DISSERTATION

QUANTUM SERRE DUALITY FOR QUASIMAPS

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ABSTRACT

QUANTUM SERRE DUALITY FOR QUASIMAPS

Let X be a smooth variety or orbifold and let $Z \subseteq X$ be a complete intersection defined by a section of a vector bundle $E \to X$. Originally proposed by Givental, quantum Serre duality refers to a precise relationship between the Gromov–Witten invariants of Z and those of the dual vector bundle $E^\vee$. In this paper we prove a quantum Serre duality statement for quasimap invariants. In shifting focus to quasimaps, we obtain a comparison which is simpler and which also holds for non-convex complete intersections. By combining our results with the wall-crossing formula developed by Zhou, we recover a quantum Serre duality statement in Gromov-Witten theory without assuming convexity.

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DEDICATION

To Mom, Dad, Olivia, Paul, Sarah, and William;

To Shirley, Ralph, Sylvia, and Lyle.

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Chapter 1

Introduction

The primary result of this paper is a relationship between quasimap invariants of a local complete intersection cut out by a section of vector bundle over a variety or orbifold to the quasimap invariants of the dual vector bundle. The origins of quasimap invariants are rooted in Gromov-Witten theory and their definitions are completely analogous to those of Gromov-Witten invariants. Before introducing quasimap invariants or the main result, we briefly describe the history of Gromov-Witten theory and this correspondence. Along with this history we motivate the study of such invariants by providing an intuitive example.

1.1 History

Gromov–Witten theory was initially motivated by a conjecture of Witten, a physicist, in [1], which states that a specific generating function of intersection numbers of the moduli space of stable curves satisfies a certain series of partial differential equations. Kontsevich's definition of the moduli space of stable maps and his work in [2] proved Witten's conjecture. Since then, Gromov–Witten theory has become an area of mathematics in its own right. It is filled with rich geometry as well as themes from algebra, topology, combinatorics, and differential equations.

1.1.1 Motivating Example

Given a variety or orbifold X, an aim of algebraic geometry is to explain how subspaces of X intersect. For instance if X is the complex projective plane $\mathbb{P}^2$, we may ask the following.

How many conics (degree two curves) pass through five points in general position in the plane?

By general position, we mean distinct and that any three points are non-collinear. A modern strategy to answering this question involves the moduli space of degree 2 maps from the complex projective line $\mathbb{P}^1$ to $\mathbb{P}^2$ with 5 marked points, denoted by $\mathcal{M}_{0,5}(\mathbb{P}^2, 2)$. A point of $\mathcal{M}_{0,5}(\mathbb{P}^2, 2)$ consists of the following data:

- the projective line $\mathbb{P}^1$,
- five distinct points $x_1, \dots, x_5$ in $\mathbb{P}^1$, and
- a degree two map $f : \mathbb{P}^1 \to \mathbb{P}^2$.

This moduli space comes equipped with evaluation maps $ev_i : \mathcal{M}_{0,5}(\mathbb{P}^2, 2) \to \mathbb{P}^2$ defined by

$$(\mathbb{P}^1, x_1, \dots, x_5, f) \mapsto f(x_i)$$

for each $i = 1, \dots, 5$. Note that the preimage of a point y in $\mathbb{P}^2$ via an evaluation map ev_i is all of the points $(\mathbb{P}^1, x_1, \dots, x_5, f)$ such that $f(x_i) = y$. Hence for five points $y_1, \dots, y_5$ of $\mathbb{P}^2$ in general position, the intersection

$$ev_1^{-1}(y_1) \cap \cdots \cap ev_5^{-1}(y_5) \tag{1.1}$$

contains the points $(\mathbb{P}^1, x_1, \dots, x_5, f) \in \mathcal{M}_{0,5}(\mathbb{P}^2, 2)$ such that the image $f(\mathbb{P}^1)$ passes through each y_i. So we can answer the initial question by counting the number of points in this intersection.

Computing the intersection (1.1) is nontrivial. One approach to consider is using the intersection product in cohomology and then pushing forward the class of the intersection to a point. Then the coefficient in the cohomology of a point would equal the number of points in the intersection. However, the pushforward via $\pi : \mathcal{M}_{0,5}(\mathbb{P}^2, 2) \to \text{pt}$ is undefined because the moduli space $\mathcal{M}_{0,5}(\mathbb{P}^2, 2)$ is not compact, so π is not proper.

There are multiple ways to compactify $\mathcal{M}_{0,5}(\mathbb{P}^2, 2)$. In this paper we will focus on the compactification by quasimaps [3–6]. However for this example we will use the more classical compactification via stable maps, which Kontsevich developed in [2]. We denote Kontsevich's compactification, called the moduli space of stable maps, by $\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 2)$. A point in this moduli space consists of the following data:

- an at worst nodal projective genus-0 curve C,

- five distinct points $x_1, \ldots, x_5$ in C, and

- a degree two map $f : C \to \mathbb{P}^2$

satisfying the condition that the rational components of C where f is degree zero must contain a combination of at least three nodes or marked points. The map $\pi : \overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 2) \to$ pt is proper. Therefore, the pushforward

$$\pi_* \left(ev_1^*(H^2) \cup \cdots \cup ev_5^*(H^2) \cap \left[\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 2)\right]\right)$$

is well defined. Here $H^2 \in H^*(\mathbb{P}^2)$ represents the class of a point, $\cup$ denotes the intersection/cup product, $\cap$ is the cap product, and $\left[\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 2)\right]$ is the fundamental class. It takes a little bit of work, but nonetheless we compute

$$\pi_* \left(ev_1^*(H^2) \cup \cdots \cup ev_5^*(H^2) \cap \left[\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 2)\right]\right) = 1 \cdot [\text{pt}].$$

Thus there is exactly one conic which passes through five points in general position.

We often use integral notation to denote pushing forward to a point and taking the coefficient– i.e.,

$$\int_{\left[\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2,2)\right]} ev_1^*(H^2) \cup \cdots \cup ev_5^*(H^2) := \pi_* \left(ev_1^*(H^2) \cup \cdots \cup ev_5^*(H^2) \cap \left[\overline{\mathcal{M}}_{0,5}(\mathbb{P}^2, 2)\right]\right).$$

The above integral is our first example of a Gromov-Witten invariant.

This method for counting conics through five points generalizes. Fix integers $d, n, r \geq 0$ with $rd+r+d+n \geq 3$ (this ensures the moduli space is nonempty) and general subvarieties $V_1, \ldots, V_n$ in $\mathbb{P}^r$ such that the codimensions of the subvarieties $V_1, \ldots, V_n$ sum to $rd + r + d + n - 3$ (so that (1.2) is finite and nonzero). Denote by $[V_i]$ the class in homology associated to the subvariety V_i and let $\gamma_i = [V_i] \cap [\mathbb{P}^r]$ (the cohomology class Poincaré dual to $[V_i]$) for $i = 1, \ldots, n$. Then the integral

$$\int_{\left[\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)\right]} ev_1^*(\gamma_1) \cup \cdots \cup ev_n^*(\gamma_n) \tag{1.2}$$

equals the number of degree d rational curves incident to the subvarieties $V_1, \ldots, V_n$.

For small values of n, r, and d, we can compute (1.2) using classical methods. In our example of counting conics, given precise coordinates of the five points we may determine the number of conics passing through those points using elementary linear algebra. However, in many of the other small cases the methods of calculation are quite technical.

1.1.2 Quantum Serre duality

Computing Gromov-Witten invariants of varieties or orbifolds different from projective space often requires more finesse. For this reason, there is value in finding relationships between the Gromov-Witten invariants of different varieties/orbifolds. The phenomena we study, called Quantum Serre duality, is an example of one such relationship.

Let X be a smooth projective variety or Deligne–Mumford stack and let $Z \subset X$ be a smooth complete intersection, defined by the vanishing of a section of a vector bundle $E \to X$. Quantum Serre duality refers to a relationship between the genus-zero Gromov-Witten invariants of Z and those of the dual vector bundle $E^\vee$.

Quantum Serre duality was first described in mathematics by Givental in [7], for the case of $X = \mathbb{P}^n$. The correspondence is given as a relation between generating functions of genus-zero Gromov–Witten invariants of Z and $E^\vee$ via a complicated change of variables and a non-equivariant limit. Since then, quantum Serre duality has been generalized and reformulated in a number of ways. In [8], Coates–Givental employ Givental's symplectic formalism [9] to show that twisted overruled Lagrangian cones for E and $E^\vee$ may be identified by a symplectic isomorphism. When E is *convex*, this implies that generating functions of Gromov–Witten invariants of Z may be recovered from those of $E^\vee$. In [10], Iritani–Mann–Mignon observe that quantum Serre duality may be cast as an isomorphism of quantum D-modules, and is compatible with a Fourier–Mukai transform in K-theory. This result was refined and extended to orbifolds by Shoemaker in [11].

The utility of the correspondence is based on the fact that in many cases the geometry of $E^\vee$ is simpler than that of Z. For instance, if X is a toric variety then $E^\vee$ is as well. It was noted in [8] that by using quantum Serre duality, the mirror theorem for Z follows from the mirror theorem for $E^\vee$. In recent years, quantum Serre duality has been employed to prove other correspondences in Gromov–Witten theory. Applications have included the crepant transformation conjecture [12, §6], the LG/CY correspondence [13, 14], and the Gromov–Witten theory of extremal transitions [15].

A theme which persists through all of the formulations of quantum Serre duality described above is that in order to observe a correspondence, the Gromov–Witten invariants of Z and $E^\vee$ must be packaged in a clever way (Lagrangian cones, D-modules, etc...). There is no simple relation between the individual Gromov–Witten invariants of Z and those of $E^\vee$.

1.2 Quasimaps

Let W be an affine variety, acted on by a reductive algebraic group G, and let θ be a character of G such that $W^{ss}(\theta) = W^s(\theta)$. Denote by X the GIT stack quotient $[W^{ss}(\theta)/G]$. The moduli stacks $Q^{0+}_{g,k}(X, \beta)$ of stable quasimaps to X (which depends implicitly on the GIT presentation) provide an alternative to Kontsevich's space of stable maps. Generalizing the stable quotient spaces of [4], quasimaps were first introduced for toric varieties in [3] and generalized to GIT quotients in [5, 6].

In contrast to stable maps, a quasimap $f : C \dashrightarrow X$ generally defines only a rational map. For $X = [W^{ss}(\theta)/G]$, a quasimap to X is a morphism from a (orbi-)curve C to the stack $[W/G]$ such that the preimage of the unstable locus is a finite set of points, disjoint from the nodes and markings of C. Under certain mild conditions on W, G, and θ, the moduli space of (0+-stable) quasimaps $Q^{0+}_{g,k}(X, \beta)$ is (relatively) proper, and carries a canonical virtual fundamental class (see §2.3 for details). As in Gromov–Witten theory, quasimap invariants may be defined by integrating over the virtual fundamental class.

The relationship between quasimap invariants and Gromov–Witten invariants is now well understood in many cases; these results are called ε-wall-crossing [16–19]. In principal, one can determine Gromov–Witten invariants from quasimap invariants and vice versa.

1.3 Results

Let $X = [W^{ss}(\theta)/G]$ be as in the previous section. Denote by $\mathfrak{X}$ the stack quotient $[W/G]$. A choice of representation τ in $\mathrm{Hom}(G, \mathrm{GL}(r, \mathbb{C}))$ determines vector bundles

$$E := [W^{ss}(\theta) \times \mathbb{C}^r/G] \to X$$
$$\mathcal{E} := [W \times \mathbb{C}^r/G] \to \mathfrak{X}.$$

Assume $\mathcal{E}$ is *weakly convex* (Definition 3.2.4). Let $s \in \Gamma(X, E)$ be a section of E defined as in §2.2.1, and let $Z = Z(s) \subset X$ be the complete intersection defined by s.

In this paper we compare the two-pointed genus-zero quasimap invariants of Z with those of $E^\vee$. In contrast to the case of Gromov–Witten theory, here we obtain a direct relation between invariants, as well as a statement at the level of virtual classes.

The first step is to identify the relevant state spaces. Consider the diagram

$$\begin{tikzcd} & IE^\vee \arrow[d, "p"', shift right] \\ IZ \arrow[r, "j"] & IX \arrow[u, bend right, "i"'] \end{tikzcd}$$

of inertia stacks. Define the *ambient cohomology* of Z by

$$H^*_{\mathrm{CR,amb}}(Z) := \mathrm{im}(j^*)$$

and the *cohomology of compact type* of $E^\vee$ by

$$H^*_{\mathrm{CR,ct}}(E^\vee) := \mathrm{im}(i_*).$$

Under mild assumptions, there exists (Lemma 6.2.3) an isomorphism

$$\tilde{\Delta} : H^*_{\mathrm{CR,ct}}(E^\vee) \to H^*_{\mathrm{CR,amb}}(Z),$$

characterized by the fact that $\tilde{\Delta}(i_*(\alpha)) = e^{\pi i \,\mathrm{age}_g(\mathcal{E})} j^*(\alpha)$ for $\alpha \in H^*(X_g)$. Our first result is that $\tilde{\Delta}$ identifies two-pointed genus-zero quasimap invariants of Z and $E^\vee$ up to a sign.

Theorem 6.2.6. Given elements $\gamma_1, \gamma_2 \in H^*_{\mathrm{CR,ct}}(E^\vee)$, we have the equality

$$\left\langle \tilde{\Delta}\left(\gamma_1\right)\psi_1^{a_1}, \tilde{\Delta}\left(\gamma_2\right)\psi_2^{a_2} \right\rangle_{0,\beta}^{Z,0+} = e^{\pi i (\beta(\det \mathcal{E}) + \mathrm{rank}(E))} \left\langle \gamma_1 \psi_1^{a_1}, \gamma_2 \psi_2^{a_2} \right\rangle_{0,\beta}^{E^\vee,0+}.$$

Our focus on two-pointed invariants arises from their role in solutions to the quantum differential equation. We use the superscripts ∞ and $0+$ to distinguish between Gromov–Witten theory and quasimap theory respectively. Recall the Dubrovin connection in Gromov–Witten theory

$$\nabla_i^{X,\infty} f := \frac{\partial}{\partial t_i} f + \frac{1}{z} T^i \bullet_{\boldsymbol{t}}^{X,\infty} f,$$

where $\{T^i\}_{i \in I}$ is a basis of $H^*_{CR}(X)$ and $\bullet_{\boldsymbol{t}}^{X,\infty}$ denotes the quantum product. The fundamental solution is given by

$$L^{X,\infty}(\boldsymbol{t}, z)(\alpha) := \alpha + \sum_{i \in I} \left\langle\!\!\left\langle \frac{\alpha}{-z-\psi}, T_i \right\rangle\!\!\right\rangle_0^{X,\infty} (\boldsymbol{t}) T^i,$$

where the double bracket is defined in Definition 2.4.1.

In quasimap theory, one can define an analogous product $\bullet_{\boldsymbol{t}}^{X,0+}$ and connection $\nabla^{X,0+}$, replacing Gromov–Witten invariants with quasimap invariants in the definition. As in Gromov-Witten theory, $\nabla^{X,0+}$ is flat and its solution is given by $L^{X,0+}(\boldsymbol{t}, z)$. Restricting to $\boldsymbol{t} = 0$, we obtain the generating function

$$L^{X,0+}(z)(\alpha) := L^{X,0+}(\mathbf{0}, z)(\alpha) = \alpha + \sum_{i \in I} \sum_{\beta \in \mathrm{Eff}} q^\beta \left\langle \frac{\alpha}{-z-\psi}, T_i \right\rangle_{0,\beta}^{X,0+} T^i, \tag{2.10}$$

of two-pointed genus-zero quasimap invariants. Our main theorem is an identification of $L^{Z,0+}(z)$ and $L^{E^\vee,0+}(z)$.

Theorem 6.2.7. The transformation $\tilde{\Delta}$ identifies the operators $L^{Z,0+}(z)$ and $L^{E^\vee,0+}(z)$ up to a change of variables in the Novikov parameter:

$$L^{Z,0+}(z) \circ \tilde{\Delta} = \tilde{\Delta} \circ L^{E^\vee,0+}(z)|_{q^\beta \mapsto e^{\pi i \beta(\det \mathcal{E})} q^\beta}.$$

This theorem provides a quasimap analogue of [11, Proposition 6.13], proven by Shoemaker in the context of Gromov–Witten theory. Note that here the comparison statement is simpler and holds for more general vector bundles $E^\vee \to X$.

Our strategy of proof is different than the arguments used to prove similar results in Gromov–Witten theory. By working with quasimaps as opposed to stable maps, we are able to recover a direct relationship between virtual classes. As in Gromov–Witten theory, we first relate the virtual fundamental classes $[Q^{0+}_{0,2}(Z,\beta)]^{\mathrm{vir}}$ and $[Q^{0+}_{0,2}(E^\vee,\beta)]^{\mathrm{vir}}$ to certain twisted virtual classes on $Q^{0+}_{0,2}(X,\beta)$. We then show, using (non-quantum) Serre duality, that these twisted virtual classes agree up to a sign (Theorem 6.1.2). The cycle-valued statement is interesting in its own right, as its failure in Gromov–Witten theory is exactly what accounts for the change of variables appearing in previous results.

