DISSERTATION

ROBUST ESTIMATION AND TESTING OF LOCATION FOR SYMMETRIC STABLE DISTRIBUTIONS

Submitted by

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WE HEREBY RECOMMEND THAT THE DISSERTATION ROBUST ESTIMATION AND TESTING OF LOCATION FOR SYMMETRIC STABLE DISTRIBUTIONS PREPARED UNDER OUR SUPERVISION BY ATEQ AHMED M. AL-GHAMEDI BE ACCEPTED AS FULFILLING IN PART REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

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ABSTRACT

ROBUST ESTIMATION AND TESTING OF LOCATION FOR SYMMETRIC STABLE DISTRIBUTIONS

Along the line of growing interest in studying stable distributions, we propose a robust estimation technique for the location parameter δ of these distributions. It will be based on using some selector statistics $\Delta = \frac{x_{1-q_1} - x_{q_1}}{x_{1-q_2} - x_{q_2}}$, where x_q is the q^{th} sample quantile and $0 < q_1 < q_2 < 1/2$, for selecting the index parameter α of a symmetric stable distribution where $0 < \alpha \leq 2$. We then use a class of robust estimators, $T = T_{2\lambda}(F_n)$, where $T_{2\lambda}(F_n)$ is 2λ−trimmed mean and $0 < \lambda \leq 1/2$, for estimating the location parameter δ. A new class of tests is then introduced and investigated $V = \frac{T_{2\lambda}(F_n)}{x_{1-q_1} - x_{q_1}}$ for testing $H_0 : \delta = 0$, against $H_A : \delta \neq 0$. For two-sample problem, we introduce the class of tests $V = \frac{T_{2\lambda_1}(F_n(x)) - T_{2\lambda_2}(F_n(y))}{\frac{1}{2}[(x_{1-q_1} - x_{q_1}) + (y_{1-q_2} - y_{q_2})]}$ for testing $H_0 : \delta_1 - \delta_2 = 0$ vs. $H_A : \delta_1 - \delta_2 \neq 0$. V statistic is similar to Student's t statistic. For each α, the asymptotic distributions of Δ and V will be derived and general formulas of means and variances of Δ and V will be given. Normal($\alpha = 2$) and Cauchy($\alpha = 1$) distributions will be used as special cases from the family of symmetric stable distributions to show the derivation of the asymptotic distributions, the means and the variances. The performance of our robust procedures for estimating and testing is investigated using both computer simulation and real examples. Also, the power of these tests is studied. Our robust tests can be adapted to investigate other symmetric distributions.

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DEDICATION

To the one who taught me how to count, **Halimah bint Hassan.**

CONTENTS

LIST OF FIGURES

LIST OF TABLES

Chapter 1

PRELIMINARIES

1.1 Introduction

The word "robust" is used to refer to someone who is strong and crude. By 1950s, the word was given its statistical meaning, the evolution of language had eliminated the negative connotation: robust meant simply strong, healthy, and sufficiently tough.

The problem of robust estimation of location and testing has a long history and recent intensive activity. For Some years, an enormous amount of effort has been devoted to robust procedures because statisticians are now concerned. The acceptance of the dogma that errors should be distributed according to the normal law without much inquiry is causing this concern for many reasons, among which are the following:

1. We never have a very accurate knowledge of the true underlying distribution.

2. In many cases, basic assumptions about the underlying model may very well dominate the analysis of the data.

3. The fact that most models rarely exactly fit the situations.

4. The performance of some of the classical tests or estimates is very unstable when small changes of the underlying distribution are involved.

5. Some alternative tests or estimates (such as the λ-trimmed mean instead of the mean or the Wilcoxon test instead of the t-test) lose very little efficiency for an exactly normal law, but show a much better and more stable performance under deviations from it.

Researchers in "robust statistics" have extended standard statistical techniques to work (a) in the presence of outliers (occasional extremely wrong values), and (b) when the real data may be generated by a mechanism which is similar to but does not belong to the class of theoretical models used to analyze the data. The robust methods are better able to cope with the types of data found in real scientific data sets.

Nowadays, more and more data sets seem to have come from distributions with heavier tails than that of the normal distribution. Thus, a more realistic approach would be to seek statistical procedures that are good for a broad class of possible underlying models, but which are not necessarily best for any one of them. Such procedures are characterized as being robust, that is, exhibiting strength.

The problem is to estimate and to test for a location parameter from a large number of independent observations $x_1, x_2, ..., x_n$ distributed according to

$$P(X_i < x) = F((x - \theta)/\sigma), \tag{1.1}$$

where the shape is not exactly known. For most good estimators T_n, $\sqrt{n}(T_n - \theta)$ will be asymptotically normal as $n \to \infty$, and we will have to judge estimators in terms of their asymptotic variance $\sigma_F^2(T)$, their relative efficiency $\sigma_F^2(T_1)/\sigma_F^2(T)$, where T_1 is another estimator, or their absolute efficiency $1/(I(F)\sigma_F^2(T))$, where the $I(F)$ is the Fisher information. Consequently a robust estimator should possess:

1. A high absolute efficiency for all suitably smooth shapes F. This can be achieved asymptotically for a large sample size, but the convergence seems to be much too slow for practical purposes (Van Eeden, 1970).

2. A high efficiency relative to the sample mean or other selected estimates for all F (Bickel, 1965).

3. A high absolute efficiency over a strategically selected finite set F_i of shapes (Crow and Siddiqui, 1967).

4. A small asymptotic variance over some neighborhood of one shape, in particular of the normal one (Huber, 1964).

5. The distribution of the estimate should change little under arbitrary small variations of the underlying distribution F (Hampel, 1969).

All the five goals are important. However, none of these goals guarantees the stability. Goal (3) is very attractive for studies after we have proposed some parametric families of estimates. Since the ordinary variance is not an adequate measure of performance of a robust estimate, it is too sensitive to the irrelevant extreme tail behavior of the estimate. It is preferable to look at selected quantiles, which will also indicate how fast the limiting normal law is approached. Crow and Siddiqui (1967) recommended that among the class $T_n(\underline{a})$ one ought to select an estimator which has the highest guaranteed efficiency (geff) relative to $\mathcal{F}$. In conclusion, we can say that whenever there is a suspicion that the data may have come from among a family of distributions, it is best to use procedures that are good for all possible underlying models, but which are not necessarily best for any one of them.

The dissertation is composed of four parts. In the first part, a review of robust estimation methods and stable distributions will be given. In the second part the asymptotic theory of estimators which are a linear combinations of the order statistics will be considered. These estimators called linear systematic statistics. Also, a function of population quantiles,

$$\Delta_1 = \frac{x_{.95} - x_{.05}}{x_{.75} - x_{.25}},$$

is suggested as a measure of tail thickness (Badahdah and Siddiqui, 1991) will be investigated. The third part will deal with the testing for location parameters. Power and critical values of the tests will be given. The study ends with a brief application using Kappa distributions, real examples, and simulations.

1.2 Robust Estimation of Location

Statisticians are often interested in the "location problem". One of the richest studies of robust estimation of location was the Princeton study (Andrews *et al.*, 1972). In this study the properties of sixty-eight estimators were investigated. Among these estimators were the M-estimators, Multi-times folded medians, various linear combinations of order statistics, and many more. Only a few of these estimators will be mentioned here. The most important application will be to estimate the location of a "nearly normal" distribution.

1.2.1 Methods of Constructing Estimates

Let $X_1, X_2, ..., X_n$ be independent random variables with common distribution

$$P(X_i < x) = F((x-\theta)/\sigma),$$

and we shall assume that F has a density f.

- **L-Estimators:**

The class of linear combinations of order statistics is called L-estimators. Let $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ be the order statistics resulting from a random sample from $F[(x-\mu)/\sigma] \in \mathcal{F}$. L-estimators will have the form

$$T(a) = \sum_{i=1}^{n} a_i x_{(i)}, \qquad a_{n-i+1} = a_i, \qquad \sum_{i=1}^{n} a_i = 1, \tag{1.2}$$

special cases are λ-trimmed means.

- **M-Estimators:**

Let F be any distribution function on the real line and let ψ be an odd nondecreasing function which is not identically zero. The value of ρ such that

$$\frac{1}{n}\sum_{i=1}^{n}\psi(x_i-\rho)=\int\psi(x-\rho)dF_n(x)=0 \tag{1.3}$$

is called a maximum likelihood type estimator. Special cases are one-step M-estimators (Staudte and Sheather, 1990).

- **R-Estimators:**

R-estimators of location are derived by inverting tests of hypotheses based on ranks. Let J be some specified function. In functional form, an R-estimator, ρ, satisfies

$$\int J\{\frac{1}{2}[q+1-F(2\rho-x_q)]\}dq=0. \tag{1.4}$$

A common choice for J is $J(x)=x-\frac{1}{2}$. This leads to the Hodges-Lehmann estimator (Staudte and Sheather, 1990).

The three estimators can be written as functionals of the empirical distribution function $T_n=T(F_n)$ (either exactly or at least approximately). In particular the L-estimator corresponds to

$$T(F)=\int J(t)F^{-1}(t)dt. \tag{1.5}$$

where $J(t)$ is any continuous function on (0,1) which is symmetrical about $\frac{1}{2}$: $J(\frac{1}{2}-t^-)=J(\frac{1}{2}+t)$, for $0<t<\frac{1}{2}$ (Staudte and Sheather, 1990).

A basic requirement of stability of the estimator is that a small change in F_n (either small changes affecting most or all observations, like rounding) should cause only small changes in $T_n=T(F_n)$ (Hampel, 1971).

1.3 Stable Distributions

Stable distributions are a rich class of distributions that allow skewness and heavy tails. A distribution is said to be heavy tailed if its tails are heavier than exponential. A consequence of heavy tails is that not all moments exist. The class of stable distributions was characterized by Paul Lévy (1954) in his study of sums of independent identically distributed random variables.

The general stable distribution is described by four parameters: an index of stability $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, scale parameter $\gamma > 0$, and a location parameter $\delta \in \mathbf{R}$. The lack of closed formulas for densities and distribution functions for all but a few stable distributions (Gaussian, Cauchy and Lévy) has been a major drawback to the use of stable distributions by practitioners.

There are several reasons for using a stable distribution to describe a system. The first is where there are solid theoretical reasons for expecting a non-Gaussian stable model. The second reason is the Generalized Central Limit Theorem which states that the only possible non-trivial limit of normalized sums of independent identically distributed random variables is stable. The third argument for modelling with stable distributions is empirical: many large data sets exhibit heavy tails and skewness.

There are many parameterizations for stable distributions. We will use the following:

Definition: A random variable X is stable $S(\alpha, \beta, \gamma, \delta)$ if and only if

$$X \stackrel{d}{=} \gamma Z + \delta, \tag{1.6}$$

where $0 < \alpha \leq 2,\ -1 \leq \beta \leq 1,\ \gamma \geq 0,\ \delta \in \mathbf{R}$. $Z = Z(\alpha, \beta)$ is a random variable with characteristic function,

$$E\ exp(iuZ) = \begin{cases} exp(-|u|^{\alpha}[1 + i\beta \tan \frac{\pi\alpha}{2}(sign\ u)(|u|^{1-\alpha} - 1)]) & \alpha \neq 1 \\ exp(-|u|[1 + i\beta \frac{2}{\pi}(sign\ u) \ln |u|]) & \alpha = 1, \end{cases} \tag{1.7}$$

and

$$sign\ u = \begin{cases} -1 & u < 0 \\ 0 & u = 0 \\ 1 & u > 0. \end{cases}$$

Properties of Stable Distributions

1. All (non-degenerate) stable distributions are continuous distributions with an infinitely differentiable density.

2. Every stable distribution has a mode.

3. It is clear that $Z(2,0) \sim N(0,2)$, $Z(1,0)$ ~Cauchy (0,1) and $Z(\frac{1}{2},0)$ ~Lévy (0,1).

4. When $Z(\alpha,-\beta) \stackrel{d}{=} -Z(\alpha,\beta)$, then the distribution is symmetric.

5. When $1 < \alpha \leq 2$, the mean of $X \sim S(\alpha,\beta,\gamma,\delta)$ is $\mu = E\,X = \delta - \beta\gamma \tan\frac{\pi\alpha}{2}$. As $\alpha \downarrow 1$, it has a mean of $\mu = \beta \tan\frac{\pi\alpha}{2}$. When $\beta = 0$, the mean is always 0. When $\beta > 0$, the mean tends to $+\infty$ because both tails are getting heavier, and the right tail is heavier than the left. By symmetry, when $\beta < 0$ the mean tends to $-\infty$. Finally, when α reaches 1, the tails are too heavy for the integral $E\,X = \int_{-\infty}^{\infty} x f(x)\, dx$ to converge.

6. Tail approximation: let $X \sim S(\alpha,\beta,1,0)$ with $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$. Then as $X \to \infty$,

$$P(X > x) = C_\alpha(1+\beta)x^{-\alpha}, \tag{1.8}$$

$$f(x/\alpha,\beta) \sim \alpha C_\alpha(1+\beta)x^{-(\alpha+1)}, \tag{1.9}$$

where $C_\alpha = \Gamma(\alpha)(\sin\frac{\pi\alpha}{2})/\pi$. Using the symmetry property, the lower tail properties are similar.

7. If $X \sim S(\alpha, \beta, \gamma, \delta)$, then for any $a \neq 0, \quad b \in \mathbf{R}$,

$$aX + b \sim S(\alpha, (sign\ a)\beta, |a|\gamma, a\delta + b). \tag{1.10}$$

8. If $X_1 \sim S(\alpha, \beta_1, \gamma_1, \delta_1)$ and $X_2 \sim S(\alpha, \beta_2, \gamma_2, \delta_2)$ are independent, then

$$X_1 + X_2 \sim S(\alpha, \beta, \gamma, \delta), \tag{1.11}$$

where

$$\beta = \frac{\beta_1 \gamma_1^\alpha + \beta_2 \gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, \qquad \gamma^\alpha = \gamma_1^\alpha + \gamma_2^\alpha,$$

$$\delta = \begin{cases} \delta_1 + \delta_2 + (\tan \frac{\pi\alpha}{2})[\beta\gamma - \beta_1\gamma_1 - \beta_2\gamma_2] & \alpha \neq 1 \\ \delta_1 + \delta_2 + \frac{2}{pi}[\beta\gamma \ln \gamma - \beta_1\gamma_1 \ln \gamma_1 - \beta_2\gamma_2 \ln \gamma_2] & \alpha = 1. \end{cases}$$

9. The class of distribution functions, D, selected for the study consists of all dfs. $F(x - \mu), \quad -\infty < \mu < \infty$, where $F(x) = S_\alpha(x)$, and $S_\alpha(x)$ is the df of the symmetric stable distribution with index of stability α. The probability density function (pdf) $f(x) = F'(x)$ is represented by the inverse Fourier transform of the cf (with the location parameter equal to zero and the scale parameter equal to one) as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{-itx} dt = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} \cos(tx) dt \quad -\infty < x < \infty. \tag{1.12}$$

Note that for $\alpha = 2$, we have a normal distribution with mean zero and variance 2.

10.

$$f'(x) = -\frac{1}{\pi} \int_0^\infty e^{-t^\alpha} sin(tx) dt. \tag{1.13}$$

11. The quantile function, $\xi(q) = F^{-1}(q), \;\; 0 < q < 1$, is uniquely defined.

Nowadays, there are reliable computer programs to simulate and compute stable densities, distribution functions, and quantiles. With these programs, it is possible to use stable models in a variety of practical problems. We use a program proposed by Nolan (1997), Maple, SAS and Splus Packages.

Table 1.1: $f(x_q)$ of Some Stable Distributions

q / α	0.50	0.75	0.95
2.00	0.282095	0.224702	0.072928
1.75	0.283492	0.217189	0.050887
1.50	0.287353	0.206242	0.030029
1.25	0.296469	0.188637	0.016226
1.00	0.318310	0.159155	0.007790
0.75	0.378992	0.116147	0.002724
0.50	0.636620	0.065480	0.000413
0.25	7.639437	0.016885	0.000003

Figure 1.1: Symmetric Stable Distribution Functions

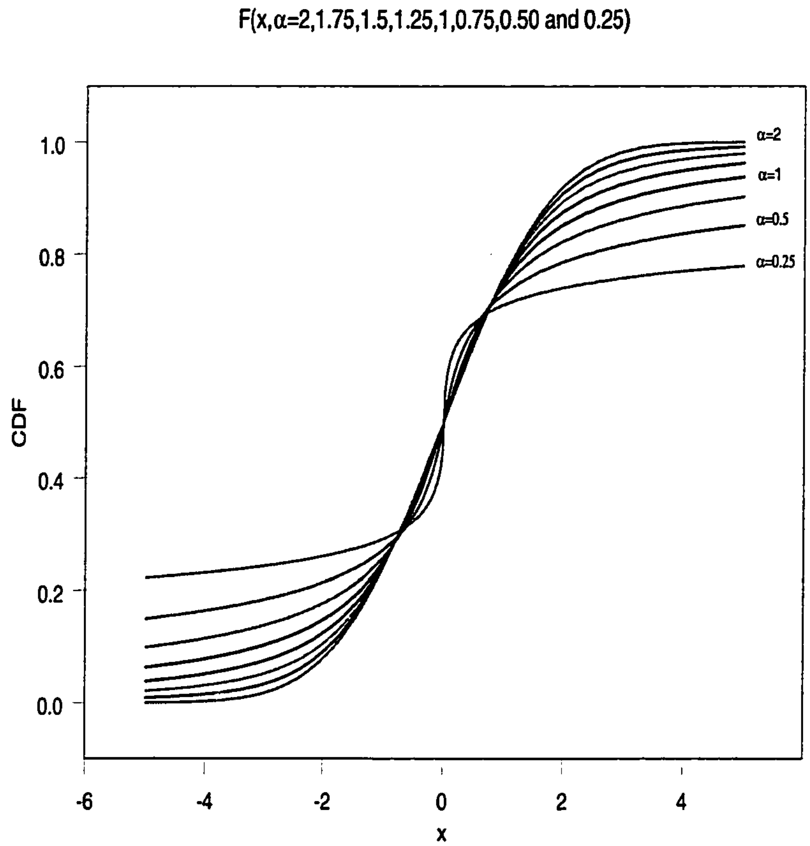


Figure 1.2: Symmetric Stable Densities

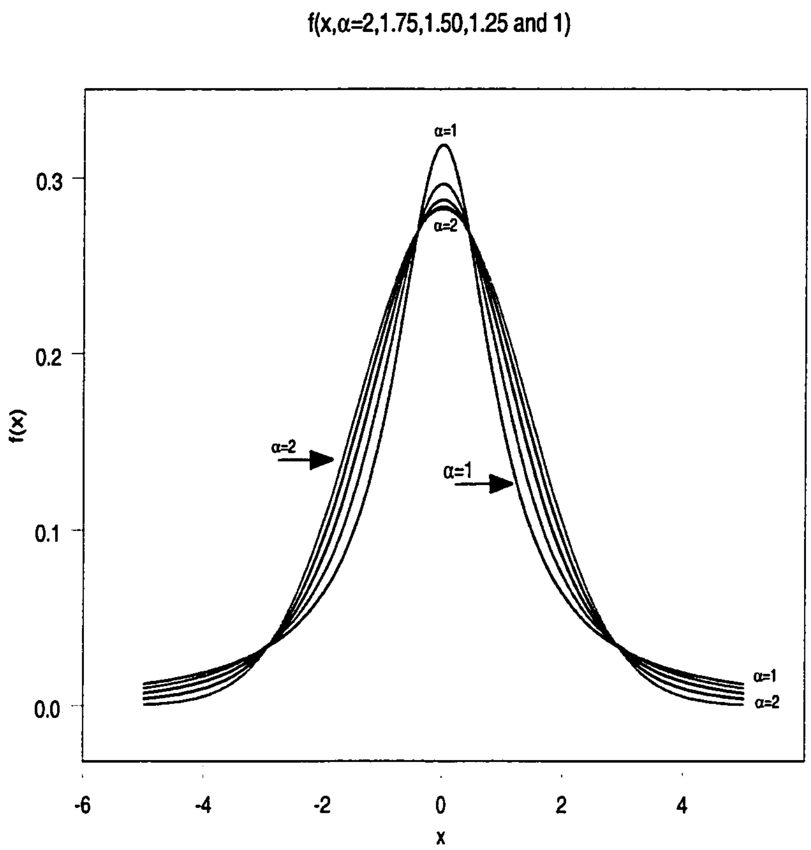


Table 1.2: Quantiles of Stable Distributions

α \ q	0.60	0.75	0.90	0.95	0.99
2.00	0.35827	0.95387	1.81238	2.32617	3.28995
1.90	0.35804	0.95680	1.84304	2.40427	3.66906
1.80	0.35752	0.95976	1.88029	2.50488	4.27679
1.75	0.35713	0.96124	1.90211	2.56640	4.68243
1.70	0.35664	0.96274	1.92654	2.63730	5.15194
1.60	0.35529	0.96577	1.98526	2.81429	6.28410
1.50	0.35334	0.96893	2.06146	3.05194	7.73644
1.40	0.35059	0.97237	2.16219	3.36986	9.65882
1.30	0.34677	0.97638	2.29714	3.79466	12.31255
1.25	0.34436	0.97876	2.38141	4.05942	14.04597
1.20	0.34155	0.98153	2.47962	4.36867	16.16006
1.10	0.33447	0.98885	2.72926	5.16465	22.07139
1.00	0.32492	1.00000	3.07768	6.31375	31.82052
0.90	0.31212	1.01758	3.58052	8.06790	49.41078
0.80	0.29515	1.04553	4.34395	10.95624	85.13934
0.75	0.28478	1.06520	4.88506	13.16324	117.79804
0.70	0.27298	1.09006	5.59179	16.23516	170.55719
0.60	0.24459	1.16210	7.86401	27.43982	429.21914
0.50	0.20889	1.28383	12.7413	57.30403	1559.72610
0.40	0.16463	1.50895	26.45989	173.58545	10812.94044
0.30	0.11086	2.00605	90.39228	1109.35983	273949.61723
0.25	0.08086	2.53608	242.84752	4915.20441	3650601.79435

Chapter 2

ASYMPTOTIC THEORY OF ESTIMATORS

2.1 Introduction

In the literature, we find varied approaches to the problem of estimating the parameters of symmetric stable distributions. Some of these techniques have utilized sample quantiles for estimators (Fama and Roll, 1971). Others have utilized distance-measure approaches for estimation (Press, 1972; Paulson *et al.*, 1975) utilized an empirical characteristic function which was integrated over the real line.

The estimation procedure that we are going to use is based on the work of Crow and Siddiqui (1967) and employs L-estimators. This procedure is for the location parameter. It is used for estimating the location parameter of the stable distribution and can be used for other families of distributions. It is a robust procedure.

