions. 69:323–348, 2016.

[17] Bumsig Kim and Hyenho Lho. Mirror theorem for elliptic quasimap invariants. *Geometry & Topology*, 22, 06 2015.

[18] Dan Abramovich, Tom Graber, and Angelo Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *American Journal of Mathematics*, 130(5):1337–1398, 2008.

[19] Luca Battistella and Navid Nabijou. A Lefschetz-type theorem for quasimap invariants via relative quasimaps, 2017. arXiv:1710.11158v1.

[20] Joachim Kock and Israel Vainsencher. *An Invitation to Quantum Cohomology: Kontsevich's Formula for Rational Plane Curves*. Birkhäuser Boston, 10 2017.

[21] William Fulton. *Intersection Theory*. Springer New York, NY, 1998.

[22] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik. Stable quasimaps to git quotients. *Journal of Geometry and Physics*, 75:17–47, 2014.

[23] Masao Jinzenji and Kohki Matsuzaka. Hori's equation for gravitational virtual structure constants of Calabi-Yau hypersurface in $\mathbb{CP}^{N-1}$. 2023.

[24] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Society, 2011.

[25] K. Behrend. Gromov-Witten invariants in algebraic geometry. *Invent. Math.*, 127(3):601–617, 1997.

[26] Mark Shoemaker. Virtual classes for hypersurfaces via two-periodic complexes. *Contemp. Math.*, 763:89–116, 2021.

[27] Andrey S. Losev and Yuri I. Manin. New moduli spaces of pointed curves and pencils of flat connections. *Michigan Mathematical Journal*, 48:443–472, 2000.

[28] Ionut Ciocan-Fontanine, David Favero, Jérémy Guéré, Bumsig Kim, and Mark Shoemaker. *Fundamental Factorization of a GLSM Part I: Construction*, volume 289 of *Memoirs*. American Mathematical Society, 2023.

[29] Jim Bryan and Rahul Pandharipande. On the rigidity of stable maps to Calabi-Yau threefolds. In *The interaction of finite-type and Gromov-Witten invariants (BIRS 2003)*, volume 8 of *Geom. Topol. Monogr.*, pages 97–104. Geom. Topol. Publ., Coventry, 2006.

[30] David Anderson and William Fulton. *Equivariant Cohomology in Algebraic Geometry*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2023.

[31] Chiu-Chu Melissa Liu. Localization in Gromov-Witten theory and orbifold Gromov-Witten theory. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 353–425. Int. Press, Somerville, MA, 2013.

[32] Tom Graber and Rahul Pandharipande. Localization of virtual classes. *Inventiones mathematicae*, 135:487–518, 1997.

[33] Dragos Oprea. *Notes on the Moduli Space of Stable Quotients*, pages 69–135. Springer Basel, Basel, 2016.

[34] Alina Marian, Dragoş Ştefan Oprea, and Rahul Pandharipande. The moduli space of stable quotients. *Geometry & Topology*, 15:1651–1706, 2009.

[35] Richard P. Stanley. *Enumerative Combinatorics: Volume 2*. Cambridge University Press, 1999.

[36] Victor V. Batyrev. On the classification of smooth projective toric varieties. *Tohoku Mathematical Journal*, 43(4):569 – 585, 1991.