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Eigenvalues and Completeness for Regular and Simply Irregular Two-Point Differential Operators

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Preface

In this monograph we develop the spectral theory for an nth order two-point differential operator L in the Hilbert space $L^2[0,1]$, where L is determined by a formal differential operator ℓ having variable coefficients and by linearly independent boundary values $B_1, \ldots, B_n$. This new work is a natural extension of our earlier work in [34]. Using the Birkhoff approximate solutions $z_k(t,\rho,m)$, $k = 0, 1, \ldots, n-1$, of the differential equation

$$(\rho^n I - \ell)u(t) = 0,$$

together with the associated approximate characteristic determinant $\widehat{\Delta}(\rho, m)$, we proceed to classify L as belonging to one of three possible classes: regular, simply irregular, or degenerate irregular. The regular class has been studied extensively, and has a more or less complete spectral theory; the simply irregular class is a new and unexplored class, and its spectral theory, together with the regular class, is the main subject of this book; the degenerate irregular class has never been studied, and is a topic for future work. Throughout we assume that L is regular or simply irregular.

Working on two sectors T_0 and T_1 having angular opening π/n, we use high order asymptotic expansions to construct two independent sets of solutions of the differential equation $(\rho^n I - \ell)u(t) = 0$:

$$v_{00}(t,\rho), v_{01}(t,\rho), \ldots, v_{0\,n-1}(t,\rho) \quad \text{for } \rho \in T_0$$

and

$$v_{10}(t,\rho), v_{11}(t,\rho), \ldots, v_{1\,n-1}(t,\rho) \quad \text{for } \rho \in T_1.$$

The $v_{0k}(t,\rho)$, $v_{1k}(t,\rho)$ behave asymptotically like the Birkhoff approximate solutions $z_k(t,\rho,m)$. They are used to construct characteristic determinants $\Delta_0(\rho)$ and $\Delta_1(\rho)$ on the sectors T_0 and T_1, and to construct representations of the resolvent $R_\lambda(L)$ and the corresponding Green's function $G(t,s;\lambda)$ for $\lambda = \rho^n$ with ρ in either T_0 or T_1. The spectrum of L is then computed; it consists of two sequences of eigenvalues,

$$\lambda'_k = (\rho'_k)^n, \qquad k = k_0, k_0 + 1, \ldots,$$

and

$$\lambda''_k = (\rho''_k)^n, \qquad k = k_0, k_0 + 1, \ldots,$$

plus a finite number of additional points, where the ρ'_k are zeros of $\varDelta_0(\rho)$ in T_0 and the ρ''_k are zeros of $\varDelta_1(\rho)$ in T_1. We establish asymptotic formulas that detail the structure of the ρ'_k, ρ''_k. After deriving growth rates for the resolvent $R_\lambda(L)$ on various regions of the λ plane, we show that the generalized eigenfunctions of L are complete in $L^2[0, 1]$.

Our method for obtaining the higher order expansion of solutions is much simpler than previous expansion methods, and the results for the eigenvalues and the completeness of the generalized eigenfunctions are the first that we are aware of for a large class of nth order irregular differential operators having variable coefficients.

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Contents

1

Introduction

In this monograph we develop the spectral theory of an nth order two-point differential operator L in the Hilbert space $L^2[0,1]$. The differential operator is determined by a formal differential operator ℓ having variable coefficients, and by independent boundary values $B_1, \ldots, B_n$ that may be either regular or irregular. This initial chapter summarizes the basic spectral properties of the differential operator, presents some historical remarks concerning earlier studies of the spectral theory, and gives an overview of the results established here. The spectral theory for these differential operators is by no means complete — there remain many unsolved problems.

1.1 Two-Point Differential Operators

Throughout we work in the complex Hilbert space $L^2[0,1]$. Let n be a positive integer with $n \geq 2$, and let $H^n[0,1]$ denote the subspace of $L^2[0,1]$ consisting of all functions $u \in C^{n-1}[0,1]$ with $u^{(n-1)}$ absolutely continuous on $[0,1]$ and with $u^{(n)} \in L^2[0,1]$. Let

$$\ell = \sum_{p=0}^{n} a_p(t) \left(\frac{d}{dt} \right)^p$$

be an nth order formal differential operator on $[0,1]$, where the leading coefficients are assumed to be $a_n(t) = 1/\,\mathrm{i}^n$ and $a_{n-1}(t) = 0$; let

$$B_i(u) = \sum_{p=0}^{n-1} \alpha_{ip} u^{(p)}(0) + \sum_{p=0}^{n-1} \beta_{ip} u^{(p)}(1), \qquad i = 1, \ldots, n,$$

be a set of n linearly independent boundary values on $H^n[0,1]$; and let L be the nth order two-point differential operator in $L^2[0,1]$ defined by

$$\mathcal{D}(L) = \{u \in H^n[0,1] \mid B_i(u) = 0,\, i = 1, \ldots, n\}, \qquad Lu = \ell u.$$

We assume that the coefficients a_p are infinitely differentiable on $[0,1]$. The basic spectral theory of the differential operator L is developed in [28, 34].

Let us review some of the spectral properties of the differential operators. First, L is a Fredholm operator in the Hilbert space $L^2[0,1]$. Consequently, as a foundation for our work, we know that (a) L is a densely defined closed linear operator in $L^2[0,1]$, (b) the range $\mathcal{R}(L)$ is a closed subspace of $L^2[0,1]$, and (c) the null spaces $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$ are finite-dimensional subspaces of $L^2[0,1]$, with their respective dimensions less than or equal to n. Since the adjoint operator L^* is also a two-point differential operator, it shares these same properties. For the dimensions of the null spaces $\mathcal{N}(L)$ and $\mathcal{N}(L^*)$, we have the relation

$$\dim \mathcal{N}(L) = \dim \mathcal{N}(L^*), \tag{1.1}$$

so L is a Fredholm operator of index 0.

Second, for each $\lambda \in \mathbb{C}$ the operator $\lambda I - L$ is again a two-point differential operator, which implies that $\lambda I - L$ is a Fredholm operator in $L^2[0,1]$, as are all its powers $(\lambda I - L)^k$, $k = 0, 1, 2, \ldots$. It follows that the Fredholm set for L is $\Phi(L) = \mathbb{C}$, and for the index

$$i(\lambda I - L) = \dim \mathcal{N}(\lambda I - L) - \dim \mathcal{N}(\overline{\lambda} I - L^*) = 0 \tag{1.2}$$

for all $\lambda \in \mathbb{C}$. These results are useful for studying the eigenspaces and generalized eigenspaces of L.

Third, for each $\lambda \in \mathbb{C}$ we can form the subspace

$$\mathcal{M}_\lambda = \bigcup_{k=1}^{\infty} \mathcal{N}((\lambda I - L)^k).$$

Relevant to this subspace is its dimension

$$\nu(\lambda) = \dim \mathcal{M}_\lambda = \lim_{k \to \infty} \dim \mathcal{N}((\lambda I - L)^k),$$

which is the algebraic multiplicity of λ; and relevant to the differential operator $\lambda I - L$ is its ascent $\alpha(\lambda I - L)$, which is the smallest integer $k \geq 0$ such that $\mathcal{N}((\lambda I - L)^k) = \mathcal{N}((\lambda I - L)^{k+1})$. In case $\mathcal{M}_\lambda \neq \{0\}$, then λ is an eigenvalue of L and $\mathcal{M}_\lambda$ is the generalized eigenspace of L corresponding to λ. The algebraic multiplicity $\nu(\lambda)$ is finite if and only if the ascent $\alpha(\lambda I - L)$ is finite.

Fourth, the spectrum $\sigma(L)$ is precisely the set of all eigenvalues of L; it is either a countable set having no limit points in $\mathbb{C}$ or it is equal to all of $\mathbb{C}$. The complement of the spectrum is the resolvent set $\rho(L)$. For each λ belonging to $\rho(L)$, the resolvent $R_\lambda(L) = (\lambda I - L)^{-1}$ is an L^2-integral operator on $L^2[0,1]$, and the Green's function $G(t,s;\lambda)$ for $\lambda I - L$ is the L^2-kernel of $R_\lambda(L)$:

$$R_\lambda(L)u(t) = \int_0^1 G(t,s;\lambda)u(s)\,ds, \qquad 0 \leq t \leq 1, \tag{1.3}$$

for $u \in L^2[0,1]$. Thus, for each $\lambda \in \rho(L)$ the resolvent $R_\lambda(L)$ is a Hilbert-Schmidt operator on $L^2[0,1]$, and the differential operator L is a Hilbert-Schmidt discrete operator in case $\rho(L) \neq \emptyset$.

Fifth, introducing the two formal differential operators

$$\tau = a_n(t)\left(\frac{d}{dt}\right)^n = \frac{1}{\mathrm{i}^n}\left(\frac{d}{dt}\right)^n \quad \text{and} \quad \sigma = \sum_{p=0}^{n-2} a_p(t)\left(\frac{d}{dt}\right)^p,$$

we can then define differential operators T and S in $L^2[0,1]$ by

$$\mathcal{D}(T) = \mathcal{D}(L) = \{u \in H^n[0,1] \mid B_i(u) = 0,\ i = 1, \ldots, n\},$$
$$Tu = \tau u = \mathrm{i}^{-n} u^{(n)},$$

and

$$\mathcal{D}(S) = H^{n-2}[0,1], \quad Su = \sigma u = \sum_{p=0}^{n-2} a_p(t) u^{(p)}.$$

Clearly $\ell = \tau + \sigma$ and $L = T + S$. The differential operator T is called the *principal part* of the differential operator L, and we can consider the operator L as a perturbation of the operator T by the operator S. The assumption that $a_n(t) = 1/\mathrm{i}^n$ produces a relatively simple principal part T, with τ being formally self-adjoint. The net effect of this will be to locate the spectrum of L near the real axis in the complex plane.

Sixth, there are several useful Banach space and Hilbert space structures available for the subspace $H^n[0,1]$. The maximal operator $T_1(\ell)$ corresponding to the formal differential operator ℓ is an nth order differential operator in $L^2[0,1]$, and hence, $T_1(\ell)$ is a Fredholm operator in $L^2[0,1]$, and its domain $H^n[0,1]$ becomes a Hilbert space under the associated graph norm structure:

$$(u,v)_{T_1(\ell)} = (u,v) + (\ell u, \ell v), \quad \|u\|_{T_1(\ell)} = (u,u)_{T_1(\ell)}^{1/2} = \left[\|u\|^2 + \|\ell u\|^2\right]^{1/2}.$$

A second norm for $H^n[0,1]$ is defined by

$$|u|_{H^n} = \sum_{p=0}^{n-1} \|u^{(p)}\|_\infty + \|u^{(n)}\|,$$

and under this norm $H^n[0,1]$ becomes a Banach space. A third inner product and norm structure for $H^n[0,1]$ is given by

$$(u,v)_n = \sum_{p=0}^{n} (u^{(p)}, v^{(p)}), \quad \|u\|_n = (u,u)_n^{1/2} = \Big[\sum_{p=0}^{n} \|u^{(p)}\|^2\Big]^{1/2},$$

and under this structure $H^n[0,1]$ becomes a Hilbert space. The norms $\|\ \|_{T_1(\ell)}$, $|\ |_{H^n}$, and $\|\ \|_n$ are equivalent norms for $H^n[0,1]$. We refer to this common topological structure as the H^n-*structure* or the H^n-*Sobolev structure* for $H^n[0,1]$. By considering the norm $|\ \ |_{H^n}$, we see immediately that convergence in the H^n-structure is uniform convergence on $[0,1]$ of the derivatives of orders $0,1,\dots,n-1$, together with L^2-convergence of the nth derivatives. It follows that each one of the boundary values $B_i\colon H^n[0,1]\to\mathbb{C}$, $i=1,\dots,n$, is a continuous linear functional on $H^n[0,1]$ under the H^n-structure.

Let us introduce the $n\times 2n$ *boundary coefficient matrix* associated with the boundary values $B_1,\dots,B_n$:

$$A := \begin{pmatrix} \alpha_{1\,n-1} & \beta_{1\,n-1} & \alpha_{1\,n-2} & \beta_{1\,n-2} & \cdots & \alpha_{1\,0} & \beta_{1\,0} \\ \alpha_{2\,n-1} & \beta_{2\,n-1} & \alpha_{2\,n-2} & \beta_{2\,n-2} & \cdots & \alpha_{2\,0} & \beta_{2\,0} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n\,n-1} & \beta_{n\,n-1} & \alpha_{n\,n-2} & \beta_{n\,n-2} & \cdots & \alpha_{n\,0} & \beta_{n\,0} \end{pmatrix}.$$

Without loss of generality we can assume that the matrix A is in reduced row echelon form with rank n. This corresponds to a normalization of the boundary values $B_1,\dots,B_n$. For $i=1,\dots,n$ let m_i denote the order of the boundary value B_i, i.e., m_i is the largest nonnegative integer such that either $\alpha_{i\,m_i}\neq 0$ or $\beta_{i\,m_i}\neq 0$. Clearly $0\le m_i\le n-1$, and from the normalization it follows that $m_1\ge m_2\ge\cdots\ge m_{n-1}\ge m_n$. Set

$$p_0 := \sum_{i=1}^{n} m_i.$$

The integer p_0 will play a central role in the classification of the differential operator L as being either regular or irregular.

1.2 Historical Remarks

The study of regular boundary value problems for nth order two-point differential operators began in 1908 in the pioneering work of Birkhoff [3, 4]. He derived approximate solutions of the differential equation

$$(\lambda I-\ell)u(t)=\lambda u(t)-\sum_{p=0}^{n} a_p(t)u^{(p)}(t)=0, \qquad 0\le t\le 1,$$

utilised the approximate solutions to construct asymptotically n independent solutions of this differential equation, formed the characteristic determinant and the Green's function, defined the class of regular boundary conditions, characterized the simple eigenvalues, and developed an expansion theorem (pointwise convergence) for a piecewise C^1 function f in terms of the eigenfunctions of L. The expansion behaves like a Fourier series except in the vicinity of the endpoints $t=0$ and $t=1$. All of his results are for regular boundary conditions.

In the paper [44] Stone generalizes Birkhoff's work from the classical C^n setting to the modern H^n setting. The first expansion theorem for solutions of the differential equation (Theorem III) is a first order expansion: it is stated and proved using translated sectors T. The second expansion theorem is a higher order version (Theorem III$'$), and it is quoted directly from Birkhoff without proof. Stone assumes regular boundary conditions. He shows that periodic boundary conditions are regular and lead to Fourier series when applied to the principal part; quotes Birkhoff on the structure of the eigenvalues for regular boundary values problems; develops the various Green's functions; and on any interval $[a, b]$ where $0 < a < b < 1$, proves that the eigenfunction expansion of any function f summable on $[0, 1]$ converges to f if and only if the Fourier series of f converges to f, i.e., the eigenfunction expansion of f is uniformly equiconvergent with the Fourier expansion of f on $[a, b]$ (see Theorem XIII on p. 723 for the case $n = 2\nu - 1$ and p. 756 for the $n = 2\nu$ analogue).

Other classical works for regular boundary value problems are Tamarkin [47], Hopkins [15], Ward [51, 52], and Coddington and Levinson [6]. For more modern treatments of regular boundary value problems using functional analysis and operator theory, see Dunford and Schwartz [8] and Locker [34]. In these modern treatments the emphasis is on the L^2-expansion problem. Naimark [36] is a mix of both classical and modern; much of his work is derived from Birkhoff [3, 4] and Stone [44].

In the paper [45] Stone examines the expansion problem for the case $n = 2$ with irregular boundary conditions. For $n = 2$ the irregular boundary conditions can have only two possible forms (Theorem I), and this leads to a simple form for the characteristic determinant. The differential operator is classified as being of finite type M, $1 \leq M < \infty$, or of type Ω. Only differential operators of type M are discussed. He then characterizes the eigenvalues (Theorem IV), and obtains a convenient description of the Green's function. The equiconvergence result is no longer true, but equiconvergence can be restored by applying the summability process of Riesz typical means to both the eigenfunction expansion and the Fourier series of f (see Theorem V and Theorem IX). This paper of Stone's appears to be the first significant contribution to the theory of irregular boundary value problems; the logarithmic case for the zeros of the characteristic determinant appears for the first time.

Benzinger [2] and Schultze [41] give partial results for irregular boundary value problems for the case of arbitrary n. They use only first order expansions of solutions, quoting Birkhoff [4], and their results apply to very special classes of irregular boundary conditions, which they refer to as *Stone regular* and *strongly irregular*, respectively. It is difficult to identify exactly what the Stone regular class is, while the strongly irregular class is a generalization of the class of all irregular decomposing boundary conditions (see Theorem 1 in [41]). They characterize the eigenvalues, and prove Riesz summability of eigenfunction expansions. See Theorem 4.2 in [2] and Theorem 6 in [41].

Y. Yakubov [55] has established the completeness of the generalized eigenfunctions for a large class of irregular boundary value problems in the special case $n = 2$. His results allow the spectral parameter to appear in both the differential operator and the boundary values, permits the boundary values to involve interior points of the interval $[0, 1]$ as well as the endpoints, and includes an abstract operator in the differential operator and abstract linear functionals in the boundary values. S. Yakubov [53, p. 124] has extablished a completeness theorem for the case of arbitrary order n and regular boundary conditions.

In [25, 26] Lang and Locker consider the second order case for $\ell = -(d/dt)^2$ and for both regular and irregular boundary values B_1, B_2. They develop the characteristic determinant and Green's function, compute the eigenvalues and the corresponding algebraic multiplicities and ascents, determine the family of projections associated with L, and solve the L^2- expansion problem. Specifically, if $S_\infty(L)$ is the subspace of $L^2[0, 1]$ containing all functions that can be expressed as an infinite series of the generalized eigenfunctions of L, then for the case of regular boundary values

$$S_\infty(L) = \overline{S_\infty(L)} = L^2[0, 1],$$

while for the case of irregular boundary values

$$S_\infty(L) \neq \overline{S_\infty(L)} = L^2[0, 1].$$

Case VIII is a regular case which is an exception: see the paper [29].

In the four part series [30, 31, 32, 33] we develop the spectral theory for the case $n = 2$ and for the general formal differential operator $\ell = -(d/dt)^2 + q(t)$ and for regular and irregular boundary values B_1, B_2. For this general case we also develop the characteristic determinant and Green's function, compute the eigenvalues and the corresponding algebraic multiplicities and ascents, determine the family of projections associated with L, and show that for the regular cases (Cases 1, 2, 3A)

$$S_\infty(L) = \overline{S_\infty(L)} = L^2[0, 1],$$

while for the irregular cases (Cases 3B, 4)

$$S_\infty(L) \neq \overline{S_\infty(L)} = L^2[0, 1].$$

There is also a degenerate Case 5 that is not discussed, where the spectrum is either $\emptyset$ or $\mathbb{C}$ when $q(t) \equiv 0$.

In Chapter 6 of the recent monograph [34], we develop the spectral theory for the general nth order two-point differential operator L determined by regular boundary values. Included is a determination of the eigenvalues and the corresponding algebraic multiplicities and ascents, a computation and bounding of the family of projections associated with L, and the L^2- expansion result

$$\mathbb{S}_\infty(L) = \overline{\mathbb{S}_\infty(L)} = L^2[0,1].$$

See Theorems 4.1, 5.1 (multiple eigenvalue case), and 6.1 of Chapter 6. Also included in this monograph are some basic results for irregular boundary values. Specifically, for the special case $\ell = \tau = \mathrm{i}^{-n}(d/dt)^n$ and $L = T$, the eigenvalues, algebraic multiplicities, and ascents of T are characterized in Sections 7 and 8 of Chapter 4; and the generalized eigenfunctions of T are shown to be complete in $L^2[0,1]$ in Theorems 9.1 and 9.2 of Chapter 4:

$$\overline{S_\infty(T)} = L^2[0,1].$$

There are no results for the general differential operator L in the case of irregular boundary values.

1.3 Summary of Results

Let us briefly outline the main features of the spectral theory presented here:

- With the Birkhoff approximate solutions $z_k(t,\rho,m)$, $k = 0,1,\dots,n-1$, and the approximate characteristic determinant $\widehat{\varDelta}(\rho,m)$ as motivation, we first calculate two sequences of constants a_κ, $\kappa = p_0, p_0-1,\dots,1,0,-1,\dots$, and b_κ, $\kappa = p_0, p_0-1,\dots,1,0,-1,\dots$, and use them to classify the differential operator L as being regular, simply irregular, or degenerate irregular.
- Assuming that L is regular or simply irregular, we determine the leading constants a_p, b_q and then fix the sectors S_0, S_1, the translated sectors T_0, T_1, and the integer m. This is a key step in the logical progession of our work.
- We use asymptotic expansions to construct independent solutions of the differential equation $(\rho^n I - \ell)u(t) = 0$:

  $$v_{00}(t,\rho), v_{01}(t,\rho), \dots, v_{0\,n-1}(t,\rho) \quad \text{for } \rho \in T_0,$$

  $$v_{10}(t,\rho), v_{11}(t,\rho), \dots, v_{1\,n-1}(t,\rho) \quad \text{for } \rho \in T_1.$$

  These are high order expansions with the solutions behaving asymptotically like the Birkhoff approximate solutions $z_k(t,\rho,m)$.
- Using these solutions, we construct the characteristic determinant $\varDelta_0(\rho)$ for $\rho \in T_0$ and the characteristic determinant $\varDelta_1(\rho)$ for $\rho \in T_1$.
- These solutions are also used to represent the resolvent $R_\lambda(L)$ and the Green's function $G(t,s;\lambda)$ for $\lambda = \rho^n$ with $\rho \in T_0$ and $\varDelta_0(\rho) \neq 0$ or with $\rho \in T_1$ and $\varDelta_1(\rho) \neq 0$.
- We then calculate the zeros of the characteristic determinants $\varDelta_0(\rho)$ and $\varDelta_1(\rho)$, producing the eigenvalues of L and their corrresponding algebraic multiplicities.
- Growth rates for the resolvent $R_\lambda(L)$ are determined on various regions of the λ plane, and the generalized eigenfunctions of L are shown to be complete in $L^2[0,1]$.

We now summarize the results of this monograph, most of them appearing for the first time. The material in Chapters 2, 3, and 4 is discussed in some detail since these chapters contain the newest and most interesting results. The development of the characteristic determinant and the Green's function, the characterization of the eigenvalues, and the establishment of the completeness of the generalized eigenfunctions given in Chapters 5–9 follow along somewhat familiar lines, although many cases need to be worked through to establish these results. Throughout we express the order n of the differential operator L in the form $n = 2\nu$ for n even and the form $n = 2\nu - 1$ for n odd, and let $\omega_k = \mathrm{e}^{\mathrm{i}2\pi k/n}$, $k = 0, \pm 1, \pm 2, \ldots$, denote the nth roots of unity.

For any complex number $\rho \neq 0$ and for integers k and m with $0 \leq k \leq n-1$ and $m > n$, we introduce in Chapter 2 the mth order Birkhoff approximate solutions

$$z_k(t, \rho, m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}$$

of the differential equation $(\rho^n I - \ell)u(t) = 0$. These approximate solutions are formed in the same way as Birkhoff first formed them in his original paper [4], viz., substitute $z_k(t, \rho, m)$ into the expression $-(\mathrm{i}^n/\rho^n)(\rho^n I - \ell)u(t)$, and then determine the coefficient functions $z_{kj}(t)$ by requiring that the coefficients of the terms $\mathrm{e}^{\mathrm{i}\rho\omega_k t}\rho^{-s}$, $s = 0, 1, \ldots, m$, all vanish identically on the interval $[0, 1]$ — the terms involving $\mathrm{e}^{\mathrm{i}\rho\omega_k t}\rho^{-s}$, $s = m+1, m+2, \ldots, n+m-1$, still remain. See Theorem 2.1.

We establish several important properties for these approximate solutions. First, the $z_k(t, \rho, m)$ satisfy the conditions

$$z_k(t, \rho, m) = z_0(t, \rho\omega_k, m), \qquad k = 1, \ldots, n-1,$$

so to calculate the Birkhoff approximate solutions, one needs only calculate $z_0(t, \rho, m)$. Second, the coefficient functions $z_{0j}(t)$ that appear in $z_0(t, \rho, m)$ are independent of the integer m, and are determined by an infinite set of recursion relations. Specifically, the initial coefficient is taken to be $z_{00}(t) \equiv 1$, and then the derivatives of the coefficient functions satisfy the recursion relations

$$z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{n-1}} \sum_{j=0}^{s-2} \ell_{s-j}\, z_{0j}(t)$$

for $2 \leq s \leq n$ and

$$z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{n-1}} \sum_{j=s-n}^{s-2} \ell_{s-j}\, z_{0j}(t)$$

for $n + 1 \leq s < \infty$, with the coefficient functions themselves calculated by taking

$$z_{0\,s-1}(t) = \int_0^t z'_{0\,s-1}(\xi)\, d\xi$$

for $0 \leq t \leq 1$. See equations (2.20) and (2.21). Here the ℓ_q, $q = 2, \ldots, n$, are known qth order formal differential operators with $\ell_n = \mathrm{i}^n \ell$. Thus, the derivative $z'_{0\,s-1}(t)$ is determined by the $n-1$ preceding functions $z_{0\,s-n}(t)$, $\ldots$, $z_{0\,s-2}(t)$ and their derivatives up to order n, and only these $n-1$ predecessors are used and not all $s-2$ predecessors. This shows that the recursion relations possess a banded structure. In forming the Birkhoff approximate solutions $z_k(t, \rho, m)$, $k = 0, 1, \ldots, n-1$, the integer m is used solely to specify the number of terms that appear in the summation.

In Chapter 3 we assume that the integer m satisfies the conditions $m > n$ and $m > p_0$, and proceed to introduce the modified Birkhoff approximate solutions

$$\begin{aligned} y_k(t, \rho, m) &= z_k(t, \rho, m), && k = 0, 1, \ldots, \nu - 1, \\ y_k(t, \rho, m) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} z_k(t, \rho, m), && k = \nu, \ldots, n - 1, \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. The approximate characteristic determinant in defined in terms of these functions by

$$\widehat{\Delta}(\rho, m) = \det(B_i(y_k(\,\cdot\,, \rho, m)))$$

for $\rho \neq 0$ in $\mathbb{C}$, where for $i = 1, \ldots, n$ the functions $B_i(y_k(\,\cdot\,, \rho, m))$ have the form

$$\begin{aligned} B_i(y_k(\,\cdot\,, \rho, m)) &= \widehat{P}_{ik}(\rho, m) + \widehat{Q}_{ik}(\rho, m)\mathrm{e}^{\mathrm{i}\rho\omega_k}, && k = 0, 1, \ldots, \nu - 1, \\ B_i(y_k(\,\cdot\,, \rho, m)) &= \widehat{P}_{ik}(\rho, m) + \widehat{Q}_{ik}(\rho, m)\mathrm{e}^{-\mathrm{i}\rho\omega_k}, && k = \nu, \ldots, n - 1. \end{aligned}$$

We show that the functions $\widehat{P}_{ik}(\rho, m)$, $\widehat{Q}_{ik}(\rho, m)$ can be expressed in the form

$$\begin{aligned} \widehat{P}_{ik}(\rho, m) &= \sum_{s=-(m-m_i-1)}^{m_i} p_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{p}_{iks}(m)\rho^s, \\ \widehat{Q}_{ik}(\rho, m) &= \sum_{s=-(m-m_i-1)}^{m_i} q_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{q}_{iks}(m)\rho^s \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the constants p_{iks}, q_{iks} are independent of the integer m and the constants $\hat{p}_{iks}(m)$, $\hat{q}_{iks}(m)$ are dependent on the integer m. For fixed i and k, we have explicit formulas for calculating the constants p_{iks}, q_{iks}, $s = m_i, m_i - 1, \ldots, 1, 0, -1, \ldots$. These sequences form invariants for the differential operator L.

Assume that n is even, $n = 2\nu$, and introduce the sectors

$$\begin{aligned} &S_0\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } 0 \leq \theta \leq \frac{\pi}{n}\,, \\ &S_1\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} \leq \theta \leq 0 \end{aligned}$$

in the ρ plane. Using the constants p_{iks}, q_{iks}, we construct three sequences of constants a_κ, b_κ, c_κ, $\kappa = p_0, p_0 - 1, \dots, 1, 0, -1, \dots$. These sequences are also independent of the integer m, and therefore form invariants for the differential operator L. The third sequence is actually determined from the first sequence by the conditions $a_\kappa = -\,\omega_\kappa c_\kappa$ for $\kappa = p_0, p_0 - 1, \dots, 1, 0, -1, \dots$. In terms of these sequences the approximate characteristic determinant can be expressed in the form

$$\begin{aligned}\widehat{\varDelta}(\rho, m) = {} & \pi_2(\rho, m)\mathrm{e}^{2\mathrm{i}\rho} + \pi_1(\rho, m)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho, m) \\ & + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widehat{\varPhi}_2(\rho, m) \Big] \mathrm{e}^{2\mathrm{i}\rho} \\ & + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widehat{\varPhi}_1(\rho, m) \Big] \mathrm{e}^{\mathrm{i}\rho} \\ & + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m)\rho^\kappa + \widehat{\varPhi}_0(\rho, m) \Big]\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where

$$\begin{aligned}\pi_2(\rho, m) &= \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa, \\ \pi_1(\rho, m) &= \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa, \\ \pi_0(\rho, m) &= \sum_{\kappa=-(m-p_0-1)}^{p_0} c_\kappa \rho^\kappa,\end{aligned}$$

where the $a_\kappa(m)$, $b_\kappa(m)$, $c_\kappa(m)$ are constants that depend on m, and where the $\widehat{\varPhi}_i(\rho, m)$, $i = 0, 1, 2$, are analytic functions depending on m that involve products of the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = 1, \dots, \nu - 1$, or the exponentials $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $k = \nu + 1, \dots, n - 1$.

Our classification scheme for the differential operator L with n even is based on the constants a_κ: L is *regular* if $a_{p_0} \neq 0$, L is *simply irregular* if $a_{p_0} = 0$ and $a_\kappa \neq 0$ for some integer κ with $-\infty < \kappa < p_0$, and L is *degenerate irregular* if $a_\kappa = 0$ for $\kappa = p_0, p_0 - 1, \dots, 1, 0, -1, \dots$. See Definition 3.2. The constant a_{p_0} is given by

$$a_{p_0} = \mathrm{i}^{p_0} \det \begin{array}{c} \begin{array}{cccc} & 1 \le k \le \nu - 1 & & \nu + 1 \le k \le n - 1 \end{array} \\ \begin{pmatrix} \beta_{1m_1} & \alpha_{1m_1}\omega_k^{m_1} & \alpha_{1m_1}(-1)^{m_1} & \beta_{1m_1}\omega_k^{m_1} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{nm_n} & \alpha_{nm_n}\omega_k^{m_n} & \alpha_{nm_n}(-1)^{m_n} & \beta_{nm_n}\omega_k^{m_n} \end{pmatrix} \end{array}.$$

Consequently, the classification of L as being regular or irregular is determined exclusively by the leading coefficients in the boundary values $B_1, \ldots, B_n$, and is not affected in any way by the coefficients in the formal differential operator ℓ. In classifying an irregular differential operator L as being simply irregular or degenerate irregular, both the boundary values $B_1, \ldots, B_n$ and the coefficients of ℓ play a role. See Example 5.3.

At this point we make the assumption that L is either regular or simply irregular. Let p be the largest integer with $a_p \neq 0$, so $-\infty < p \leq p_0$; and let q be the integer defined by $q = p$ in case $b_\kappa = 0$ for $\kappa = p+1, \ldots, p_0$, and otherwise, q is the largest integer with $b_q \neq 0$. For the case $p = q$ choose a constant $d > 0$ such that

$$|a_p|\mathrm{e}^{-2d} + |b_p|\mathrm{e}^{-d} + |c_p|\mathrm{e}^{-2d} \leq \frac{1}{4}\,|a_p| = \frac{1}{4}\,|c_p|,$$

and form the horizontal strip

$$\Gamma = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \geq -\pi \text{ and } |b| \leq d\}.$$

Then select complex constants τ_0 and τ_1 and form the translated sectors

$$T_0 = \{\rho - \tau_0 \mid \rho \in S_0\} \quad \text{and} \quad T_1 = \{\rho - \tau_1 \mid \rho \in S_1\}$$

such that S_0 and S_1 lie in the interiors of T_0 and T_1, respectively, and such that the horizontal strip Γ lies in the interiors of both T_0 and T_1 in the case $p = q$. Fix the integer m with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p \leq p_0$, and then form the corresponding Birkhoff approximate solutions $z_k(t, \rho) = z_k(t, \rho, m)$, $k = 0, 1, \ldots, n-1$.

The selection of the translated sectors T_0 and T_1 and the choice of the integer m are perhaps the most subtle features of our spectral theory. The constant a_p must first be determined, and *then* T_0, T_1, and m are selected. This part of the theory is completed before we even have any actual solutions of the differential equation $(\rho^n I - \ell)u(t) = 0$.

Assume that n is odd, $n = 2\nu - 1$, and introduce the sectors

$$\begin{aligned} S_0\colon &\ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{2n} \leq \theta \leq \frac{\pi}{2n}\,, \\ S_1\colon &\ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \pi - \frac{\pi}{2n} \leq \theta \leq \pi + \frac{\pi}{2n}\,. \end{aligned}$$

Using the constants p_{iks}, q_{iks} once more, we construct two new sequences of constants a_κ, b_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, which are independent of the integer m and form invariants for the differential operator L, and which lead to the following representation of the approximate characteristic determinant:

$$\begin{aligned}\widehat{\Delta}(\rho,m) &= \pi_1(\rho,m)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho,m) \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widehat{\Phi}_1(\rho,m)\Big]\mathrm{e}^{\mathrm{i}\rho} \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widehat{\Phi}_0(\rho,m)\Big]\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where

$$\pi_1(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa, \qquad \pi_0(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa,$$

where the $a_\kappa(m)$, $b_\kappa(m)$ are constants that depend on m, and where the $\widehat{\Phi}_i(\rho,m)$, $i = 0, 1$, are analytic functions depending on m that involve products of the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = 1, \ldots, \nu - 1$, or $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $k = \nu, \ldots, n-1$.

An alternate form for the approximate characteristic determinant is obtained by starting with the modified approximate solutions

$$\begin{aligned} x_k(t,\rho,m) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} z_k(t,\rho,m), \qquad && k = 0, 1, \ldots, \nu-1, \\ x_k(t,\rho,m) &= z_k(t,\rho,m), && k = \nu, \ldots, n-1, \end{aligned}$$

and then defining the new approximate characteristic determinant by

$$\widetilde{\Delta}(\rho,m) = \det(B_i(x_k(\,\cdot\,,\rho,m)))$$

for $\rho \neq 0$ in $\mathbb{C}$. This approximate characteristic determinant has the representation

$$\begin{aligned}\widetilde{\Delta}(\rho,m) &= \pi_1'(\rho,m)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0'(\rho,m) \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa'(m)\rho^\kappa + \widetilde{\Phi}_1(\rho,m)\Big]\mathrm{e}^{-\mathrm{i}\rho} \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa'(m)\rho^\kappa + \widetilde{\Phi}_0(\rho,m)\Big]\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where

$$\pi_1'(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa' \rho^\kappa, \qquad \pi_0'(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa' \rho^\kappa$$

with the constants a_κ', b_κ' independent of m, where the $a_\kappa'(m)$, $b_\kappa'(m)$ are constants that depend on m, and where the $\widetilde{\Phi}_1(\rho,m)$, $i = 0, 1$, are analytic functions depending on m that involve products of the exponentials $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$,

$k = 1, \dots, \nu - 1$, or $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = \nu, \dots, n-1$. The constants a_κ, b_κ and a'_κ, b'_κ are related by the equations

$$a'_\kappa = b_\kappa(\omega_\nu)^\kappa \quad \text{and} \quad b'_\kappa = a_\kappa(\omega_{\nu-1})^\kappa$$

for $\kappa = p_0, p_0 - 1, \dots, 1, 0, -1, \dots$.

The classification scheme for the differential operator L with n odd is given in Definition 3.3: L is *regular* if $a_{p_0} \neq 0$ and $b_{p_0} \neq 0$; L is *simply irregular* if either $a_{p_0} = 0$ or $b_{p_0} = 0$, and $a_\kappa \neq 0$ and $b_\ell \neq 0$ for some integers κ, ℓ with $-\infty < \kappa, \ell \leq p_0$; and L is *degenerate irregular* if either $a_\kappa = 0$ for $\kappa = p_0, p_0 - 1, \dots, 1, 0, -1, \dots$ or $b_\kappa = 0$ for $\kappa = p_0, p_0 - 1, \dots, 1, 0, -1, \dots$. Since

$$a_{p_0} = \mathrm{i}^{p_0} \det \begin{array}{c} \\ \left(\begin{array}{ccc} & 1 \leq k \leq \nu - 1 & \nu \leq k \leq n-1 \\ \beta_{1m_1} & \alpha_{1m_1}\omega_k^{m_1} & \beta_{1m_1}\omega_k^{m_1} \\ \vdots & \vdots & \vdots \\ \beta_{nm_n} & \alpha_{nm_n}\omega_k^{m_n} & \beta_{nm_n}\omega_k^{m_n} \end{array} \right) \end{array}$$

and

$$b_{p_0} = \mathrm{i}^{p_0} \det \begin{array}{c} \\ \left(\begin{array}{ccc} & 1 \leq k \leq \nu - 1 & \nu \leq k \leq n-1 \\ \alpha_{1m_1} & \alpha_{1m_1}\omega_k^{m_1} & \beta_{1m_1}\omega_k^{m_1} \\ \vdots & \vdots & \vdots \\ \alpha_{nm_n} & \alpha_{nm_n}\omega_k^{m_n} & \beta_{nm_n}\omega_k^{m_n} \end{array} \right) \end{array},$$

once more the classification of L as being regular or irregular is determined exclusively by the leading coefficients in the boundary values $B_1, \dots, B_n$. In subdividing the irregular case, both the boundary values $B_1, \dots, B_n$ and the coefficients of ℓ play a role.

Assume that L is either regular or simply irregular. Let p and q be the largest integers with $a_p \neq 0$ and $b_q \neq 0$, so $-\infty < p, q \leq p_0$. For the case $p = q$ choose a constant $d > 0$ such that

$$|a_p|\mathrm{e}^{-d} + |b_p|\mathrm{e}^{-d} \leq \frac{1}{4}\min\{|a_p|, |b_p|\},$$

and in terms of d form the horizontal strips

$$\begin{aligned} \Gamma_0 &= \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \geq -\pi \text{ and } |b| \leq d\}, \\ \Gamma_1 &= \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \leq \pi \text{ and } |b| \leq d\}. \end{aligned}$$

Select complex constants τ_0 and τ_1 and form the translated sectors

$$T_0 = \{\rho - \tau_0 \mid \rho \in S_0\} \quad \text{and} \quad T_1 = \{\rho - \tau_1 \mid \rho \in S_1\}$$

with the properties: S_0 and S_1 lie in the interiors of T_0 and T_1, respectively, and for the case $p = q$ the horizontal strips Γ_0 and Γ_1 lie in the interiors of T_0 and T_1, respectively. Fix the integer m subject to the conditions $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$, and then form the corresponding Birkhoff approximate solutions $z_k(t, \rho) = z_k(t, \rho, m)$, $k = 0, 1, \ldots, n-1$. This gives us an algorithm for choosing the translated sectors T_0 and T_1 and the integer m once the constants a_p and b_q have been determined.

In Chapter 4 we derive high order asymptotic expansions for actual solutions of the differential equation

$$(\rho^n I - \ell)u(t) = 0 \tag{$*$}$$

for ρ belonging to the sectors T_0 and T_1, where it is assumed that T_0 and T_1 and the integer m have been selected as in Chapter 3. Let us look at the expansions on the sector T_0 for the case $n = 2\nu$ even — the other expansions are similar. Choose a permutation $\omega_0^0, \omega_1^0, \ldots, \omega_{n-1}^0$ of the nth roots of unity $\omega_0, \omega_1, \ldots, \omega_{n-1}$ such that

$$\operatorname{Re}(\mathrm{i}\rho\omega_0^0) \leq \operatorname{Re}(\mathrm{i}\rho\omega_1^0) \leq \cdots \leq \operatorname{Re}(\mathrm{i}\rho\omega_{n-1}^0) \quad \text{for all } \rho \in S_0.$$

Fix an integer k with $0 \leq k \leq n-1$, and let κ be the integer satisfying $0 \leq \kappa \leq n-1$ and $\omega_\kappa^0 = \omega_k$. We proceed to construct a solution $v_{0k}(t, \rho)$ of the differential equation $(*)$ that behaves asymptotically like the Birkhoff approximate solution $z_k(t, \rho)$ on the sector T_0. Let $\eta_k(t, \rho) = \eta_k(t, \rho, m)$ be the function defined by the equation

$$(\rho^n I - \ell)z_k(t, \rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t, \rho), \qquad 0 \leq t \leq 1,$$

for $\rho \neq 0$ in $\mathbb{C}$, the so-called mth order residual function. Let k_0 be the function defined by

$$k_0(t, s, \rho) = -\frac{1}{n\rho^{n-1}} \sum_{j=0}^{\kappa} (\mathrm{i}\omega_j^0)\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}, \qquad 0 \leq s < t \leq 1,$$

$$k_0(t, s, \rho) = \frac{1}{n\rho^{n-1}} \sum_{j=\kappa+1}^{n-1} (\mathrm{i}\omega_j^0)\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}, \qquad 0 \leq t < s \leq 1,$$

for $\rho \neq 0$ in $\mathbb{C}$, and let $\mathcal{K}_{0\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$\mathcal{K}_{0\rho}u(t) = \int_0^1 k_0(t, s, \rho)u(s)\, ds, \qquad 0 \leq t \leq 1, \qquad u \in L^2[0,1],$$

for each $\rho \neq 0$ in $\mathbb{C}$.

Fix a point $\rho \neq 0$ in $\mathbb{C}$. Then a function $u(t) = z_k(t, \rho) + \phi(t)$ is a solution of the differential equation $(*)$ if and only if the function $\phi(t)$ is a solution of the integro-differential equation

$$\phi(t) = \mathcal{K}_{0\rho}\sigma\phi(t) - \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big) + \sum_{j=0}^{n-1} c_j \mathrm{e}^{\mathrm{i}\rho\omega_j t} \qquad (**)$$

for some complex constants $c_0, c_1, \ldots, c_{n-1}$. Setting all the constants c_j equal to zero in $(**)$ and defining the function $\psi(t)$ by $\sigma\phi(t) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)$, it follows that $\phi(t)$ is a solution of the integro-differential equation $(**)$ if and only if $\psi(t)$ is a solution of the integral equation

$$\psi(t) = \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)\big) - \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big). \qquad (***)$$

Equation $(***)$ is the equation that we actually solve for $\rho \in T_0$ with $|\rho|$ sufficiently large: we show that $(***)$ has a unique solution $\psi_{0k}(t,\rho)$, and then the function

$$\phi_{0k}(t,\rho) = \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi_{0k}(t,\rho) - \eta_k(t,\rho)]\big)$$

is a solution of $(**)$, and the function

$$\begin{aligned} v_{0k}(t,\rho) &= z_k(t,\rho) + \phi_{0k}(t,\rho) \\ &= z_k(t,\rho) + \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi_{0k}(t,\rho) - \eta_k(t,\rho)]\big) \end{aligned}$$

is a solution of the differential equation $(*)$.

The solution $v_{0k}(\,\cdot\,,\rho)$ exists for all $\rho \in T_0$ with $|\rho|$ sufficiently large, and its derivatives have the form

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}E_{0k\alpha}(t,\rho)\rho^{-m+\alpha}$$

for $\alpha = 0, 1, \ldots, n-1$, where the function $E_{0k\alpha}(t,\rho)$ is bounded for $0 \leq t \leq 1$ and for $\rho \in T_0$ with $|\rho|$ sufficiently large. The smoothness of $v_{0k}^{(\alpha)}(t,\rho)$ in the t and ρ variables is established using the two lemmas that are included in the Appendix (Chapter 12). Carrying out this construction for $k = 0, 1, \ldots, n-1$, we obtain a basis

$$v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots, v_{0\,n-1}(\,\cdot\,,\rho)$$

for the solution space of the differential equation $(*)$, the basis existing for all $\rho \in T_0$ with $|\rho|$ greater than some constant R_0. See Theorem 4.3. A similar construction yields the basis

$$v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \ldots, v_{1\,n-1}(\,\cdot\,,\rho)$$

for the solution space of the differential equation $(*)$ for $\rho \in T_1$ with $|\rho| > R_0$. See Theorem 4.4. For n odd the analogous theorems areTheorems 4.6 and 4.7.

The asymptotic expansion of solutions given here is much simpler than other expansions appearing in the literature. Its main features are the crucial role played by the Birkhoff approximate solutions, and the use of modern operator theory in its development. Naimark [36, pp. 53–55] and Coddington and Levinson [6, p. 184] have proposed similar high order expansions using iteration schemes. We have tried unsuccessfully to work out the details of their schemes — the first two steps are fine, but at the third step unwanted exponentials come into play that can not be eliminated.

In Chapter 5 the characteristic determinants of L are formed using the asymptotically expanded solutions of Chapter 4. Assume that n is even. Relative to the sector T_0, we first form the modified solutions

$$\begin{aligned} u_{0k}(t,\rho) &= v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad k = 0,1,\ldots,\nu-1, \\ u_{0k}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad k = \nu,\ldots,n-1, \end{aligned}$$

and then define the characteristic determinant by

$$\Delta_0(\rho) = \det(B_i(u_{0k}(\,\cdot\,,\rho)))$$

for $\rho \in G_0$, where G_0 is the open set $\{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$. For working on the sector T_1, we form the modified solutions

$$\begin{aligned} u_{10}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_0} v_{10}(t,\rho), \\ u_{1k}(t,\rho) &= v_{1k}(t,\rho), && k = 1,\ldots,\nu-1, \\ u_{1\nu}(t,\rho) &= v_{1\nu}(t,\rho), \\ u_{1k}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{1k}(t,\rho), && k = \nu+1,\ldots,n-1, \end{aligned}$$

and then define an alternate form of the characteristic determinant by

$$\Delta_1(\rho) = \det(B_i(u_{1k}(\,\cdot\,,\rho)))$$

for $\rho \in G_1$, where G_1 is the open set $\{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$.

Upon expanding the determinant for $\Delta_0(\rho)$, we obtain the representation

$$\begin{aligned} \Delta_0(\rho) = {} & \pi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) \\ & + \Phi_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \Phi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Phi_{00}(\rho) \end{aligned}$$

for $\rho \in G_0$, where the functions $\pi_i(\rho) = \pi_i(\rho,m)$, $i = 0,1,2$, are the same functions that appeared earlier in the approximate characteristic determinant $\widehat{\Delta}(\rho,m)$, and where the functions $\Phi_{0i}(\rho)$, $i = 0,1,2$, are analytic on G_0 and satisfy the estimates $|\Phi_{0i}(\rho)| \le \gamma|\rho|^{-(m-p_0)}$ for $\rho \in G_0$. This is Theorem 5.1. There is a similar representation of $\Delta_1(\rho)$ given in Theorem 5.2, viz.,

$$\begin{aligned} \Delta_1(\rho) = {} & \pi_2(\rho) + \pi_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \\ & + \Phi_{12}(\rho) + \Phi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \Phi_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \end{aligned}$$

for $\rho \in G_1$ with $|\Phi_{1i}(\rho)| \le \gamma|\rho|^{-(m-p_0)}$ for $\rho \in G_1$.

For the special case $n = 2$, $\ell = -(d/dt)^2 + q(t)$, and $m = 3$, we describe in some detail the possible forms taken by the characteristic determinant. These forms are categorized as Cases 1–5, and the regular, simply irregular, and degenerate irregular differential operators are identified in each case. The one exception is Case 5 where we can only guarantee that L is irregular. In Example 5.3 the differential operator L is determined by the boundary values

$$B_1(u) = u'(0) + u'(1), \qquad B_2(u) = u(0) - u(1),$$

and it belongs to this exceptional case. If $q(0) \neq q(1)$, then $p = q = -1$ and L is simply irregular; in Chapter 7 it is shown that the spectrum $\sigma(L)$ consists of two sequences of points $\lambda'_k = (\rho'_k)^2$, $k = k_0, k_0 + 1, \ldots$, and $\lambda''_k = (\rho''_k)^2$, $k = k_0, k_0 + 1, \ldots$, plus a finite number of additional points, where

$$\begin{aligned} \rho'_k &= (2\pi k + \pi/2) + \epsilon'_k, \quad k = k_0, k_0 + 1, \ldots, \\ \rho''_k &= (2\pi k - \pi/2) + \epsilon''_k, \quad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

with $|\epsilon'_k| \leq \gamma/k$ and $|\epsilon''_k| \leq \gamma/k$ for $k = k_0, k_0 + 1, \ldots$. On the other hand, for the principal part of L, which is the differential operator T determined by $\tau = -(d/dt)^2$ and by the same boundary values B_1, B_2, in Example 10.2 it is shown that the spectrum is $\sigma(T) = \mathbb{C}$. Consequently, the differential operators L and T have very different spectral properties.

In previous work we operated under the premise that the spectral theory of a differential operator L is a perturbation of the spectral theory of its principal part T, which was true in the cases that we studied (see [34, p. 87 and p. 212]). This example shows this premise to be false in general.

Assume that n is odd. For the sector T_0 we form the modified solutions

$$\begin{aligned} u_{0k}(t,\rho) &= v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad k = 0, 1, \ldots, \nu - 1, \\ u_{0k}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad k = \nu, \ldots, n - 1, \end{aligned}$$

and then define the characteristic determinant by

$$\Delta_0(\rho) = \det(B_i(u_{0k}(\,\cdot\,,\rho)))$$

for $\rho \in G_0$, where G_0 is the open set $\{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$. For the sector T_1 we begin with the modified solutions

$$\begin{aligned} u_{1k}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{1k}(t,\rho), \qquad k = 0, 1, \ldots, \nu - 1, \\ u_{1k}(t,\rho) &= v_{1k}(t,\rho), \qquad k = \nu, \ldots, n - 1, \end{aligned}$$

and then define the alternate form of the characteristic determinant by

$$\Delta_1(\rho) = \det(B_i(u_{1k}(\,\cdot\,,\rho)))$$

for $\rho \in G_1$, where G_1 is the open set $\{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$.

These characteristic determinants have the representations

$$\varDelta_0(\rho) = \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) + \varPhi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \varPhi_{00}(\rho)$$

for $\rho \in G_0$, and

$$\varDelta_1(\rho) = \pi_1'(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0'(\rho) + \varPhi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \varPhi_{10}(\rho)$$

for $\rho \in G_1$. Here the functions $\pi_i(\rho) = \pi_i(\rho, m)$, $i = 0, 1$, and $\pi_i'(\rho) = \pi_i'(\rho, m)$, $i = 0, 1$, are the same functions that appeared earlier in the approximate characteristic determinants $\widehat{\varDelta}(\rho, m)$ and $\widetilde{\varDelta}(\rho, m)$; and the functions $\varPhi_{0i}(\rho)$, $i = 0, 1$, are analytic on G_0 and satisfy the estimates $|\varPhi_{0i}(\rho)| \le \gamma|\rho|^{-(m-p_0)}$ for $\rho \in G_0$, while the functions $\varPhi_{1i}(\rho)$, $i = 0, 1$, are analytic on G_1 and satisfy the analogous estimates $|\varPhi_{1i}(\rho)| \le \gamma|\rho|^{-(m-p_0)}$ for $\rho \in G_1$. See Theorem 5.4 and Theorem 5.5.

In Chapter 6 we study the resolvent $R_\lambda(L) = (\lambda I - L)^{-1}$ and the associated Green's function $G(t, s; \lambda)$, where

$$R_\lambda(L)u(t) = \int_0^1 G(t, s; \lambda)u(s)\, ds, \qquad 0 \le t \le 1, \quad u \in L^2[0, 1].$$

Assume that n is even, $n = 2\nu$, and take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and $\varDelta_0(\rho) \ne 0$, so λ belongs to the resolvent set $\rho(L)$. The resolvent $R_\lambda(L)$ can be expressed as the sum of two parts. First, for the differential operator L_0 defined by

$$\mathcal{D}(L_0) = \{u \in H^n[0, 1] \mid u^{(n-i)}(0) = 0,\ i = 1, \ldots, n\}, \qquad L_0 u = \ell u,$$

the Green's function for $\lambda I - L_0$ is given by

$$\begin{aligned} g(t, s; \lambda) &= \sum_{k=0}^{n-1} v_{0k}(t, \rho)\eta_{0k}(s, \rho), \qquad && 0 \le s < t \le 1, \\ g(t, s; \lambda) &= 0, && 0 \le t < s \le 1, \end{aligned}$$

where the functions $\eta_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, are determined by the linear system

$$\sum_{k=0}^{n-1} v_{0k}^{(\alpha)}(s, \rho)\eta_{0k}(s, \rho) = -\delta_{\alpha\, n-1}\mathrm{i}^{\,n}, \qquad \alpha = 0, 1, \ldots, n-1,$$

for $0 \le s \le 1$. We modify $g(t, s; \lambda)$ by forming the function

$$\begin{aligned} k_0(t, s; \rho) &= \sum_{k=0}^{\nu-1} v_{0k}(t, \rho)\eta_{0k}(s, \rho), \qquad && 0 \le s < t \le 1, \\ k_0(t, s; \rho) &= -\sum_{k=\nu}^{n-1} v_{0k}(t, \rho)\eta_{0k}(s, \rho), && 0 \le t < s \le 1, \end{aligned}$$

and then use this function to define the integral operator $K_{0\rho}$ on $L^2[0,1]$ by

$$K_{0\rho}u(t) = \int_0^1 k_0(t,s;\rho)u(s)\,ds, \qquad 0 \le t \le 1,$$

for $u \in L^2[0,1]$. The operator $K_{0\rho}$ makes up the first part of the resolvent $R_\lambda(L)$. Its kernel satisfies the growth rate $|k_0(t,s;\rho)| \le 2/|\rho|^{n-1}$ for $\rho \in S_0$ with $|\rho|$ sufficiently large.

Second, for any function $u \in L^2[0,1]$ there exist constants $c_0, c_1, \ldots, c_{n-1}$ such that

$$R_\lambda(L)u(t) = K_{0\rho}u(t) + \sum_{k=0}^{n-1} c_k u_{0k}(t,\rho), \qquad 0 \le t \le 1,$$

where the constants are determined by the linear system

$$\sum_{k=0}^{n-1} B_i(u_{0k}(\,\cdot\,,\rho))\,c_k = -B_i(K_{0\rho}u), \qquad i = 1,\ldots,n.$$

The functions $u_{0k}(\,\cdot\,,\rho)$, $k = 0,1,\ldots,n-1$, are the modified solutions introduced in Chapter 5, in terms of which $\Delta_0(\rho)$ is defined. Upon working out the details, we arrive at the representations (6.25) and (6.26) for the resolvent $R_\lambda(L)$ and the Green's function $G(t,s;\lambda)$. This leads to our principal result (6.37) for the growth rate of the Green's function:

$$|G(t,s;\lambda)| \le \frac{2}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_0(\rho)|}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in S_0$ of sufficiently large modulus and with $\Delta_0(\rho) \neq 0$. In (6.63) and (6.64) we have similar representations for the resolvent and the Green's function that are valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\Delta_1(\rho) \neq 0$, leading to the growth rate (6.77):

$$|G(t,s;\lambda)| \le \frac{2}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_1(\rho)|}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in S_1$ of sufficiently large modulus and with $\Delta_1(\rho) \neq 0$.

The corresponding representations and growth rates for the case n odd, $n = 2\nu - 1$, are divided into four different subcases: (i) $\rho \in G_0$ with $\Delta_0(\rho) \neq 0$, and $\rho \in S_0$ with $\Delta_0(\rho) \neq 0$ and $\operatorname{Im}\rho \ge 0$, (ii) $\rho \in G_0$ with $\Delta_0(\rho) \neq 0$, and $\rho \in S_0$ with $\Delta_0(\rho) \neq 0$ and $\operatorname{Im}\rho \le 0$, (iii) $\rho \in G_1$ with $\Delta_1(\rho) \neq 0$, and $\rho \in S_1$ with $\Delta_1(\rho) \neq 0$ and $\operatorname{Im}\rho \le 0$, and (iv) $\rho \in G_1$ with $\Delta_1(\rho) \neq 0$, and $\rho \in S_1$ with $\Delta_1(\rho) \neq 0$ and $\operatorname{Im}\rho \ge 0$. In each subcase an appropriate basis for the solution space of $(*)$ is selected. The principal growth rates for the Green's function are given in (6.118), (6.148), (6.189), and (6.219).