Traditionally, the three main concerns of asymptotic theory of a sequence of estimators $\hat{\theta}_n$ of θ are:

1. Consistency: A sequence of estimator $\hat{\theta}_n$ is consistent for a parameter θ if, as $n \to \infty$,

$$\hat{\theta}_n \xrightarrow{P} \theta. \tag{2.1}$$

2. Asymptotic normality: we say that $\hat{\theta}$ is asymptotically normal with parameters $\theta,\ var(\theta)$, if, $n \to \infty$,

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, var(\theta)). \tag{2.2}$$

3. Relative efficiency: Given two sequences of estimators, say $\hat{\theta}_{1,n}$ and $\hat{\theta}_{2,n}$ which are each consistent for θ and also asymptotically normal, the relative efficiency of the first to the second is given by the ratio $var_2(\theta)/var_1(\theta)$. It often does not depend on the parameter θ.

2.2 Construction of Estimators

Let X be a random variable with a df $F(x)$, which has the following properties:

1. $F(x)$ possesses a continuous pdf $f(x)$.

2. The set $\{x : f(x) > 0\} = \{x : -\infty < x < \infty\}$.

3. $f(x)$ is symmetric about zero; i.e., $f(x) = f(-x)$.

4. $f(x)$ has a unique maximum at $x = 0$.

5. $F(x)$ is strictly increasing and has an inverse function, $\xi(q) = F^{-1}(q)$. $0 < q < 1$, which is called the quantile function (qf), and it is uniquely defined.

Now consider a location and scale transformation defined by $Y = \mu + \sigma X$, $\sigma > 0$, $-\infty < \mu < \infty$. We then have $F_Y(y) = F(\frac{y-\mu}{\sigma})$ and $f_Y(y) = \frac{1}{\sigma} f(\frac{y-\mu}{\sigma})$. Therefore $\xi_Y(q) = \mu + \sigma\xi(q)$, and hence $\xi_Y(\frac{1}{2}) = \mu$. Hence we can obtain an asymptotically unbiased estimator of μ.

The existence of moments of random variables is an important issue, especially when stable distributions are under study, because moments of certain orders do not exist for many of these distributions; e.g., neither the mean nor the variance exist for $0 < \alpha \leqslant 1$. It is commonly known that $E(x_{r:n})$ exists whenever $E(x)$ exists. Moreover, Sen (1959) showed that if $E|x|^\delta$ exists for some $\delta > 0$, then $E(x_{r:n}^k)$ exists for all r satisfying $r_0 \leq r \leq n - r + 1$ where $r_0\delta = k$.

Now let $X_1, X_2, ..., X_n$ be independent random variables having a common distribution $F(\frac{x-\mu}{\sigma})$, and let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding order statistics.

If $Y_i = (\frac{X_i-\mu}{\sigma})$, $i = 1,2,...n$, then $Y_{1:n} \leq ... < Y_{n:n}$ are order statistics of a random sample from $f(x)$. Whenever the following quantiles exist. We may denote them as follows: $EY_{i:n} = \beta_i$ and $cov(Y_{i:n}, Y_{j:n}) = V_{ij}$. Then $EX_{i:n} = \mu + \sigma\beta_i$ and $cov(X_{i:n}, X_{j:n}) = V_{ij}\sigma^2$. If the sampled df $F(\frac{x-\mu}{\sigma})$ satisfies conditions (1) through (5) with the symmetry about μ mentioned previously, then if $E|X|^\delta < \infty$ and $\frac{1}{\delta} \leqslant r_0 \leqslant i$, we have $\frac{1}{2}E(X_{i:n} + X_{n-i+1:n}) = \mu$, because of symmetry, which guarantees the existence of $EX_{n-i+1:n}$ whenever $EX_{i:n}$ exists. Therefore, any linear combination $\frac{1}{2}\sum_{i=1}^{k} a_i(X_{i:n} + X_{n-i+1:n})$ is an unbiased estimator of μ provided that $\frac{1}{2}\sum_1^k a_i = 1$.

2.3 Asymptotic Theory of L-Estimators

L-estimator provide good estimators of location and scale parameters. Among such estimators, those which give zero or negligible weight to extreme order statistics have another desirable property: the of being robust against outliers and against some deviations from the assumed form of F.

In general, the exact distribution of an L-estimator is difficult to obtain. We, therefore, look for some asymptotic theory. Asymptotic distribution of a finite number of central order statistics can be obtain using the following theorems:

Theorem 2.3.1: For the q^{th} quantile and any F that has a continuous, nonzero density at $\xi_q = F^{-1}(q)$, let $n_j = [nq_j] + 1$, and $0 < q_1 < .... < q_k < 1$, then

$$\sqrt{n}[x_{n_q} - \xi_q] \xrightarrow{d} N(0, \frac{q(1-q)}{f^2(\xi_q)}); \tag{2.3}$$

see (Staudte and Sheather, 1990).

Theorem 2.3.2: Let $x_1, x_2, ..., x_n$ be random sample from a continuous distribution with pdf $f(x)$, and let x_{n_j}, $j = 1,2,...,k$ be the sample quantiles where $n_j = [nq_j] + 1$ and $0 < q_1 < .... < q_k < 1$. If $0 < f(\xi_{q_j}) < \infty$ then the asymptotic joint distribution of

$$\sqrt{n}(x_{n_1} - \xi_{q_1}), ..., \sqrt{n}(x_{n_k} - \xi_{q_k}) \tag{2.4}$$

is k-dimensional normal with zero mean vector and covariance matrix

$$[\frac{q_i(1-q_j)}{f(\xi_{q_i})f(\xi_{q_j})}] \quad i \leq j;$$

where the ξ_{q_j} is the population quantile corresponding to x_{n_j} (David, 1981).

The joint normality is of great use in studying the asymptotic properties of estimators of location or scale parameters which are a linear function of a finite number of order statistics.

2.3.1 Asymptotic Theory of Trimmed Mean

An important L statistics is the trimmed mean, where either upper or lower or both extremes are deleted.

$$T_{(\phi,\lambda)}(F) = \int_{F^{-1}(\phi)}^{F^{-1}(1-\lambda)} \frac{xdF(x)}{1-\phi-\lambda} \tag{2.5}$$

This is a measure of location. If $0 < \phi = \lambda < 1/2$, then it is called the $2\lambda-$trimmed mean(or symmetric trimmed mean). $T_{2\lambda}(F_n)$ is the average of the observations remaining after trimming the smallest and largest $[\lambda n]$.

Theorem 2.3.3: Let F be continuous at the unique quantiles $x_\lambda = F^{-1}(\lambda)$, and $x_{1-\lambda} = F^{-1}(1-\lambda)$, and that F is without flat spots at the trimming points,

$$\sqrt{n}[T_{2\lambda}(F_n) - T_{2\lambda}(F)] \xrightarrow{d} N(0, var(T_{(2\lambda,F)})), \tag{2.6}$$

where $Var(T_{(2\lambda,F)}) = E[IF^2_{\lambda,F}(X)]$; and the influence function is given for symmetric F by:

$$(1-2\lambda)IF_{T_{2\lambda},F}(x) = \begin{cases} F^{-1}(\lambda) - W_{2\lambda}(F), & x < F^{-1}(\lambda) \\ x - W_{2\lambda}(F), & F^{-1}(\lambda) \leq x \leq F^{-1}(1-\lambda) \\ F^{-1}(1-\lambda) - W_{2\lambda}(F), & F^{-1}(1-\lambda) < x, \end{cases} \tag{2.7}$$

where $W_{2\lambda}(F)$ is the 2λ-Winsorized mean:

$$W_{2\lambda}(F) = (1-2\lambda)T_{2\lambda}(F) + \lambda F^{-1}(\lambda) + \lambda F^{-1}(1-\lambda). \tag{2.8}$$

See Staudte and Sheather (1990).

Example 2.3.1: Let $x_1, ..., x_n$ be a random sample from $N(\mu, \sigma^2)$, $0 < \lambda \leq 1/2$, and let $F^{-1}(\lambda) = \mu - b\sigma$. Then

$$T_{2\lambda}(F_n) \overset{approx.}{\backsim} N(\mu, var_1(2\lambda, F)), \tag{2.9}$$

where

$$var_1(T_{(2\lambda,F)}) = \frac{1}{n(1-2\lambda)^2}[2b^2\sigma^2\Phi(-b) + \frac{\sigma^2}{\sqrt{2\pi}}\int_{-b}^{b} t^2 e^{-t^2/2}dt]. \tag{2.10}$$

Example 2.3.2: Let $x_1, ..., x_n$ be a random sample from Cauchy(α, β), $0 < \lambda \leq 1/2$, and let $F^{-1}(\lambda) = \alpha + \beta\tan(\pi(\lambda - \frac{1}{2}))$. Then

$$T_{2\lambda}(F_n) \overset{approx.}{\backsim} N(\alpha, var_2(2\lambda, F)), \tag{2.11}$$

where

$$var_2(T_{(2\lambda,F)}) = \frac{2\beta^2}{n(1-2\lambda)^2}[\lambda\tan^2(\pi(\frac{1}{2} - \lambda)) + \frac{1}{\pi}\tan(\pi(\frac{1}{2} - \lambda)) + \lambda - \frac{1}{2}]. \tag{2.12}$$

Equations (2.10) and (2.12) are the only two closed forms in our study. When $\alpha \neq 2$ or 1, then the asymptotic variance of the trimmed mean will be evaluated using numerical integration of the stable distributions over an appropriate region using

$$\begin{aligned} var(T_{(2\lambda,F)}) = \quad & \frac{2\lambda c^2}{(1-2\lambda)^2} + \frac{2c^2}{\pi(1-2\lambda)^2} \\ & \int_0^{\infty} e^{-(y/c)^\alpha}[\frac{\sin y}{y} + 2(\frac{y\cos y - \sin y}{y^3})]dy, \end{aligned} \tag{2.13}$$

where $c = F^{-1}(1-\lambda)$ (Badahdah and Siddiqui, 1991).

Table 2.1: Asymptotic Variance of Trimmed Mean
n($var(T_{(2\lambda,F)})$)

λ / α	0.05	0.10	0.20	0.25	0.30	0.40	0.45
2.00	2.0525	2.1208	2.2894	2.3904	2.5042	2.7796	2.9473
1.75	2.3161	2.2746	2.3478	2.4213	2.5142	2.7613	2.9208
1.50	2.8856	2.5539	2.4271	2.4496	2.5052	2.7026	2.8468
1.25	4.3159	3.1647	2.5622	2.4797	2.4622	2.5661	2.6815
1.00	8.7726	4.7715	2.8722	2.5465	2.3703	2.2827	2.3408
0.75	32.3507	10.6846	3.7770	2.7975	2.2472	1.7493	1.6877
0.50	527.0943	64.2874	7.6748	3.8534	2.1974	0.9329	0.6919

2.3.2 Asymptotic Theory of Selector Statistics

In section 1.1 it was mentioned that $\Delta_1 = \frac{x_{.95}-x_{.05}}{x_{.75}-x_{.25}}$ would be used as a measure of tail thickness of the distribution under consideration after the data are observed. In general, the distribution of $\Delta = \frac{x_{1-q_1}-x_{q_1}}{x_{1-q_2}-x_{q_2}}$ will be derived. Since the sample quantiles are asymptotically normally distributed, the discussion on asymptotic theory of Δ will primarily utilize the Taylor series expansion.

Theorem 2.3.4: Let $\mathbf{Y_n} = \mathbf{a} + O_p(r_n)$, where $\mathbf{Y_n}$ is a p x 1 random vector, a is a p x 1 vector of constant, and $r_n \rightarrow 0$ as $n \rightarrow \infty$, and let

$$E\{\mathbf{Y_n}\} = \mathbf{a},$$

$$E\{(\mathbf{Y_n} - \mathbf{a})(\mathbf{Y_n} - \mathbf{a})'\} = \Sigma_\mathbf{n},$$

where $0 < |\Sigma_\mathbf{n}| < \infty$.

Then the asymptotic variance of $g(\mathbf{Y_n}) = \mathbf{d}'\mathbf{Y_n}$, to order r_n^3 is

$$\bar{E}\{(g(\mathbf{Y_n}) - g(\mathbf{a}))^2\} = \mathbf{d}\Sigma_\mathbf{n}\mathbf{d}' + O_p(r_n^3),$$

where $\mathbf{d}$ is a 1 x p vector with typical element $d_j = \frac{\partial g(\mathbf{a})}{\partial y_j}$. (The notation, $\bar{E}$, should be interpreted to mean that $(g(\mathbf{Y_n}) - g(\mathbf{a}))^2$ can be written as the sum of two random variables, say $\mathbf{X_n}$ and $\mathbf{Z_n}$, where $E\{\mathbf{X_n}\} = \mathbf{d}\Sigma_\mathbf{n}\mathbf{d}'$ and $\mathbf{Z_n} = O_p(r_n^3)$. This does not necessarily mean that $E\{(g(\mathbf{Y_n}) - g(\mathbf{a}))^2\}$ exists for any finite n (Wolter, 1985).

This reduces the problem to finding the distribution of some functions, namely Δ, of normal random variables. In general, let

$$\Delta = H(x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4}) = (x_{q_4} - x_{q_1})(x_{q_3} - x_{q_2})^{-1}, \tag{2.14}$$

where $0 < q_1 < q_2 < 1/2$, $q_4 = 1 - q_1, q_3 = 1 - q_2$ and x_q is the q^{th} sample quantile, using a Taylor series expansion to approximate the moments of Δ around the point $\mathbf{t_n} = (\xi_{q_1}, \xi_{q_2}, \xi_{q_3}, \xi_{q_4})$. Since $(\xi_{q_3} - \xi_{q_2})$ is greater than zero, the Taylor series expansion of H is appropriate, and is given by

$$\begin{aligned} H(x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4}) &= H(\xi_{q_1}, \xi_{q_2}, \xi_{q_3}, \xi_{q_4}) + [\sum_{i=1}^{4}(x_{q_i} - \xi_{q_i})\frac{\partial H}{\partial x_{q_i}}] \\ &+\frac{1}{2!}[\sum_{i=1}^{4}(x_{q_i} - \xi_{q_i})^2\frac{\partial^2 H}{\partial x_{q_i}^2} + 2\sum_{i=1}^{4}\sum_{j>i}^{4}(x_{q_i} - \xi_{q_i}) \\ &(x_{q_j} - \xi_{q_j})\frac{\partial^2 H}{\partial x_{q_i}\partial x_{q_j}}] + \frac{1}{3!}[\sum_{i=1}^{4}(x_{q_i} - \xi_{q_i})^3\frac{\partial^3 H}{\partial x_{q_i}^3} \\ &+3\sum_{i=1}^{4}\sum_{j>i}^{4}(x_{q_i} - \xi_{q_i})^2(x_{q_j} - \xi_{q_j})\frac{\partial^3 H}{\partial x_{q_i}^2\partial x_{q_j}} \\ &+3\sum_{i=1}^{4}\sum_{j>i}^{4}(x_{q_i} - \xi_{q_i})(x_{q_j} - \xi_{q_j})^2\frac{\partial^3 H}{\partial x_{q_i}\partial x_{q_j}^2}] \\ &+\ldots \end{aligned} \tag{2.15}$$

Then we evaluate the partial derivatives at $x_{q_i} = \xi_{q_i}$ for $i = 1, 2, 3, 4$ and take expectation of both sides of Equation (2.15). So, from the first expansion we obtain

$$\begin{aligned} EH(x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4}) &= E[\frac{\xi_{q_4} - \xi_{q_1}}{\xi_{q_3} - \xi_{q_2}} + \frac{x_{q_4} - \xi_{q_4}}{\xi_{q_3} - \xi_{q_2}} - \frac{x_{q_1} - \xi_{q_1}}{\xi_{q_3} - \xi_{q_2}} - \frac{(\xi_{q_4} - \xi_{q_1})}{(\xi_{q_3} - \xi_{q_2})^2} \\ &[x_{q_3} - \xi_{1-q_2}] + \frac{(\xi_{q_4} - \xi_{q_1})(x_{q_2} - \xi_{q_2})}{(\xi_{q_3} - \xi_{q_2})^2}] \\ &+o_p(\frac{1}{n^{1/2}}). \end{aligned} \tag{2.16}$$

Now observe,

$$E[\frac{x_{1-q_1} - x_{q_1}}{x_{1-q_2} - x_{q_2}} - \frac{\xi_{1-q_1} - \xi_{q_1}}{\xi_{1-q_2} - \xi_{q_2}}] \to 0 \tag{2.17}$$

as $n \to \infty$. The asymptotic variance of Δ is obtained by the following:

$$
\begin{aligned}
var(H(x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4})) &= \frac{1}{(\xi_{q_3} - \xi_{q_2})^2}[\sigma^2_{x_{q_4}} + \sigma^2_{x_{q_1}} - 2cov(x_{q_4}, x_{q_1})] \\
&+ (\frac{\xi_{q_4} - \xi_{q_1}}{(\xi_{q_3} - \xi_{q_2})^2})^2[\sigma^2_{x_{q_3}} + \sigma^2_{x_{q_2}} - 2cov(x_{q_3}, x_{q_2})] \\
&- 2(\frac{\xi_{q_4} - \xi_{q_1}}{(\xi_{q_3} - \xi_{q_2})^3})[cov(x_{q_4}, x_{q_3}) - cov(x_{q_4}, x_{q_2}) \\
&- cov(x_{q_1}, x_{q_3}) + cov(x_{q_1}, x_{q_2})] + o_p(\frac{1}{n^{\frac{3}{2}}}). \quad (2.18)
\end{aligned}
$$

Before we derive the distribution of Δ, we prove the following theorems:

Theorem 2.3.5: Let $x_1, x_2, ..., x_n$ be a random sample with common density $f(x)$ where $f(x)$ is symmetric. For $0 < q_1 < q_2 < 1/2$, the asymptotic correlation coefficient

1. $corr(x_{1-q_1} + x_{q_1}, x_{1-q_2} + x_{q_2}) \simeq \sqrt{q_1/q_2}$ (2.19)
2. $corr(x_{1-q_1} - x_{q_1}, x_{1-q_2} - x_{q_2}) \simeq \sqrt{\frac{q_1(1-2q_2)}{q_2(1-2q_1)}}.$ (2.20)

Proof: Since 1 and 2 are similar, we will prove 2 only. We know that

$$x_{nq_i:n} \overset{approx.}{\sim} N(\xi_{q_i}, \frac{q_i(1-q_i)}{nf^2(\xi_{q_i})}),$$

and the

$$cov(x_{nq_i:n}, x_{nq_j:n}) \simeq \frac{q_i(1-q_j)}{nf^2(\xi_{q_i})f^2(\xi_{q_j})}, \quad q_i < q_j.$$

and by symmetry $f(a) = f(-a)$, then

$$
\begin{aligned}
corr(x_{1-q_1} - x_{q_1}, x_{1-q_2} - x_{q_2}) &= \frac{cov(x_{1-q_1} - x_{q_1}; x_{1-q_2} - x_{q_2})}{[var(x_{1-q_1} - x_{q_1})var(x_{1-q_2} - x_{q_2})]^{\frac{1}{2}}} \\
&\simeq \frac{[2q_1(1-2q_2)]/[nf(\xi_{q_1})f(\xi_{q_2})]}{[[4q_1q_2(1-2q_1)(1-2q_2)]/[n^2f^2(\xi_{q_1})f^2(\xi_{q_2})]]^{\frac{1}{2}}} \\
&= \frac{2q_1(1-2q_2)}{\sqrt{4q_1q_2(1-2q_1)(1-2q_2)}} \\
&= \sqrt{\frac{q_1(1-2q_2)}{q_2(1-2q_1)}}. \quad (2.21)
\end{aligned}
$$

Since this feature does not depend on the distribution function, it is a good way to check the accuracy of the simulated function of symmetric distributions. For example, $corr(x_{0.95}+x_{0.05}, x_{0.75}+x_{0.25}) \simeq \sqrt{0.2}$ and $corr(x_{0.95}-x_{0.05}, x_{0.75}-x_{0.25}) \simeq \frac{1}{3}$.

Theorem 2.3.6:

Let $x_1, x_2, ..., x_n$ be a random sample from a symmetric distribution function F and let $0 < q_1 < q_2 < 1/2$ under the conditions stated in Theorems 2.3.1 and 2.3.4

$$\sqrt{n}[\frac{x_{1-q_1} - x_{q_1}}{x_{1-q_2} - x_{q_2}} - \frac{\xi_{1-q_1} - \xi_{q_1}}{\xi_{1-q_2} - \xi_{q_2}}] \xrightarrow{d} N(0, var(\Delta)), \tag{2.22}$$

where

$$\begin{aligned} var(\Delta) &= (\xi_{1-q_2} - \xi_{q_2})^{-2}[\frac{2q_1(1-2q_1)}{f^2(\xi_{q_1})}] + (\frac{(\xi_{1-q_1} - \xi_{q_1})}{(\xi_{1-q_2} - \xi_{q_2})^2})^2[\frac{2q_2(1-2q_2)}{f^2(\xi_{q_2})}] \\ &\quad -2(\frac{(\xi_{1-q_1} - \xi_{q_1})}{(\xi_{1-q_2} - \xi_{q_2})^3})[\frac{2q_1(1-2q_2)}{f(\xi_{q_1})f(\xi_{q_2})}]. \end{aligned} \tag{2.23}$$

Proof: Follows directly from Taylor series theorem.

Example 2.3.3: Let $x_1, ..., x_n$ be a random sample from $N(\mu, \sigma^2)$, and $0 < q_1 < q_2 < 1/2$. Let $F^{-1}(q_i) = \mu - a_i\sigma$, $i = 1, 2$, then

$$\Delta \overset{approx.}{\backsim} N(\mu_1, var_1(\Delta)), \tag{2.24}$$

where

$$\mu_1 = \frac{a_1}{a_2}, \tag{2.25}$$

and

$$\begin{aligned} var_1(\Delta) &= \frac{\pi}{na_2^4}[q_1(1-2q_1)a_2^2\exp(a_1^2) + q_2(1-2q_2)a_1^2\exp(a_2^2) \\ &\quad -2a_1a_2q_1(1-2q_2)\exp(\frac{1}{2}(a_1^2+a_2^2))]. \end{aligned} \tag{2.26}$$

Example 2.3.4: Let $x_1, ..., x_n$ be a random sample from Cauchy(α, β) and $0 < q_1 < q_2 < 1/2$. Let $F^{-1}(q_i) = \alpha + \beta\tan(\pi(q_i - \frac{1}{2}))$, $i = 1, 2$, then

$$\Delta \overset{approx.}{\backsim} N(\mu_2, var_2(\Delta)), \tag{2.27}$$

where

$$\mu_2 = \frac{\tan(\pi(\frac{1}{2} - q_1))}{\tan(\pi(\frac{1}{2} - q_2))}, \tag{2.28}$$

and

$$\begin{aligned} var_2(\Delta) \simeq\ & \frac{\pi^2}{2n\tan^4(\pi(\frac{1}{2} - q_2))}[q_1(1-2q_1)\tan^2(\pi(\frac{1}{2} - q_2))[1+\tan^2(\pi(\frac{1}{2} - q_1))]^2 \\ & + q_2(1-2q_2)\tan^2(\pi(\frac{1}{2} - q_1))[1+\tan^2(\pi(\frac{1}{2} - q_2))]^2 \\ & -2q_1(1-2q_2)\tan(\pi(\frac{1}{2} - q_1))\tan(\pi(\frac{1}{2} - q_2))[1+\tan^2(\pi(\frac{1}{2} - q_1))] \\ & [1+\tan^2(\pi(\frac{1}{2} - q_2))]]. \end{aligned} \tag{2.29}$$

Since the asymptotic distributions are free from the location and scale parameters, replace q_1 and q_2 by their values to get the mean and variance; e.g., if $q_1 = 0.05$ and $q_2 = 0.25$, then $\mu_1 = 2.43866, \mu_2 = 6.31375, var_1(\Delta) \simeq \frac{8.65134}{n}$ and $var_2(\Delta) \simeq \frac{341.85130}{n}$.

