DETERMINATION OF BED MATERIAL TRANSPORT WITH RADIOACTIVE TRACERS

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ABSTRACT

Since the advent of radioactive tracer techniques, application of Lagrangian experimental techniques in the investigation of sediment transport phenomena has become feasible. The results of one field experiment and two subsequent laboratory experiments in which radioactive tracer techniques were used to study the transport and longitudinal dispersion of bed-material particles is reported.

The development of a concentration distribution function, which is based on a Lagrangian probabilistic model wherein the transport of bed material particles is described as a sequence of alternating steps and rest periods, and a method for determining the discharge of bed-material particles is reviewed. The discharge is obtained from applying the continuity principle to the mean rate of movement of a group of tracer particles and the effective cross-sectional area of the bed through which they move. The concentration-distribution function and the method for determining bed-material discharge are examined in the light of the experimental data and are found to be applicable both in the field and in the laboratory.

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INTRODUCTION

The transport of sediment particles in alluvial streams can be described in either of two ways. In one of the ways, attention is focused on a particular cross section of the channel and the concentration, discharge and characteristics of the sediment as it passes the cross section are described. By analogy with classical mechanics, this can be called an Eulerian description. In the other way, attention is focused on a particular particle or group of particles, and the motion of the particles over a period of time is described. Using the same analogy, this can be called a Lagrangian description.

Although Eulerian experimental techniques provide a more direct method of determining quantities such as sediment discharge, Lagrangian techniques can yield information which is perhaps more instructive with regard to fundamental transport processes. For example, Lagrangian experimental techniques permit direct evaluations of the rates of movement and dispersion of any type or combination of types of sediment particles.

Nearly all of the theories and experimental techniques associated with current and past sediment transport research are rooted in the Eulerian system. Investigations by H. A. Einstein (1937), Crickmore and Lean (1962), and Lean and Crickmore (1960), are among the few exceptions. The reason why the Lagrangian system has not been used more often is no doubt due largely to the experimental difficulties which have been associated with tracing the motion of a given particle or group of particles. With the recent development of radioactive (and other) tracer techniques, however, Lagrangian experimental techniques have become feasible for the study of sediment transport, both in natural streams and to a greatly increased extent in the laboratory.

In November, 1960, an experiment was conducted on the North Loup River in Nebraska, in which radioactive tracer techniques were used to measure the transport and dispersion of bed-material particles. The results of this experiment, together with the development and verification of a Lagrangian probabalistic model which describes the transport and longitudinal dispersion of tracer particles released from a transverse source in the bed, and a method for determining bed-material discharge, are reported by Sayre and Hubbell (1963). Subsequent to the field study on the North Loup, two laboratory experiments in which radioactive tracer techniques were used have been conducted. This paper presents a resume of the findings from the field study together with some of the results of the laboratory experiments. In particular, the applicability of the dispersion model and the method for determining bed-material discharge is examined in the light of experimental data.

The experiments and analysis were performed under a cooperative program between the U. S. Geological Survey and the Atomic Energy Commission, the purpose of which is to investigate the transport and dispersal of waterborne radioactive waste materials which have become attached to fluvial sediments.

THEORETICAL MODELS

In an alluvial channel, the part of the total sediment discharge that is composed of particles having sizes the same as those of particles commonly found in the bed material is called the bed-material discharge. The motion of bed-material particles over a period of time consists of a series of discrete steps. A particle may roll along the bed or be entrained temporarily in the flow by a turbulent eddy, then come to rest on the bed where it will remain, usually becoming covered by other particles, until it is re-exposed and takes another step. Thus, the motion of a particle can be described by the length of the steps that it takes and the duration of the rest periods between steps, provided, of course, that the aggregate resting time is very large in comparison to the aggregate time spent in motion.

The step lengths and rest periods that a particle takes are random phenomena that depend on factors such as flow and turbulence characteristics, sediment characteristics, and bed form. Due to the randomness, the step lengths and the durations of the rest periods are variable. Consequently, if a group of particles are released simultaneously from a source that extends across the width of the channel, the particles will not move downstream in a body, but will become dispersed along the length of the channel. In addition to being distributed longitudinally, the particles will be dispersed throughout some depth into the bed; the depth depends on the characteristics of the bed forms.

The Concentration-distribution Function

Because the transport and dispersion of bed-material particles is essentially a random process it is logical to use probability theory to derive a function for describing the dispersion process. H. A. Einstein (1937) considered bed-load transport from the viewpoint of probability theory. Beginning with essentially the same assumptions, Sayre and Hubbell (1963) arrived at distribution functions which are identical to those obtained by Einstein. However, the two derivations are considerably different.

In order to acquaint the reader with the nature of the distributions, the derivation as presented by Sayre and Hubbell (1963) is reviewed.