In the last section, we combine our main theorem with the recent ε-wall-crossing results of Zhou [19, Theorem 1.12.2] to recover a quantum Serre duality statement in Gromov–Witten theory when $E^\vee$ is a GIT quotient.

Corollary 6.3.3. The operator $\tilde{\Delta}$ identifies the fundamental solutions $L^{Z,\infty}$ and $L^{E^\vee,\infty}$ up to a change of variables:

$$\begin{aligned} &L^{Z,\infty}(\mu^{Z,\geq 0+}(q,-\psi),z) \circ \tilde{\Delta} \\ =&\tilde{\Delta} \circ L^{E^\vee,\infty}(\mu^{E^\vee,\geq 0+}(q,-\psi),z)|_{q^\beta \mapsto e^{\pi i \beta(\det \mathcal{E})} q^\beta}, \end{aligned}$$

where $\mu^{Z,\geq 0+}(q,-\psi),z)$ and $\mu^{E^\vee,\geq 0+}(q,-\psi),z)$ are changes of variables defined in terms of the I-functions for Z and $E^\vee$ respectively [19, §1.11].

We may interpret Theorem 6.2.7 and Corollary 6.3.3 together as evidence that quantum Serre duality arises naturally in the setting of quasimaps, at least when the base X can be expressed as a GIT quotient. The complicated change of variables appearing in previous results in Gromov–Witten theory is not inherent to the correspondence itself, but rather a remnant of the ε-wall-crossing formula arising in the passage from $0+$-stable quasimaps to ∞-stable maps. A further benefit to this approach is that we no longer require E be convex, as we describe below.

1.4 Removing the convexity hypothesis

In previous results on quantum Serre duality in Gromov–Witten theory, the vector bundle $\mathcal{E} \to \mathfrak{X}$ was required to be convex. In the case that X is a variety this holds whenever $\mathcal{E}$ is *semi-positive* (Definition 3.2.1). However, it was observed in [20] that when X is an orbifold, semi-positivity of $\mathcal{E}$ does not imply convexity. This has been a serious obstacle in the computation of Gromov–Witten invariants of orbifold complete intersections. By working with quasimap invariants, we avoid the convexity requirement. Our results require only *weak convexity* (Definition 3.2.4), which once again holds whenever $\mathcal{E}$ is semi-positive (and in fact holds slightly more generally, see Definition 3.2.3).

Recent works by Guéré [21] and Wang [22] also address computing genus-zero Gromov–Witten theory of complete intersections when E is not convex. Wang's work also features quasimaps, and is based on a similar philosophy to that used in this paper.

Chapter 2

Preliminaries

In this section we define quasimap invariants and introduce the generating functions which we will use in the rest of the paper. This section also serves to set notation.

2.1 Orbifold cohomology

We refer the reader to [23] for an introduction to Chen-Ruan orbifold cohomology. Denote by X an oriented orbifold which admits a finite good cover.

Definition 2.1.1. Define the *inertia stack* of X, written IX, by the fiber square

$$\begin{array}{ccc} IX & \longrightarrow & X \\ \downarrow & & \downarrow \Delta \\ X & \xrightarrow{\Delta} & X \times X. \end{array}$$

For a stack quotient $X = [U/G]$ the inertia stack is composed of a disjoint union of suborbifolds $X_g = [U_g/C(g)]$ where U_g are the elements of U fixed by $g \in G$. Let S be a set of representatives of each conjugacy class of G, then we may write the inertia stack as

$$IX = \bigsqcup_{g \in S} X_g.$$

We refer to each X_g as a *twisted sector* and call the twisted sector corresponding to the identity the *untwisted sector*.

Let (x, g) be a point in a twisted sector X_g. The tangent space T_xX splits as the direct sum of eigenspaces

$$T_xX = \bigoplus_{0 \le f < 1} (T_xX)_f,$$

where g acts on the fiber $(T_xX)_f$ by multiplication by $e^{2\pi i f}$. The *age shift* for X_g is defined to be

$$\iota_{X_g} = \sum_{0 \le f < 1} f \dim_{\mathbb{C}} (T_xX)_f.$$

Definition 2.1.2. [23, 24] The *dth Chen-Ruan cohomology group* of X is

$$H^d_{\mathrm{CR}}(X) := \bigoplus_{g \in S} H^{d-2\iota_{X_g}}(X_g; \mathbb{C})$$

and

$$H^*_{\mathrm{CR}}(X) := \bigoplus_{d \in \mathbb{Q}_{\geq 0}} H^d_{\mathrm{CR}}(X).$$

Unless specified otherwise we assume cohomology groups have complex coefficients. Let ι be the involution of IX that maps X_g to $X_{g^{-1}}$. Denote the compactly supported Chen-Ruan cohomology of X by $H^*_{\mathrm{CR,cs}}(X)$.

Definition 2.1.3. [23, §4] The *Chen-Ruan Poincaré pairing* of the classes $\alpha \in H^*_{\mathrm{CR}}(X)$ and $\beta \in H^*_{\mathrm{CR,cs}}(X)$ is given by the integral

$$\langle \alpha, \beta \rangle^X := \int_{IX} \alpha \cup \iota^* \beta.$$

2.2 GIT stack quotients

We follow the notations and definitions developed in [6]. Let W be an irreducible affine variety with a right action by a reductive algebraic group G. Let θ be a character of G and $\mathbb{C}_\theta$ be the corresponding one-dimensional G-representation. We also denote the linearization $W \times \mathbb{C}_\theta$ as $\mathbb{C}_\theta$. Denote the semistable locus by $W^{ss}(\theta)$ and the stable locus by $W^s(\theta)$. Observe the following diagram of quotients:

$$\begin{tikzcd}
X := [W^{ss}(\theta)/G] \arrow[r, hook] \arrow[d] & \mathfrak{X} := [W/G] \arrow[d] \\
\underline{X} := W/\!\!/_\theta G \arrow[r] & \underline{X}_0 := \mathrm{Spec}(\mathbb{C}[W]^G).
\end{tikzcd}$$

We refer to X as the GIT stack quotient, $\mathfrak{X}$ as the stack quotient, $\underline{X}$ as the underlying coarse space or GIT quotient with respect to θ and $\underline{X}_0$ as the affine quotient.

Assumption 2.2.1. We assume $W^{ss}(\theta) = W^s(\theta)$ so that X is a quasi-compact Deligne-Mumford stack.

Definition 2.2.2. Fix a representation $\tau : G \to \mathrm{GL}(r, \mathbb{C})$ for some integer r. Define the vector bundle $\mathcal{E} = \mathcal{E}_\tau \to \mathfrak{X}$ as

$$\mathcal{E} := [W \times \mathbb{C}^r / G],$$

where $g \in G$ acts on the $\mathbb{C}^r$ factor by multiplication by $\tau(g)$. For simplicity we will omit τ from the notation for the representation $\mathbb{C}^r$ and the vector bundle $\mathcal{E}$ throughout the paper.

The vector bundle $\mathcal{E}$ restricts to a vector bundle on X, which we denote by E. It may be realized as the quotient $E := [W^{ss}(\theta) \times \mathbb{C}^r / G]$.

Definition 2.2.3. Let (x, g) be a point in a twisted sector $\mathfrak{X}_g$. The fiber $\mathcal{E}_x$ splits as the direct sum of eigenspaces

$$\mathcal{E}_x = \bigoplus_{0 \le f < 1} (\mathcal{E}_x)_f,$$

where g acts on $(\mathcal{E}_x)_f$ by multiplication by $e^{2\pi i f}$. Define the *age* of $\mathcal{E}$ at g as

$$\mathrm{age}_g(\mathcal{E}) := \sum_{0 \le f < 1} f \dim_{\mathbb{C}} (\mathcal{E}_x)_f.$$

2.2.1 A local complete intersection in X

Let $\mathcal{E}$ and E be as defined in Definition 2.2.2. We do not assume E is pulled back from the coarse space $\underline{X}$, hence the isotropy groups G_x for x in X may act nontrivially on the fibers of E.

Let s be a global section of $\Gamma(W, W \times \mathbb{C}^r)^G$ such that the zero locus $Z(s)$ is an irreducible complete intersection intersecting the semistable locus $W^{ss}(\theta)$ non-trivially. Assume $Z(s) \cap W^{ss}(\theta)$ is non singular.

Definition 2.2.4. Define $\mathfrak{Z} := [Z(s)/G]$ to be the a closed substack of $\mathfrak{X}$ given by the section s. Define Z as the GIT stack quotient

$$Z := [(Z(s) \cap W^{ss}(\theta))/G].$$

The semistable locus of $Z(s)$ is exactly the intersection of $Z(s)$ and $W^{ss}(\theta)$. The section s restricts to a section of $W^{ss}(\theta) \times \mathbb{C}^r$ whose zero locus is exactly $Z(s) \cap W^{ss}(\theta)$. Equivalently, $\mathfrak{Z}$ and Z are defined as the zero loci of the sections of $\mathcal{E}$ and E induced by s. We will also sometimes denote these sections by s.

Remark 2.2.5. If we assume the representation $\mathbb{C}^r$ splits as the direct sum of one-dimensional representations $\oplus_{i=1}^r \mathbb{C}_{\tau_i}$ for characters $\tau_i : G \to \mathbb{C}^*$, then $\mathcal{E}$ and E split and Z defines a complete intersection of X.

Definition 2.2.6. For a fixed g in S, denote

$$E_g := [(W^{ss}(\theta) \times \mathbb{C}^r)^g / C(g)].$$

The inertia stack IE may be written as $IE = \bigsqcup_{g \in S} E_g$. Note that E_g is not usually equal to $E|_{X_g}$. When g acts nontrivially on the representation $\mathbb{C}^r$, the twisted sector E_g equals X_g. If g acts trivially, then E_g is a vector bundle over X_g.

The section s induces a section $s_g \in \Gamma(X_g, E_g)$ for each $g \in S$. The twisted sector $Z_g \subset IZ$ may be realized as the vanishing locus of s_g. Denote by $j : IZ \to IX$ and $0_E : IX \to IE$ the inclusion and zero section respectively. Then we have a fiber square

$$\begin{tikzcd} IZ \arrow[r, "j"] \arrow[d, "j"] & IX \arrow[d, "s"] \\ IX \arrow[r, "0_E"] & IE. \end{tikzcd}$$

By abuse of notation, we will also denote by j, s, and 0_E the corresponding maps between the rigidified inertia stacks.

Definition 2.2.7. Define the *ambient cohomology* of Z_g, denoted $H^*_{\mathrm{amb}}(Z_g)$, as the image of the cohomology of X_g under the pullback via the inclusion $j_g : Z_g \to X_g$. The *ambient Chen-Ruan cohomology* of Z is

$$H^*_{\mathrm{CR,amb}}(Z) := \bigoplus_{g \in S} H^{*-2\iota_{Z_g}}_{\mathrm{amb}}(Z_g).$$

Restricting the computation of quasimap invariants of Z to the ambient cohomology is common and, as Proposition 4.1.5 will suggests, natural.

Assumption 2.2.8. Following [10] and [11], we assume the Poincaré pairing on $H^*_{\mathrm{CR,amb}}(Z)$ is non-degenerate. This is equivalent to assuming the cohomology of Z splits as the direct sum:

$$H^*_{\mathrm{CR}}(Z) = \mathrm{im}(j^*) \oplus \ker(j_*).$$

2.2.2 The total space $E^\vee$

With the setting as in §2.2.1, consider the vector bundles

$$\mathcal{E}^\vee := [W \times \mathbb{C}^r/G] \qquad E^\vee := [(W^{ss}(\theta) \times \mathbb{C}^r)/G].$$

Note that $IE^\vee = \bigsqcup_{g\in S} E^\vee_g$. In this section we describe the Chen–Ruan cohomology of compact type of $E^\vee$. A more detailed introduction to cohomology of compact type appears in §2 of [11]. Note that in [11], the cohomology of compact type is referred to as the narrow cohomology.

Definition 2.2.9. [11, Definition 2.1] The *cohomology of compact type* of $E^\vee_g$ is the image of the natural homomorphism

$$\phi : H^*_{\mathrm{cs}}(E^\vee_g) \to H^*(E^\vee_g)$$

from compactly supported cohomology to cohomology. The *Chen-Ruan cohomology of compact type* is

$$H^*_{\mathrm{CR,ct}}(E^\vee) := \bigoplus_{g\in S} H_{\mathrm{ct}}^{*-2\iota_{E^\vee_g}}(E^\vee_g).$$

Given a class $\gamma \in H^*_{\mathrm{CR,ct}}(E^\vee)$, we call $\overline{\gamma} \in H^*_{\mathrm{CR,cs}}(E^\vee)$ a *lift* of γ if $\phi(\overline{\gamma}) = \gamma$. It is proven in [11, Lemma 2.6] that for $\gamma \in H^*_{\mathrm{CR,ct}}(E^\vee)$ and $\kappa \in \ker(\phi) \subset H^*_{\mathrm{CR,cs}}(E^\vee)$, $\kappa \cup \gamma$ is zero. From this fact one can check that the following pairing is well-defined and nondegenerate.

Definition 2.2.10. Define the *compact type pairing* on $E^\vee$ as follows. For $\alpha, \beta \in H^*_{\mathrm{CR,ct}}(E^\vee)$,

$$\langle \alpha, \beta \rangle^{E^\vee, \mathrm{ct}} := \langle \alpha, \overline{\beta} \rangle^{E^\vee} = \int_{IE^\vee} \alpha \cup \iota^*(\overline{\beta}),$$

where $\overline{\beta} \in H^*_{\mathrm{CR,cs}}(E^\vee)$ is a lift of β.

Let $p_g : E^\vee_g \to X_g$ denote the vector bundle projection for each $g \in S$. Define the linear map

$$e(p_g^* E_g^\vee) \cup - : H^*(E_g^\vee) \to H^*(E_g^\vee)$$

by cupping with the Euler class $e(p_g^* E_g^\vee)$. Denote by $i : IX \hookrightarrow IE^\vee$ the inclusion induced by the zero section of $E^\vee$ and $i_g : X_g \hookrightarrow E_g^\vee$ the inclusion induced by the zero section of $E_g^\vee$.

Lemma 2.2.11. [11, Proposition 2.15] *If X is compact, then the following vector spaces are equal:*

$$H^*_{\mathrm{CR,ct}}(E^\vee) = \mathrm{im}\,(i_* : H^*_{\mathrm{CR}}(X) \to H^*_{\mathrm{CR}}(E^\vee)) = \bigoplus_{g \in S} \mathrm{im}\left(e(p^* E_g^\vee) \cup -\right). \tag{2.1}$$

Proof. We will prove the result for a given twisted sector $E_g^\vee$. If g fixes only the origin of $W \times \mathbb{C}^r$, then the equality holds trivially since X is compact. Let i^{cs}_{g*} denote the pushforward on compactly supported cohomology induced by i_g. Note that i_{g*} factors as

$$i_{g*} = i_{g*} \circ \phi = \phi \circ i^{\mathrm{cs}}_{g*}.$$

Thus, by [25, Equation (6.11)], i^{cs}_{g*} is an isomorphism. We obtain the first equality of (2.1):

$$\mathrm{im}\left(i_{g*} : H^*(X_g) \to H^*(E_g^\vee)\right) = \mathrm{im}\left(\phi \circ i^{\mathrm{cs}}_{g*}\right) = H^*_{\mathrm{ct}}(E_g^\vee).$$

Now, let α be an element of $H^*(E_g^\vee)$. Then

$$\begin{aligned} e(p_g^* E_g^\vee) \cup \alpha &= i_{g*}(1) \cup \alpha \\ &= i_{g*}(1 \cup i_g^*(\alpha)) \\ &= i_{g*}(i_g^*(\alpha)). \end{aligned}$$

The first equality holds because X_g can be viewed as the zero locus of the tautological section of $p_g^* E_g^\vee$ and the second is the projection formula.

Since the pullback via i_g is an isomorphism, the second equality of (2.1) holds. $\square$

Later we will use Lemma 2.2.11 to express classes in the Chen-Ruan cohomology of compact type of $E^\vee$ in terms of classes in the Chen-Ruan cohomology of X. By the lemma, for γ an element of $H^*_{\text{ct}}(E_g^\vee)$, there exist a class α in $H^*(X_g)$ such that

$$\gamma = i_{g*}(\alpha) = p_g^*(e(E_g^\vee) \cup \alpha). \tag{2.2}$$

2.3 Quasimaps to a GIT stack quotient

In this section we recall the definition of ε-stable quasimap invariants and define the generating functions that will appear in our later theorems. Let X be a GIT stack quotient as in §2.2.

Definition 2.3.1. [26, §4] Over an algebraically closed field, a *twisted curve* is a connected, one-dimensional Deligne-Mumford stack which is étale locally a nodal curve, and a scheme outside the marked points and the singular locus.

Definition 2.3.2. [6, Definition 2.1] Let $(C, x_1, \ldots, x_k)$ be a k-pointed, genus-g twisted curve and let $\varphi : (C, x_1, \ldots, x_k) \to (\underline{C}, \underline{x}_1, \ldots, \underline{x}_k)$ be its rigidification.