2.3.3 Improving the Sampling Distribution of Selector Statistics

For a small to moderate sample size, the sampling distribution of Δ needs to be improved. Using the second expansion of Taylor series, we can approximate the sampling distribution of Δ by a general 2-parameter gamma distribution, matching the first two moments of Δ to the first two moments of the gamma. Taking the expectation of both sides of Equation (2.15), we get:

$$\begin{aligned} EH(x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4}) =\ & \frac{\xi_{q_4} - \xi_{q_1}}{\xi_{q_3} - \xi_{q_2}} + \frac{\xi_{q_4} - \xi_{q_1}}{(\xi_{q_3} - \xi_{q_2})^3}[\sigma^2_{x_{q_3}} + \sigma^2_{x_{q_2}} - 2cov(x_{q_3}, x_{q_2})] \\ & - \frac{1}{(\xi_{q_3} - \xi_{q_2})^2}[cov(x_{q_4}, x_{q_3}) - cov(x_{q_4}, x_{q_2}) - cov(x_{q_1}, x_{q_3}) \\ & + cov(x_{q_1}, x_{q_2})] + o_p(\frac{1}{n^{\frac{3}{2}}}). \end{aligned} \tag{2.30}$$

The asymptotic variance of Δ is obtained by the following:

$$\begin{aligned}
Var(H(x_{q_1}, x_{q_2}, x_{q_3}, x_{q_4})) &= \frac{1}{(\xi_{q3}-\xi_{q2})^2}[\sigma^2_{x_{q4}} + \sigma^2_{x_{q1}} - 2cov(x_{q_4}, x_{q_1})] \\
&+ (\frac{\xi_{q4}-\xi_{q1}}{(\xi_{q3}-\xi_{q2})^2})^2[\sigma^2_{x_{q3}} + \sigma^2_{x_{q2}} - 2cov(x_{q_3}, x_{q_2})] \\
&-2(\frac{\xi_{q4}-\xi_{q1}}{(\xi_{q3}-\xi_{q2})^3})[cov(x_{q_4}, x_{q_3}) - cov(x_{q_4}, x_{q_2}) \\
&-cov(x_{q_1}, x_{q_3}) + cov(x_{q_1}, x_{q_2})] \\
&-(\frac{\xi_{q4}-\xi_{q1}}{(\xi_{q3}-\xi_{q2})^3})^2[\sigma^2_{x_{q3}} + \sigma^2_{x_{q2}} - 2cov(x_{q_3}, x_{q_2})]^2 \\
&-(\frac{1}{(\xi_{q3}-\xi_{q2})^4})[cov(x_{q_4}, x_{q_3}) - cov(x_{q_4}, x_{q_2}) \\
&-cov(x_{q_1}, x_{q_3}) + cov(x_{q_1}, x_{q_2})]^2 \\
&+2(\frac{\xi_{q4}-\xi_{q1}}{(\xi_{q3}-\xi_{q2})^5})[\sigma^2_{x_{q3}} + \sigma^2_{x_{q2}} - 2cov(x_{q_3}, x_{q_2})] \\
&[cov(x_{q_4}, x_{q_3}) - cov(x_{q_4}, x_{q_2}) - cov(x_{q_1}, x_{q_3}) \\
&+cov(x_{q_1}, x_{q_2})] + o_p(\frac{1}{n^{\frac{5}{2}}}). \qquad (2.31)
\end{aligned}$$

Theorem 2.3.7:

Let $x_1, x_2, ..., x_n$ be a random sample from a symmetric distribution function F, and let $0 < q_1 < q_2 < 1/2$, under the conditions stated in Theorems 2.3.1 and 2.3.4. Δ follows approximately gamma distribution with (r, η),

$$f_\Delta(y) = \frac{\eta^r}{\Gamma(r)} y^{r-1} exp^{-\eta y} I_{(0,\infty)}(y), \qquad (2.32)$$

where $r = \frac{E^2(\Delta)}{Var(\Delta)}$ is the shape parameter and $\eta = \frac{E(\Delta)}{Var(\Delta)}$ is the rate parameter. Also, using symmetry property of $f(x) = f(-x)$, we have

$$\begin{aligned}
E(\Delta) &\simeq \frac{\xi_{q4}-\xi_{q1}}{\xi_{q3}-\xi_{q2}} + \frac{\xi_{q4}-\xi_{q1}}{(\xi_{q3}-\xi_{q2})^3}[\frac{2q_2(1-2q_2)}{nf^2(\xi_2)}] \\
&- \frac{1}{(\xi_{q3}-\xi_{q2})^2}[\frac{2q_1(1-2q_2)}{nf(\xi_{q1})f(\xi_{q2})}], \qquad (2.33)
\end{aligned}$$

and

$$
\begin{aligned}
var(\Delta) \;\simeq\;& \frac{1}{(\xi_{q_3}-\xi_{q_2})^2}[\frac{2q_1(1-2q_1)}{nf^2(\xi_1)}] + (\frac{\xi_{q_4}-\xi_{q_1}}{(\xi_{q_3}-\xi_{q_2})^2})^2[\frac{2q_2(1-2q_2)}{nf^2(\xi_2)}] \\
& -2\frac{\xi_{q_4}-\xi_{q_1}}{(\xi_{q_3}-\xi_{q_2})^3}[\frac{2q_1(1-2q_2)}{nf(\xi_{q_1})f(\xi_{q_2})}] - (\frac{\xi_{q_4}-\xi_{q_1}}{(\xi_{q_3}-\xi_{q_2})^3})^2[\frac{2q_2(1-2q_2)}{nf^2(\xi_2)}]^2 \\
& -\frac{1}{(\xi_{q_3}-\xi_{q_2})^4}[\frac{2q_1(1-2q_2)}{nf(\xi_{q_1})f(\xi_{q_2})}]^2 + 2\frac{\xi_{q_4}-\xi_{q_1}}{(\xi_{q_3}-\xi_{q_2})^5}[\frac{2q_2(1-2q_2)}{nf^2(\xi_2)}] \\
& [\frac{2q_1(1-2q_2)}{nf(\xi_{q_1})f(\xi_{q_2})}] + o_p(\frac{1}{n^{\frac{5}{2}}}). \qquad (2.34)
\end{aligned}
$$

Proof: Follows directly from Taylor series theorem.

Example 2.3.5: Let $x_1, ..., x_n$ be a random sample from $N(\mu, \sigma^2)$, and $0 < q_1 < q_2 < 1/2$. Supposing $F^{-1}(q_i) = \mu - a_i\sigma$, then

$$\Delta \overset{approx.}{\backsim} Gamma(r_1, \eta_1), \qquad (2.35)$$

where

$$\frac{r_1}{\eta_1} = \frac{a_1}{a_2} + \frac{\pi(1-2q_2)[q_2 a_1 \exp(a_2^2) - q_1 a_2 \exp(\frac{1}{2}(a_1^2+a_2^2))]}{na_2^3}, \qquad (2.36)$$

and

$$
\begin{aligned}
\frac{r_1}{\eta_1^2} \;=\;& \frac{\pi}{na_2^4}[q_1(1-2q_1)a_2^2\exp(a_1^2) + q_2(1-2q_2)a_1^2\exp(a_2^2) \\
& -2a_1a_2q_1(1-2q_2)\exp(\frac{1}{2}(a_1^2+a_2^2))] - \frac{\pi^2(1-2q_2)^2}{n^2a_2^4} \\
& [\frac{a_1^2q_2^2}{a_2^2}\exp(2a_2^2) + q_1^2\exp(a_1^2+a_2^2) - \frac{2a_1q_1q_2}{a_2}\exp(\frac{a_1^2+3a_2^2}{2})]. \qquad (2.37)
\end{aligned}
$$

Example 2.3.6: Let $x_1, ..., x_n$ be a random sample from Cauchy(α, β) and $0 < q_1 < q_2 < 1/2$. Supposing $F^{-1}(q_i) = \alpha + \beta\tan(\pi(q_i - \frac{1}{2}))$, then

$$\Delta \overset{approx.}{\backsim} Gamma(r_2, \eta_2), \qquad (2.38)$$

where

$$
\begin{aligned}
\frac{r_2}{\eta_2} \;=\;& \frac{\tan(\pi(\frac{1}{2}-q_1))}{\tan(\pi(\frac{1}{2}-q_2))} + \frac{\pi^2(1-2q_2)[1+\tan^2(\pi(\frac{1}{2}-q_2))]}{2n\tan^2(\pi(\frac{1}{2}-q_2))} \\
& \{\frac{\tan(\pi(\frac{1}{2}-q_1))}{\tan(\pi(\frac{1}{2}-q_2))}q_2[1+\tan^2(\pi(\frac{1}{2}-q_2))] \\
& -q_1[1+\tan^2(\pi(\frac{1}{2}-q_1))]\}, \qquad (2.39)
\end{aligned}
$$

and

$$\frac{r_2}{\eta_2^2} = \frac{\pi^2}{2n\tan^4(\pi(\frac{1}{2}-q_2))}\{q_1(1-2q_1)\tan^2(\pi(\frac{1}{2}-q_2))[1+\tan^2(\pi(\frac{1}{2}-q_1))]^2$$
$$+q_2(1-2q_2)\tan^2(\pi(\frac{1}{2}-q_1))[1+\tan^2(\pi(\frac{1}{2}-q_2))]^2$$
$$-2q_1(1-2q_2)\tan(\pi(\frac{1}{2}-q_1))\tan(\pi(\frac{1}{2}-q_2))[1+\tan^2(\pi(\frac{1}{2}-q_1))]$$
$$[1+\tan^2(\pi(\frac{1}{2}-q_2))]\} - \frac{\pi^4(1-2q_2)^2[1+\tan^2(\pi(\frac{1}{2}-q_2))]^2}{4n^2\tan^4(\pi(\frac{1}{2}-q_2))}$$
$$\{\frac{q_2^2\tan^2(\pi(\frac{1}{2}-q_1))}{\tan^2(\pi(\frac{1}{2}-q_2))}[1+\tan^2(\pi(\frac{1}{2}-q_2))]^2 + q_1^2[1+\tan^2(\pi(\frac{1}{2}-q_1))]^2$$
$$-\frac{2q_1q_2\tan(\pi(\frac{1}{2}-q_1))}{\tan(\pi(\frac{1}{2}-q_2))}[1+\tan^2(\pi(\frac{1}{2}-q_1))][1+\tan^2(\pi(\frac{1}{2}-q_2))]\}. \quad (2.40)$$

The asymptotic distributions are free from location and scale parameters. Consequently, replace q_1 and q_2 by their values to get the mean and variance; e.g., if $q_1 = 0.05$, and $q_2 = 0.25$, then $\frac{r_1}{\eta_1} = 2.43866 + 2.47934/n$, $\frac{r_2}{\eta_2} = 6.31375 + 5.49590/n$, $\frac{r_1}{\eta_1^2} = 8.65134/n - 6.14714/n^2$, and $\frac{r_2}{\eta_2^2} = 341.851/n - 30.20495/n^2$.

Table 2.2 shows $F^{-1}(0.95)$ of the Δ_1 distributions. Figures 2.1 to 2.5 show distributions of Δ_1 and the simulated empirical gamma CDFs of sampling distribution of Δ_1 for some α, Where $N = 5000$ and $n = 20$.

Table 2.2: $F^{-1}(0.95)$ of the Δ_1 Distributions

α / n	2	1.75	1.50	1.25	1	0.75	0.5
10	4.32	4.93	6.54	10.10	18.31	44.88	221.71
20	3.71	4.24	5.45	8.14	14.50	34.97	171.74
30	3.46	3.93	4.97	7.35	12.85	30.15	147.84
40	3.30	3.70	4.72	6.88	11.87	27.65	131.70

Figure 2.1: Densities of Δ_1 When $\alpha = 2, 1.75, 1.5, 1.25, 1$ and $n = 40$.

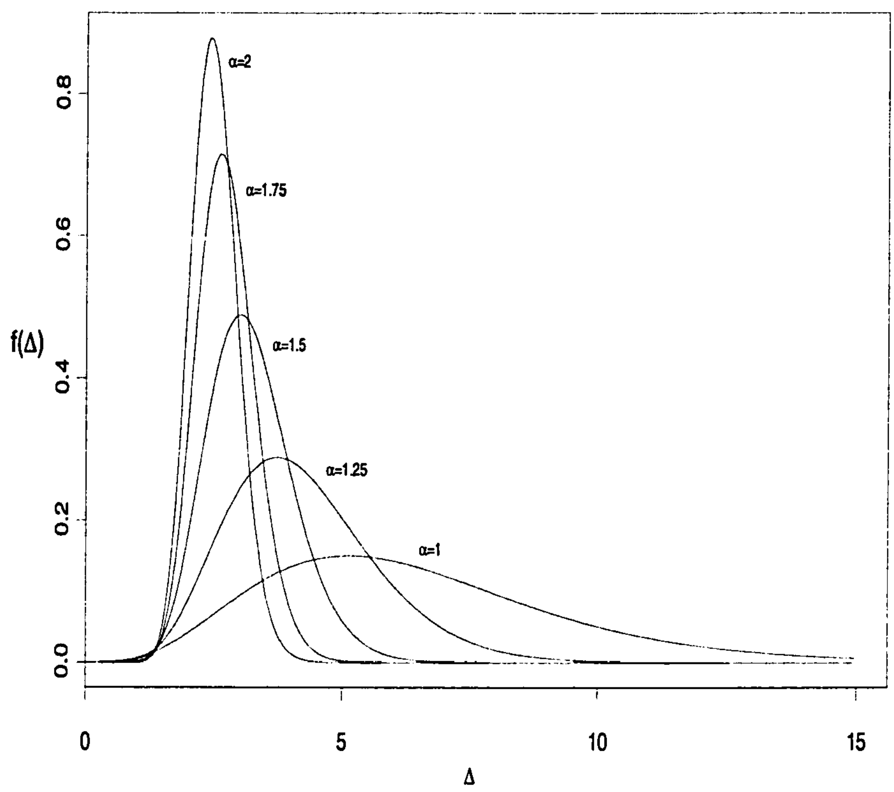


Figure 2.2: Simulation of Sampling Distribution of Δ_1 When $\alpha = 2$

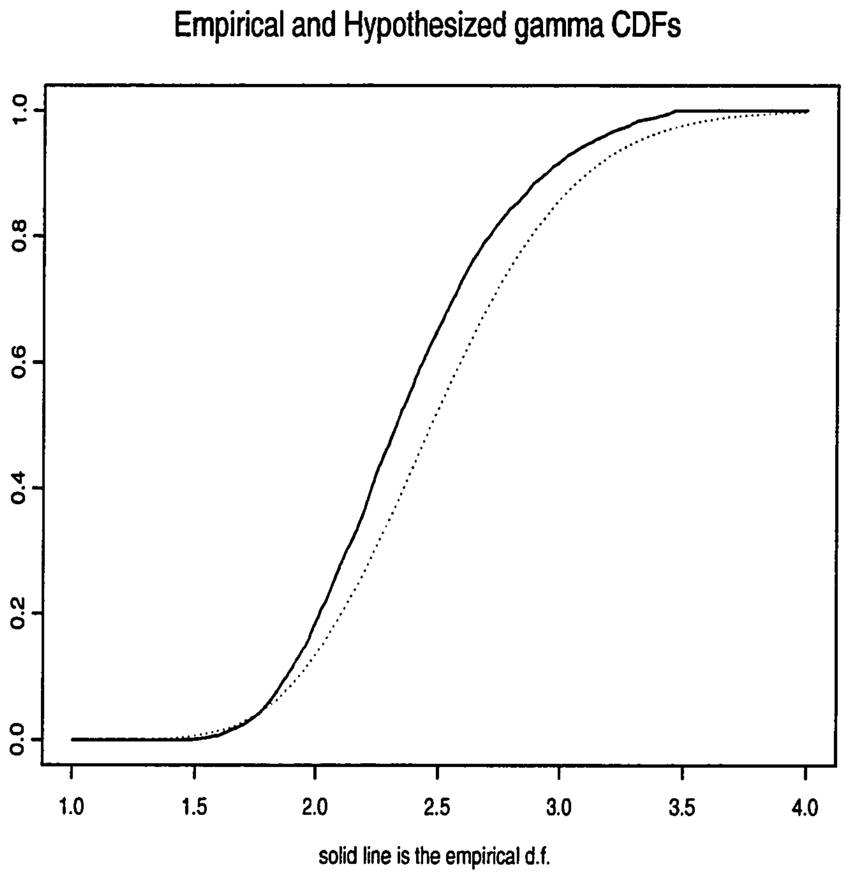


Figure 2.3: Simulation of Sampling Distribution of Δ_1 When $\alpha = 1.5$

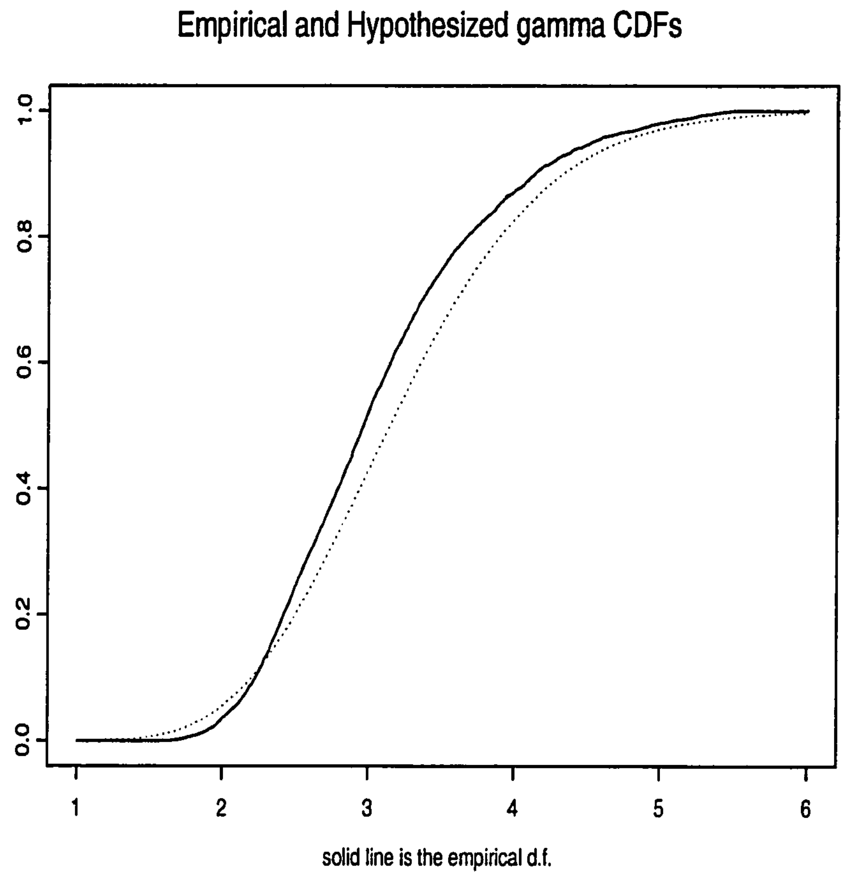


Figure 2.4: Simulation of Sampling Distribution of Δ_1 When $\alpha = 1$

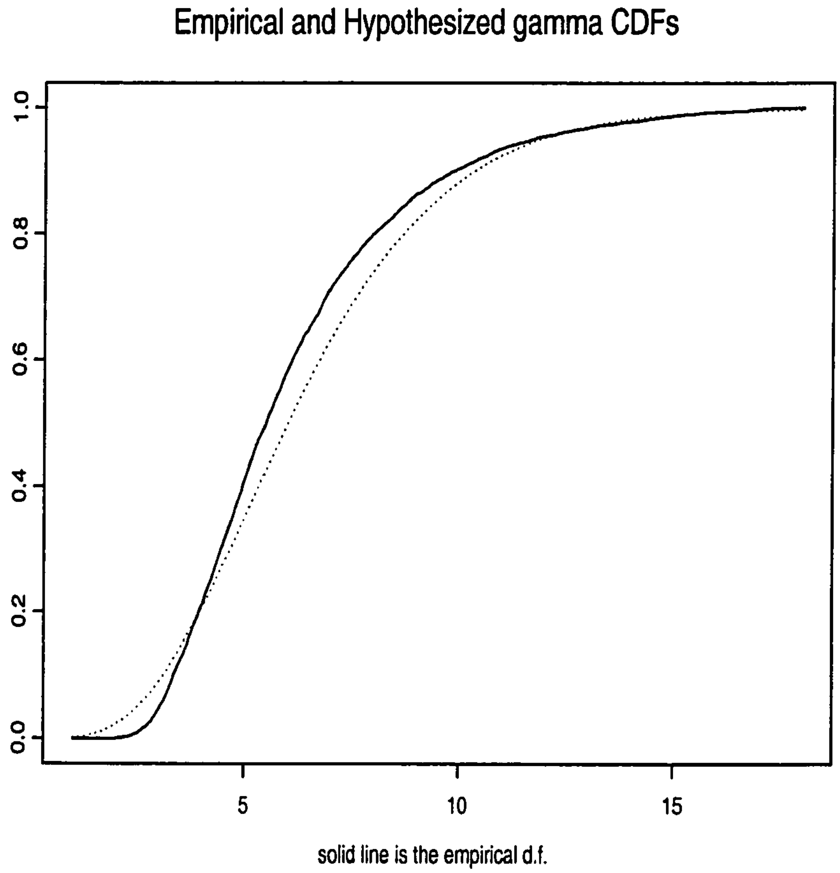
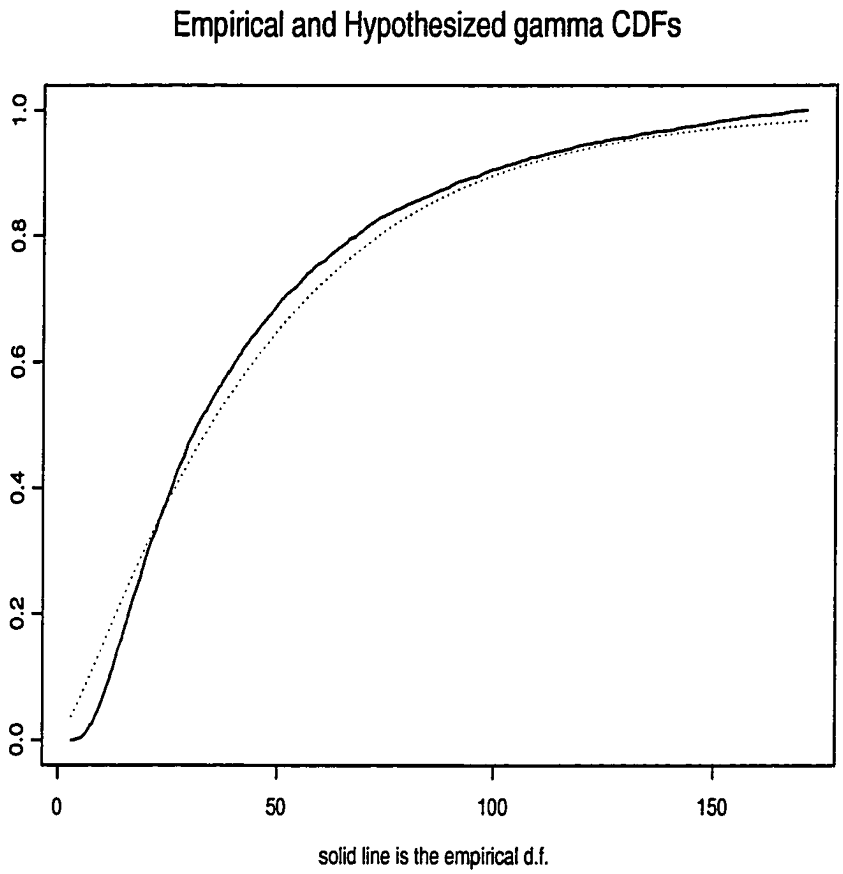


Figure 2.5: Simulation of Sampling Distribution of Δ_1 When $\alpha = 0.5$



2.4 Asymptotic Distribution of Ratio of Trimmed Mean and Central Range

One popular robust estimator of the center of a symmetric distribution is the symmetric trimmed mean. In section 2.3.1 we saw that the asymptotic distribution of trimmed mean is normal with mean θ_1 and variance σ_1^2. Also, we saw that the asymptotic distribution of the $(x_{1-q} - x_q)$ is normal with mean θ_2 and variance σ_2^2. Now consider the ratio of 2λ–trimmed mean and $(x_{1-q} - x_q)$, where $0 < \lambda < 1/2$ and $0 < q < 1/2$. Let

$$V = \frac{T_{2\lambda}(F_n)}{x_{1-q} - x_q}, \tag{2.41}$$

or

$$V = \frac{T_{2\lambda_1}(F_n(x)) - T_{2\lambda_2}(F_n(y))}{\frac{1}{2}[(x_{1-q_1} - x_{q_1}) + (y_{1-q_2} - y_{q_2})]}, \tag{2.42}$$

used as test statistics for location parameters. V is a ratio of two asymptotic normal variables. The distribution of the ratio of two normal random variables is derived by Hinkley (1969). Applying this distribution to V, we obtain the following theorem but first notice:

Lemma 2.4.1: Let $x_1, x_2, ..., x_n$ be random sample with common density $f(x)$ where $f(x)$ is symmetric. For $0 < q < 1/2$, and $0 < \lambda < 1/2$ the asymptotic covariance of:

$$Cov(T_{2\lambda}(F_n), (x_{1-q} - x_q)) \simeq 0. \tag{2.43}$$

Proof: See Siddiqui and Butler (1969).