Assume a straight, uniform alluvial channel in which the water and bed-material discharges per unit width are uniform and constant. Now, consider a group of tracer particles which have identical transport characteristics. Let these particles be released simultaneously, at time $t = 0$ and station $x = 0$, from a uniformly-distributed plane source that extends laterally across the channel and vertically from the surface of the bed to the lower limit of the zone of particle movement. Also, let the tracer particles move and disperse downstream in such a way that there is no tendency to develop a vertical concentration gradient of tracer particles within the bed. Furthermore, assume that the number of tracer particles is small enough so that the bed-material discharge is virtually unchanged by the release of the particles, yet is large enough to approximate an infinite population.

If the foregoing conditions are satisfied the stream channel can be characterized as a homogeneous probability field in space and time. If, as it is, reasonable to assume, the length of any particular step and the duration of any particular rest period of any given particle are independent of x and t and of each other, the step lengths and rest periods can each be represented as a set of independent, identically-distributed random variables. Also, the total distance from the origin travelled by a particle in n steps can be represented as the sum of n independent, identically distributed random variables.

Let it be assumed that the step lengths are exponentially distributed with mean step length $1/k_1$ and that the durations of the rest periods are exponentially distributed with mean duration $1/k_2$. With these assumptions it can be shown (Parzen, 1962) that the total distance, x, travelled by a particle in n steps is gamma distributed, and that the number of waiting periods, i, in the time interval $(0,t)$ is Poisson distributed. Expressed mathematically,

$$f(x|n) = k_1 e^{-k_1 x} \frac{(k_1 x)^{n-1}}{\Gamma(n)} \qquad 0 < x < \infty \qquad (1)$$

and

$$p_t(i) = e^{-k_2 t} \frac{(k_2 t)^i}{i!} \qquad i = 1, 2, \ldots. \qquad (2)$$

Equation 1 is the probability density function of the gamma distribution. When equation 1 is multiplied by dx, it expresses the probability that a tracer particle will be located in the length increment $(x, x + dx)$, which is to say that the particle will have been transported the exact distance (o,x), given that it has taken exactly n steps. Equation 2 is the probability mass function of the Poisson distribution and it expresses the probability that during the time interval $(0,t)$ a particle will have completed exactly i rest periods.

As a tracer particle moves downstream in a sequence of discrete steps, each rest period is followed by a step which, in turn, is followed by another rest period, etc. Therefore, assuming that the particle is initially at rest and begins its first rest period at $t = 0$, during any given rest period the particle must have undergone the same number of complete rest periods as steps. Hence, a particle which has taken n steps also will have undergone n rest periods.

From a practical standpoint, it is not possible to keep track of the number of steps (or rest periods) that a particle undergoes; consequently, equations 1 and 2 as they stand are of little practical use. However, by using the concept of conditional probability, equations 1 and 2, in effect, can be combined to form the density function $f_t(x)$ from which the probability of the location of the tracer particle in any length increment, $x + dx$, at time t, can be obtained. To this end it is convenient to express equation 1 in the form of a conditional probability distribution function as follows:

$$F(x|n) = \int_0^x f(x'|n)\,dx' = P\left[x' \le x | i = n\right] \tag{1a}$$

in which

P denotes probability,
x' is a dummy variable of integration.

Applying the definition of conditional probability,

$$F_t(x|n) = P_t\left[x' \le x|i = n\right] = \frac{P_t\left[x' \le x, i = n\right]}{P_t\left[i = n\right]} = \frac{P_t\left[x' \le x, i = n\right]}{p_t(n)} \qquad 0 < t < \infty$$

When n is given, knowledge of t adds no information, hence

$$F_t(x|n) = F(x|n)$$

and

$$P_t\left[x' \le x, i = n\right] = F(x|n)\, p_t(n) \tag{3}$$

The marginal distribution function in x corresponding to the joint probability function in equation 3 is

$$F_t(x) = \sum_{n=o}^{\infty} P_t\left[x' \leq x, i = n\right] = \sum_{n=o}^{\infty} F(x|n)\, p_t(n)$$

$$= \sum_{n=1}^{\infty} F(x|n)\, p_t(n) + F(x|0)\, p_t(0) \tag{4}$$

Differentiation of equation 4 with respect to x yields the density function for x at time t

$$f_t(x) = \frac{\partial}{\partial x}\left[F_t(x)\right] = \sum_{n=1}^{\infty} f(x|n)\, p_t(n) + p_t(0)\, \frac{\partial}{\partial x}\left[F(x|0)\right]$$

Performing the substitutions indicated by equations 1 and 2, and noting that

$$p_t(0) = e^{-k_2 t}$$

$$F(x|0) = 1 \qquad \text{when } x = 0$$

$$= 0 \qquad \text{elsewhere}$$

the density function for x at time t can be written

$$f_t(x) = k_1\, e^{-(k_1 x + k_2 t)} \sum_{n=1}^{\infty} \frac{(k_1 x)^{n-1}}{\Gamma(n)} \frac{(k_2 t)^n}{n!} \tag{5}$$

which transforms readily into

$$f_t(x) = k_1\, e^{-(k_1 x + k_2 t)} \sqrt{\frac{k_2 t}{k_1 x}}\; I_1\left(2\sqrt{k_1 x\, k_2 t}\right) \tag{5a}$$

where $I_1(2\sqrt{k_1 x\, k_2 t})$ is a modified Bessel function of the first kind of order one. Equations 5 and 5a were first derived by H. A. Einstein (1937).