- A k-pointed, genus-g *quasimap to* X is a twisted curve $(C, x_1, \ldots, x_k)$ together with a representable morphism $[u] : C \to \mathfrak{X}$ such that $[u]^{-1}(\mathfrak{X} \setminus X)$, referred to as the *base lo-*

cus, is purely zero-dimensional. We denote this quasimap as $(C, x_1, \ldots, x_k, [u])$ and call $(C, x_1, \ldots, x_k)$ or C its *source curve*.

- The *class* β of the quasimap is an element of $\mathrm{Hom}(\mathrm{Pic}\,\mathfrak{X}, \mathbb{Q})$ defined by

$$\beta : \mathrm{Pic}\,\mathfrak{X} \to \mathbb{Q}, \qquad \mathcal{L} \mapsto \deg\left([u]^*(\mathcal{L})\right).$$

  The *degree* of the quasimap $[u]$ is given by the rational number $\beta(\mathbb{C}_\theta)$. We say β is *θ-effective* if it is represented by a quasimap to X. The set of θ-effective classes forms a sub-semigroup of $\mathrm{Hom}(\mathrm{Pic}\,\mathfrak{X}, \mathbb{Q})$ denoted $\mathrm{Eff}(W, G, \theta)$. For brevity, we often denote this sub-semigroup as Eff.

- If the base locus of a quasimap $(C, x_1, \ldots, x_k, [u])$ does not contain marked or nodal gerbes, then we call the quasimap *prestable*.

- Let $\mathbf{e}$ be the least common multiple of $|\operatorname{Aut}(p)|$ for all geometric points $p \to X$ with isotropy groups $\operatorname{Aut}(p)$. Fix a rational number ε. A prestable quasimap is called *ε-stable* if

  1. the $\mathbb{Q}$-line bundle
$$\omega_{\underline{C}}\left(\sum_{i=1}^{k} \underline{x}_i\right) \otimes \left(\varphi_*([u]^*\mathbb{C}_\theta^{\otimes \mathbf{e}})\right)^\varepsilon \tag{2.3}$$
     on the coarse curve $\underline{C}$ is ample and
  2. for all $x \in C$,
$$\varepsilon l(x) \leq 1,$$
     where $l(x)$ is the *length at* x defined in [5, §7.1].

  We say a prestable quasimap is $0+$*-stable*, if (2.3) is ample for all rational $\varepsilon > 0$.

Definition 2.3.3. The *moduli space of k-pointed, genus-g, ε-stable quasimaps to X of degree β*, denoted by $Q^\varepsilon_{g,k}(X, \beta)$, is the space of isomorphism classes of k-pointed, genus-g, ε-stable quasimaps to X of degree β.

By Theorem 2.7 of [6], $Q^{\varepsilon}_{g,k}(X,\beta)$ is a proper Deligne-Mumford stack over the affine quotient $\underline{X}_0$. Furthermore, when the singularities of W are at worst local complete intersections and $W^{ss}(\theta)$ is nonsingular, $Q^{\varepsilon}_{g,k}(X,\beta)$ carries a canonical perfect obstruction theory.

When ε is sufficiently large, the moduli space of ε-stable quasimaps $Q^{\varepsilon}_{g,k}(X,\beta)$ coincides with the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,\beta)$ from Gromov-Witten theory. We adopt the notation of [5, 6] and define

$$Q^{\infty}_{g,k}(X,\beta) := \overline{\mathcal{M}}_{g,n}(X,\beta).$$

2.4 The Dubrovin connection

Since the base locus of an ε-stable quasimap is disjoint from the marked gerbes for $i = 1, \ldots, k$, there exists evaluation maps to the rigidified inertia stack $\bar{I}X$:

$$ev_i : Q^{\varepsilon}_{g,k}(X,\beta) \to \bar{I}X.$$

We refer the reader to [26, §4.4] for an introduction to such maps. There is a canonical map $\bar{\omega} : IX \to \bar{I}X$ inducing an isomorphism of cohomology $\bar{\omega}_* : H^*(IX) \to H^*(\bar{I}X)$. By composing ev_i^* with $\bar{\omega}_*^{-1}$, we may define quasimap invariants using cohomology class insertions from $H^*_{\mathrm{CR}}(X)$.

Let ψ_i represent the first Chern class of the universal cotangent line over $Q^{\varepsilon}_{g,k}(X,\beta)$ whose fiber over a point $(C, x_1, \ldots, x_k, [u])$ is given by the cotangent space of the underlying coarse curve $\underline{C}$ at the ith marked point. Denote by $[Q^{\varepsilon}_{g,k}(X,\beta)]^{\mathrm{vir}}$ the virtual fundamental class from [5, §4.5] and [6, §2.4.5]. By [6, Theorem 2.7], $Q^{\varepsilon}_{g,k}(X,\beta)$ is proper over $\underline{X}_0$. By [27, Corollary 4.8], this implies each evaluation map ev_i is proper.

Definition 2.4.1. Assume X is proper. For non-negative integers a_i and classes $\alpha_i \in H^*_{\mathrm{CR}}(X)$, *$\varepsilon$-quasimap invariants* or simply *quasimap invariants* are given by integrals

$$\langle \alpha_1\psi_1^{a_1}, \ldots, \alpha_k\psi_k^{a_k}\rangle^{X,\varepsilon}_{g,\beta} := \int_{[Q^{\varepsilon}_{g,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{k} ev_i^*(\alpha_i)\psi_i^{a_i}. \tag{2.4}$$

Fix a basis $\{T^i\}_{i\in I}$ of $H^*_{\mathrm{CR}}(X)$ and let $\boldsymbol{t} = \sum_{i\in I} t_i T^i$. Define the double-bracket

$$\langle\langle \alpha_1\psi_1^{a_1}, \ldots, \alpha_k\psi_k^{a_k}\rangle\rangle_0^{X,\varepsilon}(\boldsymbol{t}) := \sum_{\beta\in\mathrm{Eff}}\sum_{m\geq 0} \frac{q^\beta}{m!}\langle \alpha_1\psi_1^{a_1}, \ldots, \alpha_k\psi_k^{a_k}, \boldsymbol{t}, \ldots, \boldsymbol{t}\rangle_{0,\beta}^{X,\varepsilon}, \tag{2.5}$$

where there are m insertions of $\boldsymbol{t}$ in each summand.

If X is not proper (i.e. if $\underline{X}_0$ is not a point), then slightly more care must be taken to define quasimap invariants. Assume that the evaluation maps $ev_i : Q^{0+}_{0,2}(X,\beta) \to \bar{I}X$ are proper. If at least one class α_j lies in compactly supported cohomology $H^*_{\mathrm{CR,cs}}(X)$ for $1 \leq j \leq k$ then we define

$$\langle \alpha_1\psi_1^{a_1}, \ldots, \alpha_k\psi_k^{a_k}\rangle_{g,\beta}^{X,\varepsilon} \tag{2.6}$$

exactly as in (2.4).

Definition 2.4.2. We also define ε-quasimap invariants whenever at least two insertion classes are compact type. Assume α_j and α_l are in $H^*_{\mathrm{CR,ct}}(X)$ for $1 \leq j, l \leq k$ distinct. Define

$$\langle \alpha_1\psi_1^{a_1}, \ldots, \alpha_k\psi_k^{a_k}\rangle_{g,\beta}^{X,\varepsilon} := \left\langle \tilde{ev}_{l*}\left(\psi_l^{a_l} \cup \prod_{i\in\{1,\ldots,k\}\setminus\{l\}} ev_i^*(\alpha_i)\psi_i^{a_i} \cap \left[Q^\varepsilon_{0,k}(X,\beta)\right]^{\mathrm{vir}}\right), \alpha_l\right\rangle^{X,\mathrm{ct}}, \tag{2.7}$$

where $\langle -,-\rangle^{X,\mathrm{ct}}$ is the compact type pairing of Definition 2.2.9 and $\tilde{ev}_k = \iota \circ ev_k$.

By Proposition 2.5 of [11], pullback and pushforward via a proper map each preserve the compact type subspace of $H^*_{\mathrm{CR}}(X)$. Because α_j lies in $H^*_{\mathrm{CR,ct}}(X)$, the pushforward in the right hand side of (2.7) lies in $H^*_{\mathrm{CR,ct}}(X)$ as desired.

Define the double bracket exactly as in (2.5), but replacing (2.4) with (2.7).

If $\overline{\alpha}_j$ is a lift of α_j as defined in §2.2.2, then (2.7) is equal to (2.6) after replacing α_j with $\overline{\alpha}_j$.

Let $\{T_i\}_{i\in I} \subseteq H^*_{\mathrm{CR,ct}}(X)$ denote the dual basis to $\{T^i\}_{i\in I}$ in the cohomology of compact type.

Definition 2.4.3. [28, §2] The *ε-quantum product* of α_1 and α_2 in $H^*_{\mathrm{CR,ct}}(X)$, written $\alpha_1 \bullet_{\boldsymbol{t}}^{X,\varepsilon} \alpha_2$, is the sum

$$\alpha_1 \bullet_{\boldsymbol{t}}^{X,\varepsilon} \alpha_2 := \sum_{i\in I}\langle\langle \alpha_1, \alpha_2, T_i\rangle\rangle_0^{X,\varepsilon}(\boldsymbol{t})T^i.$$

Definition 2.4.4. Let z be a formal variable. The *ε-Dubrovin connection* on

$$H^*_{\mathrm{CR,ct}}(X) \otimes \mathbb{C}[[t_i, q]]_{i \in I}[z, z^{-1}]$$

is defined by

$$\nabla_i^{X,\varepsilon} = \frac{\partial}{\partial t_i} + \frac{1}{z} T_i \bullet_{\boldsymbol{t}}^{X,\varepsilon} . \tag{2.8}$$

Define the operator $L^{X,\varepsilon}(\boldsymbol{t}, z)$ by

$$L^{X,\varepsilon}(\boldsymbol{t}, z)(\alpha) := \alpha + \sum_{i \in I} \left\langle\!\!\left\langle \frac{\alpha}{-z-\psi}, T_i \right\rangle\!\!\right\rangle_0^{X,\varepsilon} (\boldsymbol{t}) T^i, \tag{2.9}$$

for all $\alpha \in H^*_{\mathrm{CR,ct}}(X)$.

Proposition 2.4.5. *The Dubrovin connection $\nabla^{X,\varepsilon}$ is flat, with fundamental solution given by the operator $L^{X,\varepsilon}(\boldsymbol{t}, z)$. For $i \in I$ and $\alpha \in H^*_{\mathrm{CR,ct}}(X)$ we have the equality*

$$\nabla_i^{X,\varepsilon} \left(L^{X,\varepsilon}(\boldsymbol{t}, z)(\alpha) \right) = 0.$$

Proof. The proof is identical to the Gromov-Witten theory case [29, 30]. The key ingredient is the topological recursion relation for quasimaps [28, Corollary 2.3.4]. $\square$

When X is proper we have

$$H^*_{\mathrm{CR}}(X) = H^*_{\mathrm{CR,ct}}(X).$$

In this case, (2.8) and (2.9) are (ϵ-stable versions of) the usual Dubrovin connection and fundamental solution.

Denote by $L^{X,\varepsilon}(z)$ the restriction of $L^{X,\varepsilon}(\boldsymbol{t}, z)$ to $\boldsymbol{t} = \boldsymbol{0}$:

$$\begin{aligned} L^{X,\varepsilon}(z)(\alpha) := L^{X,\varepsilon}(\boldsymbol{0}, z)(\alpha) &= \alpha + \sum_{i \in I} \sum_{\beta \in \mathrm{Eff}} q^\beta \left\langle \frac{\alpha}{-z-\psi}, T_i \right\rangle_{0,\beta}^{X,\varepsilon} T^i \\ &= \alpha + \sum_{\beta \in \mathrm{Eff}} q^\beta \tilde{ev}_{2*} \left(\frac{ev_1^*(\alpha)}{-z-\psi_1} \cap \left[Q^\varepsilon_{0,2}(X, \beta) \right]^{\mathrm{vir}} \right), \end{aligned} \tag{2.10}$$

for all $\alpha \in H^*_{\mathrm{CR,ct}}(X)$. The operator $L^{X,\varepsilon}(z)$ records all two-pointed genus-zero ε-stable quasimap invariants with one compact type primary insertion and one compact type descendants insertion. In this paper we restrict our attention to $L^{X,0+}(z)$.

Chapter 3

Two-pointed genus-zero quasimaps

In this section we define weak forms of convexity and concavity and show that both are equivalent. We also prove that weakly semi-positive vector bundles $\mathcal{E} \to \mathfrak{X}$ (Definition 3.2.3) are weakly convex.

3.1 Source curves

Here we describe the source curves of two-pointed genus-zero $0+$-stable quasimaps.

Lemma 3.1.1. *For a point $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X, \beta)$, the underlying coarse curve $(\underline{C}, \underline{x}_1, \underline{x}_2)$ is an at worst nodal curve such that each irreducible component has exactly two special points. Furthermore, the degree of $[u]$ is positive on every rational component.*

Proof. For a point $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X, \beta)$, the dual graph of its source curve (C, x_1, x_2) is a tree because the curve is genus zero. The stability condition (2.3) states that the line bundle

$$\omega_{\underline{C}}\left(\underline{x}_1 + \underline{x}_2\right) \otimes \left(\phi_*([u]^* \mathbb{C}_\theta^{\otimes \mathbf{e}})\right)^{\varepsilon}$$

has positive degree on each rational component of the underlying coarse curve $(\underline{C}, \underline{x}_1, \underline{x}_2)$ for all $\varepsilon > 0$. Hence, every rational component of $(\underline{C}, \underline{x}_1, \underline{x}_2)$ with a single node must contain a marked point. Since there are only two marked points, there are at most two rational components with a single node. That is, $(\underline{C}, \underline{x}_1, \underline{x}_2)$ must be a chain of rational components with marked points on the terminal components.

Therefore each rational component of $\underline{C}$ has exactly two special points. The stability condition then implies that the degree of $[u]$ must be positive on every component of (C, x_1, x_2). $\square$

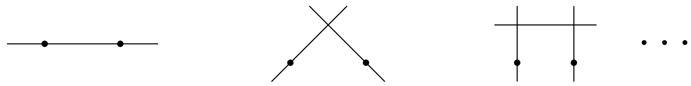

Figure 3.1: The underlying coarse curves $(\underline{C}, \underline{x}_1, \underline{x}_2)$.

For integers c, d, denote by $\mathbb{P}_{[c,d]}$ a smooth twisted curve whose coarse space is $\mathbb{P}^1$ with two marked points such that the isotropy groups at the marked points are μ_c and μ_d. By Lemma 3.1.1, each component of a source curve in $Q^{0+}_{0,2}(X, \beta)$ is given by some $\mathbb{P}_{[c,d]}$.

Denote by l, a, and b the integers $l = \gcd(c, d)$, $a = c/l$, and $b = d/l$. We can rewrite l as the product of integers l_1 and l_2 such that $\gcd(l_1, l_2) = 1$, $\gcd(l_1, b) = 1$, and $\gcd(l_2, a) = 1$. In the next lemma we give a uniform way of expressing $\mathbb{P}_{[c,d]}$ as a quotient stack.

Lemma 3.1.2. *Let $\chi : \mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^* \to \mathbb{C}^*$ be the character given by*

$$\chi\left(e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda\right) = \lambda.$$

The twisted curve $\mathbb{P}_{[c,d]}$ may be represented as a GIT stack quotient

$$\left[\mathbb{C}^2 /\!\!/_{\chi} \left(\mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^*\right)\right] \tag{3.1}$$

with action

$$\left(e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda\right) \cdot (x, y) = \left(\lambda^a x e^{2\pi i \frac{m_1}{l_1}}, \lambda^b y e^{2\pi i \frac{m_2}{l_2}}\right),$$

for $0 \leq m_1 < l_1$, $0 \leq m_2 < l_2$, $\lambda \in \mathbb{C}^*$, and $(x, y) \in \mathbb{C}^2$.

Remark 3.1.3. The presentation (3.1) is not unique. For example, $\mathbb{P}_{[c,d]}$ may also be represented by the GIT stack quotient

$$\left[\mathbb{C}^3 /\!\!/_{\sigma} \left(\mu_{l_1} \times \mu_{l_2} \times (\mathbb{C}^*)^2\right)\right]$$

where the character $\sigma : \mu_{l_1} \times \mu_{l_2} \times (\mathbb{C}^*)^2 \to \mathbb{C}^*$ is defined by

$$\sigma\left(e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda_1, \lambda_2\right) = \lambda_1 \lambda_2$$

and the action is

$$\left(e^{2\pi i\frac{m_1}{l_1}}, e^{2\pi i\frac{m_2}{l_2}}, \lambda_1, \lambda_2\right) \cdot (x, y, z) = \left(\lambda_1^a x e^{2\pi i\frac{m_1}{l_1}}, \lambda_1^b y e^{2\pi i\frac{m_2}{l_2}}, \lambda_2 z\right),$$

for $0 \leq m_1 < l_1$, $0 \leq m_2 < l_2$, $(\lambda_1, \lambda_2) \in (\mathbb{C}^*)^2$, and $(x, y, z) \in \mathbb{C}^3$.