Theorem 2.4.1:

Let $x_1, x_2, ..., x_n$ be a random sample from a symmetric distribution function F and let $0 < q < 1/2$ and $0 < \lambda < 1/2$. The pdf of V is

$$f_V(v) = \frac{b(v)d(v)}{\sqrt{(2\pi)}\sigma_1\sigma_2 a^3(v)}[\Phi\{\frac{b(v)}{a(v)}\} - \Phi\{-\frac{b(v)}{a(v)}\}] + \frac{1}{\pi\sigma_1\sigma_2 a^2(v)}exp\{-\frac{c}{2}\}, \tag{2.44}$$

where

$$a(v) = (\frac{v^2}{\sigma_1^2} + \frac{1}{\sigma_2^2})^{\frac{1}{2}},$$

$$b(v) = \frac{\theta_1 v}{\sigma_1^2} + \frac{\theta_2}{\sigma_2^2},$$

$$c = \frac{\theta_1^2}{\sigma_1^2} + \frac{\theta_2^2}{\sigma_2^2},$$

$$d(v) = exp\{\frac{b^2(v) - ca^2(v)}{2a^2(v)}\}.$$

Also,

$$\Phi(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{(2\pi)}} e^{-y^2/2} dy.$$

Note that as $\theta_2/\sigma_2 \to \infty$, i.e., as $P\{(x_{1-q} - x_q) > 0\} \to 1$,

$$F(v) \to \Phi\{\frac{\theta_2 v - \theta_1}{\sigma_1\sigma_2 a(v)}\},$$

a fact which may be used in approximating $F(v)$. For comparison of $F(v)$ and its approximation see Hinkley (1969).

The distribution of V will be used to calculate critical value and power of the test as we will see in third chapter. Graphs of $f_V(v)$ for different values of the four parameters, $\theta_1, \theta_2, \sigma_1^2$ and σ_2^2, are show in the following figures:

Figure 2.6: $f_V(v)$ When $\theta_1 = 0$, $\theta_2 = 0$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 1$

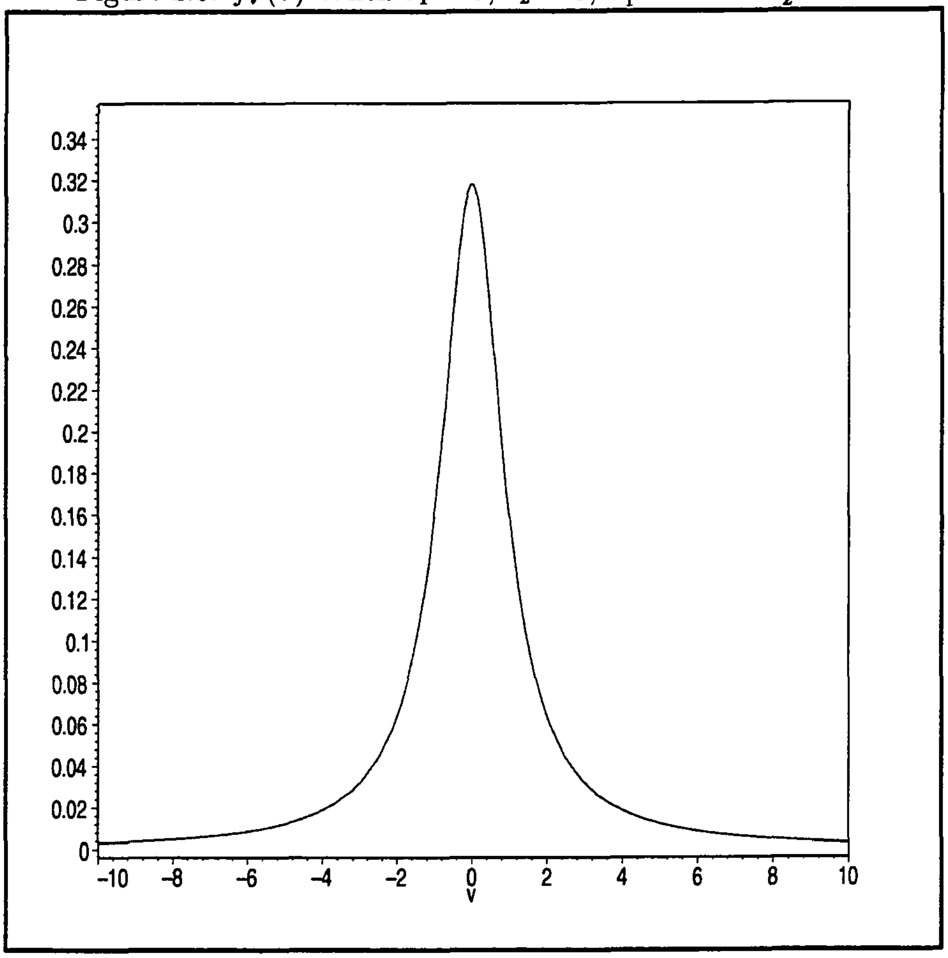


Figure 2.7: $f_V(v)$ When $\theta_1 = 3$, $\theta_2 = 0$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 1$

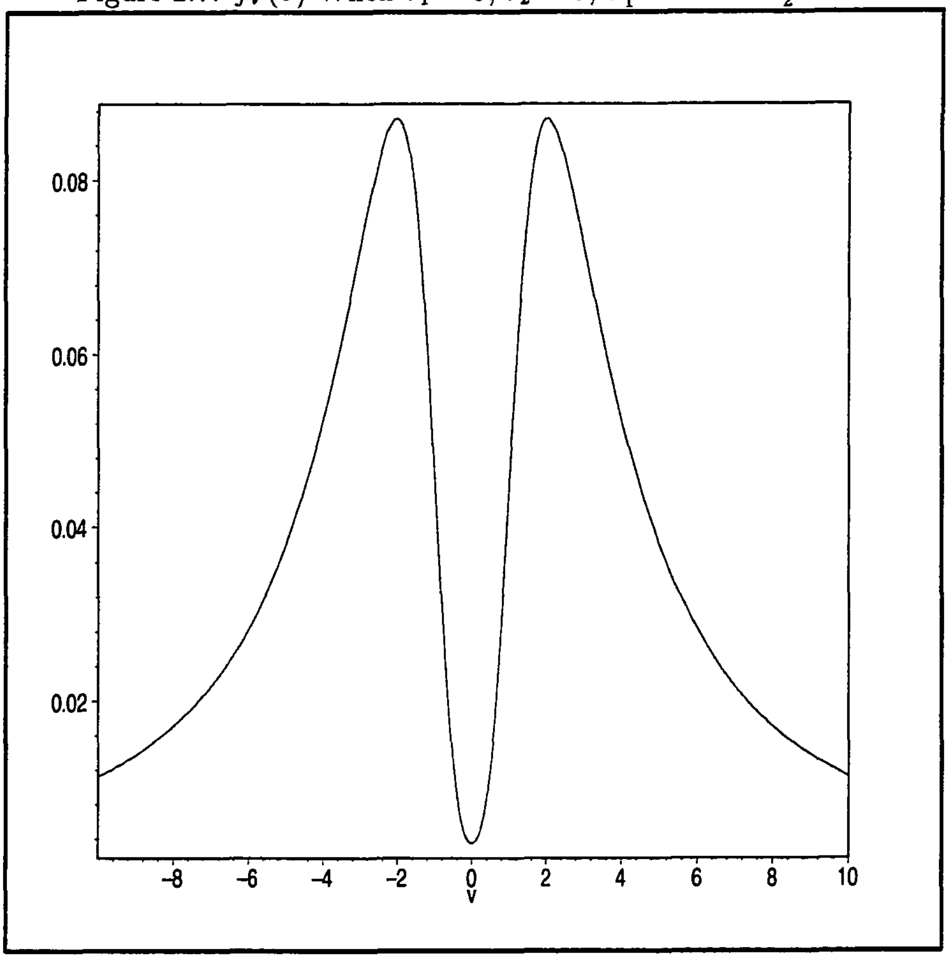


Figure 2.8: $f_V(v)$ When $\theta_1 = 0$, $\theta_2 = 2$, $\sigma_1^2 = 2$ and $\sigma_2^2 = 1$

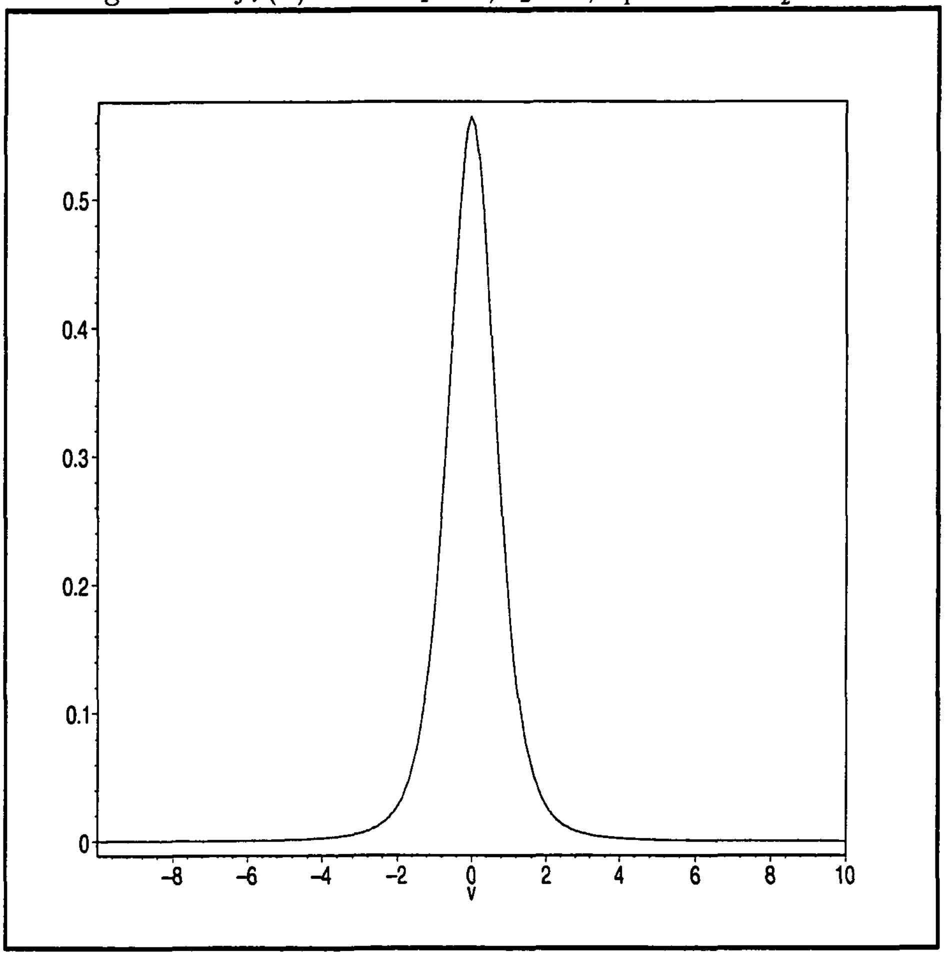


Figure 2.9: $f_V(v)$ When $\theta_1 = 5$, $\theta_2 = 1$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 1$

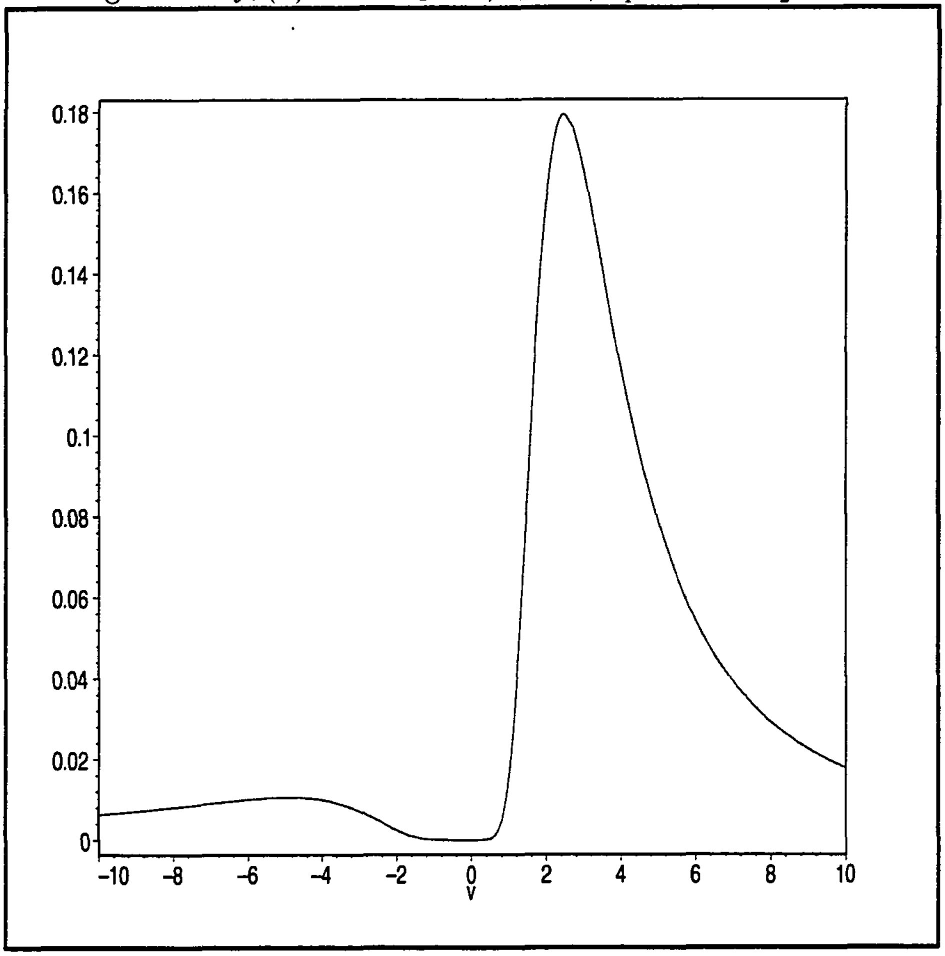
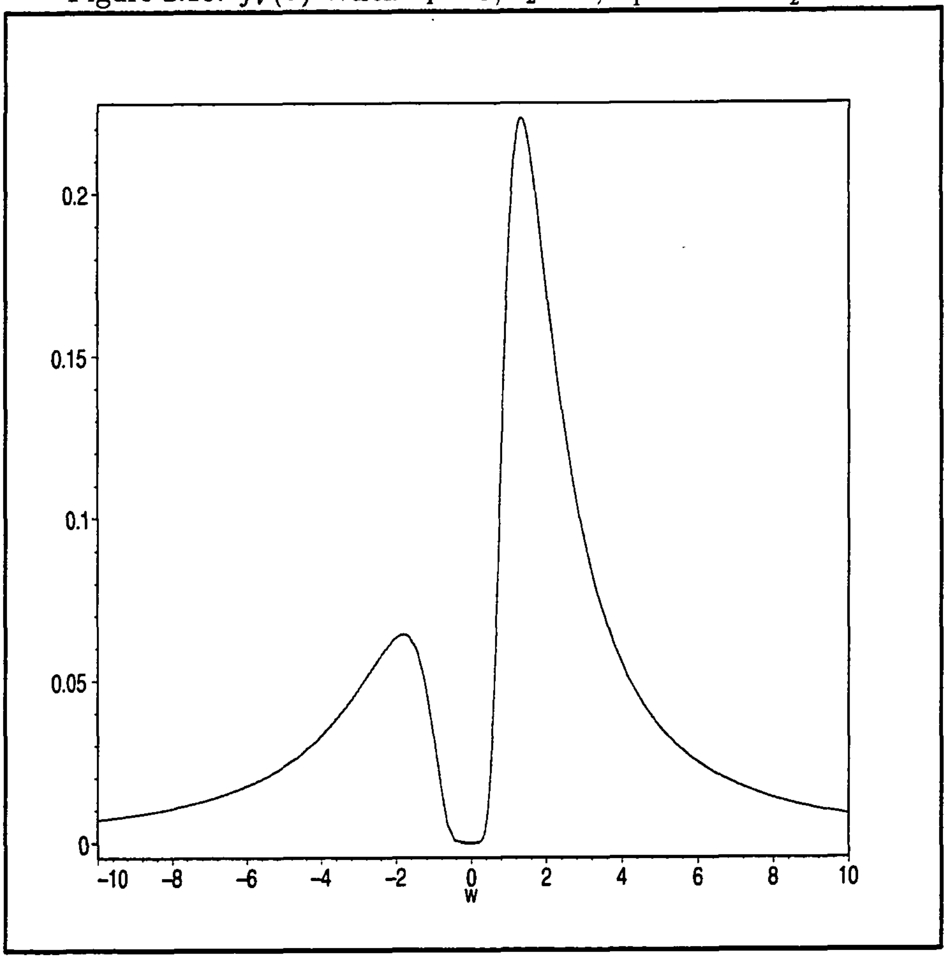


Figure 2.10: $f_V(v)$ When $\theta_1 = 5$, $\theta_2 = 1$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 5$



Chapter 3

TESTING FOR LOCATION PARAMETERS

3.1 Introduction

A statistical test is considered to be robust if the probabilities of Type I and Type II errors are basically unaffected by a test's assumption violations. Before we tests a hypotheses concerning the location of a symmetric stable distribution, we give a brief historical review of the t-test and alternatives, and the asymptotic power function for sequences of alternatives which approach the null hypotheses at the right rate.

In 1908 W. S. Gosset, writing under the name "Student", published in *Biometrika* (Student, 1908) his derivation of the t-distribution with some cumulative tables for sample sizes ranging from 4 to 10. He also studied data from normal sources and showed that the empirical distribution fitted his derived distribution. In 1945 Wilcoxon published his signed rank test, which Pitman (1949) showed to be nearly as efficient as the t-test for location of a normal mean. Hodges and Lehmann (1956) showed that the efficiency of Wilcoxon's test relative to the t-test never falls below 0.86 for asymmetric distributions but is unbounded above. The Wilcoxon test is monotone, while the t-test is not. Both tests are nonrobust to dependence and both tests are available on computer packages.

The 2λ-trimmed t-test is an alternative to the t-test which is apparently robust to $100\lambda\%$ of outliers on either side of the sample. It shares with the t-test the property of being asymptotically distribution-free in the sense that its level for large sample sizes is the desired significance level α, whether or not the observations are normally distributed. Also, it does not require finite moments for this property to hold as the t-test does. The disadvantage is that it is not as efficient as the t-test for normally distributed data.

A researcher does not know if the decisions about the hypothesis are correct. That is, is the null hypothesis being rejected because the shift in location is significantly different or because the test is not robust? If a test is not robust, there could be a higher or lower probability of a researcher's finding a significant difference (Type I) or not finding a significant difference in shift of location when there is a difference (Type II). In either case, the researcher's conclusion concerning the hypothesis is invalid.

Some factors which affect robustness of a test are the significance level, the sample size, the dependence, and the shape of the distribution. In this chapter, a new test for the location will be introduced. The power of the test will be compared with the power of the t-test under normal distribution. This power reveals a great deal about the large sample behavior of a statistical test, and it gives a good approximation to small sample power functions. Also, it provides a basis for comparing the relative efficiency of the two tests.

3.2 Testing for a Location Parameter

A question of concern is what will be a good test for location when the t-test assumptions are not met, or when the moments do not exists. In such a case, the t-test may lose its validity in the hypothesis testing and another test, with less restrictive assumptions, may be more appropriate to use than the t-test.

Throughout this chapter we assume that:

1. F has density f which is symmetrical about 0.

2. $X_1, ..., X_n$ are i.i.d with distribution function $F(x - \delta)$.

A new test for testing the hypothesis $H_0 : \delta = 0$ is using a class of statistics

$$V = \frac{T_{2\lambda}(F_n)}{x_{1-q} - x_q}, \tag{3.1}$$

where $0 < \lambda \leq 1/2$ and $0 < q < 1/2$. The distribution of the statistic V was introduced in section (2.4). After trying many values of λ and q, the values that are being used in this study are $\lambda = 0.25$ and $q = 0.25$. These value made the first test:

$$V_1 = \frac{T_{0.5}(F_n)}{x_{0.75} - x_{0.25}}, \tag{3.2}$$

another test is

$$V_2 = \frac{x_{0.5}}{x_{0.75} - x_{0.25}}. \tag{3.3}$$

These two special statistics, V_1 and V_2, are the only statistics being used in our study for testing the location of stable distribution.