If the displacement of a tracer particle from the origin at time t is considered as a stochostic process, it can be shown that equations 5 and 5a represent the density function corresponding to a compound Poisson process with exponentially distributed incremental step lengths (Parzen, 1962).

It whould be noted that the integral of the density function in equations 5 and 5a over the entire range of x is

$$\int_0^{\infty} f_t(x)\, dx = 1 - e^{-k_2 t} = 1 - p_t(0)$$

where $p_t(0)$ is the probability that the particle has not yet moved from the origin at time t . Thus the density function $f_t(x)$ applies only to particles which have taken at least one initial step.

If, as assumed initially, a very large number of tracer particles that are initially distributed uniformly in a plane extending throughout the zone of movement at $x = 0$ all begin their first rest period at $t = 0$, the density function, $f_t(x)$, as defined by equations 5 and 5a can be used to estimate the concentration, $\phi(x,t)$, of tracer particles in the bed at any distance, x , downstream from the source at time t . In fact, $\phi(x,t)$ will be directly proportional to $f_t(x)$ if the following conditions are satisfied: (1) the average depth to which the tracer particles are distributed remains constant downstream from the source; (2) the concentration of tracer particles in the bed does not, on the average, vary with the depth to which they are distributed; (3) at any given instant, no significant fraction of the tracer particles are in suspension. Given these conditions, if the concentration of tracer particles is defined as the weight of tracer particles per unit volume of bed material, the concentration is

$$\phi(x,t) = \frac{W}{Bd} f_t(x)$$

in which

- W is the total weight of tracer particles placed in the channel
- B is the width of the channel
- d is the average depth beneath the bed surface to which the tracer particles are distributed.

Thus, the concentration-distribution function at time t for tracer particles which have moved from the origin where they began their first rest period at $t = 0$ is

$$\phi(x,t) = \frac{W k_1}{Bd} e^{-(k_1 x + k_x t)} \sum_{n=1}^{\infty} \frac{(k_1 x)^{n-1}}{\Gamma(n)} \frac{(k_2 t)^n}{n!} \tag{6}$$

or

$$\phi(x,t) = \frac{W k_1}{Bd} e^{-(k_1 x + k_2 t)} \sqrt{\frac{k_2 t}{k_1 x}} I_1 (2\sqrt{k_1 x k_2 t}) \tag{6a}$$

Related Distribution Functions

If a tracer particle is introduced into the channel at $x = 0$ in such a way that it takes its first step at time $t = 0$, then during any given rest period it will have completed one less rest period than steps. In this case, equation 4 becomes

$$F_t^*(x) = \sum_{n=1}^{\infty} F(x|n)\, p_t(n-1) \tag{7}$$

where the asterisk denotes the initial condition of beginning with motion. Likewise equations 5 and 5a become

$$f_t^*(x) = k_1 e^{-(k_1 x + k_2 t)} \sum_{n=1}^{\infty} \frac{(k_1 x)^{n-1}}{\Gamma(n)} \frac{(k_2 t)^{n-1}}{(n-1)!} \tag{8}$$

and

$$f_t^*(x) = k_1 e^{-(k_1 x + k_2 t)} I_0 (2\sqrt{k_1 x k_2 t}) \tag{8a}$$

where $I_0(2\sqrt{k_1 x k_2 t})$ is a modified Bessel function of the first kind of order zero. As were equations 5 and 5a, equations 8 and 8a were first obtained by Einstein (1937).

In a manner analogous to that which was used to obtain the concentration-distribution function, $\phi(x,t)$, which specifies the concentration of tracer particles in the bed at any distance x and any time t, a discharge-distribution function, $\psi(t,x)$, which specifies the discharge of tracer particles past a cross section at any distance x downstream from the source at any time t, can be obtained from the density function $f_x(t)$. A tracer particle which begins its first rest period at the origin at $t = 0$ will, as it passes a point a distance x downstream from the origin, have completed one less step than rest periods. Thus, the probability density $f_x(t)$ which relates to the probability that the particle will pass the point x during the time interval $(t, t + dt)$ corresponds to $f_t^*(x)$ except that the variables x and t and the constants

k_1 and k_2 are interchanged so that

$$f_x(t) = k_2 \, e^{-(k_1 x + k_2 t)} \sum_{n=1}^{\infty} \frac{(k_2 t)^{n-1}}{\Gamma(n)} \, \frac{(k_1 x)^{n-1}}{(n-1)!} \tag{9}$$

or

$$f_x(t) = k_2 \, e^{-(k_1 x + k_2 t)} \, I_0 \left(2\sqrt{k_1 x \, k_2 t}\right) \tag{9a}$$