In Chapter 7 we compute the zeros of the characteristic determinants Δ_0 and Δ_1 for the case n even; this leads directly to the eigenvalues of the differential operator L. As a part of the process, growth rates are derived for Δ_0 and Δ_1 on various regions of the ρ plane; these growth rates are essential for establishing the completeness of the generalized eigenfunctions. Our methods are similar to those in [34, pp. 128–146]. The form of the zeros is determined by the constants a_p, b_q, c_p that are the leading coefficients of the functions $\pi_2(\rho)$, $\pi_1(\rho)$, $\pi_0(\rho)$ introduced in Chapter 3. The integers p and q satisfy $p = q$ or $p < q$. For the special case $p = q$, let ξ_0 and η_0 be the roots of the quadratic polynomial $Q(z) = a_p z^2 + b_p z + c_p$. The discussion divides naturally into three cases.

Case 1. $p = q$, $\xi_0 \neq \eta_0$. We prove that Δ_0 has two sequences of zeros in the horizontal strip Γ. These sequences can be expressed as

$$\begin{aligned}\rho'_k &= (2\pi k + \operatorname{Arg}\xi_0) - \mathrm{i}\ln|\xi_0| + \epsilon'_k, \qquad && k = k_0, k_0+1, \ldots,\\ \rho''_k &= (2\pi k + \operatorname{Arg}\eta_0) + \mathrm{i}\ln|\xi_0| + \epsilon''_k, \qquad && k = k_0, k_0+1, \ldots,\end{aligned}$$

with $|\epsilon'_k| \le \gamma/k$ and $|\epsilon''_k| \le \gamma/k$ for $k = k_0, k_0+1, \ldots$, and they produce the two sequences

$$\begin{aligned}\lambda'_k &= (\rho'_k)^n, \qquad && k = k_0, k_0+1, \ldots,\\ \lambda''_k &= (\rho''_k)^n, \qquad && k = k_0, k_0+1, \ldots,\end{aligned}$$

of eigenvalues for L. Each of these eigenvalues has algebraic multiplicity 1; together they account for all but a finite number of the eigenvalues of L. See Theorem 7.2. The principal growth rates for Δ_0 and Δ_1 are given in (7.9) and (7.10). These growth rates also apply to Case 2.

Case 2. $p = q$, $\xi_0 = \eta_0$. For this case Δ_0 has two sequences of zeros in the horizontal strip Γ:

$$\begin{aligned}\rho'_k &= 2\pi k + \operatorname{Arg}\xi_0 + \epsilon'_k, \quad && k = k_0, k_0+1, \ldots,\\ \rho''_k &= 2\pi k + \operatorname{Arg}\xi_0 + \epsilon''_k, \quad && k = k_0, k_0+1, \ldots,\end{aligned}$$

with $|\epsilon'_k| \le \gamma/\sqrt{k}$ and $|\epsilon''_k| \le \gamma/\sqrt{k}$ for $k = k_0, k_0+1, \ldots$. The corresponding eigenvalues

$$\begin{aligned}\lambda'_k &= (\rho'_k)^n, \qquad && k = k_0, k_0+1, \ldots,\\ \lambda''_k &= (\rho''_k)^n, \qquad && k = k_0, k_0+1, \ldots,\end{aligned}$$

have algebraic multiplicities 1 or 2, so multiple eigenvalues are possible in this case. The λ'_k, λ''_k account for all but a finite number of the points in the spectrum $\sigma(L)$. See Theorem 7.3.

Case 3. $p < q$. For this so-called logarithmic case, let $n_0 = q - p$, let $\mu_0 = -b_q/c_p \neq 0$, and let $\mu_1 = -b_q/a_p \neq 0$. Then the characteristic determinants Δ_0 and Δ_1 have sequences of zeros in the sectors S_0 and S_1:

$$\begin{aligned}\rho'_k &= (2\pi k - \operatorname{Arg}\mu_0) + \mathrm{i}n_0\ln[|\mu_0|^{1/n_0}(2\pi k - \operatorname{Arg}\mu_0)] + \epsilon'_k,\\ &\qquad\qquad k = k_0, k_0+1, \ldots,\\ \rho''_k &= (2\pi k + \operatorname{Arg}\mu_1) - \mathrm{i}n_0\ln[|\mu_0|^{1/n_0}(2\pi k + \operatorname{Arg}\mu_1)] + \epsilon''_k,\\ &\qquad\qquad k = k_0, k_0+1, \ldots,\end{aligned}$$

with $|\epsilon'_k| \le \gamma \ln k/k$ and $|\epsilon''_k| \le \gamma \ln k/k$ for $k = k_0, k_0 + 1, \ldots$. The eigenvalues

$$\begin{aligned} \lambda'_k &= (\rho'_k)^n, \qquad & k &= k_0, k_0+1, \ldots, \\ \lambda''_k &= (\rho''_k)^n, \qquad & k &= k_0, k_0+1, \ldots, \end{aligned}$$

all have algebraic multiplicity 1, and account for all but a finite number of the eigenvalues of L. See Theorem 7.5. The principal growth rates for $\varDelta_0$ and $\varDelta_1$ are given in (7.31) and (7.36).

In Chapter 8 we compute the zeros of the characteristic determinants $\varDelta_0$ and $\varDelta_1$ for the case n odd. These zeros produce the eigenvalues of the differential operator L, and give a complete description of the spectrum $\sigma(L)$. At the same time we derive growth rates for $\varDelta_0$ and $\varDelta_1$ on various regions of the ρ plane. This treatment follows along the same lines as used previously in [34, pp. 146–181]. The four constants a_p, b_q and $a'_q = b_q(\omega_\nu)^q$, $b'_p = a_p(\omega_{\nu-1})^p$ determine the form of the zeros. The case n odd divides naturally into the three cases where $p = q$, $p < q$, and $p > q$, the latter two cases being logarithmic cases.

Case 1. $p = q$. Let $\xi_0 = -b_p/a_p \neq 0$ and $\eta_0 = -b'_p/a'_p \neq 0$. Then we show that $\varDelta_0$ has a sequence of zeros in the horizontal strip $\varGamma_0$, and $\varDelta_1$ has a sequence of zeros in the horizontal strip $\varGamma_1$. These zeros take the form

$$\begin{aligned} \rho'_k &= (2\pi k + \operatorname{Arg}\xi_0) - \mathrm{i}\ln|\xi_0| + \epsilon'_k, \qquad & k &= k_0, k_0+1, \ldots, \\ \rho''_k &= -(2\pi k + \operatorname{Arg}\eta_0) - \mathrm{i}\ln|\xi_0| + \epsilon''_k, \qquad & k &= k_0, k_0+1, \ldots, \end{aligned}$$

with $|\epsilon'_k| \le \gamma/k$ and $|\epsilon''_k| \le \gamma/k$ for $k = k_0, k_0 + 1, \ldots$. The corresponding eigenvalues

$$\begin{aligned} \lambda'_k &= (\rho'_k)^n, \qquad & k &= k_0, k_0+1, \ldots, \\ \lambda''_k &= (\rho''_k)^n, \qquad & k &= k_0, k_0+1, \ldots, \end{aligned}$$

have algebraic multiplicity 1, and there are only finitely many additional eigenvalues for L. See Theorem 8.2. The principal growth rates for $\varDelta_0$ and $\varDelta_1$ are given in (8.6), (8.7) and (8.9), (8.10).

Case 2. $p < q$. Let $n_0 = q - p$, $\mu_0 = -b_q/a_p \neq 0$, and $\mu_1 = -a'_q/b'_p \neq 0$. Then $\varDelta_0$ has a sequence of zeros in the sector S_0, and $\varDelta_1$ has a sequence of zeros in the sector S_1:

$$\begin{aligned} \rho'_k &= (2\pi k + \operatorname{Arg}\mu_0) - i n_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \operatorname{Arg}\mu_0)] + \epsilon'_k, \\ &\qquad\qquad k = k_0, k_0+1, \ldots, \\ \rho''_k &= -(2\pi k - \operatorname{Arg}\mu_1 + \pi n_0) - i n_0 \ln[|\mu_0|^{1/n_0}(2\pi k - \operatorname{Arg}\mu_1 + \pi n_0)] + \epsilon''_k, \\ &\qquad\qquad k = k_0, k_0+1, \ldots, \end{aligned}$$

with $|\epsilon'_k| \le \gamma \ln k/k$ and $|\epsilon''_k| \le \gamma \ln k/k$ for $k = k_0, k_0 + 1, \ldots$. The associated eigenvalues

$$\begin{aligned}\lambda'_k &= (\rho'_k)^n, \qquad k = k_0, k_0+1, \ldots, \\ \lambda''_k &= (\rho''_k)^n, \qquad k = k_0, k_0+1, \ldots,\end{aligned}$$

all have algebraic multiplicity 1, and the spectrum $\sigma(L)$ consists of the λ'_k, λ''_k, plus a finite number of additional points. See Theorem 8.4. The principal growth rates for $\varDelta_0$ and $\varDelta_1$ are given in (8.31), (8.34) and (8.39), (8.42).

Case 3. $p > q$. Let $n_0 = p - q$, $\mu_0 = -a_p/b_q \neq 0$, and $\mu_1 = -b'_p/a'_q \neq 0$. Then $\varDelta_0$ and $\varDelta_1$ have sequences of zeros in the sectors S_0 and S_1 of the form

$$\begin{aligned}\rho'_k &= (2\pi k - \operatorname{Arg}\mu_0) + in_0 \ln[|\mu_0|^{1/n_0}(2\pi k - \operatorname{Arg}\mu_0)] + \epsilon'_k, \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = k_0, k_0+1, \ldots, \\ \rho''_k &= -(2\pi k + \operatorname{Arg}\mu_1 + \pi n_0) + in_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \operatorname{Arg}\mu_1 + \pi n_0)] + \epsilon''_k, \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = k_0, k_0+1, \ldots,\end{aligned}$$

with $|\epsilon'_k| \leq \gamma \ln k/k$ and $|\epsilon''_k| \leq \gamma \ln k/k$ for $k = k_0, k_0+1, \ldots$. These zeros produce the eigenvalues

$$\begin{aligned}\lambda'_k &= (\rho'_k)^n, \qquad k = k_0, k_0+1, \ldots, \\ \lambda''_k &= (\rho''_k)^n, \qquad k = k_0, k_0+1, \ldots,\end{aligned}$$

which are all of algebraic multiplicity 1. There are at most a finite number of additional eigenvalues. See Theorem 8.6. The principal growth rates for $\varDelta_0$ and $\varDelta_1$ are given in (8.68), (8.71) and (8.76), (8.79).

In Chapter 9 we prove that the generalized eigenfunctions of L are complete in the Hilbert space $L^2[0,1]$. Under our assumption that L is either regular or simply irregular, the spectrum $\sigma(L)$ is an infinite countable set with no limit points in $\mathbb{C}$. Let $\sigma(L) = \{\lambda_i\}_{i=1}^{\infty}$ be any enumeration of $\sigma(L)$, let m_i $(0 < m_i < \infty)$ denote the ascent of the operator $\lambda_i I - L$ for $i = 1, 2, \ldots$, and let P_i, $i = 1, 2, \ldots$, denote the projection of $L^2[0,1]$ onto the generalized eigenspace $\mathcal{N}((\lambda_i I - L)^{m_i})$ along the range $\mathcal{R}((\lambda_i I - L)^{m_i})$. Also, let $\operatorname{sp}(L)$ denote the subspace of $L^2[0,1]$ spanned by the generalized eigenfunctions of L, and introduce the subspaces

$$S_\infty(L) = \Big\{u \in L^2[0,1] \;\Big|\; u = \sum_{i=1}^{\infty} P_i u\Big\}$$

and

$$M_\infty(L) = \big\{u \in L^2[0,1] \mid P_i u = 0 \text{ for } i = 1, 2, \ldots\big\}.$$

Then $M_\infty(L)$ is closed, $\operatorname{sp}(L) \subseteq S_\infty(L)$, and $\overline{\operatorname{sp}(L)} = \overline{S_\infty(L)}$.

In Theorem 9.1 and Theorem 9.2 we show that

$$\overline{\operatorname{sp}(L)} = \overline{S_\infty(L)} = L^2[0,1] \quad \text{and} \quad M_\infty(L) = \{0\},$$

with the first theorem treating the case n even and the second the case n odd. The proof combines the growth rates for the Green's function $G(t,s;\lambda)$ established in Chapter 6 with the growth rates for the characteristic determinants

Δ_0 and Δ_1 established in Chapters 7 and 8, thereby obtaining growth rates for the resolvent $R_\lambda(L)$ on various regions of the λ plane. Included in these regions are five equally spaced rays $\mathcal{R}_j$, $j = 1, \ldots, 5$, with

$$\|R_\lambda(L)\| = O\big(|\lambda|^N\big) \quad \text{as } \lambda \to \infty \text{ along each ray } \mathcal{R}_j,$$

where N is a positive integer satisfying the conditions

$$N \geq (p_0 - p - n + 1)/n \quad \text{and} \quad N \geq (p_0 - q - n + 1)/n.$$

The completeness results are then an immediate consequence of Theorem 6.2, Chapter 2 of [34] or of Corollary XI.6.31 of [8].

In Chapter 10 we present several examples of degenerate irregular differential operators for the special case where L is equal to its principal part T. When $L = T$, the mth order Birkhoff approximate solutions $z_k(t, \rho, m)$, $k = 0, 1, \ldots, n-1$, reduce to the actual solutions

$$z_k(t, \rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}, \qquad k = 0, 1, \ldots, n-1,$$

and the approximate characteristic determinant $\widehat{\Delta}(\rho, m)$ is independent of the integer m and is identical to the characteristic determinant $\Delta(\rho)$ defined for the differential operator $L = T$ in [34, p. 100]. Example 10.1 examines the nth order differential operator $L = T$ determined by initial conditions at the endpoint $t = 0$:

$$B_i(u) = u^{(n-i)}(0), \qquad i = 1, \ldots, n.$$

This model is indeed degenerate irregular with $\sigma(L) = \emptyset$ and $\rho(L) = \mathbb{C}$. For n even Example 10.2 studies the differential operator $L = T$ determined by the boundary values

$$B_i(u) = u^{(n-i)}(0) + (-1)^{i+1} u^{(n-i)}(1), \qquad i = 1, \ldots, n.$$

This model is also degenerate irregular, but with $\sigma(L) = \mathbb{C}$ and $\rho(L) = \emptyset$. In both examples the characteristic determinant $\Delta(\rho) = \widehat{\Delta}(\rho)$ is computed explicitly.

For the special case $n = 4$, $L = T$, we carry out an explicit calculation of the characteristic determinant:

$$\begin{aligned}\Delta(\rho) = \mathbb{P}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \mathbb{Q}_0(\rho)\mathrm{e}^{\mathrm{i}\rho} + \mathbb{P}_0(\rho) + [\mathbb{P}_2(\rho)\mathrm{e}^{-2\rho} + \mathbb{Q}_1(\rho)\mathrm{e}^{-\rho}]\mathrm{e}^{2\mathrm{i}\rho} \\ + [\mathbb{Q}_2(\rho)\mathrm{e}^{-2\rho} + \mathbb{D}(\rho)\mathrm{e}^{-\rho}]\mathrm{e}^{\mathrm{i}\rho} + [\mathbb{P}_3(\rho)\mathrm{e}^{-2\rho} + \mathbb{Q}_3(\rho)\mathrm{e}^{-\rho}]\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the polynomials $\mathbb{P}_i(\rho)$, $\mathbb{Q}_i(\rho)$, and $\mathbb{D}(\rho)$ are given explicitly by the formulas (10.39)–(10.43). We then reexamine our classification scheme of Chapter 3 by exploiting the explicit forms of the polynomials $\mathbb{P}_0(\rho)$, $\mathbb{Q}_0(\rho)$, and $\mathbb{D}(\rho)$. The classification takes the form of three cases.

Case I. $n = 4$, $\mathbb{P}_0(\rho) \not\equiv 0$. In this case the fourth order differential operator $L = T$ is either regular or simply irregular, and it is studied in the previous chapters.

Case II. $n = 4$, $\mathbb{P}_0(\rho) \equiv 0$, $\mathbb{Q}_0(\rho) \not\equiv 0$. For this case the differential operator $L = T$ is degenerate irregular. The spectrum $\sigma(L)$ is quite unusual in that it consists of a single sequence

$$\lambda_k = (\rho_k)^4, \qquad k = k_0, k_0 + 1, \dots,$$

of eigenvalues, plus a finite number of additional points. The λ_k have algebraic multiplicity 1, and they approach the *negative* real axis as $k \to \infty$. A model for this case is given in Example 10.3, where the 4th order differential operator $L = T$ is determined by the boundary values

$$B_1(u) = u'''(0) + 6u(0),\ B_2(u) = u''(0),\ B_3(u) = u'(0),\ B_4(u) = u(1).$$

Case III. $n = 4$, $\mathbb{P}_0(\rho) \equiv 0$, $\mathbb{Q}_0(\rho) \equiv 0$. For this case the differential operator $L = T$ is again degenerate irregular. The characteristic determinant simplifies to

$$\varDelta(\rho) = \mathbb{D}(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho} = 8\mathrm{i}\gamma_0\rho^6\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}$$

for $\rho \neq 0$ in $\mathbb{C}$, where γ_0 is a constant, and we show that either $\sigma(L) = \emptyset$ and $\rho(L) = \mathbb{C}$, or $\sigma(L) = \mathbb{C}$ and $\rho(L) = \emptyset$. This third case is truly degenerate.

For the case $n = 2\nu \geq 6$, $L = T$, we establish some results that are analogous to Case I and Case II for $n = 4$. In this new Case II the differential operator is again degenerate irregular, and the spectrum $\sigma(L)$ consists of a single sequence that lies near the negative real axis.

In Chapter 11 we present a list of unsolved problems in the spectral theory of nth order two-point differential operators. Some of the problems are for general L, and others are for the special case $L = T$. There still remain fundamental unsolved problems even for the simplest two-point differential operators, e.g., what is the subspace $S_\infty(L)$ when $n = 2$, $L = T$, and L is simply irregular?

2

Approximate Solutions and Formal Solutions

In this chapter we construct approximate solutions and formal solutions to the differential equation

$$(\rho^n I - \ell)u(t) = \rho^n u(t) - \sum_{p=0}^{n} a_p(t)u^{(p)}(t) = 0, \qquad 0 \le t \le 1, \tag{2.1}$$

for $\rho \neq 0$ in $\mathbb{C}$. In this equation the two highest order coefficients are $a_n(t) = 1/\mathrm{i}^{\,n}$ and $a_{n-1}(t) = 0$. We will follow the pioneering work of Birkhoff [4]. Set

$$b_p(t,\rho) := \frac{\mathrm{i}^{\,n} a_p(t)}{\rho^{n-p}}, \quad p = 1, \dots, n, \qquad b_0(t,\rho) := \frac{\mathrm{i}^{\,n} a_0(t)}{\rho^n} - \mathrm{i}^{\,n},$$

so $b_n(t,\rho) = 1$, $b_{n-1}(t,\rho) = 0$, and

$$-\frac{\mathrm{i}^n}{\rho^{\,n}}\,(\rho^n I - \ell)u(t) = \sum_{p=0}^{n} \frac{\mathrm{i}^{\,n} a_p(t)}{\rho^{n-p}}\,\frac{u^{(p)}(t)}{\rho^p} - \mathrm{i}^{\,n} u(t) = \sum_{p=0}^{n} b_p(t,\rho)\frac{u^{(p)}(t)}{\rho^p}\,.$$

Thus, for $\rho \neq 0$ in $\mathbb{C}$ we can replace (2.1) by the equivalent differential equation

$$\sum_{p=0}^{n} b_p(t,\rho)\frac{u^{(p)}(t)}{\rho^p} = 0, \qquad 0 \le t \le 1. \tag{2.2}$$

2.1 Birkhoff Approximate Solutions

Let the positive integer n be written in the form $n := 2\nu$ for n even and in the form $n := 2\nu - 1$ for n odd, and let $\omega_k := \mathrm{e}^{\mathrm{i}2\pi k/n}$ for $k = 0, \pm 1, \pm 2, \dots$. The constants ω_k are just the nth roots of unity with $\omega_0 = \omega_n = 1$ and with $\omega_\nu = -1$ for the case $n = 2\nu$. Fix $\rho \neq 0$ in $\mathbb{C}$, and fix an integer k with $0 \le k \le n-1$ and an integer m with $m > n$. We look for an approximate solution to the differential equation (2.2) of the form

$$z_k(t,\rho) = z_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j},$$

where the coefficient functions $z_{kj}(t)$, $j = 0,1,\ldots,m-1$, are to be determined. In our notation for the approximate solution $z_k(t,\rho) = z_k(t,\rho,m)$, we always display the ρ dependence, but generally surpress the m dependence, using the simpler notation $z_k(t,\rho)$. The coefficient functions $z_{kj}(t)$, $j = 0,1,\ldots,m-1$, are to be selected independent of both ρ and m.

Substituting $z_k(t,\rho)$ into the left side of (2.2), we see that the quantities $b_p(t,\rho)$ and $z_k^{(p)}(t,\rho)/\rho^p$ appear as finite sums of the powers $\rho^0, \rho^{-1}, \rho^{-2}, \ldots$, and upon collecting like powers of ρ, the left side of (2.2) becomes a finite sum of the powers $\rho^0, \rho^{-1}, \rho^{-2}, \ldots, \rho^{-(n+m-1)}$. The algorithm for calculating the coefficients $z_{kj}(t)$, $j = 0,1,\ldots,m-1$, is to force the coefficients of the powers $\rho^0, \rho^{-1}, \rho^{-2}, \ldots, \rho^{-m}$ in the collected sum to be identically zero on the interval $[0,1]$; the terms involving the powers $\rho^{-(m+1)}, \rho^{-(m+2)}, \ldots, \rho^{-(n+m-1)}$ are not eliminated and form residual quantities.

Here are the details of the algorithm. For $p = 0,1,\ldots,n$ and for $\ell = 0,1,\ldots,p$, set $\alpha_{kp\ell} := \binom{p}{\ell}(\mathrm{i}\omega_k)^{p-\ell}$, and for $\ell = 1,\ldots,n$ set

$$C_{k\ell}(t,\rho) := \sum_{p=\ell}^{n} b_p(t,\rho)\alpha_{kp\ell} = \alpha_{kn\ell} + \frac{\mathrm{i}^n a_{n-2}(t)\alpha_{k\,n-2\,\ell}}{\rho^2} + \cdots + \frac{\mathrm{i}^n a_\ell(t)\alpha_{k\ell\ell}}{\rho^{n-\ell}}$$

and set

$$C_{k0}(t,\rho) := \sum_{p=0}^{n} b_p(t,\rho)\alpha_{kp0} = \frac{\mathrm{i}^n a_{n-2}(t)\alpha_{k\,n-2\,0}}{\rho^2} + \cdots + \frac{\mathrm{i}^n a_0(t)\alpha_{k00}}{\rho^n}\,.$$

By Leibniz's rule $z_k^{(p)}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{\ell=0}^{p}\sum_{j=0}^{m-1}\alpha_{kp\ell}\, z_{kj}^{(\ell)}(t)\rho^{p-\ell-j}$, and hence,

$$\begin{aligned}\sum_{p=0}^{n} b_p(t,\rho)\frac{z_k^{(p)}(t,\rho)}{\rho^p} &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{p=0}^{n}\sum_{\ell=0}^{p}\sum_{j=0}^{m-1} b_p(t,\rho)\alpha_{kp\ell} z_{kj}^{(\ell)}(t)\rho^{-\ell-j}\\ &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{\ell=0}^{n}\sum_{p=\ell}^{n} b_p(t,\rho)\alpha_{kp\ell}\cdot\sum_{j=0}^{m-1} z_{kj}^{(\ell)}(t)\rho^{-\ell-j}\\ &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{\ell=0}^{n}\sum_{j=0}^{m-1} C_{k\ell}(t,\rho) z_{kj}^{(\ell)}(t)\rho^{-\ell-j}.\end{aligned}$$

Introducing the notation $C_{k\ell}(t,\rho) := \sum_{p=0}^{n-\ell} c_{k\ell p}(t)\rho^{-p}$, $\ell = 0,1,\ldots,n$, where $c_{k\ell 0}(t) = \alpha_{kn\ell}$ for $\ell = 1,\ldots,n$ and $c_{k00}(t) = 0$, we can rewrite the last equation as

$$\begin{aligned}\sum_{p=0}^{n} b_p(t,\rho)\frac{z_k^{(p)}(t,\rho)}{\rho^p} &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{\ell=0}^{n}\sum_{j=0}^{m-1}\sum_{p=0}^{n-\ell} c_{k\ell p}(t) z_{kj}^{(\ell)}(t)\rho^{-\ell-p-j} \\ &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{s=0}^{n+m-1}\left[\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t)\right]\rho^{-s}.\end{aligned} \tag{2.3}$$

Thus, the conditions determining the coefficients in the approximate solution $z_k(t,\rho)$ are

$$\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t) = 0, \qquad s = 0,1,\dots,m. \tag{2.4}$$

As we proceed to solve the system (2.4), it will be convenient to set

$$\begin{aligned} c_{k\ell p}(t) &:= 0, \qquad \ell = 0,1,\dots,n, \qquad &p = n-\ell+1, n-\ell+2,\dots, \\ c_{k\ell p}(t) &:= 0, \qquad \ell = n+1, n+2,\dots, \qquad &p = 0,1,2,\dots. \end{aligned}$$

Then in (2.4) we allow the ranges $\ell \geq 0$, $p \geq 0$, and $0 \leq j \leq m-1$.

At first glance the system (2.4) appears to be an over-determined system of $m+1$ equations for the m unknown functions $z_{kj}(t)$, $j = 0,1,\dots,m-1$. But for $s = 0$ the corresponding equation in (2.4) is simply $c_{k00}(t)z_{k0}(t) = 0$, which is a valid equation because $c_{k00}(t) \equiv 0$. For $s = 1$ the corresponding equation in (2.4) reads

$$\underbrace{c_{k10}(t)}_{\alpha_{kn1}} z_{k0}'(t) + \underbrace{c_{k01}(t)}_{0} z_{k0}(t) + \underbrace{c_{k00}(t)}_{0} z_{k1}(t) = 0$$

or $n(\mathrm{i}\omega_k)^{n-1} z_{k0}'(t) = 0$. This implies that $z_{k0}(t)$ must be a constant: we will choose $z_{k0}(t) \equiv 1$. With this choice we can replace (2.4) by the reduced system

$$\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t) = 0, \qquad s = 2,\dots,m, \tag{2.5}$$

where it is assumed that $z_{k0}(t) \equiv 1$. In (2.5) we have a system of $m-1$ equations for the $m-1$ unknown functions $z_{kj}(t)$, $j = 1,\dots,m-1$.

Next, let us consider the system (2.5). Specifically, for the integer s with $2 \leq s \leq m-1$, the corresponding equation in (2.5) can be written as

$$\sum_{j=0}^{s}\sum_{\ell=0}^{s-j} c_{k\,\ell\, s-\ell-j}(t) z_{kj}^{(\ell)}(t) = 0, \tag{2.6}$$

with $z_{k0}(t) \equiv 1$. When $j = s$ in (2.6), we obtain the term $c_{k00}(t)z_{ks}(t)$, which is identically zero because $c_{k00}(t) \equiv 0$. On the other hand, when $j = s-1$ in (2.6), we get the terms

$$\underbrace{c_{k01}(t)}_{0} z_{k\,s-1}(t) + \underbrace{c_{k10}(t)}_{\alpha_{kn1}} z_{k\,s-1}'(t) = \alpha_{kn1}\, z_{k\,s-1}'(t).$$

Thus, (2.6) simplifies to

$$\alpha_{kn1}\, z'_{k\,s-1}(t) + \sum_{j=0}^{s-2}\sum_{\ell=0}^{s-j} c_{k\,\ell\;s-\ell-j}(t) z_{kj}^{(\ell)}(t) = 0, \tag{2.7}$$

valid for $s = 2, \dots, m-1$ with $z_{k0}(t) \equiv 1$. It is simple to check that (2.7) is also valid for $s = m$.

Finally, we rewrite the system (2.7) as

$$z'_{k\,s-1}(t) = -\frac{1}{\alpha_{kn1}} \sum_{j=0}^{s-2}\sum_{\ell=0}^{s-j} c_{k\,\ell\;s-\ell-j}(t)\, z_{kj}^{(\ell)}(t), \qquad s = 2, \dots, m, \tag{2.8}$$

where $z_{k0}(t) \equiv 1$ and $\alpha_{kn1} = n(\mathrm{i}\omega_k)^{n-1} \neq 0$. Equation (2.8) expresses the derivative $z'_{k\,s-1}(t)$ in terms of the functions $z_{k0}(t), z_{k1}(t), \dots, z_{k\,s-2}(t)$ and their derivatives, and hence, we have a recursive scheme going here. If we set

$$\begin{aligned} z_{k\,s-1}(t) &:= \int_0^t z'_{k\,s-1}(\xi)\, d\xi \\ &= -\frac{1}{\alpha_{kn1}} \sum_{j=0}^{s-2}\sum_{\ell=0}^{s-j} \int_0^t c_{k\,\ell\;s-\ell-j}(\xi)\, z_{kj}^{(\ell)}(\xi)\, d\xi \end{aligned} \tag{2.9}$$

for $s = 2, \dots, m$, with $z_{k0}(t) \equiv 1$, then the functions $z_{k1}(t), \dots, z_{k\,m-1}(t)$ are uniquely determined by equation (2.9). Since the functions $c_{k\ell p}(t)$ are infinitely differentiable on the interval $[0, 1]$, it follows that the functions $z_{kj}(t)$, $j = 0, 1, \dots, m-1$, are also infinitely differentiable on $[0, 1]$.

Note that the recursive schemes (2.8) and (2.9) are not restricted in any way to the range $s = 2, \dots, m$, but can be used for all values $s = 2, 3, \dots$. Thus, we can use (2.8) and (2.9) to construct an infinite sequence of functions $z_{k0}(t), z_{k1}(t), z_{k2}(t), \dots$. These functions are independent of both ρ and m. In forming the approximate solution $z_k(t, \rho) = z_k(t, \rho, m)$, only the terms $z_{k0}(t), z_{k1}(t), \dots, z_{k\,m-1}(t)$ are used from the infinite sequence.

For $s = 2$ in (2.8), we have

$$\begin{aligned} z'_{k1}(t) &= -\frac{1}{\alpha_{kn1}} \left[\, c_{k02}(t) z_{k0}(t) + c_{k11}(t) z'_{k0}(t) + c_{k20}(t) z''_{k0}(t) \,\right] \\ &= -\frac{\mathrm{i}^{\,n-1} a_{n-2}(t)}{n\omega_k}, \end{aligned}$$

and hence,

$$z_{k1}(t) = -\frac{\mathrm{i}^{\,n-1}}{n\omega_k} \int_0^t a_{n-2}(\xi)\, d\xi, \qquad 0 \le t \le 1. \tag{2.10}$$

Let us summarize the above results as a theorem; this result is a refinement of Lemma I appearing in [4].

Theorem 2.1. *For each integer k with $0 \le k \le n-1$, there exists a sequence of functions $z_{k0}(t), z_{k1}(t), z_{k2}(t), \dots$, which are infinitely differentiable on $[0,1]$, such that $z_{k0}(t) \equiv 1$ and such that for any $\rho \neq 0$ in $\mathbb{C}$ and for any integer m with $m > n$, if the function*

$$z_k(t,\rho) = z_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}$$

is substituted into the differential expression $-(\mathrm{i}^n/\rho^n)(\rho^n I - \ell)u(t)$, then the coefficients of the terms $\mathrm{e}^{\mathrm{i}\rho\omega_k t}\rho^{-s}$, $s = 0,1,\dots,m$, all vanish identically on the interval $[0,1]$ (the terms involving $\mathrm{e}^{\mathrm{i}\rho\omega_k t}\rho^{-s}$, $s = m+1, m+2, \dots, n+m-1$, still remain). Moreover, the functions $z_{kj}(t)$, $j = 1,2,\dots$, can be calculated recursively using equation (2.9).

Fix a positive integer m with $m > n$, and let us consider the n functions

$$z_k(t,\rho) = z_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0,1,\dots,n-1,$$

determined by Theorem 2.1 and defined for $0 \le t \le 1$ and for $\rho \neq 0$ in $\mathbb{C}$. We will refer to these functions as the *mth order Birkhoff approximate solutions* of the differential equation (2.1). From (2.3) and (2.4) we have

$$\begin{aligned}(\rho^n I - \ell)z_k(t,\rho) &= -\frac{\rho^n}{\mathrm{i}^n} \sum_{p=0}^{n} b_p(t,\rho) \frac{z_k^{(p)}(t,\rho)}{\rho^p} \\ &= -\frac{\rho^n}{\mathrm{i}^n} \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{s=m+1}^{n+m-1} \left[\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t) \right] \rho^{-s}\end{aligned}$$

for $k = 0,1,\dots,n-1$. For $\rho \neq 0$ in $\mathbb{C}$ and for $k = 0,1,\dots,n-1$, introduce the functions

$$\eta_k(t,\rho) = \eta_k(t,\rho,m) := -\frac{\rho^n}{\mathrm{i}^n} \sum_{s=m+1}^{n+m-1} \left[\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t) \right] \rho^{-s}.$$

In terms of these functions the last equation can be rewritten as

$$(\rho^n I - \ell)z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho), \qquad 0 \le t \le 1, \tag{2.11}$$

for $\rho \neq 0$ in $\mathbb{C}$ and for $k = 0,1,\dots,n-1$. The functions $\eta_k(t,\rho)$, $k = 0,1,\dots,n-1$, will be referred to as the *mth order residual functions*. They are linear combinations of the powers $\rho^{-(m+1-n)}, \rho^{-(m+2-n)}, \dots, \rho^{-(m+n-1-n)}$. In Chapter 4 we will construct actual solutions of the differential equation (2.1), and in this construction both the Birkhoff approximate solutions and the residual functions will play important roles.

Several remarks are in order that can greatly simplify the calculation of the Birkhoff approximate solutions.

Remark 2.2. Fix the integer m with $m > n$, and consider the Birkhoff approximate solutions $z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}$, $k = 0, 1, \ldots, n-1$. We assert that

$$z_k(t,\rho) = z_0(t,\rho\omega_k) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{0j}(t)(\rho\omega_k)^{-j}, \qquad k = 1, \ldots, n-1. \quad (2.12)$$

Thus, to calculate the Birkhoff approximate solutions, we need only calculate $z_0(t,\rho)$. This result has been pointed out by Stone [44, p. 707].

Take any integer k with $1 \le k \le n-1$. To establish (2.12), it is sufficient to show that

$$z_{kj}(t) = z_{0j}(t)(\omega_k)^{-j} \quad \text{for } j = 0, 1, \ldots, m-1. \quad (2.13)$$

We will use induction on j. For $j = 0$ we have $z_{k0}(t) = z_{00}(t) \equiv 1$, so (2.13) is certainly true for this case. Assume that (2.13) is true for the values $j = 0, 1, \ldots, s-2$, which implies that

$$z_{kj}^{(\ell)}(t) = z_{0j}^{(\ell)}(t)(\omega_k)^{-j} \quad \text{for } j = 0, 1, \ldots, s-2, \quad \ell = 0, 1, 2, \ldots. \quad (2.14)$$

Now from the definitions of the constants α_{kn1}, we have

$$\alpha_{kn1} = \alpha_{0n1}(\omega_k)^{n-1}. \quad (2.15)$$

By treating the three cases $\ell = 0$, $\ell = 1, \ldots, n$, and $\ell = n+1, n+2, \ldots$ separately, it is immediate from the definitions that

$$c_{k\ell p}(t) = c_{0\ell p}(t)(\omega_k)^{n-p-\ell}, \qquad \ell = 0, 1, 2, \ldots, \quad p = 0, 1, 2, \ldots. \quad (2.16)$$

Substituting (2.14), (2.15), and (2.16) into (2.8), we have

$$\begin{aligned} z'_{k\,s-1}(t) &= -\frac{1}{\alpha_{kn1}}\sum_{j=0}^{s-2}\sum_{\ell=0}^{s-j} c_{k\,\ell\,s-\ell-j}(t)\, z_{kj}^{(\ell)}(t) \\ &= -\frac{1}{\alpha_{0n1}(\omega_k)^{n-1}}\sum_{j=0}^{s-2}\sum_{\ell=0}^{s-j} c_{0\,\ell\,s-\ell-j}(t)\,(\omega_k)^{n-s+j} z_{0j}^{(\ell)}(t)(\omega_k)^{-j} \\ &= \frac{(\omega_k)^{n-s}}{(\omega_k)^{n-1}}\, z'_{0\,s-1}(t) = z'_{0\,s-1}(t)(\omega_k)^{-(s-1)}. \end{aligned}$$

From (2.9) we conclude that

$$\begin{aligned} z_{k\,s-1}(t) &= \int_0^t z'_{k\,s-1}(\xi)\,d\xi = \int_0^t z'_{0\,s-1}(\xi)(\omega_k)^{-(s-1)}\,d\xi \\ &= z_{0\,s-1}(t)(\omega_k)^{-(s-1)}. \end{aligned}$$

This shows that (2.13) is true for the value $j = s-1$, and the proof of (2.12) is complete.

Remark 2.3. Let us look more carefully at the system (2.8) for the case $k = 0$, which in combination with (2.9) determines the functions $z_{0j}(t)$, $j = 0, 1, \ldots, m-1$. Of course we start with $z_{00}(t) \equiv 1$. From the definitions: $\alpha_{0n1} = n\mathrm{i}^{\,n-1}$ and

$$\begin{aligned}
&c_{0n0}(t) \equiv 1, \qquad c_{0\,n-1\,0}(t) \equiv n\mathrm{i}, \\
&c_{0\ell 0}(t) \equiv \binom{n}{\ell}\mathrm{i}^{\,n-\ell}, \quad \ell = 1, \ldots, n-2, \\
&c_{0\ell p}(t) = a_{n-p}(t)\binom{n-p}{\ell}\mathrm{i}^{\,2n-\ell-p}, \quad \ell = 0, 1, \ldots, n-2,\ p = 2, 3, \ldots, n-\ell,
\end{aligned} \tag{2.17}$$

with *all other* $c_{0\ell p}(t)$ identically zero on the interval $[0,1]$. In particular, $c_{0\ell p}(t) \equiv 0$ when $\ell + p > n$. Consider the equation appearing in (2.8) for s satisfying $n+1 \le s \le m$. For the indices j and ℓ with $0 \le j \le s-n-1$ and $0 \le \ell \le s-j$, we have

$$\ell + (s-\ell-j) = s-j \ge n+1 \quad \text{and} \quad c_{0\,\ell\,s-\ell-j}(t) \equiv 0,$$

and hence, the corresponding equation in (2.8) simplifies to

$$z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{\,n-1}} \sum_{j=s-n}^{s-2} \sum_{\ell=0}^{s-j} c_{0\,\ell\ s-\ell-j}(t)\, z_{0j}^{(\ell)}(t) \tag{2.18}$$

for the case $n+1 \le s \le m$.

For the case $2 \le s \le n$, for any j with $0 \le j \le s-2$ we have $2 \le s-j \le n$, while for the case $n+1 \le s \le m$, for any j with $s-n \le j \le s-2$ we also have $2 \le s-j \le n$. For $q = 2, \ldots, n$ let ℓ_q be the qth order formal differential operator defined by

$$\ell_q u(t) := \sum_{\ell=0}^{q} c_{0\,\ell\,q-\ell}(t) u^{(\ell)}(t).$$

Note that the leading coefficient of ℓ_q is $c_{0q0}(t) = \binom{n}{q}\mathrm{i}^{\,n-q}$, and for $q = n$ we have

$$\begin{aligned}
\ell_n u(t) &= \sum_{\ell=0}^{n} c_{0\,\ell\,n-\ell}(t) u^{(\ell)}(t) = u^{(n)}(t) + \sum_{\ell=0}^{n-2} a_\ell(t)\binom{\ell}{\ell}\mathrm{i}^{\,2n-\ell-(n-\ell)}\, u^{(\ell)}(t) \\
&= u^{(n)}(t) + \mathrm{i}^{\,n} \sum_{\ell=0}^{n-2} a_\ell(t) u^{(\ell)}(t) = \mathrm{i}^{\,n}\, \ell u(t)
\end{aligned}$$

or

$$\ell_n = \mathrm{i}^{\,n} \ell\,. \tag{2.19}$$

Then for $2 \le s \le n$ equation (2.8) yields the recursion equation

$$z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{\,n-1}} \sum_{j=0}^{s-2} \ell_{s-j}\, z_{0j}(t). \tag{2.20}$$

Thus, for this case the derivative $z'_{0\,s-1}(t)$ is determined by the functions $z_{00}(t)$, $z_{01}(t)$, …, $z_{0\,s-2}(t)$ and their derivatives up to order n (there are at most $n-1$ functions in this list). For $n+1 \le s \le m$ equation (2.18), which is the simplified version of (2.8), yields the recursion equation

$$z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{\,n-1}} \sum_{j=s-n}^{s-2} \ell_{s-j}\, z_{0j}(t). \tag{2.21}$$

Therefore, for this latter case the derivative $z'_{0\,s-1}(t)$ is now determined by the $n-1$ preceding functions $z_{0\,s-n}(t)$, $z_{0\,s-n+1}(t)$, …, $z_{0\,s-2}(t)$ and their derivatives up to order n (there are $n-1$ functions in this list).

In equations (2.20) and (2.21) we have obtained our most simplified form of the system (2.8) for the index $k=0$. This final system has a banded structure, with at most $n-1$ terms appearing in the sum on the right side, and once again we see that the functions $z_{0j}(t)$, $j = 0, 1, 2, \ldots$, are independent of ρ and m. The integer m specifies how many terms to include in forming the Birkhoff approximate solutions; it does not affect the values of the coefficients in these approximate solutions. In the next chapter we will see how to choose m; its selection is determined by both the formal differential operator ℓ and by the boundary values $B_1, \ldots, B_n$. The selection of the integer m is a very subtle feature in our development of the spectral theory.

Example 2.4. Consider the special case $\ell = \tau = \mathrm{i}^{-n}(d/dt)^n$, $\sigma = 0$, and $L = T$. Here we have $a_p(t) \equiv 0$ for $p = 0, 1, \ldots, n-2$, and from equation (2.17)

$$c_{0\ell 0}(t) \equiv \binom{n}{\ell} \mathrm{i}^{\,n-\ell}, \qquad \ell = 1, \ldots, n,$$

and all other $c_{0\ell p}(t)$ are identically zero on $[0,1]$. Thus, for $q = 2, \ldots, n$ the differential operators ℓ_q are given by

$$\ell_q u(t) = c_{0q0}(t) u^{(q)}(t) = \binom{n}{q} \mathrm{i}^{\,n-q} u^{(q)}(t).$$

If $u(t)$ is a constant function, then clearly $\ell_q u(t) \equiv 0$ for $q = 2, \ldots, n$. From (2.20) and (2.9) it then follows that $z_{01}(t) \equiv 0$, …, $z_{0\,n-1}(t) \equiv 0$, and from (2.21) and (2.9) that $z_{0n}(t) \equiv 0$, $z_{0\,n+1}(t) \equiv 0$, …. We conclude that $z_0(t,\rho) = \mathrm{e}^{\mathrm{i}\rho t}$, and hence, by (2.12)

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}, \qquad k = 0, 1, \ldots, n-1. \tag{2.22}$$

This is the expected result for this special case.

Example 2.5. Let us consider the case $n = 2$ and $\ell = -(d/dt)^2 + q(t)$, where the coefficient $q(t) := a_0(t)$ is infinitely differentiable on $[0, 1]$. We proceed to calculate the two Birkhoff approximate solutions

$$z_0(t,\rho) = \mathrm{e}^{\mathrm{i}\rho t} \sum_{j=0}^{m-1} z_{0j}(t)\rho^{-j}, \quad z_1(t,\rho) = z_0(t,-\rho) = \mathrm{e}^{-\mathrm{i}\rho t} \sum_{j=0}^{m-1} z_{0j}(t)(-\rho)^{-j}.$$

For $n = 2$ we have only the differential operator $\ell_2 = -\ell = (d/dt)^2 - q(t)$. Thus, starting with $z_{00}(t) \equiv 1$, equation (2.20) gives

$$z'_{01}(t) = -\frac{1}{2\mathrm{i}}\,\ell_2\, z_{00}(t) = \frac{1}{2\mathrm{i}}\, q(t),$$

and hence,

$$z_{01}(t) = \frac{1}{2\mathrm{i}} \int_0^t q(\xi)\, d\xi := \frac{1}{2\mathrm{i}}\, Q(t). \tag{2.23}$$

Cf. equation (2.10). For $3 \le s \le m$ equation (2.21) gives the recursion equation

$$z'_{0\,s-1}(t) = -\frac{1}{2\mathrm{i}}\,\ell_2\, z_{0\,s-2}(t) = -\frac{1}{2\mathrm{i}}\left[z''_{0\,s-2}(t) - q(t)z_{0\,s-2}(t)\right]. \tag{2.24}$$

Thus, for $s = 3$ we get

$$\begin{aligned} z'_{02}(t) &= -\frac{1}{2\mathrm{i}}\,[z''_{01}(t) - q(t)z_{01}(t)] = \frac{1}{4}\,[q'(t) - q(t)Q(t)], \\ z_{02}(t) &= \frac{1}{4}\int_0^t [q'(\xi) - q(\xi)Q(\xi)]\, d\xi = \frac{1}{4}\,[q(t) - q(0)] - \frac{1}{8}\, Q(t)^2, \end{aligned} \tag{2.25}$$

and for $s = 4$

$$\begin{aligned} z'_{03}(t) &= -\frac{1}{2\mathrm{i}}\left[z''_{02}(t) - q(t)z_{02}(t)\right] \\ &= -\frac{1}{8\mathrm{i}}\left[q''(t) - q'(t)Q(t) - 2q(t)^2 + q(t)q(0) + \frac{1}{2}\, q(t)Q(t)^2\right], \\ z_{03}(t) &= -\frac{1}{8\mathrm{i}}\int_0^t \left[q''(\xi) - q'(\xi)Q(\xi) - 2q(\xi)^2 + q(\xi)q(0) + \frac{1}{2}\, q(\xi)Q(\xi)^2\right] d\xi. \end{aligned} \tag{2.26}$$

Therefore, for $m = 3$ the Birkhoff approximate solutions $z_0(t,\rho)$, $z_1(t,\rho)$ are

$$\begin{aligned} z_0(t,\rho) &= e^{\mathrm{i}\rho t}\left\{1 + \frac{1}{2i}\, Q(t)\rho^{-1} + \left[\frac{1}{4}\, q(t) - \frac{1}{4}\, q(0) - \frac{1}{8}\, Q(t)^2\right]\rho^{-2}\right\}, \\ z_1(t,\rho) &= z_0(t,-\rho) \\ &= e^{-\mathrm{i}\rho t}\left\{1 - \frac{1}{2i}\, Q(t)\rho^{-1} + \left[\frac{1}{4}\, q(t) - \frac{1}{4}\, q(0) - \frac{1}{8}\, Q(t)^2\right]\rho^{-2}\right\} \end{aligned} \tag{2.27}$$

for $0 \le t \le 1$ and for $\rho \ne 0$ in $\mathbb{C}$.

2.2 Formal Solutions

The previous material in this chapter is easily modified to obtain formal solutions of the differential equation (2.1), or the equivalent differential equation (2.2). Since the treatment of formal solutions follows along the same lines as the treatment of approximate solutions, we will simply sketch the results. Formal solutions will not be used in the sequel.

Fix $\rho \neq 0$ in $\mathbb{C}$, and fix an integer k with $0 \leq k \leq n-1$. We look for a formal solution to the differential equation (2.2) of the form

$$Z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{\infty} Z_{kj}(t)\rho^{-j},$$

where the coefficient functions $Z_{kj}(t)$ are to be determined. The formal derivatives are given by $Z_k^{(p)}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{\ell=0}^{p} \sum_{j=0}^{\infty} \alpha_{kp\ell}\, Z_{kj}^{(\ell)}(t)\rho^{p-\ell-j}$ for $p = 0, 1, \ldots, n$, and hence, upon formally substituting $Z_k(t,\rho)$ into the left side of (2.2) and collecting like powers of ρ, we obtain the equation

$$\sum_{p=0}^{n} b_p(t,\rho)\frac{Z_k^{(p)}(t,\rho)}{\rho^p} = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{s=0}^{\infty} \left[\sum_{\ell+p+j=s} c_{k\ell p}(t) Z_{kj}^{(\ell)}(t) \right] \rho^{-s} = 0. \qquad (2.28)$$

The coefficients $Z_{kj}(t)$, $j = 0, 1, 2, \ldots$, are calculated by setting the coefficients of the powers $\rho^0, \rho^{-1}, \rho^{-2}, \ldots$ in the collected sum equal to 0. This produces the infinite system of equations

$$\sum_{\ell+p+j=s} c_{k\ell p}(t) Z_{kj}^{(\ell)}(t) = 0, \qquad s = 0, 1, 2, \ldots. \qquad (2.29)$$

In this system we allow the ranges $\ell \geq 0$, $p \geq 0$, and $j \geq 0$.

For $s = 0$ the corresponding equation in the system (2.29) is simply $c_{k00}(t)Z_{k0}(t) = 0$, which is automatically valid because $c_{k00}(t) \equiv 0$. For $s = 1$ the corresponding equation becomes $n(\mathrm{i}\omega_k)^{n-1}Z_{k0}'(t) = 0$. Thus, $Z_{k0}(t)$ must be a constant: we will choose $Z_{k0}(t) \equiv 1$. With this choice we can replace (2.29) by the reduced system

$$\sum_{\ell+p+j=s} c_{k\ell p}(t) Z_{kj}^{(\ell)}(t) = 0, \qquad s = 2, 3, \ldots, \qquad (2.30)$$

where it is assumed that $Z_{k0}(t) \equiv 1$. In (2.30) we have an infinite system of equations for the unknown functions $Z_{kj}(t)$, $j = 1, 2, \ldots$.

Next, for $2 \leq s < \infty$ the corresponding equation in (2.30) simplifies to

$$\alpha_{kn1}\, Z_{k\,s-1}'(t) + \sum_{j=0}^{s-2} \sum_{\ell=0}^{s-j} c_{k\,\ell\,s-\ell-j}(t) Z_{kj}^{(\ell)}(t) = 0,$$

and hence,

$$Z'_{k\,s-1}(t) = -\frac{1}{\alpha_{kn1}} \sum_{j=0}^{s-2} \sum_{\ell=0}^{s-j} c_{k\,\ell\;s-\ell-j}(t)\, Z_{kj}^{(\ell)}(t), \qquad s = 2, 3, \ldots, \tag{2.31}$$

where $Z_{k0}(t) \equiv 1$. Equation (2.31) is a recursive scheme which expresses the derivative $Z'_{k\,s-1}(t)$ in terms of the functions $Z_{k0}(t), Z_{k1}(t), \ldots, Z_{k\,s-2}(t)$ and their derivatives. Setting

$$Z_{k\,s-1}(t) := \int_0^t Z'_{k\,s-1}(\xi)\, d\xi = -\frac{1}{\alpha_{kn1}} \sum_{j=0}^{s-2} \sum_{\ell=0}^{s-j} \int_0^t c_{k\,\ell\;s-\ell-j}(\xi)\, Z_{kj}^{(\ell)}(\xi)\, d\xi, \tag{2.32}$$

$0 \le t \le 1$ and $s = 2, 3, \ldots$, with $Z_{k0}(t) \equiv 1$, the functions $Z_{k1}(t), z_{k2}(t), \ldots$ are uniquely determined by the recursive scheme (2.32). Clearly these functions are infinitely differentiable on $[0, 1]$. Comparing equations (2.31) and (2.32) to equations (2.8) and (2.9), we see that the coefficients $Z_{kj}(t)$ are identical to the coefficients $z_{kj}(t)$ given in Theorem 2.1:

$$Z_{kj}(t) = z_{kj}(t), \qquad 0 \le t \le 1, \tag{2.33}$$

for $k = 0, 1, \ldots, n-1$ and for $j = 0, 1, 2, \ldots$. Thus, for any positive integer $m > n$ and for any index k with $0 \le k \le n-1$, the corresponding Birkhoff approximate solution $z_k(t, \rho) = z_k(t, \rho, m)$ is obtained from the formal solution $Z_k(t, \rho)$ by taking just the first m terms in the infinite series for $Z_k(t, \rho)$.

Finally, the formal solutions $Z_k(t, \rho)$, $k = 0, 1, \ldots, n-1$, are related by the equation

$$Z_k(t, \rho) = Z_0(t, \rho\omega_k) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{\infty} Z_{0j}(t)(\rho\omega_k)^{-j}, \qquad k = 1, \ldots, n-1. \tag{2.34}$$

Consequently, to calculate the formal solutions, we need only calculate $Z_0(t, \rho)$. Also, upon careful examination, we see that for $k = 0$ the system (2.31) can be expressed as

$$Z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{\,n-1}} \sum_{j=0}^{s-2} \ell_{s-j}\, Z_{0j}(t), \qquad 2 \le s \le n, \tag{2.35}$$

$$Z'_{0\,s-1}(t) = -\frac{1}{n\mathrm{i}^{\,n-1}} \sum_{j=s-n}^{s-2} \ell_{s-j}\, Z_{0j}(t), \qquad n+1 \le s < \infty. \tag{2.36}$$

Equations (2.35) and (2.36) give a simplified form of the system (2.31) for the index $k = 0$. This final system has a banded structure, with at most $n-1$ terms appearing in the sums on the right side.

3

The Approximate Characteristic Determinant: Classification

For the nth order differential operator L in $L^2[0,1]$, we assume that n is expressed in the form $n = 2\nu$ for n even and in the form $n = 2\nu - 1$ for n odd, that the boundary values $B_1, \ldots, B_n$ have been normalized so that the boundary coefficient matrix A is in reduced row echelon form, and that $p_0 = \sum_{i=1}^{n} m_i$ where m_i is the order of the boundary value B_i. Fix an integer m with $m > n$, and form the mth order Birkhoff approximate solutions

$$z_k(t,\rho) = z_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

as in Theorem 2.1. At this point we add the additional condition that m be greater than p_0: $m > n$ and $m > p_0$.

3.1 The Approximate Characteristic Determinant

To simplify the discussion, we modify the Birkhoff approximate solutions by introducing the functions

$$y_k(t,\rho) := z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, \nu - 1,$$

$$y_k(t,\rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k} z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \quad k = \nu, \ldots, n-1,$$

for $0 \le t \le 1$ and for $\rho \ne 0$ in $\mathbb{C}$. These functions are again approximate solutions of the differential equation (2.1) in the sense of Theorem 2.1. In terms of the boundary values and these modified approximate solutions, we then form the functions $B_i(y_k(\,\cdot\,,\rho))$. Indeed, for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu-1$, we have

$$y_k^{(p)}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(t) \rho^{p-\ell-j}, \quad p = 0,1,\ldots,n-1,$$

and

$$\begin{aligned} B_i(y_k(\,\cdot\,,\rho)) &= \sum_{p=0}^{m_i} \alpha_{ip} y_k^{(p)}(0,\rho) + \sum_{p=0}^{m_i} \beta_{ip} y_k^{(p)}(1,\rho) \\ &= \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(0) \rho^{p-\ell-j} \\ &\quad + \mathrm{e}^{\mathrm{i}\rho\omega_k} \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(1) \rho^{p-\ell-j} \\ &:= \widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho) \mathrm{e}^{\mathrm{i}\rho\omega_k} \end{aligned} \tag{3.1}$$

for $\rho \neq 0$ in $\mathbb{C}$, while for $i = 1,\ldots,n$ and $k = \nu,\ldots,n-1$

$$y_k^{(p)}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(t) \rho^{p-\ell-j}, \quad p = 0,1,\ldots,n-1,$$

and

$$\begin{aligned} B_i(y_k(\,\cdot\,,\rho)) &= \sum_{p=0}^{m_i} \alpha_{ip} y_k^{(p)}(0,\rho) + \sum_{p=0}^{m_i} \beta_{ip} y_k^{(p)}(1,\rho) \\ &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(0) \rho^{p-\ell-j} \\ &\quad + \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(1) \rho^{p-\ell-j} \\ &:= \widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho) \mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{aligned} \tag{3.2}$$

for $\rho \neq 0$ in $\mathbb{C}$. The functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ are defined and analytic for $\rho \neq 0$ in $\mathbb{C}$, and they can be calculated explicitly once the integer m has been selected.