Proof. It suffices to show that the orbifold curve (3.1) contains exactly two orbifold points with isotropy groups of orders c and d.

We first calculate the generic isotropy. Fix $x, y \neq 0$ and let $\left(e^{2\pi i\frac{m_1}{l_1}}, e^{2\pi i\frac{m_2}{l_2}}, \lambda\right)$ be an element of $\mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^*$ which fixes (x, y). Then we have

$$\lambda^a e^{2\pi i\frac{m_1}{l_1}} = 1 \tag{3.2}$$

and

$$\lambda^b e^{2\pi i\frac{m_2}{l_2}} = 1. \tag{3.3}$$

Equality (3.2) implies $\lambda = e^{2\pi i\frac{l_1 n - m_1}{a l_1}}$ for some integer n. So, the product $\lambda^b e^{2\pi i\frac{m_2}{l_2}}$ equals $e^{2\pi i\left(\frac{m_2}{l_2}+\frac{b l_1 n - b m_1}{a l_1}\right)}$ and (3.3) implies

$$\frac{m_2}{l_2} + \frac{b l_1 n - b m_1}{a l_1} = \frac{a l_1 m_2 + b l_1 l_2 n - b l_2 m_1}{a l_1 l_2}$$

is an integer.

If $a l_1 l_2$ divides $a l_1 m_2 + b l_1 l_2 n - b l_2 m_1$, then l_2 divides $a l_1 m_2$. However, the greatest common divisor of $a l_1$ and l_2 is one, and m_2 is a non-negative integer strictly less than l_2. Thus, m_2 must equal 0. Similar reasoning shows m_1 also equals 0.

In this case, (3.2) and (3.3) reduce to the equation $\lambda^a = \lambda^b = 1$. By assumption, the greatest common divisor of a and b is one. Therefore (3.2) holds if and only if $\left(e^{2\pi i\frac{m_1}{l_1}}, e^{2\pi i\frac{m_2}{l_2}}, \lambda\right)$ is the identity.

An element $\left(e^{2\pi i \frac{m_1}{l_1}}, e^{2\pi i \frac{m_2}{l_2}}, \lambda\right)$ in $\mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^*$ fixes $(1,0)$ if and only if $\lambda^a e^{2\pi i \frac{m_1}{l_1}} = 1$. Hence, μ_{l_2} fixes $(1,0)$ and the subgroup of $\mu_{l_1} \times \mathbb{C}^*$ which fixes $(1,0)$ is

$$\left\langle \left(e^{-2\pi i \frac{1}{l_1}}, e^{2\pi i \frac{1}{a l_1}}\right) \right\rangle \cong \mu_{a l_1}. \tag{3.4}$$

This gives us the following isotropy group:

$$G_{(1,0)} \cong \mu_{a l_1} \times \mu_{l_2} \cong \mu_{a l_1 l_2} = \mu_c.$$

The second isomorphism follows from the fact that $\gcd(a l_1, l_2) = 1$. The final equality results from the equalities $l = l_1 l_2$ and $a = c/l$.

An identical argument shows that the order of the isotropy group at $(0,1)$ is d. One easily checks that the χ-stable locus is $\mathbb{C}^2 \setminus \{(0,0)\}$ as desired. $\square$

The GIT stack quotient $\left[\mathbb{C}^2 /\!\!/_{\chi} (\mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^*)\right]$ may be written as a global quotient of a weighted projective space by a finite cyclic group. Recall that $l = l_1 l_2$ and $\gcd(l_1, l_2) = 1$. We will sometimes write $\left[\mathbb{C}^2 /\!\!/_{\chi} (\mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^*)\right]$ simply as $\mathbb{P}(a,b)/\mu_l$, where the action of $\mu_l = \mu_{l_1} \times \mu_{l_2}$ is understood to be as described.

Corollary 3.1.4. *Let (C, x_1, x_2) be a source curve of a two-pointed genus-zero $0+$-stable quasimap to a GIT stack quotient. Then the rational components of (C, x_1, x_2) are isomorphic to a quotient of a weighted projective $\mathbb{P}(a,b)/\mu_l$ as above.*

Proof. The result follows immediately from Lemmas 3.1.1 and 3.1.2. $\square$

We conclude our discussion of two-pointed genus-zero $(0+)$-stable quasimap source curves with a remark about line bundles over such curves.

Remark 3.1.5. Let $(C, x_1, x_2, [u])$ be a quasimap in the moduli space $Q^{0+}_{0,2}(X, \beta)$. By Lemma 3.1.1, Lemma 3.1.2, and [31, Proposition 2.2], line bundles over C restricted to an irreducible component are isomorphic to a GIT stack quotient

$$\left[\left(\mathbb{C}^2 \setminus \{\overline{0}\} \times \mathbb{C}\right) /\!\!/_{\chi} \left(\mu_{l_1} \times \mu_{l_2} \times \mathbb{C}^*\right)\right]$$

with the action given by

$$\left(e^{2\pi i\frac{m_1}{l_1}}, e^{2\pi i\frac{m_2}{l_2}}, \lambda\right) \cdot (x, y, z) = \left(\lambda^a x e^{2\pi i\frac{m_1}{l_1}}, \lambda^b y e^{2\pi i\frac{m_2}{l_2}}, \lambda^d z e^{2\pi i\frac{k_1 l_2 m_1 + k_2 l_1 m_2}{l_1 l_2}}\right),$$

for some integers k_1, k_2, and d. Denote this GIT quotient by $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$. We omit the superscripts k_1 and k_2 when l equals 1.

3.2 Weak convexity

We now define (weak) semi-positivity for a vector bundle over $\mathfrak{X}$ and prove that it implies a weak form for convexity.

Definition 3.2.1. For a character $\tau : G \to \mathbb{C}^*$ with one-dimensional representation $\mathbb{C}_\tau$, denote by $\mathcal{L} = [W \times \mathbb{C}_\tau / G]$ the corresponding line bundle over $\mathfrak{X}$. We say $\mathcal{L}$ is *positive* if $\beta(\mathcal{L}) > 0$ for all $\beta \in \mathrm{Eff}(W, G, \theta)$ and *semi-positive* if $\beta(\mathcal{L}) \geq 0$ for all $\beta \in \mathrm{Eff}(W, G, \theta)$.

A vector bundle $\mathcal{E}$ over $\mathfrak{X}$ is *positive* (*semi-positive*) if it splits as the direct sum of positive (semi-positive) line bundles.

Remark 3.2.2. Our definition of a positive (semi-positive) line bundle $\mathcal{L}$ agrees with the definition in [5, §6.2] of a positive (semi-positive) character τ.

The authors of [31] generalize the Birkhoff-Grothendieck theorem to orbifolds whose coarse space is $\mathbb{P}^1$ with only two points with nontrivial isotropy and chains of projective lines meeting at nodal singularities. These are exactly the source curves of $Q^{0+}_{0,2}(X, \beta)$ that we are considering.

Fix a quasimap $(C, x_1, x_2, [u]) \in Q^{0+}_{0,2}(X, \beta)$. By Lemma 3.1.1 and [31], vector bundles over C split as line bundles. Hence, for any vector bundle $\mathcal{E} \to \mathfrak{X}$ as in Definition 2.2.2, the pullback

$[u]^*\mathcal{E}$ splits as the direct sum of line bundles $\oplus_{i=1}^r \mathcal{L}_i$ regardless of whether $\mathcal{E}$ splits. This allows us to make the following definition:

Definition 3.2.3. The vector bundle $\mathcal{E}$ is *weakly semi-positive* if, for any $\beta \in \mathrm{Eff}(W, G, \theta)$ and $(C, x_1, x_2, [u]) \in Q^{0+}_{0,2}(X, \beta)$, the pullback $[u]^*\mathcal{E}$ splits as the direct sum of line bundles $\oplus_{i=1}^r \mathcal{L}_i$ such that $\deg(\mathcal{L}_i) \geq 0$ for all $1 \leq i \leq r$.

If X is a smooth variety, convexity follows from semi-positivity of E. This is no longer the case when X is an orbifold (See Remark 4.0.1 for more details). In this section we consider a weaker notion, which we term weak convexity.

Definition 3.2.4. A vector bundle $\mathcal{E}$ over $\mathfrak{X}$ is *weakly convex* if $H^1(C, [u]^*\mathcal{E}(-x_2))$ vanishes for all $(C, x_1, x_2, [u]) \in Q^{0+}_{0,2}(X, \beta)$ and $\beta \in \mathrm{Eff}(W, G, \theta)$.

Assume the vector bundle $\mathcal{E}$ is weakly semi-positive. For a given quasimap $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X, \beta)$, the group $H^1(C, [u]^*\mathcal{E})$ splits as

$$H^1(C, [u]^*\mathcal{E}) = \bigoplus_{i=1}^{r} H^1(C, \mathcal{L}_i).$$

The line bundles $\mathcal{L}_i$ have non-negative degree by assumption. Furthermore, each $\mathcal{L}_i$ has non-negative degree on every irreducible component of C. Let l, a, and b be as in §3.1. As noted in Remark 3.1.5, the restriction of $\mathcal{L}_i$ to an irreducible component of C is then isomorphic to $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$ for some integers k_1, k_2, and d with $d \geq 0$.

We will show that $H^1(C, \mathcal{L})$ vanishes whenever C is a two-pointed genus-zero $0+$-stable quasimap source curve and $\mathcal{L} \to C$ has non-negative degree on each irreducible component. We begin with the case that C is smooth.

For Lemmas 3.2.5 and 3.2.6, let (x, y) be the homogeneous coordinates of $\mathbb{P}(a, b)/\mu_l$ and label $x_1 = (1, 0)$ and $x_2 = (0, 1)$. Also fix a non-negative integer d.

Lemma 3.2.5. *The cohomology group* $H^1\left(\mathbb{P}(a, b)/\mu_l, \mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)\right)$ *vanishes.*

Proof. First consider the case μ_l is trivial. Then $\mathbb{P}(a,b)/\mu_l$ is a weighted projective space $\mathbb{P}(a,b)$ and $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$ is simply $\mathcal{O}_{\mathbb{P}(a,b)}(d)$. Let φ denote the rigidification map from $\mathbb{P}(a,b)$ to $\mathbb{P}^1$.

The points x_1 and x_2 are the only points of $\mathbb{P}(a,b)$ with nontrivial isotropy. The generator of the isotropy group at $L|_{x_j}$ acts by multiplication by $e^{2\pi i \frac{w_j}{ab}}$ for some weight $0 \leq w_j < ab$. Orbifold Reimann-Roch gives the following:

$$\begin{aligned}
\deg(\varphi_* \mathcal{O}_{\mathbb{P}(a,b)}(d)) &= \frac{d}{ab} - \left(\frac{w_1}{ab} + \frac{w_2}{ab}\right) \\
&> \frac{d}{ab} - 2 \\
&> -2.
\end{aligned}$$

Since the degree of $\varphi_* \mathcal{O}_{\mathbb{P}(a,b)}(d)$ is an integer no less than -1, we conclude that

$$H^1(\mathbb{P}(a,b), \mathcal{O}_{\mathbb{P}(a,b)}(d)) = H^1(\mathbb{P}^1, \varphi_* \mathcal{O}_{\mathbb{P}(a,b)}(d)) = 0. \tag{3.5}$$

Now let μ_l be an arbitrary finite cyclic group. We have the fiber diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(a,b)}(d) \arrow[r] \arrow[d] & \mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d) \arrow[d] \\
\mathbb{P}(a,b) \arrow[r] & \mathbb{P}(a,b)/\mu_l.
\end{tikzcd}$$

So, $H^1\left(\mathbb{P}(a,b)/\mu_l, \mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)\right)$ is the μ_l-invariant part of $H^1\left(\mathbb{P}(a,b), \mathcal{O}_{\mathbb{P}(a,b)}(d)\right)$. Equation (3.5) implies $H^1\left(\mathbb{P}(a,b), \mathcal{O}_{\mathbb{P}(a,b)}(d)\right)$ vanishes when $d \geq 0$. Therefore the cohomology group $H^1\left(\mathbb{P}(a,b)/\mu_l, \mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)\right)$ vanishes. □

Lemma 3.2.6. *If the isotropy at x_j acts nontrivially on the fibers of the line bundle $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$, then $H^0\left(x_j, \mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)|_{x_j}\right)$ vanishes. Otherwise, there exists a section of $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$ that is nonzero at x_j and zero at the other point.*

Proof. The first claim is immediate. For the second, assume the isotropy at x_1 acts trivially.

By (3.4), the isotropy group at x_1 is

$$G_{x_1} = \left\langle \left(e^{2\pi i \frac{1}{l_1}}, e^{2\pi i \frac{1}{l_2}}, e^{-2\pi i \frac{1}{al_1}} \right) \right\rangle.$$

The action of $\left(e^{2\pi i \frac{1}{l_1}}, e^{2\pi i \frac{1}{l_2}}, e^{-2\pi i \frac{1}{al_1}} \right)$ on $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)|_{x_1}$ is given by

$$\left(e^{2\pi i \frac{1}{l_1}}, e^{2\pi i \frac{1}{l_2}}, e^{-2\pi i \frac{1}{al_1}} \right) \cdot (1, 0, z) = \left(1, 0, z e^{2\pi i \frac{al_2k_1 + al_1k_2 - l_2d}{al_1l_2}} \right).$$

This action is trivial, therefore al_1l_2 divides $al_2k_1 + al_1k_2 - l_2d$. Thus a divides l_2d. Since l_2 and a are coprime by the assumption preceding Lemma 3.1.2, this implies a divides d. Rewrite d as the product ac for some integer c. Similar arguments show that l_2 divides k_2 and l_1 divides $(k_1 - c)$. The latter ensures k_1 is congruent to c modulo l_1.

Consider the map $f : \mathbb{C}^2 \to \mathbb{C}$ defined by $f(x, y) = x^c$. We claim f descends to a section of $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$. Note that

$$\begin{aligned} f\left(\lambda^a x e^{2\pi i \frac{m_1}{l_1}}, \lambda^b y e^{2\pi i \frac{m_2}{l_2}} \right) &= \lambda^{ac} x^c e^{2\pi i \frac{cm_1}{l_1}} \\ &= \lambda^d e^{2\pi i \frac{k_1l_2m_1 + k_2l_1m_2}{l_1l_2}} f(x, y). \end{aligned}$$

The second equality holds because k_1 is congruent to c modulo l_1, d equals ac, and l_2 divides k_2 so $\frac{k_2l_1m_2}{l_1l_2}$ is an integer for all integers $0 \leq m_2 < l_2$.

This verifies the map $(x, y) \mapsto (x, y, f(x, y))$ is G-equivariant, and therefore f descends to a section s of $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$. One sees immediately that $s(x_1) \neq 0$ and $s(x_2) = 0$.

An identical argument shows that if the action of the isotropy group at x_2 is trivial, then there exists a section $\tilde{s}$ of $\mathcal{O}^{k_1,k_2}_{\mathbb{P}(a,b)/\mu_l}(d)$ such that $\tilde{s}(x_1) = 0$ and $\tilde{s}(x_2) \neq 0$. □

Proposition 3.2.7. *Let (C, x_1, x_2) be the source curve of a genus-zero $0+$-stable quasimap to a GIT stack quotient and $\mathcal{L} \to C$ a line bundle with non-negative degree on each irreducible component of C. Then $H^1(C, \mathcal{L}(-x_2))$ vanishes.*

Proof. By Lemma 3.1.1, the underlying coarse curve $\underline{C}$ is a chain of rational components such that the marked points $\underline{x}_1$ and $\underline{x}_2$ lie on the end components. Suppose C has k irreducible components $\{C_j\}_{j=1}^k$, labeled so that x_1 lies on C_1, x_2 lies on C_k, and the curves C_j and C_{j+1} intersect at the node n_j.

We first show $H^1(C, \mathcal{L})$ vanishes. Consider the normalization sequence:

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \bigoplus_{j=1}^{k} \mathcal{O}_{C_j} \longrightarrow \bigoplus_{j=1}^{k-1} \mathcal{O}_C|_{n_j} \longrightarrow 0. \tag{3.6}$$

Tensoring by $\mathcal{L}$ and taking cohomology, we obtain

$$0 \longrightarrow H^0(C, \mathcal{L}) \longrightarrow \bigoplus_{j=1}^{k} H^0(C_j, \mathcal{L}|_{C_j}) \xrightarrow{F} \bigoplus_{j=1}^{k-1} H^0(n_j, \mathcal{L}|_{n_j}) \longrightarrow$$

$$\longrightarrow H^1(C, \mathcal{L}) \longrightarrow 0.$$

Here $\bigoplus_{j=1}^k H^1(C_j, \mathcal{L}|_{C_j})$ is zero by Corollary 3.1.4 and Lemma 3.2.5.