The corresponding discharge-distribution function in weight per unit time per unit of cross-sectional area is

$$\psi(t,x) = \frac{W k_2}{Bd} \, e^{-(k_1 x + k_2 t)} \sum_{n=1}^{\infty} \frac{(k_2 t)^{n-1}}{\Gamma(n)} \, \frac{(k_1 x)^{n-1}}{(n-1)!} \tag{10}$$

or

$$\psi(t,x) = \frac{W k_2}{Bd} \, e^{-(k_1 x + k_2 t)} \, I_0 \left(2\sqrt{k_1 x \, k_2 t}\right) \tag{10a}$$

Bed-material Transport

In addition to presenting the concentration-distribution function, Sayre and Hubbell (1963) presented a method for the computation of bed-material discharge from measurements made by the radioactive tracer technique or, for that matter, by any Lagrangian experimental technique. The rudiments of the method are as follows. Consider, within the framework of the same restrictions that were assumed in the derivation of the theoretical concentration-distribution function, a group of tracer particles released from a plane source in the bed of a channel. Assume, now, however, that the transport characteristics of the tracer particles simulate those of the bed material. As the tracer particles move downstream they disperse so that any any time, t , after release they are distributed about some mean distance from the source, $\bar{x}$. Hence, the ratio, $\bar{x}/t$, which defines the mean velocity of the tracer particles in the downstream direction, also defines the mean velocity of the bed-material particles.

Thus, the bed-material discharge, Q_s , in weight per unit time is,

$$Q_s = \gamma_s (1-\gamma) Bd \, \frac{\bar{x}}{t} = \gamma_s (1-\gamma) \frac{W}{A} \, \frac{\bar{x}}{t} \tag{11}$$

in which

γ_s is the specific weight of the bed material.

λ is the fraction of a volume not occupied by sediment particles (porosity).

If all the bed-material particles have essentially the same transport characteristics, the tracer particles will be distributed according to equation 6 and k_2/k_1 may be substituted for $\bar{x}/t$ in equation 11.

Although equation 11 has the form of a continuity equation, the concept of continuity applies only in a statistical sense. Actually, particles move only when they are exposed on the surface of the bed or when they are in suspension. The particles move in a series of discrete steps and are buried in the bed, for varying periods of time, between the steps. Thus, the mean particle velocity, $\bar{x}/t$, represents the average rate of movement during a total elapsed time. The use of the cross-sectional area Bd as the zone of movement is based on the assumption that it is the area which, in correspondence with $\bar{x}/t$, satisfies continuity.

The discharge of separate fractions of the bed material that possess certain characteristics such as the same particle size, shape or specific gravity also can be computed with the continuity equation. For such applications, the tracer particles must be prepared to simulate only the particles that possess the characteristic of interest and equation 11 must be used in its generalized form:

$$(Q_s)_c = i_c(\gamma_s)_c\ (1-\lambda)\ Bd\left(\frac{\bar{x}}{t}\right)_c = i_c\ (\gamma_s)_c\ (1-\lambda)\ \left(\frac{W}{A}\right)_c\left(\frac{\bar{x}}{t}\right)_c \qquad (12)$$

in which

i_c is the ratio of the volume of particles possessing the characteristic to the volume of bed material particles in the zone of particle movement.

e is a subscript that denotes terms associated with the particles possessing the characteristic.

EXPERIMENTS

The transport of bed material has been investigated by means of radioactive tracer techniques in one field experiment and two laboratory experiments. In each of the experiments the hydraulic and sediment conditions were different. The field experiment was conducted in an 1800 ft by 50 ft straight reach of the North Loup River near Purdum, Nebraska. At the time of the experiment the bed material, which has a median fall diameter of about 0.29 mm, was formed into large dunes about 1.5 ft high and 15 ft long; the water discharge was about 260 cfs, the mean depth was about 2.5 ft and the slope was about 0.00083. In contrast, the first laboratory experiment was conducted in a 150-ft long, 8-ft wide, recirculating-type flume with a bed material having a median fall diameter of about 0.93 mm. For this experiment, the mean depth was 1.04 ft, the discharge was 16 cfs, and the slope was about 0.0013. The bed form consisted of relatively long low dunes that averaged 4.0 ft in length and 0.13 ft in height and travelled along the channel at about 7.6 ft per hr. The third and most recent experiment was conducted in the 8-ft wide flume with sand having a median diameter of about 0.19 mm. The flow depth in this experiment also was 1.04 ft, the discharge was 8.6 cfs and the slope was about 0.00033. The bed form in this experiment consisted of ripples about 0.04 ft high and 0.5 ft long. As a result of the diversity of the three experiments, data is available for a fairly wide range of transport and dispersion rates.