Fix the integer m with $m > n$ and $m > p_0$. Let i and k be indices with $1 \le i \le n$ and $0 \le k \le \nu - 1$, and let us consider the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ defined by equation (3.1), where we express them in a format that incorporates their dependence on the integer m:

$$\begin{aligned} \widehat{P}_{ik}(\rho,m) &= \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(0) \rho^{p-\ell-j} := \sum_{s=-(m-1)}^{m_i} \hat{p}_{iks}(m) \rho^s, \\ \widehat{Q}_{ik}(\rho,m) &= \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(1) \rho^{p-\ell-j} := \sum_{s=-(m-1)}^{m_i} \hat{q}_{iks}(m) \rho^s \end{aligned} \tag{3.3}$$

for $\rho \neq 0$ in $\mathbb{C}$. We are going to derive precise formulas for the constants $\hat{p}_{iks}(m)$, $\hat{q}_{iks}(m)$, determining their exact dependence on the integer m. Many of them turn out to be independent of m. This analysis involves looking at three cases.

Case 1. Fix the integer s with $0 \leq s \leq m_i$. Suppose p, ℓ, j are integers satisfying $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, $0 \leq j \leq m-1$, and $p - \ell - j = s$. Then $p = s + \ell + j \geq s$, so $s \leq p \leq m_i$; $\ell = p - s - j \leq p - s$, so $0 \leq \ell \leq p - s$; and $p - \ell - j = s$. Conversely, suppose p, ℓ, j are integers satisfying $s \leq p \leq m_i$, $0 \leq \ell \leq p - s$, and $p - \ell - j = s$. Then $0 \leq s \leq p$, so $0 \leq p \leq m_i$; $\ell \leq p - s \leq p$, so $0 \leq \ell \leq p$; $j = p - s - \ell \geq 0$ and $j = p - s - \ell \leq p \leq m_i \leq p_0 \leq m - 1$, so $0 \leq j \leq m - 1$; and $p - \ell - j = s$. Thus, the conditions $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, $0 \leq j \leq m - 1$, $p - \ell - j = s$ are equivalent to the conditions $s \leq p \leq m_i$, $0 \leq \ell \leq p - s$, $p - \ell - j = s$. It follows that the coefficients $\hat{p}_{iks}(m)$, $\hat{q}_{iks}(m)$ are given by

$$\begin{aligned} \hat{p}_{iks}(m) &= \sum_{p=s}^{m_i} \sum_{\ell=0}^{p-s} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(0), \\ \hat{q}_{iks}(m) &= \sum_{p=s}^{m_i} \sum_{\ell=0}^{p-s} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(1) \end{aligned} \tag{3.4}$$

for the case $s = 0, 1, \ldots, m_i$. These coefficients are *independent* of the integer m. In terms of these results we introduce the constants

$$\begin{aligned} p_{iks} &:= \sum_{p=s}^{m_i} \sum_{\ell=0}^{p-s} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(0), \\ q_{iks} &:= \sum_{p=s}^{m_i} \sum_{\ell=0}^{p-s} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(1) \end{aligned}$$

for $s = 0, 1, \ldots, m_i$. The p_{iks}, q_{iks} are constants that are independent of the integer m, and from the above $\hat{p}_{iks}(m) = p_{iks}$, $\hat{q}_{iks}(m) = q_{iks}$ for $s = 0, 1, \ldots, m_i$.

Case 2. Fix the integer s with $-(m - m_i - 1) \leq s \leq -1$. Suppose p, ℓ, j are integers satisfying $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, $0 \leq j \leq m - 1$, and $p - \ell - j = s$. Then trivially $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, and $p - \ell - j = s$. Conversely, suppose p, ℓ, j are integers satisfying $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, and $p - \ell - j = s$. Then $j = p - \ell - s \geq 0 + 1 \geq 0$ and $j = p - \ell - s \leq m_i + (m - m_i - 1) = m - 1$, so $0 \leq j \leq m - 1$. Thus, the conditions $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, $0 \leq j \leq m - 1$, $p - \ell - j = s$ are equivalent to the smaller set of conditions $0 \leq p \leq m_i$, $0 \leq \ell \leq p$, $p - \ell - j = s$. It follows that the coefficients $\hat{p}_{iks}(m)$, $\hat{q}_{iks}(m)$ are given by

$$\begin{aligned}\hat{p}_{iks}(m) &= \sum_{p=0}^{m_i}\sum_{\ell=0}^{p}\alpha_{ip}\binom{p}{\ell}(\mathrm{i}\omega_k)^{p-\ell}z^{(\ell)}_{k\,p-\ell-s}(0),\\ \hat{q}_{iks}(m) &= \sum_{p=0}^{m_i}\sum_{\ell=0}^{p}\beta_{ip}\binom{p}{\ell}(\mathrm{i}\omega_k)^{p-\ell}z^{(\ell)}_{k\,p-\ell-s}(1)\end{aligned} \tag{3.5}$$

for the case $s = -(m - m_i - 1), \ldots, -2, -1$. These coefficients are also *independent* of the integer m. Based on this analysis we introduce the constants

$$\begin{aligned}p_{iks} &:= \sum_{p=0}^{m_i}\sum_{\ell=0}^{p}\alpha_{ip}\binom{p}{\ell}(\mathrm{i}\omega_k)^{p-\ell}z^{(\ell)}_{k\,p-\ell-s}(0),\\ q_{iks} &:= \sum_{p=0}^{m_i}\sum_{\ell=0}^{p}\beta_{ip}\binom{p}{\ell}(\mathrm{i}\omega_k)^{p-\ell}z^{(\ell)}_{k\,p-\ell-s}(1)\end{aligned}$$

for $s = -1, -2, \ldots$. Then the sequences p_{iks}, $s = -1, -2, \ldots$, and q_{iks}, $s = -1, -2, \ldots$, are infinite sequences of constants that are independent of the integer m, and $\hat{p}_{iks}(m) = p_{iks}$, $\hat{q}_{iks}(m) = q_{iks}$ for $s = -(m-m_i-1), \ldots, -2, -1$. A finite number of terms from these sequences appear in (3.3).

Case 3. Fix the integer s with $-(m-1) \le s \le -(m-m_i)$. Suppose p, ℓ, j are integers satisfying $0 \le p \le m_i$, $0 \le \ell \le p$, $0 \le j \le m-1$, and $p-\ell-j = s$. Clearly $0 \le s + (m-1) \le m_i - 1$, so $0 \le s + (m-1) < m_i$. There are two possibilities for the integer p: $0 \le p \le s + (m-1)$ or $s + m \le p \le m_i$. First, assume that $0 \le p \le s + (m-1)$. Then we have $0 \le p \le s + (m-1)$, $0 \le \ell \le p$, and $p - \ell - j = s$. Second, assume that $s + m \le p \le m_i$. Then we have $s+m \le p \le m_i$; $\ell = p-s-j \ge p-s-(m-1)$, so $p-s-(m-1) \le \ell \le p$; and $p - \ell - j = s$. Conversely, suppose p, ℓ, j are integers satisfying either the conditions $0 \le p \le s + (m-1)$, $0 \le \ell \le p$, $p - \ell - j = s$ or the conditions $s + m \le p \le m_i$, $p - s - (m-1) \le \ell \le p$, $p - \ell - j = s$. First, assume that $0 \le p \le s + (m-1)$, $0 \le \ell \le p$, $p - \ell - j = s$. Then $0 \le p \le s + (m-1) < m_i$, so $0 \le p \le m_i$; $0 \le \ell \le p$; $j = p-\ell-s \ge 0 + (m-m_i) > 0$ and $j = p-\ell-s \le s+(m-1)+0-s = m-1$, so $0 \le j \le m-1$; and $p-\ell-j = s$. Second, assume that $s+m \le p \le m_i$, $p-s-(m-1) \le \ell \le p$, $p-\ell-j = s$. Then $p \ge s+m \ge 1$, so $0 \le p \le m_i$; $\ell \ge p - s - (m-1) \ge s + m - s - (m-1) = 1$, so $0 \le \ell \le p$; $j = p-\ell-s \ge 0+(m-m_i) > 0$ and $j = p-\ell-s \le s+(m-1)-s = m-1$, so $0 \le j \le m-1$; and $p - \ell - j = s$. Thus, the conditions $0 \le p \le m_i$, $0 \le \ell \le p$, $0 \le j \le m-1$, $p - \ell - j = s$ are equivalent to the pair of conditions $0 \le p \le s + (m-1)$, $0 \le \ell \le p$, $p - \ell - j = s$ or $s + m \le p \le m_i$, $p - s - (m-1) \le \ell \le p$, $p - \ell - j = s$. It follows that the coefficients $\hat{p}_{iks}(m)$, $\hat{q}_{iks}(m)$ are given by

$$\begin{aligned}\hat{p}_{iks}(m) = &\sum_{p=0}^{s+(m-1)} \sum_{\ell=0}^{p} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(0) \\ &+ \sum_{p=s+m}^{m_i} \sum_{\ell=p-s-(m-1)}^{p} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(0), \\ \hat{q}_{iks}(m) = &\sum_{p=0}^{s+(m-1)} \sum_{\ell=0}^{p} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(1) \\ &+ \sum_{p=s+m}^{m_i} \sum_{\ell=p-s-(m-1)}^{p} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(1)\end{aligned} \tag{3.6}$$

for the case $s = -(m-1), -(m-2), \ldots, -(m-m_i)$. These coefficients are *dependent* on the integer m.

We remark that it is easy to develop these three cases geometrically by visualizing the triangular cylinder $0 \le p \le m_i$, $0 \le \ell \le p$, $0 \le j \le m-1$ in 3-space being cut by the family of planes $j = p - \ell - s$ obtained by setting $s = -(m-1), \ldots, m_i$.

Applying the results from the three cases, we see that (3.3) can be rewritten in terms of the constants p_{iks}, q_{iks} that are independent of the integer m as follows:

$$\begin{aligned}\widehat{P}_{ik}(\rho, m) &= \sum_{s=-(m-m_i-1)}^{m_i} p_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{p}_{iks}(m)\rho^s, \\ \widehat{Q}_{ik}(\rho, m) &= \sum_{s=-(m-m_i-1)}^{m_i} q_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{q}_{iks}(m)\rho^s\end{aligned} \tag{3.7}$$

for $\rho \neq 0$ in $\mathbb{C}$. These are our results for the structure of the functions $\widehat{P}_{ik}(\rho, m)$, $\widehat{Q}_{ik}(\rho, m)$ for the indices $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$.

To complete this part of the discussion, fix i and k with $1 \le i \le n$ and $\nu \le k \le n-1$. Then the corresponding functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ defined by (3.2) can be expressed in the form

$$\begin{aligned}\widehat{P}_{ik}(\rho, m) &= \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(1) \rho^{p-\ell-j} := \sum_{s=-(m-1)}^{m_i} \hat{p}_{iks}(m)\rho^s, \\ \widehat{Q}_{ik}(\rho, m) &= \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(0) \rho^{p-\ell-j} := \sum_{s=-(m-1)}^{m_i} \hat{q}_{iks}(m)\rho^s\end{aligned} \tag{3.8}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the m-dependence has been incorporated into these equations. Proceeding as above, set

$$p_{iks} := \sum_{p=s}^{m_i} \sum_{\ell=0}^{p-s} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(1),$$

$$q_{iks} := \sum_{p=s}^{m_i} \sum_{\ell=0}^{p-s} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(0)$$

for $s = 0, 1, \ldots, m_i$; the p_{iks}, q_{iks} are constants that are independent of the integer m. Then introduce the sequences

$$p_{iks} := \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(1),$$

$$q_{iks} := \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{k\,p-\ell-s}^{(\ell)}(0)$$

for $s = -1, -2, \ldots$. These are again infinite sequences whose terms are constants independent of the integer m, and $\hat{p}_{iks}(m) = p_{iks}$, $\hat{q}_{iks}(m) = q_{iks}$ for $s = -(m - m_i - 1), \ldots, m_i$. In terms of the constants p_{iks}, q_{iks}, the functions appearing in (3.8) can then be expressed in the form

$$\begin{aligned} \widehat{P}_{ik}(\rho, m) &= \sum_{s=-(m-m_i-1)}^{m_i} p_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{p}_{iks}(m)\rho^s, \\ \widehat{Q}_{ik}(\rho, m) &= \sum_{s=-(m-m_i-1)}^{m_i} q_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{q}_{iks}(m)\rho^s \end{aligned} \tag{3.9}$$

for $\rho \neq 0$ in $\mathbb{C}$. These are our results for $\widehat{P}_{ik}(\rho, m)$, $\widehat{Q}_{ik}(\rho, m)$ for $i = 1, \ldots, n$, $k = \nu, \ldots, n-1$.

Remark 3.1. For any integer m with $m > n$ and $m > p_0$, we can form the corresponding Birkhoff approximate solutions $z_k(t, \rho) = z_k(t, \rho, m)$, $k = 0, 1, \ldots, n-1$, and at the same time form the functions $z_{kj}(t)$, $k = 0, 1, \ldots$, $n-1$, $j = 0, 1, \ldots, m-1$. The constants p_{iks}, q_{iks} can then be computed from their definitions for any value of s with $-(m - m_i - 1) \leq s \leq m_i$. Thus, by taking the integer m sufficiently large, any of the constants p_{iks}, q_{iks} can be calculated explicitly.

In terms of the functions $B_i(y_k(\,\cdot\,, \rho))$ and the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, the *approximate characteristic determinant* is defined by

$$\widehat{\Delta}(\rho) = \widehat{\Delta}(\rho, m) := \det(B_i(y_k(\,\cdot\,, \rho)))$$

for $\rho \neq 0$ in $\mathbb{C}$. It is also defined and analytic for $\rho \neq 0$ in $\mathbb{C}$, and can be calculated explicitly once m has been chosen. For the case $n = 2\nu$ even, we express the approximate characteristic determinant in the form

$$\widehat{\Delta}(\rho) = \det(B_i(y_k(\,\cdot\,,\rho)))$$

$$= \det \begin{array}{c} \begin{array}{cccc} & 1\le k\le \nu-1 & & \nu+1\le k\le n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)+\widehat{Q}_{10}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & \widehat{P}_{1\nu}(\rho)+\widehat{Q}_{1\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widehat{Q}_{n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & \widehat{P}_{n\nu}(\rho)+\widehat{Q}_{n\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix} \end{array} \tag{3.10}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the future emphasis is on the 0th and the νth columns of this matrix. Similarly, for the case $n = 2\nu - 1$ odd, we express it in the form

$$\widehat{\Delta}(\rho) = \det(B_i(y_k(\,\cdot\,,\rho)))$$

$$= \det \begin{array}{c} \begin{array}{ccc} & 1\le k\le \nu-1 & \nu\le k\le n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)+\widehat{Q}_{10}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widehat{Q}_{n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix} \end{array} \tag{3.11}$$

for $\rho \neq 0$ in $\mathbb{C}$, with the future emphasis focusing on the 0th column of this matrix.

While the analysis of the structure of $\widehat{\Delta}(\rho)$ is slightly different for the two cases $n = 2\nu$ and $n = 2\nu - 1$, the general approach is the same. Consequently, we examine the case $n = 2\nu$ in detail, and then simply outline the case $n = 2\nu - 1$.

3.2 Classification for n Even

Assume that n is even: $n = 2\nu$. Let us begin by expanding the determinant in equation (3.10) using the linearity of the determinant function in the 0th and νth columns:

$$\widehat{\Delta}(\rho) = \widehat{D}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \widehat{D}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{D}_0(\rho) \tag{3.12}$$

for $\rho \neq 0$ in $\mathbb{C}$, where

$$\widehat{D}_2(\rho) := \det \begin{array}{c} \begin{array}{cccc} & 1\le k\le \nu-1 & & \nu+1\le k\le n-1 \end{array} \\ \begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & \widehat{Q}_{1\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & \widehat{Q}_{n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix} \end{array},$$

$$\widehat{D}_1(\rho) := \det \begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ \left(\begin{array}{c} \widehat{P}_{10}(\rho) \\ \vdots \\ \widehat{P}_{n0}(\rho) \end{array}\right. & \begin{array}{c} \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ \vdots \\ \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \end{array} & \begin{array}{c} \widehat{Q}_{1\nu}(\rho) \\ \vdots \\ \widehat{Q}_{n\nu}(\rho) \end{array} & \left.\begin{array}{c} \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots \\ \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{array}\right) \end{array}$$

$$+ \det \begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ \left(\begin{array}{c} \widehat{Q}_{10}(\rho) \\ \vdots \\ \widehat{Q}_{n0}(\rho) \end{array}\right. & \begin{array}{c} \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ \vdots \\ \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \end{array} & \begin{array}{c} \widehat{P}_{1\nu}(\rho) \\ \vdots \\ \widehat{P}_{n\nu}(\rho) \end{array} & \left.\begin{array}{c} \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots \\ \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{array}\right) \end{array},$$

and

$$\widehat{D}_0(\rho) := \det \begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ \left(\begin{array}{c} \widehat{P}_{10}(\rho) \\ \vdots \\ \widehat{P}_{n0}(\rho) \end{array}\right. & \begin{array}{c} \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ \vdots \\ \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \end{array} & \begin{array}{c} \widehat{P}_{1\nu}(\rho) \\ \vdots \\ \widehat{P}_{n\nu}(\rho) \end{array} & \left.\begin{array}{c} \widehat{P}_{1k}(\rho)+\widehat{Q}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots \\ \widehat{P}_{nk}(\rho)+\widehat{Q}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{array}\right) \end{array}$$

for $\rho \neq 0$ in $\mathbb{C}$.

Now consider the analytic functions $\widehat{D}_i(\rho)$, $i = 0, 1, 2$. Suppose we expand $\widehat{D}_2(\rho)$ using the linearity of the determinant in the columns with indices 1 through $\nu - 1$ and $\nu + 1$ through $n - 1$. Then $\widehat{D}_2(\rho)$ becomes the sum of 2^{n-2} determinants, starting with the determinant

$$\widehat{\pi}_2(\rho) := \det\begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \widehat{P}_{1\nu+1}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \widehat{P}_{n\nu+1}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix},$$

which is defined and analytic for $\rho \neq 0$ in $\mathbb{C}$. Thus, $\widehat{D}_2(\rho)$ can be expressed in the form

$$\widehat{D}_2(\rho) = \widehat{\pi}_2(\rho) + \widehat{\varPhi}_2(\rho), \qquad \rho \neq 0 \text{ in } \mathbb{C},$$

where the function $\widehat{\varPhi}_2(\rho)$ is defined and analytic for $\rho \neq 0$ in $\mathbb{C}$, and where $\widehat{\varPhi}_2(\rho)$ is the sum of $2^{n-2} - 1$ determinants with each determinant expressible in the form of a product of some of the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = 1, \ldots, \nu - 1$, or $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $k = \nu + 1, \ldots, n - 1$, (at least one of these exponentials appears in each product) times the determinant of an $n \times n$ matrix whose entries are selected from among the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$. Similarly, the functions $\widehat{D}_1(\rho)$, $\widehat{D}_0(\rho)$ can be expressed in the form

$$\widehat{D}_1(\rho) = \widehat{\pi}_1(\rho) + \widehat{\varPhi}_1(\rho), \qquad \widehat{D}_0(\rho) = \widehat{\pi}_0(\rho) + \widehat{\varPhi}_0(\rho)$$

for $\rho \neq 0$ in $\mathbb{C}$, where

$$\widehat{\pi}_1(\rho) := \det\begin{pmatrix} \widehat{P}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \widehat{P}_{1\nu+1}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widehat{P}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \widehat{P}_{n\nu+1}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix}$$

$$+ \det\begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{P}_{1\nu}(\rho) & \widehat{P}_{1\nu+1}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{P}_{n\nu}(\rho) & \widehat{P}_{n\nu+1}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix},$$

$$\widehat{\pi}_0(\rho) := \det\begin{pmatrix} \widehat{P}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{P}_{1\nu}(\rho) & \widehat{P}_{1\nu+1}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widehat{P}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{P}_{n\nu}(\rho) & \widehat{P}_{n\nu+1}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix}$$

for $\rho \neq 0$ in $\mathbb{C}$, and where the functions $\widehat{\varPhi}_1(\rho)$, $\widehat{\varPhi}_0(\rho)$ have the same structure as the function $\widehat{\varPhi}_2(\rho)$. We can now rewrite the approximate characteristic determinant in the form

$$\widehat{\Delta}(\rho) = \widehat{\pi}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \widehat{\pi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\pi}_0(\rho) + \widehat{\varPhi}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \widehat{\varPhi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\varPhi}_0(\rho) \tag{3.13}$$

for $\rho \neq 0$ in $\mathbb{C}$.

We emphasize again that the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, the approximate characteristic determinant $\widehat{\Delta}(\rho)$, and the functions $\widehat{\pi}_i(\rho)$, $\widehat{\varPhi}_i(\rho)$ are all defined and analytic for $\rho \neq 0$ in $\mathbb{C}$, and these functions can be calculated explicitly once the integer m has been fixed.

The functions $\widehat{\pi}_2(\rho)$ and $\widehat{\pi}_0(\rho)$ appearing in (3.13) are intemately related to each other. To see this, note that $\omega_0 = \omega_n = 1$ and $\omega_1\omega_k = \omega_{k+1}$ for $k = 0, 1, \ldots, n-1$. Consider the matrix that appears in the definition of $\widehat{\pi}_0(\rho)$. Now for $k = 0, 1, \ldots, n-2$ equation (2.13) shows that

$$z^{(\ell)}_{k+1\,j}(t) = z^{(\ell)}_{0j}(t)(\omega_1\omega_k)^{-j} = z^{(\ell)}_{kj}(t)(\omega_1)^{-j} \quad \text{for } j, \ell = 0, 1, 2, \ldots,$$

while

$$z^{(\ell)}_{0j}(t) = z^{(\ell)}_{0j}(t)(\omega_1\omega_{n-1})^{-j} = z^{(\ell)}_{n-1\,j}(t)(\omega_1)^{-j} \quad \text{for } j, \ell = 0, 1, 2, \ldots.$$

Consequently, from the definitions of the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, we obtain the following results: for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu-2$

$$\widehat{P}_{ik}(\rho\omega_1) = \sum_{p=0}^{m_i}\sum_{\ell=0}^{p}\sum_{j=0}^{m-1} \alpha_{ip}\binom{p}{\ell}(\mathrm{i}\omega_k)^{p-\ell} z^{(\ell)}_{kj}(0)(\rho\omega_1)^{p-\ell-j} = \widehat{P}_{i\,k+1}(\rho) \tag{3.14}$$

for $\rho \neq 0$ in $\mathbb{C}$; for $i = 1, \ldots, n$ and $k = \nu - 1$

$$\widehat{P}_{i\,\nu-1}(\rho\omega_1) = \sum_{p=0}^{m_i}\sum_{\ell=0}^{p}\sum_{j=0}^{m-1} \alpha_{ip}\binom{p}{\ell}(\mathrm{i}\omega_{\nu-1})^{p-\ell} z^{(\ell)}_{\nu-1\,j}(0)(\rho\omega_1)^{p-\ell-j} = \widehat{Q}_{i\nu}(\rho) \tag{3.15}$$

for $\rho \neq 0$ in $\mathbb{C}$; for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-2$

$$\widehat{P}_{ik}(\rho\omega_1) = \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z_{kj}^{(\ell)}(1)(\rho\omega_1)^{p-\ell-j} = \widehat{P}_{i\,k+1}(\rho) \tag{3.16}$$

for $\rho \neq 0$ in $\mathbb{C}$; and for $i = 1, \ldots, n$ and $k = n-1$

$$\widehat{P}_{i\,n-1}(\rho\omega_1) = \sum_{p=0}^{m_i} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_{n-1})^{p-\ell} z_{n-1\,j}^{(\ell)}(1)(\rho\omega_1)^{p-\ell-j} = \widehat{Q}_{i0}(\rho) \tag{3.17}$$

for $\rho \neq 0$ in $\mathbb{C}$. If we substitute (3.14)–(3.17) into the definition of $\widehat{\pi}_0(\rho)$, then we get

$$\begin{aligned} \widehat{\pi}_0(\rho\omega_1) &= \det \begin{pmatrix} \widehat{P}_{11}(\rho) & \widehat{P}_{12}(\rho) & \cdots & \widehat{P}_{1\,\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \widehat{P}_{1\,\nu+1}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) & \widehat{Q}_{10}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \widehat{P}_{n1}(\rho) & \widehat{P}_{n2}(\rho) & \cdots & \widehat{P}_{n\,\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \widehat{P}_{n\,\nu+1}(\rho) & \cdots & \widehat{P}_{n\,n-1}(\rho) & \widehat{Q}_{n0}(\rho) \end{pmatrix} \\ &= (-1)^{n-1}\widehat{\pi}_2(\rho) \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. Thus,

$$\widehat{\pi}_2(\rho) = -\,\widehat{\pi}_0(\rho\omega_1) \quad \text{for } \rho \neq 0 \text{ in } \mathbb{C}. \tag{3.18}$$

Next, we examine in detail the structure of the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, determining their dependence on the integer m. We will show that associated with each of these functions is an infinite sequence of constants that are independent of m. These sequences can be calculated explicitly using the functions $z_{k0}(t)$, $z_{k1}(t)$, $z_{k2}(t)$, …, $k = 0, 1, \ldots, n-1$, and the boundary values $B_1, \ldots, B_n$; they form important invariants for the differential operator L. In terms of these sequences of constants we will (a) classify the differential operator L as being regular, simply irregular, or degenerate irregular (this section); (b) determine the exact sectors T_0, T_1 to be used in the sequel to develop asymptotic expansions for actual solutions of the differential equation (2.1) (Chapter 4); (c) develop the characteristic determinant (Chapter 5); and (d) derive the basic theory for the eigenvalues of the differential operator L (Chapter 7).

Consider the function $\widehat{\pi}_2(\rho) = \widehat{\pi}_2(\rho, m)$ that appears in the representation (3.13) of the approximate characteristic determinant. Upon substituting the representations (3.3) and (3.8) into the matrix appearing in the definition of $\widehat{\pi}_2(\rho, m)$, we observe that each row of this matrix is a linear combination of row vectors. Hence, appealing to the linearity of the determinant function in its rows, we see that

$$\widehat{\pi}_2(\rho, m) = \sum_{s_1=-(m-1)}^{m_1} \cdots \sum_{s_n=-(m-1)}^{m_n} \rho^{s_1+\cdots+s_n} \det \mathit{\Pi}_2(s_1, \ldots, s_n, m), \tag{3.19}$$

where $\widehat{\varPi}_2(s_1, \ldots, s_n, m)$ is the $n \times n$ matrix of constants defined by

$$\widehat{\varPi}_2(s_1, \ldots, s_n, m)$$

$$:= \begin{pmatrix} \hat{q}_{1\,0\,s_1}(m) & \hat{p}_{1\,1\,s_1}(m) & \cdots & \hat{p}_{1\,\nu-1\,s_1}(m) & \hat{q}_{1\,\nu\,s_1}(m) & \hat{p}_{1\,\nu+1\,s_1}(m) & \cdots & \hat{p}_{1\,n-1\,s_1}(m) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \hat{q}_{n\,0\,s_n}(m) & \hat{p}_{n\,1\,s_n}(m) & \cdots & \hat{p}_{n\,\nu-1\,s_n}(m) & \hat{q}_{n\,\nu\,s_n}(m) & \hat{p}_{n\,\nu+1\,s_n}(m) & \cdots & \hat{p}_{n\,n-1\,s_n}(m) \end{pmatrix}.$$

It is immediate that $\widehat{\pi}_2(\rho, m)$ can be represented in the form

$$\widehat{\pi}_2(\rho, m) = \sum_{\kappa=-n(m-1)}^{p_0} a_\kappa(m)\rho^\kappa \tag{3.20}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the $a_\kappa(m)$ are constants that depend on the integer m. Specifically, for $\kappa = -n(m-1), \ldots, p_0$ we have

$$a_\kappa(m) = \sum_{s_1+\cdots+s_n=\kappa} \det \widehat{\varPi}_2(s_1, \ldots, s_n, m), \tag{3.21}$$

where the indices $s_1, \ldots, s_n$ satisfy the conditions $-(m-1) \leq s_i \leq m_i$ for $i = 1, \ldots, n$. We will show that many of the $a_\kappa(m)$ are actually independent of m.

Indeed, in equation (3.21) the constant $a_\kappa(m)$ depends upon the integer m in two ways: first, through the entries of the matrices $\widehat{\varPi}_2(s_1, \ldots, s_n, m)$, and second, through the conditions $-(m-1) \leq s_i \leq m_i$ on the indices $s_1, \ldots, s_n$. Fix an index κ that satisfies the condition $-(m - p_0 - 1) \leq \kappa \leq p_0$. Let us examine how the constant $a_\kappa(m)$ is formed. How small can the index s_1 be and still make a contribution to forming $a_\kappa(m)$? Initially we have the condition $-(m-1) \leq s_1 \leq m_1$. Note that

$$\kappa - (p_0 - m_1) \geq -(m - p_0 - 1) - (p_0 - m_1) = -(m-1) + m_1 \geq -(m-1).$$

Consider an index s_1 with $-(m-1) \leq s_1 \leq \kappa - (p_0 - m_1) - 1$. Then the largest possible power of $\rho^{s_1+\cdots+s_n}$ that can be produced is

$$s_1 + m_2 + \cdots + m_n \leq \kappa - (p_0 - m_1) - 1 + m_2 + \cdots + m_n = \kappa - 1,$$

and hence, these powers make no contribution to the computation of $a_\kappa(m)$. Thus, in forming $a_\kappa(m)$ it can be assumed that $\kappa - (p_0 - m_1) \leq s_1 \leq m_1$. We obtain similar conditions for the indices $s_2, \ldots, s_n$. Summarizing, for any index κ with $-(m - p_0 - 1) \leq \kappa \leq p_0$, in forming the constant $a_\kappa(m)$ by means of equation (3.21), the indices $s_1, \ldots, s_n$ can be assumed to satisfy the more restrictive conditions $\kappa - (p_0 - m_i) \leq s_i \leq m_i$ for $i = 1, \ldots, n$; these conditions no longer depend on the integer m.

For integers $s_1, \ldots, s_n$ with $-\infty < s_i \leq m_i$ for $i = 1, \ldots, n$, introduce the matrices

$$\Pi_2(s_1,\dots,s_n) := \begin{pmatrix} q_{1\,0\,s_1} & p_{1\,1\,s_1} \cdots p_{1\,\nu-1\,s_1} & q_{1\,\nu\,s_1} & p_{1\,\nu+1\,s_1} \cdots p_{1\,n-1\,s_1} \\ \vdots & \vdots \qquad\quad \vdots & \vdots & \vdots \qquad\quad \vdots \\ q_{n\,0\,s_n} & p_{n\,1\,s_n} \cdots p_{n\,\nu-1\,s_n} & q_{n\,\nu\,s_n} & p_{n\,\nu+1\,s_n} \cdots p_{n\,n-1\,s_n} \end{pmatrix},$$

which are matrices with constant entries independent of the integer m. Now continuing the discussion for κ with $-(m-p_0-1) \le \kappa \le p_0$, for integers i, k, s_i with $1 \le i \le n$, $0 \le k \le n-1$, and $\kappa - (p_0 - m_i) \le s_i \le m_i$, we have

$$\kappa - (p_0 - m_i) \ge -(m - p_0 - 1) - (p_0 - m_i) = -(m - m_i - 1),$$

and hence, by the representations (3.7) and (3.9),

$$\hat{p}_{i\,k\,s_i}(m) = p_{i\,k\,s_i} \qquad \hat{q}_{i\,k\,s_i}(m) = q_{i\,k\,s_i}, \tag{3.22}$$

which are constants independent of m. Thus, the matrix $\widehat{\Pi}_2(s_1,\dots,s_n,m)$ simplifies to

$$\widehat{\Pi}_2(s_1,\dots,s_n,m) = \Pi_2(s_1,\dots,s_n),$$

and (3.21) in turn simplifies to

$$a_\kappa(m) = \sum_{s_1+\cdots+s_n=\kappa} \det \Pi_2(s_1,\dots,s_n), \tag{3.23}$$

where the indices $s_1,\dots,s_n$ satisfy the restricted conditions $\kappa - (p_0 - m_i) \le s_i \le m_i$ for $i = 1,\dots,n$. Thus, the constants $a_\kappa(m)$, $\kappa = -(m-p_0-1),\dots,p_0$, are *independent* of the integer m, and at the same time we have obtained an explicit formula for calculating these constants.

In terms of the above analysis, set

$$a_\kappa := \sum_{s_1+\cdots+s_n=\kappa} \det \Pi_2(s_1,\dots,s_n)$$

for $\kappa = p_0, p_0-1,\dots,1,0,-1,\dots$, where the indices $s_1,\dots,s_n$ are restricted to satisfy the conditions $\kappa - (p_0 - m_i) \le s_i \le m_i$ for $i = 1,\dots,n$. Clearly this yields an infinite sequence of constants that are independent of the integer m, and from the above $a_\kappa(m) = a_\kappa$ for $\kappa = -(m-p_0-1),\dots,p_0$. Therefore, the function $\widehat{\pi}_2(\rho,m)$ has the representation

$$\widehat{\pi}_2(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa \tag{3.24}$$

for $\rho \ne 0$ in $\mathbb{C}$. We emphasize again that the constants a_κ, $\kappa = p_0, p_0-1, \dots,1,0,-1,\dots$, can be calculated explicitly using the functions $z_{k0}(t)$, $z_{k1}(t)$, $z_{k2}(t)$, …, $k = 0,1,\dots,n-1$, and the boundary values $B_1,\dots,B_n$. The infinite sequence a_κ, $\kappa = p_0, p_0-1,\dots,1,0,-1,\dots$, forms an important invariant for the differential operator L.

A similar discussion can be carried out for the functions $\widehat{\pi}_1(\rho, m)$ and $\widehat{\pi}_0(\rho, m)$, and we will simply state the results. For integers $s_1, \ldots, s_n$ with $-\infty < s_i \leq m_i$ for $i = 1, \ldots, n$, introduce the matrices

$$\mathit{\Pi}_1(s_1, \ldots, s_n) := \begin{pmatrix} p_{1\,0\,s_1} & p_{1\,1\,s_1} \cdots p_{1\,\nu-1\,s_1} & q_{1\,\nu\,s_1} & p_{1\,\nu+1\,s_1} \cdots p_{1\,n-1\,s_1} \\ \vdots & \vdots \qquad\quad \vdots & \vdots & \vdots \qquad\quad \vdots \\ p_{n\,0\,s_n} & p_{n\,1\,s_n} \cdots p_{n\,\nu-1\,s_n} & q_{n\,\nu\,s_n} & p_{n\,\nu+1\,s_n} \cdots p_{n\,n-1\,s_n} \end{pmatrix},$$

$$\mathit{\Pi}^1(s_1, \ldots, s_n) := \begin{pmatrix} q_{1\,0\,s_1} & p_{1\,1\,s_1} \cdots p_{1\,\nu-1\,s_1} & p_{1\,\nu\,s_1} & p_{1\,\nu+1\,s_1} \cdots p_{1\,n-1\,s_1} \\ \vdots & \vdots \qquad\quad \vdots & \vdots & \vdots \qquad\quad \vdots \\ q_{n\,0\,s_n} & p_{n\,1\,s_n} \cdots p_{n\,\nu-1\,s_n} & p_{n\,\nu\,s_n} & p_{n\,\nu+1\,s_n} \cdots p_{n\,n-1\,s_n} \end{pmatrix},$$

and

$$\mathit{\Pi}_0(s_1, \ldots, s_n) := \begin{pmatrix} p_{1\,0\,s_1} & p_{1\,1\,s_1} \cdots p_{1\,\nu-1\,s_1} & p_{1\,\nu\,s_1} & p_{1\,\nu+1\,s_1} \cdots p_{1\,n-1\,s_1} \\ \vdots & \vdots \qquad\quad \vdots & \vdots & \vdots \qquad\quad \vdots \\ p_{n\,0\,s_n} & p_{n\,1\,s_n} \cdots p_{n\,\nu-1\,s_n} & p_{n\,\nu\,s_n} & p_{n\,\nu+1\,s_n} \cdots p_{n\,n-1\,s_n} \end{pmatrix},$$

which are matrices with constant entries independent of the integer m. Then form the infinite sequences of constants

$$\begin{aligned} b_\kappa &:= \sum_{s_1+\cdots+s_n=\kappa} [\det \mathit{\Pi}_1(s_1, \ldots, s_n) + \det \mathit{\Pi}^1(s_1, \ldots, s_n)], \\ c_\kappa &:= \sum_{s_1+\cdots+s_n=\kappa} \det \mathit{\Pi}_0(s_1, \ldots, s_n), \end{aligned}$$

$\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, where the indices $s_1, \ldots, s_n$ are restricted to satisfy the conditions $\kappa - (p_0 - m_i) \leq s_i \leq m_i$ for $i = 1, \ldots, n$. The functions $\widehat{\pi}_1(\rho, m)$, $\widehat{\pi}_0(\rho, m)$ can be represented in the form

$$\widehat{\pi}_1(\rho, m) = \sum_{\kappa=-n(m-1)}^{p_0} b_\kappa(m) \rho^\kappa, \tag{3.25}$$

$$\widehat{\pi}_0(\rho, m) = \sum_{\kappa=-n(m-1)}^{p_0} c_\kappa(m) \rho^\kappa \tag{3.26}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the $b_\kappa(m)$, $c_\kappa(m)$ are constants that depend on the integer m. Since $b_\kappa(m) = b_\kappa$ and $c_\kappa(m) = c_\kappa$ for $\kappa = -(m - p_0 - 1), \ldots, p_0$, this produces the representations

$$\widehat{\pi}_1(\rho, m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m) \rho^\kappa, \tag{3.27}$$

$$\widehat{\pi}_0(\rho, m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} c_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m) \rho^\kappa \tag{3.28}$$

for $\rho \neq 0$ in $\mathbb{C}$. The infinite sequences b_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, and c_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, are also important invariants for the differential operator L.

From equation (3.18) we see that

$$a_\kappa(m) = -\,\omega_1^\kappa c_\kappa(m) = -\,\omega_\kappa c_\kappa(m) \quad \text{for } \kappa = -n(m-1), \ldots, p_0, \tag{3.29}$$

and hence, $a_\kappa = -\,\omega_\kappa c_\kappa$ for $\kappa = -(m - p_0 - 1), \ldots, p_0$. Since m can be chosen arbitrarily large, it follows that

$$a_\kappa = -\,\omega_\kappa c_\kappa \quad \text{for } \kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots. \tag{3.30}$$

We conclude this discussion of the m-dependence by defining the functions

$$\begin{aligned} \pi_2(\rho, m) &:= \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa, \\ \pi_1(\rho, m) &:= \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa, \\ \pi_0(\rho, m) &:= \sum_{\kappa=-(m-p_0-1)}^{p_0} c_\kappa \rho^\kappa \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. The only dependence of these functions on the integer m is in the lower limit $-(m - p_0 - 1)$ of the summations, i.e., m determines how many terms to use in forming the functions $\pi_i(\rho, m)$, $i = 0, 1, 2$.

Finally, we use the three sequences $a_\kappa, b_\kappa, c_\kappa$, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, to formulate our classification scheme for the differential operator L.

Definition 3.2. *In the case $n = 2\nu$ with the boundary values $B_1, \ldots, B_n$ in normalized form and $p_0 = m_1 + \cdots + m_n$, the differential operator L is said to be:*

(*i*) regular *if $a_{p_0} \neq 0$.*

(*ii*) simply irregular *if $a_{p_0} = 0$ and $a_\kappa \neq 0$ for some integer κ with $-\infty < \kappa < p_0$.*

(*iii*) degenerate irregular *if $a_\kappa = 0$ for $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$.*

Henceforth, we will assume that for the case n even, $n = 2\nu \geq 2$, the differential operator L is either regular or simply irregular. Let p be the largest integer with $a_p \neq 0$, so $-\infty < p \leq p_0$. From (3.30) we have $c_p = -\,(\omega_p)^{-1} a_p \neq 0$ and $a_\kappa = c_\kappa = 0$ for $\kappa = p + 1, \ldots, p_0$. At this point we define a second integer q as follows: First, if $b_\kappa = 0$ for $\kappa = p + 1, \ldots, p_0$, then set $q := p$, so $b_q = b_p$ in this case and the constant b_q may be either zero or nonzero. Second, if $b_\kappa \neq 0$ for some integer κ with $p+1 \leq \kappa \leq p_0$, then let q be defined to be the largest such integer, so $p < q \leq p_0$ in this case and the constant b_q is nonzero. In either case we clearly have $b_\kappa = 0$ for $\kappa = q+1, \ldots, p_0$. The integers p, q and the constants a_p, b_q, c_p will play crucial roles in the sequel.

In the ρ plane let us introduce the two sectors

$$\begin{aligned} S_0 &: \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } 0 \leq \theta \leq \frac{\pi}{n}\,, \\ S_1 &: \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} \leq \theta \leq 0\,. \end{aligned}$$

Each of these sectors has angular opening π/n. In Chapter 5 we will show that the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_1}, \ldots, \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \mathrm{e}^{-\mathrm{i}\rho\omega_{\nu+1}}, \ldots, \mathrm{e}^{-\mathrm{i}\rho\omega_{n-1}}$ go to zero very rapidly on the sectors S_0 and S_1. For the case $p = q$, choose a constant $d > 0$ such that

$$|a_p|\mathrm{e}^{-2d} + |b_p|\mathrm{e}^{-d} + |c_p|\mathrm{e}^{-2d} \leq \frac{1}{4}\,|a_p| = \frac{1}{4}\,|c_p|, \tag{3.31}$$

and in terms of the constant d introduce the horizontal strip

$$\varGamma := \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \geq -\pi \text{ and } |b| \leq d\}.$$

Then select complex constants τ_0 and τ_1 and form the translated sectors

$$T_0 := \{\rho - \tau_0 \mid \rho \in S_0\} \quad \text{and} \quad T_1 := \{\rho - \tau_1 \mid \rho \in S_1\}$$

with the following properties: for the case $p = q$ we require that the sectors S_0, S_1 lie in the interiors of T_0, T_1, respectively, and that the horizontal strip $\varGamma$ lies in the interiors of both T_0 and T_1; for the case $p < q$ we require only that the sectors S_0, S_1 lie in the interiors of T_0, T_1, respectively. The translated sectors T_0 and T_1 also have angular opening π/n. They have been constructed by utilizing the constants or invariants a_p, b_q, c_p determined by the differential operator L, and their construction is *independent* of the integer m. The asymptotic expansions developed in the sequel will take place in the sectors T_0 and T_1.

Fix an integer m with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p \leq p_0$, and then form the Birkhoff approximate solutions corresponding to this choice of m:

$$z_k(t,\rho) := z_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$. As earlier in this chapter introduce the modified functions

$$\begin{aligned} y_k(t,\rho) &:= y_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad && k = 0, 1, \ldots, \nu-1, \\ y_k(t,\rho) &:= y_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad && k = \nu, \ldots, n-1, \end{aligned}$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$, and form the approximate characteristic determinant

$$\widehat{\Delta}(\rho) := \widehat{\Delta}(\rho, m) = \det(B_i(y_k(\,\cdot\,, \rho, m)))$$

for $\rho \neq 0$ in $\mathbb{C}$. Lastly introduce the functions

$$\begin{aligned}
\pi_2(\rho) := \pi_2(\rho, m) &= \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \\
\pi_1(\rho) := \pi_1(\rho, m) &= \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa, \\
\pi_0(\rho) := \pi_0(\rho, m) &= \sum_{\kappa=-(m-p_0-1)}^{p} c_\kappa \rho^\kappa
\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. In anticipation of our future work, we rewrite the representation (3.13) of the approximate characteristic determinant in its final form:

$$\begin{aligned}
\widehat{\Delta}(\rho) = {} & \pi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) \\
& + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widehat{\Phi}_2(\rho, m) \Big] \mathrm{e}^{2\mathrm{i}\rho} \\
& + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widehat{\Phi}_1(\rho, m) \Big] \mathrm{e}^{\mathrm{i}\rho} \\
& + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m)\rho^\kappa + \widehat{\Phi}_0(\rho, m) \Big]
\end{aligned} \tag{3.32}$$

for $\rho \neq 0$ in $\mathbb{C}$.

For the case $n = 2\nu$ even, all of the quantities introduced in the last three paragraphs will remain fixed throughout the sequel. In particular, the integer m will be held fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p \leq p_0$.

3.3 Classification for n Odd

Assume that n is odd: $n = 2\nu - 1$. Let us proceed to outline the analogous theory for this case. Recall that we are assuming $n \geq 3$, so $\nu \geq 2$. The starting point for the odd order case is the representation (3.11) of the approximate characteristic determinant $\widehat{\Delta}(\rho)$. Expanding the determinant for $\widehat{\Delta}(\rho)$ using linearity in the 0th column, $\widehat{\Delta}(\rho)$ can be expressed in the form

$$\widehat{\Delta}(\rho) = \widehat{D}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{D}_0(\rho) \tag{3.33}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the functions $\widehat{D}_1(\rho)$, $\widehat{D}_0(\rho)$ are given by

$$\widehat{D}_1(\rho) = \widehat{\pi}_1(\rho) + \widehat{\Phi}_1(\rho), \qquad \widehat{D}_0(\rho) = \widehat{\pi}_0(\rho) + \widehat{\Phi}_0(\rho)$$

for $\rho \neq 0$ in $\mathbb{C}$. In this last equation the functions $\widehat{\pi}_1(\rho)$, $\widehat{\pi}_0(\rho)$ are defined by

$$\widehat{\pi}_1(\rho) := \det\begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{P}_{1\nu}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{P}_{n\nu}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix},$$

$$\widehat{\pi}_0(\rho) := \det\begin{pmatrix} \widehat{P}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{P}_{1\nu}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{P}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{P}_{n\nu}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix}$$

for $\rho \neq 0$ in $\mathbb{C}$, and the functions $\widehat{\Phi}_1(\rho)$, $\widehat{\Phi}_0(\rho)$ are the sums of $2^{n-1} - 1$ determinants with each determinant expressible in the form of a product of some of the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = 1, \ldots, \nu - 1$, or $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $k = \nu, \ldots, n-1$, (at least one of these exponentials appears in each product) times the determinant of an $n \times n$ matrix whose entries are selected from among the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$. Thus, the approximate characteristic determinant can be rewritten in the form

$$\widehat{\Delta}(\rho) = \widehat{\pi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\pi}_0(\rho) + \widehat{\Phi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\Phi}_0(\rho) \tag{3.34}$$

for $\rho \neq 0$ in $\mathbb{C}$.

The approximate characteristic determinant $\widehat{\Delta}(\rho)$ and the functions $\widehat{\pi}_i(\rho)$, $\widehat{\Phi}_i(\rho)$ are all defined and analytic for $\rho \neq 0$ in $\mathbb{C}$, and these functions can be calculated explicitly once the integer m has been fixed.

Next, we examine the structure of the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1$, determining how each of these functions depends on the integer m. We will show that associated with each of these functions is an infinite sequence of constants that are independent of m. In terms of these sequences of constants we will (a) classify the differential operator L as being regular, simply irregular, or degenerate irregular (this section); (b) determine the exact sectors T_0, T_1 to be used in the sequel to develop asymptotic expansions for actual solutions of the differential equation (2.1) (Chapter 4); (c) develop the characteristic determinant (Chapter 5); and (d) derive the basic theory for the eigenvalues of the differential operator L (Chapter 8).

Consider the function $\widehat{\pi}_1(\rho) = \widehat{\pi}_1(\rho, m)$. Appealing to the linearity of the determinant function in its rows, we see that

$$\begin{aligned} \widehat{\pi}_1(\rho, m) &= \sum_{s_1=-(m-1)}^{m_1} \cdots \sum_{s_n=-(m-1)}^{m_n} \rho^{s_1+\cdots+s_n} \det \widehat{\Pi}_1(s_1, \ldots, s_n, m) \\ &= \sum_{\kappa=-n(m-1)}^{p_0} a_\kappa(m)\rho^\kappa \end{aligned} \tag{3.35}$$

for $\rho \neq 0$ in $\mathbb{C}$, where $\widehat{\Pi}_1(s_1, \ldots, s_n, m)$ is the $n \times n$ matrix

$$\widehat{\Pi}_1(s_1, \ldots, s_n, m) \\ := \begin{pmatrix} \hat{q}_{1\,0\,s_1}(m) & \hat{p}_{1\,1\,s_1}(m) & \cdots & \hat{p}_{1\,\nu-1\,s_1}(m) & \hat{p}_{1\,\nu\,s_1}(m) & \cdots & \hat{p}_{1\,n-1\,s_1}(m) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \hat{q}_{n\,0\,s_n}(m) & \hat{p}_{n\,1\,s_n}(m) & \cdots & \hat{p}_{n\,\nu-1\,s_n}(m) & \hat{p}_{n\,\nu\,s_n}(m) & \cdots & \hat{p}_{n\,n-1\,s_n}(m) \end{pmatrix}$$

and the $a_\kappa(m)$ are the constants given by

$$a_\kappa(m) := \sum_{s_1+\cdots+s_n=\kappa} \det \widehat{\Pi}_1(s_1, \ldots, s_n, m) \tag{3.36}$$

for $\kappa = -n(m-1), \ldots, p_0$. Many of the constants $a_\kappa(m)$ turn out to be independent of m.

Indeed, let us introduce the matrices

$$\Pi_1(s_1, \ldots, s_n) := \begin{pmatrix} q_{1\,0\,s_1} & p_{1\,1\,s_1} & \cdots & p_{1\,\nu-1\,s_1} & p_{1\,\nu\,s_1} & \cdots & p_{1\,n-1\,s_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{n\,0\,s_n} & p_{n\,1\,s_n} & \cdots & p_{n\,\nu-1\,s_n} & p_{n\,\nu\,s_n} & \cdots & p_{n\,n-1\,s_n} \end{pmatrix}$$

for integers $s_1, \ldots, s_n$, where $-\infty < s_i \leq m_i$ for $i = 1, \ldots, n$, and then set

$$a_\kappa := \sum_{s_1+\cdots+s_n=\kappa} \det \Pi_1(s_1, \ldots, s_n)$$

for $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, where the indices $s_1, \ldots, s_n$ are restricted to satisfy the conditions $\kappa - (p_0 - m_i) \leq s_i \leq m_i$ for $i = 1, \ldots, n$. The a_κ form an infinite sequence of constants that are independent of m, and $a_\kappa(m) = a_\kappa$ for $\kappa = -(m - p_0 - 1), \ldots, p_0$. Therefore, the function $\widehat{\pi}_1(\rho, m)$ has the representation

$$\widehat{\pi}_1(\rho, m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m) \rho^\kappa \tag{3.37}$$

for $\rho \neq 0$ in $\mathbb{C}$. The constants a_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, can be calculated explicitly using the functions $z_{k0}(t)$, $z_{k1}(t)$, $z_{k2}(t), \ldots$, $k = 0, 1, \ldots, n-1$, and the boundary values $B_1, \ldots, B_n$. This infinite sequence is an important invariant for the differential operator L.

A similar discussion can be carried out for the function $\widehat{\pi}_0(\rho, m)$. For integers $s_1, \ldots, s_n$ with $-\infty < s_i \leq m_i$ for $i = 1, \ldots, n$, introduce the $n \times n$ matrices

$$\Pi_0(s_1, \ldots, s_n) := \begin{pmatrix} p_{1\,0\,s_1} & p_{1\,1\,s_1} & \cdots & p_{1\,\nu-1\,s_1} & p_{1\,\nu\,s_1} & \cdots & p_{1\,n-1\,s_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_{n\,0\,s_n} & p_{n\,1\,s_n} & \cdots & p_{n\,\nu-1\,s_n} & p_{n\,\nu\,s_n} & \cdots & p_{n\,n-1\,s_n} \end{pmatrix},$$

and then form the infinite sequence of constants

$$b_\kappa := \sum_{s_1+\cdots+s_n=\kappa} \det \Pi_0(s_1, \ldots, s_n),$$

$\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, where the indices $s_1, \ldots, s_n$ are restricted to satisfy the conditions $\kappa - (p_0 - m_i) \leq s_i \leq m_i$ for $i = 1, \ldots, n$. The b_κ are independent of the integer m. In terms of these constants the function $\widehat{\pi}_0(\rho, m)$ can be represented in the form

$$\widehat{\pi}_0(\rho, m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m) \rho^\kappa \tag{3.38}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the $b_\kappa(m)$ are constants that do depend on m. The infinite sequence b_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, forms another important invariant for the differential operator L. To conclude this discussion of the m-dependence, we form the two functions

$$\pi_1(\rho, m) := \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa, \qquad \pi_0(\rho, m) := \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. Their only dependence on the integer m is in the lower limit $-(m - p_0 - 1)$ of the summations.

The sequences a_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, and b_κ, $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$, are used to formulate our classification scheme for the differential operator L.

Definition 3.3. *In the case $n = 2\nu - 1$ with the boundary values $B_1, \ldots, B_n$ in normalized form and $p_0 = m_1 + \cdots + m_n$, the differential operator L is said to be:*

(*i*) regular *if $a_{p_0} \neq 0$ and $b_{p_0} \neq 0$.*

(*ii*) simply irregular *if either $a_{p_0} = 0$ or $b_{p_0} = 0$, and $a_\kappa \neq 0$ and $b_\ell \neq 0$ for some integers κ, ℓ with $-\infty < \kappa, \ell \leq p_0$.*

(*iii*) degenerate irregular *if either $a_\kappa = 0$ for $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$ or $b_\kappa = 0$ for $\kappa = p_0, p_0 - 1, \ldots, 1, 0, -1, \ldots$.*

Next, in the ρ plane we introduce the three sectors

$$\begin{aligned} S_0 &: \text{ all } \rho = |\rho| e^{i\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{2n} \leq \theta \leq \frac{\pi}{2n}, \\ S_\diamond &: \text{ all } \rho = |\rho| e^{i\theta} \in \mathbb{C} \text{ with } \frac{\pi}{2n} \leq \theta \leq \frac{3\pi}{2n}, \\ S_1 &: \text{ all } \rho = |\rho| e^{i\theta} \in \mathbb{C} \text{ with } \pi - \frac{\pi}{2n} \leq \theta \leq \pi + \frac{\pi}{2n}. \end{aligned}$$

Each of these sectors has angular opening π/n, with the sectors S_0 and S_1 symmetric in the real axis. Observe that the sector S_0 has been altered from its earlier form for the case n even. The reason for this change is to establish a sector where the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_1}, \ldots, \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \mathrm{e}^{-\mathrm{i}\rho\omega_\nu}, \ldots, \mathrm{e}^{-\mathrm{i}\rho\omega_{n-1}}$ go to zero very rapidly. Indeed, on the original sector S_0 the exponential $\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$ does not go to zero when n is odd: for any point $\rho = |\rho|\mathrm{e}^{\mathrm{i}\pi/n}$ we have

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\right| = \mathrm{e}^{|\rho|\cos[\arg(\mathrm{i}\rho\omega_{\nu-1})]} = \mathrm{e}^{|\rho|\cos\left[\frac{\pi}{2}+\frac{\pi}{n}+\frac{2\pi(\nu-1)}{n}\right]} = \mathrm{e}^{|\rho|\cos[3\pi/2]} = 1.$$

The new sector S_0 corrects this problem (see Chapter 5).