The hypothesis $H_0 : \delta = 0$ is rejected in favor of $H_A : \delta \geq 0$, when a test statistic V_i, $i = 1, 2$. exceeds a critical point:

$$V_{i_n} = V_{i_n}(X_1, ..., X_n) \geq c_{i_n}.$$

The critical point c_{i_n} is chosen so that the test has size not exceeding a preassigned significance level a :

$$P_0\{V_{i_n} \geq c_{i_n}\} \leq a.$$

The fact that $(\frac{T_{0.5}(F_n)-\delta}{x_{0.75}-x_{0.25}})$and $(\frac{x_{0.5}-\delta}{x_{0.75}-x_{0.25}})$ both have V distributions with different parameters allows us to compute v_l and v_u for any a, $0 < a < 1$, in the following:

$$P(v_l \leq \frac{T_{0.5}(F_n) - \delta}{x_{0.75} - x_{0.25}} \leq v_u) = 1 - a, \tag{3.4}$$

and

$$P(v_l \leq \frac{x_{0.5} - \delta}{x_{0.75} - x_{0.25}} \leq v_u) = 1 - a, \tag{3.5}$$

or, equivalently, that a $100(1-a)\%$ confidence interval for δ is given by

$$[T_{0.5}(F_n) - v_u(x_{0.75} - x_{0.25}), T_{0.5}(F_n) - v_l(x_{0.75} - x_{0.25})],$$

and

$$[x_{0.5} - v_u(x_{0.75} - x_{0.25}), x_{0.5} - v_l(x_{0.75} - x_{0.25})].$$

Table 3.1 and 3.2 show the critical points of V_1 and V_2 for some stable distributions:

Table 3.1: Critical Point of V_i $i = 1,2$ When $n = 20$.

	V_1			V_2		
α	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
2.00	0.330	0.413	0.530	0.378	0.474	0.608
1.75	0.332	0.417	0.539	0.376	0.473	0.610
1.50	0.335	0.423	0.553	0.372	0.471	0.615
1.25	0.341	0.438	0.588	0.366	0.470	0.630
1.00	0.359	0.479	0.699	0.353	0.469	0.688
0.75	0.413	0.626	1.210	0.326	0.464	0.960
0.50	0.569	1.080	2.660	0.227	0.434	1.070

Table 3.2: Critical Point of V_i $i = 1,2$ When $n = 40$.

	V_1			V_2		
α	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
2.00	0.221	0.270	0.331	0.254	0.309	0.379
1.75	0.222	0.271	0.332	0.251	0.306	0.376
1.50	0.223	0.272	0.335	0.247	0.302	0.373
1.25	0.224	0.275	0.341	0.240	0.294	0.366
1.00	0.228	0.283	0.360	0.224	0.279	0.354
0.75	0.240	0.312	0.430	0.189	0.246	0.339
0.50	0.294	0.457	0.929	0.118	0.183	0.374

3.2.1 Comparing the Power of V_1-test and V_2-test

The power of the test is a function of the difference between the value of δ stated in the null hypothesis and the value of δ_1. The test's power should increase as the absolute value of the difference between the value of δ and δ_1 increases. Also, it should increase as the sample size increases.

In testing the hypothesis $H_0 : \delta = 0$, against $H_A : \delta \geq 0$, the power with sample size n is defined by

$$\Pi_n(\delta) = P_\delta\{V_{i_n} \geq c_{i_n}\}, \tag{3.6}$$

ideally $\Pi_n(\delta)$ is small ($\leq \alpha$) at $\delta = 0$ and large ($\simeq 1$) for all $\delta > 0$.

To compute the power of the V_i-test, $i = 1, 2$, we use the distribution of V from section 2.4, using $f(v_i)$ with θ_{i_1}, σ^2_{i1}, θ_{i_2}, and σ^2_{i2}. When θ_{i_1} is zero, we use $f(v_i)$ distribution to define a rejection region, say $v_i > v_i(a)$. The power for the v_i-test is defined by

$$P(\delta) = \{\int_{v_i(a)}^{\infty} f(v_i, \theta_{i_1}) dv_i\}. \tag{3.7}$$

The powers of the V_1-test and V_2-test were compared to each other in order to examine the maximum difference in the power of the two tests when the sample size is fixed $n = 20$ and index of stability is $\alpha \in (0, 2]$. When $\alpha = 2$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 0.85$, the $max(|p_{v_1} - p_{v_2}|) = 0.102$ (see Figure 3.1). When $\alpha = 1.5$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 0.9$, the $max(|p_{v_1} - p_{v_2}|) = 0.08$ (see Figure 3.2). When $\alpha = 1$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 0.9$, the $max(|p_{v_1} - p_{v_2}|) = 0.012$ (see Figure 3.3). When $\alpha = 0.75$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 1$, the $max(|p_{v_1} - p_{v_2}|) = 0.17$ (see Figure 3.4). When

Figure 3.1: Power of the V_1-test and V_2-test When $\alpha = 2$.

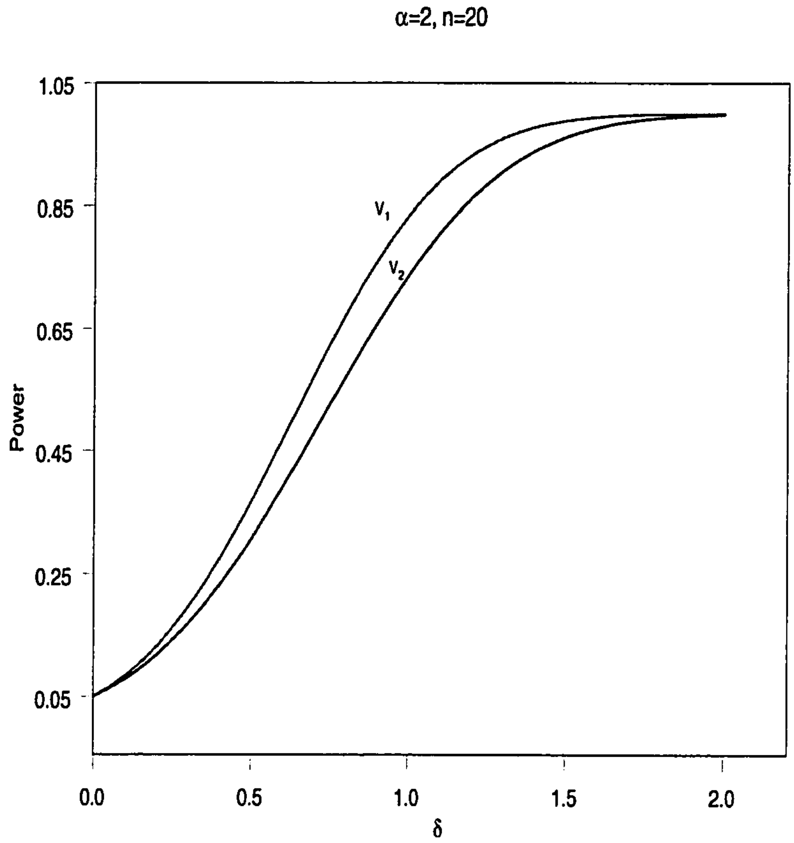


Figure 3.2: Power of the V_1-test and V_2-test When $\alpha = 1.5$.

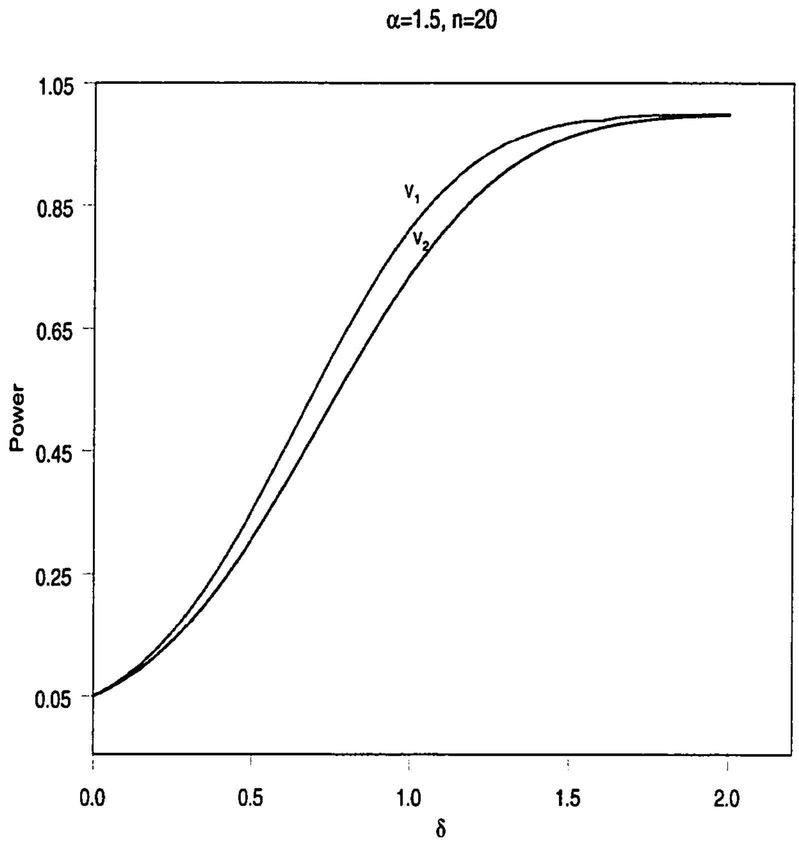


Figure 3.3: Power of the V_1-test and V_2-test When $\alpha = 1$.

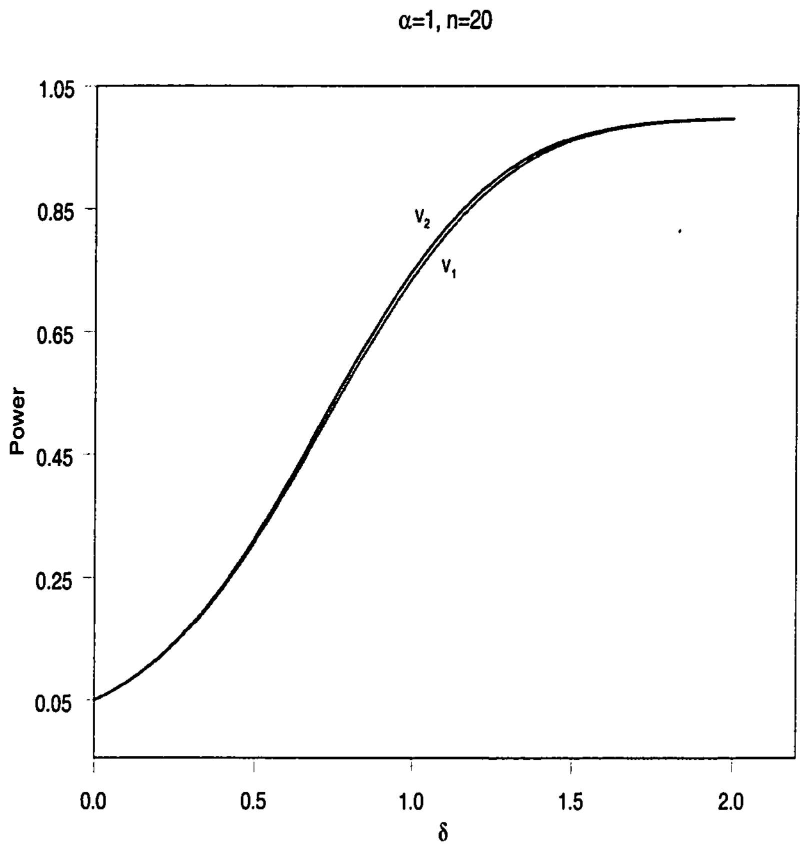


Figure 3.4: Power of the V_1-test and V_2-test When $\alpha = 0.75$.

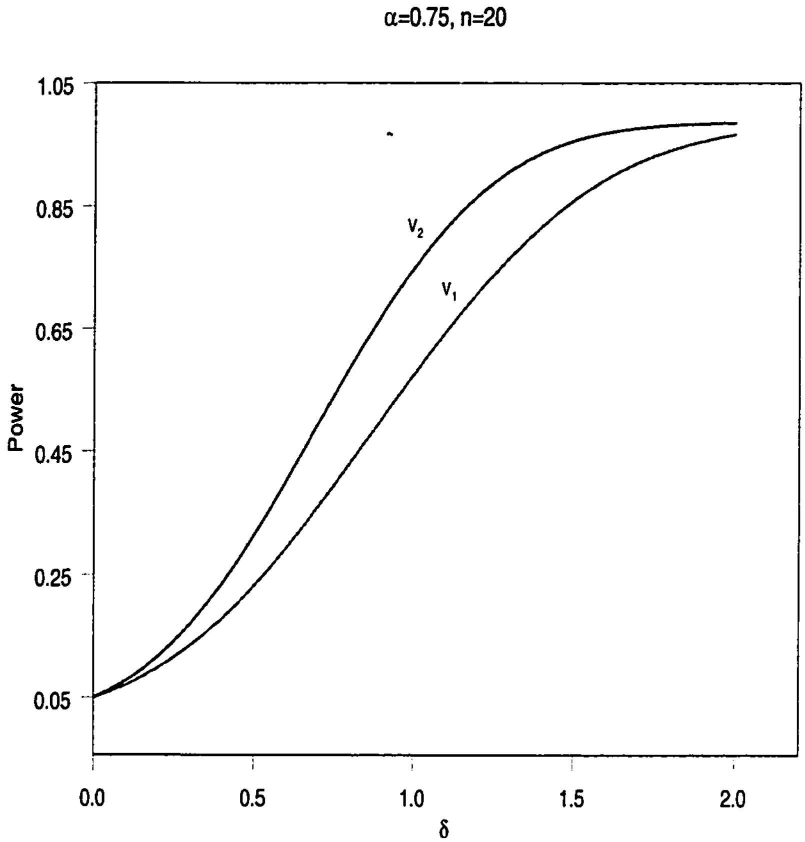
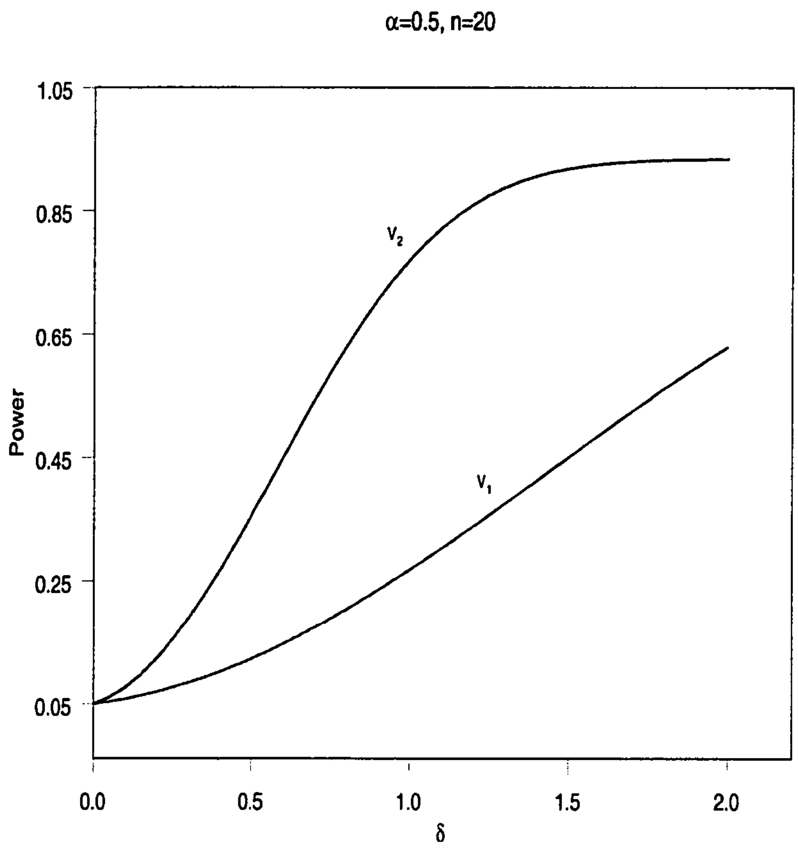


Figure 3.5: Power of the V_1-test and V_2-test When $\alpha = 0.5$.



$\alpha = 0.5$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 1.15$, the $max(|p_{v_1} - p_{v_2}|) = 0.52$ (see Figure 3.5).

All the above results and plots of the power which are shown in Figures 3.1 to 3.5 indicate that generally the V_1-test has better power advantages than the V_2-test when the index of stability of stable distributions $\alpha \in (1, 2]$. However, when the index of stability of stable distributions $\alpha \in (0, 1)$, the V_2-test has more significant power advantages than the V_1-test. [It is clear from Figure 3.3 that the V_1-test and V_2-test are not much different from each other, since $max(|p_{v_1} - p_{v_2}|) = 0.012 < 0.05$.]

3.2.2 Comparing the Power of the t-test and V_1-test under Normality

It has long been part of statistical folklore that the t-test is conservative for a sample from a long-tailed symmetric distribution. This conservatism means that the Type I error of the t-test is smaller under a long-tailed symmetric distribution than it is under normality. It is also well known that the usual Student's t-test is conservative and hence less powerful when the underlying distribution is long-tailed.

The decision on whether to use the t-test or V_1-test can depend on whether the t-test's assumptions hold. For samples of larger size, the t-test can be used at the usual levels; otherwise the parent distribution should be examined more closely. The proportion of rejections of the t-test was compared to the proportion of rejections of the V_1-test when the null hypothesis was false in order to investigate the power of the two tests.

To compute the power of the t-test we use the noncentral t distribution, that $t \backsim t(n-1, \tau)$, where

$$\tau = \frac{\delta - \delta_0}{\sqrt{\sigma^2/n}}, \tag{3.8}$$

is the noncentrality parameter. When the noncentrality parameter is zero, we use the central t statistic to define a rejection region, say $t > t(a, n-1)$. The power for the t-test is defined by

$$P(\tau) = Pr\{t(n-1,\delta) > t(a,n-1)\}. \tag{3.9}$$

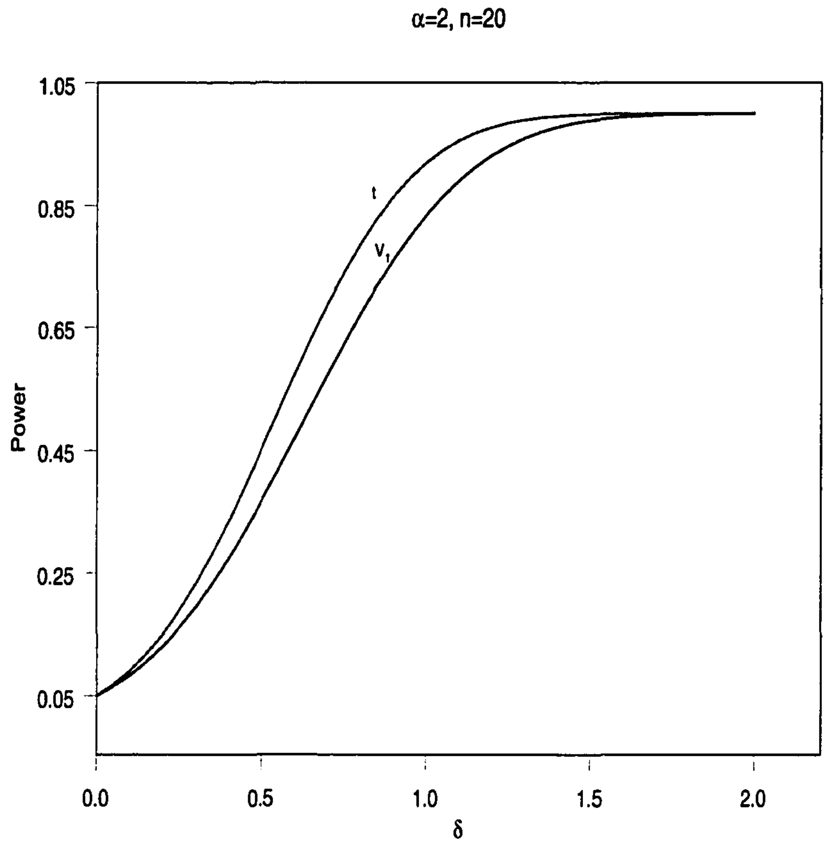
The power of the t-test and V_1-test were compared to each other in order to examine the maximum difference in the power of the two tests when the sample size is fixed $n = 20$. The largest power difference between the t-test and the V_1-test occurred when $\delta = 0.75$, $max(p_t - p_{v_1}) = 0.115$. If the $(p_t - p_{v_1})$ is greater than 0.05, then the t-test is considered the more powerful statistic to use. Therefore, the t-test generally has more power than the V_i-tests, $i = 1,2$, under normality. The plot of the power shown in Figure 3.7 indicates that the t-test has significant power advantages over the V_1-test under normality.

Figure 3.6 shows that the power of the V_1-test competes very favorably with the t-test under normality. If the mean used in this case was $V_3 = \frac{\bar{x}}{x_{0.75}-x_{0.25}}$, then the power will be better than the power of the V_1-test and less than the power of the t-test, since the the mean has a smaller variance then the trimmed mean.

In general, the V_i-test is more favorable than t-test over a wide range of distributions since the V_i-test assumes only that F is symmetric and continuous. The t-test assumes that F has a second moment. Also, V_i-test is more robust to outliers than the t-test and easier to compute than some other nonparametric tests.

It appears that researchers need to be cautious in claiming that the t-test is robust to its assumption violations when they compare the power of this parametric test to the power of a nonparametric test because there is conflicting evidence as to whether the t-test or the V_i-test is more powerful under the t-test's assumption violations. A solution would be to examine the power advantages of these tests under specific conditions rather than generalizing from one specific condition to another.

Figure 3.6: Power of the t-test and V_1-test



3.3 Testing for the Two-Sample Location Parameters

Suppose that we have two populations with location parameters δ_1 and δ_2 respectively. A random sample is drawn from each population to test the null hypothesis $H_0 : \delta_1 - \delta_2 = 0$ vs. $H_A : \delta_1 - \delta_2 \geq 0$. For the null hypothesis, V is defined by

$$V = \frac{T_{2\lambda_1}(F_n(x)) - T_{2\lambda_2}(F_n(y))}{\frac{1}{2}[(x_{1-q_1} - x_{q_1}) + (y_{1-q_2} - y_{q_2})]}, \tag{3.10}$$

where $0 < \lambda_i < 1/2$ and $0 < q_i < 1/2, \quad i = 1, 2.$

The distribution of the statistic V was introduced in section (2.4). V statistics were designed for testing differences between two location parameters for the following reasons:

1. Quantile functions reflect a common location shift characteristics of all the symmetric distributions.

2. Quantiles exist regardless of the existence of moments.

3. Shifts expressed in terms of quantiles are clearly defined for all absolutely continuous distributions, in contrast to those shifts expressed in terms of moments.