For sections s_j in $H^0(C_j, \mathcal{L}|_{C_j})$, the map F is defined as

$$F(s_1, \ldots, s_k) = (s_1(n_1) - s_2(n_1), s_2(n_2) - s_3(n_2), \ldots, s_{k-1}(n_{k-1}) - s_k(n_{k-1}))\,.$$

If the isotropy group G_{n_j} acts nontrivially on the fiber $\mathcal{L}|_{n_j}$, then $H^0(n_j, \mathcal{L}|_{n_j})$ is zero. Otherwise, Lemma 3.2.6 implies there exists a section $\tilde{s}_j$ in $H^0(C_j, \mathcal{L}|_{C_j})$ such that $\tilde{s}_j(n_j) \neq 0$ and, if $j > 1$, $\tilde{s}_j(n_{j-1}) = 0$. Thus

$$F\left(0, \ldots, \frac{1}{\tilde{s}_j(n_j)}\tilde{s}_j, \ldots, 0\right) = (0, \ldots, 1, \ldots, 0),$$

where the vectors above have a zero in every entry except the ith one. This shows F is surjective. As a result, $H^1(C, \mathcal{L})$ vanishes.

Tensoring the divisor exact sequence of C for the marked point x_2 by $\mathcal{L}$ and pushing forward gives the long exact sequence:

$$0 \longrightarrow H^0(C, \mathcal{L}(-x_2)) \longrightarrow H^0(C, \mathcal{L}) \longrightarrow H^0(x_2, \mathcal{L}|_{x_2}) \longrightarrow$$

$$\longrightarrow H^1(C, \mathcal{L}(-x_2)) \longrightarrow 0.$$

If G_{x_2} acts nontrivially on the fiber of $\mathcal{L}$ at x_2, then $H^0(x_2, \mathcal{L}|_{x_2})$ vanishes. So, the cohomology group $H^1(C, \mathcal{L}(-x_2))$ also vanishes.

If G_{x_2} acts trivially on the fiber of $\mathcal{L}$ at x_2, we must show that the map $H^0(C, \mathcal{L}) \to H^0(x_2, \mathcal{L}|_{x_2})$ is surjective. By Lemma 3.2.6, there exists a section s of $\mathcal{L}|_{C_k} \to C_k$ such that $s(n_{k-1})$ is zero and $s(x_2)$ is nonzero. The section s can be extended by the zero section on the other components to get a section in $H^0(C, \mathcal{L})$ that does not vanish at x_2. Therefore the map $H^0(C, \mathcal{L}) \to H^0(x_2, \mathcal{L}|_{x_2})$ is surjective, so $H^1(C, \mathcal{L}(-x_2))$ is zero. □

Theorem 3.2.8. *If a vector bundle $\mathcal{E} \to \mathfrak{X}$ is weakly semi-positive, then it is weakly convex.*

Proof. Consider a $0+$-stable quasimap $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X, \beta)$. Since $[u]^*\mathcal{E}$ is a direct sum of line bundles, we may write

$$H^1(C, [u]^*\mathcal{E}(-x_2)) = \bigoplus_{i=1}^{r} H^1(C, \mathcal{L}_i(-x_2)).$$

Each $\mathcal{L}_i$ satisfies the hypothesis of Proposition 3.2.7. Thus $H^1(C, [u]^*\mathcal{E}(-x_2))$ vanishes. □

3.3 Weak concavity

In this section we define weak concavity. We then prove the relative log canonical bundle over the universal curve $\pi : \mathcal{C} \to Q^{0+}_{0,2}(X, \beta)$ is trivial and use this result to prove a vector bundle $\mathcal{E} \to \mathfrak{X}$ is weakly convex if and only if its dual $\mathcal{E}^\vee \to \mathfrak{X}$ is weakly concave.

Definition 3.3.1. A vector bundle $\mathcal{E}^\vee$ over $\mathfrak{X}$ is *weakly concave* if $H^0(C, [u]^*\mathcal{E}^\vee(-x_1))$ vanishes for all $0+$-stable quasimaps $(C, x_1, x_2, [u]) \in Q^{0+}_{0,2}(X, \beta)$ and $\beta \in \mathrm{Eff}(W, G, \theta)$.

Lemma 3.3.2. *The log canonical bundle over a source curve (C, x_1, x_2) of a $0+$-stable quasimap $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X, \beta)$ is trivial.*

Proof. If C is smooth, the canonical bundle of $\mathbb{P}(a, b)/\mu_l$ is isomorphic to $\mathcal{O}(-[0] - [\infty])$. The result is then immediate. When the source curve is nodal, we again use Corollary 3.1.4. In this case we will show that $\omega_C(x_1 + x_2)$ has a nowhere vanishing section, which gives an isomorphism $\omega_C(x_1 + x_2) \cong \mathcal{O}_C$.

Let C have k rational components and $k - 1$ nodes, labeled as $\{C_i\}_{i=1}^k$ and $\{n_i\}_{i=1}^{k-1}$ as in the proof of Proposition 3.2.7. For convenience, relabel x_1 by n_0 and x_2 by n_k. Tensoring the normalization sequence (3.6) by the log canonical bundle gives rise to the following long exact sequence:

$$0 \to H^0(C, \omega_C(x_1 + x_2)) \xrightarrow{G} \bigoplus_{i=1}^{k} H^0(C_i, \omega_{C_i}(n_{i-1} + n_i)) \xrightarrow{F} \bigoplus_{i=1}^{k-1} H^0(n_i, \omega_C(x_1 + x_2)|_{n_i}) \to \cdots .$$

Note that the map F is a sum of residue maps. There exists nowhere vanishing sections s_i of each $\omega_{C_i}(n_{i-1} + n_i)$ that have residue 1 near n_{i-1} and -1 near n_i. The tuple

$$(s_1, \dots, s_k) \in \bigoplus_{i=1}^{k} H^0(C_i, \omega_{C_i}(n_{i-1} + n_i))$$

lies in kernel of F, which is isomorphic to $H^0(C, \omega_C(x_1 + x_2))$. Hence, the sections $s_1, \dots, s_k$ glue together to give a nowhere vanishing global section of $\omega_C(x_1 + x_2)$. Thus the log canonical bundle is trivial. $\square$

Proposition 3.3.3. *The relative log canonical bundle over the universal curve $\pi : \mathcal{C} \to Q^{0+}_{0,2}(X, \beta)$ is trivial.*

Proof. Tensoring the divisor exact sequence of $\mathcal{C}$ for the divisor corresponding to the first marked point by $\omega_\pi(x_2)$ and pushing forward, we obtain the long exact sequence

$$0 \longrightarrow \mathbb{R}^0\pi_*\omega_\pi(x_2) \longrightarrow \mathbb{R}^0\pi_*\omega_\pi(x_1+x_2) \longrightarrow \mathbb{R}^0\pi_*\left(\omega_\pi(x_1+x_2)|_{x_1}\right) \longrightarrow$$

$$\longrightarrow \mathbb{R}^1\pi_*\omega_\pi(x_2) \longrightarrow \mathbb{R}^1\pi_*\omega_\pi(x_1+x_2) \longrightarrow 0.$$

Lemma 3.3.2 and Serre duality imply that for every $0+$-stable quasimap $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X,\beta)$, the cohomology groups $H^0(C, \omega_C(x_2))$ and $H^1(C, \omega_C(x_2))$ vanish. Therefore the sheaves $\mathbb{R}^0\pi_*\omega_\pi(x_2)$ and $\mathbb{R}^1\pi_*\omega_\pi(x_2)$ are both zero. Hence, the map

$$\mathbb{R}^0\pi_*\omega_\pi(x_1+x_2) \longrightarrow \mathbb{R}^0\pi_*\left(\omega_\pi(x_1+x_2)|_{x_1}\right)$$

is an isomorphism.

The sheaf $\mathbb{R}^0\pi_*(\omega_\pi(x_1+x_2)|_{x_1})$ is canonically trivialized by the residue map. Therefore $\mathbb{R}^0\pi_*\omega_\pi(x_1+x_2)$ is a trivial line bundle.

The constant function 1 in $\Gamma(Q^{0+}_{0,2}(X,\beta), \mathbb{R}^0\pi_*(\omega_\pi(x_1+x_2)|_{x_1}))$ is the image of a section s in $\Gamma(Q^{0+}_{0,2}(X,\beta), \mathbb{R}^0\pi_*\omega_\pi(x_1+x_2))$. The section s corresponds to a nonzero section $\tilde{s}$ of $\omega_\pi(x_1+x_2)$. The restriction of $\omega_\pi(x_1+x_2)$ to a fiber (C, x_1, x_2) of the universal curve $\mathcal{C}$ is canonically trivial by Lemma 3.3.2. The section $\tilde{s}|_C$ is nonzero and hence nowhere vanishing on each fiber (C, x_1, x_2) in $\mathcal{C}$. Thus $\tilde{s}$ is nowhere vanishing.

Hence, $\omega_\pi(x_1+x_2)$ has a nowhere vanishing section and so it is trivial. $\square$

Theorem 3.3.4. *The vector bundle $\mathcal{E}$ is weakly convex if and only if $\mathcal{E}^\vee$ is weakly concave.*

Proof. By Lemma 3.3.2, the log canonical bundle $\omega_C(x_1+x_2)$ is canonically trivial for all points $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X,\beta)$. Hence, the canonical bundle ω_C is isomorphic to $\mathcal{O}_C(-x_1-x_2)$. We now have the following:

$$
\begin{aligned}
H^0(C,[u]^*\mathcal{E}^\vee(-x_1)) &= H^1(C,[u]^*\mathcal{E}(x_1)\otimes\omega_C)^\vee \\
&= H^1(C,[u]^*\mathcal{E}(-x_2))^\vee.
\end{aligned}
$$

The first equality is Serre duality and the second is due to the isomorphism $\omega_C \cong \mathcal{O}_C(-x_1-x_2)$. If the left hand side vanishes for all $0+$-stable two-pointed quasimaps then so does the right, and vice versa. This completes the proof. $\square$

Chapter 4

Quantum Lefschetz

The quantum Lefschetz hyperplane theorem compares the genus-zero Gromov-Witten theory of a space X with that of a complete intersection $Z \subset X$ defined by a section of a vector bundle $E \to X$ [8, 32–36]. The same proof applies to quasimaps under certain conditions on E [5].

In this section we use a modification of the quantum Lefschetz theorem for two-pointed genus-zero quasimaps to relate the generating function $L^{Z,0+}(z)$ to a twisted version of $L^{X,0+}(z)$.

Remark 4.0.1 (Quantum Lefschetz for orbifolds)**.** Let Z be a closed subset of X cut out by a section of a vector bundle $E \to X$. In the case when X is a smooth variety, the quantum Lefschetz theorem holds as long as E is the direct sum of semi-positive line bundles. However this can fail when X is an orbifold. For example, the line bundle $\mathcal{O}_{\mathbb{P}(1,1,2,2)}(1) \to \mathbb{P}(1,1,2,2)$ is positive, but quantum Lefschetz fails for the hypersurface $\mathbb{P}(1,2,2)$ defined by a section of $\mathcal{O}_{\mathbb{P}(1,1,2,2)}(1)$ [20].

As observed in [20], the general setting in which one should expect a quantum Lefschetz statement is not when E is semi-positive, but rather when E is *convex*, that is, when for every stable map $f : C \to X$ from a genus-zero curve C, the cohomology group $H^1(C, f^*(E))$ is zero. If X is a variety, then if $E \to X$ is semi-positive it is also convex. This no longer holds when X is an orbifold.

For two-pointed genus-zero quasimaps, we will see that in fact weak convexity is sufficient for a quantum Lefschetz statement.

4.1 Weak convexity and twisted invariants

Let X, $\mathfrak{X}$, and $\mathcal{E}$ be as in §2.2.1. For the remainder of the paper we assume that X is proper and that $\mathcal{E}$ is weakly convex.

Lemma 4.1.1. *The cohomology group $H^1(C, [u]^*\mathcal{E})$ vanishes for all points $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X, \beta)$ and classes $\beta \in \mathrm{Eff}(W, G, \theta)$.*

Proof. Fix a point $(C, x_1, x_2, [u])$ in $Q^{0+}_{0,2}(X,\beta)$. Consider the divisor exact sequence

$$0 \longrightarrow \mathcal{O}_C(-x_2) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C|_{x_2} \longrightarrow 0.$$

Tensoring this by $[u]^*\mathcal{E}$ and pushing forward gives rise to a long exact sequence in cohomology

$$0 \longrightarrow H^0(C, [u]^*\mathcal{E}(-x_2)) \longrightarrow H^0(C, [u]^*\mathcal{E}) \longrightarrow H^0(x_2, [u]^*\mathcal{E}|_{x_2}) \longrightarrow$$

$$\longrightarrow H^1(C, [u]^*\mathcal{E}(-x_2)) \longrightarrow H^1(C, [u]^*\mathcal{E}) \longrightarrow 0.$$

By Theorem 3.2.8, $H^1(C, [u]^*\mathcal{E}(-x_2))$ vanishes. Therefore $H^1(C, [u]^*\mathcal{E})$ also vanishes. $\square$

Denote by π the projection from the universal curve $\mathcal{C}$ to $Q^{0+}_{0,2}(X,\beta)$. Let $[u]$ be the universal map from $\mathcal{C}$ to $\mathfrak{X}$. By Lemma 4.1.1, $\mathbb{R}^0\pi_*[u]^*\mathcal{E}$ is a vector bundle on $Q^{0+}_{0,2}(X,\beta)$. Let 0_X denote the zero section of $\mathbb{R}^0\pi_*[u]^*\mathcal{E}$ and $\tilde{s} \in \Gamma(Q^{0+}_{0,2}(X,\beta), \mathbb{R}^0\pi_*[u]^*\mathcal{E})$ denote the section induced by s.

Theorem 4.1.2. *There is a fiber square*

$$\begin{tikzcd}
Q^{0+}_{0,2}(Z,\beta) \arrow[r, hook, "\tilde{j}"] \arrow[d, "\tilde{j}"] & Q^{0+}_{0,2}(X,\beta) \arrow[d, "\tilde{s}"] \\
Q^{0+}_{0,2}(X,\beta) \arrow[r, hook, "0_X"'] & \mathbb{R}^0\pi_*[u]^*\mathcal{E}.
\end{tikzcd}$$

The virtual classes are related by

$$0_X^! \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}} = \left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}.$$

Proof. The proof of the theorem in [37] for Gromov-Witten theory extends to quasimaps. $\square$

Theorem 4.1.2 implies the more familiar quantum Lefschetz statement from [5].

Proposition 4.1.3. [5] *The following classes are equal in the Chow group of $Q^{0+}_{0,2}(X,\beta)$:*

$$\tilde{j}_* \left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}} = e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}.$$

The definitions and arguments that follow are similar to those appearing in [38, §2.1] to define a twisted quantum product.

Definition 4.1.4. In the Chow group of $Q^{0+}_{0,2}(X,\beta)$, define the twisted virtual class

$$\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2} := e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}. \tag{4.1}$$

Define the operator $L^{X/Z,0+}(z)$ by

$$L^{X/Z,0+}(z)(\alpha) := \alpha + \sum_{\beta \in \mathrm{Eff}} q^\beta \tilde{ev}_{2*}\left(\frac{ev_1^*(\alpha)}{-z-\psi_1} \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2}\right)$$

for $\alpha \in H^*_{\mathrm{CR}}(X)$.

The subscript 2 following X/Z in (4.1) indicates the twist down by the second marked point. Now, we compare $L^{X/Z,0+}(z)$ and $L^{Z,0+}(z)$.

Proposition 4.1.5. *The operators $L^{X/Z,0+}(z)$ and $L^{Z,0+}(z)$ are related as follows*

$$j^* \circ L^{X/Z,0+}(z) = L^{Z,0+}(z) \circ j^*.$$

The analogous statement in Gromov-Witten theory appears in [30, Proposition 2.4]. The proof in this case is very similar.

Proof. To avoid confusion, we use superscripts to distinguish between the evaluation maps $ev_i^X : Q^{0+}_{0,2}(X,\beta) \to \bar{I}X$ and $ev_i^Z : Q^{0+}_{0,2}(Z,\beta) \to \bar{I}Z$. For α in $H^*(\bar{I}X)$, we can express $j^* \circ L^{X/Z,0+}(z)(\alpha)$ as

$$j^*\alpha + \sum_{\beta \in \mathrm{Eff}} q^\beta j^* \tilde{ev}^X_{2*}\left(\frac{ev_1^{X*}(\alpha)}{-z-\psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}\right) \tag{4.2}$$

and $L^{Z,0+}(z) \circ j^*(\alpha)$ as

$$j^*(\alpha) + \sum_{\beta \in \mathrm{Eff}} q^\beta \tilde{ev}^Z_{2*} \left(\frac{ev_1^{Z*} j^*(\alpha)}{-z-\psi_1} \cap \left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}} \right). \tag{4.3}$$

To prove the claim we compare (4.2) and (4.3) term by term.