Now consider the sectors $S_\diamond$ and S_1. For $\rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \neq 0$ we have

$$\begin{aligned} \rho \in S_\diamond &\iff \frac{\pi}{2n} \le \theta \le \frac{3\pi}{2n} \\ &\iff \frac{\pi}{2n} + \frac{2\pi}{n}(\nu-1) \le \theta + \frac{2\pi}{n}(\nu-1) \le \frac{3\pi}{2n} + \frac{2\pi}{n}(\nu-1) \\ &\iff \frac{\pi}{2n} + \frac{\pi}{n}(n-1) \le \arg(\rho\omega_{\nu-1}) \le \frac{3\pi}{2n} + \frac{\pi}{n}(n-1) \\ &\iff \pi - \frac{\pi}{2n} \le \arg(\rho\omega_{\nu-1}) \le \pi + \frac{\pi}{2n} \iff \rho\omega_{\nu-1} \in S_1. \end{aligned}$$

Equivalently, $\rho \in S_1 \iff \rho\omega_{\nu-1}^{-1} \in S_\diamond$. Thus, $\rho \in S_\diamond$ and ρ^n is an eigenvalue of L if and only if $\rho\omega_{\nu-1} \in S_1$ and $(\rho\omega_{\nu-1})^n = \rho^n$ is an eigenvalue of L. It follows that finding the eigenvalues $\lambda = \rho^n$ of L with $\rho \in S_\diamond$ is equivalent to finding the eigenvalues $\lambda = \rho^n$ of L with $\rho \in S_1$. In the sequel we choose to work on the sector S_1 because of the simpler geometry.

The modified Birkhoff approximate solutions $y_k(t,\rho)$, $k = 0, 1, \ldots, n-1$, and the approximate characteristic determinant $\widehat{\Delta}(\rho)$ are designed to develop the spectral theory of the differential operator L relative to the sector S_0. In working on the sector S_1, we must replace these quantities with alternate forms. We begin by introducing the functions

$$\begin{aligned} x_k(t,\rho) &:= \mathrm{e}^{-\mathrm{i}\rho\omega_k} z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \quad && k = 0, 1, \ldots, \nu-1, \\ x_k(t,\rho) &:= z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, && k = \nu, \ldots, n-1, \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \neq 0$ in $\mathbb{C}$. These functions are again approximate solutions of the differential equation (2.1) in the sense of Theorem 2.1. They are related to the earlier modified Birkhoff approximate solutions by the equations

$$\begin{aligned} x_k(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} y_k(t,\rho), \qquad && k = 0, 1, \ldots, \nu-1, \\ x_k(t,\rho) &= \mathrm{e}^{\mathrm{i}\rho\omega_k} y_k(t,\rho), && k = \nu, \ldots, n-1, \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \neq 0$ in $\mathbb{C}$.

Applying the boundary values, for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu-1$ we have

$$B_i(x_k(\,\cdot\,,\rho)) = \mathrm{e}^{-\mathrm{i}\rho\omega_k} B_i(y_k(\,\cdot\,,\rho)) = \widehat{P}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widehat{Q}_{ik}(\rho) \tag{3.39}$$

for $\rho \neq 0$ in $\mathbb{C}$, while for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$

$$B_i(x_k(\,\cdot\,,\rho)) = \mathrm{e}^{\mathrm{i}\rho\omega_k} B_i(y_k(\,\cdot\,,\rho)) = \widehat{P}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} + \widehat{Q}_{ik}(\rho) \tag{3.40}$$

for $\rho \neq 0$ in $\mathbb{C}$. In terms of these functions we then form a new *approximate characteristic determinant* by defining

$$\widetilde{\Delta}(\rho) = \widetilde{\Delta}(\rho, m) := \det(B_i(x_k(\,\cdot\,,\rho)))$$

for $\rho \neq 0$ in $\mathbb{C}$. It is also defined and analytic for $\rho \neq 0$ in $\mathbb{C}$; it can be calculated explicitly once m has been chosen; and it is particularly well suited for work on the sector S_1.

The new approximate characteristic determinant has the representation

$$\begin{aligned}\widetilde{\Delta}(\rho) &= \det(B_i(x_k(\,\cdot\,,\rho))) \\ &= \det \begin{array}{c} \begin{array}{ccc} & 1\le k\le \nu-1 & \nu\le k\le n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+\widehat{Q}_{10}(\rho) & \widehat{P}_{1k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}+\widehat{Q}_{1k}(\rho) & \widehat{P}_{1k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}+\widehat{Q}_{1k}(\rho) \\ \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+\widehat{Q}_{n0}(\rho) & \widehat{P}_{nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}+\widehat{Q}_{nk}(\rho) & \widehat{P}_{nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}+\widehat{Q}_{nk}(\rho) \end{pmatrix} \end{array}\end{aligned} \tag{3.41}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the future emphasis will focus on the 0th column of this matrix. Setting $\eta = 1 + \omega_1 + \cdots + \omega_{\nu-1} - \omega_\nu - \cdots - \omega_{n-1}$, we can factor out the exponentials appearing in the columns of the matrix in equation (3.41):

$$\widetilde{\Delta}(\rho) = \mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{-\mathrm{i}\rho\omega_1} \cdots \mathrm{e}^{-\mathrm{i}\rho\omega_{\upsilon-1}}\mathrm{e}^{\mathrm{i}\rho\omega_\nu} \cdots \mathrm{e}^{\mathrm{i}\rho\omega_{n-1}}\, \widehat{\Delta}(\rho) = \mathrm{e}^{-\mathrm{i}\rho\eta}\, \widehat{\Delta}(\rho) \tag{3.42}$$

for $\rho \neq 0$ in $\mathbb{C}$.

Using equation (3.41), we proceed as above to expand the determinant for $\widetilde{\Delta}(\rho)$ using the linearity in the 0th column:

$$\widetilde{\Delta}(\rho) = \widetilde{D}_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{D}_0(\rho) \tag{3.43}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the functions $\widetilde{D}_1(\rho)$, $\widetilde{D}_0(\rho)$ are given by

$$\widetilde{D}_1(\rho) = \widetilde{\pi}_1(\rho) + \widetilde{\Phi}_1(\rho), \qquad \widetilde{D}_0(\rho) = \widetilde{\pi}_0(\rho) + \widetilde{\Phi}_0(\rho)$$

for $\rho \neq 0$ in $\mathbb{C}$. Here the functions $\widetilde{\pi}_1(\rho)$, $\widetilde{\pi}_0(\rho)$ are defined by

$$\widetilde{\pi}_1(\rho) := \det \begin{pmatrix} \widehat{P}_{10}(\rho) & \widehat{Q}_{11}(\rho) & \cdots & \widehat{Q}_{1\,\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \cdots & \widehat{Q}_{1\,n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{P}_{n0}(\rho) & \widehat{Q}_{n1}(\rho) & \cdots & \widehat{Q}_{n\,\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \cdots & \widehat{Q}_{n\,n-1}(\rho) \end{pmatrix},$$

$$\widetilde{\pi}_0(\rho) := \det\begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{Q}_{11}(\rho) & \cdots & \widehat{Q}_{1\,\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \cdots & \widehat{Q}_{1\,n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{Q}_{n1}(\rho) & \cdots & \widehat{Q}_{n\,\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \cdots & \widehat{Q}_{n\,n-1}(\rho) \end{pmatrix}$$

for $\rho \neq 0$ in $\mathbb{C}$, and the functions $\widetilde{\Phi}_1(\rho)$, $\widetilde{\Phi}_0(\rho)$ are the sums of $2^{n-1}-1$ determinants with each determinant expressible in the form of a product of some of the exponentials $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $k = 1, \ldots, \nu-1$, or some of the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = \nu, \ldots, n-1$, (at least one of these exponentials appears in each product), times the determinant of an $n \times n$ matrix whose entries are selected from among the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$. Thus, the approximate characteristic determinant $\widetilde{\Delta}(\rho)$ can be expressed in the form

$$\widetilde{\Delta}(\rho) = \widetilde{\pi}_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\pi}_0(\rho) + \widetilde{\Phi}_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\Phi}_0(\rho) \tag{3.44}$$

for $\rho \neq 0$ in $\mathbb{C}$. Again all these functions are defined and analytic for $\rho \neq 0$ in $\mathbb{C}$, and they can be calculated explicitly once the integer m has been fixed.

Let us consider the function $\widetilde{\pi}_1(\rho) = \widetilde{\pi}_1(\rho, m)$. Proceeding as above, we see that

$$\begin{aligned} \widetilde{\pi}_1(\rho, m) &= \sum_{s_1=-(m-1)}^{m_1} \cdots \sum_{s_n=-(m-1)}^{m_n} \rho^{s_1+\cdots+s_n} \det \widetilde{\Pi}_1(s_1, \ldots, s_n, m) \\ &= \sum_{\kappa=-n(m-1)}^{p_0} a'_\kappa(m)\rho^\kappa \end{aligned} \tag{3.45}$$

for $\rho \neq 0$ in $\mathbb{C}$, where $\widetilde{\Pi}_1(s_1, \ldots, s_n, m)$ is the $n \times n$ matrix

$$\begin{aligned} &\widetilde{\Pi}_1(s_1, \ldots, s_n, m) \\ &:= \begin{pmatrix} \hat{p}_{1\,0\,s_1}(m) & \hat{q}_{1\,1\,s_1}(m) & \cdots & \hat{q}_{1\,\nu-1\,s_1}(m) & \hat{q}_{1\,\nu\,s_1}(m) & \cdots & \hat{q}_{1\,n-1\,s_1}(m) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \hat{p}_{n\,0\,s_n}(m) & \hat{q}_{n\,1\,s_n}(m) & \cdots & \hat{q}_{n\,\nu-1\,s_n}(m) & \hat{q}_{n\,\nu\,s_n}(m) & \cdots & \hat{q}_{n\,n-1\,s_n}(m) \end{pmatrix} \end{aligned}$$

and the $a'_\kappa(m)$ are the constants given by

$$a'_\kappa(m) := \sum_{s_1+\cdots+s_n=\kappa} \det \widetilde{\Pi}_1(s_1, \ldots, s_n, m), \tag{3.46}$$

for $\kappa = -n(m-1), \ldots, p_0$.

For integers $s_1, \ldots, s_n$ with $-\infty < s_i \leq m_i$ for $i = 1, \ldots, n$, introduce the $n \times n$ matrices

$$\Pi'_1(s_1, \ldots, s_n) := \begin{pmatrix} p_{1\,0\,s_1} & q_{1\,1\,s_1} & \cdots & q_{1\,\nu-1\,s_1} & q_{1\,\nu\,s_1} & \cdots & q_{1\,n-1\,s_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ p_{n\,0\,s_n} & q_{n\,1\,s_n} & \cdots & q_{n\,\nu-1\,s_n} & q_{n\,\nu\,s_n} & \cdots & q_{n\,n-1\,s_n} \end{pmatrix},$$

and then set

$$a'_\kappa := \sum_{s_1+\cdots+s_n=\kappa} \det \Pi'_1(s_1,\ldots,s_n)$$

for $\kappa = p_0, p_0-1,\ldots,1,0,-1,\ldots$, where the indices $s_1,\ldots,s_n$ are restricted to satisfy the conditions $\kappa-(p_0-m_i) \le s_i \le m_i$ for $i=1,\ldots,n$. The a'_κ form an infinite sequence of constants that are independent of the integer m, $a'_\kappa(m) = a'_\kappa$ for $\kappa = -(m-p_0-1),\ldots,p_0$, and the function $\widetilde{\pi}_1(\rho,m)$ has the representation

$$\widetilde{\pi}_1(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a'_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a'_\kappa(m)\rho^\kappa \tag{3.47}$$

for $\rho \ne 0$ in $\mathbb{C}$. The constants a'_κ, $\kappa = p_0, p_0-1,\ldots,1,0,-1,\ldots$, can be calculated explicitly, and they form an invariant for the differential operator L.

A similar discussion can be carried out for the function $\widetilde{\pi}_0(\rho,m)$. For integers $s_1,\ldots,s_n$ with $-\infty < s_i \le m_i$ for $i=1,\ldots,n$, introduce the $n\times n$ matrices

$$\Pi'_0(s_1,\ldots,s_n) := \begin{pmatrix} q_{1\,0\,s_1} & q_{1\,1\,s_1} & \cdots & q_{1\,\nu-1\,s_1} & q_{1\,\nu\,s_1} & \cdots & q_{1\,n-1\,s_1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ q_{n\,0\,s_n} & q_{n\,1\,s_n} & \cdots & q_{n\,\nu-1\,s_n} & q_{n\,\nu\,s_n} & \cdots & q_{n\,n-1\,s_n} \end{pmatrix},$$

which have constant entries independent of the integer m. Then form the infinite sequence of constants

$$b'_\kappa := \sum_{s_1+\cdots+s_n=\kappa} \det \Pi'_0(s_1,\ldots,s_n),$$

$\kappa = p_0, p_0-1,\ldots,1,0,-1,\ldots$, where the indices $s_1,\ldots,s_n$ are restricted to satisfy the conditions $\kappa-(p_0-m_i) \le s_i \le m_i$ for $i=1,\ldots,n$. The constants b'_κ are independent of the integer m, and in terms of them the function $\widetilde{\pi}_0(\rho,m)$ can be written in the form

$$\widetilde{\pi}_0(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b'_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b'_\kappa(m)\rho^\kappa \tag{3.48}$$

for $\rho \ne 0$ in $\mathbb{C}$, where the $b'_\kappa(m)$ are constants that do depend on m. The infinite sequence b'_κ, $\kappa = p_0, p_0-1,\ldots,1,0,-1,\ldots$, forms an additional invariant for the differential operator L. We conclude the discussion of m-dependence by introducing the functions

$$\pi'_1(\rho,m) := \sum_{\kappa=-(m-p_0-1)}^{p_0} a'_\kappa \rho^\kappa, \qquad \pi'_0(\rho,m) := \sum_{\kappa=-(m-p_0-1)}^{p_0} b'_\kappa \rho^\kappa$$

for $\rho \ne 0$ in $\mathbb{C}$.

There is a close relationship between the functions $\widehat{\pi}_1(\rho)$ and $\widehat{\pi}_0(\rho)$ appearing in (3.34) and the functions $\widetilde{\pi}_1(\rho)$ and $\widetilde{\pi}_0(\rho)$ appearing in (3.44). Indeed, consider the matrix that appears in the definition of $\widehat{\pi}_1(\rho)$. From equation (2.13) and the definitions of the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, we have $\widehat{Q}_{i0}(\rho\omega_{\nu-1}) = \widehat{Q}_{i\,\nu-1}(\rho)$ for $i = 1, \dots, n$; $\widehat{P}_{ik}(\rho\omega_{\nu-1}) = \widehat{Q}_{i\,k+\nu-1}(\rho)$ for $i = 1, \dots, n$ and $k = 1, \dots, \nu - 1$; and $\widehat{P}_{ik}(\rho\omega_{\nu-1}) = \widehat{Q}_{i\,k-\nu}(\rho)$ for $i = 1, \dots, n$ and $k = \nu, \dots, n-1$. Thus, from the definition of $\widehat{\pi}_1(\rho)$ we get

$$\begin{aligned}\widehat{\pi}_1(\rho\omega_{\nu-1}) &= \det\begin{pmatrix} \widehat{Q}_{1\,\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \cdots & \widehat{Q}_{1\,n-1}(\rho) & \widehat{Q}_{10}(\rho) & \cdots & \widehat{Q}_{1\,\nu-2}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n\,\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \cdots & \widehat{Q}_{n\,n-1}(\rho) & \widehat{Q}_{n0}(\rho) & \cdots & \widehat{Q}_{n\,\nu-2}(\rho) \end{pmatrix} \\ &= (-1)^{\nu(\nu-1)}\widetilde{\pi}_0(\rho) = \widetilde{\pi}_0(\rho),\end{aligned}$$

or

$$\widetilde{\pi}_0(\rho) = \widehat{\pi}_1(\rho\omega_{\nu-1}) \quad \text{for } \rho \neq 0 \text{ in } \mathbb{C}. \tag{3.49}$$

A similar argument shows that

$$\widetilde{\pi}_1(\rho) = \widehat{\pi}_0(\rho\omega_{\nu}) \quad \text{for } \rho \neq 0 \text{ in } \mathbb{C}. \tag{3.50}$$

Henceforth, we will assume that for the case n odd, $n = 2\nu - 1 \geq 3$, the differential operator L is either regular or simply irregular. Let p be the largest integer with $a_p \neq 0$, so $-\infty < p \leq p_0$, and let q be the largest integer with $b_q \neq 0$, so $-\infty < q \leq p_0$. Then $a_\kappa = 0$ for $\kappa = p+1, \dots, p_0$, and $b_\kappa = 0$ for $\kappa = q+1, \dots, p_0$. The integers p, q and the nonzero constants a_p, b_q will play major roles in the sequel.

For the case $p = q$ choose a constant $d > 0$ such that

$$|a_p|\mathrm{e}^{-d} + |b_p|\mathrm{e}^{-d} \leq \frac{1}{4}\min\{|a_p|, |b_p|\}, \tag{3.51}$$

and in terms of the constant d introduce the horizontal strips

$$\begin{aligned}\mathit{\Gamma}_0 &:= \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \geq -\pi \text{ and } |b| \leq d\}, \\ \mathit{\Gamma}_1 &:= \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \leq \pi \text{ and } |b| \leq d\}.\end{aligned}$$

Then select complex constants τ_0 and τ_1 and form the translated sectors

$$T_0 := \{\rho - \tau_0 \mid \rho \in S_0\}, \qquad T_1 := \{\rho - \tau_1 \mid \rho \in S_1\}$$

with the following properties: for the case $p = q$ we require that the sectors S_0, S_1 lie in the interiors of T_0, T_1, respectively, and that the horizontal strips $\mathit{\Gamma}_0$, $\mathit{\Gamma}_1$ lie in the interiors of T_0, T_1, respectively; for the cases $p < q$ and $p > q$ we require only that the sectors S_0, S_1 lie in the interiors of T_0, T_1, respectively. The translated sectors T_0 and T_1 have been constructed by utilizing the constants or invariants a_p and b_q determined by the differential operator L, and their construction is *independent* of the integer m. The asymptotic expansions developed in the next chapter will take place in the sectors T_0 and T_1.

Fix an integer m with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$, and then form the Birkhoff approximate solutions corresponding to this choice of m:

$$z_k(t,\rho) := z_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0,1,\dots,n-1,$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$. As earlier in this chapter, introduce the modified Birkhoff approximate solutions

$$\begin{aligned} y_k(t,\rho) &:= y_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0,1,\dots,\nu-1, \\ y_k(t,\rho) &:= y_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \quad k = \nu,\dots,n-1, \end{aligned}$$

and

$$\begin{aligned} x_k(t,\rho) &:= x_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \quad k = 0,1,\dots,\nu-1, \\ x_k(t,\rho) &:= x_k(t,\rho,m) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = \nu,\dots,n-1, \end{aligned}$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$, and form the approximate characteristic determinants

$$\begin{aligned} \widehat{\Delta}(\rho) &:= \widehat{\Delta}(\rho,m) = \det(B_i(y_k(\,\cdot\,,\rho,m))), \\ \widetilde{\Delta}(\rho) &:= \widetilde{\Delta}(\rho,m) = \det(B_i(x_k(\,\cdot\,,\rho,m))) \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. Lastly introduce the functions

$$\pi_1(\rho) := \pi_1(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa\rho^\kappa, \quad \pi_0(\rho) := \pi_0(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa\rho^\kappa$$

and

$$\pi_1'(\rho) := \pi_1'(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa'\rho^\kappa, \quad \pi_0'(\rho) := \pi_0'(\rho,m) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa'\rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. From (3.50) and (3.49) we have

$$a_\kappa' = b_\kappa(\omega_\nu)^\kappa \quad \text{and} \quad b_\kappa' = a_\kappa(\omega_{\nu-1})^\kappa \tag{3.52}$$

for $\kappa = p_0, p_0 - 1, \ldots, -(m - p_0 - 1)$, and hence, $a'_q \neq 0$ and $a'_\kappa = 0$ for $\kappa = q + 1, \ldots, p_0$, and $b'_p \neq 0$ and $b'_\kappa = 0$ for $\kappa = p + 1, \ldots, p_0$. It follows that the functions $\pi'_1(\rho)$, $\pi'_0(\rho)$ simplify to

$$\pi'_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} a'_\kappa \rho^\kappa, \qquad \pi'_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} b'_\kappa \rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. Also, in the case $p = q$ we have $|a'_p| = |b_p|$ and $|b'_p| = |a_p|$, and by (3.51)

$$|a'_p| \mathrm{e}^{-d} + |b'_p| \mathrm{e}^{-d} \leq \frac{1}{4} \min\{|a'_p|, |b'_p|\}. \tag{3.53}$$

For our future work we rewrite the representations (3.34) and (3.44) of the approximate characteristic determinants in their final forms:

$$\begin{aligned} \widehat{\Delta}(\rho) &= \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widehat{\Phi}_1(\rho, m) \Big] \mathrm{e}^{\mathrm{i}\rho} \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widehat{\Phi}_0(\rho, m) \Big], \end{aligned} \tag{3.54}$$

$$\begin{aligned} \widetilde{\Delta}(\rho) &= \pi'_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi'_0(\rho) \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} a'_\kappa(m)\rho^\kappa + \widetilde{\Phi}_1(\rho, m) \Big] \mathrm{e}^{-\mathrm{i}\rho} \\ &\quad + \Big[\sum_{\kappa=-n(m-1)}^{-(m-p_0)} b'_\kappa(m)\rho^\kappa + \widetilde{\Phi}_0(\rho, m) \Big] \end{aligned} \tag{3.55}$$

for $\rho \neq 0$ in $\mathbb{C}$.

For the case $n = 2\nu - 1$ odd, the quantities introduced in the last four paragraphs will remain fixed throughout the sequel. Specifically, the integer m will be held fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$.

3.4 Tests for Regularity

Some remarks are in order regarding the conditions for the differential operator L to be *regular*. These conditions are stated in Definition 3.2 and Definition 3.3 for the cases n even and n odd. It is important that these definitions be consistent with the earlier definitions given in the monograph [34]. See Definiton 5.1 and Definition 6.1 in Chapter 4 of [34] for the definitions of the principal part T being determined by regular boundary values $B_1, \dots, B_n$, and then see p. 212 for the general differential operator L, where we assume that the boundary values $B_1, \dots, B_n$ are regular relative to the principal part T. Let us examine this question of consistency.

From equations (3.7) and (3.9) the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, $i = 1, \dots, n$, $k = 0, 1, \dots, n-1$, have the representations

$$\widehat{P}_{ik}(\rho) = \sum_{s=-(m-m_i-1)}^{m_i} p_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{p}_{iks}(m)\rho^s,$$

$$\widehat{Q}_{ik}(\rho) = \sum_{s=-(m-m_i-1)}^{m_i} q_{iks}\rho^s + \sum_{s=-(m-1)}^{-(m-m_i)} \hat{q}_{iks}(m)\rho^s$$

for $\rho \neq 0$ in $\mathbb{C}$. In these formulas the leading coefficients are given as follows: for $i = 1, \dots, n$ and $k = 0, 1, \dots, \nu - 1$,

$$p_{ikm_i} = \sum_{p=m_i}^{m_i} \sum_{\ell=0}^{0} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z^{(\ell)}_{k\,p-\ell-s}(0) = \alpha_{im_i} (\mathrm{i}\omega_k)^{m_i},$$

$$q_{ikm_i} = \sum_{p=m_i}^{m_i} \sum_{\ell=0}^{0} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z^{(\ell)}_{k\,p-\ell-s}(1) = \beta_{im_i} (\mathrm{i}\omega_k)^{m_i},$$

while for $i = 1, \dots, n$ and $k = \nu, \dots, n-1$,

$$p_{ikm_i} = \sum_{p=m_i}^{m_i} \sum_{\ell=0}^{0} \beta_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z^{(\ell)}_{k\,p-\ell-s}(1) = \beta_{im_i} (\mathrm{i}\omega_k)^{m_i},$$

$$q_{ikm_i} = \sum_{p=m_i}^{m_i} \sum_{\ell=0}^{0} \alpha_{ip} \binom{p}{\ell} (\mathrm{i}\omega_k)^{p-\ell} z^{(\ell)}_{k\,p-\ell-s}(0) = \alpha_{im_i} (\mathrm{i}\omega_k)^{m_i}.$$

Observe that these leading coefficients are determined exclusively by the normalized boundary values $B_1, \dots, B_n$. They are independent of the formal differential operator ℓ that determines L, and they are identical for both L and for its principal part T.

Assume that n is even: $n = 2\nu$. In our current work the differential operator L is defined to be regular if and only if the constant a_{p_0} is nonzero. See Definition 3.2. It is possible to obtain an explicit formula for the constant a_{p_0}. Indeed, for $i = 1, \dots, n$ define

$$\begin{aligned}
N_{i0}(\rho) &:= \widehat{P}_{i0}(\rho) + \widehat{Q}_{i0}(\rho)\mathrm{e}^{\mathrm{i}\rho}, \\
N_{i\nu}(\rho) &:= \widehat{P}_{i\nu}(\rho) + \widehat{Q}_{i\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho}, \\
N_{ik}(\rho) &:= \widehat{P}_{ik}(\rho), \qquad 1 \le k \le \nu - 1, \\
N_{ik}(\rho) &:= \widehat{P}_{ik}(\rho), \qquad \nu + 1 \le k \le n - 1,
\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. From the definitions of the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, we see that

$$\det\left(N_{ik}(\rho)\right) = \widehat{\pi}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \widehat{\pi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\pi}_0(\rho) \tag{3.56}$$

for $\rho \neq 0$ in $\mathbb{C}$, where by (3.24), (3.27), and (3.28) the functions $\widehat{\pi}_i(\rho)$ have the form

$$\widehat{\pi}_2(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa,$$

$$\widehat{\pi}_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa,$$

$$\widehat{\pi}_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p_0} c_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m)\rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. The constants a_{p_0}, b_{p_0}, c_{p_0} are the leading coefficients of the functions $\widehat{\pi}_2(\rho)$, $\widehat{\pi}_1(\rho)$, $\widehat{\pi}_0(\rho)$, respectively, and we have the important relation $a_{p_0} = -\omega_{p_0} c_{p_0}$ given in equation (3.30).

Now for $i = 1, \ldots, n$ define

$$\begin{aligned}
\mathcal{N}_{i0}(\rho) &:= \alpha_{im_i} + \beta_{im_i}\mathrm{e}^{\mathrm{i}\rho}, \\
\mathcal{N}_{i\nu}(\rho) &:= \beta_{im_i}(-1)^{m_i} + \alpha_{im_i}(-1)^{m_i}\mathrm{e}^{\mathrm{i}\rho}, \\
\mathcal{N}_{ik}(\rho) &:= \alpha_{im_i}\omega_k^{m_i}, \qquad 1 \leq k \leq \nu - 1, \\
\mathcal{N}_{ik}(\rho) &:= \beta_{im_i}\omega_k^{m_i}, \qquad \nu + 1 \leq k \leq n - 1,
\end{aligned}$$

for $\rho \in \mathbb{C}$. If we expand the determinant on the left side of (3.56) using linearity in all n columns and then pull out the factors of $\mathrm{i}\rho$ from the rows of the resulting determinants, then it follows that

$$\begin{aligned}
&(\mathrm{i}\rho)^{p_0} \det\left(\mathcal{N}_{ik}(\rho)\right) + \text{ lower powers of } \rho \;= \\
&\qquad a_{p_0}\rho^{p_0}\mathrm{e}^{2\mathrm{i}\rho} + b_{p_0}\rho^{p_0}\mathrm{e}^{\mathrm{i}\rho} + c_{p_0}\rho^{p_0} + \text{ lower powers of } \rho
\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, and hence,

$$\det\left(\mathcal{N}_{ik}(\rho)\right) = \mathrm{i}^{\,-p_0}\left(a_{p_0}\mathrm{e}^{2\mathrm{i}\rho} + b_{p_0}\mathrm{e}^{\mathrm{i}\rho} + c_{p_0}\right) \tag{3.57}$$

for $\rho \in \mathbb{C}$. The justification for matching powers of ρ follows from simple limit arguments and the facts that if $\rho = \mathrm{i}b$ is a pure imaginary, then

$$|\rho^k \mathrm{e}^{2\mathrm{i}\rho}| = |b|^k \mathrm{e}^{-2b} \to 0, \qquad |\rho^k \mathrm{e}^{\mathrm{i}\rho}| = |b|^k \mathrm{e}^{-b} \to 0, \qquad |1| = 1 \to 1$$

as $b \to \infty$ for $k = 0, 1, 2, \ldots$. Equation (3.57) is identical to equation (5.20) in Chapter 4 of the monograph [34], and it shows that the constant a_{p_0} occurring in our current work is equal to the constant a_{p_0} that occurs in [34]. Specifically,

$$a_{p_0} = \mathrm{i}^{p_0} \det \begin{array}{c} \begin{array}{cccc} & 1 \le k \le \nu - 1 & & \nu + 1 \le k \le n - 1 \end{array} \\ \begin{pmatrix} \beta_{1m_1} & \alpha_{1m_1}\omega_k^{m_1} & \alpha_{1m_1}(-1)^{m_1} & \beta_{1m_1}\omega_k^{m_1} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{nm_n} & \alpha_{nm_n}\omega_k^{m_n} & \alpha_{nm_n}(-1)^{m_n} & \beta_{nm_n}\omega_k^{m_n} \end{pmatrix} \end{array}. \tag{3.58}$$

Cf. Naimark [36, p. 57].

From the above we draw the following conclusions for the case n even: (i) the definition for the differential operator L being regular given in our current work is consistent with the earlier definition of regular boundary values given in [34]; (ii) the differential operator L is regular if and only if its principal part T is regular; and (iii) in checking the condition $a_{p_0} \neq 0$ for regularity, only the boundary values $B_1, \ldots, B_n$ are relevant — the coefficients determining L play no role. The constant a_{p_0} can be calculated explicitly using equation (3.58), and does not require the calculation of any solutions of the differential equation (2.1), nor the calculation of any characteristic determinants.

Next, assume that n is odd: $n = 2\nu - 1$. In our current work, the differential operator L is defined to be regular if and only if the constants a_{p_0} and b_{p_0} are both nonzero. See Definition 3.3. Explicit formulas for these constants are available. For $i = 1, \ldots, n$ define

$$\begin{aligned} N_{i0}(\rho) &:= \widehat{P}_{i0}(\rho) + \widehat{Q}_{i0}(\rho)\mathrm{e}^{\mathrm{i}\rho}, \\ N_{ik}(\rho) &:= \widehat{P}_{ik}(\rho), \qquad 1 \le k \le \nu - 1, \\ N_{ik}(\rho) &:= \widehat{P}_{ik}(\rho), \qquad \nu \le k \le n - 1, \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. From the definitions of the functions $\widehat{\pi}_1(\rho)$, $\widehat{\pi}_0(\rho)$, we see that

$$\det\left(N_{ik}(\rho)\right) = \widehat{\pi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\pi}_0(\rho) \tag{3.59}$$

for $\rho \neq 0$ in $\mathbb{C}$, where from (3.37) and (3.38) we have

$$\widehat{\pi}_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p_0} a_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa,$$

$$\widehat{\pi}_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p_0} b_\kappa \rho^\kappa + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. The constants a_{p_0}, b_{p_0} are the leading coefficients of the functions $\widehat{\pi}_1(\rho)$, $\widehat{\pi}_0(\rho)$, respectively.

Now for $i = 1, \ldots, n$ define

$$\begin{aligned} \mathcal{N}_{i0}(\rho) &:= \alpha_{im_i} + \beta_{im_i}\mathrm{e}^{\mathrm{i}\rho}, \\ \mathcal{N}_{ik}(\rho) &:= \alpha_{im_i}\omega_k^{m_i}, \qquad 1 \le k \le \nu - 1, \\ \mathcal{N}_{ik}(\rho) &:= \beta_{im_i}\omega_k^{m_i}, \qquad \nu \le k \le n - 1, \end{aligned}$$

for $\rho \in \mathbb{C}$. If we expand the determinant on the left side of (3.59) using linearity in all n columns and then pull out the factors of $\mathrm{i}\rho$ from the rows of the resulting determinants, then it follows that

$$(\mathrm{i}\rho)^{p_0} \det\left(\mathcal{N}_{ik}(\rho)\right) + \text{lower powers of } \rho = \\ a_{p_0}\rho^{p_0}\mathrm{e}^{\mathrm{i}\rho} + b_{p_0}\rho^{p_0} + \text{ lower powers of } \rho$$

for $\rho \neq 0$ in $\mathbb{C}$, and hence,

$$\det\left(\mathcal{N}_{ik}(\rho)\right) = \mathrm{i}^{-p_0}\left(a_{p_0}\mathrm{e}^{\mathrm{i}\rho} + b_{p_0}\right) \tag{3.60}$$

for $\rho \in \mathbb{C}$. Equation (3.60) is identical to equation (6.16) in Chapter 4 of [34], and it shows that the constants a_{p_0}, b_{p_0} occurring in our current work are equal to the constants a_{p_0}, b_{p_0} that occur in [34]. Specifically,

$$a_{p_0} = \mathrm{i}^{p_0} \det \begin{array}{c} \qquad\quad 1 \leq k \leq \nu - 1 \quad \nu \leq k \leq n-1 \\ \begin{pmatrix} \beta_{1m_1} & \alpha_{1m_1}\omega_k^{m_1} & \beta_{1m_1}\omega_k^{m_1} \\ \vdots & \vdots & \vdots \\ \beta_{nm_n} & \alpha_{nm_n}\omega_k^{m_n} & \beta_{nm_n}\omega_k^{m_n} \end{pmatrix} \end{array}, \tag{3.61}$$

$$b_{p_0} = \mathrm{i}^{p_0} \det \begin{array}{c} \qquad\quad 1 \leq k \leq \nu - 1 \quad \nu \leq k \leq n-1 \\ \begin{pmatrix} \alpha_{1m_1} & \alpha_{1m_1}\omega_k^{m_1} & \beta_{1m_1}\omega_k^{m_1} \\ \vdots & \vdots & \vdots \\ \alpha_{nm_n} & \alpha_{nm_n}\omega_k^{m_n} & \beta_{nm_n}\omega_k^{m_n} \end{pmatrix} \end{array}. \tag{3.62}$$

Cf. Naimark [36, p. 56].

For the case n odd, we draw the same conclusions as before: (i) the definition for the differential operator L being regular given in our current work is consistent with the earlier definition of regular boundary values given in [34]; (ii) the differential operator L is regular if and only if its principal part T is regular; and (iii) in checking the conditions $a_{p_0} \neq 0$ and $b_{p_0} \neq 0$ for regularity, only the boundary values $B_1, \ldots, B_n$ are relevant — the coefficients determining L play no role. The constants a_{p_0} and b_{p_0} can be calculated explicitly using (3.61) and (3.62), and do not require the calculation of any solutions of the diffferential equation (2.1), nor the calculation of any characteristic determinants.

4

Asymptotic Expansion of Solutions

In this chapter we construct actual solutions of the differential equation (2.1) for ρ belonging to the sectors T_0 or T_1. These solutions behave asymptotically like the Birkhoff approximate solutions $z_k(t, \rho) = z_k(t, \rho, m)$, where the integer m has been fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$. Again the discussion divides naturally into the two cases n even and n odd.

4.1 Expansions for n Even

Assume that n is even: $n = 2\nu \geq 2$. We begin by examining the relationship between the complex numbers $\mathrm{i}\rho\omega_0$, $\mathrm{i}\rho\omega_1$, …, $\mathrm{i}\rho\omega_{n-1}$ for ρ belonging to the sectors S_0 or S_1, where

$$\begin{aligned} S_0 &: \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } 0 \leq \theta \leq \frac{\pi}{n}\,, \\ S_1 &: \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} \leq \theta \leq 0\,. \end{aligned}$$

The results in the following lemma are variations of some well-known results (see [4, p. 220] or [36, pp. 43–45]).

Lemma 4.1. *Let n be even: $n = 2\nu$. Then there exist permutations*

$$\omega_0^0, \omega_1^0, \ldots, \omega_{n-1}^0 \quad \textit{and} \quad \omega_0^1, \omega_1^1, \ldots, \omega_{n-1}^1$$

of the nth roots of unity $\omega_0, \omega_1, \ldots, \omega_{n-1}$ such that

$$\begin{aligned} \mathrm{Re}(\mathrm{i}\rho\omega_0^0) \leq \mathrm{Re}(\mathrm{i}\rho\omega_1^0) \leq \cdots \leq \mathrm{Re}(\mathrm{i}\rho\omega_{n-1}^0) \quad & \textit{for all } \rho \in S_0, \\ \mathrm{Re}(\mathrm{i}\rho\omega_0^1) \leq \mathrm{Re}(\mathrm{i}\rho\omega_1^1) \leq \cdots \leq \mathrm{Re}(\mathrm{i}\rho\omega_{n-1}^1) \quad & \textit{for all } \rho \in S_1. \end{aligned}$$

Proof. Note that the list $\omega_0, \omega_1^{-1}, \omega_1, \omega_2^{-1}, \ldots, \omega_{\nu-1}, \omega_\nu^{-1}$ is a permutation of $\omega_0, \omega_1, \ldots, \omega_{n-1}$, and for $\rho \in S_0$ we have

$$0 \le \arg(\rho\omega_0) \le \frac{\pi}{n}, \qquad -\frac{2\pi}{n} \le \arg(\rho\omega_1^{-1}) \le -\frac{\pi}{n},$$

$$\frac{2\pi}{n} \le \arg(\rho\omega_1) \le \frac{3\pi}{n}, \qquad -\frac{4\pi}{n} \le \arg(\rho\omega_2^{-1}) \le -\frac{3\pi}{n}, \quad \dots \quad ,$$

$$\frac{(2\nu-2)\pi}{n} \le \arg(\rho\omega_{\nu-1}) \le \frac{(2\nu-1)\pi}{n},$$

$$-\pi = -\frac{(2\nu)\pi}{n} \le \arg(\rho\omega_\nu^{-1}) \le -\frac{(2\nu-1)\pi}{n}.$$

Since the cosine is an even function and is decreasing on the interval $[0,\pi]$, it follows that

$$\begin{aligned}\cos[\arg(\rho\omega_\nu^{-1})] &\le \cos[\arg(\rho\omega_{\nu-1})] \le \dots \le \cos[\arg(\rho\omega_2^{-1})] \\ &\le \cos[\arg(\rho\omega_1)] \le \cos[\arg(\rho\omega_1^{-1})] \le \cos[\arg(\rho\omega_0)]\end{aligned}$$

for $\rho \in S_0$, and hence,

$$\begin{aligned}\mathrm{Re}(\rho\omega_\nu^{-1}) &\le \mathrm{Re}(\rho\omega_{\nu-1}) \le \dots \le \mathrm{Re}(\rho\omega_2^{-1}) \\ &\le \mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_1^{-1}) \le \mathrm{Re}(\rho\omega_0)\end{aligned} \tag{$*$}$$

for all $\rho \in S_0$. It is easy to visualize these inequalities geometrically by simply plotting the complex numbers $\rho\omega_0, \rho\omega_1^{-1}, \rho\omega_1, \rho\omega_2^{-1}, \dots, \rho\omega_{\nu-1}, \rho\omega_\nu^{-1}$ as points on the circle of radius $r = |\rho|$. Similarly, $\omega_0^{-1}, \omega_1, \omega_1^{-1}, \omega_2, \dots, \omega_{\nu-1}^{-1}, \omega_\nu$ is a permutation of $\omega_0, \omega_1, \dots, \omega_{n-1}$, and

$$\begin{aligned}\mathrm{Re}(\rho\omega_\nu) &\le \mathrm{Re}(\rho\omega_{\nu-1}^{-1}) \le \dots \le \mathrm{Re}(\rho\omega_2) \\ &\le \mathrm{Re}(\rho\omega_1^{-1}) \le \mathrm{Re}(\rho\omega_1) \le \mathrm{Re}(\rho\omega_0^{-1})\end{aligned} \tag{$**$}$$

for all $\rho \in S_1$.

Next, consider the cases where ν is odd or even. First, assume ν is odd: $\nu = 2\mu - 1$. Then $n = 4\mu - 2$, $\mu = (n+2)/4$, and

$$\begin{aligned}\arg(\mathrm{i}\omega_{3\mu-2}) &= \frac{\pi}{2} + \frac{2\pi}{n}\left[3\left(\frac{n+2}{4}\right) - 2\right] = 2\pi - \frac{\pi}{n}, \\ \arg(\mathrm{i}\omega_{3\mu-1}) &= \frac{\pi}{2} + \frac{2\pi}{n}\left[3\left(\frac{n+2}{4}\right) - 1\right] = 2\pi + \frac{\pi}{n}.\end{aligned}$$

Thus, for any point $\rho \in S_0$, the point $\mathrm{i}\rho\omega_{3\mu-2}$ belongs to S_1, and by $(**)$

$$\begin{aligned}\mathrm{Re}(\mathrm{i}\rho\omega_{3\mu-2}\omega_\nu) &\le \mathrm{Re}(\mathrm{i}\rho\omega_{3\mu-2}\omega_{\nu-1}^{-1}) \\ &\le \dots \le \mathrm{Re}(\mathrm{i}\rho\omega_{3\mu-2}\omega_1) \le \mathrm{Re}(\mathrm{i}\rho\omega_{3\mu-2}\omega_0^{-1}).\end{aligned}$$

On the other hand, for any point $\rho \in S_1$, the point $\mathrm{i}\rho\omega_{3\mu-1}$ belongs to S_0, and by $(*)$

$$\begin{aligned}\operatorname{Re}(\mathrm{i}\rho\omega_{3\mu-1}\omega_\nu^{-1}) &\leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu-1}\omega_{\nu-1})\\ &\leq \cdots \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu-1}\omega_1^{-1}) \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu-1}\omega_0).\end{aligned}$$

This establishes the desired inequalities for the case ν odd.

Second, assume that ν is even: $\nu = 2\mu$. Then $n = 4\mu$, $\mu = n/4$, and

$$\arg(\mathrm{i}\omega_{3\mu}) = \frac{\pi}{2} + \frac{2\pi}{n} \cdot \frac{3n}{4} = 2\pi.$$

Therefore, for any $\rho \in S_0$, we have $\mathrm{i}\rho\omega_{3\mu} = \rho \in S_0$, and by $(*)$

$$\operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_\nu^{-1}) \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_{\nu-1}) \leq \cdots \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_1^{-1}) \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_0);$$

for any $\rho \in S_1$, $\mathrm{i}\rho\omega_{3\mu} = \rho \in S_1$, and by $(**)$

$$\operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_\nu) \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_{\nu-1}^{-1}) \leq \cdots \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_1) \leq \operatorname{Re}(\mathrm{i}\rho\omega_{3\mu}\omega_0^{-1}).$$

This completes the proof of the lemma. □

Let $\omega_0^0, \omega_1^0, \ldots, \omega_{n-1}^0$ and $\omega_0^1, \omega_1^1, \ldots, \omega_{n-1}^1$ be the permutations of $\omega_0, \omega_1, \ldots, \omega_{n-1}$ determined in the last lemma. In this section we will select constants $C_0, C_1, M_0, M_1, r_0, \gamma_0, R_0$ that can be used concurrently for the translated sectors T_0 and T_1. Indeed, for the first two constants we choose $C_0 \geq 0$ such that $|a_\alpha(t)| \leq C_0$ for $0 \leq t \leq 1$ and for $\alpha = 0, 1, \ldots, n-2$, and in terms of the sectors $T_0 = \{\rho - \tau_0 \mid \rho \in S_0\}$ and $T_1 = \{\rho - \tau_1 \mid \rho \in S_1\}$, we choose $C_1 > 0$ such that

$$\left|\mathrm{e}^{\mathrm{i}\tau_0(\omega_\mathrm{i}^0 - \omega_j^0)(t-s)}\right| \leq C_1 \quad \text{and} \quad \left|\mathrm{e}^{\mathrm{i}\tau_1(\omega_\mathrm{i}^1 - \omega_j^1)(t-s)}\right| \leq C_1$$

for all $t, s \in [0,1]$ and for $i, j = 0, 1, \ldots, n-1$. At this point we fix our attention upon the sector T_0, returning later in the section to a discussion of the sector T_1.

Fix any integer k with $0 \leq k \leq n-1$, and let κ be the integer satisfying $0 \leq \kappa \leq n-1$ and $\omega_\kappa^0 = \omega_k$. Let us reconsider the Birkhoff approximate solution

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j} = \mathrm{e}^{\mathrm{i}\rho\omega_\kappa^0 t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad 0 \leq t \leq 1,$$

on the sector T_0. Our goal in the first part of this section is to construct an actual solution $v_{0k}(t,\rho)$ of the differential equation (2.1) for $\rho \in T_0$ with $|\rho|$ sufficiently large, with $v_{0k}(t,\rho)$ behaving asymptotically like $z_k(t,\rho)$ on the sector T_0 and with $v_{0k}(t,\rho)$ having nice regularity properties on T_0 relative to the t and ρ variables.

Applying Lemma 4.1, for any point $\rho \in T_0$ we have

$$\operatorname{Re}(\mathrm{i}(\rho + \tau_0)\omega_0^0) \leq \operatorname{Re}(\mathrm{i}(\rho + \tau_0)\omega_1^0) \leq \cdots \leq \operatorname{Re}(\mathrm{i}(\rho + \tau_0)\omega_{n-1}^0). \tag{4.1}$$

Now take any point $\rho \in T_0$. Then for $j = 0, 1, \dots, \kappa$ the estimates (4.1) give

$$\mathrm{Re}(\mathrm{i}\rho\omega_j^0) \le \mathrm{Re}(\mathrm{i}\rho\omega_j^0 + \mathrm{i}(\rho + \tau_0)(\omega_\kappa^0 - \omega_j^0)) = \mathrm{Re}(\mathrm{i}\tau_0(\omega_\kappa^0 - \omega_j^0) + \mathrm{i}\rho\omega_\kappa^0),$$

so for $0 \le s < t \le 1$ we obtain the estimates

$$\begin{aligned}\left|\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}\right| &= \mathrm{e}^{\mathrm{Re}(\mathrm{i}\rho\omega_j^0(t-s))} \le \mathrm{e}^{\mathrm{Re}(\mathrm{i}\tau_0(\omega_\kappa^0-\omega_j^0)(t-s)+\mathrm{i}\rho\omega_\kappa^0(t-s))} \\ &= \left|\mathrm{e}^{\mathrm{i}\tau_0(\omega_\kappa^0-\omega_j^0)(t-s)}\right|\left|\mathrm{e}^{\mathrm{i}\rho\omega_\kappa^0(t-s)}\right| \le C_1\left|\mathrm{e}^{\mathrm{i}\rho\omega_\kappa^0(t-s)}\right|,\end{aligned}$$

or

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}\right| \le C_1\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \tag{4.2}$$

for $0 \le s < t \le 1$, for $\rho \in T_0$, and for $j = 0, 1, \dots, \kappa$. Similarly, for $j = \kappa + 1, \dots, n-1$ the estimates (4.1) yield

$$\mathrm{Re}(-\mathrm{i}\rho\omega_j^0) \le \mathrm{Re}(-\mathrm{i}\rho\omega_j^0 + \mathrm{i}(\rho + \tau_0)(\omega_j^0 - \omega_\kappa^0)) = \mathrm{Re}(\mathrm{i}\tau_0(\omega_j^0 - \omega_\kappa^0) - \mathrm{i}\rho\omega_\kappa^0),$$

so for $0 \le t < s \le 1$

$$\begin{aligned}\left|\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}\right| &= \mathrm{e}^{\mathrm{Re}(-\mathrm{i}\rho\omega_j^0(s-t))} \le \mathrm{e}^{\mathrm{Re}(\mathrm{i}\tau_0(\omega_j^0-\omega_\kappa^0)(s-t)-\mathrm{i}\rho\omega_\kappa^0(s-t))} \\ &= \left|\mathrm{e}^{\mathrm{i}\tau_0(\omega_\kappa^0-\omega_j^0)(t-s)}\right|\left|\mathrm{e}^{\mathrm{i}\rho\omega_\kappa^0(t-s)}\right| \le C_1\left|\mathrm{e}^{\mathrm{i}\rho\omega_\kappa^0(t-s)}\right|,\end{aligned}$$

or

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}\right| \le C_1\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \tag{4.3}$$

for $0 \le t < s \le 1$, for $\rho \in T_0$, and for $j = \kappa + 1, \dots, n-1$.

Let K_{01} and K_{02} be the functions defined by

$$K_{01}(t, s, \rho) := -\sum_{j=0}^{\kappa}(\mathrm{i}\omega_j^0)\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}, \quad K_{02}(t, s, \rho) := \sum_{j=\kappa+1}^{n-1}(\mathrm{i}\omega_j^0)\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}$$

for $t, s \in [0, 1]$ and for $\rho \in \mathbb{C}$; let k_0 be the function defined by

$$\begin{aligned} k_0(t, s, \rho) &:= \frac{1}{n\rho^{n-1}}\, K_{01}(t, s, \rho), \qquad 0 \le s < t \le 1, \\ k_0(t, s, \rho) &:= \frac{1}{n\rho^{n-1}}\, K_{02}(t, s, \rho), \qquad 0 \le t < s \le 1, \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$; and let $\mathcal{K}_{0\rho}$ be the integral operator on $L^2[0, 1]$ defined by

$$\mathcal{K}_{0\rho}u(t) := \int_0^1 k_0(t, s, \rho)u(s)\, ds, \qquad 0 \le t \le 1, \quad u \in L^2[0, 1],$$

for each $\rho \neq 0$ in $\mathbb{C}$. From our earlier work [34, pp. 103–105], we know that if $u \in L^2[0, 1]$ and $v = \mathcal{K}_{0\rho}u$, then $v \in H^n[0, 1]$ and $(\rho^n I - \tau)v = u$. Also, the derivatives of v are given by

$$\begin{aligned} v^{(\alpha)}(t) &= \int_0^1 \frac{\partial^\alpha k_0}{\partial t^\alpha}(t,s,\rho)u(s)\,ds \\ &= -\frac{1}{n\rho^{n-1}}\sum_{j=0}^{\kappa}(\mathrm{i}\omega_j^0)^{\alpha+1}\rho^\alpha\int_0^t \mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}u(s)\,ds \\ &\quad + \frac{1}{n\rho^{n-1}}\sum_{j=\kappa+1}^{n-1}(\mathrm{i}\omega_j^0)^{\alpha+1}\rho^\alpha\int_t^1 \mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}u(s)\,ds \end{aligned} \tag{4.4}$$

for $0 \le t \le 1$ and for $\alpha = 0, 1, \ldots, n-1$. We will use the integral operator $\mathcal{K}_{0\rho}$ extensively in deriving our asymptotic expansions, with the estimates (4.2) and (4.3) producing bounds for the various operators relative to the sector T_0.

Next, we begin the construction of actual solutions of the differential equation (2.1), looking for solutions of the form $u(t,\rho) = z_k(t,\rho) + \phi(t,\rho)$ where $z_k(t,\rho)$ is the Birkhoff approximate solution. From equation (2.11) we know that

$$(\rho^n I - \ell)z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho) \tag{4.5}$$

for $0 \le t \le 1$ and $\rho \neq 0$ in $\mathbb{C}$, where the residual function $\eta_k(t,\rho)$ is given by

$$\eta_k(t,\rho) = -\frac{\rho^n}{\mathrm{i}^n}\sum_{s=m+1}^{n+m-1}\left[\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t)\right]\rho^{-s}$$

for $0 \le t \le 1$ and $\rho \neq 0$ in $\mathbb{C}$. The function $\eta_k(t,\rho)$ depends on the fixed integers m and k. Choose a constant $M_0 \ge 0$ such that

$$|\eta_k(t,\rho)| \le M_0|\rho|^{-(m+1-n)} \tag{4.6}$$

for $0 \le t \le 1$ and for $\rho \in \mathbb{C}$ with $|\rho| \ge 1$ (recall that $m > n$). Without loss of generality we can assume that the constant M_0 is independent of the integer k.

Fix any point $\rho \neq 0$ in $\mathbb{C}$. As we determine solutions to (2.1) and to other equivalent equations, the various functions u, ϕ, ψ and constants $c_0, c_1, \ldots, c_{n-1}$ that appear will all depend upon the fixed integer k and the fixed parameter ρ. For the time being we will surpress this dependence on k and ρ in our notation. The only exceptions are in the Birkhoff approximate solution $z_k(\,\cdot\,,\rho)$ and the associated residual function $\eta_k(\,\cdot\,,\rho)$, where we continue to display both the k and ρ dependence, and in the integral operator $\mathcal{K}_{0\rho}$, where we continue to show the ρ dependence but surpress the k dependence.

Suppose $u \in H^n[0,1]$ is a solution of the differential equation (2.1), and let $\phi \in H^n[0,1]$ be the function defined by the equation $u(t) := z_k(t,\rho) + \phi(t)$ for $0 \le t \le 1$. Then by (4.5) we have

$$0 = (\rho^n I - \ell)u(t) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho) + (\rho^n I - \tau)\phi(t) - \sigma\phi(t)$$

or

$$(\rho^n I - \tau)\phi(t) = \sigma\phi(t) - \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)$$

for $0 \le t \le 1$. Set $v(t) := \mathcal{K}_{0\rho}\big(\sigma\phi(t) - \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big)$ for $0 \le t \le 1$. Then v belongs to $H^n[0,1]$ and $(\rho^n I - \tau)v(t) = \sigma\phi(t) - \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)$, and hence, $(\rho^n I - \tau)\big(\phi(t) - v(t)\big) = 0$ for $0 \le t \le 1$. It follows that there exist complex constants $c_0, c_1, \ldots, c_{n-1}$ such that

$$\phi(t) - v(t) = c_0\mathrm{e}^{\mathrm{i}\rho\omega_0 t} + c_1\mathrm{e}^{\mathrm{i}\rho\omega_1 t} + \cdots + c_{n-1}\mathrm{e}^{\mathrm{i}\rho\omega_{n-1} t},$$

or

$$\phi(t) = \mathcal{K}_{0\rho}\sigma\phi(t) - \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big) + \sum_{j=0}^{n-1} c_j\mathrm{e}^{\mathrm{i}\rho\omega_j t} \tag{4.7}$$

for $0 \le t \le 1$. Equation (4.7) is an integro-differential equation for the function ϕ.

Conversely, suppose $\phi \in H^n[0,1]$ is a solution of (4.7) for some constants $c_0, c_1, \ldots, c_{n-1}$. We assert that the function $u(t) := z_k(t,\rho) + \phi(t)$ is a solution of the differential equation (2.1). Clearly u belongs to $H^n[0,1]$, $(\rho^n I - \tau)\phi(t) = \sigma\phi(t) - \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)$ for $0 \le t \le 1$, and by (4.5) again

$$\begin{aligned}(\rho^n I - \ell)u(t) &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho) + (\rho^n I - \tau)\phi(t) - \sigma\phi(t)\\ &= \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho) + \sigma\phi(t) - \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho) - \sigma\phi(t) = 0\end{aligned}$$

for $0 \le t \le 1$. This establishes the assertion. We conclude that finding solutions u to the differential equation (2.1) is equivalent to finding solutions ϕ to the integro-differential equation (4.7) with arbitrary constants $c_0, c_1, \ldots, c_{n-1}$.

Next, consider the special case of equation (4.7) where all the constants c_j are set equal to zero:

$$\phi(t) = \mathcal{K}_{0\rho}\sigma\phi(t) - \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big) \tag{4.8}$$

for $0 \le t \le 1$. Suppose $\phi \in H^n[0,1]$ is a solution of equation (4.8). Clearly

$$\sigma\phi(t) = \sigma\mathcal{K}_{0\rho}\sigma\phi(t) - \sigma\mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big)$$

for $0 \le t \le 1$. Let ψ be the function defined by the equation

$$\sigma\phi(t) := \mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t), \qquad 0 \le t \le 1.$$

Clearly $\psi \in H^2[0,1]$ and $\psi \in C[0,1]$, and

$$\psi(t) = \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)\big) - \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big)$$

for $0 \le t \le 1$. Writing out the last equation in detail, we get

$$\begin{aligned}\psi(t) = \frac{1}{n\rho^{n-1}} &\int_0^t \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds\\ &+ \frac{1}{n\rho^{n-1}}\int_t^1 \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds\end{aligned} \tag{4.9}$$

for $0 \le t \le 1$. Equation (4.9) is an integral equation for the function ψ.