4. The use of symmetric distributions for the location shift alternative is justified by the fact that when the distribution F is symmetric, it is understood that the point of symmetry is the center of the distribution. Consequently, any location shift of a symmetric distribution can be most easily visualized in terms of the shift of a symmetry point of the distribution.

After trying many values of λ_i and q_i, the values that are being used in this study are $\lambda_1 = \lambda_2 = 0.25$ and $q_1 = q_2 = 0.25$. These values made the first test:

$$V_1 = \frac{T_{0.5}(F_n(x)) - T_{0.5}(F_n(y))}{\frac{1}{2}[(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})]}, \tag{3.11}$$

and another test is

$$V_2 = \frac{x_{0.5} - y_{0.5}}{\frac{1}{2}[(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})]}. \tag{3.12}$$

These two special statistics V_1 and V_2 tests are the only statistics being used for testing differences between two location parameters of stable distributions in our study.

The hypothesis H_0 is rejected in favor of H_A when a test statistic $V_i,\ i = 1, 2.$ exceeds a critical point $V_{i_n} \geq c_{i_n}$. The critical point c_{i_n} is chosen so that the size of test does not exceed a preassigned significance level a :

$$P_0\{V_{i_n} \geq c_{i_n}\} \leq a.$$

A $100(1-a)\%$ confidence interval for $\delta_1 - \delta_2$ is given by

$$P\{T_{0.5}(F_n(x)) - T_{0.5}(F_n(y)) - v_u[\frac{(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})}{2}] \leq \delta_1 - \delta_2 \leq T_{0.5}(F_n(x)) - T_{0.5}(F_n(y)) - v_l[\frac{(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})}{2}]\} = 1 - a, \tag{3.13}$$

and

$$P\ \{\ (x_{0.5} - y_{0.5}) - v_u[\frac{(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})}{2}] \leq \delta_1 - \delta_2 \leq (x_{0.5} - y_{0.5}) - v_l[\frac{(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})}{2}]\} = 1 - a. \tag{3.14}$$

Table 3.3, Table 3.4 and Table 3.5 show the critical points of V_1 and V_2 of some stable distributions.

3.3.1 Comparing the Power of the V_1-test and V_2-test

The power of the test is a function of the difference between the value of δ_1 and the value of δ_2 stated in the null hypothesis. The test's power should increase as the

Table 3.3: Critical Point of V_i $i = 1, 2$, When $n = 20$.

	V_1			V_2		
α	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
2.00	0.442	0.539	0.661	0.507	0.617	0.757
1.75	0.443	0.541	0.665	0.502	0.612	0.753
1.50	0.445	0.543	0.674	0.494	0.604	0.745
1.25	0.448	0.549	0.683	0.479	0.588	0.731
1.00	0.455	0.567	0.720	0.448	0.557	0.708
0.75	0.480	0.623	0.860	0.379	0.492	0.677
0.50	0.587	0.916	1.870	0.235	0.367	0.745

Table 3.4: Critical Point of V_i $i = 1, 2$, When $n = 40$.

	V_1			V_2		
α	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
2.00	0.305	0.367	0.442	0.350	0.421	0.508
1.75	0.306	0.368	0.443	0.346	0.417	0.503
1.50	0.307	0.368	0.447	0.340	0.409	0.494
1.25	0.307	0.369	0.448	0.327	0.396	0.479
1.00	0.307	0.373	0.450	0.302	0.367	0.448
0.75	0.311	0.384	0.481	0.246	0.303	0.379
0.50	0.335	0.440	0.620	0.134	0.177	0.248

Table 3.5: Critical Point of V_i $i = 1, 2$, When $\alpha_1 = 2$ and $\alpha_2 = 1$

	V_1			V_2		
$n_1 = n_2$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
10	0.689	0.888	1.210	0.735	0.947	1.301
15	0.532	0.664	0.845	0.567	0.707	0.905
20	0.449	0.552	0.690	0.479	0.589	0.735
25	0.399	0.483	0.596	0.422	0.515	0.635
30	0.357	0.435	0.532	0.381	0.463	0.568
40	0.306	0.370	0.450	0.327	0.395	0.479

absolute value of the difference between the value of δ_1 and δ_2 increases. Also, it should increases as the sample size n increases. To compute the power of the V_1-test and V_2-test, we use the same method as was used in section 3.2.

The power of the V_1-test and V_2-test were compared to each other in order to examine the maximum difference in the power of the two tests when the sample size is fixed $n = 20$ and index of stability $\alpha \in (0, 2]$. When $\alpha = 2$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta_1 - \delta_2 = 1.15$, the $max(|p_{v_1} - p_{v_2}|) = 0.103$ (see Figure 3.7). When $\alpha = 1.5$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta_1 - \delta_2 = 1.15$, the $max(|p_{v_1} - p_{v_2}|) = 0.079$ (see Figure 3.8). When $\alpha = 1$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta_1 - \delta_2 = 1.20$, the $max(|p_{v_1} - p_{v_2}|) = 0.012$ (see Figure 3.9). When $\alpha = 0.75$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta_1 - \delta_2 = 1.15$, the $max(|p_{v_1} - p_{v_2}|) = 0.177$ (see Figure 3.10).

All the above results and plots of the power which are shown in Figures 3.7 to 3.10 indicate that generally the V_1-test has more significant power advantages than the V_2-test when index of stability of stable distributions $\alpha \in (1, 2]$. However, when the index of stability of stable distributions $\alpha \in (0, 1)$, the V_2-test has more significant power advantages than the V_1-test. It is clear from Figure 3.9 that the V_1-test and V_2-test are not much different from each other since $max(|p_{v_1} - p_{v_2}|) = 0.012 < 0.05$.

3.3.2 Comparing the Power of the t-test and V_1-test under Normality

To compute the power of the t-test for a two-sample problem, we use the non-central t distribution, that $t \backsim t(n_1 + n_2 - 2, \tau)$, where

$$\tau = \frac{\delta_1 - \delta_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}, \tag{3.15}$$

Figure 3.7: Power of the V_1-test and V_2-test When $\alpha = 2$.

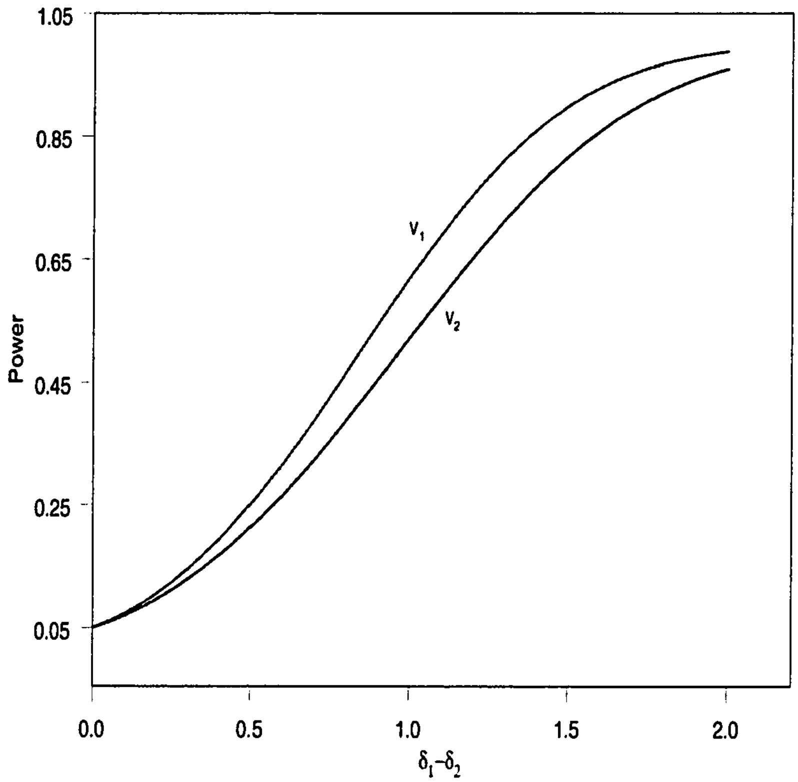


Figure 3.8: Power of the V_1-test and V_2-test When $\alpha = 1.5$.

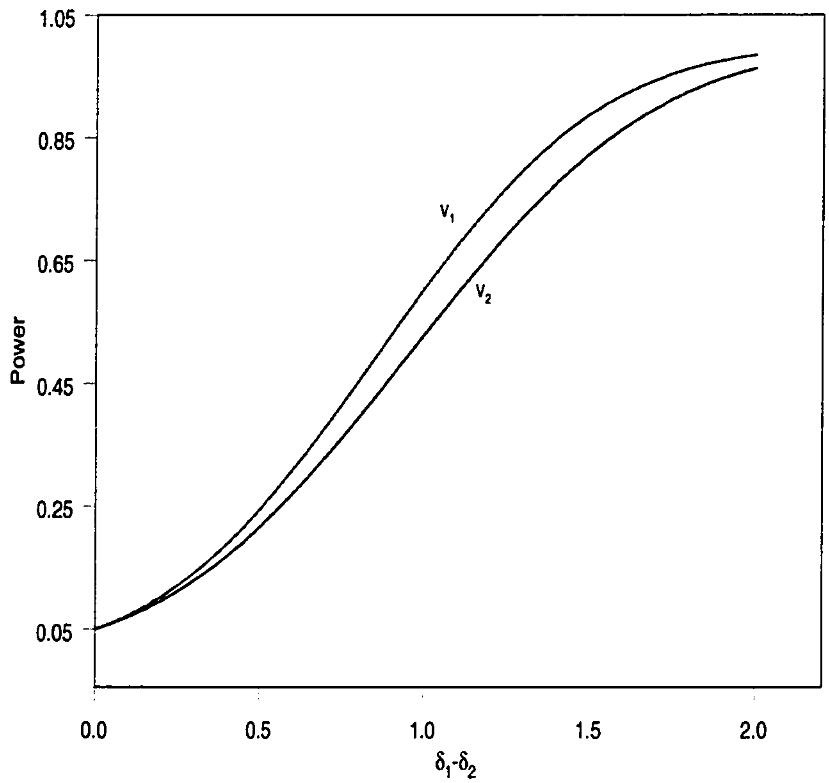


Figure 3.9: Power of the V_1-test and V_2-test When $\alpha = 1$.

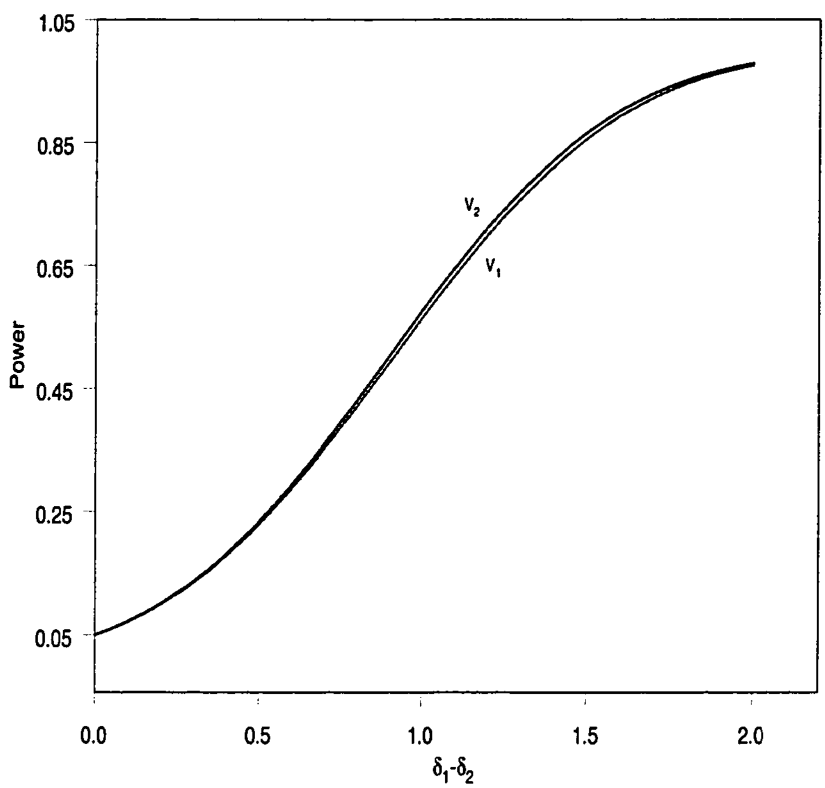
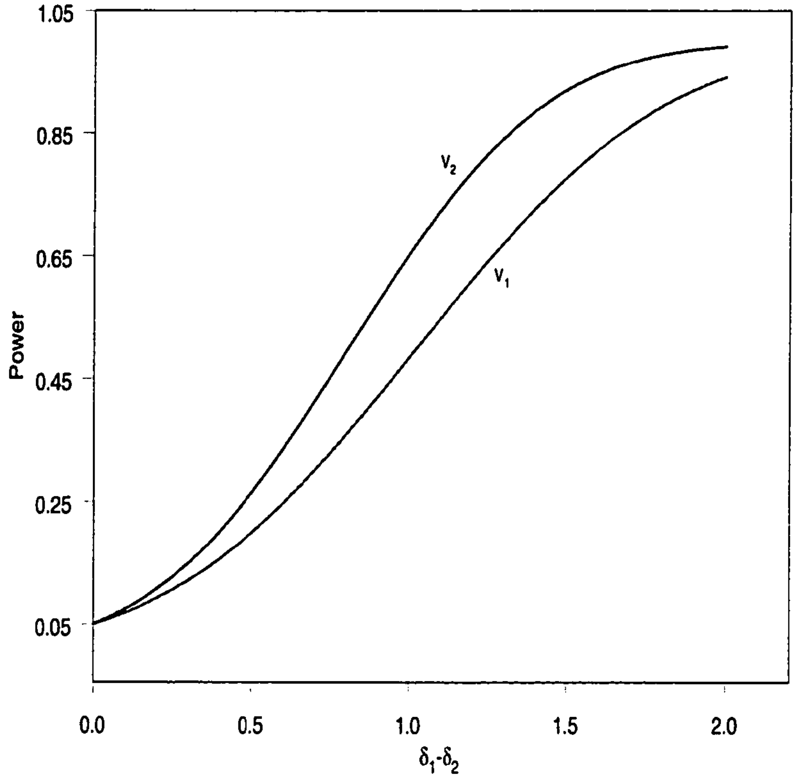


Figure 3.10: Power of the V_1-test and V_2-test When $\alpha = 0.75$.



is the noncentrality parameter. When the noncentrality parameter is zero, we use the central t statistics to define a rejection region, say $t > t(a, n_1 + n_2 - 2)$.

The power of the t-test and V_1-test were compared to each other in order to examine the maximum difference in the power of the two tests when the sample size is fixed $n = 20$. The largest power difference between the t-test and the V_1-test occurred when $\delta_1 - \delta_2 = 1$, $max(p_t - p_{v_1}) = 0.09$. If the $(p_t - p_{v_1})$ is greater than 0.05, then the t-test is considered the more powerful statistic to use. Therefore, the t-test generally has more power than the V_i-test under normality. The plot of the power shown in Figure 3.11 indicates that the t-test has significant power advantages over the V_1-test under normality.

Figure 3.11 shows that the power of of V_1-test competes very favorably with the t-test under normality. If the difference between the two means is used here

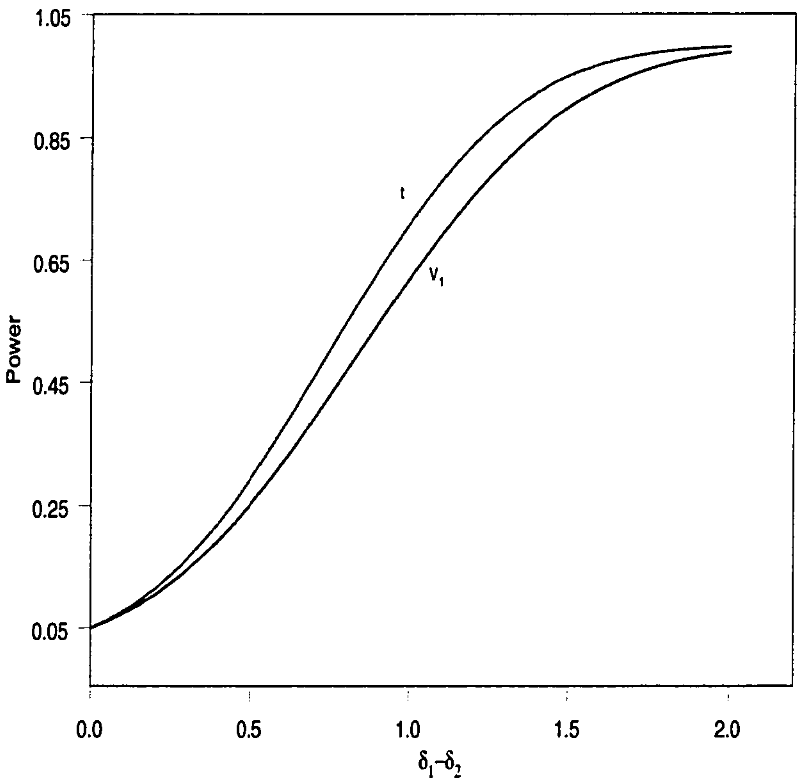
$$V_3 = \frac{\bar{x} - \bar{y}}{\frac{1}{2}[(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})]},$$

then the power will be more than the power of the V_1-test and less than power of the t-test since the mean in normal case has the smallest variance.

In general, the V_i-test is more favorable than the t-test over a wide range of distributions since the V_i-test assumes only that F is symmetric and continuous. The t-test assumes that F has a second moment. Also, the V_i-test is more robust to outliers then t-test and easier to compute than some other nonparametric tests.

It appears that researchers need to be cautious in claiming that the t-test is robust to its assumption violations when they compare the power of this parametric test to the power of a nonparametric test because there is conflicting evidence as to whether the t-test or the V_i-test is more powerful under the t-test's assumption violations. A solution would be to examine the power advantages of these tests under specific conditions rather than generalizing from one specific condition to another.

Figure 3.11: Power of the t-test and V_1-test



Chapter 4

APPLICATION

4.1 Introduction

The first section of this chapter includes another family of symmetric distributions and a discussion of the use of the tests of location and measuring the heaviness of the tail. Section Tow applies the test to a real example. Third section summarizes the results of a simulation study of the tests of locations and a brief conclusion.

4.2 Symmetric Kappa Distributions

Symmetric kappa distributions are a rich class of distributions that allow heavy tails. The class of Kappa distributions was characterized by Mielke (1972) in his study of asymptotic behavior of two-sample tests based on powers of ranks for detecting scale and location alternatives. The distribution function of symmetric kappa distribution is given by

$$F(x) = \begin{cases} (1 + [(x^r/r)/(1 + x^r/r)]^{1/r})/2 & if\ \ 0 \leq x \\ (1 - [(|x|^r/r)/(1 + |x|^r/r)]^{1/r})/2 & if\ \ x \leq 0, \end{cases} \tag{4.1}$$

and the density is given by

$$f(x) = r^{-1/r}(1 + |x|^r/r)^{-(r+1)/r}/2, \tag{4.2}$$

where $r > 0$. The parameter r determines the shape of the distribution. For very small values of r, the density tends to be peaked at the origin with heavy tails. In contrast, $f(x)$ approaches the uniform density function with increasingly large values of r. For $r = 2$, $F(x)$ happens to be the t distribution with 2 degrees of freedom. Table 4.1 shows quantiles of symmetric kappa distributions for some values of r. Figures 4.1 to 4.4 show the shape of the densities for some values of r.

Table 4.1: Quantiles of Symmetric Kappa Distributions

r \ q	0.60	0.75	0.90	0.95	0.99
100	0.2094	0.5236	0.8377	0.9424	1.0276
10	0.2518	0.6295	1.0186	1.1827	1.4622
2	0.2888	0.8165	1.8856	2.920	6.9646
1	0.2500	1	4	9	49
0.75	0.2189	1.1356	6.5982	19.0575	179.9125
0.50	0.1636	1.4571	17.9443	85.4407	2425.4381
0.25	0.0649	3.0480	360.5646	7697.6734	5942550.5833

Figure 4.1: Kappa Distribution ($r = 0.5$)

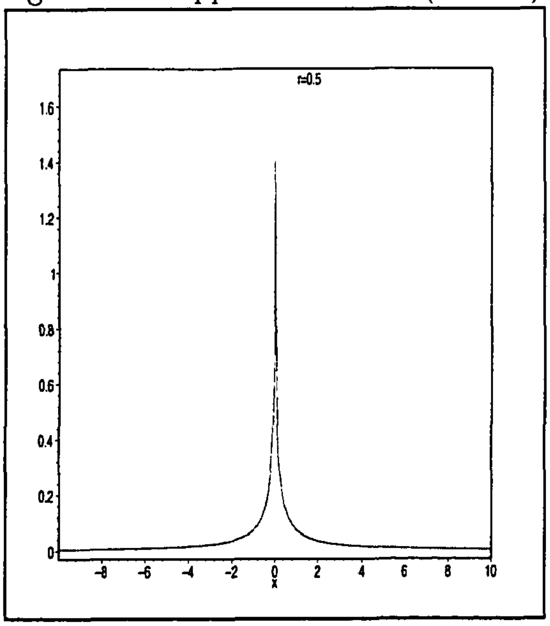


Figure 4.2: Kappa Distribution ($r = 1$)

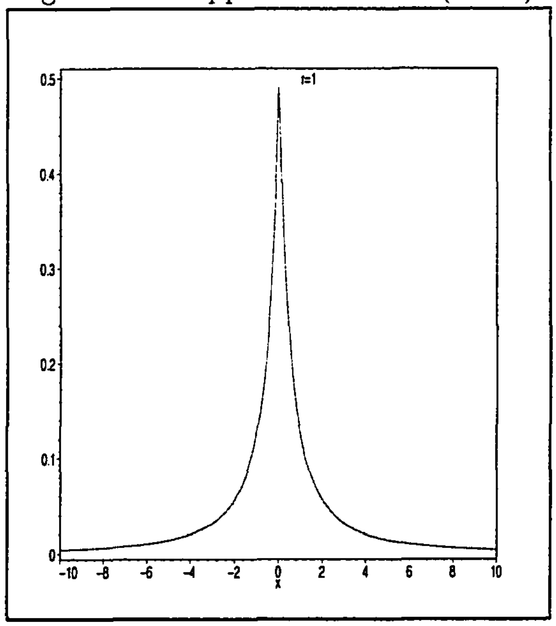


Figure 4.3: Kappa Distribution ($r = 10$)

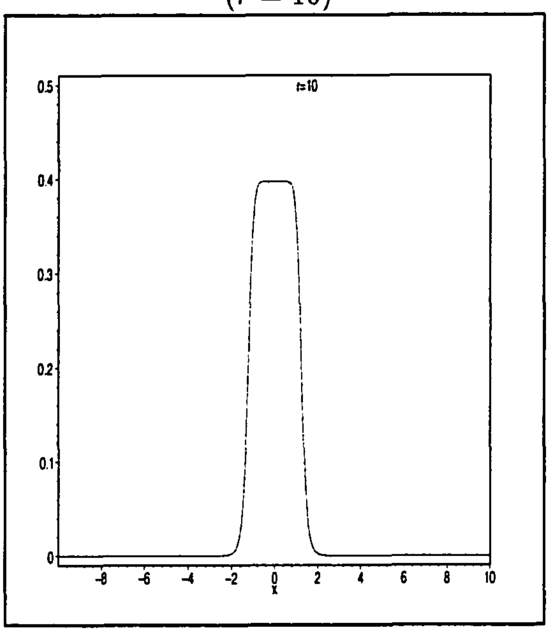
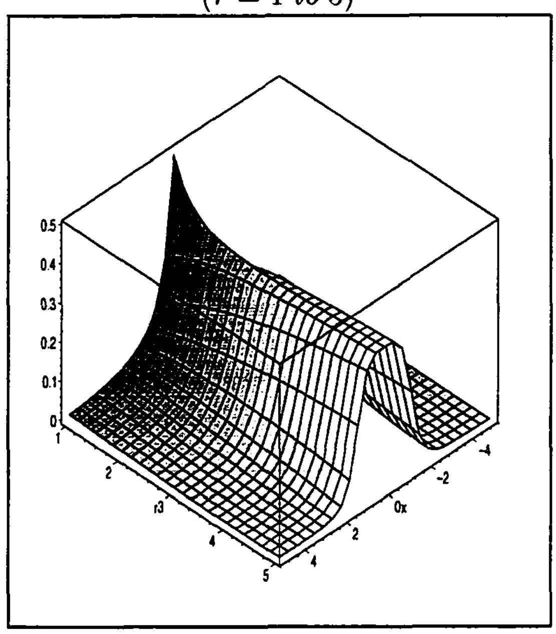


Figure 4.4: Kappa Distribution ($r = 1$ *to* 5)



4.2.1 Testing for Location Parameters

Let $X_1, ..., X_n$ be i.i.d with distribution function $F(x-\delta)$, where F is kappa distribution with shape parameter r. We test the hypothesis $H_0 : \delta = 0$, using the statistics $V_1 = \frac{T_{0.5}(F_n)}{x_{0.75}-x_{0.25}}$, or $V_2 = \frac{x_{0.5}}{x_{0.75}-x_{0.25}}$, the hypothesis $H_0 : \delta = 0$ is rejected in favor of $H_A : \delta \geq 0$, when a test statistic V_i, $i = 1, 2$, exceeds a critical point:

$$V_{i_n} = V_{i_n}(X_1, ..., X_n) \geq c_{i_n}.$$

The critical point c_{i_n} is chosen so that the size of the test does not exceed a preassigned significance level a. Tables 4.2 shows the critical points of V_1 and V_2 for some kappa distributions.