In each of the experiments, natural sand particles labeled with a radio nuclide were released on or in the stream bed and the subsequent transport and dispersion of the particles was measured periodically by monitoring the radioactivity along the channel with radiation detection equipment. In order to convert the observed levels of radioactivity into a concentration that expressed the weight of tracer particles per unit volume of bed material, detailed calibrations were made for each experiment. In the calibrations, known weights of particles labeled with known amounts of radioactivity were mixed homogeneously with the natural bed material and the resultant mixtures were monitored under conditions that simulated the actual experimental environment. Although fundamentally all of the experiments have been the same, some of the equipment and procedures were varied between experiments because of the different hydraulic and sediment conditions.

More detailed information concerning the design and conduct of the field experiment is given in publications by Sayre and Hubbell (1963), and Hubbell and Sayre (1963).

Experimental Equipment and Procedures

In the field experiment, 40 lbs of tracer particles labeled with 40 millicuries of Iridium-192 were introduced into the stream by placing 2-lb lots on the bed surface at 2-ft intervals across the width of the stream. Placement was accomplished by pouring the particles down a funnel tube which had a bell-shaped bottom and could be adjusted so that it rested on the bed (see fig. 1). In the laboratory experiment with 0.93 mm sand, 2400 grams of tracer particles labeled with 600 microcuries of Gold-198 were introduced into the flow by releasing them from an 8-ft long split-pipe apparatus that simultaneously deposited the particles across the entire width of the flume on the bed surface. In the most recent experiment, 25 grams of tracer particles labeled with 250 microcuries of Antimony-122 were placed in a narrow trough in the bed that extended the width of the channel and to a depth approximating the depth of particle movement. The particles were buried when the flume was drained after the bed slope and bed form had been established. For protection, the tracer particles were covered with a loose-fitting cloth that was anchored in the bed in such a way that it conformed to the bed configuration. The particles were released simply by removing the protective cloth after the flow had been reestablished.

In the field and the first laboratory experiment, the distribution of the tracer particles was defined by traversing the channel longitudinally with a scintillation detector that was supported directly on the bed by a dish-shaped, wooden bearing plate. The detector was housed in a water-tight casing and mounted by a linkage that permitted vertical motion. For the field, the mounting linkage was attached to a sled that was dragged along the stream bottom, (see figs. 2 and 3); in the laboratory, the linkage was attached to the flume carriage. Although the wooden plate that supported the detector tended to disturb the bed, the magnitude of the disturbance was insignificant in comparison to the natural forces acting on the bed. However, because the bed form in the second laboratory experiment was ripples, and the natural forces were small, the detector in its water-tight casing was mounted a short distance off the bed in order to reduce any disturbing effects of the detector. The mounting was accomplished with a geared rack and pinion mechanism actuated by a reversing motor. The mechanism was used to raise and lower the detector so as to maintain a mean distance of 3 in. between the bed surface and the bottom of the casing. The distance between the bed and the detector was continuously monitored with an ultrasonic depth sounder (Karaki and others, 1961) and adjustments in the detector position were made in response to the sounder record (see fig. 4).

Experimental results

Several typical longitudinal distribution curves obtained in the experiments are shown in figures 5 and 6. The curves from the field experiment were obtained by converting the field record to concentrations by means of the calibration curves. The individual points represent average values over short lengths of channel. The curves from the laboratory experiment have been adjusted to compensate for radioactive decay but have not been converted to concentrations; however, the conversion is linear so that shapes of the curves are not distorted. The individual points represent averages determined by monitoring the entire width of the channel at each of the indicated stations.

In addition to the longitudinal distribution, typical vertical distributions that show the variation of radioactivity (tracer particles) with depth into the bed are given in figs. 7 and 8. The common characteristic of the illustrated, as well as most other vertical distributions, is that the activity varies irregularly with depth and that no continuous vertical distribution pattern is definable.

APPLICATION OF CONCENTRATION-DISTRIBUTION FUNCTION TO EXPERIMENTAL DATA

In order to fit the concentration-distribution function to observed data for a given set of experimental conditions, the coefficients k_1 and k_2 must be evaluated. In the discussion which follows this is accomplished by equating the rate of movement of the mode and the rate of attenuation of the peak concentration indicated by the theory to the corresponding rates indicated by the observed longitudinal concentration-distribution curves. The method is illustrated for the field data.

According to the theory, the rate of movement of the mode is approximately equal to the rate of movement of the mean for $k_2 t \gtrsim$ about 4, in which case

$$\frac{d\,x_m}{dt} \simeq \frac{d\,\bar{x}}{d\,t} = \frac{k_2}{k_1} \tag{13}$$

In figure 9, the position of the means and the modes of the observed distributions from the field experiment are plotted as a function of the dispersion time. The slope of the straight portion of the line that defines the rate of movement of the mode is

$$\frac{d\,x_m}{dt} = 3.00 \text{ ft/hr} \cong \frac{k_2}{k_1}$$

Corresponding rates for the first and second laboratory experiments are 6.7 ft/hr and 0.070 ft/hr , respectively.