Conversely, suppose $\psi \in C[0,1]$ is a solution of the integral equation (4.9). Let ϕ and u be the functions in $H^n[0,1]$ defined by

$$\begin{aligned}
&\phi(t) := \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\big), \\
&u(t) := z_k(t,\rho) + \phi(t) = z_k(t,\rho) + \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\big)
\end{aligned}$$

for $0 \le t \le 1$.

First, we assert that $\sigma\phi(t) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)$ for $0 \le t \le 1$. Indeed, from the definition of ϕ we have

$$\begin{aligned}
\phi(t) = {} & \frac{1}{n\rho^{n-1}} \int_0^t K_{01}(t,s,\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k s}[\psi(s) - \eta_k(s,\rho)]\,ds \\
& + \frac{1}{n\rho^{n-1}} \int_t^1 K_{02}(t,s,\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k s}[\psi(s) - \eta_k(s,\rho)]\,ds
\end{aligned}$$

for $0 \le t \le 1$, and hence, by (4.9)

$$\begin{aligned}
\sigma\phi(t) = {} & \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds \\
& + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds \\
= {} & \mathrm{e}^{\mathrm{i}\rho\omega_k t}\,\psi(t)
\end{aligned}$$

for $0 \le t \le 1$.

Second, again from the definition of ϕ we have

$$\phi(t) = \mathcal{K}_{0\rho}\sigma\phi(t) - \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big)$$

for $0 \le t \le 1$, and hence, ϕ is a solution of (4.8). By our earlier work it follows that the function u is a solution of the differential equation (2.1). Summarizing, if $\psi \in C[0,1]$ is a solution of the integral equation (4.9), then the function

$$\phi(t) = \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\big), \qquad 0 \le t \le 1, \tag{4.10}$$

belongs to the space $H^n[0,1]$ and is a solution of the integro-differential equation (4.8), $\sigma\phi(t) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)$ for $0 \le t \le 1$, and the function

$$\begin{aligned}
u(t) &= z_k(t,\rho) + \phi(t) \\
&= z_k(t,\rho) + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} K_{01}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds \\
&\quad + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} K_{02}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds,
\end{aligned} \tag{4.11}$$

$0 \le t \le 1$, belongs to $H^n[0,1]$ and is a solution of the differential equation (2.1). Moreover, the derivatives of u are given by

$$\begin{aligned} u^{(\alpha)}(t) = z_k^{(\alpha)}(t,\rho) &+ \frac{e^{i\rho\omega_k t}}{n\rho^{n-1}} \int_0^t e^{-i\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds \\ &+ \frac{e^{i\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 e^{-i\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds \end{aligned} \tag{4.12}$$

for $0 \le t \le 1$ and for $\alpha = 0, 1, \ldots, n-1$. We conclude that finding solutions ϕ to equation (4.8) is equivalent to finding solutions ψ to equation (4.9). In view of this fact, the emphasis now shifts to solving the integral equation (4.9) for the function ψ. At this point the sector T_0 comes into play. We will show that equation (4.9) is uniquely solvable for all $\rho \in T_0$ with $|\rho|$ sufficiently large.

For fixed $\rho \ne 0$ in $\mathbb{C}$, let $\mathcal{A}_{0\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{A}_{0\rho}u(t) := \int_0^1 g_0(t,s,\rho)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in C[0,1],$$

where the kernel is defined by

$$g_0(t,s,\rho) := \frac{1}{n}\sum_{\alpha=0}^{n-2} a_\alpha(t)\rho^{-(n-2)} e^{-i\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho), \quad 0 \le s < t \le 1,$$
$$g_0(t,s,\rho) := \frac{1}{n}\sum_{\alpha=0}^{n-2} a_\alpha(t)\rho^{-(n-2)} e^{-i\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho), \quad 0 \le t < s \le 1.$$

Then the integral equation (4.9) can be written in the operator form

$$\big[I - \frac{1}{\rho}\mathcal{A}_{0\rho}\big]\psi(t) = -\frac{1}{\rho}\mathcal{A}_{0\rho}\,\eta_k(t,\rho), \qquad 0 \le t \le 1, \tag{4.13}$$

in the setting of the Banach space $C[0,1]$.

Lemma 4.2. *For any point* $\rho \in T_0$ *with* $|\rho| \ge 1$*,*

$$\left|\frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)\right| \le C_1 n|\rho|^\alpha \left|e^{i\rho\omega_k(t-s)}\right|$$

for $0 \le s < t \le 1$ *and for* $\alpha = 0, 1, 2, \ldots$*;*

$$\left|\frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)\right| \le C_1 n|\rho|^\alpha \left|e^{i\rho\omega_k(t-s)}\right|$$

for $0 \le t < s \le 1$ *and for* $\alpha = 0, 1, 2, \ldots$*; and* $|g_0(t,s,\rho)| \le C_0C_1n := M_1$ *for* $t \ne s$ *in* $[0,1]$.

Proof. Take any point $\rho \in T_0$ with $|\rho| \geq 1$. Then from (4.2) we have

$$\left|\frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)\right| = \left|\sum_{j=0}^{\kappa}(\mathrm{i}\omega_j^0)^{\alpha+1}\rho^\alpha \mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}\right| \leq n|\rho|^\alpha C_1\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \qquad (*)$$

for $0 \leq s < t \leq 1$ and for $\alpha = 0, 1, 2, \ldots$. Similarly, by (4.3)

$$\left|\frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)\right| = \left|\sum_{j=\kappa+1}^{n-1}(\mathrm{i}\omega_j^0)^{\alpha+1}\rho^\alpha \mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}\right| \leq n|\rho|^\alpha C_1\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \qquad (**)$$

for $0 \leq t < s \leq 1$ and for $\alpha = 0, 1, 2, \ldots$. The estimate for $|g_0(t,s,\rho)|$ follows immediately from $(*)$ and $(**)$. □

By the last lemma the norm of the operator $\mathcal{A}_{0\rho}$ satisfies the condition $\|\mathcal{A}_{0\rho}\| \leq M_1$ for all $\rho \in T_0$ with $|\rho| \geq 1$. Set $r_0 := \max\{1, M_1\} \geq 1$. Take any point $\rho \in T_0$ with $|\rho| > r_0$. Then $|\rho| > 1$, $|\rho| > M_1$, and $\|(1/\rho)\mathcal{A}_{0\rho}\| < 1$, and hence, in the operator space $\mathcal{B}(C[0,1])$ the operator $I - (1/\rho)\mathcal{A}_{0\rho}$ is invertible with its inverse given by the Neumann expansion

$$\left[I - \frac{1}{\rho}\mathcal{A}_{0\rho}\right]^{-1} = \sum_{j=0}^{\infty}\frac{1}{\rho^j}\mathcal{A}_{0\rho}^j\,.$$

It follows that the integral equation (4.9), or the equivalent operator equation (4.13), has a unique solution $\psi_{0k}(\,\cdot\,,\rho)$ in $C[0,1]$ given by

$$\psi_{0k}(t,\rho) = -\frac{1}{\rho}\left[I - \frac{1}{\rho}\mathcal{A}_{0\rho}\right]^{-1}\mathcal{A}_{0\rho}\,\eta_k(t,\rho) = -\sum_{j=0}^{\infty}\frac{1}{\rho^{j+1}}\mathcal{A}_{0\rho}^{j+1}\eta_k(t,\rho) \qquad (4.14)$$

for $0 \leq t \leq 1$, where the series in (4.14) is converging in $C[0,1]$ under the supremum norm. In denoting the solution $\psi_{0k}(\,\cdot\,,\rho)$, we have now displayed the full dependence upon the integer k and the parameter ρ, with ρ restricted to belong to the sector T_0 with $|\rho| > r_0$.

Now use (4.11) to define the function $v_{0k}(\,\cdot\,,\rho) \in H^n[0,1]$ by

$$\begin{aligned} v_{0k}(t,\rho) := z_k(t,\rho) &+ \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}}\int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}K_{01}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \\ &+ \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}}\int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}K_{02}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \end{aligned} \qquad (4.15)$$

for $0 \leq t \leq 1$ and for $\rho \in T_0$ with $|\rho| > r_0$. By our earlier work the function $v_{0k}(\,\cdot\,,\rho)$ is a solution of the differential equation (2.1), and by (4.12) the derivatives of $v_{0k}(\,\cdot\,,\rho)$ are given by

$$\begin{aligned} v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) &\\ &+ \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \\ &+ \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \end{aligned} \tag{4.16}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$.

For $\rho \neq 0$ in $\mathbb{C}$ and for $\alpha = 0, 1, \ldots, n-1$ let

$$\begin{aligned} h_{0\alpha}(t,s,\rho) &:= \frac{1}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho) \quad \text{for } 0 \le s < t \le 1, \\ h_{0\alpha}(t,s,\rho) &:= \frac{1}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho) \quad \text{for } 0 \le t < s \le 1, \end{aligned}$$

and let $\mathcal{B}_{0\alpha\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{B}_{0\alpha\rho}u(t) := \int_0^1 h_{0\alpha}(t,s,\rho)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in C[0,1].$$

From Lemma 4.2 we have

$$|h_{0\alpha}(t,s,\rho)| \le \frac{1}{n|\rho|^{n-1}}\,|\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}| \cdot C_1 n|\rho|^\alpha |\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}| = C_1|\rho|^{-(n-1-\alpha)}$$

for $t \neq s$ in $[0,1]$, for $\rho \in T_0$ with $|\rho| \ge 1$, and for $\alpha = 0, 1, \ldots, n-1$, and hence, the norm of the operator $\mathcal{B}_{0\alpha\rho}$ satisfies the estimate

$$\|\mathcal{B}_{0\alpha\rho}\| \le C_1|\rho|^{-(n-1-\alpha)}$$

for $\rho \in T_0$ with $|\rho| \ge 1$ and for $\alpha = 0, 1, \ldots, n-1$. In terms of $\mathcal{B}_{0\alpha\rho}$ equation (4.16) can then be written as

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}\,\mathcal{B}_{0\alpha\rho}[\psi_{0k}(t,\rho) - \eta_k(t,\rho)] \tag{4.17}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$.

Note that

$$\begin{aligned} \mathcal{B}_{0\alpha\rho}u(t) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k t} \frac{\partial^\alpha}{\partial t^\alpha} \left\{ \frac{1}{n\rho^{n-1}} \int_0^t K_{01}(t,s,\rho)[\mathrm{e}^{\mathrm{i}\rho\omega_k s}u(s)]\,ds \right. \\ &\qquad \left. + \frac{1}{n\rho^{n-1}} \int_t^1 K_{02}(t,s,\rho)[\mathrm{e}^{\mathrm{i}\rho\omega_k s}u(s)]\,ds \right\} \\ &= \mathrm{e}^{-\mathrm{i}\rho\omega_k t} \frac{\partial^\alpha}{\partial t^\alpha}\,\mathcal{K}_{0\rho}[\mathrm{e}^{\mathrm{i}\rho\omega_k t}u(t)], \end{aligned}$$

which shows that the operator $\mathcal{B}_{0\alpha\rho}$ maps $C[0,1]$ into $H^{n-\alpha}[0,1]$, and is continuous from the sup norm-structure to the $H^{n-\alpha}$-structure. Combining this fact with (4.17) and (4.14), we conclude that

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) - \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{\infty} \frac{1}{\rho^j}\, \mathcal{B}_{0\alpha\rho}\, \mathcal{A}_{0\rho}^j\, \eta_k(t,\rho) \tag{4.18}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$, where the series converges in the space $H^{n-\alpha}[0,1]$ under the norm $|\ \ |_{H^{n-\alpha}}$.

For $\rho \in T_0$ with $|\rho| > r_0$ and for $\alpha = 0, 1, \ldots, n-1$, set

$$E_{0k\alpha}(t,\rho) := \rho^{m-\alpha}\, \mathcal{B}_{0\alpha\rho}[\psi_{0k}(t,\rho) - \eta_k(t,\rho)], \qquad 0 \le t \le 1.$$

The function $E_{0k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$, and (4.17) can be rewritten as

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\alpha}(t,\rho)\rho^{-m+\alpha} \tag{4.19}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$. To estimate the growth rate of the function $E_{0k\alpha}(\,\cdot\,,\rho)$, consider any point $\rho \in T_0$ with $|\rho| > 2r_0$. Then

$$\Big\| \frac{1}{\rho}\, \mathcal{A}_{0\rho} \Big\| \le \frac{M_1}{|\rho|} < \frac{1}{2}\,,$$

and from (4.13) and (4.6) we have

$$\|\psi_{0k}(\,\cdot\,,\rho)\|_\infty \le \frac{1}{2}\, \|\psi_{0k}(\,\cdot\,,\rho)\|_\infty + \frac{1}{2}\, M_0 |\rho|^{-(m+1-n)},$$

or

$$\|\psi_{0k}(\,\cdot\,,\rho)\|_\infty \le M_0 |\rho|^{-(m+1-n)} \tag{4.20}$$

for $\rho \in T_0$ with $|\rho| > 2r_0$. Cf. equation (4.6). It follows that

$$\|E_{0k\alpha}(\,\cdot\,,\rho)\|_\infty \le |\rho|^{m-\alpha} \cdot C_1 |\rho|^{-(n-1-\alpha)} \cdot 2M_0 |\rho|^{-(m+1-n)} = 2C_1 M_0,$$

or

$$\|E_{0k\alpha}(\,\cdot\,,\rho)\|_\infty \le 2C_1 M_0 := \gamma_0 \qquad \text{(a constant)} \tag{4.21}$$

for $\rho \in T_0$ with $|\rho| > 2r_0$ and for $\alpha = 0, 1, \ldots, n-1$. In (4.19) and (4.21) we have established our asymptotic formulas for the solution $v_{0k}(\,\cdot\,,\rho)$ and its derivatives relative to the sector T_0.

Next, we apply the above construction to create a basis for the solution space of the differential equation (2.1). For $k = 0, 1, \ldots, n-1$ and $\rho \in T_0$ with $|\rho| > r_0$, let

$$v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots, v_{0\,n-1}(\,\cdot\,,\rho) \in H^n[0,1]$$

be the solutions of the differential equation (2.1) constructed above. We assert that these solutions are linearly independent for $|\rho|$ sufficiently large. Indeed, from Chapter 2 the derivatives of the Birkhoff approximate solutions are given by

$$z_k^{(p)}(t,\rho) = \rho^p \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{\ell=0}^{p} \sum_{j=0}^{m-1} \alpha_{kp\ell}\, z_{kj}^{(\ell)}(t) \rho^{-\ell-j} := \rho^p \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{s=0}^{m+p-1} f_{kps}(t) \rho^{-s}$$

for $k, p = 0, 1, \ldots, n-1$, where $f_{kp0}(t) = \alpha_{kp0} = (\mathrm{i}\omega_k)^p$. Combining this result with equation (4.19), we have

$$v_{0k}^{(\alpha)}(t,\rho) = \rho^\alpha \mathrm{e}^{\mathrm{i}\rho\omega_k t}\Big[(\mathrm{i}\omega_k)^\alpha + \sum_{s=1}^{m+\alpha-1} f_{k\alpha s}(t)\rho^{-s} + E_{0k\alpha}(t,\rho)\rho^{-m}\Big]$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $k, \alpha = 0, 1, \ldots, n-1$, where by the bound given in equation (4.21)

$$(\mathrm{i}\omega_k)^\alpha + \sum_{s=1}^{m+\alpha-1} f_{k\alpha s}(t)\rho^{-s} + E_{0k\alpha}(t,\rho)\rho^{-m} \to (\mathrm{i}\omega_k)^\alpha$$

uniformly on $[0,1] \times T_0$ as $|\rho| \to \infty$ for $k, \alpha = 0, 1, \ldots, n-1$. Since the Vandermonde matrix $\big((\mathrm{i}\omega_k)^\alpha\big)$ is nonsingular, using the continuity of the determinant, we can choose a constant $R_0 \ge 2r_0$ such that the matrix

$$V(t,\rho) := \Big[(\mathrm{i}\omega_k)^\alpha + \sum_{s=1}^{m+\alpha-1} f_{k\alpha s}(t)\rho^{-s} + E_{0k\alpha}(t,\rho)\rho^{-m}\Big]$$

is nonsingular for $0 \le t \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. It follows that the Wronskian

$$\begin{aligned} W(v_{00}, v_{01}, \ldots, v_{0\,n-1})(t,\rho) &= \det\big[v_{0k}^{(\alpha)}(t,\rho)\big] \\ &= \rho^{1+2+\cdots+(n-1)} \mathrm{e}^{\mathrm{i}\rho(\omega_0+\omega_1+\cdots+\omega_{n-1})t} \det V(t,\rho) \\ &= \rho^{n(n-1)/2} \det V(t,\rho) \end{aligned}$$

is nonzero for $0 \le t \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. This proves that the solutions $v_{0k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, are indeed linearly independent for $\rho \in T_0$ with $|\rho| > R_0$.

To develop the regularity properties of the functions $\psi_{0k}(t,\rho)$ and $v_{0k}(t,\rho)$, we will use Lemma 12.1 and Lemma 12.2 of the Appendix (Chapter 12). Again fix the integer k with $0 \le k \le n-1$. Set

$$G_0 := \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\},$$

an open set in the ρ plane. We know that

$$\eta_k(t,\rho) = -\frac{\rho^n}{\mathrm{i}^n} \sum_{s=m+1}^{n+m-1} \Bigg[\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t)\Bigg]\rho^{-s}$$

for $0 \le t \le 1$ and $\rho \neq 0$ in $\mathbb{C}$, and hence, the function $\eta_k(t,\rho)$ is continuous on $[0,1] \times G_0$, and $\frac{\partial}{\partial\rho}\eta_k(t,\rho)$ exists and is continuous on $[0,1] \times G_0$. For $\rho \neq 0$ in $\mathbb{C}$ define

$$\phi_{01}(t,s,\rho) := \frac{1}{n}\sum_{\alpha=0}^{n-2} a_\alpha(t)\rho^{-(n-2)}\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)\,\eta_k(s,\rho), \qquad 0 \le s < t \le 1,$$

$$\phi_{02}(t,s,\rho) := \frac{1}{n}\sum_{\alpha=0}^{n-2} a_\alpha(t)\rho^{-(n-2)}\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)\,\eta_k(s,\rho), \qquad 0 \le t < s \le 1,$$

and set

$$\begin{aligned}\phi_0(t,s,\rho) &:= \phi_{01}(t,s,\rho) \qquad \text{for } 0 \le s < t \le 1, \quad \rho \in G_0,\\ \phi_0(t,s,\rho) &:= \phi_{02}(t,s,\rho) \qquad \text{for } 0 \le t < s \le 1, \quad \rho \in G_0.\end{aligned}$$

Clearly $\phi_0(t,s,\rho) = g_0(t,s,\rho)\eta_k(s,\rho)$ for $t \ne s$ in $[0,1]$ and $\rho \in G_0$, and

$$\mathcal{A}_{0\rho}\eta_k(t,\rho) = \int_0^1 \phi_0(t,s,\rho)\,ds, \qquad 0 \le t \le 1, \quad \rho \in G_0.$$

From Lemma 12.1 we conclude that $\mathcal{A}_{0\rho}\eta_k(t,\rho)$ is continuous on $[0,1]\times G_0$, and $\frac{\partial}{\partial\rho}\,[\mathcal{A}_{0\rho}\eta_k(t,\rho)]$ exists and is continuous on $[0,1]\times G_0$. Proceeding by induction, for $j = 0,1,2,\ldots$ the function $\mathcal{A}_{0\rho}^j\eta_k(t,\rho)$ is continuous on $[0,1]\times G_0$, and the derivative $\frac{\partial}{\partial\rho}\,[\mathcal{A}_{0\rho}^j\eta_k(t,\rho)]$ exists and is continuous on $[0,1]\times G_0$.

Next, for each $\rho \in G_0$ let us examine the unique solution $\psi_{0k}(t,\rho)$ of the integral equation (4.9), where by equation (4.14)

$$\psi_{0k}(t,\rho) = -\sum_{j=1}^{\infty}\frac{1}{\rho^j}\,\mathcal{A}_{0\rho}^j\eta_k(t,\rho)$$

for $0 \le t \le 1$, with the series converging in $C[0,1]$ for each $\rho \in G_0$. For the partial sums of the series,

$$S_{0N}(t,\rho) = -\sum_{j=1}^{N}\frac{1}{\rho^j}\,\mathcal{A}_{0\rho}^j\eta_k(t,\rho),$$

$N = 1,2,\ldots$, it follows that $S_{0N}(t,\rho)$ is continuous on $[0,1]\times G_0$, and $\frac{\partial}{\partial\rho}\,[S_{0N}(t,\rho)]$ exists and is continuous on $[0,1]\times G_0$.

Since $\|\mathcal{A}_{0\rho}\| \le M_1$ and $|\rho| > R_0 \ge 2M_1$ for $\rho \in G_0$, we have

$$\begin{aligned}\|\psi_{0k}(\,\cdot\,,\rho) - S_{0N}(\,\cdot\,,\rho)\|_\infty &= \Big\| \sum_{j=N+1}^{\infty} \frac{1}{\rho^j}\, \mathcal{A}_{0\rho}^j \eta_k(\,\cdot\,,\rho) \Big\|_\infty \\ &\le \sum_{j=N+1}^{\infty} \frac{1}{|\rho|^j}\, (M_1)^j \cdot M_0 |\rho|^{-(m+1-n)} \\ &= M_0 (M_1)^{-(m+1-n)} \sum_{j=N+1}^{\infty} \frac{M_1^{j+m+1-n}}{|\rho|^{j+m+1-n}} \\ &\le M_0 (M_1)^{-(m+1-n)} \sum_{j=N+1}^{\infty} \frac{1}{2^{j+m+1-n}} \\ &= \frac{2^{-(m+1-n)} M_0 (M_1)^{-(m+1-n)}}{2^N}\end{aligned}$$

for each $\rho \in G_0$, and hence,

$$\|\psi_{0k}(\,\cdot\,,\rho) - S_{0N}(\,\cdot\,,\rho)\|_\infty \le \frac{M_0 (2M_1)^{-(m+1-n)}}{2^N} \tag{4.22}$$

for $\rho \in G_0$ and for $N = 1, 2, \ldots$. This result shows that the functions $S_{0N}(t,\rho)$ converge uniformly on $[0,1] \times G_0$ to $\psi_{0k}(t,\rho)$ as $N \to \infty$. From Lemma 12.2 we conclude that $\psi_{0k}(t,\rho)$ is continuous on $[0,1] \times G_0$, and the derivative $\frac{\partial}{\partial\rho}\,[\psi_{0k}(t,\rho)]$ exists and is continuous on $[0,1] \times G_0$. It is immediate that the function

$$\psi_{0k}(t,\rho) - \eta_k(t,\rho) = -\sum_{j=0}^{\infty} \frac{1}{\rho^j}\, \mathcal{A}_{0\rho}^j \eta_k(t,\rho)$$

is continuous on $[0,1] \times G_0$, and the derivative $\frac{\partial}{\partial\rho}\,[\psi_{0k}(t,\rho) - \eta_k(t,\rho)]$ exists and is continuous on $[0,1] \times G_0$.

Finally, consider the derivative functions $v_{0k}^{(\alpha)}(t,\rho)$ given by equation (4.16) for $0 \le t \le 1$, for $\rho \in G_0$, and for $\alpha = 0, 1, \ldots, n-1$. Again by Lemma 12.1 we see that $v_{0k}^{(\alpha)}(t,\rho)$ is continuous on $[0,1] \times G_0$, and the derivative $\frac{\partial}{\partial\rho}\,\big[v_{0k}^{(\alpha)}(t,\rho)\big]$ exists and is continuous on $[0,1] \times G_0$.

We summarize these results for the sector T_0 in the following theorem.

Theorem 4.3. *Let n be even: $n = 2\nu$; let $T_0 = \{\rho - \tau_0 \mid \rho \in S_0\}$ be the translation of the sector S_0 selected in the last chapter; let m be the integer chosen in the last chapter with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \le p \le p_0$; and for each $\rho \neq 0$ in $\mathbb{C}$ let*

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t) \rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

be the Birkhoff approximate solutions of the differential equation (2.1) *that are determined by Theorem 2.1. Then there exists a constant $R_0 > 0$ such* *that for* $\rho \in T_0$ *with* $|\rho| > R_0$*, there exist* n *functions* $v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \dots,$ $v_{0\,n-1}(\,\cdot\,,\rho)$ *in* $H^n[0,1]$ *that are linearly independent solutions of the differential equation* (2.1)*, with*

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\alpha}(t,\rho)\rho^{-m+\alpha}$$

for $0 \le t \le 1$*, for* $\rho \in T_0$ *with* $|\rho| > R_0$*, and for* $k, \alpha = 0, 1, \dots, n-1$*. The function* $E_{0k\alpha}(\,\cdot\,,\rho)$ *belongs to* $H^{n-\alpha}[0,1]$ *with* $|E_{0k\alpha}(t,\rho)| \le \gamma_0$ *(a constant) for* $0 \le t \le 1$*, for* $\rho \in T_0$ *with* $|\rho| > R_0$*, and for* $k, \alpha = 0, 1, \dots, n-1$*. For the open set* $G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$ *and for* $k, \alpha = 0, 1, \dots, n-1$*, the function* $v_{0k}^{(\alpha)}(t,\rho)$ *is continuous on* $[0,1] \times G_0$*, and the derivative* $\frac{\partial}{\partial \rho}\left[v_{0k}^{(\alpha)}(t,\rho)\right]$ *exists and is continuous on* $[0,1] \times G_0$*.*

To construct solutions of the differential equation (2.1) on the sector T_1, we will follow the procedure used above for the sector T_0. Since the development is so similar, we simply sketch the results. In working on the sector T_1, we will utilize the permutation $\omega_0^1, \omega_1^1, \dots, \omega_{n-1}^1$ of $\omega_0, \omega_1, \dots, \omega_{n-1}$ determined by Lemma 4.1. Fix any integer k with $0 \le k \le n-1$, and let κ be the unique integer satisfying $0 \le \kappa \le n-1$ and $\omega_\kappa^1 = \omega_k$. Then as an application of Lemma 4.1 we obtain the estimates

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_j^1(t-s)}\right| \le C_1 \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \tag{4.23}$$

for $0 \le s < t \le 1$, for $\rho \in T_1$, and for $j = 0, 1, \dots, \kappa$; and

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_j^1(t-s)}\right| \le C_1 \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \tag{4.24}$$

for $0 \le t < s \le 1$, for $\rho \in T_1$, and for $j = \kappa+1, \dots, n-1$.

Let K_{11} and K_{12} be the functions defined by

$$K_{11}(t,s,\rho) := -\sum_{j=0}^{\kappa} (\mathrm{i}\omega_j^1)\mathrm{e}^{\mathrm{i}\rho\omega_j^1(t-s)}, \quad K_{12}(t,s,\rho) := \sum_{j=\kappa+1}^{n-1} (\mathrm{i}\omega_j^1)\mathrm{e}^{\mathrm{i}\rho\omega_j^1(t-s)}$$

for $t, s \in [0,1]$ and for $\rho \in \mathbb{C}$; let k_1 be the function defined by

$$\begin{aligned} k_1(t,s,\rho) &:= \frac{1}{n\rho^{n-1}} K_{11}(t,s,\rho), \qquad 0 \le s < t \le 1, \\ k_1(t,s,\rho) &:= \frac{1}{n\rho^{n-1}} K_{12}(t,s,\rho), \qquad 0 \le t < s \le 1, \end{aligned}$$

for $\rho \ne 0$ in $\mathbb{C}$; and let $\mathcal{K}_{1\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$\mathcal{K}_{1\rho} u(t) := \int_0^1 k_1(t,s,\rho) u(s)\, ds, \qquad 0 \le t \le 1, \quad u \in L^2[0,1],$$

for each $\rho \ne 0$ in $\mathbb{C}$. Again let $z_k(t,\rho)$ be the Birkhoff approximate solution, and let $\eta_k(t,\rho)$ be the associated residual function. These functions have been used extensively in our work on the sector T_0, and they will be used again here for our work on the sector T_1. They are independent of the sectors.

Fix any point $\rho \neq 0$ in $\mathbb{C}$. Let $u, \phi \in H^n[0,1]$ be functions related by the equation

$$u(t) = z_k(t,\rho) + \phi(t) \quad \text{for } 0 \le t \le 1.$$

Then it follows that u is a solution of the differential equation (2.1) if and only if there exist constants $c_0, c_1, \dots, c_{n-1}$ such that

$$\phi(t) = \mathcal{K}_{1\rho}\sigma\phi(t) - \mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big) + \sum_{j=0}^{n-1} c_j \mathrm{e}^{\mathrm{i}\rho\omega_j t} \quad \text{for } 0 \le t \le 1. \tag{4.25}$$

This is an integro-differential equation for the function ϕ.

Next, we consider the special case of (4.25) where all the c_j are set equal to zero:

$$\phi(t) = \mathcal{K}_{1\rho}\sigma\phi(t) - \mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big) \quad \text{for } 0 \le t \le 1. \tag{4.26}$$

Suppose $\phi \in H^n[0,1]$ is a solution of equation (4.26), and let $\psi \in H^2[0,1]$ be the function defined by the equation

$$\sigma\phi(t) := \mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t) \quad \text{for } 0 \le t \le 1.$$

Then ψ belongs to $C[0,1]$, and ψ is a solution of the equation

$$\psi(t) = \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)\big) - \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big),$$

which reduces to the integral equation

$$\begin{aligned}\psi(t) = {} & \frac{1}{n\rho^{n-1}} \int_0^t \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{11}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds \\ & + \frac{1}{n\rho^{n-1}} \int_t^1 \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{12}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds\end{aligned} \tag{4.27}$$

for $0 \le t \le 1$.

On the other hand, suppose $\psi \in C[0,1]$ is a solution to the integral equation (4.27). Let $\phi \in H^n[0,1]$ and $u \in H^n[0,1]$ be the functions defined by

$$\phi(t) := \mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\big), \tag{4.28}$$

$$u(t) := z_k(t,\rho) + \phi(t) = z_k(t,\rho) + \mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\big) \tag{4.29}$$

for $0 \le t \le 1$. Then $\sigma\phi(t) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\psi(t)$ for $0 \le t \le 1$, ϕ is a solution to equation (4.26), and u is a solution of the differential equation (2.1). Consequently, our attention now focuses on solving the integral equation (4.27) for the function ψ.

For fixed $\rho \neq 0$ in $\mathbb{C}$, let $\mathcal{A}_{1\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{A}_{1\rho} u(t) := \int_0^1 g_1(t,s,\rho) u(s)\, ds, \qquad 0 \le t \le 1, \quad u \in C[0,1],$$

where the kernel is defined by

$$g_1(t,s,\rho) := \frac{1}{n} \sum_{\alpha=0}^{n-2} a_\alpha(t) \rho^{-(n-2)} \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{11}}{\partial t^\alpha}(t,s,\rho), \quad 0 \le s < t \le 1,$$

$$g_1(t,s,\rho) := \frac{1}{n} \sum_{\alpha=0}^{n-2} a_\alpha(t) \rho^{-(n-2)} \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{12}}{\partial t^\alpha}(t,s,\rho), \quad 0 \le t < s \le 1.$$

The integral equation (4.27) can then be written in the operator form

$$\big[I - \frac{1}{\rho} \mathcal{A}_{1\rho}\big] \psi(t) = -\frac{1}{\rho} \mathcal{A}_{1\rho}\, \eta_k(t,\rho), \qquad 0 \le t \le 1, \tag{4.30}$$

in the setting of the Banach space $C[0,1]$. Up to this point the sector T_1 has not been a factor — the situation is about to change.

Indeed, for $\rho \in T_1$ with $|\rho| \ge 1$, the operator $\mathcal{A}_{1\rho}$ satisfies the bound $\|\mathcal{A}_{1\rho}\| \le M_1$, where $M_1 = C_0 C_1 n$ is the constant introduced previously. As earlier set $r_0 = \max\{1, M_1\} \ge 1$. Then for any $\rho \in T_1$ with $|\rho| > r_0$, $\|(1/\rho)\mathcal{A}_{1\rho}\| < 1$, the operator $I - (1/\rho)\mathcal{A}_{1\rho}$ is invertible with its inverse given by

$$\big[I - \frac{1}{\rho} \mathcal{A}_{1\rho}\big]^{-1} = \sum_{j=0}^{\infty} \frac{1}{\rho^j} \mathcal{A}_{1\rho}^j\,,$$

and the integral equation (4.27), or the equivalent operator equation (4.30), has a unique solution $\psi_{1k}(\,\cdot\,,\rho)$ in $C[0,1]$ given by

$$\psi_{1k}(t,\rho) = -\frac{1}{\rho} \big[I - \frac{1}{\rho} \mathcal{A}_{1\rho}\big]^{-1} \mathcal{A}_{1\rho}\, \eta_k(t,\rho) = -\sum_{j=0}^{\infty} \frac{1}{\rho^{j+1}} \mathcal{A}_{1\rho}^{j+1} \eta_k(t,\rho) \tag{4.31}$$

for $0 \le t \le 1$, where the series in (4.31) is converging in $C[0,1]$. Applying (4.29), the function $v_{1k}(\,\cdot\,,\rho) \in H^n[0,1]$ defined by

$$v_{1k}(t,\rho) := z_k(t,\rho) + \mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi_{1k}(t,\rho) - \eta_k(t,\rho)]\big) \tag{4.32}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > r_0$, is a solution of the differential equation (2.1). The derivatives of $v_{1k}(\,\cdot\,,\rho)$ are given by

$$\begin{aligned} v_{1k}^{(\alpha)}(t,\rho) &= z_k^{(\alpha)}(t,\rho) \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{11}}{\partial t^\alpha}(t,s,\rho)[\psi_{1k}(s,\rho) - \eta_k(s,\rho)]\, ds \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{12}}{\partial t^\alpha}(t,s,\rho)[\psi_{1k}(s,\rho) - \eta_k(s,\rho)]\, ds \end{aligned} \tag{4.33}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \dots, n-1$.

For $\rho \neq 0$ in $\mathbb{C}$ and for $\alpha = 0, 1, \dots, n-1$ let

$$h_{1\alpha}(t,s,\rho) := \frac{1}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{11}}{\partial t^\alpha}(t,s,\rho) \quad \text{for } 0 \le s < t \le 1,$$
$$h_{1\alpha}(t,s,\rho) := \frac{1}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{12}}{\partial t^\alpha}(t,s,\rho) \quad \text{for } 0 \le t < s \le 1,$$

and let $\mathcal{B}_{1\alpha\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{B}_{1\alpha\rho}u(t) := \int_0^1 h_{1\alpha}(t,s,\rho)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in C[0,1].$$

The operator $\mathcal{B}_{1\alpha\rho}$ satisfies the bound

$$\|\mathcal{B}_{1\alpha\rho}\| \le C_1|\rho|^{-(n-1-\alpha)}$$

for $\rho \in T_1$ with $|\rho| \ge 1$ and for $\alpha = 0, 1, \dots, n-1$, and equation (4.33) can be written as

$$v_{1k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}\,\mathcal{B}_{1\alpha\rho}[\psi_{1k}(t,\rho) - \eta_k(t,\rho)] \tag{4.34}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \dots, n-1$.

Note that the operator $\mathcal{B}_{1\alpha\rho}$ maps $C[0,1]$ into $H^{n-\alpha}[0,1]$, and is continuous from the sup norm-structure to the $H^{n-\alpha}$-structure. Combining this fact with (4.34) and (4.31), we conclude that

$$v_{1k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) - \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{\infty}\frac{1}{\rho^j}\,\mathcal{B}_{1\alpha\rho}\,\mathcal{A}_{1\rho}^j\,\eta_k(t,\rho) \tag{4.35}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \dots, n-1$, where the series converges in the space $H^{n-\alpha}[0,1]$ under the norm $|\ \ |_{H^{n-\alpha}}$.

For $\rho \in T_1$ with $|\rho| > r_0$ and for $\alpha = 0, 1, \dots, n-1$, set

$$E_{1k\alpha}(t,\rho) := \rho^{m-\alpha}\,\mathcal{B}_{1\alpha\rho}[\psi_{1k}(t,\rho) - \eta_k(t,\rho)], \qquad 0 \le t \le 1.$$

The function $E_{1k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$, and (4.34) can be rewritten as

$$v_{1k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}E_{1k\alpha}(t,\rho)\rho^{-m+\alpha} \tag{4.36}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \dots, n-1$. As in our previous work we obtain the estimates (see (4.6))

$$\|\psi_{1k}(\,\cdot\,,\rho)\|_\infty \le M_0|\rho|^{-(m+1-n)} \tag{4.37}$$

for $\rho \in T_1$ with $|\rho| > 2r_0$, and

$$\|E_{1k\alpha}(\,\cdot\,,\rho)\|_\infty \le 2C_1M_0 = \gamma_0 \tag{4.38}$$

for $\rho \in T_1$ with $|\rho| > 2r_0$ and for $\alpha = 0, 1, \ldots, n-1$. In (4.36) and (4.38) we have established our asymptotic formulas for the solution $v_{1k}(\,\cdot\,,\rho)$ and its derivatives relative to the sector T_1.

Next, for $k = 0, 1, \ldots, n-1$ and for $\rho \in T_1$ with $|\rho| > r_0$, let

$$v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \ldots, v_{1\,n-1}(\,\cdot\,,\rho) \in H^n[0,1]$$

be the solutions of the differential equation (2.1) constructed above. Applying the argument used earlier to determine the constant R_0, we can choose a constant $R_1 \geq 2r_0$ such that the Wronskian

$$W(v_{10}, v_{11}, \ldots, v_{1\,n-1})(t,\rho) = \det\bigl[v_{1k}^{(\alpha)}(t,\rho)\bigr]$$

is nonzero for $0 \leq t \leq 1$ and for $\rho \in T_1$ with $|\rho| > R_1$. Thus, the solutions $v_{1k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, of (2.1) are linearly independent for $\rho \in T_1$ with $|\rho| > R_1$. By choosing the maximum of R_0 and R_1, we can assume that $R_1 = R_0$ in the sequel.

The regularity properties of the functions $\psi_{1k}(t,\rho)$ and $v_{1k}(t,\rho)$ are identical to those for the functions $\psi_{0k}(t,\rho)$ and $v_{0k}(t,\rho)$; they are also based on Lemma 12.1 and Lemma 12.2 of the Appendix (Chapter 12). Fix the integer k with $0 \leq k \leq n-1$, and set

$$G_1 := \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\},$$

an open set in the ρ plane. The function $\psi_{1k}(t,\rho)$ is continuous on $[0,1] \times G_1$, and the derivative $\frac{\partial}{\partial\rho}\,[\psi_{1k}(t,\rho)]$ exists and is continuous on $[0,1] \times G_1$; for $\alpha = 0, 1, \ldots, n-1$ the derivative $v_{1k}^{(\alpha)}(t,\rho)$ is continuous on $[0,1] \times G_1$, and the derivative $\frac{\partial}{\partial\rho}\,\bigl[v_{1k}^{(\alpha)}(t,\rho)\bigr]$ exists and is continuous on $[0,1] \times G_1$.

These results for the sector T_1 are summarized below in a theorem.

Theorem 4.4. *Let n be even: $n = 2\nu$; let $T_1 = \{\rho - \tau_1 \mid \rho \in S_1\}$ be the translation of the sector S_1 selected in the last chapter; let m be the integer chosen in the last chapter with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p \leq p_0$; and for each $\rho \neq 0$ in $\mathbb{C}$ let*

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

be the Birkhoff approximate solutions of the differential equation (2.1) that are determined by Theorem 2.1. Then there exists a constant $R_0 > 0$ such that for $\rho \in T_1$ with $|\rho| > R_0$, there exist n functions $v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \ldots,$ $v_{1\,n-1}(\,\cdot\,,\rho)$ in $H^n[0,1]$ that are linearly independent solutions of the differential equation (2.1), with

$$v_{1k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{1k\alpha}(t,\rho)\rho^{-m+\alpha}$$

for $0 \le t \le 1$*, for* $\rho \in T_1$ *with* $|\rho| > R_0$*, and for* $k, \alpha = 0, 1, \ldots, n-1$*. The function* $E_{1k\alpha}(\,\cdot\,,\rho)$ *belongs to* $H^{n-\alpha}[0,1]$ *with* $|E_{1k\alpha}(t,\rho)| \le \gamma_0$ *(a constant) for* $0 \le t \le 1$*, for* $\rho \in T_1$ *with* $|\rho| > R_0$*, and for* $k, \alpha = 0, 1, \ldots, n-1$*. For the open set* $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$ *and for* $k, \alpha = 0, 1, \ldots, n-1$*, the function* $v_{1k}^{(\alpha)}(t,\rho)$ *is continuous on* $[0,1] \times G_1$*, and the derivative* $\frac{\partial}{\partial \rho}\left[v_{1k}^{(\alpha)}(t,\rho)\right]$ *exists and is continuous on* $[0,1] \times G_1$*.*

4.2 Expansions for *n* Odd

In the last part of this chapter, we establish the asymptotic expansion of solutions to the differential equation (2.1) for the case n odd. The development here is identical to the case n even, and hence, we simply indicate the main features of the theory. Assume that n is odd: $n = 2\nu - 1 \ge 3$. Relative to the sectors S_0, S_1 given by

$$\begin{aligned} S_0\colon &\ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{2n} \le \theta \le \frac{\pi}{2n}, \\ S_1\colon &\ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \pi - \frac{\pi}{2n} \le \theta \le \pi + \frac{\pi}{2n}, \end{aligned}$$

we have the following analogue of Lemma 4.1.

Lemma 4.5. *Let* n *be odd:* $n = 2\nu - 1$*. Then there exist permutations*

$$\omega_0^0, \omega_1^0, \ldots, \omega_{n-1}^0 \quad \textit{and} \quad \omega_0^1, \omega_1^1, \ldots, \omega_{n-1}^1$$

of the nth roots of unity $\omega_0, \omega_1, \ldots, \omega_{n-1}$ *such that*

$$\begin{aligned} \operatorname{Re}(\mathrm{i}\rho\omega_0^0) \le \operatorname{Re}(\mathrm{i}\rho\omega_1^0) \le \cdots \le \operatorname{Re}(\mathrm{i}\rho\omega_{n-1}^0) \quad &\textit{for all } \rho \in S_0, \\ \operatorname{Re}(\mathrm{i}\rho\omega_0^1) \le \operatorname{Re}(\mathrm{i}\rho\omega_1^1) \le \cdots \le \operatorname{Re}(\mathrm{i}\rho\omega_{n-1}^1) \quad &\textit{for all } \rho \in S_1. \end{aligned}$$

Proof. Let Σ_0 and Σ_1 be the sectors in the ρ plane defined by

$$\begin{aligned} \Sigma_0\colon &\text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } 0 \le \theta \le \frac{\pi}{n}\,, \\ \Sigma_1\colon &\text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} \le \theta \le 0\,. \end{aligned}$$

The two lists $\omega_0, \omega_1^{-1}, \omega_1, \omega_2^{-1}, \omega_2, \ldots, \omega_{\nu-1}^{-1}, \omega_{\nu-1}$ and $\omega_0^{-1}, \omega_1, \omega_1^{-1}, \omega_2, \omega_2^{-1}, \ldots, \omega_{\nu-1}, \omega_{\nu-1}^{-1}$ are permutations of $\omega_0, \omega_1, \ldots, \omega_{n-1}$ with

$$\begin{aligned} \operatorname{Re}(\rho\omega_{\nu-1}) \le \operatorname{Re}(\rho\omega_{\nu-1}^{-1}) \le \cdots \le \operatorname{Re}(\rho\omega_2) \le \operatorname{Re}(\rho\omega_2^{-1}) \\ \le \operatorname{Re}(\rho\omega_1) \le \operatorname{Re}(\rho\omega_1^{-1}) \le \operatorname{Re}(\rho\omega_0) \end{aligned} \tag{$*$}$$

for $\rho \in \Sigma_0$, and

$$\begin{aligned} \operatorname{Re}(\rho\omega_{\nu-1}^{-1}) \le \operatorname{Re}(\rho\omega_{\nu-1}) \le \cdots \le \operatorname{Re}(\rho\omega_2^{-1}) \le \operatorname{Re}(\rho\omega_2) \\ \le \operatorname{Re}(\rho\omega_1^{-1}) \le \operatorname{Re}(\rho\omega_1) \le \operatorname{Re}(\rho\omega_0^{-1}) \end{aligned} \tag{$**$}$$

for $\rho \in \varSigma_1$.

Assume ν is odd: $\nu = 2\mu - 1$. Then $\arg(\mathrm{i}\omega_{1-\mu}) = \pi/(2n)$ and $\arg(\mathrm{i}\omega_{2-3\mu}) = -\pi - \pi/(2n)$. Thus, for any point $\rho \in S_0$, the point $\mathrm{i}\rho\omega_{1-\mu}$ belongs to $\varSigma_0$, and by $(*)$

$$\begin{aligned}\operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_{\nu-1}) &\le \operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_{\nu-1}^{-1}) \le \cdots \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_2) \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_2^{-1}) \\ &\le \operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_1) \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_1^{-1}) \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-\mu}\omega_0).\end{aligned}$$

Similarly, for any point $\rho \in S_1$, the point $\mathrm{i}\rho\omega_{2-3\mu}$ belongs to $\varSigma_1$, and by $(**)$

$$\begin{aligned}\operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_{\nu-1}^{-1}) &\le \operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_{\nu-1}) \le \cdots \le \operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_2^{-1}) \\ &\le \operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_2) \le \operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_1^{-1}) \\ &\le \operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_1) \le \operatorname{Re}(\mathrm{i}\rho\omega_{2-3\mu}\omega_0^{-1}).\end{aligned}$$

Assume ν is even: $\nu = 2\mu$. Then $\arg(\mathrm{i}\omega_{-\mu}) = -\pi/(2n)$ and $\arg(\mathrm{i}\omega_{1-3\mu}) = -\pi + \pi/(2n)$. Thus, for any point $\rho \in S_0$, we have $\mathrm{i}\rho\omega_{-\mu} \in \varSigma_1$, and by $(**)$

$$\begin{aligned}\operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_{\nu-1}^{-1}) &\le \operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_{\nu-1}) \le \cdots \le \operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_2^{-1}) \le \operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_2) \\ &\le \operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_1^{-1}) \le \operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_1) \le \operatorname{Re}(\mathrm{i}\rho\omega_{-\mu}\omega_0^{-1}).\end{aligned}$$

Also, for any point $\rho \in S_1$, the point $\mathrm{i}\rho\omega_{1-3\mu}$ belongs to $\varSigma_0$, and by $(*)$

$$\begin{aligned}\operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_{\nu-1}) &\le \operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_{\nu-1}^{-1}) \le \cdots \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_2) \\ &\le \operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_2^{-1}) \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_1) \\ &\le \operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_1^{-1}) \le \operatorname{Re}(\mathrm{i}\rho\omega_{1-3\mu}\omega_0).\end{aligned}$$

This completes the proof of the lemma. □

Let $\omega_0^0, \omega_1^0, \ldots, \omega_{n-1}^0$ and $\omega_0^1, \omega_1^1, \ldots, \omega_{n-1}^1$ be the permutations of $\omega_0, \omega_1, \ldots, \omega_{n-1}$ determined in Lemma 4.5. We begin our expansion of solutions for ρ belonging to the translated sector T_0. Fix any integer k with $0 \le k \le n-1$, and let κ be the integer satisfying $0 \le \kappa \le n-1$ and $\omega_\kappa^0 = \omega_k$. Let K_{01} and K_{02} be the functions defined by

$$K_{01}(t,s,\rho) := -\sum_{j=0}^{\kappa}(\mathrm{i}\omega_j^0)\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}, \quad K_{02}(t,s,\rho) := \sum_{j=\kappa+1}^{n-1}(\mathrm{i}\omega_j^0)\mathrm{e}^{\mathrm{i}\rho\omega_j^0(t-s)}$$

for $t, s \in [0,1]$ and for $\rho \in \mathbb{C}$; let k_0 be the function defined by

$$\begin{aligned}k_0(t,s,\rho) &:= \frac{1}{n\rho^{n-1}}K_{01}(t,s,\rho), \qquad 0 \le s < t \le 1, \\ k_0(t,s,\rho) &:= \frac{1}{n\rho^{n-1}}K_{02}(t,s,\rho), \qquad 0 \le t < s \le 1,\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$; and let $\mathcal{K}_{0\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$\mathcal{K}_{0\rho}u(t) := \int_0^1 k_0(t,s,\rho)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in L^2[0,1],$$

for each $\rho \neq 0$ in $\mathbb{C}$. The operator $\mathcal{K}_{0\rho}$ has the same properties as its earlier version for the case n even.

For the fixed integer k, we form the Birkhoff approximate solution

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j} = \mathrm{e}^{\mathrm{i}\rho\omega_\kappa^0 t}\sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad 0 \le t \le 1,$$

and the corresponding residual function

$$\eta_k(t,\rho) = -\frac{\rho^n}{\mathrm{i}^n}\sum_{s=m+1}^{n+m-1}\left[\sum_{\ell+p+j=s} c_{k\ell p}(t) z_{kj}^{(\ell)}(t)\right]\rho^{-s}, \qquad 0 \le t \le 1,$$

with

$$(\rho^n I - \ell)z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho) \tag{4.39}$$

for $0 \le t \le 1$ and $\rho \neq 0$ in $\mathbb{C}$, and with

$$|\eta_k(t,\rho)| \le M_0|\rho|^{-(m+1-n)} \tag{4.40}$$

for $0 \le t \le 1$ and for $\rho \in \mathbb{C}$ with $|\rho| \ge 1$. Fix any point $\rho \neq 0$ in $\mathbb{C}$. A function $u(t)$ belonging to $H^n[0,1]$ is a solution of the differential equation (2.1) if and only if the function $\phi(t) = u(t) - z_k(t,\rho)$ is a solution of the integro-differential equation

$$\phi(t) = \mathcal{K}_{0\rho}\sigma\phi(t) - \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big) + \sum_{j=0}^{n-1} c_j \mathrm{e}^{\mathrm{i}\rho\omega_j t}, \qquad 0 \le t \le 1, \tag{4.41}$$

for some constants $c_0, c_1, \ldots, c_{n-1}$.

Consider the special case of equation (4.41) where all the constants c_j are equal to zero:

$$\phi(t) = \mathcal{K}_{0\rho}\sigma\phi(t) - \mathcal{K}_{0\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}\eta_k(t,\rho)\big), \qquad 0 \le t \le 1. \tag{4.42}$$

If $\phi(t)$ is a function in $H^n[0,1]$ that is a solution of equation (4.42), then the function $\psi(t) = \mathrm{e}^{-\mathrm{i}\rho\omega_k t}\sigma\phi(t)$ is a solution of the integral equation

$$\begin{aligned}\psi(t) = {} & \frac{1}{n\rho^{n-1}}\int_0^t \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds \\ & + \frac{1}{n\rho^{n-1}}\int_t^1 \sum_{\alpha=0}^{n-2} a_\alpha(t)\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)}\frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)\big[\psi(s) - \eta_k(s,\rho)\big]\,ds,\end{aligned} \tag{4.43}$$

$0 \le t \le 1$. Conversely, if $\psi(t)$ is a function in $C[0,1]$ that is a solution of the integral equation (4.43), then the function $\phi(t) = \mathcal{K}_{0\rho}\bigl(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\bigr)$ is a solution of equation (4.42), and the function

$$\begin{aligned} u(t) &= z_k(t,\rho) + \phi(t) = z_k(t,\rho) + \mathcal{K}_{0\rho}\bigl(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi(t) - \eta_k(t,\rho)]\bigr) \\ &= z_k(t,\rho) + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} K_{01}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} K_{02}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds, \end{aligned} \tag{4.44}$$

$0 \le t \le 1$, is a solution of the differential equation (2.1), where the derivatives of $u(t)$ are given by

$$\begin{aligned} u^{(\alpha)}(t) = z_k^{(\alpha)}(t,\rho) &+ \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds \\ &+ \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)[\psi(s) - \eta_k(s,\rho)]\,ds \end{aligned} \tag{4.45}$$

for $0 \le t \le 1$ and for $\alpha = 0, 1, \ldots, n-1$. It is in solving equation (4.43) that the sector T_0 comes into play.

For each $\rho \ne 0$ in $\mathbb{C}$, let $\mathcal{A}_{0\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{A}_{0\rho}u(t) := \int_0^1 g_0(t,s,\rho)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in C[0,1],$$

where the kernel is defined by

$$\begin{aligned} g_0(t,s,\rho) &:= \frac{1}{n}\sum_{\alpha=0}^{n-2} a_\alpha(t)\rho^{-(n-2)}\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho), \quad 0 \le s < t \le 1, \\ g_0(t,s,\rho) &:= \frac{1}{n}\sum_{\alpha=0}^{n-2} a_\alpha(t)\rho^{-(n-2)}\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho), \quad 0 \le t < s \le 1. \end{aligned}$$

Then the integral equation (4.43) can be written in the operator form

$$\bigl[I - \frac{1}{\rho}\mathcal{A}_{0\rho}\bigr]\psi(t) = -\frac{1}{\rho}\mathcal{A}_{0\rho}\,\eta_k(t,\rho), \qquad 0 \le t \le 1, \tag{4.46}$$

in the setting of the Banach space $C[0,1]$.

It follows that the operator $\mathcal{A}_{0\rho}$ is bounded, $\|\mathcal{A}_{0\rho}\| \le M_1$, for all $\rho \in T_0$ with $|\rho| \ge 1$. Setting $r_0 := \max\{1, M_1\} \ge 1$, for any $\rho \in T_0$ with $|\rho| > r_0$ the integral equation (4.43), or the equivalent operator equation (4.46), has a unique solution $\psi_{0k}(\,\cdot\,,\rho)$ in $C[0,1]$ given by

$$\psi_{0k}(t,\rho) := -\frac{1}{\rho}\bigl[I - \frac{1}{\rho}\mathcal{A}_{0\rho}\bigr]^{-1}\mathcal{A}_{0\rho}\,\eta_k(t,\rho) = -\sum_{j=0}^{\infty}\frac{1}{\rho^{j+1}}\mathcal{A}_{0\rho}^{j+1}\eta_k(t,\rho) \tag{4.47}$$

for $0 \le t \le 1$, where the series in (4.47) is converging in $C[0,1]$ under the supremum norm. Now use (4.44) to define the function $v_{0k}(\,\cdot\,,\rho) \in H^n[0,1]$ by

$$\begin{aligned} v_{0k}(t,\rho) :=\ & z_k(t,\rho) \\ & + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} K_{01}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \\ & + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} K_{02}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \end{aligned} \tag{4.48}$$

for $0 \le t \le 1$ and for $\rho \in T_0$ with $|\rho| > r_0$. The function $v_{0k}(\,\cdot\,,\rho)$ is a solution of the differential equation (2.1), and by (4.45) the derivatives of $v_{0k}(\,\cdot\,,\rho)$ are given by

$$\begin{aligned} v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) & \\ + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} & \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \\ + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} & \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho)[\psi_{0k}(s,\rho) - \eta_k(s,\rho)]\,ds \end{aligned} \tag{4.49}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$.

Next, for $\rho \ne 0$ in $\mathbb{C}$ and for $\alpha = 0, 1, \ldots, n-1$ let

$$\begin{aligned} h_{0\alpha}(t,s,\rho) &:= \frac{1}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{01}}{\partial t^\alpha}(t,s,\rho) \quad \text{for } 0 \le s < t \le 1, \\ h_{0\alpha}(t,s,\rho) &:= \frac{1}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{02}}{\partial t^\alpha}(t,s,\rho) \quad \text{for } 0 \le t < s \le 1, \end{aligned}$$

and let $\mathcal{B}_{0\alpha\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{B}_{0\alpha\rho}u(t) := \int_0^1 h_{0\alpha}(t,s,\rho)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in C[0,1].$$

The norm of $\mathcal{B}_{0\alpha\rho}$ satisfies the estimate $\|\mathcal{B}_{0\alpha\rho}\| \le C_1|\rho|^{-(n-1-\alpha)}$ for $\rho \in T_0$ with $|\rho| \ge 1$ and for $\alpha = 0, 1, \ldots, n-1$, and in terms of this operator we can rewrite (4.49) as

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}\,\mathcal{B}_{0\alpha\rho}[\psi_{0k}(t,\rho) - \eta_k(t,\rho)] \tag{4.50}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$.