Table 4.2: Critical Point of V_i $i = 1, 2$, When $n = 20$.

	V_1			V_2		
r	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
10	0.325	0.398	0.50	0.396	0.488	0.610
2	0.340	0.434	0.58	0.366	0.468	0.625
1	0.408	0.615	1.18	0.261	0.393	0.750
0.75	0.485	0.847	2	0.19	0.329	0.770
0.50	0.615	1.206	3	0.07	0.137	0.350

To compute the power of the V_1-test and V_2-test, we use the distribution of V from section 2.4. The power of the V_1-test and V_2-test were compared to each other in order to examine the maximum difference in the power of the two tests when the sample size is fixed $n = 20$, and some values of the shape parameter r. When $r = 2$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 0.75$, the $max(|p_{v_1} - p_{v_2}|) = 0.06$ (see Figure 4.5). When $r = 1$, the largest power difference between the V_1-test and the V_2-test occurred when $\delta = 0.85$, the $max(|p_{v_1} - p_{v_2}|) = 0.32$. (see Figure 4.6).

Figure 4.5: Power of V_1-test and V_2-test When $r = 2$.

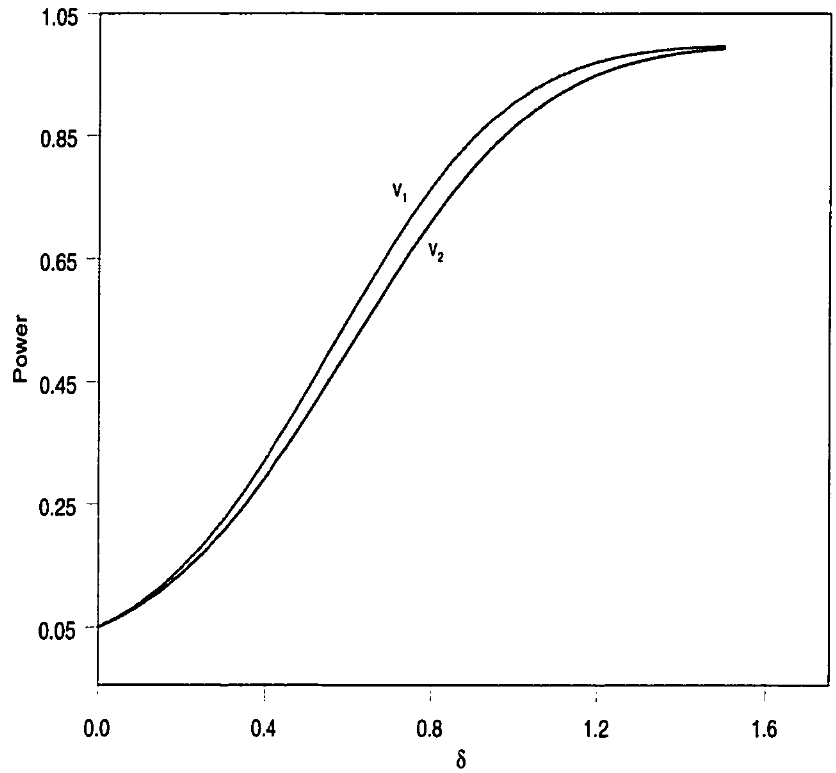
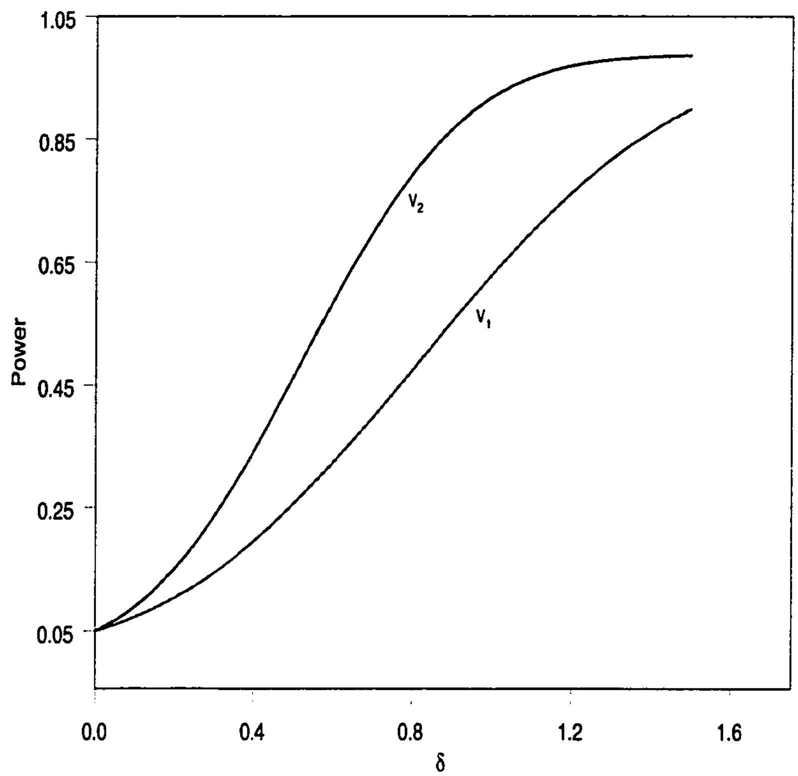


Figure 4.6: Power of V_1-test and V_2-test When $r = 1$.



All the above results and plots of power which are shown in Figures 4.5 to 4.6 indicate that generally the V_1-test has more significant power advantages than the V_2-test when the shape parameter of kappa distributions $r \geq 2$. However, when the shape parameter of kappa distributions $r < 2$, V_2-test has more significant power advantages than the V_1-test.

Testing for the two-sample location parameters, requires that we have two populations with location parameters δ_1 and δ_2. Random samples $x_1, ..., x_{n_1}$ and $y_1, ..., y_{n_2}$ are drawn from each population to test the null hypothesis $H_0 : \delta_1 - \delta_2 = 0$ vs. $H_A : \delta_1 - \delta_2 > 0$, using the statistics

$$V_1 = \frac{T_{0.5}(F_n(x)) - T_{0.5}(F_n(y))}{\frac{1}{2}[(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})]},$$

and

$$V_2 = \frac{x_{0.5} - y_{0.5}}{\frac{1}{2}[(x_{0.75} - x_{0.25}) + (y_{0.75} - y_{0.25})]}.$$

The hypothesis H_0 is rejected in favor of H_A when a test statistic V_i, $i = 1, 2$, exceeds a critical point $V_{i_n} \geq c_{i_n}$. The critical point c_{i_n} is chosen so that the size of the test does not exceed a preassigned significance level a. Table 4.3 shows the critical points of V_1 and V_2 in some kappa distributions:

Table 4.3: Critical Point of V_i $i = 1, 2$, When $n_1 = n_2 = 20$.

	V_1			V_2		
$r_1 = r_2$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
10	0.440	0.532	0.645	0.539	0.652	0.792
2	0.447	0.548	0.680	0.480	0.590	0.730
1	0.477	0.618	0.850	0.304	0.394	0.540
0.75	0.518	0.723	1.165	0.202	0.281	0.452
0.50	0.660	1.130	2.600	0.075	0.128	0.295

The power of the V_1-test and V_2-test were compared to each other in order to examine the maximum difference in the power of the two tests when the sample size is fixed $n = 20$, and some values of the shape parameter r. When $r_1 = r_2 = 2$, the largest power difference between the V_1-test and V_2-test occurred when $\delta_1 - \delta_2 = 0.95$, with $max(|p_{v_1} - p_{v_2}|) = 0.055$ (see Figure 4.7). When $r_1 = r_2 = 1$, the largest power difference between the V_1-test and V_2-test occurred when $\delta_1 - \delta_2 = 1$, with $max(|p_{v_1} - p_{v_2}|) = 0.323$ (see Figure 4.8).

All the above results and plots of power which are shown in Figures 4.7 to 4.8 indicate that generally the V_1-test has more significant power advantages than the V_2-test when the shape parameter of kappa distributions $(r_1 = r_2) \geq 2$. However, when the shape parameter of kappa distributions $(r_1 = r_2) < 2$, V_2-test has more significant power advantages than the V_1-test.

4.3 Examples

Example A: The example that will be used to illustrate the new tests of location parameters is a set of data from a randomized airborne pyrotechnic seeding experiment on 52 isolated cumulus clouds in south Florida from 1968 to 1970 described by Simpson and Olsen (1975).

Rainfall was measured from 52 clouds, of which 26 were chosen randomly to be seeded with silver iodode smoke. The data are the amounts of rainfall in acre-feet from the 52 clouds. The objective is to describe the effect that seeding has on rainfall. The data for this example is given in Table 4.4.

Figure 4.9 shows the empirical quantile-quantile plot for the rainfall data. There are several things about the rainfall data. Nearly all the points lie above the $y = x$ line, and that the larger the values, the further the points are from the line. Looking more closely, we notice that the points at the upper right are much more widely

Figure 4.7: Power of the V_1-test and V_2-test When $r_1 = r_2 = 2$

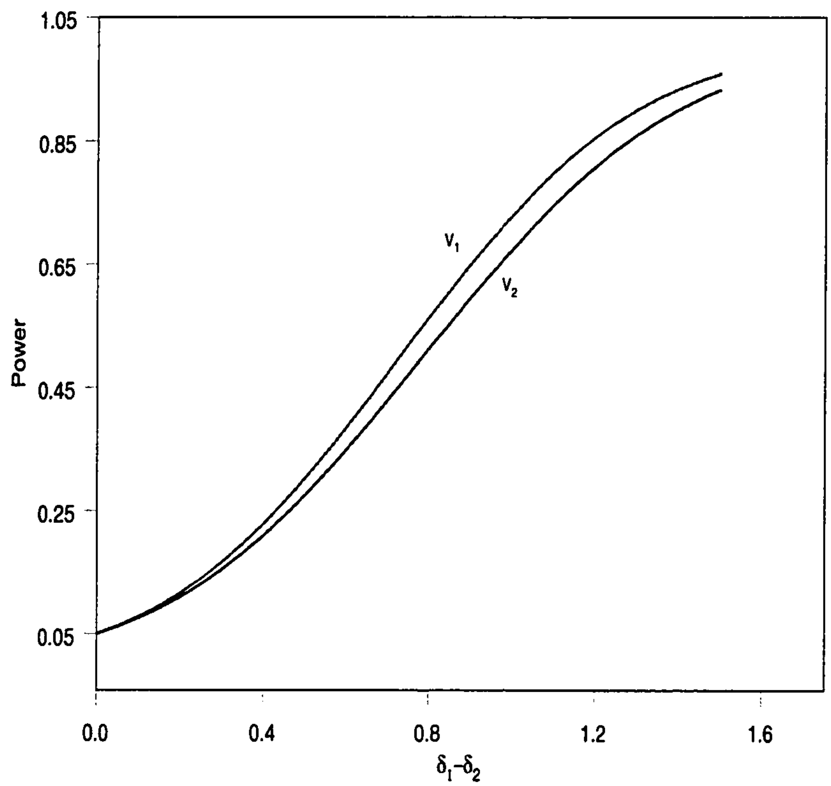
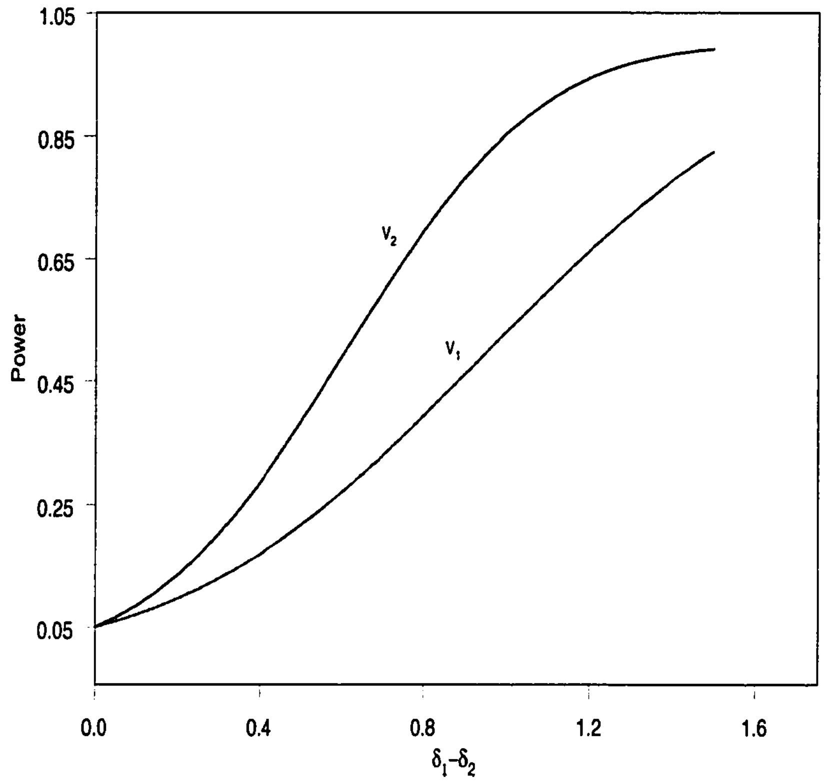


Figure 4.8: Power of the V_1-test and V_2-test When $r_1 = r_2 = 1$



scattered than those at the lower left. This suggests taking logarithms. Figure 4.10 is the logarithms of the empirical quantile-quantile plot. The points are now more uniformly spread out diagonally across the page. Also, two pair of box plots for the rainfall data and the logarithms of it are shown in Figures 4.11 and 4.12.

Table 4.4: Rainfall from Cloud-Seeding

Control Clouds			Seeded Clouds		
26.1	321.2	4.9	129.6	17.5	1697.8
26.3	68.5	4.9	31.4	200.7	334.1
87.0	81.2	41.1	2745.6	274.7	118.3
95.0	47.3	29.0	489.1	274.7	255.0
372.4	28.6	163.0	430.0	7.7	115.3
1.0	830.1	244.3	302.8	1656.0	242.5
17.3	345.5	147.8	119.0	978.0	32.7
24.4	1202.6	21.7	4.1	198.6	40.6
11.5	36.6		92.4	703.4	

Now the question arises: should the difference in the locations of the distributions be regarded as meaningful or simply a result of random fluctuations that occur in the data? The answer depends critically on the result from testing the difference in the locations. Four tests, t-test, Mann-Whitney test, V_1-test and V_2-test, have been chosen to test H_0: The difference in the locations is not statistically significant. versus H_A: The rainfall amounts for the seeded clouds are greater than the rainfall amounts for the control clouds.

Table 4.6 shows descriptive statistics of the rainfall data. Δ_1 statistics are bigger than 3.71 for both samples which indicate that the distributions of the samples have tails heavier than the normal distribution. Also, Table 4.5 shows the critical values of the V_1-test and V_2-test when $n = 26$ for three different distributions.

The results of the tests for the rainfall data are shown in Table 4.7. Only t-test fails to reject the null hypothesis. The shortest 95% confidence interval is the one from V_1-test. The results of the tests for the logarithms of the rainfall data are shown in Table 4.8. All the tests reject the null hypothesis. Now based on the results, we conclude that the rainfall amounts for the seeded clouds are more than than the rainfall amounts for the control clouds.

Although we have focused on one particular example, the V_1 and V_2 tests can be applied broadly to the testing of location parameter problems for most of data sets. Knowledge of the heaviness of the tails of the distributions, which the data come from, and the sample size will help to choose an appropriate test.

Table 4.5: Critical Point of V_i $i = 1, 2$, When $n_1 = n_2 = 26$.

Distributions	V_1			V_2		
	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$	$v_{0.95}$	$v_{0.975}$	$v_{0.99}$
Normal	0.384	0.465	0.57	0.44	0.533	0.648
Kappa(r=2)	0.386	0.47	0.576	0.415	0.506	0.62
Cauchy	0.39	0.48	0.60	0.384	0.472	0.588

Table 4.6: Descriptive Statistics of the Rainfall Data

Statistic	Seeded Clouds	Control Clouds	Log(Seeded)	Log(Control)
N	26	26	26	26
Min	4.1	1	1.41	0
Q_1	98.13	24.82	4.58	3.21
Median	221.6	44.2	5.4	3.79
Mean	442	164.6	5.13	3.99
$T_{0.5}(F_n)$	220.55	64.12	5.30	3.95
Q_3	406	159.2	6.0	5.07
Max	2746	1203	7.92	7.09
StDev	650.76	278.43	1.60	1.64
Δ_1	5.45	5.29	3.65	2.65

Figure 4.9: Empirical quantile-quantile Plot of the Rainfall Data

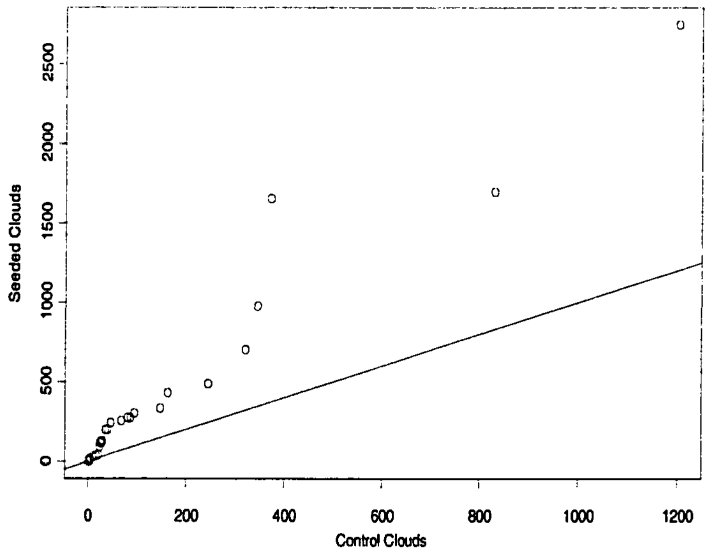


Figure 4.10: Empirical quantile-quantile Plot of the Log of the Rainfall Data

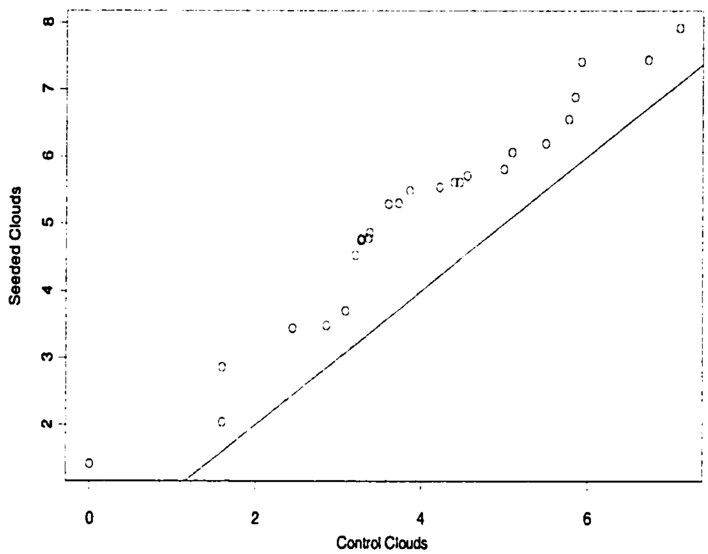


Figure 4.11: Box Plots of the Rainfall Data

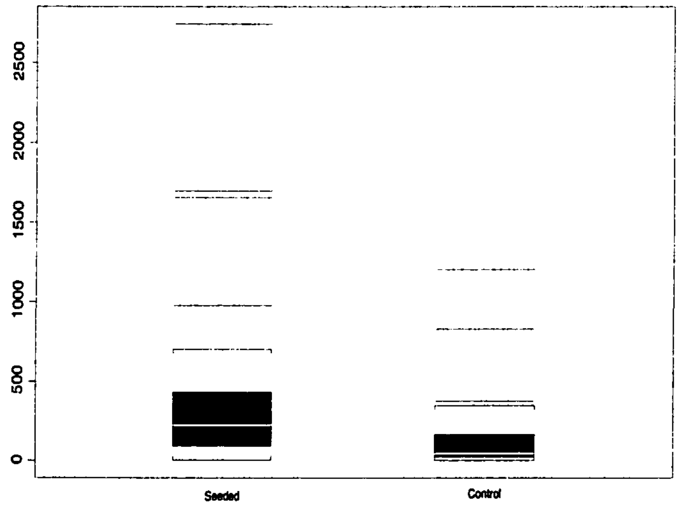


Figure 4.12: Box Plots of the Log of the Rainfall Data

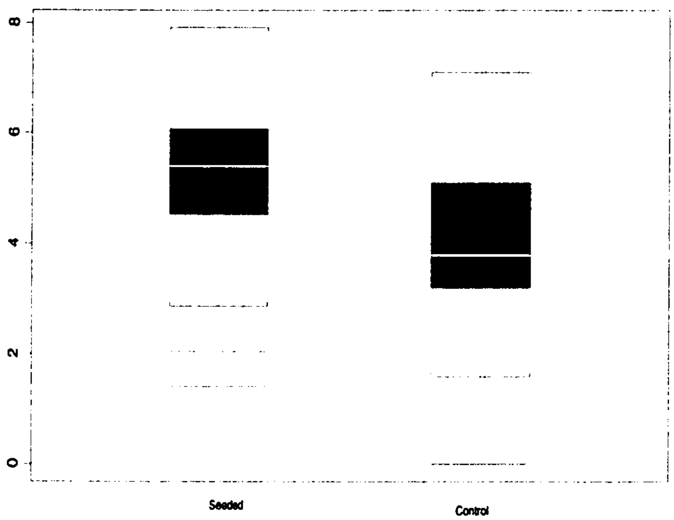


Table 4.7: Results of the Rainfall Data

Test	Test-value	95% CI	P-value
t	2.00	(-4.76, 559.56)	0.054
U	2.46	(14.10, 237.70)	0.014
V_1	0.71	(53.30, 258.96)	0.009
V_2	0.80	(59.53, 295.27)	0.003

Table 4.8: Results of Log of the Rainfall Data

Test	Test-value	95% CI	P-value
t	2.54	(0.24, 2.05)	0.014
U	2.46	(0.28, 2.10)	0.014
V_1	0.82	(0.58, 2.10)	0.004
V_2	0.98	(0.74, 2.48)	0.001

Example B: Children with Down syndrome generally show a pattern of retarded mental development, although some achieve higher intellectual levels than others. Two groups were compared in the article "Mental Development in Down Syndrome Mosaicism" (Fishler and Koch, (1991). *Amer. Journal on Mental Retardation, 96:345-351).*

"Thirty children with mosaic Down syndrome who were treated at USC Medical Center were selected to participate in the research project. The investigators then chose 30 children with trisomy 21 Down syndrome from among 350 who had been seen at the medical center. The 30 chosen were selected (with the help of a computer) to achieve the best possible matches for children in the mosaic group, using age, gender, and parental socioeconomic status as the criteria for matching. The result was 30 matched pairs of children. IQ levels for all of the children were determined." (Devore and Peack, 2001).