Also, according to the theory, the peak relative concentration is given by

$$\frac{\phi(x,t)_{max}}{A} = \frac{\phi(x,t)_{max}}{\frac{W}{Bd}\left(1-e^{-k_2 t}\right)} = \frac{f_t(x)_{max}}{\left(1-e^{-k_2 t}\right)} \tag{14}$$

where $f_t(x)_{max}$ is equal to $f_t(x)$ evaluated for x such that $\frac{\partial f_t(x)}{\partial x} = 0$

in which case

$$\frac{k_1 x}{k_2 t} = \left\{\frac{I_2\left(2\sqrt{k_1 x\, k_2 t}\right)}{I_1\left(2\sqrt{k_1 x\, k_2 t}\right)}\right\}^2$$

As the dispersion time increases the probability that a particle is still at the origin, $p_t(0) = e^{-k_2 t}$, approaches zero so that for $k_2 t \geq$ about 3

$$\frac{\phi(x,t)_{max}}{A} \cong f_t(x)_{max} \tag{14a}$$

In figure 10, the values of the observed peak relative concentration, ϕ_{max}/A, from the field experiment are plotted on logarithmic coordinates as a function of the dispersion time. In figure 11 the ratio of the theoretical peak relative concentration (as given in equation 14a) to k_1 is plotted, also on logarithmic coordinates, as a function of $k_2 t$. Because the respective abscissas and ordinates of figures 10 and 11 differ only by the constants of proportionality k_1 and k_2, these constants may be evaluated by superimposing figure 10 over figure 11 in the position where (1) the experimental data fit the theoretical curve and (2) k_2/k_1 equals the rate of movement of the mode (3.00 feet per hour). When the field data are superimposed over the theoretical curve in this position,

$$k_1 = \frac{\phi_{max}/A}{\frac{1}{k_1} f_t(x)_{max}} = \frac{1}{36}\ ft^{-1}$$

and

$$k_2 = \frac{k_2 t}{t} = \frac{1}{12}\ hr^{-1}$$

The theoretical attenuation curve for these values of k_1 and k_2 is shown on figure 10. Values of k_1 and k_2 for the second laboratory experiment were determined in the same way and are respectively 0.60 ft^{-1} and 0.042 hr^{-1}.

Several theoretical concentration-distribution curves, constructed from equation 6 with the values of k_1 and k_2 determined above, are shown in figures 12 and 13 along with the corresponding observed distribution curves. The differences between the theoretical and the observed curves for the field experiment are discussed at length by Sayre and Hubbell (1963). It is concluded that the major differences are attributable to discrepancies between some of the physical features of the experiment and the corresponding assumptions on which the theory is based.

Unfortunately, only a preliminary analysis of the longitudinal distribution data from the laboratory experiments has been completed at this time. However, as shown in figure 13, the preliminary analysis indicates that the data are consistent and that they will compare with the theoretical distribution function at least as well as do the field data.

On the basis of a limited comparitive analysis of the field and laboratory data together with data from experiments with coarse gravel reported by Einstein (1937), some interesting observations can be made. First, data from all the experiments indicate that the theoretical model applies to a wide range of conditions which includes 0.19 mm sand transported at an average rate of 0.07 ft/hr at one extreme and 30 mm gravel transported at an average rate of 120 ft/hr at the other extreme. Secondly, the data indicate that there is a considerable variation in $1/k_1$, the mean step length, and $1/k_2$, the mean rest period. The extreme values of $1/k_1$ were 1.67 ft for 0.19 mm sand when the bed form consisted of ripples and 36 ft for 0.29 mm sand when the bed form consisted of large dunes. These step lengths correspond, respectively, to approximately 3.4 times the average ripple length and 2.4 times the average dune length. Values of $1/k_2$ varied from about 10 seconds in one of Einstein's (1937) experiments with coarse gravel to about 24 hours in the experiment with 0.19 mm sand.

The physical significance of the values of the mean step length and the mean rest period in the theoretical model is not clear at this time. The model is based on the assumption that both the step length and the duration of the rest periods are exponentially distributed. Whether or not the exponential distribution actually is the closest approximation is subject to conjecture, and, no doubt, will continue to be until experiments are made in which the motion of individual particles over a continuous period of time is studied in detail.

Casual observations of the movement of particles tend to lead one to the conclusion that probably the duration of the rest periods is exponentially distributed but that there may be some minimum step length on the order of a few diameters. However, even if step lengths do not approach zero, the exponential distribution may still be the best approximation. For the present, the only assurance of the validity of the assumptions is that the theoretical approximations fit the observed data reasonably well. Additional experiments are being planned with the object of relating k_1 and k_2 to sediment and hydraulic parameters.