For $\rho \in T_0$ with $|\rho| > r_0$ and for $\alpha = 0, 1, \ldots, n-1$, set

$$E_{0k\alpha}(t,\rho) := \rho^{m-\alpha}\,\mathcal{B}_{0\alpha\rho}[\psi_{0k}(t,\rho) - \eta_k(t,\rho)], \qquad 0 \le t \le 1.$$

The function $E_{0k\alpha}(\,\cdot\,,\rho)$ belongs to the space $H^{n-\alpha}[0,1]$; equation (4.50) can be expressed as

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\alpha}(t,\rho)\rho^{-m+\alpha} \tag{4.51}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$; and

$$\|E_{0k\alpha}(\,\cdot\,,\rho)\|_\infty \le \gamma_0 \qquad \text{(a constant)} \tag{4.52}$$

for $\rho \in T_0$ with $|\rho| > 2r_0$ and for $\alpha = 0, 1, \ldots, n-1$. Equations (4.51) and (4.52) are our asymptotic formulas for the solution $v_{0k}(\,\cdot\,,\rho)$ and its derivatives relative to the sector T_0.

Finally, we repeat the above construction for $k = 0, 1, \ldots, n-1$ and for $\rho \in T_0$ with $|\rho| > r_0$, producing the solutions

$$v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots, v_{0\,n-1}(\,\cdot\,,\rho) \in H^n[0,1]$$

of the differential equation (2.1). These solutions are linearly independent for $\rho \in T_0$ with $|\rho|$ sufficiently large. The regularity properties of the solutions $v_{0k}(t,\rho)$ then follow from the lemmas in the Appendix (Chapter 12).

These results for the sector T_0 are collected below in a basic theorem. Cf. Theorem 4.3.

Theorem 4.6. *Let n be odd: $n = 2\nu - 1$; let $T_0 = \{\rho - \tau_0 \mid \rho \in S_0\}$ be the translation of the sector S_0 selected in the last chapter; let m be the integer chosen in the last chapter with $m > n$, $m > p_0$, and $-(m-p_0-1) \le p, q \le p_0$; and for each $\rho \neq 0$ in $\mathbb{C}$ let*

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

be the Birkhoff approximate solutions of the differential equation (2.1) *that are determined by Theorem 2.1. Then there exists a constant $R_0 > 0$ such that for $\rho \in T_0$ with $|\rho| > R_0$, there exist n functions $v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots,$ $v_{0\,n-1}(\,\cdot\,,\rho)$ in $H^n[0,1]$ that are linearly independent solutions of the differential equation* (2.1)*, with*

$$v_{0k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\alpha}(t,\rho)\rho^{-m+\alpha}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. The function $E_{0k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$ with $|E_{0k\alpha}(t,\rho)| \le \gamma_0$ (a constant) for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. For the open set $G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$ and for $k, \alpha = 0, 1, \ldots, n-1$, the function $v_{0k}^{(\alpha)}(t,\rho)$ is continuous on $[0,1] \times G_0$, and the derivative $\frac{\partial}{\partial\rho}\left[v_{0k}^{(\alpha)}(t,\rho)\right]$ exists and is continuous on $[0,1] \times G_0$.

To obtain solutions of the differential equation (2.1) for ρ belonging to the sector T_1, we use the exact same argument used earlier for the case n even. Indeed, fix any integer k with $0 \le k \le n-1$, and let κ be the unique integer satisfying $0 \leq \kappa \leq n-1$ and $\omega_\kappa^1 = \omega_k$. The key to this work is to introduce the integral operators $\mathcal{K}_{1\rho}$ and $\mathcal{A}_{1\rho}$. First, let K_{11} and K_{12} be the functions defined by

$$K_{11}(t,s,\rho) := -\sum_{j=0}^{\kappa} (\mathrm{i}\omega_j^1) \mathrm{e}^{\mathrm{i}\rho\omega_j^1(t-s)}, \quad K_{12}(t,s,\rho) := \sum_{j=\kappa+1}^{n-1} (\mathrm{i}\omega_j^1) \mathrm{e}^{\mathrm{i}\rho\omega_j^1(t-s)}$$

for $t, s \in [0,1]$ and for $\rho \in \mathbb{C}$; let k_1 be the function defined by

$$\begin{aligned} k_1(t,s,\rho) &:= \frac{1}{n\rho^{n-1}} K_{11}(t,s,\rho), \qquad 0 \leq s < t \leq 1, \\ k_1(t,s,\rho) &:= \frac{1}{n\rho^{n-1}} K_{12}(t,s,\rho), \qquad 0 \leq t < s \leq 1, \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$; and let $\mathcal{K}_{1\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$\mathcal{K}_{1\rho} u(t) := \int_0^1 k_1(t,s,\rho) u(s)\, ds, \qquad 0 \leq t \leq 1, \quad u \in L^2[0,1],$$

for each $\rho \neq 0$ in $\mathbb{C}$. Second, for each $\rho \neq 0$ in $\mathbb{C}$, let $\mathcal{A}_{1\rho}$ be the integral operator on $C[0,1]$ defined by

$$\mathcal{A}_{1\rho} u(t) := \int_0^1 g_1(t,s,\rho) u(s)\, ds, \qquad 0 \leq t \leq 1, \quad u \in C[0,1],$$

where the kernel is defined by

$$\begin{aligned} g_1(t,s,\rho) &:= \frac{1}{n} \sum_{\alpha=0}^{n-2} a_\alpha(t) \rho^{-(n-2)} \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{11}}{\partial t^\alpha}(t,s,\rho), \quad 0 \leq s < t \leq 1, \\ g_1(t,s,\rho) &:= \frac{1}{n} \sum_{\alpha=0}^{n-2} a_\alpha(t) \rho^{-(n-2)} \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{12}}{\partial t^\alpha}(t,s,\rho), \quad 0 \leq t < s \leq 1. \end{aligned}$$

Then for $\rho \in T_1$ with $|\rho| > r_0$, r_0 sufficiently large, the integral equation

$$\big[I - \frac{1}{\rho} \mathcal{A}_{1\rho}\big] \psi(t) = -\frac{1}{\rho} \mathcal{A}_{1\rho}\, \eta_k(t,\rho), \qquad 0 \leq t \leq 1, \tag{4.53}$$

has a unique solution $\psi_{1k}(\,\cdot\,,\rho)$ in $C[0,1]$ given by

$$\psi_{1k}(t,\rho) := -\frac{1}{\rho} \big[I - \frac{1}{\rho} \mathcal{A}_{1\rho}\big]^{-1} \mathcal{A}_{1\rho}\, \eta_k(t,\rho) = -\sum_{j=0}^{\infty} \frac{1}{\rho^{j+1}} \mathcal{A}_{1\rho}^{j+1} \eta_k(t,\rho) \tag{4.54}$$

for $0 \leq t \leq 1$, where the series is converging in $C[0,1]$. The function $v_{1k}(\,\cdot\,,\rho)$ in $H^n[0,1]$ defined by

$$v_{1k}(t,\rho) := z_k(t,\rho) + \mathcal{K}_{1\rho}\big(\mathrm{e}^{\mathrm{i}\rho\omega_k t}[\psi_{1k}(t,\rho) - \eta_k(t,\rho)]\big) \tag{4.55}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > r_0$, is then a solution of the differential equation (2.1), with the derivatives of $v_{1k}(\,\cdot\,,\rho)$ given by

$$\begin{aligned} v_{1k}^{(\alpha)}(t,\rho) &= z_k^{(\alpha)}(t,\rho) \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_0^t \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{11}}{\partial t^\alpha}(t,s,\rho)[\psi_{1k}(s,\rho) - \eta_k(s,\rho)]\,ds \\ &\quad + \frac{\mathrm{e}^{\mathrm{i}\rho\omega_k t}}{n\rho^{n-1}} \int_t^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k(t-s)} \frac{\partial^\alpha K_{12}}{\partial t^\alpha}(t,s,\rho)[\psi_{1k}(s,\rho) - \eta_k(s,\rho)]\,ds \end{aligned} \tag{4.56}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > r_0$, and for $\alpha = 0, 1, \ldots, n-1$.

Carrying out this construction for $k = 0, 1, \ldots, n-1$ and for $\rho \in T_1$ with $|\rho| > r_0$, we are lead to the following basic theorem for the sector T_1. Cf. Theorem 4.4.

Theorem 4.7. *Let n be odd: $n = 2\nu - 1$; let $T_1 = \{\rho - \tau_1 \mid \rho \in S_1\}$ be the translation of the sector S_1 selected in the last chapter; let m be the integer chosen in the last chapter with $m > n$, $m > p_0$, and $-(m-p_0-1) \le p, q \le p_0$; and for each $\rho \ne 0$ in $\mathbb{C}$ let*

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

be the Birkhoff approximate solutions of the differential equation (2.1) *that are determined by Theorem 2.1. Then there exists a constant $R_0 > 0$ such that for $\rho \in T_1$ with $|\rho| > R_0$, there exist n functions $v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \ldots, v_{1\,n-1}(\,\cdot\,,\rho)$ in $H^n[0,1]$ that are linearly independent solutions of the differential equation* (2.1)*, with*

$$v_{1k}^{(\alpha)}(t,\rho) = z_k^{(\alpha)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{1k\alpha}(t,\rho)\rho^{-m+\alpha}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. The function $E_{1k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$ with $|E_{1k\alpha}(t,\rho)| \le \gamma_0$ (a constant) for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. For the open set $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$ and for $k, \alpha = 0, 1, \ldots, n-1$, the function $v_{1k}^{(\alpha)}(t,\rho)$ is continuous on $[0,1] \times G_1$, and the derivative $\frac{\partial}{\partial\rho}\left[v_{1k}^{(\alpha)}(t,\rho)\right]$ exists and is continuous on $[0,1] \times G_1$.

5

The Characteristic Determinant

Now that we have constructed independent solutions of the differential equation (2.1), the next step is to develop the characteristic determinant on the sectors T_0 and T_1.

5.1 The Characteristic Determinant for n Even

Assume that n is even: $n = 2\nu \geq 2$. Consider the sectors

$$S_0\colon \text{ all } \rho = |\rho| \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } 0 \leq \theta \leq \frac{\pi}{n}\,,$$

$$S_1\colon \text{ all } \rho = |\rho| \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} \leq \theta \leq 0\,,$$

and the corresponding translated sectors T_0, T_1. For each point $\rho \in T_0$ with $|\rho| > R_0$ let $v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots, v_{0\,n-1}(\,\cdot\,,\rho)$ be the n linearly independent solutions of the differential equation (2.1) constructed in Theorem 4.3. For fixed ρ the function $v_{0k}(\,\cdot\,,\rho)$ belongs to $H^n[0,1]$, and $v_{0k}(\,\cdot\,,\rho)$ and its derivatives have the asymptotic expansions

$$v_{0k}^{(\eta)}(t,\rho) = z_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\eta}(t,\rho)\rho^{-m+\eta} \tag{5.1}$$

for $0 \leq t \leq 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \ldots, n-1$. In this representation the function $z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{m-1} z_{kj}(t)\rho^{-j}$ is the Birkhoff approximate solution of (2.1) constructed in Chapter 2, and the function $E_{0k\eta}(\,\cdot\,,\rho)$ belongs to $H^{n-\eta}[0,1]$ and satisfies the bound $|E_{0k\eta}(t,\rho)| \leq \gamma_0$ for $0 \leq t \leq 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \ldots, n-1$. The functions $v_{0k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, form a basis for the solution space of the differential equation (2.1) for $\rho \in T_0$ satisfying $|\rho| > R_0$.

Similarly, for each point $\rho \in T_1$ with $|\rho| > R_0$ let $v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \ldots,$ $v_{1\,n-1}(\,\cdot\,,\rho)$ be the linearly independent solutions of the differential equation (2.1) constructed in Theorem 4.4. Again the function $v_{1k}(\,\cdot\,,\rho)$ belongs to $H^n[0,1]$ with

$$v_{1k}^{(\eta)}(t,\rho) = z_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{1k\eta}(t,\rho)\rho^{-m+\eta} \tag{5.2}$$

for $0 \leq t \leq 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \ldots, n-1$. Here the function $E_{1k\eta}(\,\cdot\,,\rho)$ belongs to the space $H^{n-\eta}[0,1]$ and satisfies the bound $|E_{1k\eta}(t,\rho)| \leq \gamma_0$ for $0 \leq t \leq 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \ldots, n-1$. The functions $v_{1k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, also form a basis for the solution space of the differential equation (2.1) for $\rho \in T_1$ with $|\rho| > R_0$.

We will use these two bases to construct the characteristic determinant of the differential operator L. The first basis is used for the construction on the sector T_0, and the second for the construction on T_1. For the sector T_0 we develop the theory in detail. Since the development for the sector T_1 is similar, we simply sketch the theory for T_1.

In Chapter 4 we showed that

$$\begin{aligned} v_{0k}^{(\eta)}(t,\rho) &= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k t}\Big[(\mathrm{i}\omega_k)^\eta + \sum_{\ell=1}^{m+\eta-1} f_{k\eta\ell}(t)\rho^{-\ell} + E_{0k\eta}(t,\rho)\rho^{-m}\Big] \\ &:= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k t}\Big[(\mathrm{i}\omega_k)^\eta + F_{0k\eta}(t,\rho)\Big] \end{aligned} \tag{5.3}$$

for $0 \leq t \leq 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \ldots, n-1$. The function $F_{0k\eta}(\,\cdot\,,\rho)$ belongs to $H^{n-\eta}[0,1]$ with $F_{0k\eta}(t,\rho) \to 0$ uniformly on $[0,1] \times T_0$ as $|\rho| \to \infty$ for $k, \eta = 0, 1, \ldots, n-1$. Choose a constant $R_1 \geq R_0$ such that

$$|F_{0k\eta}(t,\rho)| \leq 1 \tag{5.4}$$

for $0 \leq t \leq 1$, for $\rho \in T_0$ with $|\rho| > R_1$, and for $k, \eta = 0, 1, \ldots, n-1$.

Relative to the sector T_0, we form the *modified solutions* of the differential equation (2.1):

$$\begin{aligned} u_{0k}(t,\rho) &:= v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad k = 0, 1, \ldots, \nu-1, \\ u_{0k}(t,\rho) &:= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad k = \nu, \ldots, n-1, \end{aligned}$$

for $0 \leq t \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. Cf. the definitions of the functions $y_k(\,\cdot\,,\rho)$ given at the beginning of Chapter 3. Clearly these functions also form a basis for the solution space of the differential equation (2.1) for $\rho \in T_0$ with $|\rho| > R_0$. It will be convenient to use these modified solutions in the sequel.

Using (5.1) and (5.3), the derivatives of the solutions $u_{0k}(\,\cdot\,,\rho)$ can be expressed as

$$\begin{aligned} u_{0k}^{(\eta)}(t,\rho) &= y_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\eta}(t,\rho)\rho^{-m+\eta} \\ &= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^\eta + F_{0k\eta}(t,\rho)\big], \qquad k = 0, 1, \ldots, \nu-1, \\ u_{0k}^{(\eta)}(t,\rho) &= y_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)} E_{0k\eta}(t,\rho)\rho^{-m+\eta} \\ &= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}\big[(\mathrm{i}\omega_k)^\eta + F_{0k\eta}(t,\rho)\big], \quad k = \nu, \ldots, n-1, \end{aligned} \tag{5.5}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $\eta = 0, 1, \ldots, n-1$. Applying the regularity results of Theorem 4.3, for fixed $t \in [0, 1]$ the functions $u_{0k}^{(\eta)}(t, \rho)$ and $F_{0k\eta}(t, \rho)$ are analytic functions of the ρ variable on the open set

$$G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}.$$

It follows that the boundary data $u_{0k}^{(\eta)}(0, \rho)$, $u_{0k}^{(\eta)}(1, \rho)$, $k, \eta = 0, 1, \ldots, n-1$, consists of functions of ρ that are analytic on G_0.

Next, we establish various bounds and growth rates relative to the sector T_0 for the functions appearing above. Fix a real number σ_0 with $0 < \sigma_0 < \pi/10$, set $\alpha := \sin(\sigma_0/n) > 0$, and then form the sector

$$\Sigma\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{2\pi}{n} + \frac{\sigma_0}{n} \le \theta \le \frac{2\pi}{n} - \frac{\sigma_0}{n}$$

in the ρ plane. The reason for the constant $\pi/10$ is that eventually we will apply the basic completeness theorem [34, p. 80] in which we need five rays with the angles between adjacent rays being less than $\pi/2$ (see Chapter 9). Clearly $-2\pi/n + \sigma_0/n < -\pi/n < \pi/n < 2\pi/n - \sigma_0/n$, and hence, any ρ in $T_0 \cup T_1$ with $|\rho|$ sufficiently large lies in the sector Σ. Without loss of generality we can assume that the constant $R_0 > 0$ chosen earlier (see Theorem 4.3 and Theorem 4.4) has the additional property that $\rho \in T_0 \cup T_1$ with $|\rho| > R_0$ implies $\rho \in \Sigma$.

Take any point $\rho = a + \mathrm{i}b \in \Sigma$. Then

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0 t}\right| = \left|\mathrm{e}^{\mathrm{i}\rho t}\right| = \mathrm{e}^{-bt}, \qquad 0 \le t \le 1, \tag{5.6}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_\nu (t-1)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho(1-t)}\right| = \mathrm{e}^{-b(1-t)}, \qquad 0 \le t \le 1. \tag{5.7}$$

Clearly these two exponential functions are unbounded on the sector Σ. For $k = 1, \ldots, \nu - 1$ we have $\sigma_0/n < 2\pi/n - \sigma_0/n$ and

$$\frac{\pi}{2} + \left(-\frac{2\pi}{n} + \frac{\sigma_0}{n}\right) + \frac{2\pi}{n} \le \arg(\mathrm{i}\rho\omega_k t) \le \frac{\pi}{2} + \left(\frac{2\pi}{n} - \frac{\sigma_0}{n}\right) + \frac{2\pi(\nu-1)}{n},$$

and hence, $\pi/2 + \sigma_0/n \le \arg(\mathrm{i}\rho\omega_k t) \le 3\pi/2 - \sigma_0/n$. It follows that

$$\begin{aligned}\operatorname{Re}(\mathrm{i}\rho\omega_k t) &= |\mathrm{i}\rho\omega_k t| \cos[\arg(\mathrm{i}\rho\omega_k t)] \\ &\le t|\rho| \cos\left(\frac{\pi}{2} + \frac{\sigma_0}{n}\right) = -\,t|\rho| \sin\frac{\sigma_0}{n}\end{aligned}$$

and $\left|\mathrm{e}^{\mathrm{i}\rho\omega_k t}\right| = \mathrm{e}^{\operatorname{Re}(\mathrm{i}\rho\omega_k t)} \le \mathrm{e}^{-t\alpha|\rho|} \le 1$, or

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k t}| \le \mathrm{e}^{-t\alpha|\rho|} \le 1, \qquad 0 \le t \le 1, \tag{5.8}$$

for $\rho = a + \mathrm{i}b \in \Sigma$ and for $k = 1, \ldots, \nu - 1$. Similarly, for $k = \nu + 1, \ldots, n - 1$

$$-\frac{\pi}{2}+\left(-\frac{2\pi}{n}+\frac{\sigma_0}{n}\right)+\frac{2\pi(\nu+1)}{n}\le \arg(\mathrm{i}\rho\omega_k(t-1))$$
$$\le -\frac{\pi}{2}+\left(\frac{2\pi}{n}-\frac{\sigma_0}{n}\right)+\frac{2\pi(n-1)}{n},$$

so $\pi/2+\sigma_0/n \le \arg(\mathrm{i}\rho\omega_k(t-1)) \le 3\pi/2-\sigma_0/n$. This leads to the estimate

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}| \le \mathrm{e}^{-(1-t)\alpha|\rho|} \le 1, \qquad 0\le t\le 1, \tag{5.9}$$

for $\rho = a+\mathrm{i}b \in \Sigma$ and for $k=\nu+1,\dots,n-1$.

From the estimates (5.6)–(5.9) it is immediate that

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0}\right| = \left|\mathrm{e}^{\mathrm{i}\rho}\right| = \mathrm{e}^{-b}, \tag{5.10}$$
$$\left|\mathrm{e}^{-\mathrm{i}\rho\omega_\nu}\right| = \left|\mathrm{e}^{\mathrm{i}\rho}\right| = \mathrm{e}^{-b}, \tag{5.11}$$
$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k}\right| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k=1,\dots,\nu-1, \tag{5.12}$$
$$\left|\mathrm{e}^{-\mathrm{i}\rho\omega_k}\right| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k=\nu+1,\dots,n-1, \tag{5.13}$$

for all $\rho = a+\mathrm{i}b \in \Sigma$. Thus, the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_0} = \mathrm{e}^{-\mathrm{i}\rho\omega_\nu} = \mathrm{e}^{\mathrm{i}\rho}$ are unbounded on Σ as $b\to-\infty$, while the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $1\le k\le \nu-1$, and $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $\nu+1\le k\le n-1$, go to 0 very rapidly on Σ as $|\rho|\to\infty$. Note that the estimates (5.6)–(5.13) are also valid for $\rho\in T_0\cup T_1$ with $|\rho|>R_0$, and in particular, they are valid for $\rho\in S_0\cup S_1$ with $|\rho|>R_0$ or for $\rho\in G_0$ or $\rho\in G_1$.

Applying the estimates (5.6)–(5.9) and (5.4) to the representations (5.5) with $\eta=0$, it follows that

$$|u_{00}(t,\rho)| \le 2\mathrm{e}^{-bt}, \tag{5.14}$$
$$|u_{0\nu}(t,\rho)| \le 2\mathrm{e}^{-b(1-t)}, \tag{5.15}$$
$$|u_{0k}(t,\rho)| \le 2\mathrm{e}^{-t\alpha|\rho|} \le 2, \qquad k=1,\dots,\nu-1, \tag{5.16}$$
$$|u_{0k}(t,\rho)| \le 2\mathrm{e}^{-(1-t)\alpha|\rho|} \le 2, \qquad k=\nu+1,\dots,n-1, \tag{5.17}$$

for $0\le t\le 1$ and for $\rho=a+\mathrm{i}b\in T_0$ with $|\rho|>R_1$. In particular, for $\rho=a+\mathrm{i}b$ belonging to the sector S_0 (where $b\ge 0$) with $|\rho|>R_1$, we have

$$|u_{0k}(t,\rho)| \le 2, \qquad 0\le t\le 1, \quad k=0,1,\dots,n-1. \tag{5.18}$$

These bounds will be used in Chapter 6 to determine the growth rate of the Green's function and the resolvent.

Using the modified solutions $u_{0k}(t,\rho)$, $k=0,1,\dots,n-1$, we form the functions

$$M_{0ik}(\rho) := B_i(u_{0k}(\,\cdot\,,\rho)) = \sum_{\eta=0}^{m_i}\alpha_{i\eta}u_{0k}^{(\eta)}(0,\rho) + \sum_{\eta=0}^{m_i}\beta_{i\eta}u_{0k}^{(\eta)}(1,\rho)$$

for $\rho\in T_0$ with $|\rho|>R_0$ and for $i=1,\dots,n$ and $k=0,1,\dots,n-1$. Clearly these functions are analytic on the open set G_0. For $i=1,\dots,n$ and $k=0,1,\dots,\nu-1$ define

$$\widetilde{P}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{0k\eta}(0,\rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{0k\eta}(1,\rho)\rho^{-m+\eta}$$

for $\rho \in T_0$ with $|\rho| > R_0$; for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ define

$$\widetilde{P}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{0k\eta}(1,\rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{0k\eta}(0,\rho)\rho^{-m+\eta}$$

for $\rho \in T_0$ with $|\rho| > R_0$; and in terms of these functions and the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ appearing in equations (3.1) and (3.2), for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$ define

$$P_{0ik}(\rho) := \widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho), \quad Q_{0ik}(\rho) := \widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)$$

for $\rho \in T_0$ with $|\rho| > R_0$. Clearly these functions are analytic on the open set G_0. We can then express the functions M_{0ik} as follows: for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$

$$\begin{aligned} M_{0ik}(\rho) &= \sum_{\eta=0}^{m_i} \alpha_{i\eta}\, [y_k^{(\eta)}(0,\rho) + E_{0k\eta}(0,\rho)\rho^{-m+\eta}] \\ &\quad + \sum_{\eta=0}^{m_i} \beta_{i\eta}\, [y_k^{(\eta)}(1,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k} E_{0k\eta}(1,\rho)\rho^{-m+\eta}] \\ &= \widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} + \widetilde{Q}_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}, \end{aligned}$$

while for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$

$$\begin{aligned} M_{0ik}(\rho) &= \sum_{\eta=0}^{m_i} \alpha_{i\eta}\, [y_k^{(\eta)}(0,\rho) + \mathrm{e}^{-\mathrm{i}\rho\omega_k} E_{0k\eta}(0,\rho)\rho^{-m+\eta}] \\ &\quad + \sum_{\eta=0}^{m_i} \beta_{i\eta}\, [y_k^{(\eta)}(1,\rho) + E_{0k\eta}(1,\rho)\rho^{-m+\eta}] \\ &= \widehat{Q}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widetilde{Q}_{0ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho). \end{aligned}$$

Thus, for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$ we have

$$\begin{aligned} M_{0ik}(\rho) &= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho)] + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ &= P_{0ik}(\rho) + Q_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \end{aligned} \tag{5.19}$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ we have

$$\begin{aligned} M_{0ik}(\rho) &= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho)] + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ &= P_{0ik}(\rho) + Q_{0ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{aligned} \tag{5.20}$$

for $\rho \in T_0$ with $|\rho| > R_0$.

The *characteristic determinant* of the differential operator L relative to the sector T_0 is the analytic function Δ_0 defined by

$$\Delta_0(\rho) := \det(B_i(u_{0k}(\,\cdot\,,\rho))) = \det(M_{0ik}(\rho)) \quad \text{for } \rho \in G_0.$$

For any complex number $\lambda = \rho^n$ with $\rho \in G_0$, we know that λ is an eigenvalue of L if and only if $\Delta_0(\rho) = 0$, and hence, in Chapter 7 we will proceed to compute the zeros of Δ_0 in the sector T_0. Applying (5.19) and (5.20), we can express the characteristic determinant in the form

$$\Delta_0(\rho) =$$

$$\det\begin{array}{c} \begin{array}{cccc} & 1\le k\le \nu-1 & & \nu+1\le k\le n-1 \end{array} \\ \begin{pmatrix} P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{01k}(\rho)+Q_{01k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & P_{01\nu}(\rho)+Q_{01\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{01k}(\rho)+Q_{01k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots & \vdots & \vdots \\ P_{0n0}(\rho)+Q_{0n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{0nk}(\rho)+Q_{0nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & P_{0n\nu}(\rho)+Q_{0n\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{0nk}(\rho)+Q_{0nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix} \end{array} \tag{5.21}$$

for $\rho \in G_0$. Cf. equation (3.10) for the approximate characteristic determinant.

For the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, we see from either (3.1), (3.2) or (3.7), (3.8) that the powers of ρ appearing in the sums are $\rho^{m_i}, \rho^{m_i-1}, \dots, \rho^{-(m-1)}$. Hence, there exists a constant $\gamma_0 > 0$ such that

$$|\widehat{P}_{ik}(\rho)| \le \gamma_0 |\rho|^{m_i}, \qquad |\widehat{Q}_{ik}(\rho)| \le \gamma_0 |\rho|^{m_i} \tag{5.22}$$

for $\rho \in \mathbb{C}$ with $|\rho| \ge 1$ and for $i = 1, \dots, n$ and $k = 0, 1, \dots, n-1$. In the definitions of the functions $\widetilde{P}_{0ik}(\rho)$, $\widetilde{Q}_{0ik}(\rho)$, we see that the functions $E_{0k\eta}(0,\rho)$, $E_{0k\eta}(1,\rho)$ are bounded for $\rho \in T_0$ with $|\rho| > R_0$ by Theorem 4.3, and that the powers of ρ appearing in the sums are $\rho^{-(m-m_i)}, \rho^{-(m-m_i+1)}, \dots, \rho^{-m}$. Thus, there exists a constant $\gamma_1 > 0$ such that

$$|\widetilde{P}_{0ik}(\rho)| \le \gamma_1 |\rho|^{-(m-m_i)}, \qquad |\widetilde{Q}_{0ik}(\rho)| \le \gamma_1 |\rho|^{-(m-m_i)} \tag{5.23}$$

for $\rho \in T_0$ with $|\rho| > R_0$ and for $i = 1, \dots, n$ and $k = 0, 1, \dots, n-1$.

For $i = 1, \dots, n$ define

$$\begin{aligned} \widetilde{F}_{0ik}(\rho) &:= \widetilde{P}_{0ik}(\rho) + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k}, \qquad k = 1, \dots, \nu-1, \\ \widetilde{F}_{0ik}(\rho) &:= \widetilde{P}_{0ik}(\rho) + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k}, \quad k = \nu+1, \dots, n-1, \end{aligned}$$

for $\rho \in T_0$ with $|\rho| > R_0$. Clearly these functions are analytic on the open set G_0. From the estimates (5.12), (5.13) and (5.22), (5.23) we have

$$\begin{aligned} |\widetilde{F}_{0ik}(\rho)| &\le \gamma_1 |\rho|^{-(m-m_i)} + [\gamma_0 |\rho|^{m_i} + \gamma_1 |\rho|^{-(m-m_i)}]\, \mathrm{e}^{-\alpha|\rho|} \\ &\le \gamma_2 |\rho|^{-(m-m_i)} \end{aligned} \tag{5.24}$$

for $\rho \in T_0$ with $|\rho| > R_0$, for $i = 1, \ldots, n$, and for $k = 1, \ldots, \nu - 1$ and $k = \nu + 1, \ldots, n - 1$. In terms of these functions we can rewrite the representation (5.21) of the characteristic determinant in the form

$$\varDelta_0(\rho) =$$

$$\det\left(\begin{array}{cccc} & 1\leq k\leq \nu-1 & & \nu+1\leq k\leq n-1 \\ P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & P_{01\nu}(\rho)+Q_{01\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ P_{0n0}(\rho)+Q_{0n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & P_{0n\nu}(\rho)+Q_{0n\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{array}\right) \tag{5.25}$$

for $\rho \in G_0$.

We now proceed to expand the determinant for $\varDelta_0(\rho)$ that appears in equation (5.25). These expansions parallel the ones used earlier in equations (3.10)–(3.13) for the approximate characteristic determinant $\widehat{\varDelta}(\rho)$, and in fact, the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, that were introduced in Chapter 3 will also appear in these new expansions for $\varDelta_0(\rho)$.

Indeed, suppose we expand the determinant in (5.25) using the linearity of the determinant in the 0th and νth columns:

$$\varDelta_0(\rho) = D_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + D_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + D_{00}(\rho) \tag{5.26}$$

for $\rho \in G_0$, where

$$D_{02}(\rho) := \det\left(\begin{array}{cccc} & 1\leq k\leq \nu-1 & & \nu+1\leq k\leq n-1 \\ \widehat{Q}_{10}(\rho)+\widetilde{Q}_{010}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{Q}_{1\nu}(\rho)+\widetilde{Q}_{01\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho)+\widetilde{Q}_{0n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{Q}_{n\nu}(\rho)+\widetilde{Q}_{0n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{array}\right),$$

$$\begin{aligned} D_{01}(\rho) := {} & \det\left(\begin{array}{cccc} & 1\leq k\leq \nu-1 & & \nu+1\leq k\leq n-1 \\ \widehat{P}_{10}(\rho)+\widetilde{P}_{010}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{Q}_{1\nu}(\rho)+\widetilde{Q}_{01\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widetilde{P}_{0n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{Q}_{n\nu}(\rho)+\widetilde{Q}_{0n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{array}\right) \\ & + \det\left(\begin{array}{cccc} & 1\leq k\leq \nu-1 & & \nu+1\leq k\leq n-1 \\ \widehat{Q}_{10}(\rho)+\widetilde{Q}_{010}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1\nu}(\rho)+\widetilde{P}_{01\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho)+\widetilde{Q}_{0n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{n\nu}(\rho)+\widetilde{P}_{0n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{array}\right), \end{aligned}$$

and

$$D_{00}(\rho) := \det \begin{array}{c} \begin{array}{cccc} & 1\leq k\leq \nu-1 & & \nu+1\leq k\leq n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)+\widetilde{P}_{010}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1\nu}(\rho)+\widetilde{P}_{01\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widetilde{P}_{0n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{n\nu}(\rho)+\widetilde{P}_{0n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{pmatrix} \end{array}$$

for $\rho \in G_0$. Clearly the functions $D_{0i}(\rho)$, $i = 0, 1, 2$, are analytic on the open set G_0. In these representations we will treat the "hat terms" as principal terms and the "tilde terms" as small perturbation terms. See (5.22)–(5.24).

Now consider the function $D_{02}(\rho)$. If we expand $D_{02}(\rho)$ using the linearity of the determinant in its n columns, then $D_{02}(\rho)$ becomes the sum of 2^n determinants, starting with the determinant

$$\widehat{\pi}_2(\rho) = \det \begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{Q}_{1\nu}(\rho) & \widehat{P}_{1\nu+1}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{Q}_{n\nu}(\rho) & \widehat{P}_{n\nu+1}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix},$$

which is precisely the function introduced in Chapter 3. Each of the remaining $2^n - 1$ determinants contains at least one column consisting of the functions $\widetilde{Q}_{0i0}(\rho)$, $i = 1, \ldots, n$, or of the functions $\widetilde{F}_{0ik}(\rho)$, $i = 1, \ldots, n$, or of the functions $\widetilde{Q}_{0i\nu}(\rho)$, $i = 1, \ldots, n$. When such a determinant is expanded, it becomes the sum of $n!$ products each having modulus less than or equal to

$$\gamma|\rho|^{m_1}|\rho|^{m_2}\cdots|\rho|^{m_{i-1}}|\rho|^{-(m-m_i)}|\rho|^{m_{i+1}}\cdots|\rho|^{m_n} = \gamma|\rho|^{-(m-p_0)}$$

by virtue of the estimates (5.22)–(5.24). It follows that the function $D_{02}(\rho)$ can be expressed as

$$D_{02}(\rho) = \widehat{\pi}_2(\rho) + \widetilde{\Phi}_{02}(\rho)$$

for $\rho \in G_0$, where the function $\widetilde{\Phi}_{02}(\rho)$ is analytic on the open set G_0 and satisfies the estimate

$$|\widetilde{\Phi}_{02}(\rho)| \leq \gamma_3|\rho|^{-(m-p_0)}$$

for $\rho \in G_0$. Similarly, we can express $D_{01}(\rho)$ and $D_{00}(\rho)$ as

$$D_{01}(\rho) = \widehat{\pi}_1(\rho) + \widetilde{\Phi}_{01}(\rho), \qquad D_{00}(\rho) = \widehat{\pi}_0(\rho) + \widetilde{\Phi}_{00}(\rho)$$

for $\rho \in G_0$, where the functions $\widetilde{\Phi}_{01}(\rho)$, $\widetilde{\Phi}_{00}(\rho)$ are analytic on the open set G_0 with

$$|\widetilde{\Phi}_{01}(\rho)| \leq \gamma_3|\rho|^{-(m-p_0)}, \qquad |\widetilde{\Phi}_{00}(\rho)| \leq \gamma_3|\rho|^{-(m-p_0)}$$

for $\rho \in G_0$.

Combining the above results, we obtain our principal representation of the characteristic determinant Δ_0 relative to the sector T_0:

$$\Delta_0(\rho) = \widehat{\pi}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \widehat{\pi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\pi}_0(\rho) + \widetilde{\Phi}_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \widetilde{\Phi}_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widetilde{\Phi}_{00}(\rho) \quad (5.27)$$

for $\rho \in G_0$, where the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, are analytic for $\rho \neq 0$ in $\mathbb{C}$, and the functions $\widetilde{\Phi}_{0i}(\rho)$, $i = 0, 1, 2$, are analytic on the open set G_0 with

$$|\widetilde{\Phi}_{0i}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, 2, \tag{5.28}$$

for $\rho \in G_0$. The functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, are the functions introduced in Chapter 3 in our formation of the approximate characteristic determinant $\widehat{\Delta}(\rho) = \widehat{\Delta}(\rho, m)$; they are determined completely by the Birkhoff approximate solutions. The functions $\widetilde{\Phi}_{0i}(\rho)$, $i = 0, 1, 2$, contain all the perturbation terms that are produced in constructing the actual solutions of the differential equation (2.1).

Let us recall some of the results of Chapter 3 where we constructed the approximate characteristic determinant and classified the differential operator L. First, we are assuming that n is even and that the differential operator L is either regular or simply irregular. This yields the integer p with $-\infty < p \leq p_0$, $a_p \neq 0$ and $c_p \neq 0$, and $a_\kappa = c_\kappa = 0$ for $\kappa = p + 1, \ldots, p_0$. Second, the integer q is defined as follows: if $b_\kappa = 0$ for $\kappa = p + 1, \ldots, p_0$, then $q = p$ and $b_q = b_p$ can be either zero or nonzero; if $b_\kappa \neq 0$ for some κ with $p + 1 \leq \kappa \leq p_0$, then q is the largest such integer and b_q is nonzero. In either case we have $p \leq q \leq p_0$ and $b_\kappa = 0$ for $\kappa = q + 1, \ldots, p_0$. Third, the translated sectors T_0 and T_1 are formed subject to the condition (3.31). Fourth, the integer m is fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p \leq p_0$, and then the corresponding Birkhoff approximate solutions $z_k(t, \rho)$, $k = 0, 1, \ldots, n - 1$, are formed, and the modified Birkhoff approximate solutions $y_k(t, \rho)$, $k = 0, 1, \ldots, n - 1$, are determined. Fifth, the functions $\pi_i(\rho)$, $i = 0, 1, 2$, are defined by

$$\pi_2(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \quad \pi_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa, \quad \pi_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} c_\kappa \rho^\kappa \tag{5.29}$$

for $\rho \neq 0$ in $\mathbb{C}$. From equations (3.24), (3.27), and (3.28) it is immediate that

$$\begin{aligned} \widehat{\pi}_2(\rho) &= \pi_2(\rho) \ + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m) \rho^\kappa, \\ \widehat{\pi}_1(\rho) &= \pi_1(\rho) \ + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m) \rho^\kappa, \\ \widehat{\pi}_0(\rho) &= \pi_0(\rho) \ + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m) \rho^\kappa \end{aligned} \tag{5.30}$$

for $\rho \neq 0$ in $\mathbb{C}$.

Finally, in terms of (5.27) and (5.30) we define the functions

$$\begin{aligned}
\Phi_{02}(\rho) &:= \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widetilde{\Phi}_{02}(\rho), \\
\Phi_{01}(\rho) &:= \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widetilde{\Phi}_{01}(\rho), \\
\Phi_{00}(\rho) &:= \sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m)\rho^\kappa + \widetilde{\Phi}_{00}(\rho)
\end{aligned}$$

for $\rho \in G_0$. These functions are clearly analytic on the open set G_0, and by using them we can rewrite the representation (5.27) in the simpler form

$$\Delta_0(\rho) = \pi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) + \Phi_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \Phi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Phi_{00}(\rho) \quad (5.31)$$

for $\rho \in G_0$. In equation (5.31) the functions $\pi_i(\rho)$, $i = 0, 1, 2$, are given by (5.29), they are analytic for $\rho \neq 0$ in $\mathbb{C}$, and they are determined exclusively by the Birkhoff approximate solutions and the boundary values $B_1, \ldots, B_n$; the functions $\Phi_{0i}(\rho)$, $i = 0, 1, 2$, are analytic on the open set G_0 and satisfy the growth rates

$$|\Phi_{0i}(\rho)| \leq \gamma_4 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, 2, \quad (5.32)$$

for $\rho \in G_0$. The representation (5.31) is our working form for the characteristic determinant Δ_0 relative to the sector T_0. Compare the representation (5.31) for the characteristic determinant Δ_0 to the representation (3.32) for the approximate characteristic determinant $\widehat{\Delta}$. In Chapter 7 we will determine the zeros of Δ_0 in the open set G_0, and hence, determine eigenvalues for the differential operator L.

The above results are summarized in the following theorem. Here we assume the conditions set forth in Chapter 3: (i) $n = 2\nu$ is even; (ii) the differential operator L is either regular or simply irregular; (iii) the integers p and q have been determined with $-\infty < p \leq q \leq p_0$ and with $a_p \neq 0$, $c_p \neq 0$, and $a_\kappa = c_\kappa = 0$ for $\kappa = p+1, \ldots, p_0$ and $b_\kappa = 0$ for $\kappa = q+1, \ldots, p_0$; (iv) the translated sectors T_0 and T_1 have been chosen; (v) the integer m has been fixed with $m > n$, $m > p_0$, and $-(m-p_0-1) \leq p \leq p_0$; and (vi) the functions $\pi_i(\rho)$, $i = 0, 1, 2$, have been determined as per Chapter 3 or equation (5.29).

Theorem 5.1. *Let n be even: $n = 2\nu$. Under the above assumptions* (i)–(vi)*, let $v_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the linearly independent solutions of the differential equation* (2.1) *constructed in Theorem 4.3 for $\rho \in T_0$ with $|\rho| > R_0$, let $u_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the modified solutions of* (2.1) *defined above for $\rho \in T_0$ with $|\rho| > R_0$, and let Δ_0 be the characteristic determinant of the differential operator L given by*

$$\Delta_0(\rho) = \det(B_i(u_{0k}(\,\cdot\,, \rho))) \quad \text{for } \rho \in G_0,$$

where $G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$. *Then* Δ_0 *is analytic on the open set* G_0, *and* Δ_0 *has the representation*

$$\Delta_0(\rho) = \pi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) + \Phi_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \Phi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Phi_{00}(\rho)$$

for $\rho \in G_0$, *where the functions* $\Phi_{0i}(\rho)$, $i = 0, 1, 2$, *are analytic on* G_0 *and satisfy the estimates* $|\Phi_{0i}(\rho)| \le \gamma|\rho|^{-(m-p_0)}$ *for* $\rho \in G_0$ *and for* $i = 0, 1, 2$.

Next, for the case $n = 2\nu$ even, we form the characteristic determinant on the sector T_1. The starting point for the discussion is the set of functions $v_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, which form a basis for the solution space of the differential equation (2.1) for $\rho \in T_1$ with $|\rho| > R_0$. We can rewrite (5.2) in the form

$$\begin{aligned} v_{1k}^{(\eta)}(t,\rho) &= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\eta} + \sum_{\ell=1}^{m+\eta-1} f_{k\eta\ell}(t)\rho^{-\ell} + E_{1k\eta}(t,\rho)\rho^{-m}\big] \\ &:= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\eta} + F_{1k\eta}(t,\rho)\big] \end{aligned} \tag{5.33}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \ldots, n-1$. The function $F_{1k\eta}(\,\cdot\,, \rho)$ belongs to $H^{n-\eta}[0,1]$ with $F_{1k\eta}(t,\rho) \to 0$ uniformly on $[0,1] \times T_1$ as $|\rho| \to \infty$ for $k, \eta = 0, 1, \ldots, n-1$. Without loss of generality we can assume that the constant R_1 chosen earlier in this section satisfies the additional condition that

$$|F_{1k\eta}(t,\rho)| \le 1 \tag{5.34}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_1$, and for $k, \eta = 0, 1, \ldots, n-1$. Let us now proceed to construct the characteristic determinant on the sector T_1.

First, we first introduce the *modified solutions* of differential equation (2.1) relative to the sector T_1:

$$\begin{aligned} u_{10}(t,\rho) &:= \mathrm{e}^{-\mathrm{i}\rho\omega_0}v_{10}(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho\omega_0}[y_0(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_0 t}E_{100}(t,\rho)\rho^{-m}], \\ u_{1k}(t,\rho) &:= v_{1k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}E_{1k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \ldots, \nu-1, \\ u_{1\nu}(t,\rho) &:= v_{1\nu}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_\nu}[y_\nu(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_\nu(t-1)}E_{1\nu0}(t,\rho)\rho^{-m}], \\ u_{1k}(t,\rho) &:= \mathrm{e}^{-\mathrm{i}\rho\omega_k}v_{1k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}E_{1k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = \nu+1, \ldots, n-1, \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, where the functions $y_k(\,\cdot\,, \rho)$ are defined at the beginning of Chapter 3. The functions $u_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, also form a basis for the solution space of the differential equation (2.1).

Applying (5.2) and (5.33), we can express the derivatives of the solutions $u_{1k}(\,\cdot\,, \rho)$ in the form

$$\begin{aligned}
u_{10}^{(\eta)}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_0}\bigl[y_0^{(\eta)}(t,\rho)+\mathrm{e}^{\mathrm{i}\rho\omega_0 t}E_{10\eta}(t,\rho)\rho^{-m+\eta}\bigr]\\
&= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_0(t-1)}\bigl[(\mathrm{i}\omega_0)^{\eta}+F_{10\eta}(t,\rho)\bigr],\\
u_{1k}^{(\eta)}(t,\rho) &= y_k^{(\eta)}(t,\rho)+\mathrm{e}^{\mathrm{i}\rho\omega_k t}E_{1k\eta}(t,\rho)\rho^{-m+\eta}\\
&= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\bigl[(\mathrm{i}\omega_k)^{\eta}+F_{1k\eta}(t,\rho)\bigr], \qquad k=1,\ldots,\nu-1,\\
u_{1\nu}^{(\eta)}(t,\rho) &= \mathrm{e}^{\mathrm{i}\rho\omega_\nu}\bigl[y_\nu^{(\eta)}(t,\rho)+\mathrm{e}^{\mathrm{i}\rho\omega_\nu(t-1)}E_{1\nu\eta}(t,\rho)\rho^{-m+\eta}\bigr]\\
&= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_\nu t}\bigl[(\mathrm{i}\omega_\nu)^{\eta}+F_{1\nu\eta}(t,\rho)\bigr],\\
u_{1k}^{(\eta)}(t,\rho) &= y_k^{(\eta)}(t,\rho)+\mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}E_{1k\eta}(t,\rho)\rho^{-m+\eta}\\
&= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}\bigl[(\mathrm{i}\omega_k)^{\eta}+F_{1k\eta}(t,\rho)\bigr], \quad k=\nu+1,\ldots,n-1,
\end{aligned} \tag{5.35}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, and for $\eta = 0,1,\ldots,n-1$. From the regularity results of Theorem 4.4, for $t \in [0,1]$ fixed the functions $u_{1k}^{(\eta)}(t,\rho)$ and $F_{1k\eta}(t,\rho)$ are analytic functions of the ρ variable on the open set $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$. Thus, the boundary data $u_{1k}^{(\eta)}(0,\rho)$, $u_{1k}^{(\eta)}(1,\rho)$, $k,\eta = 0,1,\ldots,n-1$, consists of functions of ρ that are analytic on G_1.

Take any point $\rho = a + \mathrm{i}b$ in the sector Σ. Then by direct calculation and by the previous estimates (5.8) and (5.9):

$$\bigl|\mathrm{e}^{\mathrm{i}\rho\omega_0(t-1)}\bigr| = \bigl|\mathrm{e}^{\mathrm{i}\rho(t-1)}\bigr| = \mathrm{e}^{b(1-t)}, \quad 0 \le t \le 1, \tag{5.36}$$

$$\bigl|\mathrm{e}^{\mathrm{i}\rho\omega_\nu t}\bigr| = \bigl|\mathrm{e}^{-\mathrm{i}\rho t}\bigr| = \mathrm{e}^{bt}, \quad 0 \le t \le 1, \tag{5.37}$$

$$\bigl|\mathrm{e}^{\mathrm{i}\rho\omega_k t}\bigr| \le \mathrm{e}^{-t\alpha|\rho|} \le 1, \qquad 0 \le t \le 1, \quad k = 1,\ldots,\nu-1, \tag{5.38}$$

$$\bigl|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}\bigr| \le \mathrm{e}^{-(1-t)\alpha|\rho|} \le 1, \quad 0 \le t \le 1, \quad k = \nu+1,\ldots,n-1. \tag{5.39}$$

From these estimates it is immediate that

$$\bigl|\mathrm{e}^{-\mathrm{i}\rho\omega_0}\bigr| = \bigl|\mathrm{e}^{-\mathrm{i}\rho}\bigr| = \mathrm{e}^{b}, \tag{5.40}$$

$$\bigl|\mathrm{e}^{\mathrm{i}\rho\omega_\nu}\bigr| = \bigl|\mathrm{e}^{-\mathrm{i}\rho}\bigr| = \mathrm{e}^{b}, \tag{5.41}$$

$$\bigl|\mathrm{e}^{\mathrm{i}\rho\omega_k}\bigr| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k = 1,\ldots,\nu-1, \tag{5.42}$$

$$\bigl|\mathrm{e}^{-\mathrm{i}\rho\omega_k}\bigr| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \quad k = \nu+1,\ldots,n-1, \tag{5.43}$$

for all $\rho = a + \mathrm{i}b \in \Sigma$. Thus, the exponentials $\mathrm{e}^{-\mathrm{i}\rho\omega_0} = \mathrm{e}^{\mathrm{i}\rho\omega_\nu} = \mathrm{e}^{-\mathrm{i}\rho}$ are unbounded on Σ as $b \to \infty$, while the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $1 \le k \le \nu-1$, and $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $\nu+1 \le k \le n-1$, go to 0 very rapidly on Σ as $|\rho| \to \infty$. Clearly the estimates (5.36)–(5.43) are also valid for $\rho \in T_1$ with $|\rho| > R_0$, and in particular, they are valid $\rho \in S_1$ with $|\rho| > R_0$ or for $\rho \in G_1$.

Applying the estimates (5.36)–(5.39) and (5.34) to the representation (5.35) with $\eta = 0$, it is immediate that

$$|u_{10}(t,\rho)| \le 2\mathrm{e}^{b(1-t)}, \tag{5.44}$$

$$|u_{1\nu}(t,\rho)| \le 2\mathrm{e}^{bt}, \tag{5.45}$$

$$|u_{1k}(t,\rho)| \le 2\mathrm{e}^{-t\alpha|\rho|} \le 2, \qquad k = 1,\ldots,\nu-1, \tag{5.46}$$

$$|u_{1k}(t,\rho)| \le 2\mathrm{e}^{-(1-t)\alpha|\rho|} \le 2, \qquad k = \nu+1,\ldots,n-1, \tag{5.47}$$

for $0 \leq t \leq 1$ and for $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_1 (where $b \leq 0$) with $|\rho| > R_1$, we have

$$|u_{1k}(t,\rho)| \leq 2, \qquad 0 \leq t \leq 1, \quad k = 0, 1, \ldots, n-1. \tag{5.48}$$

These bounds will also be used in Chapter 6 to determine the growth rate of the Green's function and the resolvent.

Second, in terms of the modified solutions $u_{1k}(t,\rho)$, $k = 0, 1, \ldots, n-1$, we form the functions

$$M_{1ik}(\rho) := B_i(u_{1k}(\,\cdot\,,\rho)) = \sum_{\eta=0}^{m_i} \alpha_{i\eta} u_{1k}^{(\eta)}(0,\rho) + \sum_{\eta=0}^{m_i} \beta_{i\eta} u_{1k}^{(\eta)}(1,\rho)$$

for $\rho \in T_1$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$. These functions are analytic on the open set G_1. For $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$ define

$$\widetilde{P}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{1k\eta}(0,\rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{1k\eta}(1,\rho)\rho^{-m+\eta}$$

for $\rho \in T_1$ with $|\rho| > R_0$; for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ define

$$\widetilde{P}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{1k\eta}(1,\rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{1k\eta}(0,\rho)\rho^{-m+\eta}$$

for $\rho \in T_1$ with $|\rho| > R_0$; and in terms of these functions and the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ (see equations (3.1) and (3.2)), for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$ define

$$P_{1ik}(\rho) := \widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho), \quad Q_{1ik}(\rho) := \widehat{Q}_{ik}(\rho) + \widetilde{Q}_{1ik}(\rho)$$

for $\rho \in T_1$ with $|\rho| > R_0$. All of these functions are analytic on G_1, and the functions $M_{1ik}(\rho)$ can then be expressed as follows: for $i = 1, \ldots, n$

$$\begin{aligned}
M_{1i0}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}\big[\widehat{P}_{i0}(\rho) + \widetilde{P}_{1i0}(\rho) + \widehat{Q}_{i0}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_0} + \widetilde{Q}_{1i0}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_0}\big],\\
M_{1ik}(\rho) &= \widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} + \widetilde{Q}_{1ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k},\\
&\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \ldots, \nu - 1,\\
M_{1i\nu}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}\big[\widehat{Q}_{i\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_\nu} + \widetilde{Q}_{1i\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_\nu} + \widehat{P}_{i\nu}(\rho) + \widetilde{P}_{1i\nu}(\rho)\big],\\
M_{1ik}(\rho) &= \widehat{Q}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widetilde{Q}_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho),\\
&\qquad\qquad\qquad\qquad\qquad\qquad k = \nu + 1, \ldots, n - 1.
\end{aligned}$$

Thus, for $\rho \in T_1$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$, we have

$$\begin{aligned} M_{1i0}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}\big\{[\widehat{P}_{i0}(\rho) + \widetilde{P}_{1i0}(\rho)] + [\widehat{Q}_{i0}(\rho) + \widetilde{Q}_{1i0}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_0}\big\} \\ &= \mathrm{e}^{-\mathrm{i}\rho}\big[P_{1i0}(\rho) + Q_{1i0}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_0}\big], \end{aligned} \tag{5.49}$$

$$\begin{aligned} M_{1ik}(\rho) &= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho)] + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{1ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ &= P_{1ik}(\rho) + Q_{1ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}, \qquad k = 1, \ldots, \nu - 1, \end{aligned} \tag{5.50}$$

$$\begin{aligned} M_{1i\nu}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}\big\{[\widehat{P}_{i\nu}(\rho) + \widetilde{P}_{1i\nu}(\rho)] + [\widehat{Q}_{i\nu}(\rho) + \widetilde{Q}_{1i\nu}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_\nu}\big\} \\ &= \mathrm{e}^{-\mathrm{i}\rho}\big[P_{1i\nu}(\rho) + Q_{1i\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_\nu}\big], \end{aligned} \tag{5.51}$$

$$\begin{aligned} M_{1ik}(\rho) &= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho)] + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{1ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ &= P_{1ik}(\rho) + Q_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}, \qquad k = \nu + 1, \ldots, n - 1. \end{aligned} \tag{5.52}$$

Third, the *characteristic determinant* of the differential operator L relative to the sector T_1 is the analytic function Δ_1 defined by

$$\Delta_1(\rho) := \det(B_i(u_{1k}(\,\cdot\,,\rho))) = \det(M_{1ik}(\rho)) \quad \text{for } \rho \in G_1.$$

For any complex number $\lambda = \rho^n$ with $\rho \in G_1$, we know that λ is an eigenvalue of L if and only if $\Delta_1(\rho) = 0$. In Chapter 7 we will compute the zeros of Δ_1 in the sector T_1. Applying (5.49)–(5.52), we can express the characteristic determinant in the form

$$\Delta_1(\rho) = \mathrm{e}^{-2\mathrm{i}\rho} \times$$

$$\det\begin{array}{c} \begin{array}{cccc} & 1\le k\le\nu-1 & & \nu+1\le k\le n-1 \end{array} \\ \begin{pmatrix} P_{110}(\rho)+Q_{110}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{11k}(\rho)+Q_{11k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & P_{11\nu}(\rho)+Q_{11\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{11k}(\rho)+Q_{11k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots & \vdots & \vdots \\ P_{1n0}(\rho)+Q_{1n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{1nk}(\rho)+Q_{1nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & P_{1n\nu}(\rho)+Q_{1n\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{1nk}(\rho)+Q_{1nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix} \end{array} \tag{5.53}$$

for $\rho \in G_1$. Cf. equation (3.10) for the approximate characteristic determinant. Observe that the determinant in (5.53) that produces Δ_1 has exactly the same form as the determinant in (5.21) that produces Δ_0. Consequently, in expanding this determinant to obtain Δ_1, we will use the same procedures that were used earlier to obtain Δ_0.