Define Q to be the zero locus of the section $ev_2^{X*}s \in \Gamma(Q^{0+}_{0,2}(X,\beta), ev_2^{X*}E)$ and denote the evaluation maps of Q by $ev_1^Q : Q \to \bar{I}X$ and $ev_2^Q : Q \to \bar{I}Z$. Label the following projections $\pi_1^X, \pi_2^X : \bar{I}X \times \bar{I}X \to \bar{I}X$, $\pi_1 : \bar{I}X \times \bar{I}Z \to \bar{I}X$, and $\pi_2 : \bar{I}X \times \bar{I}Z \to \bar{I}Z$. Consider the following diagram, where the rectangles are fiber squares

$$\begin{tikzcd}
Q^{0+}_{0,2}(Z,\beta) \arrow[r, "j''"] \arrow[dr, "\underline{ev}^Z"'] & Q \arrow[r, "j'"] \arrow[d, "\underline{ev}^Q"] & Q^{0+}_{0,2}(X,\beta) \arrow[d, "\underline{ev}^X"] \\
& \bar{I}X \times \bar{I}Z \arrow[r, "\mathrm{id}\times j"] \arrow[d, "\pi_2"] & \bar{I}X \times \bar{I}X \arrow[d, "\pi_2^X"] \\
& \bar{I}Z \arrow[r, "j"] & \bar{I}X,
\end{tikzcd} \tag{4.4}$$

where $\underline{ev}^X = (ev_1^X, ev_2^X)$, $\underline{ev}^Q = (ev_1^Q, ev_2^Q)$, and $\underline{ev}^Z = (j \circ ev_1^Z, ev_2^Z)$.

Using the diagram above, we rewrite each term in the sum of (4.2):

$$\begin{aligned}
&j^* \tilde{ev}^X_{2*} \left(\frac{ev_1^{X*}(\alpha)}{-z-\psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}} \right) \\
=&\tilde{ev}^Q_{2*} j^! \left(\frac{ev_1^{X*}(\alpha)}{-z-\psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}} \right) \\
=&\iota_*\pi_{2*}\underline{ev}^Q_*(\mathrm{id}\times j)^! \left(\frac{\underline{ev}^{X*}\pi_1^{X*}(\alpha)}{-z-\psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}} \right) \\
=&\iota_*\pi_{2*}(\mathrm{id}\times j)^*\underline{ev}^X_* \left(\frac{\underline{ev}^{X*}\pi_1^{X*}(\alpha)}{-z-\psi_1} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}} \right) \\
=&\iota_*\pi_{2*} \left((\mathrm{id}\times j)^*\pi_1^{X*}(\alpha) \cup (\mathrm{id}\times j)^*\underline{ev}^X_* \left(\frac{e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}}{-z-\psi_1} \right) \right) \\
=&\iota_*\pi_{2*} \left(\pi_1^*(\alpha) \cup \underline{ev}^Q_* j^! \left(\frac{e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}}{-z-\psi_1} \right) \right).
\end{aligned} \tag{4.5}$$

Equalities one, three, and five follow from [39, Theorem 6.2(a)] and (4.4). Equality two is by [39, Theorem 6.2(c)] and (4.4). Equality four is the projection formula.

Let 0_X denote the zero section of $\mathbb{R}^0\pi_*[u]^*\mathcal{E}$ over $Q^{0+}_{0,2}(X,\beta)$. We rewrite each term in (4.3):

$$\begin{aligned}
&\tilde{ev}^Z_{2*}\left(\frac{ev_1^{Z*}j^*(\alpha)}{-z-\psi_1}\cap\left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}\right)\\
=&\iota_*\pi_{2*}\underline{ev}^Z_*\left(\frac{\underline{ev}^{Z*}\pi_1^*(\alpha)}{-z-\psi_1}\cap\left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}\right)\\
=&\iota_*\pi_{2*}\left(\pi_1^*(\alpha)\cup\underline{ev}^Z_*\left(\frac{\left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}}{-z-\psi_1}\right)\right)\\
=&\iota_*\pi_{2*}\left(\pi_1^*(\alpha)\cup\underline{ev}^Q_*j''_*0^!_X\left(\frac{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}}{-z-\psi_1}\right)\right).
\end{aligned}\tag{4.6}$$

The third equality follows by factoring $\underline{ev}^Z$ as $\underline{ev}^Q\circ j''$, Theorem 4.1.2, and [39, Proposition 6.3].

Compare (4.5) and (4.6). We may factor ψ-classes out of Gysin maps by [39, Proposition 6.3]. Hence, to complete the proof it suffices to show the following equality in the Chow group of Q:

$$j''_*0^!_X\left(\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}\right)=j^!\left(e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2))\cap\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}\right).\tag{4.7}$$

Consider the fiber diagram (also constructed in [30, §2]):

$$\begin{tikzcd}
Q^{0+}_{0,2}(Z,\beta) \arrow[r, "j''"] \arrow[d, "j''"] & Q \arrow[r] \arrow[d, "\tilde{s}_Q"] & Q^{0+}_{0,2}(X,\beta) \arrow[dd, "\tilde{s}"] \\
Q \arrow[r, "0_Q"] \arrow[d] & \mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)|_Q \arrow[d, "f"] & \\
Q^{0+}_{0,2}(X,\beta) \arrow[r, "0'_X"] & \mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2) \arrow[r, "h"] & \mathbb{R}^0\pi_*[u]^*\mathcal{E}.
\end{tikzcd}\tag{4.8}$$

The morphisms 0_Q and $0'_X$ are the zero sections, f and h are the natural inclusions, and $\tilde{s}_Q$ is the section induced by s. We then have the following:

$$
\begin{aligned}
j''_* 0^!_X \left(\left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right) &= j''_* 0'^!_X h^! \left(\left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right) \\
&= 0^*_Q \tilde{s}_{Q*} h^! \left(\left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right) \\
&= h^! \left(e(\mathbb{R}^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right).
\end{aligned}
$$

The first equality is [39, Theorem 6.5]. The second is [39, Theorem 6.2(a)].

To verify (4.7) it now suffices to show that $h^!$ equals $j^!$. This can be seen from the following commutative diagram, where the front, back, top, and bottom faces are fiber squares:

$$
\begin{tikzcd}
 & Q \arrow[rr, "j'"] \arrow[dl, "f\circ\tilde{s}_Q"'] \arrow[dd] & & Q^{0+}_{0,2}(X,\beta) \arrow[dl, "\tilde{s}"] \arrow[dd, "ev_2^X"] \\
\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2) \arrow[rr, "h"] \arrow[dd] & & \mathbb{R}^0\pi_*[u]^*\mathcal{E} \arrow[dd, "\tilde{s}|_{x_2}"] & \\
 & \bar{I}Z \arrow[rr, "j"] \arrow[dl, "j"] & & \bar{I}X \arrow[dl, "s"] \\
\bar{I}X \arrow[rr, "0_E"] & & \bar{I}E. &
\end{tikzcd}
$$

The morphism $\tilde{s}|_{x_2}$ is defined by the composition

$$
\mathbb{R}^0\pi_*[u]^*\mathcal{E} \longrightarrow ev_2^{X*}E \longrightarrow \bar{I}E.
$$

Applying [39, Theorem 6.2(c)] twice completes the proof,

$$
\begin{aligned}
& h^! \left(e(\mathbb{R}^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right) \\
=& 0^!_E \left(e(\mathbb{R}^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right) \\
=& j^! \left(e(\mathbb{R}^0 \pi_* [u]^* \mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,2}(X,\beta) \right]^{\mathrm{vir}} \right).
\end{aligned}
$$

□

Chapter 5

The total space

In this section we consider quasimaps to $E^\vee$. Similar to the previous section, we will compute the quasimap invariants of $E^\vee$ in terms of integrals over the moduli space of $0+$-stable quasimaps to X.

Definition 5.0.1. Let $\mathcal{E}^\vee \to \mathfrak{X}$ be a vector bundle. Define the moduli space of k-pointed, genus-g, $0+$-stable quasimaps to $E^\vee$ of degree β as

$$Q^{0+}_{g,k}(E^\vee, \beta) := \operatorname{tot}(\pi_*[u]^*\mathcal{E}^\vee) := \operatorname{Spec}\operatorname{Sym}\mathbb{R}^1\pi_*\left([u]^*\mathcal{E}\otimes\omega_\pi\right).$$

There is a natural perfect obstruction theory on $Q^{0+}_{g,k}(E^\vee, \beta)$ obtained by pulling back the obstruction theory on $Q^{0+}_{g,k}(X, \beta)$ and taking the direct sum with $(\mathbb{R}^\bullet\pi_*[u]^*\mathcal{E}^\vee)^\vee$. This yields a virtual fundamental class $[Q^{0+}_{g,k}(E^\vee, \beta)]^{\mathrm{vir}}$. When the evaluation maps

$$ev_i : Q^{0+}_{g,k}(E^\vee, \beta) \to \bar{I}E^\vee$$

are proper, we can define quasimap invariants as in Definition 2.4.2.

As in the previous section we will assume that $\mathcal{E}$ is weakly convex. Then recall by Theorem 3.3.4 that $\mathcal{E}^\vee \to \mathfrak{X}$ is weakly concave.

On the universal curve over $Q^{0+}_{0,2}(X, \beta)$, consider the divisor given by the first marked point. Tensor the divisor exact sequence by $[u]^*\mathcal{E}^\vee$ and apply $\mathbb{R}\pi_*(-)$ to get the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{R}^0\pi_*[u]^*\mathcal{E}^\vee \longrightarrow \mathbb{R}^0\pi_*[u]^*\mathcal{E}^\vee|_{x_1} \longrightarrow \\ \longrightarrow \mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1) \longrightarrow \mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee \longrightarrow 0. \end{aligned} \tag{5.1}$$

Fix an open and closed subspace $ev_1^{-1}(X_g^\vee) \subset Q_{0,2}^{0+}(X, \beta)$. Let G_{x_1} denote the isotropy group at x_1 for a point $(C, x_1, x_2, [u])$ in this subspace. Recall that $[u]^*\mathcal{E}$ splits as a direct sum of line bundles $\oplus_{i=1}^r \mathcal{L}_i$. The line bundle $\mathbb{R}^0\pi_*\mathcal{L}_i^\vee|_{x_1}$ is nonzero if and only if G_{x_1} acts trivially on the fiber of $\mathcal{L}_i^\vee$ at x_1. We conclude that on a given open and closed subset $ev_1^{-1}(X_g^\vee) \subset Q_{0,2}^{0+}(X, \beta)$,

$$\mathbb{R}^0\pi_*[u]^*\mathcal{E}^\vee|_{x_1} = ev_1^* E_g^\vee. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain the following equality in K-theory:

$$ev_1^* E_g^\vee \ominus \mathbb{R}\pi_*[u]^*\mathcal{E}^\vee = \mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1). \tag{5.3}$$

Proposition 5.0.2. *If $\mathcal{E}^\vee$ is weakly concave, then the evaluation maps*

$$ev_i : Q_{0,2}^{0+}(E^\vee, \beta) \to \bar{I}E^\vee$$

are proper.

Proof. For notational simplicity we consider ev_1. By (5.1) and (5.2), there is a closed immersion

$$Q_{g,k}^{0+}(E^\vee, \beta) = \operatorname{tot}(\pi_*[u]^*\mathcal{E}^\vee) \hookrightarrow \operatorname{tot}(\pi_*([u]^*\mathcal{E}^\vee|_{x_1})) = (ev_1^X)^*\bar{I}E^\vee,$$

where $(ev_1^X)^*\bar{I}E^\vee$ denotes the fiber product of $ev_1^X : Q_{g,k}^{0+}(X, \beta) \to \bar{I}X$ and $\bar{I}E^\vee \to \bar{I}X$. The evaluation map $ev_1^{E^\vee}$ factors as

$$Q_{g,k}^{0+}(E^\vee, \beta) \hookrightarrow (ev_1^X)^*\bar{I}E^\vee \to \bar{I}E^\vee.$$

This composition is proper because ev_1^X is. $\square$

By weak concavity, the sheaf $\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)$ is a vector bundle on $Q_{0,2}^{0+}(X, \beta)$. This allows the following definition.

Definition 5.0.3. Define the $\mathcal{E}^\vee$-twisted virtual class to be

$$\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1} := e(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}. \tag{5.4}$$

Define the operator $L^{X/E^\vee,0+}(z)$ by

$$L^{X/E^\vee,0+}(z)(\alpha) := \alpha + \sum_{\beta\in\mathrm{Eff}} q^\beta \tilde{ev}_{2*}\left(\frac{ev_1^*(\alpha)}{-z-\psi} \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}\right),$$

for $\alpha \in H^*_{\mathrm{CR}}(X)$.

In (5.4), the subscript 1 following $X/E^\vee$ indicates the twist down by the first marked point.

We will make use of the commutative diagram

$$\begin{tikzcd} Q^{0+}_{0,2}(X,\beta) \arrow[r, "\tilde{i}"] \arrow[d, "ev_i^X"] & Q^{0+}_{0,2}(E^\vee,\beta) \arrow[d, "ev_i^{E^\vee}"] \\ \bar{I}X \arrow[r, hook, "i"] & \bar{I}E^\vee, \end{tikzcd} \tag{5.5}$$

where the ith evaluation maps on $Q^{0+}_{0,2}(X,\beta)$ and $Q^{0+}_{0,2}(E^\vee,\beta)$ are denoted by ev_i^X and $ev_i^{E^\vee}$ respectively.

Proposition 5.0.4. *Given $\gamma_1,\gamma_2 \in H^*_{CR,\mathrm{ct}}(E^\vee)$, choose $\alpha_1,\alpha_2 \in H^*_{CR}(X)$ such that $\gamma_i = i_*(\alpha_i)$. We have the following equality:*

$$\langle \gamma_1\psi_1^{a_1}, \gamma_2\psi_2^{a_2}\rangle^{E^\vee,0+}_{0,\beta} = \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e(E^\vee_{g_2}))\psi_2^{a_2}. \tag{5.6}$$

Proof. The claim is most easily seen via virtual localization. Consider the $\mathbb{C}^*$-action on $\mathcal{E}^\vee$ given by scaling fibers. This induces actions on $\bar{I}E^\vee$ and $Q^{0+}_{0,2}(E^\vee,\beta)$, with fixed loci $\bar{I}X$ and $Q^{0+}_{0,2}(X,\beta)$ respectively. All the maps in Diagram (5.5) are $\mathbb{C}^*$-equivariant. By (1) of [40], we have the equality

$$\left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}} = \tilde{i}_*\left(\frac{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*\mathcal{E}^\vee)}\right), \tag{5.7}$$

where $[-]^{\mathbb{C}^*,\mathrm{vir}}$ denotes the $\mathbb{C}^*$-equivariant virtual fundamental class.

Choose equivariant lifts of γ_1, γ_2, α_1, and α_2 (by abuse of notation we will not change their labels). Using the fact that γ_2 is the pushforward of a class α_2 supported on IX, a localization argument shows that the left hand side of (5.6) is equal to the non-equivariant limit of

$$\int_{\left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}} ev_1^{E^\vee *}(\gamma_1)\psi_1^{a_1} \cup ev_2^{E^\vee *}(\gamma_2)\psi_2^{a_2}.$$

This can be rewritten as

$$\begin{aligned}
&\int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}} \frac{ev_1^{X*}(\alpha_1 \cup e_{\mathbb{C}^*}(E_g^\vee))\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e_{\mathbb{C}^*}(E_g^\vee))\psi_2^{a_2}}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*\mathcal{E}^\vee)} \\
&= \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e_{\mathbb{C}^*}(E_g^\vee))\psi_2^{a_2} \cup \frac{e_{\mathbb{C}^*}(ev_1^{X*}E_g^\vee)}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*\mathcal{E}^\vee)} \\
&= \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e_{\mathbb{C}^*}(E_g^\vee))\psi_2^{a_2} \cup e_{\mathbb{C}^*}(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)).
\end{aligned}$$

The first line is a result of (5.7), the projection formula, and (5.5). The second is immediate. The third line follows from (5.3). Taking the non-equivariant limit yields the desired equality. □

A similar argument yields the following.

Proposition 5.0.5. *The operators $L^{X/E^\vee,0+}(z)$ and $L^{E^\vee,0+}(z)$ are related by pushforward along i,*

$$i_* \circ L^{X/E^\vee,0+}(z) = L^{E^\vee,0+}(z) \circ i_*. \tag{5.8}$$

Proof. For $\alpha \in H^*_{\mathrm{CR}}(X)$, we can rewrite the left-hand-side of (5.8) applied to α as

$$i_* \circ L^{X/E^\vee,0+}(z)(\alpha) = i_*(\alpha) + \sum_{\beta\in\mathrm{Eff}} q^\beta i_* \tilde{ev}_{2*}^X \left(\frac{ev_1^*(\alpha)}{-z-\psi} \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}\right) \tag{5.9}$$

and the right-hand-side as

$$L^{E^\vee,0+}(z) \circ i_*(\alpha) = i_*(\alpha) + \sum_{\beta\in\mathrm{Eff}} q^\beta \tilde{ev}_{2*}^{E^\vee} \left(\frac{ev_1^* i_*(\alpha)}{-z-\psi} \cap \left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathrm{vir}}\right). \tag{5.10}$$

We equate (5.9) and (5.10) term by term. Again we apply virtual localization. Fix a degree β and choose a class α in $H^*_{\mathrm{CR},\mathbb{C}^*}(X)$.