COMPUTATIONS OF BED-MATERIAL DISCHARGE

The accuracy of equations 12 and 13 depends mainly on the degree to which the mean particle velocity, $\bar{x}/t$, and the depth of the zone of particle movement, d, can be estimated correctly. Regardless of the measurement technique, accurate estimates of $\bar{x}/t$ depend on how well the observed distribution curves represent, on the average, the actual concentration distribution. With the radioactive tracer technique, assuming suitable calibrations, the observed distributions will be representative of the actual distributions if (1) the concentration of tracer particles in the bed does not vary systematically with the width or with the depth into the bed, (2) the depth of particle movement downstream from the source is constant on the average, and (3) the number of tracer particles in motion either as suspended load or bed load at the time of a distribution measurement is not sufficient to affect the distribution. Segmented core samples collected during all of the experiments (see figs. 7 and 8) demonstrate that the first two conditions ordinarily will be satisfied in channels which are relatively straight and uniform. To be sure, the concentration of tracer particles will vary with depth into the bed and the depth of particle movement will vary somewhat; however, the variations can be expected to be random rather than systematic because they are functions of the statistical variations in the elevations of the troughs and crest of the bed forms in association with the distribution of the tracer particles. The third condition can be expected to be satisfied at all times except when the transport rate is exceedingly high because the rate of change of distribution curves is low and the time required to define a distribution is short.

Three different methods can be used to estimate the depth of the zone of particle movement when the radioactive tracer technique is used. If the bed configuration consists of dunes or ripples the depth of the zone of particle movement, d, can be estimated from depth-sounding data. For this purpose continuous longitudinal profiles of the bed surface are required. The method for estimating d is illustrated in figure 14, and corresponds to the method presented by D. W. Hubbell [1]. The length of the reach for which d

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[1] Apparatus and techniques for measuring bed-load. (Written communication).

is to be determined is divided into sections. Starting from the upstream end, each section of length l_i extends from the dune trough at which the section begins to the first trough downstream which is deeper. After sectioning, a mean depth for each section, d_i , is determined and the d for the total reach length, L , is computed as the weighted average of the d_i's for each section. Expressed mathematically (see fig. 14),

$$d = \frac{1}{L} \sum_{i=1}^{n} l_i d_i$$

The reasoning behind this procedure is based upon the assumption that although the individual bed forms may change shape as they progress down the stream, there is a statistical constancy of form over a long reach. Hence, quantitatively the particles subject to movement are those that would move if the entire profile were to progress downstream without changing form, and the depth of particle movement is defined by lines that are parallel to the mean bed surface and extend downstream from the deepest trough as illustrated.

A second method for estimating d involves the monitoring of core samples to establish the mean depth to which the tracer particles have been transported. The most satisfactory method for determining the mean depth from the separate cores is to average the sum of the depth of flow and depth of activity at each sampling point and then to deduct from this average the overall mean depth of flow.

The third method for estimating d is to compute it from $d = W/BA$. With this method, minor discrepancies in the measurements tend to be averaged out because the concentration in each increment of length is included in the computation.

In order to ascertain the accuracy of the continuity equations, computations of discharge have been made. Because the tracer particles used in each experiment were selected to represent only a single size range within the bed material, the computations, which were made with equation 13, represent only the discharge of particles within the size range of the tracer particles. Since the specific gravity of all of the bed material was the same, i_c was taken as the fraction, by weight, of bed material particles in the size range simulated by the tracer particles. Also, $(\gamma_s)_c$ $(1 - \lambda)$ was taken as 100 lbs/ft^3. The results of the computations for the three experiments by each of the three methods for estimating d are given in table 1 along with data for comparison. The measured bed-material

Table 1.--Computed discharges of bed-material particles represented by radioactive tracer particles and data for comparison.

xperi- ent	Fraction of Bed Material Represented by Tracer Particles	Particle-Size Range Repres- ented by Tracer Particles (mm)	Mean Particle Velocity, $\frac{\bar{x}}{t}$ (Ft/Hr)	Channel Width, B (Ft)	Depth of Zone of Particle Movement Com- puted From			Computed Discharge of Sediment Repre- sented by Tracer Particles From			Measured Discharge of Sedi- ment Re- presented by Tracer Particles (lbs/hr)	Measured Total Bed- Material Discharge (lbs/hr)
					Bed Form (ft)	Core Samples (ft)	W/BA (ft)	Bed Form (lbs/hr)	Core Samples (lbs/hr)	W/BA (lbs/hr)		
ield: orth oup iver	0.65	0.225-0.420	3.00	50	1.41	1.45	1.46	13,760	14,150	14,240	16,000	---
abora- ory: .93mm and	0.16	0.99-1.17	6.68	8	0.147	0.210	----	94.2	134	----	98.4	615
abora- ory: .19mm and	0.50	0.185-0.265	0.07	8	0.106	0.096	0.103	3.0	2.7	2.9	2.3	6.6

discharges listed for comparison with the computed discharges from the flume data are based on samples of the total bed-material concentration collected at the end of the flume. Because the bed-material discharge could not be measured in the field due to the unavailability of equipment for measuring the sediment that is transported on or very near the bed, the discharge computed from the field data is compared with a discharge determined with the modified Einstein procedure (Colby and Hembree, 1955). Inasmuch as the modified Einstein procedure was developed with data from Nebraska sandhill streams and its applicability to shallow sand bed streams has been verified, the determined discharge probably is quite accurate.