Fourth, the estimates (5.6)–(5.13) and the estimates (5.36)–(5.43) are valid for $\rho \in T_1$ with $|\rho| > R_0$ and for $\rho \in G_1$ as was shown earlier, and the estimates (5.22) for the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ remain valid for $\rho \in \mathbb{C}$ with $|\rho| \ge 1$. In the definitions of the functions $\widetilde{P}_{1ik}(\rho)$, $\widetilde{Q}_{1ik}(\rho)$, we see that the functions $E_{1k\eta}(0,\rho)$, $E_{1k\eta}(1,\rho)$ are bounded for ρ in T_1 with $|\rho| > R_0$ by Theorem 4.4, and that the powers of ρ appearing in the sums are $\rho^{-(m-m_i)}, \rho^{-(m-m_i+1)}, \ldots, \rho^{-m}$. Hence, there exists a constant $\gamma_1 > 0$ such that

$$|\widetilde{P}_{1ik}(\rho)| \le \gamma_1|\rho|^{-(m-m_i)}, \qquad |\widetilde{Q}_{1ik}(\rho)| \le \gamma_1|\rho|^{-(m-m_i)} \tag{5.54}$$

for $\rho \in T_1$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$.

For $i = 1, \ldots, n$ define

$$\begin{aligned} \widetilde{F}_{1ik}(\rho) &:= \widetilde{P}_{1ik}(\rho) + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{1ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k}, \qquad & k = 1, \ldots, \nu-1, \\ \widetilde{F}_{1ik}(\rho) &:= \widetilde{P}_{1ik}(\rho) + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{1ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k}, \qquad & k = \nu+1, \ldots, n-1, \end{aligned}$$

for $\rho \in T_1$ with $|\rho| > R_0$. These functions are analytic on the open set G_1 with

$$|\widetilde{F}_{1ik}(\rho)| \le \gamma_2 |\rho|^{-(m-m_i)} \tag{5.55}$$

for $\rho \in T_1$ with $|\rho| > R_0$, for $i = 1, \ldots, n$, and for $k = 1, \ldots, \nu-1$ and $k = \nu+1, \ldots, n-1$. In terms of these functions we can rewrite (5.53) as

$$\varDelta_1(\rho) = \mathrm{e}^{-2\mathrm{i}\rho} \times$$

$$\det \left(\begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ P_{110}(\rho)+Q_{110}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) & P_{11\nu}(\rho)+Q_{11\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ P_{1n0}(\rho)+Q_{1n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) & P_{1n\nu}(\rho)+Q_{1n\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) \end{array} \right) \tag{5.56}$$

for $\rho \in G_1$.

Fifth, expanding the determinant in (5.56) using the 0th and νth columns, we get

$$\varDelta_1(\rho) = D_{12}(\rho) + D_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + D_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \tag{5.57}$$

for $\rho \in G_1$, where

$$D_{12}(\rho) := \det \left(\begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ \widehat{Q}_{10}(\rho)+\widetilde{Q}_{110}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) & \widehat{Q}_{1\nu}(\rho)+\widetilde{Q}_{11\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho)+\widetilde{Q}_{1n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) & \widehat{Q}_{n\nu}(\rho)+\widetilde{Q}_{1n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) \end{array} \right),$$

$$\begin{aligned} D_{11}(\rho) \;:=\; & \det \left(\begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ \widehat{P}_{10}(\rho)+\widetilde{P}_{110}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) & \widehat{Q}_{1\nu}(\rho)+\widetilde{Q}_{11\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widetilde{P}_{1n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) & \widehat{Q}_{n\nu}(\rho)+\widetilde{Q}_{1n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) \end{array} \right) \\ & + \det \left(\begin{array}{cccc} & {\scriptstyle 1\le k\le \nu-1} & & {\scriptstyle \nu+1\le k\le n-1} \\ \widehat{Q}_{10}(\rho)+\widetilde{Q}_{110}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) & \widehat{P}_{1\nu}(\rho)+\widetilde{P}_{11\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho)+\widetilde{Q}_{1n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) & \widehat{P}_{n\nu}(\rho)+\widetilde{P}_{1n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) \end{array} \right), \end{aligned}$$

and

$$D_{10}(\rho) := \det \begin{array}{c} \begin{array}{cccc} & 1\leq k\leq \nu-1 & & \nu+1\leq k\leq n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)+\widetilde{P}_{110}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) & \widehat{P}_{1\nu}(\rho)+\widetilde{P}_{11\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widetilde{P}_{1n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) & \widehat{P}_{n\nu}(\rho)+\widetilde{P}_{1n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) \end{pmatrix} \end{array}$$

for $\rho \in G_1$. Clearly the functions $D_{1i}(\rho)$, $i = 0, 1, 2$, are analytic on the open set G_1. In these representations we treat the “hat terms” as principal terms and the “tilde terms” as small perturbation terms.

Sixth, if we expand the determinants for the functions $D_{1i}(\rho)$, $i = 0, 1, 2$, using the linearity of the determinant function in its n columns, then we can express these functions in the form

$$D_{1i}(\rho) = \widehat{\pi}_i(\rho) + \widetilde{\varPhi}_{1i}(\rho), \qquad i = 0, 1, 2,$$

for $\rho \in G_1$, where the functions $\widetilde{\varPhi}_{1i}(\rho)$ are analytic on the open set G_1 with

$$|\widetilde{\varPhi}_{1i}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, 2,$$

for $\rho \in G_1$. For the characteristic determinant $\varDelta_1$ we obtain the representation

$$\begin{aligned} \varDelta_1(\rho) = \widehat{\pi}_2(\rho) + \widehat{\pi}_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widehat{\pi}_0(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \\ + \widetilde{\varPhi}_{12}(\rho) + \widetilde{\varPhi}_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\varPhi}_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \end{aligned} \tag{5.58}$$

for $\rho \in G_1$, where the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, are analytic for $\rho \neq 0$ in $\mathbb{C}$ and are the functions introduced earlier, and where the functions $\widetilde{\varPhi}_{1i}(\rho)$, $i = 0, 1, 2$, are analytic on the open set G_1 with

$$|\widetilde{\varPhi}_{1i}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, 2, \tag{5.59}$$

for $\rho \in G_1$.

Seventh, equations (5.29) and (5.30) are still valid for $\rho \neq 0$ in $\mathbb{C}$, and hence, setting

$$\begin{aligned} \varPhi_{12}(\rho) &:= \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widetilde{\varPhi}_{12}(\rho), \\ \varPhi_{11}(\rho) &:= \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widetilde{\varPhi}_{11}(\rho), \\ \varPhi_{10}(\rho) &:= \sum_{\kappa=-n(m-1)}^{-(m-p_0)} c_\kappa(m)\rho^\kappa + \widetilde{\varPhi}_{10}(\rho) \end{aligned}$$

for $\rho \in G_1$, we can rewrite (5.58) in the final form

$$\begin{aligned}\mathit{\Delta}_1(\rho) &= \pi_2(\rho) + \pi_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \\ &\quad + \mathit{\Phi}_{12}(\rho) + \mathit{\Phi}_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \mathit{\Phi}_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho}\end{aligned} \tag{5.60}$$

for $\rho \in G_1$. In this representation the functions $\pi_i(\rho)$, $i = 0, 1, 2$, are given by (5.29), and they are analytic for $\rho \neq 0$ in $\mathbb{C}$; the functions $\mathit{\Phi}_{1i}(\rho)$, $i = 0, 1, 2$, are analytic on the open set G_1 and satisfy the growth rates

$$|\mathit{\Phi}_{1i}(\rho)| \leq \gamma_4 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, 2, \tag{5.61}$$

for $\rho \in G_1$. The representation (5.60) is our working form for the characteristic determinant $\mathit{\Delta}_1$ relative to the sector T_1.

Let us summarize the above results for the characteristic determinant $\mathit{\Delta}_1$ in a theorem.

Theorem 5.2. *Let n be even: $n = 2\nu$. Under the above assumptions* (i)–(vi)*, let $v_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the linearly independent solutions of the differential equation* (2.1) *constructed in Theorem 4.4 for $\rho \in T_1$ with $|\rho| > R_0$, let $u_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the modified solutions of* (2.1) *defined above for $\rho \in T_1$ with $|\rho| > R_0$, and let $\mathit{\Delta}_1$ be the characteristic determinant of the differential operator L given by*

$$\mathit{\Delta}_1(\rho) = \det(B_i(u_{1k}(\,\cdot\,, \rho))) \quad \textit{for } \rho \in G_1,$$

where $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$. Then $\mathit{\Delta}_1$ is analytic on the open set G_1, and $\mathit{\Delta}_1$ has the representation

$$\begin{aligned}\mathit{\Delta}_1(\rho) &= \pi_2(\rho) + \pi_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \\ &\quad + \mathit{\Phi}_{12}(\rho) + \mathit{\Phi}_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \mathit{\Phi}_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho}\end{aligned}$$

for $\rho \in G_1$, where the functions $\mathit{\Phi}_{1i}(\rho)$, $i = 0, 1, 2$, are analytic on G_1 and satisfy the estimates $|\mathit{\Phi}_{1i}(\rho)| \leq \gamma |\rho|^{-(m-p_0)}$ for $\rho \in G_1$ and for $i = 0, 1, 2$.

In Chapter 7 we will calculate the zeros of $\mathit{\Delta}_0$ and $\mathit{\Delta}_1$, showing that they consist of two sequences in the ρ plane. These sequences lie near the positive real axis $\arg \rho = 0$, and they produce two sequences of eigenvalues for the differential operator L that lie near the positive real axis in the λ plane.

5.2 Special Case: $n = 2$

To illustrate these ideas, let us examine the special case $n = 2$, where the differential operator L is determined by the formal differential operator

$$\ell = -\left(\frac{d}{dt}\right)^2 + q(t), \qquad q(t) = a_0(t),$$

and by the boundary values

$$\begin{aligned}B_1(u) &= a_1u'(0) + b_1u'(1) + a_0u(0) + b_0u(1),\\ B_2(u) &= c_1u'(0) + d_1u'(1) + c_0u(0) + d_0u(1).\end{aligned}$$

The boundary coefficient matrix

$$A = \begin{pmatrix} a_1 & b_1 & a_0 & b_0 \\ c_1 & d_1 & c_0 & d_0 \end{pmatrix}$$

is assumed to be in reduced row echelon form with rank 2, and $p_0 = m_1 + m_2$ can have the values 2, 1, or 0. Let $A_{ij}, 1 \leq i < j \leq 4$, denote the determinant of the 2×2 submatrix of A obtained by retaining the ith and jth columns. The six parameters A_{ij} play a very prominent role in the characteristic determinants of L.

Fix the integer m at the value $m = 3$. In Example 2.5 the Birkhoff approximate solutions $z_0(t,\rho)$, $z_1(t,\rho)$ were calculated for $m = 3$:

$$\begin{aligned}z_0(t,\rho) &= \mathrm{e}^{\mathrm{i}\rho t}\Big\{1 + \frac{1}{2\mathrm{i}}\,Q(t)\rho^{-1} + \Big[\frac{1}{4}\,q(t) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(t)^2\Big]\rho^{-2}\Big\},\\ z_1(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho t}\Big\{1 - \frac{1}{2\mathrm{i}}\,Q(t)\rho^{-1} + \Big[\frac{1}{4}\,q(t) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(t)^2\Big]\rho^{-2}\Big\}\end{aligned}$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$, where $Q(t) = \int_0^t q(\xi)\,d\xi$ for $0 \leq t \leq 1$. The corresponding modified Birkhoff approximate solutions are

$$\begin{aligned}y_0(t,\rho) &= \mathrm{e}^{\mathrm{i}\rho t}\Big\{1 + \frac{1}{2\mathrm{i}}\,Q(t)\rho^{-1} + \Big[\frac{1}{4}\,q(t) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(t)^2\Big]\rho^{-2}\Big\},\\ y_1(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho(t-1)}\Big\{1 - \frac{1}{2\mathrm{i}}\,Q(t)\rho^{-1} + \Big[\frac{1}{4}\,q(t) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(t)^2\Big]\rho^{-2}\Big\}\end{aligned} \tag{5.62}$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$, with derivatives

$$\begin{aligned}y_0'(t,\rho) &= \mathrm{i}\rho\mathrm{e}^{\mathrm{i}\rho t}\Big\{1 + \frac{1}{2\mathrm{i}}\,Q(t)\rho^{-1} + \Big[\frac{1}{4}\,q(t) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(t)^2\Big]\rho^{-2}\Big\}\\ &\quad + \mathrm{e}^{\mathrm{i}\rho t}\Big\{\frac{1}{2\mathrm{i}}\,q(t)\rho^{-1} + \Big[\frac{1}{4}\,q'(t) - \frac{1}{4}\,q(t)Q(t)\Big]\rho^{-2}\Big\},\\ y_1'(t,\rho) &= -\mathrm{i}\rho\mathrm{e}^{-\mathrm{i}\rho(t-1)}\Big\{1 - \frac{1}{2\mathrm{i}}\,Q(t)\rho^{-1} + \Big[\frac{1}{4}\,q(t) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(t)^2\Big]\rho^{-2}\Big\}\\ &\quad + \mathrm{e}^{-\mathrm{i}\rho(t-1)}\Big\{-\frac{1}{2\mathrm{i}}\,q(t)\rho^{-1} + \Big[\frac{1}{4}\,q'(t) - \frac{1}{4}\,q(t)Q(t)\Big]\rho^{-2}\Big\}\end{aligned} \tag{5.63}$$

for $0 \leq t \leq 1$ and for $\rho \neq 0$ in $\mathbb{C}$. We can then form the functions

$$B_i(y_k(\,\cdot\,,\rho)) = \widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} \tag{5.64}$$

for $\rho \neq 0$ in $\mathbb{C}$ and for $i = 1, 2$, $k = 0, 1$, and hence, obtain the approximate characteristic determinant:

$$\begin{aligned}\widehat{\Delta}(\rho) &= \det\begin{pmatrix}\widehat{P}_{10}(\rho)+\widehat{Q}_{10}(\rho)e^{i\rho} & \widehat{P}_{11}(\rho)+\widehat{Q}_{11}(\rho)e^{i\rho}\\ \widehat{P}_{20}(\rho)+\widehat{Q}_{20}(\rho)e^{i\rho} & \widehat{P}_{21}(\rho)+\widehat{Q}_{21}(\rho)e^{i\rho}\end{pmatrix}\\ &= \widehat{\pi}_2(\rho)e^{2i\rho}+\widehat{\pi}_1(\rho)e^{i\rho}+\widehat{\pi}_0(\rho)\end{aligned} \tag{5.65}$$

for $\rho \neq 0$ in $\mathbb{C}$. Cf. equation (3.13). The functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, are given by

$$\widehat{\pi}_2(\rho) = \det\begin{pmatrix}\widehat{Q}_{10}(\rho) & \widehat{Q}_{11}(\rho)\\ \widehat{Q}_{20}(\rho) & \widehat{Q}_{21}(\rho)\end{pmatrix}, \quad \widehat{\pi}_0(\rho) = -\widehat{\pi}_2(-\rho) = \det\begin{pmatrix}\widehat{P}_{10}(\rho) & \widehat{P}_{11}(\rho)\\ \widehat{P}_{20}(\rho) & \widehat{P}_{21}(\rho)\end{pmatrix}, \tag{5.66}$$

$$\widehat{\pi}_1(\rho) = \det\begin{pmatrix}\widehat{P}_{10}(\rho) & \widehat{Q}_{11}(\rho)\\ \widehat{P}_{20}(\rho) & \widehat{Q}_{21}(\rho)\end{pmatrix} + \det\begin{pmatrix}\widehat{Q}_{10}(\rho) & \widehat{P}_{11}(\rho)\\ \widehat{Q}_{20}(\rho) & \widehat{P}_{21}(\rho)\end{pmatrix} \tag{5.67}$$

for $\rho \neq 0$ in $\mathbb{C}$.

From equations (3.24), (3.27), and (3.28) we have

$$\begin{aligned}\widehat{\pi}_2(\rho) &= \sum_{\kappa=-(2-p_0)}^{p_0} a_\kappa \rho^\kappa + \sum_{\kappa=-4}^{-(3-p_0)} a_\kappa(3)\rho^\kappa,\\ \widehat{\pi}_1(\rho) &= \sum_{\kappa=-(2-p_0)}^{p_0} b_\kappa \rho^\kappa + \sum_{\kappa=-4}^{-(3-p_0)} b_\kappa(3)\rho^\kappa,\\ \widehat{\pi}_0(\rho) &= \sum_{\kappa=-(2-p_0)}^{p_0} c_\kappa \rho^\kappa + \sum_{\kappa=-4}^{-(3-p_0)} c_\kappa(3)\rho^\kappa\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, with

$$\pi_2(\rho) = \sum_{\kappa=-(2-p_0)}^{p_0} a_\kappa \rho^\kappa, \quad \pi_1(\rho) = \sum_{\kappa=-(2-p_0)}^{p_0} b_\kappa \rho^\kappa, \quad \pi_0(\rho) = \sum_{\kappa=-(2-p_0)}^{p_0} c_\kappa \rho^\kappa \tag{5.68}$$

for $\rho \neq 0$ in $\mathbb{C}$. Therefore, the powers of ρ appearing in the functions $\pi_i(\rho)$, $i = 0, 1, 2$, are $p_0 = 2 : \rho^2, \rho^1, \rho^0$; $p_0 = 1 : \rho^1, \rho^0, \rho^{-1}$; $p_0 = 0 : \rho^0, \rho^{-1}, \rho^{-2}$. It follows that in calculating the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, if we compute the coefficients of the powers $\rho^2, \rho^1, \rho^0, \rho^{-1}, \rho^{-2}$ explicitly, then the corresponding functions $\pi_i(\rho)$, $i = 0, 1, 2$, can be read off.

Next, we calculate the functions appearing in (5.65). Substituting (5.62) and (5.63) into (5.64), the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ are first determined. Then using (5.66) and (5.67), the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, are computed. After a lengthy calculation we arrive at the results:

$$\begin{aligned}
\widehat{\pi}_2(\rho) = &-A_{12}\rho^2 + \Big\{\frac{\mathrm{i}}{2}A_{12}Q(1) + \mathrm{i}(A_{14}+A_{23})\Big\}\rho \\
&+ \Big\{A_{12}\Big[\frac{1}{4}\,q(1) + \frac{3}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big] + \frac{1}{2}(A_{14}+A_{23})Q(1) - A_{34}\Big\}\rho^0 \\
&+ \Big\{-A_{12}\Big[\frac{\mathrm{i}}{4}\,q'(0) + \frac{\mathrm{i}}{4}\,q(0)Q(1) - \frac{\mathrm{i}}{4}\,q'(1) + \frac{\mathrm{i}}{4}\,q(1)Q(1)\Big] \\
&+ \mathrm{i}A_{14}\Big[\frac{1}{4}\,q(1) - \frac{3}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big] \\
&- \mathrm{i}A_{23}\Big[\frac{1}{4}\,q(1) + \frac{1}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big] + \frac{\mathrm{i}}{2}\,A_{34}Q(1)\Big\}\rho^{-1} \\
&+ \Big\{-A_{12}\Big[\frac{1}{8}\,q'(0)Q(1) + \frac{1}{8}\,q(0)q(1) + \frac{1}{8}\,q(0)^2 + \frac{1}{16}\,q(0)Q(1)^2\Big] \\
&+ \mathrm{i}A_{14}\Big[\frac{\mathrm{i}}{4}\,q'(0) + \frac{\mathrm{i}}{4}\,q(0)Q(1)\Big] - \mathrm{i}A_{23}\Big[\frac{\mathrm{i}}{4}\,q'(1) - \frac{\mathrm{i}}{4}\,q(1)Q(1)\Big] \\
&- A_{34}\Big[\frac{1}{4}\,q(1) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big]\Big\}\rho^{-2} \ + \ \rho^{-3}, \rho^{-4} \text{ terms,}
\end{aligned} \tag{5.69}$$

$$\widehat{\pi}_1(\rho) = 2\mathrm{i}(A_{13}+A_{24})\rho - \mathrm{i}(A_{13}+A_{24})q(0)\rho^{-1} \ + \ \rho^{-3}, \rho^{-4} \text{ terms,} \tag{5.70}$$

$$\begin{aligned}
\widehat{\pi}_0(\rho) = &\ A_{12}\rho^2 + \Big\{\frac{\mathrm{i}}{2}A_{12}Q(1) + \mathrm{i}(A_{14}+A_{23})\Big\}\rho \\
&- \Big\{A_{12}\Big[\frac{1}{4}\,q(1) + \frac{3}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big] + \frac{1}{2}(A_{14}+A_{23})Q(1) - A_{34}\Big\}\rho^0 \\
&+ \Big\{-A_{12}\Big[\frac{\mathrm{i}}{4}\,q'(0) + \frac{\mathrm{i}}{4}\,q(0)Q(1) - \frac{\mathrm{i}}{4}\,q'(1) + \frac{\mathrm{i}}{4}\,q(1)Q(1)\Big] \\
&+ \mathrm{i}A_{14}\Big[\frac{1}{4}\,q(1) - \frac{3}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big] \\
&- \mathrm{i}A_{23}\Big[\frac{1}{4}\,q(1) + \frac{1}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big] + \frac{\mathrm{i}}{2}\,A_{34}Q(1)\Big\}\rho^{-1} \\
&+ \Big\{A_{12}\Big[\frac{1}{8}\,q'(0)Q(1) + \frac{1}{8}\,q(0)q(1) + \frac{1}{8}\,q(0)^2 + \frac{1}{16}\,q(0)Q(1)^2\Big] \\
&- \mathrm{i}A_{14}\Big[\frac{\mathrm{i}}{4}\,q'(0) + \frac{\mathrm{i}}{4}\,q(0)Q(1)\Big] + \mathrm{i}A_{23}\Big[\frac{\mathrm{i}}{4}\,q'(1) - \frac{\mathrm{i}}{4}\,q(1)Q(1)\Big] \\
&+ A_{34}\Big[\frac{1}{4}\,q(1) - \frac{1}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big]\Big\}\rho^{-2} \ + \ \rho^{-3}, \rho^{-4} \text{ terms}
\end{aligned} \tag{5.71}$$

for $\rho \neq 0$ in $\mathbb{C}$. Equations (5.69)–(5.71) uniquely determine the key functions $\pi_i(\rho)$, $i = 0, 1, 2$, and then equations (5.31) and (5.60) lead immediately to the asymptotic expansions of the characteristic determinants $\mathit{\Delta}_0$ and $\mathit{\Delta}_1$.

Let us look at the possible forms of the characteristic determinants. We will follow the classification scheme given in the four part series [30, 31, 32, 33]. See [30, p. 280].

Case 1. $A_{12} \neq 0$. For this case the boundary coefficient matrix A must have the form

$$\begin{pmatrix} 1 & 0 & a_0 & b_0 \\ 0 & 1 & c_0 & d_0 \end{pmatrix},$$

and hence, $A_{12} = 1$, $m_1 = m_2 = 1$, and $p_0 = 2$. From (5.69)–(5.71) we see that $p = q = 2$, $a_2 = -c_2 = -1$ (see equation (3.30)), and $b_2 = 0$. In Case 1 the differential operator L is regular. From (5.31) and (5.60) the characteristic determinants have the asymptotic expansions

$$\begin{aligned} \varDelta_0(\rho) &= [-\rho^2 + O(\rho)]\mathrm{e}^{2\mathrm{i}\rho} + O(\rho)\mathrm{e}^{\mathrm{i}\rho} + [\rho^2 + O(\rho)] && \text{for } \rho \in G_0, \\ \varDelta_1(\rho) &= [-\rho^2 + O(\rho)] + O(\rho)\mathrm{e}^{-\mathrm{i}\rho} + [\rho^2 + O(\rho)]\mathrm{e}^{-2\mathrm{i}\rho} && \text{for } \rho \in G_1. \end{aligned} \tag{5.72}$$

Case 2. $A_{12} = 0$, $A_{14} + A_{23} \neq 0$. Here there are three possible forms for the boundary coefficient matrix A:

$$\begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix}, \quad \begin{pmatrix} 1 & b_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix},$$

and for each of these cases it is clear that $p_0 = 1$. The forms

$$\begin{pmatrix} 0 & 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are not possible because of the condition $A_{14} + A_{23} \neq 0$. Also, we have $p = q = 1$, $a_1 = c_1 = \mathrm{i}(A_{14} + A_{23})$, and $b_1 = 2\mathrm{i}(A_{13} + A_{24})$. The differential operator L is again regular in this case. For the characteristic determinants we have the asymptotic expansions

$$\begin{aligned} \varDelta_0(\rho) &= [\mathrm{i}(A_{14} + A_{23})\rho + O(1)]\mathrm{e}^{2\mathrm{i}\rho} + [2\mathrm{i}(A_{13} + A_{24})\rho + O(1)]\mathrm{e}^{\mathrm{i}\rho} \\ &\quad + [\mathrm{i}(A_{14} + A_{23})\rho + O(1)] \qquad \text{for } \rho \in G_0, \\ \varDelta_1(\rho) &= [\mathrm{i}(A_{14} + A_{23})\rho + O(1)] + [2\mathrm{i}(A_{13} + A_{24})\rho + O(1)]\mathrm{e}^{-\mathrm{i}\rho} \\ &\quad + [\mathrm{i}(A_{14} + A_{23})\rho + O(1)]\mathrm{e}^{-2\mathrm{i}\rho} \quad \text{for } \rho \in G_1. \end{aligned} \tag{5.73}$$

Case 3. $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} \neq 0$, $A_{13} + A_{24} = 0$. The boundary coefficient matrix A must have either the form

$$\begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix},$$

where $p_0 = 1$, or the simple form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $p_0 = 0$ and the boundary conditions are Dirichlet boundary conditions; the forms

$$\begin{pmatrix} 1 & b_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are impossible because of the conditions on the A_{ij}. From (5.69)–(5.71) we see that $p = q = 0$, $a_0 = -c_0 = -A_{34}$, and $b_0 = 0$. Thus, the differential operator L where $p_0 = 1$ is simply irregular, while the differential operator L where $p_0 = 0$ (Dirichlet) is regular. For the characteristic determinants we obtain the asymptotic expansions

$$\begin{aligned} \Delta_0(\rho) &= [-A_{34} + O(\rho^{-1})]\mathrm{e}^{2\mathrm{i}\rho} + O(\rho^{-1})\mathrm{e}^{\mathrm{i}\rho} + [A_{34} + O(\rho^{-1})] \qquad \text{for } \rho \in G_0, \\ \Delta_1(\rho) &= [-A_{34} + O(\rho^{-1})] + O(\rho^{-1})\mathrm{e}^{-\mathrm{i}\rho} + [A_{34} + O(\rho^{-1})]\mathrm{e}^{-2\mathrm{i}\rho} \quad \text{for } \rho \in G_1. \end{aligned} \tag{5.74}$$

Case 4. $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} \neq 0$, $A_{13} + A_{24} \neq 0$. For the fourth case the boundary coefficient matrix A has one of the two forms

$$\begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so $p_0 = 1$. Again the forms

$$\begin{pmatrix} 1 & b_1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are impossible because of the conditions on the A_{ij}. From (5.69)–(5.71) we see that $p = 0$, $q = 1$, $a_0 = -c_0 = -A_{34}$, and $b_1 = 2\mathrm{i}(A_{13} + A_{24})$, and hence, the differential operator L is always simply irregular in this case. For the characteristic determinants we have the representations

$$\begin{aligned} \Delta_0(\rho) &= [-A_{34} + O(\rho^{-1})]\mathrm{e}^{2\mathrm{i}\rho} + [2\mathrm{i}(A_{13} + A_{24})\rho + O(\rho^{-1})]\mathrm{e}^{\mathrm{i}\rho} \\ &\qquad + [A_{34} + O(\rho^{-1})] \qquad \text{for } \rho \in G_0, \\ \Delta_1(\rho) &= [-A_{34} + O(\rho^{-1})] + [2\mathrm{i}(A_{13} + A_{24})\rho + O(\rho^{-1})]\mathrm{e}^{-\mathrm{i}\rho} \\ &\qquad + [A_{34} + O(\rho^{-1})]\mathrm{e}^{-2\mathrm{i}\rho} \quad \text{for } \rho \in G_1. \end{aligned} \tag{5.75}$$

Case 5. $A_{12} = 0$, $A_{14} + A_{23} = 0$, $A_{34} = 0$. In this last case the boundary coefficient matrix A must have the same form as in Case 4:

$$\begin{pmatrix} 1 & b_1 & 0 & b_0 \\ 0 & 0 & 1 & d_0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 & a_0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and hence, $p_0 = 1$. The functions $\pi_i(\rho)$, $i = 0, 1, 2$, are extracted from (5.69)–(5.71) by keeping only the powers $\rho^1, \rho^0, \rho^{-1}$. Thus,

$$\begin{aligned}\pi_2(\rho) &= \Big\{\mathrm{i}A_{14}\Big[\frac{1}{4}\,q(1) - \frac{3}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big]\\&\qquad - \mathrm{i}A_{23}\Big[\frac{1}{4}\,q(1) + \frac{1}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big]\Big\}\rho^{-1},\\ \pi_1(\rho) &= 2\mathrm{i}(A_{13} + A_{24})\rho - \mathrm{i}(A_{13} + A_{24})q(0)\rho^{-1},\\ \pi_0(\rho) &= \Big\{\mathrm{i}A_{14}\Big[\frac{1}{4}\,q(1) - \frac{3}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big]\\&\qquad - \mathrm{i}A_{23}\Big[\frac{1}{4}\,q(1) + \frac{1}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big]\Big\}\rho^{-1}\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, and for the invariants we have $a_1 = c_1 = 0$, $a_0 = -c_0 = 0$,

$$\begin{aligned}a_{-1} = c_{-1} = \mathrm{i}A_{14}&\Big[\frac{1}{4}\,q(1) - \frac{3}{4}\,q(0) - \frac{1}{8}\,Q(1)^2\Big]\\ &- \mathrm{i}A_{23}\Big[\frac{1}{4}\,q(1) + \frac{1}{4}\,q(0) + \frac{1}{8}\,Q(1)^2\Big],\end{aligned}$$

and

$$b_1 = 2\mathrm{i}(A_{13} + A_{24}), \quad b_0 = 0, \quad b_{-1} = -\mathrm{i}(A_{13} + A_{24})q(0).$$

This shows that the differential operator L is irregular. More precisely, if $a_{-1} \neq 0$, then $p = -1$, $q = 1$ or $q = -1$, and L is simply irregular. On the other hand, if $a_{-1} = 0$, then we can not determine whether L is simply irregular or degenerate irregular. To get a definitive classification, we must use a larger integer m, thus enlarging our "window" for viewing the constants a_κ, b_κ, c_κ. In the simply irregular case where $a_{-1} \neq 0$, the characteristic determinants have the form

$$\begin{aligned}\Delta_0(\rho) &= [a_{-1}\rho^{-1} + O(\rho^{-2})]\mathrm{e}^{2\mathrm{i}\rho} + [b_1\rho + b_{-1}\rho^{-1} + O(\rho^{-2})]\mathrm{e}^{\mathrm{i}\rho}\\&\qquad + [c_{-1}\rho^{-1} + O(\rho^{-2})] \qquad \text{for } \rho \in G_0,\\ \Delta_1(\rho) &= [a_{-1}\rho^{-1} + O(\rho^{-2})] + [b_1\rho + b_{-1}\rho^{-1} + O(\rho^{-2})]\mathrm{e}^{-\mathrm{i}\rho}\\&\qquad + [c_{-1}\rho^{-1} + O(\rho^{-2})]\mathrm{e}^{-2\mathrm{i}\rho} \qquad \text{for } \rho \in G_1.\end{aligned} \tag{5.76}$$

Example 5.3. Consider the 2nd order differential operator L determined by the formal differential operator $\ell = -(d/dt)^2 + q(t)$ and the boundary values

$$B_1(u) = u'(0) + u'(1), \qquad B_2(u) = u(0) - u(1),$$

where the boundary coefficient matrix is

$$A = \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\end{pmatrix}.$$

Clearly $A_{12} = 0$, $A_{13} = 1$, $A_{14} = -1$, $A_{23} = 1$, $A_{24} = -1$, $A_{34} = 0$, and

$$A_{12} = 0, \quad A_{14} + A_{23} = 0, \quad A_{34} = 0, \quad A_{13} + A_{24} = 0.$$

Hence, L belongs to Case 5.

Assume that $q(0) \neq q(1)$. Then

$$a_{-1} = c_{-1} = -\frac{\mathrm{i}}{2}\left[q(1) - q(0)\right] \neq 0, \qquad b_1 = b_{-1} = 0,$$

and hence, $p = q = -1$, the differential operator L is simply irregular, and the characteristic determinants have the asymptotic expansions

$$\begin{aligned} \Delta_0(\rho) &= \Big\{-\frac{\mathrm{i}}{2}\left[q(1) - q(0)\right]\rho^{-1} + O(\rho^{-2})\Big\}\mathrm{e}^{2\mathrm{i}\rho} + O(\rho^{-2})\mathrm{e}^{\mathrm{i}\rho} \\ &\quad + \Big\{-\frac{\mathrm{i}}{2}\left[q(1) - q(0)\right]\rho^{-1} + O(\rho^{-2})\Big\} \qquad \text{for } \rho \in G_0, \\ \Delta_1(\rho) &= \Big\{-\frac{\mathrm{i}}{2}\left[q(1) - q(0)\right]\rho^{-1} + O(\rho^{-2})\Big\} + O(\rho^{-2})\mathrm{e}^{-\mathrm{i}\rho} \\ &\quad + \Big\{-\frac{\mathrm{i}}{2}\left[q(1) - q(0)\right]\rho^{-1} + O(\rho^{-2})\Big\}\mathrm{e}^{-2\mathrm{i}\rho} \quad \text{for } \rho \in G_1. \end{aligned} \tag{5.77}$$

From these representations it follows that the spectrum $\sigma(L)$ consists of two sequences of points $\lambda'_k = (\rho'_k)^2$, $k = k_0, k_0 + 1, \ldots$, and $\lambda''_k = (\rho''_k)^2$, $k = k_0, k_0 + 1, \ldots$, plus a finite number of additional points, where

$$\begin{aligned} \rho'_k &= (2\pi k + \pi/2) + \epsilon'_k, \quad k = k_0, k_0 + 1, \ldots, \\ \rho''_k &= (2\pi k - \pi/2) + \epsilon''_k, \quad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

with $|\epsilon'_k| \leq \gamma/k$ and $|\epsilon''_k| \leq \gamma/k$ for $k = k_0, k_0 + 1, \ldots$. See Chapter 7. On the other hand, for the principal part of L, which is the differential operator T determined by $\tau = -(d/dt)^2$ and by the same boundary values B_1, B_2, we know that the characteristic determinant is given by $\Delta(\rho) \equiv 0$ and that the spectrum is

$$\sigma(T) = \mathbb{C}.$$

See Example 10.2 or [34, p. 28]. Consequently, the differential operators L and T have very different spectral properties. Some additional remarks on Case 5 are given in Section 8 of [30].

5.3 The Characteristic Determinant for n Odd

Assume that n is odd: $n = 2\nu - 1 \geq 3$, and consider the sectors

$$\begin{aligned} S_0\colon &\ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{2n} \leq \theta \leq \frac{\pi}{2n}, \\ S_1\colon &\ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \pi - \frac{\pi}{2n} \leq \theta \leq \pi + \frac{\pi}{2n}, \end{aligned}$$

and the corresponding translated sectors T_0, T_1. The development of the characteristic determinants for the case n odd is almost identical to the development for the case n even. Consequently, we will indicate only the highlights of this theory.

For each $\rho \in T_0$ with $|\rho| > R_0$ let $v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \dots, v_{0\,n-1}(\,\cdot\,,\rho)$ be the linearly independent solutions of the differential equation (2.1) constructed in Theorem 4.6. The function $v_{0k}(\,\cdot\,,\rho)$ belongs to $H^n[0,1]$, and $v_{0k}(\,\cdot\,,\rho)$ and its derivatives have the asymptotic expansions

$$v_{0k}^{(\eta)}(t,\rho) = z_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\eta}(t,\rho)\rho^{-m+\eta} \tag{5.78}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \dots, n-1$. Similarly, for each $\rho \in T_1$ with $|\rho| > R_0$ let $v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \dots, v_{1\,n-1}(\,\cdot\,,\rho)$ be the linearly independent solutions of the differential equation (2.1) constructed in Theorem 4.7. Again each function $v_{1k}(\,\cdot\,,\rho)$ belongs to $H^n[0,1]$ with

$$v_{1k}^{(\eta)}(t,\rho) = z_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{1k\eta}(t,\rho)\rho^{-m+\eta} \tag{5.79}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \dots, n-1$. In these expansions the function $z_k(t,\rho)$ is the Birkhoff approximate solution constructed in Chapter 2.

Let us begin by developing the characteristic determinant of the differential operator L for ρ belonging to the sector T_0. From Chapter 4 we have

$$\begin{aligned} v_{0k}^{(\eta)}(t,\rho) &= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k t}\Big[(\mathrm{i}\omega_k)^\eta + \sum_{\ell=1}^{m+\eta-1} f_{k\eta\ell}(t)\rho^{-\ell} + E_{0k\eta}(t,\rho)\rho^{-m}\Big] \\ &:= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^\eta + F_{0k\eta}(t,\rho)\big] \end{aligned} \tag{5.80}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \eta = 0, 1, \dots, n-1$, with $F_{0k\eta}(t,\rho) \to 0$ uniformly on $[0,1] \times T_0$ as $|\rho| \to \infty$ for $k, \eta = 0, 1, \dots, n-1$. Choose a constant $R_1 \ge R_0$ such that

$$|F_{0k\eta}(t,\rho)| \le 1 \tag{5.81}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_1$, and for $k, \eta = 0, 1, \dots, n-1$. Relative to the sector T_0, form the *modified solutions* of the differential equation (2.1):

$$\begin{aligned} u_{0k}(t,\rho) &:= v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad k = 0, 1, \dots, \nu-1, \\ u_{0k}(t,\rho) &:= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{0k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} E_{0k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad\qquad\qquad k = \nu, \dots, n-1, \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$, where the $y_k(t,\rho)$ are the functions defined at the beginning of Chapter 3. These modified solutions form a basis for the solution space of the differential equation (2.1) for $\rho \in T_0$ with $|\rho| > R_0$, and their derivatives can be expressed as

$$\begin{aligned} u_{0k}^{(\eta)}(t,\rho) &= y_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{0k\eta}(t,\rho)\rho^{-m+\eta} \\ &= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^\eta + F_{0k\eta}(t,\rho)\big], \qquad k = 0, 1, \dots, \nu-1, \\ u_{0k}^{(\eta)}(t,\rho) &= y_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} E_{0k\eta}(t,\rho)\rho^{-m+\eta} \\ &= \rho^\eta \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}\big[(\mathrm{i}\omega_k)^\eta + F_{0k\eta}(t,\rho)\big], \quad k = \nu, \dots, n-1, \end{aligned} \tag{5.82}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $\eta = 0, 1, \ldots, n-1$. Applying Theorem 4.6, for fixed $t \in [0,1]$ the functions $u_{0k}^{(\eta)}(t,\rho)$ and $F_{0k\eta}(t,\rho)$ are analytic functions of the ρ variable on the open set

$$G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}.$$

Hence, the boundary data $u_{0k}^{(\eta)}(0,\rho)$, $u_{0k}^{(\eta)}(1,\rho)$, $k, \eta = 0, 1, \ldots, n-1$, consists of functions of ρ that are analytic on G_0.

Next, fix σ_0 with $0 < \sigma_0 < \pi/10$, set $\alpha := \sin(\sigma_0/n) > 0$, and form the sector

$$\Sigma_0\colon \text{ all } \rho = |\rho| \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} + \frac{\sigma_0}{n} \le \theta \le \frac{\pi}{n} - \frac{\sigma_0}{n}$$

in the ρ plane. Clearly any point ρ in T_0 with $|\rho|$ sufficiently large lies in the sector Σ_0. Without loss of generality we can assume that the constant $R_0 > 0$ chosen earlier (see Theorem 4.6) has the additional property that $\rho \in T_0$ with $|\rho| > R_0$ implies $\rho \in \Sigma_0$. Now take any point $\rho = a + \mathrm{i}b \in \Sigma_0$. Then arguing as in the even order case, we obtain the following crucial estimates:

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0 t}\right| = \left|\mathrm{e}^{\mathrm{i}\rho t}\right| = \mathrm{e}^{-bt}, \qquad 0 \le t \le 1, \tag{5.83}$$

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k t}| \le \mathrm{e}^{-t\alpha|\rho|} \le 1, \qquad 0 \le t \le 1, \quad k = 1, \ldots, \nu - 1, \tag{5.84}$$

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}| \le \mathrm{e}^{-(1-t)\alpha|\rho|} \le 1, \qquad 0 \le t \le 1, \quad k = \nu, \ldots, n-1. \tag{5.85}$$

It is immediate that

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0}\right| = \left|\mathrm{e}^{\mathrm{i}\rho}\right| = \mathrm{e}^{-b}, \tag{5.86}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k}\right| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k = 1, \ldots, \nu - 1, \tag{5.87}$$

$$\left|\mathrm{e}^{-\mathrm{i}\rho\omega_k}\right| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k = \nu, \ldots, n-1, \tag{5.88}$$

for all $\rho = a + \mathrm{i}b \in \Sigma_0$. Thus, the exponential $\mathrm{e}^{\mathrm{i}\rho\omega_0} = \mathrm{e}^{\mathrm{i}\rho}$ is unbounded on the sector Σ_0 as $b \to -\infty$, while the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $1 \le k \le \nu - 1$, and $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $\nu \le k \le n-1$, go to zero very rapidly on Σ_0 as $|\rho| \to \infty$. The estimates (5.83)–(5.88) are also valid for $\rho \in T_0$ with $|\rho| > R_0$, and in particular, they are valid for $\rho \in S_0$ with $|\rho| > R_0$ or for $\rho \in G_0$.

Applying the estimates (5.83)–(5.85) and (5.81) to the representation (5.82) with $\eta = 0$, it is immediate that

$$|u_{00}(t,\rho)| \le 2\mathrm{e}^{-bt}, \tag{5.89}$$

$$|u_{0k}(t,\rho)| \le 2\mathrm{e}^{-t\alpha|\rho|} \le 2, \qquad k = 1, \ldots, \nu - 1, \tag{5.90}$$

$$|u_{0k}(t,\rho)| \le 2\mathrm{e}^{-(1-t)\alpha|\rho|} \le 2, \qquad k = \nu, \ldots, n-1, \tag{5.91}$$

for $0 \le t \le 1$ and for $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_0 with $|\rho| > R_1$ and $b \ge 0$, we have

$$|u_{0k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 0, 1, \ldots, n-1. \tag{5.92}$$

Using the modified solutions $u_{0k}(t,\rho)$, $k=0,1,\dots,n-1$, form the functions

$$M_{0ik}(\rho) := B_i(u_{0k}(\,\cdot\,,\rho)) = \sum_{\eta=0}^{m_i} \alpha_{i\eta} u_{0k}^{(\eta)}(0,\rho) + \sum_{\eta=0}^{m_i} \beta_{i\eta} u_{0k}^{(\eta)}(1,\rho)$$

for $\rho \in T_0$ with $|\rho| > R_0$ and for $i=1,\dots,n$ and $k=0,1,\dots,n-1$. These functions are analytic on the open set G_0. For $i=1,\dots,n$ and $k=0,1,\dots,\nu-1$ define

$$\widetilde{P}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{0k\eta}(0,\rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{0k\eta}(1,\rho)\rho^{-m+\eta}$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $i=1,\dots,n$ and $k=\nu,\dots,n-1$ define

$$\widetilde{P}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{0k\eta}(1,\rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{0ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{0k\eta}(0,\rho)\rho^{-m+\eta}$$

for $\rho \in T_0$ with $|\rho| > R_0$; and in terms of these functions and the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ appearing in equations (3.1) and (3.2), for $i=1,\dots,n$ and $k=0,1,\dots,n-1$ define

$$P_{0ik}(\rho) := \widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho), \quad Q_{0ik}(\rho) := \widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)$$

for $\rho \in T_0$ with $|\rho| > R_0$. Clearly these functions are analytic on the open set G_0. Using these functions, we can express the functions $M_{0ik}(\rho)$ as follows: for $i=1,\dots,n$

$$\begin{aligned} M_{0ik}(\rho) &= \widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} + \widetilde{Q}_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}, \\ &\qquad\qquad k=0,1,\dots,\nu-1, \\ M_{0ik}(\rho) &= \widehat{Q}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widetilde{Q}_{0ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho), \\ &\qquad\qquad k=\nu,\dots,n-1. \end{aligned}$$

Therefore, for $\rho \in T_0$ with $|\rho| > R_0$ and for $i=1,\dots,n$, we have

$$\begin{aligned} M_{0ik}(\rho) &= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho)] + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ &= P_{0ik}(\rho) + Q_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}, \qquad k=0,1,\dots,\nu-1, \end{aligned} \tag{5.93}$$

$$\begin{aligned} M_{0ik}(\rho) &= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{0ik}(\rho)] + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ &= P_{0ik}(\rho) + Q_{0ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}, \qquad k=\nu,\dots,n-1. \end{aligned} \tag{5.94}$$

The *characteristic determinant* of the differential operator L relative to the sector T_0 is the analytic function $\varDelta_0$ defined by

$$\varDelta_0(\rho) := \det(B_i(u_{0k}(\,\cdot\,,\rho))) = \det(M_{0ik}(\rho)) \quad \text{for } \rho \in G_0.$$

For any complex number $\lambda = \rho^n$ with $\rho \in G_0$, we know that λ is an eigenvalue of L if and only if $\varDelta_0(\rho) = 0$. Applying (5.93) and (5.94), we can express the characteristic determinant in the form

$$\varDelta_0(\rho) = \det \begin{array}{c} \begin{array}{ccc} & 1\le k\le \nu-1 & \nu\le k\le n-1 \end{array} \\ \begin{pmatrix} P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{01k}(\rho)+Q_{01k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & P_{01k}(\rho)+Q_{01k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots & \vdots \\ P_{0n0}(\rho)+Q_{0n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & P_{0nk}(\rho)+Q_{0nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} & P_{0nk}(\rho)+Q_{0nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix} \end{array} \tag{5.95}$$

for $\rho \in G_0$. Cf. equation (3.11) for the approximate characteristic determinant.

For the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$, we obtain the estimates

$$|\widehat{P}_{ik}(\rho)| \le \gamma_0 |\rho|^{m_i}, \qquad |\widehat{Q}_{ik}(\rho)| \le \gamma_0 |\rho|^{m_i} \tag{5.96}$$

for $\rho \in \mathbb{C}$ with $|\rho| \ge 1$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$, while for the functions $\widetilde{P}_{0ik}(\rho)$, $\widetilde{Q}_{0ik}(\rho)$, we obtain the analogous estimates

$$|\widetilde{P}_{0ik}(\rho)| \le \gamma_1 |\rho|^{-(m-m_i)}, \qquad |\widetilde{Q}_{0ik}(\rho)| \le \gamma_1 |\rho|^{-(m-m_i)} \tag{5.97}$$

for $\rho \in T_0$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$. For $i = 1, \ldots, n$ define

$$\begin{aligned} \widetilde{F}_{0ik}(\rho) &:= \widetilde{P}_{0ik}(\rho) + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k}, \qquad && k = 1, \ldots, \nu-1, \\ \widetilde{F}_{0ik}(\rho) &:= \widetilde{P}_{0ik}(\rho) + [\widehat{Q}_{ik}(\rho) + \widetilde{Q}_{0ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k}, \qquad && k = \nu, \ldots, n-1, \end{aligned}$$

for $\rho \in T_0$ with $|\rho| > R_0$. These functions are analytic on the open set G_0, and for them the following estimates are obtained:

$$|\widetilde{F}_{0ik}(\rho)| \le \gamma_2 |\rho|^{-(m-m_i)} \tag{5.98}$$

for $\rho \in T_0$ with $|\rho| > R_0$, for $i = 1, \ldots, n$, and for $k = 1, \ldots, \nu-1$ and $k = \nu, \ldots, n-1$. In terms of these functions we can rewrite the representation (5.95) of the characteristic determinant in the form

$$\varDelta_0(\rho) = \det \begin{array}{c} \begin{array}{ccc} & 1\le k\le \nu-1 & \nu\le k\le n-1 \end{array} \\ \begin{pmatrix} P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots \\ P_{0n0}(\rho)+Q_{0n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{pmatrix} \end{array} \tag{5.99}$$

for $\rho \in G_0$.

We now proceed to expand the determinant for $\varDelta_0(\rho)$ that appears in equation (5.99). These expansions parallel the ones used earlier in equations

(3.11), (3.33), and (3.34) for the approximate characteristic determinant $\widehat{\Delta}(\rho)$, and in fact, the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1$, that were introduced in Chapter 3 will also appear in these new expansions for $\Delta_0(\rho)$.

Indeed, suppose we expand the determinant in (5.99) using the linearity of the determinant in the 0th column:

$$\Delta_0(\rho) = D_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + D_{00}(\rho) \tag{5.100}$$

for $\rho \in G_0$, where

$$D_{01}(\rho) := \det \begin{array}{c} \begin{array}{ccc} \phantom{\widehat{Q}_{10}(\rho)+\widetilde{Q}_{010}(\rho)} & 1\leq k\leq \nu-1 & \nu\leq k\leq n-1 \end{array} \\ \begin{pmatrix} \widehat{Q}_{10}(\rho)+\widetilde{Q}_{010}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho)+\widetilde{Q}_{0n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{pmatrix} \end{array}$$

and

$$D_{00}(\rho) := \det \begin{array}{c} \begin{array}{ccc} \phantom{\widehat{P}_{10}(\rho)+\widetilde{P}_{010}(\rho)} & 1\leq k\leq \nu-1 & \nu\leq k\leq n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)+\widetilde{P}_{010}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widetilde{P}_{0n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{pmatrix} \end{array}$$

for $\rho \in G_0$. Clearly the functions $D_{0i}(\rho)$, $i = 0, 1$, are analytic on the open set G_0.

If we expand $D_{01}(\rho)$ using the linearity of the determinant in its n columns, then it can be expressed in the form

$$D_{01}(\rho) = \widehat{\pi}_1(\rho) + \widetilde{\Phi}_{01}(\rho)$$

for $\rho \in G_0$, where

$$\widehat{\pi}_1(\rho) = \det \begin{pmatrix} \widehat{Q}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{P}_{1\nu}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{Q}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{P}_{n\nu}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix}$$

is the function introduced in Chapter 3, and where $\widetilde{\Phi}_{01}(\rho)$ is the sum of $2^n - 1$ determinants, with each determinant containing at least one column consisting of the functions $\widetilde{Q}_{0i0}(\rho)$, $i = 1, \ldots, n$, or of the functions $\widetilde{F}_{0ik}(\rho)$, $i = 1, \ldots, n$. The function $\widetilde{\Phi}_{01}(\rho)$ is analytic on the open set G_0 and satisfies the estimate $|\widetilde{\Phi}_{01}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}$ for $\rho \in G_0$. Similarly, the function $D_{00}(\rho)$ can be expressed as

$$D_{00}(\rho) = \widehat{\pi}_0(\rho) + \widetilde{\Phi}_{00}(\rho)$$

for $\rho \in G_0$, where

$$\widehat{\pi}_0(\rho) = \det \begin{pmatrix} \widehat{P}_{10}(\rho) & \widehat{P}_{11}(\rho) & \cdots & \widehat{P}_{1\nu-1}(\rho) & \widehat{P}_{1\nu}(\rho) & \cdots & \widehat{P}_{1n-1}(\rho) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \widehat{P}_{n0}(\rho) & \widehat{P}_{n1}(\rho) & \cdots & \widehat{P}_{n\nu-1}(\rho) & \widehat{P}_{n\nu}(\rho) & \cdots & \widehat{P}_{nn-1}(\rho) \end{pmatrix}$$

is the function defined in Chapter 3 and where the function $\widetilde{\Phi}_{00}(\rho)$ is analytic on G_0 with $|\widetilde{\Phi}_{00}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}$ for $\rho \in G_0$.

Combining the above results, we obtain our principal representation of the characteristic determinant Δ_0 relative to the sector T_0:

$$\Delta_0(\rho) = \widehat{\pi}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widehat{\pi}_0(\rho) + \widetilde{\Phi}_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widetilde{\Phi}_{00}(\rho) \tag{5.101}$$

for $\rho \in G_0$, where the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1$, are analytic for $\rho \neq 0$ in $\mathbb{C}$, and the functions $\widetilde{\Phi}_{0i}(\rho)$, $i = 0, 1$, are analytic on the open set G_0 with

$$|\widetilde{\Phi}_{0i}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, \tag{5.102}$$

for $\rho \in G_0$. The functions $\widehat{\pi}_i(\rho)$, $i = 0, 1$, are the functions introduced in Chapter 3 in our formation of the approximate characteristic determinant $\widehat{\Delta}(\rho)$; the functions $\widetilde{\Phi}_{0i}(\rho)$, $i = 0, 1$, contain all the perturbation terms that are produced in constructing the actual solutions of the differential equation (2.1). Compare equation (5.101) to equation (3.34).

Let us recall some of the results from Chapter 3 for the case n odd. First, we are assuming that the differential operator L is either regular or simply irregular. This identifies the integer p with $a_p \neq 0$ and $a_\kappa = 0$ for $\kappa = p+1, \ldots, p_0$, and the integer q with $b_q \neq 0$ and $b_\kappa = 0$ for $\kappa = q+1, \ldots, p_0$. Second, the translated sectors T_0 and T_1 are formed subject to condition (3.51) in case $p = q$. Third, the integer m is fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$, and then the corresponding Birkhoff approximate solutions $z_k(t, \rho)$, $k = 0, 1, \ldots, n-1$, are formed, and the modified Birkhoff approximate solutions $y_k(t, \rho)$, $k = 0, 1, \ldots, n-1$, are determined. Fourth, the functions $\pi_i(\rho)$, $i = 0, 1$, are defined by

$$\pi_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \quad \pi_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa \tag{5.103}$$

for $\rho \neq 0$ in $\mathbb{C}$. From equations (3.37) and (3.38) it is immediate that

$$\widehat{\pi}_1(\rho) = \pi_1(\rho) + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa, \quad \widehat{\pi}_0(\rho) = \pi_0(\rho) + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa \tag{5.104}$$

for $\rho \neq 0$ in $\mathbb{C}$.

Finally, in terms of (5.101) and (5.104) we define the functions

$$\Phi_{01}(\rho) := \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa(m)\rho^\kappa + \widetilde{\Phi}_{01}(\rho),$$

$$\Phi_{00}(\rho) := \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa(m)\rho^\kappa + \widetilde{\Phi}_{00}(\rho)$$

for $\rho \in G_0$. These functions are analytic on the open set G_0, and using them we can rewrite (5.101) as

$$\Delta_0(\rho) = \pi_1(\rho)e^{i\rho} + \pi_0(\rho) + \Phi_{01}(\rho)e^{i\rho} + \Phi_{00}(\rho) \tag{5.105}$$

for $\rho \in G_0$. In equation (5.105) the functions $\pi_i(\rho)$, $i = 0, 1$, are given by (5.103) and are analytic for $\rho \neq 0$ in $\mathbb{C}$; the functions $\Phi_{0i}(\rho)$, $i = 0, 1$, are analytic on the open set G_0 and satisfy the growth rates

$$|\Phi_{0i}(\rho)| \leq \gamma_4 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, \tag{5.106}$$

for $\rho \in G_0$.

The representation (5.105) is our working form for the characteristic determinant Δ_0 relative to the sector T_0. Compare the representation (5.105) for the characteristic determinant Δ_0 to the representation (3.54) for the approximate characteristic determinant $\widehat{\Delta}$.

The above results are summarized in the following theorem. We assume the conditions set forth in Chapter 3: (i) $n = 2\nu - 1$ is odd; (ii) the differential operator L is either regular or simply irregular; (iii) the integers p and q have been determined with $-\infty < p, q \leq p_0$ and with $a_p \neq 0$, $b_q \neq 0$, and $a_\kappa = 0$ for $\kappa = p+1, \ldots, p_0$ and $b_\kappa = 0$ for $\kappa = q+1, \ldots, p_0$; (iv) the translated sectors T_0 and T_1 have been chosen; (v) the integer m has been fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$; and (vi) the functions $\pi_i(\rho)$, $i = 0, 1$, have been determined as per Chapter 3 or equation (5.103).