Consider the following equalities in the localized equivariant cohomology ring of $\bar{I}E^\vee$:

$$\begin{aligned}
&\tilde{ev}^{E^\vee}_{2*}\left(\frac{ev_1^{E^\vee *} i_*(\alpha)}{-z-\psi} \cap \left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}\right)\\
=&\tilde{ev}^{E^\vee}_{2*}\left(\frac{ev_1^{E^\vee *} i_*(\alpha)}{-z-\psi} \cap \tilde{i}_*\left(\frac{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*\mathcal{E}^\vee)}\right)\right)\\
=&i_*\tilde{ev}^{X}_{2*}\left(\frac{ev_1^{X*}(\alpha \cup e_{\mathbb{C}^*}(E^\vee_g))}{-z-\psi} \cap \frac{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*\mathcal{E}^\vee)}\right)\\
=&i_*\tilde{ev}^{X}_{2*}\left(\frac{ev_1^{X*}(\alpha)}{-z-\psi} \cup \frac{e_{\mathbb{C}^*}(ev_1^{X*}E^\vee_g)}{e_{\mathbb{C}^*}(\mathbb{R}\pi_*[u]^*\mathcal{E}^\vee)} \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}\right)\\
=&i_*\tilde{ev}^{X}_{2*}\left(\frac{ev_1^{X*}(\alpha)}{-z-\psi} \cup e_{\mathbb{C}^*}(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)) \cap \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathbb{C}^*,\mathrm{vir}}\right).
\end{aligned}$$

The first equality is (5.7). The second line is obtained by two applications of the projection formula. The third line is immediate. The forth follows from (5.3).

Taking the non-equivariant limit of the first and last terms in the chain of equalities and recalling Definition 5.0.3 completes the proof. $\square$

Chapter 6

Quasimap quantum Serre duality

In this section we prove a quantum Serre duality statement in three contexts. Specifically, we compare the twisted virtual classes

$$\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2} \qquad \text{and} \qquad \left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}$$

defined in Sections 4 and 5 to get a cycle-valued statement. From this we compute a direct relationship between the two-pointed genus-zero quasimap invariants of Z and $E^\vee$. We then rephrase this as a comparison between the operators $L^{Z,0+}(z)$ and $L^{E^\vee,0+}(z)$.

6.1 Cycle-valued statement

Let $\mathcal{E} \to \mathfrak{X}$ be a weakly convex vector bundle. For elements g_1 and g_2 in S, denote by $Q^{0+}_{0,g_1,g_2}(X,\beta)$ the open and closed subset of $Q^{0+}_{0,2}(X,\beta)$ defined by the conditions $\mathrm{im}(ev_1) \subset X_{g_1}$ and $\mathrm{im}(ev_2) \subset X_{g_2}$.

Lemma 6.1.1. *On $Q^{0+}_{0,g_1,g_2}(X,\beta)$ the vector bundle $\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)$ has rank*

$$\beta(\det \mathcal{E}) - \mathrm{age}_{g_1}(\mathcal{E}) + \mathrm{age}_{g_2}(\mathcal{E}^\vee),$$

where $\det \mathcal{E} = \wedge^r_{i=1}\mathcal{E}$ is the determinant line bundle.

Proof. Fix a quasimap $(C, x_1, x_2, [u]) \in Q^{0+}_{0,g_1,g_2}(X,\beta)$ and let r_j be the order of the isotropy group at x_j for $j = 1, 2$. Recall from §3.2 that there exists line bundles $\{\mathcal{L}_i \to C\}^r_{i=1}$ such that $[u]^*\mathcal{E} = \oplus^r_{i=1}\mathcal{L}_i$. Write $\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)$ as the direct sum $\oplus^r_{i=1}\mathbb{R}^1\pi_*\mathcal{L}^\vee_i(-x_1)$. Using orbifold

Riemann-Roch we compute:

$$\begin{aligned}
\operatorname{rank}\left(\mathbb{R}^1\pi_*\mathcal{L}_i^\vee(-x_1)\right) &= h^1\left(C, \mathcal{L}_i^\vee(-x_1)\right) \\
&= h^0\left(C, \mathcal{L}_i(-x_2)\right) \\
&= \deg(\mathcal{L}_i(-x_2)) + 1 - \mathrm{age}_{x_1}(\mathcal{L}_i) - \mathrm{age}_{x_2}(\mathcal{L}_i(-x_2)) \\
&= \deg(\mathcal{L}_i) - \frac{1}{r_2} + 1 - \mathrm{age}_{x_1}(\mathcal{L}_i) - \mathrm{age}_{x_2}(\mathcal{L}_i(-x_2)), \qquad (6.1)
\end{aligned}$$

where $\mathrm{age}_{x_j}(-)$ denotes the age with respect to cyclic generator which acts on a local chart by multiplication by $e^{2\pi i \frac{1}{r_j}}$. The second equality is Serre duality and Lemma 3.3.2. The third equality follows from assuming $\mathcal{E}$ is weakly convex.

We claim that

$$-\frac{1}{r_2} + 1 - \mathrm{age}_{x_2}(\mathcal{L}_i(-x_2)) = \mathrm{age}_{x_2}(\mathcal{L}_i^\vee). \qquad (6.2)$$

To see this, consider three cases.

First, assume G_{x_2} acts nontrivially on the fiber $\mathcal{L}_i(-x_2)|_{x_2}$. Then we observe that

$$\begin{aligned}
-\frac{1}{r_2} + 1 - \mathrm{age}_{x_2}(\mathcal{L}_i(-x_2)) &= -\frac{1}{r_2} + \mathrm{age}_{x_2}(\mathcal{L}_i^\vee(x_2)) \\
&= \mathrm{age}_{x_2}(\mathcal{L}_i^\vee),
\end{aligned}$$

where the second equality follows because the age of $\mathcal{O}_C(x_2)$ at x_2 is $1/r_2$.

For the second case, assume G_{x_2} is nontrivial but acts trivially on the fiber $\mathcal{L}_i(-x_2)|_{x_2}$. Then the age of $\mathcal{L}_i$ and $\mathcal{L}_i^\vee$ at x_2 is $1/r_2$ and $(r_2-1)/r_2$ respectively. We have the following:

$$\begin{aligned}
-\frac{1}{r_2} + 1 - \mathrm{age}_{x_2}(\mathcal{L}_i(-x_2)) &= \frac{r_2 - 1}{r_2} \\
&= \mathrm{age}_{x_2}(\mathcal{L}_i^\vee).
\end{aligned}$$

Finally, consider the case g_2 equals the identity. Then G_{x_2} is trivial, hence r_2 equals 1. We obtain

$$-\frac{1}{r_2} + 1 - \mathrm{age}_{x_2}(\mathcal{L}_i(-x_2)) = 0 = \mathrm{age}_{x_2}(\mathcal{L}_i^\vee).$$

This proves the claim.

Note that

$$\sum_{i=1}^{r} \deg(\mathcal{L}_i) = \deg([u]^*(\det \mathcal{E})) = \beta(\det \mathcal{E})$$

and

$$\sum_{i=1}^{r} \mathrm{age}_{x_j}(\mathcal{L}_i) = \mathrm{age}_{x_j}([u]^*\mathcal{E}) = \mathrm{age}_{g_j}(\mathcal{E}).$$

Summing (6.1) and (6.2) over i then completes the proof. □

We conclude this section with the following equality of twisted virtual classes. This may be interpreted as a cycle-valued formulation of quantum Serre duality.

Theorem 6.1.2. [Cycle-valued quantum Serre duality.] *Let $\mathcal{E}$ be weakly convex. On the connected component $Q^{0+}_{0,g_1,g_2}(X,\beta)$, the virtual classes $\left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2}$ and $\left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}$ are equal up to sign,*

$$\left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2} = (-1)^{\beta(\det \mathcal{E}) - \mathrm{age}_{g_1}(\mathcal{E}) + \mathrm{age}_{g_2}(\mathcal{E}^\vee)} \left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}.$$

Proof. We defined the twisted virtual classes in 4.1.4 and 5.0.3 as

$$\begin{aligned}\left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2} &:= e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)) \cap \left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}} \\ \left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1} &:= e(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)) \cap \left[Q^{0+}_{0,g_1,g_2}(X,\beta)\right]^{\mathrm{vir}}.\end{aligned}$$

Via Proposition 3.3.3,

$$\mathcal{O}_{\mathcal{C}} = \omega_\pi(x_1 + x_2). \tag{6.3}$$

Tensoring (6.3) by $[u]^*\mathcal{E}(-x_2)$, pushing forward via π_*, and applying Serre duality yields:

$$\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2) = \left(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)\right)^\vee.$$

This implies the Euler class identity

$$\begin{aligned} e\left(\mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2)\right) &= e\left(\left(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)\right)^\vee\right) \\ &= (-1)^{\beta(\det\mathcal{E})-\mathrm{age}_{g_1}(\mathcal{E})+\mathrm{age}_{g_2}(\mathcal{E}^\vee)} e\left(\mathbb{R}^1\pi_*[u]^*\mathcal{E}^\vee(-x_1)\right), \end{aligned}$$

where the second equality follows from Lemma 6.1.1. □

6.2 Quantum Serre duality for quasimap invariants

In this section we use the comparison of twisted virtual cycles of Theorem 6.1.2 to compare quasimap invariants for Z and $E^\vee$ when $\mathcal{E}$ is weakly convex. We obtain a simple relationship between the generating functions $L^{Z,0+}(z)$ and $L^{E^\vee,0+}(z)$.

For the remainder of the paper we assume:

- Assumption 2.2.8;
- the GIT stack quotient X is proper;
- the vector bundle $\mathcal{E}$ is weakly convex.

In particular, the third case holds whenever $\mathcal{E}$ is weakly semi-positive.

Definition 6.2.1. [11, Definition 6.9] Given a class $\gamma \in H^*_{\mathrm{CR,ct}}(E^\vee)$, denote by $\overline{\gamma}$ a lift of γ to the compactly supported cohomology. Define the linear map $\Delta : H^*_{\mathrm{CR,ct}}(E^\vee) \to H^*_{\mathrm{CR,amb}}(Z)$ by

$$\Delta(\gamma) := j^* \circ p_*^{\mathrm{cs}}(\overline{\gamma}),$$

where $p_*^{\mathrm{cs}} : H^*_{\mathrm{CR,cs}}(E^\vee) \to H^*_{\mathrm{CR}}(X)$ is the pushforward of compactly supported cohomology.

Lemma 6.2.2. [11, Lemma 6.10] *Assuming 2.2.8, the transformation*

$$\Delta : H^*_{\mathrm{CR,ct}}(E^\vee) \to H^*_{\mathrm{CR,amb}}(Z)$$

is well defined.

Proof. We work over a given twisted sector X_g for $g \in S$. Let γ be an element of $H^*_{\mathrm{ct}}(E_g^\vee)$. A lift $\overline{\gamma} \subset H^*(E_g^\vee)$ is defined up to an element of the kernel of $\phi_g : H^*_{\mathrm{cs}}(E_g^\vee) \to H^*(E_g^\vee)$. To show that Δ is well defined, we must show $p^{\mathrm{cs}}_{g*}(\ker(\phi_g)) \subseteq \ker(j_g^*)$.

Let κ be an element of $\ker(\phi_g)$. The pushforward $i^{\mathrm{cs}}_{g*} : H^*(X_g) \to H^*_{\mathrm{cs}}(E_g^\vee)$ is an isomorphism. By the compactness of X, there exists $\alpha \in H^*(X_g)$ such that $i^{\mathrm{cs}}_{g*}(\alpha) = \kappa$. Note that $p^{\mathrm{cs}}_{g*}(\kappa) = p^{\mathrm{cs}}_{g*} \circ i^{\mathrm{cs}}_{g*}(\alpha) = \alpha$. We see that

$$0 = \phi_g(\kappa) = \phi_g \circ i^{\mathrm{cs}}_{g*}(\alpha) = i_{g*}(\alpha) = p^*(e(E_g^\vee) \cup \alpha),$$

where the last equality is (2.2). Assumption 2.2.8 implies $j_g^*(\alpha) = 0$ if and only if $j_{g*} \circ j_g^*(\alpha) = 0$. The projection formula implies $j_{g*} \circ j_g^*(\alpha) = e(E_g) \cup \alpha$. Up to a sign this is equal to

$$e(E_g^\vee) \cup \alpha = i_g^*(p_g^*(e(E_g^\vee) \cup \alpha)) = 0.$$

Therefore $p^{\mathrm{cs}}_{g*}(\kappa) = \alpha$ lies in $\ker(j_g^*)$. □

Recall Lemma 2.2.11, which states that we may write any element γ in $H^*_{\mathrm{ct}}(E_g^\vee)$ as $p_g^*(e(E_g^\vee) \cup \alpha)$ for some α in $H^*(X_g)$.

Lemma 6.2.3. [11, Lemma 6.11] *Given a class $\gamma \in H^*_{\mathrm{ct}}(E_g^\vee)$, choose an $\alpha \in H^*(X_g)$ such that $\gamma = p_g^*(e(E_g^\vee) \cup \alpha)$, then*

$$\Delta(\gamma) = j_g^*(\alpha).$$

Furthermore, the transformation $\Delta : H^*_{\mathrm{CR,ct}}(E^\vee) \to H^*_{\mathrm{CR,amb}}(Z)$ is an isomorphism.

Proof. If g fixes only the origin of $W \times \mathbb{C}^r$, then $E_g^\vee$ equals X_g and the result is immediate. When that is not the case, the proof is similar to that in Lemma 6.11 of [11].

By Equation (2.2), we have

$$\gamma = p_g^* \left(e(E_g^\vee) \cup \alpha \right) = i_{g*}(\alpha).$$

Noting that i_{g*} factors as $\phi \circ i_{g*}^{\text{cs}}$ gives the following:

$$\begin{aligned}
\Delta(\gamma) &= \Delta \circ \phi_g \circ i_{g*}^{\text{cs}}(\alpha) \\
&= j_g^* \circ p_{g*}^{\text{cs}} \circ i_{g*}^{\text{cs}}(\alpha) \\
&= j_g^*(\alpha).
\end{aligned}$$

To prove the second claim, we observe

$$\begin{aligned}
H_{\text{ct}}^*(E_g^\vee) &= p_g^* \left(\operatorname{im}(e(E_g^\vee) \cup -) \right) \\
&= \operatorname{im}(e(E_g^\vee) \cup -) \\
&= \operatorname{im}(e(E_g) \cup -) \\
&\cong j_{g*} \left(\operatorname{im}(j_g^*) \right) \\
&\cong \operatorname{im}(j_g^*) \\
&= H_{\text{amb}}^*(Z_g).
\end{aligned}$$

The first equality follows from Lemma 2.2.11, the isomorphism in line four is from the projection formula, and the fifth isomorphism is from Assumption 2.2.8. □

It will be useful to consider a modification of Δ.

Definition 6.2.4. Define the linear transformation $\tilde{\Delta}$ by

$$\tilde{\Delta}|_{H_{\text{ct}}^*(E_g^\vee)} := e^{\pi i \operatorname{age}_g(\mathcal{E})} \Delta|_{H_{\text{ct}}^*(E_g^\vee)}.$$

Lemma 6.2.5. *The transformation $\tilde{\Delta}$ identifies the Chen-Ruan Poincaré pairings up to a sign:*

$$\langle \tilde{\Delta}(\gamma_1), \tilde{\Delta}(\gamma_2)\rangle^Z = (-1)^{\mathrm{rank}(E)} \langle \gamma_1, \gamma_2\rangle^{E^\vee, \mathrm{ct}},$$

for any γ_1 and γ_2 in $H^*_{\mathrm{CR,ct}}(E^\vee)$.

Proof. Assume γ_1 is supported on $E^\vee_{g_1}$ and γ_2 is supported on $E^\vee_{g_2}$. The pairings $\langle \tilde{\Delta}(\gamma_1), \tilde{\Delta}(\gamma_2)\rangle^Z$ and $\langle \gamma_1, \gamma_2\rangle^{E^\vee, \mathrm{ct}}$ equal zero unless $\iota(E^\vee_{g_1}) = E^\vee_{g_2}$.