Table 4.9 shows the data and computed differences. The researchers proposed using this data to test the theory that children with mosaic Down syndrome generally achieve higher intellectual levels. The difference seems to come from a normal distribution $\Delta_1 = 2$. Table 4.10 shows descriptive statistics of the Down syndrome data and Figures 4.13 show plots of the differences. This will help us to see how robust V_1 and V_2 tests compare to t-test.

Table 4.9: Down Syndrome Data

Pair	Mosaic	Trisomy 21	Difference	Pair	Mosaic	Trisomy 21	Difference
1	73	71	2	16	55	60	-5
2	43	53	-10	17	61	48	13
3	69	58	11	18	63	55	8
4	89	71	18	19	87	59	28
5	53	50	3	20	64	78	-14
6	81	70	11	21	63	55	8
7	59	55	4	22	58	51	7
8	71	18	53	23	50	55	-5
9	65	31	34	24	59	53	6
10	53	57	-4	25	75	47	28
11	58	63	-5	26	61	50	11
12	71	47	24	27	91	63	28
13	92	44	48	28	43	46	-3
14	57	28	29	29	55	54	1
15	76	75	1	30	88	48	40

Table 4.10: Descriptive Statistics of the Down Syndrome Data

N	Min	Q_1	Median	Mean	$T_{0.5}(F_n)$	Q_3	Max	StDev	Δ_1
30	-14	1	8	12.33	9.75	27	53	17.19	2

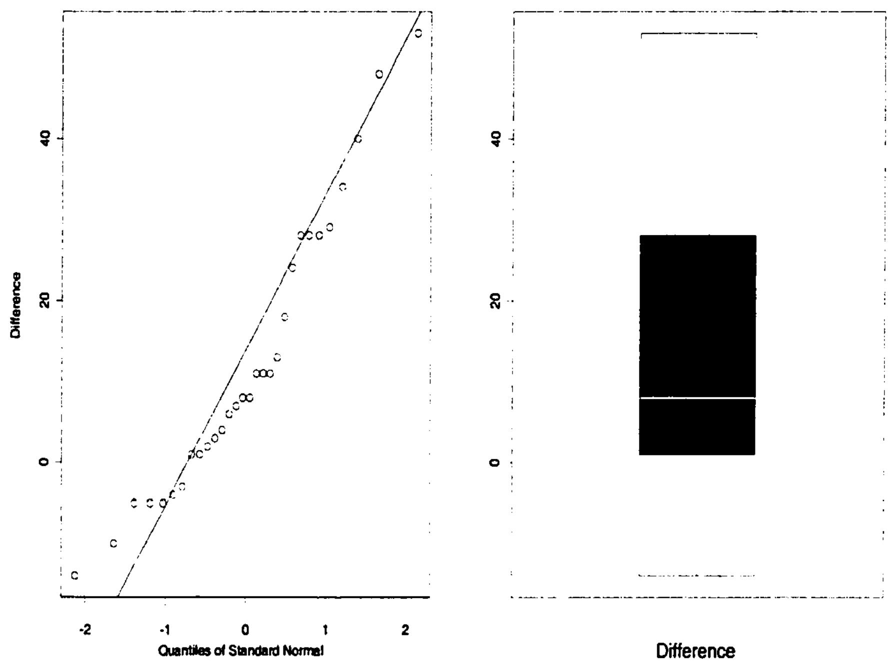
The question of interest can be answered by testing the hapothesis $H_0 : \delta_d = 0$ versus $H_A : \delta_d \geq 0$. Using the four tests as an example A, to carry out the hypothesis test. The results are shown in Table 4.11. As expected t-test has the shortest 95% confidence interval. Note that $v_{0.975}$ for the V_1 and V_2 tests under normal distribution with $n = 30$, are 0.306 and 0.35 respectively.

All tests reject H_0 except V_2 test. The data support the theory that the mean IQ for mosaic Down syndrome children is higher than the mean IQ for trisomy 21 Down syndrome children.

Table 4.11: Results of Testing Down Syndrome Data

Test	Test-value	95% CI	P-value
t	3.93	(5.91, 18.75)	0.0002
U	3.27	(4.00, 18.00)	0.010
V_1	0.375	(1.80, 17.71)	0.010
V_2	0.31	(-1.10, 17.10)	0.044

Figure 4.13: Plots of the Down Syndrome Data



4.4 Simulation

Sets i.i.d random samples of n_i, $i = 1, 2$; from the desired distribution $F(.)$ are generated using the Splus. Once the two samples are drawn, V_i, $i = 1, 2$, and t tests are calculated. The procedure is repeated N times at each of the 31 differences in location $\delta_1 - \delta_2 = 0$ to 1.5 by 0.05.

The power of each test is estimated by counting the number of the values of the test larger than the critical point of the null distribution and then dividing this number by N.

The simulation study was implemented separately (one at a time) for each of the five cases for a wide range of distribution. For each case , ($N = 2000$) samples of size ($n_1 = n_2 = 20$) were generated. The results were plotted.

Figure 4.14 shows the simulated power of the three tests when sampling from two standard normal distributions. As expected the t-test has the highest power of all, and both V_1 and V_2 were a good competitors.

The remainder of the plots show that the t-test has less much power than both the V_1 and V_2 tests. The simulated powers of the V_1 and V_2 tests are close to the theoretical powers. For example, when sampling from two Cauchy distributions V_1 and V_2 tests have almost the same power. This suggests using the V_1 and V_2 tests when the data is not normal.

Figure 4.14: Simulated Power When Sampling from Two Standard Normal Distributions

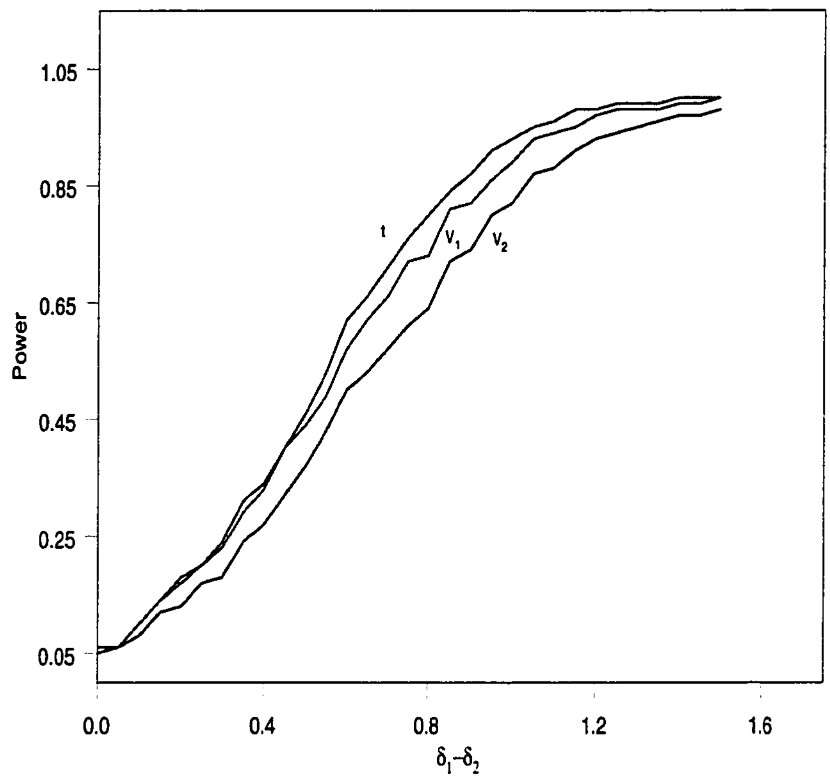


Figure 4.15: Simulated Power When Sampling from Two Standard Cauchy Distributions

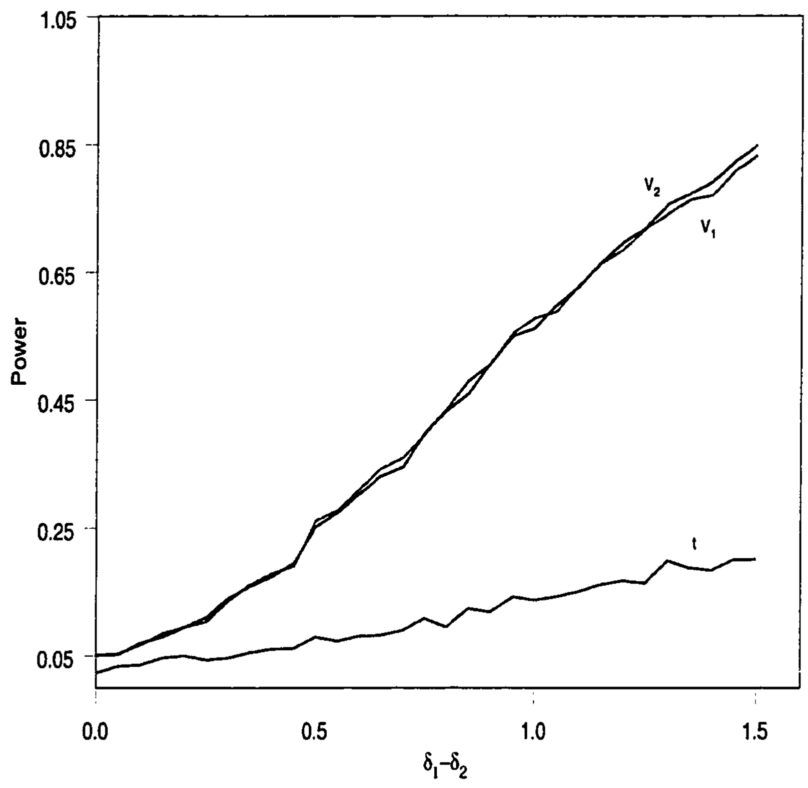


Figure 4.16: Simulated Power When Sampling from Kappa(r = 2) and Kappa ($r = 1$) Distributions

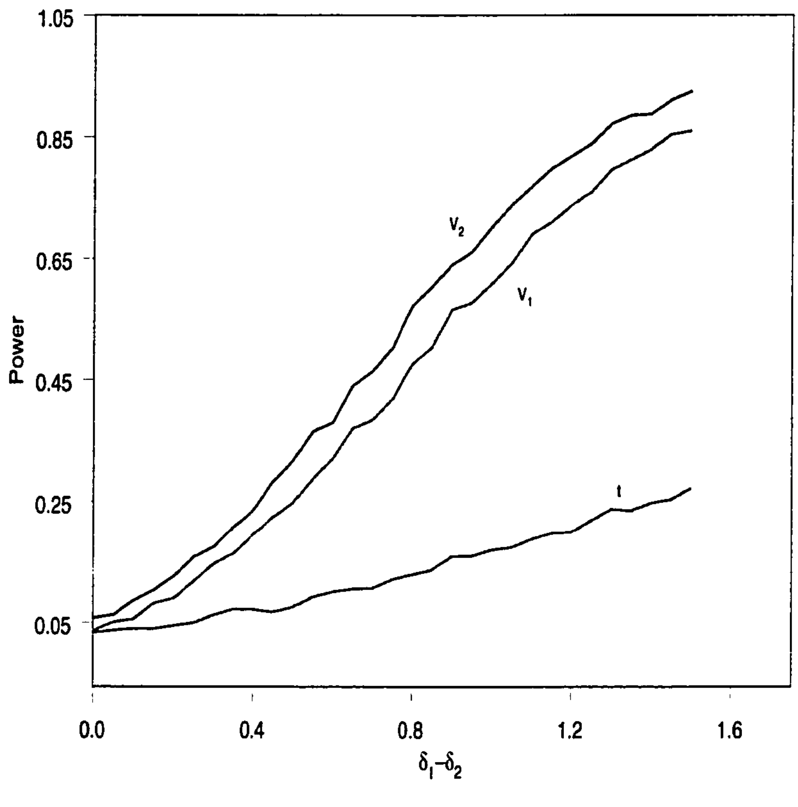


Figure 4.17: Simulated Power When Sampling from Normal(0, 2) and Cauchy(0, 1) Distributions

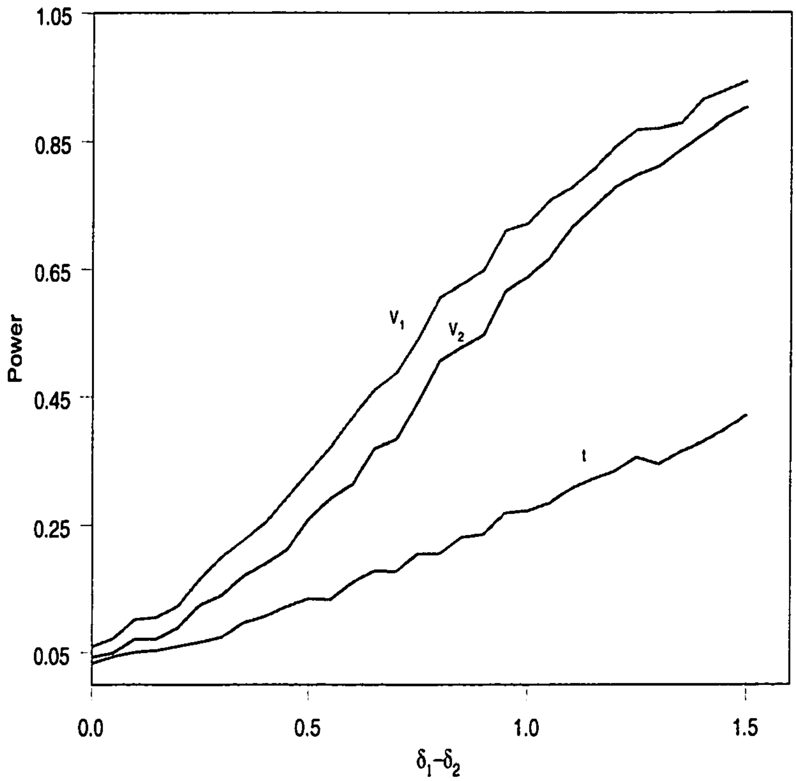
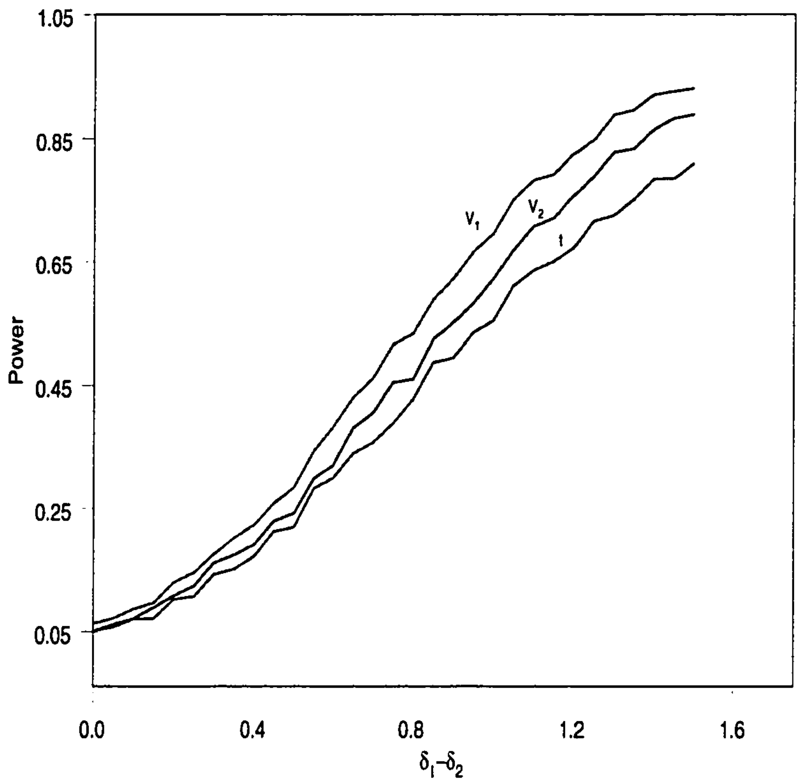


Figure 4.18: Simulated Power When Sampling from Normal(0, 2) and Kappa($r = 2$) Distributions



Chapter 5

CONCLUSIONS

Most real world data sets in statistics are not normally distributed and can only be dealt with by using approximations. This study introduced and discussed new, simple and easy class for testing location under symmetric distributions. From this study we can note the following:

1. Quantile functions reflect a common location shift characteristics of all the symmetric distributions.

2. Quantiles exist regardless of the existence of moments.

3. Shifts expressed in terms of quantiles are clearly defined for all absolutely continuous distributions, in contrast to those shifts expressed in terms of moments.

4. The use of symmetric distributions for the location shift alternative is justified by the fact that when the distribution F is symmetric, it is understood that the point of symmetry is the center of the distribution. Consequently, any location shift of a symmetric distribution can be most easily visualized in terms of the shift of a symmetry point of the distribution.

5. Trimming is always recommended for non-normal data.

6. The results obtained for the two classes Δ and V can be generalized to any symmetric distribution with a tail thickness in the neighborhood of the tail thickness of a corresponding distribution.

7. For stable distributions $\alpha > 1$ and kappa distributions $r \geq 2$, the V_1-test has more significant power advantages than the V_2-test.

8. For stable distributions $\alpha \leq 1$ and kappa distributions $r < 2$, the V_2-test has more significant power advantages than the V_1-test.

9. The V_1-test and V_2-test have more simulated power than the t-test in all cases except the normal.

10. Confidence intervals from the V_1 and V_2 tests are narrower than confidence intervals from the t-test when the data is not normal.

11. Although we have focused on two examples, the V_1 and V_2 tests can be applied broadly to the testing of location parameters problem for most of data sets.

12. Knowledge of the heaviness of the tails of the distributions, which the data come from, and the sample size will help to choose an appropriate test.

13. The V_1-test and V_2-test can be used to test a location parameter for difference on paired observations.

14. The power comparisons for location shift detection indicate that some tests of the class provide superior detection efficiency for location shifts of heavy tailed distributions.

15. The $V_1$and V_2 tests possess substantial computational advantages over other nonparametric tests.

Bibliography

Andrews, D. F., Bickel, P. J., Hampel, F. R., Huber, P. J., H., R. W., and Tukey, J. W. (1972). *Robust Estimates of Location.* Princeton University Press, Princeton, New Jersey.

Badahdah, S. O. and Siddiqui, M. M. (1991). Adaptive robust estimation of location for stable distributions. *Communications in Statistics-Theory and Methods*, 20:631–652.

Bickel, P. J. (1965). On some robust estimates of location. *Ann. Math. Statist.*, 36:847–858.

Crow, E. L. and Siddiqui, M. M. (1967). Robust estimation of location. *Journal of the American Statistical Association*, 62:353–389.

David, H. A. (1981). *Order Statistics.* J. Wiley & Sons, New York.

Devore, J. and Peack, R. (2001). *Statistics The Exploration and Analysis of Data.* Duxbury, Pacific Grove, CA.

Fama, E. F. and Roll, R. (1971). Parameter estimates of symmetric stable distributions. *Journal of the American Statistical Association*, 66:331–338.

Hampel, F. R. (1969). Contributions to the theory of robust estimation.

Hampel, F. R. (1971). A general qualitative definition of robustness. *Ann. Math. Statist.*, 42:1887–1896.

Hinkley, D. V. (1969). On the ratio of two correlated normal random variables. *Biometrika*, 56:635–639.

Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist.*, 35:73–101.

Lehmann, E. L. (1997). *Testing Statistical Hypotheses (Second Edition)*. Springer-Verlag.

Mielke, P. W. (1972). Asymptotic behavior of two-sample tests based on powers of rank for detecting scale and location alternatives. *Journal of the American Statistical Association*, 67:850–854.

Nolan, J. P. (1997). Numerical calculation of stable densities and distribution functions. *Communications in Statistics-Stochastic Models*, 13:759–774.

Paulson, A. S., Holcomb, E. W., and Leitch, R. A. (1975). The estimates of the parameters of the stable laws. *Biometrika*, 62:163–170.

Pitman, E. J. G. (1949). *Lecture Notes on Nonparametric Statistics*. Columbia Univ. Press, New York.

Press, S. J. (1972). Estimation in univariate and multivariate stable distributions. *Journal of the American Statistical Association*, 67:842–846.

Sen, P. K. (1959). On the moments of the sample quantiles. *Calcutta Statistical Association Bulletin*, 9:1–19.

Siddiqui, M. M. and Butler, C. (1969). Asymptotic joint distribution of linear systematic statistics from multivariate distributions. *Journal of the American Statistical Association*, 64:300–305.

Simpson, J. and Olsen, A. (1975). A Bayesian analysis of a multiplicative treatment effect in weather modification. *Technometrics*, 17:161–166.

Staudte, R. G. and Sheather, S. J. (1990). *Robust Estimation And Testing*. J. Wiley & Sons, New York.

Student, W. G. (1908). The probable error of a mean. *Biometrika*, 6:1–25.

Van Eeden, C. (1970). Efficiency-robust estimation of location. *Ann. Math. Statist.*, 41:172–181.

Wolter, K. M. (1985). *Introduction to Variance Estimation*. Springer-Verlag, New York.