The discharges computed with the 0.19 mm sand data are about 30 percent higher than the comparable measured discharge; however, the measured total sediment concentration was only 3.4 ppm so that the accuracy of the measured discharge is probably only about $\pm$ 60 percent. The result which deviates the most from the measured discharge is the one obtained from the core sample data for the 0.93 mm sand experiment. One explanation for the fairly large deviation is that the number of core samples collected was insufficient to define a representative value for d . If this is the case, it emphasizes the necessity of obtaining numerous random samples.

Overall, the results of the computations indicate that the measurement of bed-material discharge by using radioactive tracer techniques is entirely feasible and that reasonably accurate discharges can be obtained if the measurements are made with care. However, one restriction of the method as a practical field technique is that the measurements must be made over an extended period of time when the flow conditions remain essentially constant.

CONCLUSIONS

1. The results of the experiments reported herein demonstrate that radioactive tracer techniques provide an entirely feasible means for applying Lagrangian experimental techniques to the observation of sediment transport processes both in the field and in the laboratory.

2. The transport of bed-material particles can be described as a sequence of alternating steps and rest periods of random length and duration. The assumption of exponential, identically-distributed incremental step lengths and rest periods leads to a longitudinal concentration-distribution function which agrees reasonably well with experimental results obtained in the North Loup River, Nebraska, and a laboratory flume.

3. The results of the laboratory and field experiments demonstrate that the discharge of bed-material particles can be computed by using data collected

by the radioactive tracer technique in conjunction with a continuity equation. In the equation, the velocity is defined by the rate of movement of the mean (center of concentration) of a group of radioactive tracer particles that simulate the bed material and the area is defined as the effective cross-sectional area of the bed through which the tracer particles are distributed. This area corresponds to the zone of particle movement, the average depth of which can be determined from a knowledge of the bed configuration, core samples, or the area under concentration-distribution curves. If the tracer particles are selected to simulate only bed-material particles that possess a certain characteristic, such as the same size, shape, or specific gravity, the discharge of these particles can be determined by the same method.

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APPENDIX - NOTATION

Symbol	Definition	Dimensions	Units
A	Area under concentration-distribution curve	F/L^2	gms/ft^2
B	Width of stream bed	L	ft
c	Subscript that denotes association with particles having a common characteristic		
d	Average depth beneath the surface of the bed through which tracer particles are distributed; also the average depth of the zone in which particle movement occurs	L	ft
d_i	Mean distance between the bed surface and the corresponding line of length l_i	L	ft
f( )	Probability-density function	L^{-1} or T^{-1}	- -
F( )	Distribution function (cumulative probability distribution function)	- -	- -
i	Number of rest periods	- -	- -
i_c	Fraction, by volume, of bed material particles possessing a common characteristic	- -	- -
I_0( )	Modified Bessel function of the first kind of order 0	- -	- -
I_1( )	Modified Bessel function of the first kind of order 1	- -	- -
I_2( )	Modified Bessel function of the first kind of order 2	- -	- -

Symbol	Definition	Dimensions	Units
k_1	Proportionality constant in concentration-distribution and related functions; may also be interpreted as the mean number of steps per unit of longitudinal distance	L^{-1}	ft^{-1}
k_2	Proportionality constant in concentration-distribution and related functions; may also be interpreted as the mean number of waiting periods per unit of time	T^{-1}	hr^{-1}
l_i	Length of a section that is defined by the lowest line which can be drawn in such a way that it is parallel to the mean bed surface and is tangent to the bottom of a trough at its upstream end	L	ft
n	Particular number of steps	- -	- -
p( )	Probability mass function	- -	- -
P[ ]	Probability	- -	- -
Q_s	Bed-material discharge	F/T	lbs/hr
t	Time	T	min, hrs, or days
W	Weight of tracer particles released from source	F	gms
x	Distance downstream from source	L	ft
x'	Dummy variable of integration	L	- -
x_m	Distance that mode of concentration-distribution curve is downstream from source	L	ft

Symbol	Definition	Dimensions	Units
$\bar{x}$	Distance that mean of concentration-distribution curve is downstream from source	L	ft
γ_s	Specific weight of bed material	F/L^3	lbs/ft^3
λ	Fraction of bed material volume void of sediment particles	- -	- -
ϕ	Observed concentration of tracer particles in the bed material	F/L^3	gms/ft^3
ϕ_{max}	Observed peak concentration of particles in the bed material	F/L^3	gms/ft^3
$\phi(x, t)$	Theoretical concentration of tracer particles in the bed material	F/L^3	- -
$\psi(t, x)$	Theoretical discharge of tracer particles through a cross section	F/T	- -