It therefore suffices to consider the case where $\gamma_1 \in H^*_{\mathrm{ct}}(E^\vee_g)$ and $\gamma_2 \in H^*_{\mathrm{ct}}(E^\vee_{g^{-1}})$. Choose $\alpha_1, \alpha_2 \in H^*_{\mathrm{CR}}(X)$ such that $\gamma_1 = i_{g*}(\alpha_1)$ and $\gamma_2 = i_{g^{-1}*}(\alpha_2)$. Then, applying the projection formula and Lemma 6.2.3, we obtain

$$\begin{aligned}
\langle \tilde{\Delta}(\gamma_1), \tilde{\Delta}(\gamma_2)\rangle^Z &= (-1)^{\mathrm{age}_g(\mathcal{E}) + \mathrm{age}_{g^{-1}}(\mathcal{E})} \int_{IZ} j_g^*(\alpha_1) \cup \iota^*(j^*_{g^{-1}}(\alpha_2)) \\
&= (-1)^{\mathrm{rank}(E) - \mathrm{rank}(E_g)} \int_{IX} \alpha_1 \cup \iota^*(\alpha_2 \cup e(E_g)) \\
&= (-1)^{\mathrm{rank}(E)} \int_{IX} \alpha_1 \cup \iota^*(i^*_{g^{-1}} i_{g^{-1}*}(\alpha_2)) \\
&= (-1)^{\mathrm{rank}(E)} \int_{IE^\vee} i^{\mathrm{cs}}_{g*}(\alpha_1) \cup \iota^*(i_{g^{-1}*}(\alpha_2)) \\
&= (-1)^{\mathrm{rank}(E)} \langle \gamma_1, \gamma_2\rangle^{E^\vee, \mathrm{ct}},
\end{aligned}$$

where the second equality uses the fact [23, Lemma 4.6] that for all g in S

$$\mathrm{age}_g(\mathcal{E}) + \mathrm{age}_g(\mathcal{E}^\vee) = \mathrm{rank}(E) - \mathrm{rank}(E_g). \tag{6.4}$$

$\square$

Theorem 6.2.6. *Given elements $\gamma_1, \gamma_2 \in H^*_{\mathrm{CR,ct}}(E^\vee)$, we have the equality*

$$\left\langle \tilde{\Delta}\left(\gamma_1\right) \psi_1^{a_1}, \tilde{\Delta}\left(\gamma_2\right) \psi_2^{a_2} \right\rangle^{Z, 0+}_{0,\beta} = e^{\pi i (\beta(\det \mathcal{E}) + \mathrm{rank}(E))} \left\langle \gamma_1 \psi_1^{a_1}, \gamma_2 \psi_2^{a_2} \right\rangle^{E^\vee, 0+}_{0,\beta}.$$

Proof. Let g_1 and g_2 be elements in S. Without loss of generality, choose $\gamma_1 \in H^*_{\mathrm{ct}}(E^\vee_{g_1})$ and $\gamma_2 \in H^*_{\mathrm{ct}}(E^\vee_{g_2})$. Then choose $\alpha_1 \in H^*(X_{g_1})$, and $\alpha_2 \in H^*(X_{g_2})$ such that

$$\gamma_1 = p^*_{g_1}(\alpha_1 \cup e(E^\vee_{g_1})) = i_{g_1 *}(\alpha_1)$$
$$\gamma_2 = p^*_{g_2}(\alpha_2 \cup e(E^\vee_{g_2})) = i_{g_2 *}(\alpha_2).$$

By Proposition 3.2.7, our assumption that $\mathcal{E}$ is weakly convex, and (5.2) there is a short exact sequence

$$0 \longrightarrow \mathbb{R}^0\pi_*[u]^*\mathcal{E}(-x_2) \longrightarrow \mathbb{R}^0\pi_*[u]^*\mathcal{E} \longrightarrow ev_2^{X*}(E_{g_2}) \longrightarrow 0. \tag{6.5}$$

Also note that the following diagram commutes:

$$\begin{tikzcd} Q^{0+}_{0,2}(Z,\beta) \arrow[r, "\tilde{j}"] \arrow[d, "ev_i^Z"] & Q^{0+}_{0,2}(X,\beta) \arrow[d, "ev_i^X"] \\ \bar{I}Z \arrow[r, "j"] & \bar{I}X. \end{tikzcd} \tag{6.6}$$

Expanding the integral in the left-hand-side of the statement gives us

$$\begin{aligned}
&\left\langle \tilde{\Delta}\left(\gamma_1\right)\psi_1^{a_1}, \tilde{\Delta}\left(\gamma_2\right)\psi_2^{a_2} \right\rangle^{Z,0+}_{0,\beta} \\
=&e^{\pi i(\mathrm{age}_{g_1}(\mathcal{E})+\mathrm{age}_{g_2}(\mathcal{E}))} \int_{\left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}} ev_1^{Z*}j^*_{g_1}(\alpha_1)\psi_1^{a_1} \cup ev_2^{Z*}j^*_{g_2}(\alpha_2)\psi_2^{a_2} \\
=&e^{\pi i(\mathrm{age}_{g_1}(\mathcal{E})+\mathrm{age}_{g_2}(\mathcal{E}))} \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2)\psi_2^{a_2} \cup e(\mathbb{R}^0\pi_*[u]^*\mathcal{E}) \\
=&e^{\pi i(\mathrm{age}_{g_1}(\mathcal{E})+\mathrm{age}_{g_2}(\mathcal{E}))} \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e(E_{g_2}))\psi_2^{a_2} \\
=&e^{\pi i(\mathrm{age}_{g_1}(\mathcal{E})+\mathrm{age}_{g_2}(\mathcal{E})+\mathrm{rank}(E_{g_2}))} \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/Z,2}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e(E^\vee_{g_2}))\psi_2^{a_2} \\
=&e^{\pi i(\beta(\det\mathcal{E})+\mathrm{rank}(E))} \int_{\left[Q^{0+}_{0,2}(X,\beta)\right]^{\mathrm{vir}}_{X/E^\vee,1}} ev_1^{X*}(\alpha_1)\psi_1^{a_1} \cup ev_2^{X*}(\alpha_2 \cup e(E^\vee_{g_2}))\psi_2^{a_2} \\
=&e^{\pi i(\beta(\det\mathcal{E})+\mathrm{rank}(E))} \left\langle \gamma_1\psi_1^{a_1}, \gamma_2\psi_2^{a_2} \right\rangle^{E^\vee,0+}_{0,\beta}.
\end{aligned}$$

The second equality follows from (6.6), the projection formula, and Proposition 4.1.3. The third equality is by (6.5). The fifth equality is by (6.4) and Theorem 6.1.2. The final equality is by Proposition 5.0.4. □

Theorem 6.2.7. *The transformation $\tilde{\Delta}$ identifies the operators $L^{Z,0+}(z)$ and $L^{E^\vee,0+}(z)$ up to a change of variables in the Novikov parameter:*

$$L^{Z,0+}(z)\circ\tilde{\Delta} = \tilde{\Delta}\circ L^{E^\vee,0+}(z)|_{q^\beta\mapsto e^{\pi i\beta(\det\mathcal{E})}q^\beta}.$$

Proof. By Assumption 2.2.8, the Chen-Ruan Poincaré pairing on the ambient cohomology of Z is non-degenerate and by Lemma 6.2.3, Δ is an isomorphism. It therefore suffices to show that

$$\left\langle\tilde{\Delta}(\gamma_2), L^{Z,0+}(z)\circ\tilde{\Delta}(\gamma_1)\right\rangle^Z = \left\langle\tilde{\Delta}(\gamma_2), \tilde{\Delta}\circ L^{E^\vee,0+}(z)(\gamma_1)|_{q^\beta\mapsto e^{\pi i\beta(\det\mathcal{E})}q^\beta}\right\rangle^Z, \tag{6.7}$$

for any γ_1 and γ_2 in $H^*_{\mathrm{CR,ct}}(E^\vee)$.

Assume γ_1 is in $H^*_{\mathrm{ct}}(E^\vee_{g_1})$ and γ_2 is in $H^*_{\mathrm{ct}}(E^\vee_{g_2})$. Expand the left-hand-side of (6.7) to obtain

$$\begin{aligned}&\langle\tilde{\Delta}(\gamma_2),\tilde{\Delta}(\gamma_1)\rangle^Z + \\ &\sum_{\beta\in\mathrm{Eff}} q^\beta\left\langle\tilde{\Delta}(\gamma_2), \tilde{ev}_{2*}\left(\frac{ev_1^*\left(\tilde{\Delta}(\gamma_1)\right)}{-z-\psi_1}\cap\left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}\right)\right\rangle^Z.\end{aligned} \tag{6.8}$$

The right-hand-side of (6.7) expands as

$$\begin{aligned}&\langle\tilde{\Delta}(\gamma_2),\tilde{\Delta}(\gamma_1)\rangle^Z + \\ &\sum_{\beta\in\mathrm{Eff}} e^{\pi i\beta(\det\mathcal{E})}q^\beta\left\langle\tilde{\Delta}(\gamma_2), \tilde{\Delta}\circ\tilde{ev}_{2*}\left(\frac{ev_1^*(\gamma_1)}{-z-\psi_1}\cap\left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathrm{vir}}\right)\right\rangle^Z.\end{aligned} \tag{6.9}$$

For $\beta \in \mathrm{Eff}(W, G, \theta)$, we have

$$
\begin{aligned}
&\left\langle \tilde{\Delta}(\gamma_2), \tilde{ev}_{2*}\left(\frac{ev_1^*\left(\tilde{\Delta}(\gamma_1)\right)}{-z-\psi_1} \cap \left[Q^{0+}_{0,2}(Z,\beta)\right]^{\mathrm{vir}}\right)\right\rangle^{Z} \\
=&\left\langle \frac{\tilde{\Delta}(\gamma_1)}{-z-\psi_1}, \tilde{\Delta}(\gamma_2)\right\rangle^{Z,0+}_{0,\beta} \\
=&e^{\pi i(\beta(\det \mathcal{E})+\mathrm{rank}(E))}\left\langle \frac{\gamma_1}{-z-\psi_1}, \gamma_2\right\rangle^{E^\vee,0+}_{0,\beta} \\
=&e^{\pi i(\beta(\det \mathcal{E})+\mathrm{rank}(E))}\left\langle \gamma_2, \tilde{ev}_{2*}\left(\frac{ev_1^*(\gamma_1)}{-z-\psi_1} \cap \left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathrm{vir}}\right)\right\rangle^{E^\vee,\mathrm{ct}} \\
=&e^{\pi i\beta(\det \mathcal{E})}\left\langle \tilde{\Delta}(\gamma_2), \tilde{\Delta}\circ\tilde{ev}_{2*}\left(\frac{ev_1^*(\gamma_1)}{-z-\psi_1} \cap \left[Q^{0+}_{0,2}(E^\vee,\beta)\right]^{\mathrm{vir}}\right)\right\rangle^{Z}.
\end{aligned}
$$

The first equality is the projection formula. The second is Theorem 6.2.6. The third is another application of the projection formula. The fourth is Lemma 6.2.5.

Each coefficient of q^β in (6.8) and (6.9) is equal, completing the proof. □

6.3 Quantum Serre duality for Gromov-Witten invariants

In this section we combine the above results with the wall-crossing formulas proven by Zhou in [19] to prove a quantum Serre duality statement for Gromov-Witten invariants without assuming convexity.

Theorem 1.12.2 of [19] states

$$
J^{X,\infty}(\boldsymbol{t} + \mu^{X,\geq 0+}(q,-z), q, z) = J^{X,0+}(\boldsymbol{t}, q, z), \tag{6.10}
$$

where $J^{X,\varepsilon}(\boldsymbol{t}, q, z)$ is defined in [19, §1.12] and $\mu^{X,\geq 0+}(q,-z)$ is defined in [19, §1.11]. Similar to the $J^{X,\varepsilon}$-function defined in [19], we may generalize the operator $L^{X,\varepsilon}(\boldsymbol{t}, z)$ of (2.9) by including ψ-classes in the $\boldsymbol{t}$ insertions of (2.5) and (2.9). Let f be a cohomology-valued polynomial. We redefine the double-bracket as

$$
\begin{aligned}
&\langle\langle \alpha_1\psi_1^{a_1}, \dots, \alpha_k\psi_k^{a_k}\rangle\rangle_0^{X,\varepsilon}(f(\psi)) \\
:=& \sum_{\beta\in\mathrm{Eff}}\sum_{m\geq 0}\frac{q^\beta}{m!}\langle \alpha_1\psi_1^{a_1}, \dots, \alpha_k\psi_k^{a_k}, f(\psi_{k+1}), \dots, f(\psi_{k+m})\rangle_{0,\beta}^{X,\varepsilon}
\end{aligned}
$$

for classes $\alpha_1, \dots, \alpha_k \in H^*_{\mathrm{CR}}(X)$ and non-negative integers $a_1, \dots, a_k$. Define $L^{X,\varepsilon}(f(\psi), z)$ as in (2.9) by replacing $\boldsymbol{t}$ with $f(\psi)$. We now use (6.10) to relate $L^{X,0+}(z)$ and $L^{X,\infty}(\mu^{X,\geq 0+}(q,-\psi), z)$.

Lemma 6.3.1. *The operators $L^{X,0+}(z)$ and $L^{X,\infty}(\mu^{X,\geq 0+}(q,-\psi), z)$ are equal.*

Proof. Let $\{T_i\}_{i\in I}$ be a basis of $H^*_{\mathrm{CR}}(X)$ with dual basis $\{T^i\}_{i\in I}$. For $p \in I$, observe that

$$
z\frac{\partial}{\partial t_p}J^{X,0+}(\boldsymbol{t}, q, z)|_{t=0} = T_p + \sum_{\beta\in\mathrm{Eff}}\sum_{i\in I} q^\beta \left\langle T_p, \frac{T^i}{z-\psi}\right\rangle_{0,\beta}^{X,0+} T_i. \tag{6.11}
$$

Pairing (6.11) with $\alpha \in H^*_{\mathrm{CR}}(X)$, we obtain

$$
\begin{aligned}
\left\langle z\frac{\partial}{\partial t_p}J^{X,0+}(\boldsymbol{t}, q, z)|_{t=0}, \alpha\right\rangle^X &= \langle T_p, \alpha\rangle + \sum_{\beta\in\mathrm{Eff}} q^\beta \left\langle T_p, \frac{\alpha}{z-\psi}\right\rangle_{0,\beta}^{X,0+} \\
&= \left\langle T_p, L^{X,0+}(-z)(\alpha)\right\rangle^X .
\end{aligned}
\tag{6.12}
$$

Similarly, note that

$$
\begin{aligned}
&z\frac{\partial}{\partial t_p}J^{X,\infty}(\boldsymbol{t} + \mu^{X,\geq 0+}(q,-z), q, z)|_{t=0} \\
=&T_p + \sum_{i\in I}\sum_{\beta\in\mathrm{Eff}}\sum_{m\geq 0}\frac{q^\beta}{m!}\left\langle T_p, \frac{T^i}{z-\psi}, \mu^{X,\geq 0+}(q,-\psi), \dots, \mu^{X,\geq 0+}(q,-\psi)\right\rangle_{0,m+2,\beta}^{X,\infty} T_i
\end{aligned}
\tag{6.13}
$$

and

$$\begin{aligned}
&\left\langle z\frac{\partial}{\partial t_p}J^{X,\infty}(\boldsymbol{t}+\mu^{X,\geq 0+}(q,-z),q,z)|_{t=0},\alpha\right\rangle^X \qquad (6.14)\\
=&\langle T_p,\alpha\rangle+\sum_{\beta\in\mathrm{Eff}}\sum_{m\geq 0}\frac{q^\beta}{m!}\left\langle T_p,\frac{\alpha}{z-\psi},\mu^{X,\geq 0+}(q,-\psi),\ldots,\mu^{X,\geq 0+}(q,-\psi)\right\rangle^{X,\infty}_{0,m+2,\beta}\\
=&\left\langle T_p,L^{X,\infty}(\mu^{X,\geq 0+}(q,-\psi),-z)(\alpha)\right\rangle^X.
\end{aligned}$$

By (6.10), equations (6.12) and (6.14) are equal for all $p\in I$. By the nondegeneracy of the Poincaré pairing on X, it follows that

$$L^{X,0+}(z)(\alpha)=L^{X,\infty}(\mu^{X,\geq 0+}(q,-\psi),z)(\alpha).$$

□

Proposition 4.1.5 and Lemma 6.3.1 imply the following relationship between two-pointed quasimap invariants of X and Gromov-Witten invariants of Z.

Corollary 6.3.2. *The operators $L^{X/Z,0+}(z)$ and $L^{Z,\infty}(\boldsymbol{t},z)$ are related by*

$$j^*\circ L^{X/Z,0+}(z)=L^{Z,\infty}(\mu^{Z,\geq 0+}(q,-\psi),z)\circ j^*.$$

Assume that the total space of the vector bundle $E^\vee\to X$ is realized as the GIT stack quotient

$$[W\times\mathbb{C}^r /\!\!/_\theta G].$$

In other words, we require that

$$(W\times\mathbb{C}^r)^{ss}=W^{ss}(\theta)\times\mathbb{C}^r.$$

This guarantees that the moduli space $Q^{0+}_{g,k}(E^\vee,\beta)$ of Definition 5.0.1 coincides with the space $Q^{0+}_{g,k}([W\times\mathbb{C}^r /\!\!/_\theta G],\beta)$ of Definition 2.3.3 (as do their respective virtual classes). In particular, this allows us to combine the wall-crossing results of [19] with Theorem 6.2.7 to deduce the following corollary.

Corollary 6.3.3. *The operator $\tilde\Delta$ identifies the fundamental solutions $L^{Z,\infty}$ and $L^{E^\vee,\infty}$ up to a change of variables:*

$$\begin{aligned}&L^{Z,\infty}(\mu^{Z,\geq 0+}(q,-\psi),z)\circ\tilde\Delta\\ =&\tilde\Delta\circ L^{E^\vee,\infty}(\mu^{E^\vee,\geq 0+}(q,-\psi),z)\big|_{q^\beta\mapsto e^{\pi i\beta(\det\mathcal{E})}q^\beta},\end{aligned}$$

where $\mu^{Z,\geq 0+}(q,-\psi)$ and $\mu^{E^\vee,\geq 0+}(q,-\psi)$ are the changes of variables from [19, §1.11] for Z and $E^\vee$ respectively.

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