Theorem 5.4. *Let n be odd: $n = 2\nu - 1$. Under the above assumptions* (i)–(vi)*, let $v_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the linearly independent solutions of the differential equation* (2.1) *constructed in Theorem 4.6 for $\rho \in T_0$ with $|\rho| > R_0$, let $u_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the modified solutions of* (2.1) *defined above for $\rho \in T_0$ with $|\rho| > R_0$, and let Δ_0 be the characteristic determinant of the differential operator L given by*

$$\Delta_0(\rho) = \det(B_i(u_{0k}(\,\cdot\,, \rho))) \quad \textit{for } \rho \in G_0,$$

where $G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$. Then Δ_0 is analytic on the open set G_0, and Δ_0 has the representation

$$\Delta_0(\rho) = \pi_1(\rho)e^{i\rho} + \pi_0(\rho) + \Phi_{01}(\rho)e^{i\rho} + \Phi_{00}(\rho)$$

for $\rho \in G_0$, where the functions $\Phi_{0i}(\rho)$, $i = 0, 1$, are analytic on G_0 and satisfy the estimates $|\Phi_{0i}(\rho)| \leq \gamma |\rho|^{-(m-p_0)}$ for $\rho \in G_0$ and for $i = 0, 1$.

In the final part of this section we form the characteristic determinant on the sector T_1. The starting point for the discussion is the set of functions $v_{1k}(\,\cdot\,,\rho)$, $k=0,1,\ldots,n-1$, which form a basis for the solution space of the differential equation (2.1) for $\rho \in T_1$ with $|\rho| > R_0$. Using the representation of $z_k^{(\eta)}(t,\rho)$ given in Chapter 4, we can rewrite (5.79) in the form

$$\begin{aligned} v_{1k}^{(\eta)}(t,\rho) &= \rho^{\eta} \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\eta} + \sum_{\ell=1}^{m+\eta-1} f_{k\eta\ell}(t)\rho^{-\ell} + E_{1k\eta}(t,\rho)\rho^{-m}\big] \\ &:= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\eta} + F_{1k\eta}(t,\rho)\big] \end{aligned} \tag{5.107}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k,\eta = 0,1,\ldots,n-1$, with $F_{1k\eta}(t,\rho) \to 0$ uniformly on $[0,1]\times T_1$ as $|\rho| \to \infty$ for $k,\eta = 0,1,\ldots,n-1$. Without loss of generality we can assume that the constant R_1 chosen earlier in this section satisfies the additional condition that

$$|F_{1k\eta}(t,\rho)| \le 1 \tag{5.108}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_1$, and for $k,\eta = 0,1,\ldots,n-1$. Let us now proceed to construct the characteristic determinant on the sector T_1.

First, we introduce the *modified solutions* of differential equation (2.1) relative to the sector T_1:

$$\begin{aligned} u_{1k}(t,\rho) &:= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{1k}(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho\omega_k} y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}E_{1k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad k = 0,1,\ldots,\nu-1, \\ u_{1k}(t,\rho) &:= v_{1k}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k} y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}E_{1k0}(t,\rho)\rho^{-m}, \\ &\qquad\qquad k = \nu,\ldots,n-1, \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, where the functions $y_k(\,\cdot\,,\rho)$ are defined at the beginning of Chapter 3. The functions $u_{1k}(\,\cdot\,,\rho)$, $k=0,1,\ldots,n-1$, also form a basis for the solution space of the differential equation (2.1).

Applying (5.79) and (5.107), we can express the derivatives of the solutions $u_{1k}(\,\cdot\,,\rho)$ in the form

$$\begin{aligned} u_{1k}^{(\eta)}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} y_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}E_{1k\eta}(t,\rho)\rho^{-m+\eta} \\ &= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}\big[(\mathrm{i}\omega_k)^{\eta} + F_{1k\eta}(t,\rho)\big], \qquad k = 0,1,\ldots,\nu-1, \\ u_{1k}^{(\eta)}(t,\rho) &= \mathrm{e}^{\mathrm{i}\rho\omega_k} y_k^{(\eta)}(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t}E_{1k\eta}(t,\rho)\rho^{-m+\eta} \\ &= \rho^{\eta}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\eta} + F_{1k\eta}(t,\rho)\big], \qquad k = \nu,\ldots,n-1, \end{aligned} \tag{5.109}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, and for $\eta = 0,1,\ldots,n-1$. From the regularity results of Theorem 4.7, for $t \in [0,1]$ fixed the functions $u_{1k}^{(\eta)}(t,\rho)$ and $F_{1k\eta}(t,\rho)$ are analytic functions of the ρ variable on the open set

$$G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}.$$

Thus, the boundary data $u_{1k}^{(\eta)}(0,\rho)$, $u_{1k}^{(\eta)}(1,\rho)$, $k,\eta = 0,1,\dots,n-1$, consists of functions of ρ that are analytic on G_1.

Second, in the ρ plane we introduce the sector

$$\Sigma_1\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \pi - \frac{\pi}{n} + \frac{\sigma_0}{n} \le \theta \le \pi + \frac{\pi}{n} - \frac{\sigma_0}{n}.$$

Clearly any point ρ in the sector T_1 with $|\rho|$ sufficiently large lies in the sector Σ_1. Without loss of generality we can assume that the constant $R_0 > 0$ chosen earlier (see Theorems 4.6 and 4.7) has the additional property that $\rho \in T_1$ with $|\rho| > R_0$ implies $\rho \in \Sigma_1$.

Take any point $\rho = a + \mathrm{i}b \in \Sigma_1$. Arguing as above, we obtain the following key estimates:

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0(t-1)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho(t-1)}\right| = \mathrm{e}^{-b(t-1)}, \qquad 0 \le t \le 1, \tag{5.110}$$

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-1)}| \le \mathrm{e}^{-(1-t)\alpha|\rho|} \le 1, \qquad 0 \le t \le 1, \quad k = 1,\dots,\nu-1, \tag{5.111}$$

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k t}| \le \mathrm{e}^{-t\alpha|\rho|} \le 1, \qquad 0 \le t \le 1, \quad k = \nu,\dots,n-1. \tag{5.112}$$

It follows that

$$\left|\mathrm{e}^{-\mathrm{i}\rho\omega_0}\right| = \left|\mathrm{e}^{-\mathrm{i}\rho}\right| = \mathrm{e}^{b}, \tag{5.113}$$

$$\left|\mathrm{e}^{-\mathrm{i}\rho\omega_k}\right| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k = 1,\dots,\nu-1, \tag{5.114}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k}\right| \le \mathrm{e}^{-\alpha|\rho|} \le 1, \qquad k = \nu,\dots,n-1, \tag{5.115}$$

for all $\rho = a + \mathrm{i}b \in \Sigma_1$. Therefore, the exponential $\mathrm{e}^{-\mathrm{i}\rho\omega_0} = \mathrm{e}^{-\mathrm{i}\rho}$ is unbounded on the sector Σ_1 as $b \to \infty$, while the exponentials $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $1 \le k \le \nu - 1$, and $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $\nu \le k \le n-1$, go to 0 very rapidly on Σ_1 as $|\rho| \to \infty$. The estimates (5.110)–(5.115) are also valid for $\rho \in T_1$ with $|\rho| > R_0$, and in particular, they are valid for $\rho \in S_1$ with $|\rho| > R_0$ or for $\rho \in G_1$.

Applying the estimates (5.110)–(5.112) and (5.108) to the representation (5.109) with $\eta = 0$, it is follows that

$$|u_{10}(t,\rho)| \le 2\mathrm{e}^{-b(t-1)}, \tag{5.116}$$

$$|u_{1k}(t,\rho)| \le 2\mathrm{e}^{-(1-t)\alpha|\rho|} \le 2, \qquad k = 1,\dots,\nu-1, \tag{5.117}$$

$$|u_{1k}(t,\rho)| \le 2\mathrm{e}^{-t\alpha|\rho|} \le 2, \qquad k = \nu,\dots,n-1, \tag{5.118}$$

for $0 \le t \le 1$ and for $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_1 with $|\rho| > R_1$ and $b \le 0$, we have

$$|u_{1k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 0,1,\dots,n-1. \tag{5.119}$$

Third, in terms of the modified solutions $u_{1k}(t,\rho)$, $k = 0,1,\dots,n-1$, we form the functions

$$M_{1ik}(\rho) := B_i(u_{1k}(\,\cdot\,,\rho)) = \sum_{\eta=0}^{m_i} \alpha_{i\eta} u_{1k}^{(\eta)}(0,\rho) + \sum_{\eta=0}^{m_i} \beta_{i\eta} u_{1k}^{(\eta)}(1,\rho)$$

for $\rho \in T_1$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$. These functions are analytic on the open set G_1. For $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$ define

$$\widetilde{P}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{1k\eta}(0, \rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{1k\eta}(1, \rho)\rho^{-m+\eta}$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ define

$$\widetilde{P}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \beta_{i\eta} E_{1k\eta}(1, \rho)\rho^{-m+\eta}, \quad \widetilde{Q}_{1ik}(\rho) := \sum_{\eta=0}^{m_i} \alpha_{i\eta} E_{1k\eta}(0, \rho)\rho^{-m+\eta}$$

for $\rho \in T_1$ with $|\rho| > R_0$; and in terms of these functions and the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ (see equations (3.1) and (3.2)), for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$ define

$$P_{1ik}(\rho) := \widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho), \quad Q_{1ik}(\rho) := \widehat{Q}_{ik}(\rho) + \widetilde{Q}_{1ik}(\rho)$$

for $\rho \in T_1$ with $|\rho| > R_0$. All of these functions are analytic on the open set G_1. In terms of them, the functions $M_{1ik}(\rho)$ can be expressed as follows: for $i = 1, \ldots, n$

$$\begin{aligned} M_{1ik}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k}[\widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} + \widetilde{Q}_{1ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}], \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = 0, 1, \ldots, \nu - 1, \\ M_{1ik}(\rho) &= \mathrm{e}^{\mathrm{i}\rho\omega_k}[\widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widetilde{Q}_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}], \\ &\qquad\qquad\qquad\qquad\qquad\qquad k = \nu, \ldots, n - 1. \end{aligned}$$

Thus, for $\rho \in T_1$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$, we have

$$M_{1ik}(\rho) = \mathrm{e}^{-\mathrm{i}\rho\omega_k}\big[P_{1ik}(\rho) + Q_{1ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}\big], \qquad k = 0, 1, \ldots, \nu - 1, \tag{5.120}$$

$$M_{1ik}(\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k}\big[P_{1ik}(\rho) + Q_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}\big], \qquad k = \nu, \ldots, n - 1. \tag{5.121}$$

Fourth, the *characteristic determinant* of the differential operator L relative to the sector T_1 is the analytic function Δ_1 defined by

$$\Delta_1(\rho) := \det(B_i(u_{1k}(\,\cdot\,, \rho))) = \det(M_{1ik}(\rho)) \quad \text{for } \rho \in G_1.$$

A complex number $\lambda = \rho^n$ with $\rho \in G_1$ is an eigenvalue of L if and only if $\Delta_1(\rho) = 0$. Applying (5.120) and (5.121), the characteristic determinant can be expressed in the form

$$\Delta_1(\rho) = \det \begin{array}{c} \begin{array}{ccc} \phantom{P_{110}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{110}(\rho)} & 1\le k\le\nu-1 & \nu\le k\le n-1 \end{array} \\ \begin{pmatrix} P_{110}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{110}(\rho) & P_{11k}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}+Q_{11k}(\rho) & P_{11k}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}+Q_{11k}(\rho) \\ \vdots & \vdots & \vdots \\ P_{1n0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{1n0}(\rho) & P_{1nk}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}+Q_{1nk}(\rho) & P_{1nk}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}+Q_{1nk}(\rho) \end{pmatrix} \end{array} \tag{5.122}$$

for $\rho \in G_1$. Compare this representation to the representation (3.41) for the approximate characteristic determinant $\widetilde{\Delta}$.

Fifth, the estimates (5.96) for the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ remain valid for $\rho \in \mathbb{C}$ with $|\rho| \geq 1$. For the functions $\widetilde{P}_{1ik}(\rho)$, $\widetilde{Q}_{1ik}(\rho)$, we obtain the estimates

$$|\widetilde{P}_{1ik}(\rho)| \leq \gamma_1 |\rho|^{-(m-m_i)}, \qquad |\widetilde{Q}_{1ik}(\rho)| \leq \gamma_1 |\rho|^{-(m-m_i)} \tag{5.123}$$

for $\rho \in T_1$ with $|\rho| > R_0$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$. For $i = 1, \ldots, n$ define

$$\begin{aligned} \widetilde{F}_{1ik}(\rho) &:= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_k} + \widetilde{Q}_{1ik}(\rho), \qquad & k = 1, \ldots, \nu-1, \\ \widetilde{F}_{1ik}(\rho) &:= [\widehat{P}_{ik}(\rho) + \widetilde{P}_{1ik}(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_k} + \widetilde{Q}_{1ik}(\rho), \qquad & k = \nu, \ldots, n-1, \end{aligned}$$

for $\rho \in T_1$ with $|\rho| > R_0$. These functions are analytic on the open set G_1 with

$$|\widetilde{F}_{1ik}(\rho)| \leq \gamma_2 |\rho|^{-(m-m_i)} \tag{5.124}$$

for $\rho \in T_1$ with $|\rho| > R_0$, for $i = 1, \ldots, n$, and for $k = 1, \ldots, \nu-1$ and $k = \nu, \ldots, n-1$. Using these functions, we can rewrite (5.122) as

$$\Delta_1(\rho) = \det \begin{array}{c} \begin{array}{ccc} \phantom{P_{110}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{110}(\rho)} & 1\leq k\leq \nu-1 & \nu\leq k\leq n-1 \end{array} \\ \begin{pmatrix} P_{110}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{110}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) \\ \vdots & \vdots & \vdots \\ P_{1n0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{1n0}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) \end{pmatrix} \end{array} \tag{5.125}$$

for $\rho \in G_1$.

Sixth, expanding the determinant in (5.125) using the 0th column, we get

$$\Delta_1(\rho) = D_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + D_{10}(\rho) \tag{5.126}$$

for $\rho \in G_1$, where

$$D_{11}(\rho) := \det \begin{array}{c} \begin{array}{ccc} \phantom{\widehat{P}_{10}(\rho)+\widetilde{P}_{110}(\rho)} & 1\leq k\leq \nu-1 & \nu\leq k\leq n-1 \end{array} \\ \begin{pmatrix} \widehat{P}_{10}(\rho)+\widetilde{P}_{110}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) \\ \vdots & \vdots & \vdots \\ \widehat{P}_{n0}(\rho)+\widetilde{P}_{1n0}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) \end{pmatrix} \end{array}$$

and

$$D_{10}(\rho) := \det \begin{array}{c} \begin{array}{ccc} \phantom{\widehat{Q}_{10}(\rho)+\widetilde{Q}_{110}(\rho)} & 1\leq k\leq \nu-1 & \nu\leq k\leq n-1 \end{array} \\ \begin{pmatrix} \widehat{Q}_{10}(\rho)+\widetilde{Q}_{110}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) \\ \vdots & \vdots & \vdots \\ \widehat{Q}_{n0}(\rho)+\widetilde{Q}_{1n0}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) \end{pmatrix} \end{array}$$

for $\rho \in G_1$. Clearly the functions $D_{1i}(\rho)$, $i = 0, 1$, are analytic on the open set G_1.

Seventh, if we expand the determinants for the functions $D_{1i}(\rho)$, $i = 0, 1$, using the linearity of the determinant function in its n columns, then we can express these functions in the form

$$D_{1i}(\rho) = \widetilde{\pi}_i(\rho) + \widetilde{\Phi}_{1i}(\rho), \qquad i = 0, 1,$$

for $\rho \in G_1$, where the functions $\widetilde{\pi}_i(\rho)$, $i = 0, 1$, are analytic for $\rho \neq 0$ in $\mathbb{C}$ and are the functions introduced earlier in Chapter 3, and where the functions $\widetilde{\Phi}_{1i}(\rho)$ are analytic on the open set G_1 with $|\widetilde{\Phi}_{1i}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}$, $i = 0, 1$, for $\rho \in G_1$. Thus, for the characteristic determinant Δ_1 we obtain the representation

$$\Delta_1(\rho) = \widetilde{\pi}_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\pi}_0(\rho) + \widetilde{\Phi}_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\Phi}_{10}(\rho) \tag{5.127}$$

for $\rho \in G_1$, where the functions $\widetilde{\pi}_i(\rho)$, $i = 0, 1$, are analytic for $\rho \neq 0$ in $\mathbb{C}$ and the functions $\widetilde{\Phi}_{1i}(\rho)$, $i = 0, 1$, are analytic on the open set G_1 with

$$|\widetilde{\Phi}_{1i}(\rho)| \leq \gamma_3 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, \tag{5.128}$$

for $\rho \in G_1$. Compare equation (5.127) to equation (3.44).

Eighth, the functions $\pi_i'(\rho)$, $i = 0, 1$, introduced in Chapter 3, are given by

$$\pi_1'(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} a_\kappa' \rho^\kappa, \quad \pi_0'(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} b_\kappa' \rho^\kappa \tag{5.129}$$

for $\rho \neq 0$ in $\mathbb{C}$. From equations (3.47) and (3.48) it is immediate that

$$\widetilde{\pi}_1(\rho) = \pi_1'(\rho) + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa'(m)\rho^\kappa, \quad \widetilde{\pi}_0(\rho) = \pi_0'(\rho) + \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa'(m)\rho^\kappa \tag{5.130}$$

for $\rho \neq 0$ in $\mathbb{C}$. In terms of (5.127) and (5.130) we define the functions

$$\Phi_{11}(\rho) := \sum_{\kappa=-n(m-1)}^{-(m-p_0)} a_\kappa'(m)\rho^\kappa + \widetilde{\Phi}_{11}(\rho), \quad \Phi_{10}(\rho) := \sum_{\kappa=-n(m-1)}^{-(m-p_0)} b_\kappa'(m)\rho^\kappa + \widetilde{\Phi}_{10}(\rho)$$

for $\rho \in G_1$. These functions are clearly analytic on the open set G_1, and by using them we can rewrite the representation (5.127) in the simpler form

$$\Delta_1(\rho) = \pi_1'(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0'(\rho) + \Phi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \Phi_{10}(\rho) \tag{5.131}$$

for $\rho \in G_1$. In equation (5.131) the functions $\pi_i'(\rho)$, $i = 0, 1$, are given by (5.129), and they are analytic for $\rho \neq 0$ in $\mathbb{C}$; the functions $\Phi_{1i}(\rho)$, $i = 0, 1$, are analytic on the open set G_1 and satisfy the growth rates

$$|\Phi_{1i}(\rho)| \le \gamma_4 |\rho|^{-(m-p_0)}, \qquad i = 0, 1, \tag{5.132}$$

for $\rho \in G_1$. Recall that the functions $\pi_i(\rho)$, $i = 0, 1$, appearing in the representation (5.105) of the characteristic determinant $\Delta_0(\rho)$ are related to the functions $\pi'_i(\rho)$, $i = 0, 1$, appearing in the representation (5.131) of the characteristic determinant $\Delta_1(\rho)$ by equations (3.49) and (3.50), namely

$$\pi'_1(\rho) = \pi_0(\rho\omega_\nu), \qquad \pi'_0(\rho) = \pi_1(\rho\omega_{\nu-1}) \tag{5.133}$$

for $\rho \ne 0$ in $\mathbb{C}$.

The representation (5.131) is our working form for the characteristic determinant Δ_1 relative to the sector T_1. Compare the representation (5.131) for the characteristic determinant Δ_1 to the representation (3.55) for the approximate characteristic determinant $\widetilde{\Delta}$.

The above results are summarized in the following theorem.

Theorem 5.5. *Let n be odd: $n = 2\nu - 1$. Under the above assumptions* (i)–(vi)*, let $v_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the linearly independent solutions of the differential equation* (2.1) *constructed in Theorem 4.7 for $\rho \in T_1$ with $|\rho| > R_0$, let $u_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, be the modified solutions of* (2.1) *defined above for $\rho \in T_1$ with $|\rho| > R_0$, and let Δ_1 be the characteristic determinant of the differential operator L given by*

$$\Delta_1(\rho) = \det(B_i(u_{1k}(\,\cdot\,, \rho))) \quad \text{for } \rho \in G_1,$$

where $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$. Then Δ_1 is analytic on the open set G_1, and Δ_1 has the representation

$$\Delta_1(\rho) = \pi'_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi'_0(\rho) + \Phi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \Phi_{10}(\rho)$$

for $\rho \in G_1$, where the functions $\Phi_{1i}(\rho)$, $i = 0, 1$, are analytic on G_1 and satisfy the estimates $|\Phi_{1i}(\rho)| \le \gamma|\rho|^{-(m-p_0)}$ for $\rho \in G_1$ and for $i = 0, 1$.

6

The Green's Function

For both n even and n odd, if we start with any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and $\Delta_0(\rho) \neq 0$ or with $\rho \in G_1$ and $\Delta_1(\rho) \neq 0$, then λ belongs to the resolvent set $\rho(L)$ and the resolvent $R_\lambda(L) = (\lambda I - L)^{-1}$ exists as an integral operator on $L^2[0,1]$ with the Green's function $G(t,s;\lambda)$ as its kernel:

$$R_\lambda(L)u(t) = \int_0^1 G(t,s;\lambda)u(s)\,ds, \qquad 0 \le t \le 1, \quad u \in L^2[0,1].$$

In this chapter we construct important representations of the resolvent and the Green's function, and then use these representations to derive their growth rates for ρ belonging to the sectors S_0 and S_1. Our representations are first developed for ρ belonging to the open set G_0, and then analogous representations are established for ρ in the open set G_1.

6.1 The Green's Function for n Even

Assume that n is even: $n = 2\nu \ge 2$. For the even order case recall that the sectors S_0 and S_1 are given by

$$\begin{aligned} S_0 &: \ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } 0 \le \theta \le \frac{\pi}{n}\,, \\ S_1 &: \ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} \le \theta \le 0\,, \end{aligned}$$

with T_0 and T_1 the corresponding translated sectors. For each $\rho \in T_0$ with $|\rho| > R_0$, let us consider the basis $v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots, v_{0\,n-1}(\,\cdot\,,\rho)$ for the solution space of the differential equation (2.1) determined in Theorem 4.3. In Chapter 5 we showed that

$$v_{0k}^{(\alpha)}(t,\rho) = \rho^\alpha \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^\alpha + F_{0k\alpha}(t,\rho)\big] \tag{6.1}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$, where the function $F_{0k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$ with

$$|F_{0k\alpha}(t,\rho)| \le 1 \tag{6.2}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_1 \ge R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. Also in Chapter 5, for $\rho \in T_0$ with $|\rho| > R_0$ we formed the modified solutions $u_{00}(\,\cdot\,,\rho), u_{01}(\,\cdot\,,\rho), \ldots, u_{0\,n-1}(\,\cdot\,,\rho)$ of the differential equation (2.1). These functions have the representations

$$\begin{aligned} u_{0k}^{(\alpha)}(t,\rho) &= \rho^\alpha e^{i\rho\omega_k t}\big[(i\omega_k)^\alpha + F_{0k\alpha}(t,\rho)\big], &\quad k &= 0, 1, \ldots, \nu-1, \\ u_{0k}^{(\alpha)}(t,\rho) &= \rho^\alpha e^{i\rho\omega_k (t-1)}\big[(i\omega_k)^\alpha + F_{0k\alpha}(t,\rho)\big], &\quad k &= \nu, \ldots, n-1, \end{aligned} \tag{6.3}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $\alpha = 0, 1, \ldots, n-1$. From equation (5.18) we have the bounds

$$|u_{0k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 0, 1, \ldots, n-1, \tag{6.4}$$

for $\rho = a + ib$ belonging to the sector S_0 (where $b \ge 0$) with $|\rho| > R_1$.

Let L_0 be the nth order differential operator in $L^2[0,1]$ defined by

$$\mathcal{D}(L_0) = \{u \in H^n[0,1] \mid u^{(n-i)}(0) = 0,\ i = 1, \ldots, n\}, \qquad L_0 u = \ell u.$$

Clearly the resolvent set $\rho(L_0)$ is equal to $\mathbb{C}$ due to the initial value conditions at $t = 0$. We begin by computing the Green's function $g(t,s;\lambda)$ of the differential operator $\lambda I - L_0$, and then use it to compute the Green's function $G(t,s;\lambda)$ of the differential operator $\lambda I - L$. The algorithm in Example III.3.18 of [28] will be employed.

For $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$, our earlier work has shown that the Green's function $g(t,s;\lambda)$ is given by

$$\begin{aligned} g(t,s;\lambda) &= \sum_{k=0}^{n-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), &\quad 0 &\le s < t \le 1, \\ g(t,s;\lambda) &= 0, &\quad 0 &\le t < s \le 1, \end{aligned} \tag{6.5}$$

where the functions $\eta_{0k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, belong to $H^n[0,1]$ and are determined by the linear system

$$\sum_{k=0}^{n-1} v_{0k}^{(\alpha)}(s,\rho)\eta_{0k}(s,\rho) = -\delta_{\alpha\,n-1} i^n, \qquad \alpha = 0, 1, \ldots, n-1, \tag{6.6}$$

for $0 \le s \le 1$. Using equation (6.1), the system (6.6) can be rewritten in the form

$$\begin{aligned} \sum_{k=0}^{n-1} \big[(i\omega_k)^\alpha + F_{0k\alpha}(s,\rho)\big] e^{i\rho\omega_k s}\eta_{0k}(s,\rho) &= -\delta_{\alpha\,n-1} i^n \rho^{-(n-1)}, \\ \alpha &= 0, 1, \ldots, n-1, \end{aligned} \tag{6.7}$$

for $0 \leq s \leq 1$. We will treat (6.7) as a linear system for the unknowns $\mathrm{e}^{\mathrm{i}\rho\omega_k s}\eta_{0k}(s,\rho)$, $k = 0, 1, \ldots, n-1$, with $n \times n$ coefficient matrix $A_0(s,\rho) := \big((\mathrm{i}\omega_k)^\alpha + F_{0k\alpha}(s,\rho)\big)$. Previously in Chapter 4 this matrix was shown to be nonsingular for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_0$.

For the Vandermonde matrix

$$V := \big((\mathrm{i}\omega_k)^\alpha\big) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mathrm{i}\omega_0 & \mathrm{i}\omega_1 & \cdots & \mathrm{i}\omega_{n-1} \\ \vdots & \vdots & & \vdots \\ (\mathrm{i}\omega_0)^{n-1} & (\mathrm{i}\omega_1)^{n-1} & \cdots & (\mathrm{i}\omega_{n-1})^{n-1} \end{pmatrix},$$

the inverse is given by

$$\begin{aligned} V^{-1} &= \frac{1}{n\mathrm{i}^{\,n-1}} \left(\mathrm{i}^{\,n-k-1}\omega_\alpha^{n-k}\right) = \left(\frac{1}{n\mathrm{i}^{\,k}}\,\omega_\alpha^{n-k}\right) \\ &= \frac{1}{n\mathrm{i}^{\,n-1}} \begin{pmatrix} \mathrm{i}^{\,n-1}\omega_0^n & \mathrm{i}^{\,n-2}\omega_0^{n-1} & \cdots & \mathrm{i}\omega_0^2 & \omega_0 \\ \mathrm{i}^{\,n-1}\omega_1^n & \mathrm{i}^{\,n-2}\omega_1^{n-1} & \cdots & \mathrm{i}\omega_1^2 & \omega_1 \\ \vdots & \vdots & & \vdots & \vdots \\ \mathrm{i}^{\,n-1}\omega_{n-1}^n & \mathrm{i}^{\,n-2}\omega_{n-1}^{n-1} & \cdots & \mathrm{i}\omega_{n-1}^2 & \omega_{n-1} \end{pmatrix}. \end{aligned}$$

Let us write the inverse of $A_0(s,\rho)$ in the form

$$A_0(s,\rho)^{-1} := \left(\frac{1}{n\mathrm{i}^{\,k}}\,\omega_\alpha^{n-k}\,[1 + G_{0k\alpha}(s,\rho)]\right)$$

for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. Then from (6.7) it follows that

$$\begin{aligned} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho\omega_0 s}\eta_{00}(s,\rho) \\ \mathrm{e}^{\mathrm{i}\rho\omega_1 s}\eta_{01}(s,\rho) \\ \vdots \\ \mathrm{e}^{\mathrm{i}\rho\omega_{n-1} s}\eta_{0\,n-1}(s,\rho) \end{pmatrix} &= A_0(s,\rho)^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\mathrm{i}^{\,n}\rho^{-(n-1)} \end{pmatrix} \\ &= \frac{-\mathrm{i}}{n\rho^{n-1}} \begin{pmatrix} \omega_0[1 + G_{0\,n-1\,0}(s,\rho)] \\ \omega_1[1 + G_{0\,n-1\,1}(s,\rho)] \\ \vdots \\ \omega_{n-1}[1 + G_{0\,n-1\,n-1}(s,\rho)] \end{pmatrix}, \end{aligned}$$

and hence,

$$\eta_{0k}(s,\rho) = -\frac{\mathrm{i}\omega_k}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{0\,n-1\,k}(s,\rho)] \tag{6.8}$$

for $0 \leq s \leq 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k = 0, 1, \ldots, n-1$. Since the functions $\eta_{0k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$, the functions $G_{0\,n-1\,k}(\,\cdot\,,\rho)$ also belong to $H^n[0,1]$.

We assert that the functions $G_{0k\alpha}(s,\rho) \to 0$ uniformly on $[0,1] \times T_0$ as $|\rho| \to \infty$ for $k, \alpha = 0, 1, \ldots, n-1$. Take any $\epsilon > 0$. Using the continuity of matrix inversion and the fact that the Vandermonde matrix V is nonsingular, choose a number $\delta > 0$ such that if $W = (w_{\alpha k})$ is an $n \times n$ matrix with $|(\mathrm{i}\omega_k)^\alpha - w_{\alpha k}| < \delta$ for $\alpha, k = 0, 1, \dots, n-1$, then W is nonsingular, and the inverse $W^{-1} = (x_{\alpha k})$ satisfies

$$\left|\frac{1}{n\mathrm{i}^k}\,\omega_\alpha^{n-k} - x_{\alpha k}\right| < \epsilon \qquad \text{for } \alpha, k = 0, 1, \dots, n-1.$$

Choose a constant $R_\delta \geq R_0$ such that $|F_{0k\alpha}(s,\rho)| < \delta$ for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_\delta$, for $\alpha, k = 0, 1, \dots, n-1$. Then for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_\delta$, we have

$$|(\mathrm{i}\omega_k)^\alpha - [(\mathrm{i}\omega_k)^\alpha + F_{0k\alpha}(s,\rho)]| = |F_{0k\alpha}(s,\rho)| < \delta$$

for $\alpha, k = 0, 1, \dots, n-1$, and hence, from the above

$$\left|\frac{1}{n\mathrm{i}^k}\,\omega_\alpha^{n-k} - \frac{1}{n\mathrm{i}^k}\,\omega_\alpha^{n-k}\,[1 + G_{0k\alpha}(s,\rho)]\right| = \frac{1}{n}\,|G_{0k\alpha}(s,\rho)| < \epsilon$$

for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_\delta$, for $k, \alpha = 0, 1, \dots, n-1$. This establishes the assertion. Based on this assertion, without loss of generality we can assume that the constant R_1 chosen earlier also yields the bound

$$|G_{0k\alpha}(s,\rho)| \leq 1 \tag{6.9}$$

for $0 \leq s \leq 1$, for $\rho \in T_0$ with $|\rho| > R_1$, and for $k, \alpha = 0, 1, \dots, n-1$.

Summarizing, for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$, the Green's function for the differential operator $\lambda I - L_0$ is given by

$$\begin{aligned} g(t,s;\lambda) &= -\frac{1}{n\rho^{n-1}}\sum_{k=0}^{n-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{0k0}(t,\rho)][1 + G_{0\,n-1\,k}(s,\rho)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s < t \leq 1, \\ g(t,s;\lambda) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\quad\; 0 \leq t < s \leq 1, \end{aligned} \tag{6.10}$$

where the functions $F_{0k0}(\,\cdot\,,\rho)$ and $G_{0\,n-1\,k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$ and satisfy the bounds given in equations (6.2) and (6.9).

Next, we rewrite (6.5) or (6.10) in the form

$$g(t,s;\lambda) = k_0(t,s;\rho) + \ell_0(t,s;\rho)$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$, where

$$\begin{aligned} k_0(t,s;\rho) &:= \sum_{k=0}^{\nu-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), \qquad 0 \leq s < t \leq 1, \\ k_0(t,s;\rho) &:= -\sum_{k=\nu}^{n-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), \qquad 0 \leq t < s \leq 1, \end{aligned} \tag{6.11}$$

and

$$\ell_0(t, s; \rho) := \sum_{k=\nu}^{n-1} v_{0k}(t, \rho)\eta_{0k}(s, \rho), \qquad 0 \leq t, s \leq 1.$$

The Green's function $g(t, s; \lambda)$ is the kernel of the integral operator $R_\lambda(L_0)$, which assigns to each u in $L^2[0, 1]$ the unique solution $z \in \mathcal{D}(L_0)$ of the initial value problem $(\lambda I - L_0)z = u$. Also, the function $\ell_0(t, s; \rho)$ is the kernel of an integral operator which maps $L^2[0, 1]$ into the solution space of the differential equation $(\rho^n I - \ell)u = 0$.

For each $\rho \in T_0$ with $|\rho| > R_0$, let $K_{0\rho}$ be the integral operator on $L^2[0, 1]$ defined by

$$K_{0\rho}u(t) := \int_0^1 k_0(t, s; \rho)u(s)\, ds, \qquad 0 \leq t \leq 1,$$

for $u \in L^2[0, 1]$. From the above remarks it follows that if $u \in L^2[0, 1]$ and $v = K_{0\rho}u$, then v belongs to $H^n[0, 1]$ and $(\rho^n I - \ell)v = u$. By direct calculation we have

$$\begin{aligned} v(t) &= \int_0^1 k_0(t, s; \rho)u(s)\, ds \\ &= \sum_{k=0}^{\nu-1} v_{0k}(t, \rho)\int_0^t \eta_{0k}(s, \rho)u(s)\, ds - \sum_{k=\nu}^{n-1} v_{0k}(t, \rho)\int_t^1 \eta_{0k}(s, \rho)u(s)\, ds \end{aligned}$$

for $0 \leq t \leq 1$, and using equation (6.6) with $\alpha = 0$, we get

$$\begin{aligned} v'(t) &= \sum_{k=0}^{\nu-1} v'_{0k}(t, \rho)\int_0^t \eta_{0k}(s, \rho)u(s)\, ds + \sum_{k=0}^{\nu-1} v_{0k}(t, \rho)\eta_{0k}(t, \rho)u(t) \\ &\quad - \sum_{k=\nu}^{n-1} v'_{0k}(t, \rho)\int_t^1 \eta_{0k}(s, \rho)u(s)\, ds + \sum_{k=\nu}^{n-1} v_{0k}(t, \rho)\eta_{0k}(t, \rho)u(t) \\ &= \sum_{k=0}^{\nu-1} v'_{0k}(t, \rho)\int_0^t \eta_{0k}(s, \rho)u(s)\, ds - \sum_{k=\nu}^{n-1} v'_{0k}(t, \rho)\int_t^1 \eta_{0k}(s, \rho)u(s)\, ds \\ &= \int_0^1 \frac{\partial k_0}{\partial t}(t, s; \rho)u(s)\, ds \end{aligned}$$

for $0 \leq t \leq 1$. Proceeding by induction, we see immediately that the derivatives of $v = K_{0\rho}u$ satisfy the equations

$$\begin{aligned} v^{(j)}(t) &= \int_0^1 \frac{\partial^j k_0}{\partial t^j}(t, s; \rho)u(s)\, ds \\ &= \sum_{k=0}^{\nu-1} v_{0k}^{(j)}(t, \rho)\int_0^t \eta_{0k}(s, \rho)u(s)\, ds - \sum_{k=\nu}^{n-1} v_{0k}^{(j)}(t, \rho)\int_t^1 \eta_{0k}(s, \rho)u(s)\, ds \end{aligned} \tag{6.12}$$

for $0 \le t \le 1$ and for $j = 0, 1, \ldots, n-1$, valid for each $\rho \in T_0$ with $|\rho| > R_0$.

Using (6.1) and (6.8), for $\rho \in T_0$ with $|\rho| > R_0$ the kernel $k_0(t, s; \rho)$ can be expressed as

$$\begin{aligned} k_0(t,s;\rho) &= -\frac{1}{n\rho^{n-1}} \sum_{k=0}^{\nu-1} (\mathrm{i}\omega_k) \mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)} [1 + F_{0k0}(t,\rho)][1 + G_{0\,n-1\,k}(s,\rho)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s < t \le 1, \\ k_0(t,s;\rho) &= \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n-1} (\mathrm{i}\omega_k) \mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)} [1 + F_{0k0}(t,\rho)][1 + G_{0\,n-1\,k}(s,\rho)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < s \le 1. \end{aligned} \tag{6.13}$$

Now take any point $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_0$. If t, s are real numbers with $0 \le s < t \le 1$, then $0 < t - s \le 1$, and by (5.8) we obtain the estimates

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0(t-s)}\right| = \mathrm{e}^{-b(t-s)}, \tag{6.14}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le \mathrm{e}^{-(t-s)\alpha|\rho|} \le 1, \qquad k = 1, \ldots, \nu - 1. \tag{6.15}$$

On the other hand, for real numbers t, s with $0 \le t < s \le 1$, we have

$$0 < s - t \le 1, \qquad -1 \le t - s < 0, \qquad 0 \le 1 + t - s < 1,$$

and hence, by (5.9)

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_\nu(t-s)}\right| = \mathrm{e}^{-b(s-t)}, \tag{6.16}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(1+t-s-1)}\right| \le \mathrm{e}^{-(s-t)\alpha|\rho|} \le 1, \quad k = \nu + 1, \ldots, n-1. \tag{6.17}$$

In particular, if $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| > R_1$, then these estimates give

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le s < t \le 1, \quad k = 0, 1, \ldots, \nu - 1, \tag{6.18}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le t < s \le 1, \quad k = \nu, \ldots, n-1. \tag{6.19}$$

Applying (6.18), (6.19) and (6.2), (6.9) to the representation (6.13), it follows that

$$|k_0(t,s;\rho)| \le \frac{2}{|\rho|^{n-1}} \tag{6.20}$$

for $t \ne s$ in $[0, 1]$ and for $\rho \in S_0$ with $|\rho| > R_1$.

Finally, fix any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$, and assume that $\Delta_0(\rho) \ne 0$, so the point λ belongs to the resolvent set $\rho(L)$. Using the integral operator $K_{0\rho}$, we can establish our representations of the resolvent $R_\lambda(L)$ and the associated Green's function $G(t, s; \lambda)$. Indeed, take any function $u \in L^2[0, 1]$, and set

$$v = K_{0\rho} u \quad \text{and} \quad w = R_\lambda(L)u.$$

Clearly the functions v and w belong to $H^n[0,1]$, and

$$(\lambda I - \ell)v = u = (\lambda I - \ell)w.$$

Thus, there exist constants $c_0, c_1, \ldots, c_{n-1}$ (depending on ρ) such that

$$R_\lambda(L)u(t) = w(t) = v(t) + \sum_{k=0}^{n-1} c_k u_{0k}(t,\rho), \qquad 0 \le t \le 1.$$

The functions $u_{0k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, are the modified solutions of the differential equation (2.1) introduced earlier. They form a basis for the solution space of (2.1), and the characteristic determinant $\varDelta_0(\rho)$ is defined in terms of them.

Applying the boundary value B_i to both sides of the last equation, we obtain the linear system

$$\sum_{k=0}^{n-1} M_{0ik}(\rho)c_k = -B_i(v), \qquad i = 1, \ldots, n, \tag{6.21}$$

for the constants $c_0, c_1, \ldots, c_{n-1}$, where as in Chapter 5

$$M_{0ik}(\rho) = B_i(u_{0k}(\,\cdot\,,\rho)) = \sum_{j=0}^{m_i} \alpha_{ij} u_{0k}^{(j)}(0,\rho) + \sum_{j=0}^{m_i} \beta_{ij} u_{0k}^{(j)}(1,\rho)$$

for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$. The $n \times n$ coefficient matrix $(M_{0ik}(\rho))$ in (6.21) is nonsingular because $\det(M_{0ik}(\rho)) = \varDelta_0(\rho) \neq 0$.

Fix an index i with $1 \le i \le n$, and consider the quantity $B_i(v) = B_i(K_{0\rho}u)$. From equation (6.12) and equations (6.1), (6.8), we have

$$\begin{aligned} v^{(j)}(0) &= -\sum_{k=\nu}^{n-1} v_{0k}^{(j)}(0,\rho) \int_0^1 \eta_{0k}(s,\rho)u(s)\,ds \\ &= \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n-1} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j \\ &\qquad + F_{0kj}(0,\rho)] \int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{0\,n-1\,k}(s,\rho)]u(s)\,ds \end{aligned}$$

and

$$\begin{aligned} v^{(j)}(1) &= \sum_{k=0}^{\nu-1} v_{0k}^{(j)}(1,\rho) \int_0^1 \eta_{0k}(s,\rho)u(s)\,ds \\ &= -\frac{1}{n\rho^{n-1}} \sum_{k=0}^{\nu-1} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j \\ &\qquad + F_{0kj}(1,\rho)] \int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)}[1 + G_{0\,n-1\,k}(s,\rho)]u(s)\,ds \end{aligned}$$

for $j = 0, 1, \ldots, n-1$, and hence,

$$\begin{aligned} B_i(v) &= \sum_{j=0}^{m_i} \alpha_{ij} v^{(j)}(0) + \sum_{j=0}^{m_i} \beta_{ij} v^{(j)}(1) \\ &= \frac{1}{n\rho^{n-1}} \sum_{k=0}^{\nu-1} \Big\{ \sum_{j=0}^{m_i} (-1) \beta_{ij} (\mathrm{i}\omega_k) \rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(1,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)} [1 + G_{0\,n-1\,k}(s,\rho)] u(s)\, ds \qquad (6.22) \\ &\quad + \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n-1} \Big\{ \sum_{j=0}^{m_i} \alpha_{ij} (\mathrm{i}\omega_k) \rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(0,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{0\,n-1\,k}(s,\rho)] u(s)\, ds \end{aligned}$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$ define

$$\mathcal{T}_{0ik}(\rho) := \sum_{j=0}^{m_i} (-1) \beta_{ij} (\mathrm{i}\omega_k) \rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(1,\rho)]$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ define

$$\mathcal{T}_{0ik}(\rho) := \sum_{j=0}^{m_i} \alpha_{ij} (\mathrm{i}\omega_k) \rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(0,\rho)]$$

for $\rho \in T_0$ with $|\rho| > R_0$; and for $k = 0, 1, \ldots, \nu - 1$ define

$$U_{0k}(s,\rho) := \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)} [1 + G_{0\,n-1\,k}(s,\rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $k = \nu, \ldots, n-1$ define

$$U_{0k}(s,\rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{0\,n-1\,k}(s,\rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_0$ with $|\rho| > R_0$. Then we can express (6.22) in the simpler form

$$B_i(v) = \frac{1}{n\rho^{n-1}} \sum_{k=0}^{n-1} \mathcal{T}_{0ik}(\rho) \int_0^1 U_{0k}(s,\rho) u(s)\, ds \qquad (6.23)$$

for $i = 1, \ldots, n$, where the functions $\mathcal{T}_{0ik}(\rho)$ are analytic functions of ρ on G_0, and for fixed ρ in T_0 with $|\rho| > R_0$ the functions $U_{0k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$.

In terms of the matrix $(M_{0ik}(\rho))$, let $\widetilde{M}_{0ik}(\rho)$ denote the cofactor of the entry $M_{0ik}(\rho)$:

$\widetilde{M}_{0ik}(\rho) := (-1)^{i+k+1}$ times the determinant of the $(n-1) \times (n-1)$
submatrix of $(M_{0j\ell}(\rho))$ obtained by deleting
the ith row and kth column.

Clearly the cofactors $\widetilde{M_{0ik}}(\rho)$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$, are analytic functions in the ρ variable on the open set G_0. These cofactors arise naturally when we solve for the constants $c_0, c_1, \ldots, c_{n-1}$ in the linear system (6.21) by means of Cramer's rule:

$$\begin{aligned} c_k &= \frac{-1}{\Delta_0(\rho)} \det \begin{array}{c} \\ \begin{pmatrix} M_{010}(\rho) & \cdots & B_1(v) & \cdots & M_{01n-1}(\rho) \\ \vdots & & \vdots & & \vdots \\ M_{0n0}(\rho) & \cdots & B_n(v) & \cdots & M_{0nn-1}(\rho) \end{pmatrix} \end{array} \\ &= \frac{-1}{\Delta_0(\rho)} \sum_{j=1}^{n} \widetilde{M_{0jk}}(\rho) B_j(v), \end{aligned}$$

(the column containing $B_1(v), \ldots, B_n(v)$ is labelled kth column)

or

$$c_k = \frac{-1}{n\rho^{n-1}\Delta_0(\rho)} \sum_{l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M_{0jk}}(\rho) \mathcal{T}_{0jl}(\rho) \int_0^1 U_{0l}(s,\rho) u(s)\, ds \tag{6.24}$$

for $k = 0, 1, \ldots, n-1$.

Combining the above results, we conclude that

$$\begin{aligned} R_\lambda(L)u(t) &= K_{0\rho}u(t) \\ &- \frac{1}{n\rho^{n-1}\Delta_0(\rho)} \sum_{k,l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M_{0jk}}(\rho) \mathcal{T}_{0jl}(\rho) u_{0k}(t,\rho) \int_0^1 U_{0l}(s,\rho) u(s)\, ds \end{aligned} \tag{6.25}$$

for $0 \le t \le 1$ and for $u \in L^2[0,1]$, where (6.25) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\Delta_0(\rho) \neq 0$. Also, from (6.25) the associated Green's function is given by

$$\begin{aligned} G(t,s;\lambda) &= k_0(t,s;\rho) \\ &- \frac{1}{n\rho^{n-1}\Delta_0(\rho)} \sum_{k,l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M_{0jk}}(\rho) \mathcal{T}_{0jl}(\rho) u_{0k}(t,\rho) U_{0l}(s,\rho) \end{aligned} \tag{6.26}$$

for $t \neq s$ in $[0,1]$, where (6.26) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\Delta_0(\rho) \neq 0$.

To effectively use the representations (6.25) and (6.26), we must determine bounds or growth rates for the various functions appearing there. For the basis functions $u_{0k}(t,\rho)$ we have already established the necessary bounds in equation (6.4). For the kernel $k_0(t,s;\rho)$ the required growth rate is given in equation (6.20). Let us proceed to calculate bounds and growth rates for the functions $\mathcal{T}_{0ik}(\rho)$, $U_{0k}(s,\rho)$, and $\widetilde{M_{0ik}}(\rho)$.

First, consider the functions $\mathcal{T}_{0ik}(\rho)$. From their definitions and the estimate (6.2), it is immediate that

$$|\mathcal{T}_{0ik}(\rho)| \le \gamma_1 |\rho|^{m_i} \tag{6.27}$$

for $\rho \in T_0$ with $|\rho| > R_1$ and for $i = 1, \dots, n$, $k = 0, 1, \dots, n-1$. Second, for the functions $U_{0k}(s, \rho)$, from their definitions and the earlier estimates (5.8), (5.9), together with (6.9), we obtain the estimates (replace t by $1 - s$)

$$|U_{00}(s, \rho)| \le 2\mathrm{e}^{-b(1-s)}, \tag{6.28}$$

$$|U_{0\nu}(s, \rho)| \le 2\mathrm{e}^{-bs}, \tag{6.29}$$

$$|U_{0k}(s, \rho)| \le 2\mathrm{e}^{-(1-s)\alpha|\rho|} \le 2, \qquad k = 1, \dots, \nu - 1, \tag{6.30}$$

$$|U_{0k}(s, \rho)| \le 2\mathrm{e}^{-s\alpha|\rho|} \le 2, \qquad k = \nu + 1, \dots, n - 1, \tag{6.31}$$

for $0 \le s \le 1$ and for $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_0 (where $b \ge 0$) with $|\rho| > R_1$, we have

$$|U_{0k}(s, \rho)| \le 2, \qquad 0 \le s \le 1, \quad k = 0, 1, \dots, n - 1. \tag{6.32}$$

Third, fix indices i and k with $1 \le i \le n$ and $0 \le k \le n - 1$, and let us consider the cofactor $\widetilde{M}_{0ik}(\rho)$. It is formed by taking $(-1)^{i+k+1}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the kth column of the matrix

$$\begin{array}{c}
\begin{array}{cccc} \qquad\qquad\qquad & 1\le k\le \nu-1 & \qquad\qquad\qquad & \nu+1\le k\le n-1 \end{array} \\
\begin{pmatrix}
P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & P_{01\nu}(\rho)+Q_{01\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\
\vdots & \vdots & \vdots & \vdots \\
P_{0n0}(\rho)+Q_{0n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & P_{0n\nu}(\rho)+Q_{0n\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho)
\end{pmatrix}
\end{array}.$$

Suppose we proceed to expand the determinant for $\widetilde{M}_{0ik}(\rho)$ in the same manner as $\Delta_0(\rho)$ was expanded in Chapter 5. See equation (5.25). If $1 \le k \le \nu - 1$ or $\nu + 1 \le k \le n - 1$, then the determinant of the $(n-1) \times (n-1)$ submatrix that leads to $\widetilde{M}_{0ik}(\rho)$ is first expanded using linearity in both the 0th and νth columns; this yields the terms $\mathrm{e}^{2\mathrm{i}\rho}$, $\mathrm{e}^{\mathrm{i}\rho}$, 1. On the other hand, if $k = 0$, then the initial expansion takes place in only the νth column; this gives the terms $\mathrm{e}^{\mathrm{i}\rho}$, 1. And if $k = \nu$, then the expansion initially is in the 0th column, leading to the terms $\mathrm{e}^{\mathrm{i}\rho}$, 1.

Next, all the $(n-1) \times (n-1)$ determinants are expanded using linearity in all $n - 1$ columns. This expansion produces the representation

$$\begin{aligned}
\widetilde{M}_{0ik}(\rho) = {} & \tilde{\pi}''_{ik}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \tilde{\pi}'_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\pi}_{ik}(\rho) \\
& + \tilde{\phi}''_{0ik}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \tilde{\phi}'_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\phi}_{0ik}(\rho)
\end{aligned} \tag{6.33}$$

for $\rho \in G_0$. In this equation the functions $\tilde{\pi}''_{ik}$, $\tilde{\pi}'_{ik}$, $\tilde{\pi}_{ik}$ are formed from the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$ introduced in Chapter 3 (see equations (3.1) and (3.2)); they are analytic for $\rho \ne 0$ in $\mathbb{C}$, and each one has the simple form

$$\sum_{j=-(n-1)(m-1)}^{p_0-m_i} A_{ikj}\rho^j, \qquad \rho \neq 0 \text{ in } \mathbb{C}.$$

Cf. (3.3); (3.20), (3.25), (3.26); and (5.30). All the perturbation terms are contained in the functions $\tilde{\phi}''_{0ik}$, $\tilde{\phi}'_{0ik}$, $\tilde{\phi}_{0ik}$; they are analytic for ρ in the open set G_0, and satisfy the growth rates

$$\begin{gathered} |\tilde{\phi}''_{0ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)}, \qquad |\tilde{\phi}'_{0ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)}, \\ |\tilde{\phi}_{0ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)} \end{gathered} \tag{6.34}$$

for $\rho \in G_0$. In the special cases $k = 0$ or $k = \nu$, then $\tilde{\pi}''_{ik}(\rho) \equiv 0$ and $\tilde{\phi}''_{0ik}(\rho) \equiv 0$.

Take any point $\rho = a + ib$ in the sector S_0 with $|\rho| > R_1$. Clearly $b \geq 0$, $|e^{i\rho}| = e^{-b} \leq 1$, and $|e^{2i\rho}| \leq 1$, and hence, by (6.33) and (6.34)

$$|\widetilde{M}_{0ik}(\rho)| \leq 3\gamma_3|\rho|^{p_0-m_i} + 3\gamma_2|\rho|^{-(m-p_0+m_i)} \leq \gamma_4|\rho|^{p_0-m_i} \tag{6.35}$$

for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$. Combining this result with (6.27), we obtain the estimate

$$\Big|\sum_{j=1}^{n} \widetilde{M}_{0jk}(\rho)\mathcal{T}_{0jl}(\rho)\Big| \leq \sum_{j=1}^{n} \gamma_4|\rho|^{p_0-m_j} \cdot \gamma_1|\rho|^{m_j} \leq \gamma_5|\rho|^{p_0} \tag{6.36}$$

for $\rho \in S_0$ with $|\rho| > R_1$ and for $k, l = 0, 1, \ldots, n-1$.

Finally, take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_0$, with $|\rho| > R_1$, and with $\Delta_0(\rho) \neq 0$. Clearly $\rho \in T_0$ with $|\rho| > R_1 \geq R_0$, $\rho \in G_0$, and $b \geq 0$, and λ belongs to the resolvent set $\rho(L)$ with the Green's function $G(t, s; \lambda)$ given by (6.26). Applying the estimates (6.20), (6.36), (6.4), and (6.32) to (6.26), we see that

$$|G(t, s; \lambda)| \leq \frac{2}{|\rho|^{n-1}} + \frac{1}{n|\rho|^{n-1}|\Delta_0(\rho)|} \sum_{k,l=0}^{n-1} \gamma_5|\rho|^{p_0} \cdot 2 \cdot 2,$$

or

$$|G(t, s; \lambda)| \leq \frac{2}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_0(\rho)|} \tag{6.37}$$

for $t \neq s$ in $[0, 1]$, where (6.37) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in S_0$ and $|\rho| > R_1$ and with $\Delta_0(\rho) \neq 0$. Cf. equation (9.8) in Chapter 4 of [34]. This is our principal result for the growth rate of the Green's function $G(t, s; \lambda)$ relative to the sector S_0 when $\lambda = \rho^n$ and $\rho \in S_0$.

We can derive alternate representations of the resolvent $R_\lambda(L)$ and the Green's function $G(t, s; \lambda)$ that are in terms of the characteristic determinant $\Delta_1(\rho)$ and the sectors S_1, T_1. The treatment for Δ_1 and S_1, T_1 is similar to the one for Δ_0 and S_0, T_0, but there are some subtle differences. For

$\rho \in T_1$ with $|\rho| > R_0$, let $v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \dots, v_{1\,n-1}(\,\cdot\,,\rho)$ be the basis for the solution space of the differential equation (2.1) determined in Theorem 4.4. In Chapter 5 we showed that

$$v_{1k}^{(\alpha)}(t,\rho) = \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{1k\alpha}(t,\rho)\big] \tag{6.38}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \dots, n-1$, where the function $F_{1k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$ with

$$|F_{1k\alpha}(t,\rho)| \le 1 \tag{6.39}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_1 \ge R_0$, and for $k, \alpha = 0, 1, \dots, n-1$. Also in Chapter 5, for $\rho \in T_1$ with $|\rho| > R_0$ we formed the modified solutions $u_{10}(\,\cdot\,,\rho), u_{11}(\,\cdot\,,\rho), \dots, u_{1\,n-1}(\,\cdot\,,\rho)$ of the differential equation (2.1):

$$\begin{aligned}
u_{10}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_0} v_{10}(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho\omega_0}[y_0(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_0 t} E_{100}(t,\rho)\rho^{-m}],\\
u_{1k}(t,\rho) &= v_{1k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k t} E_{1k0}(t,\rho)\rho^{-m},\\
&\qquad\qquad\qquad\qquad\qquad\qquad k = 1, \dots, \nu-1,\\
u_{1\nu}(t,\rho) &= v_{1\nu}(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_\nu}[y_\nu(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_\nu (t-1)} E_{1\nu 0}(t,\rho)\rho^{-m}],\\
u_{1k}(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} v_{1k}(t,\rho) = y_k(t,\rho) + \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)} E_{1k0}(t,\rho)\rho^{-m},\\
&\qquad\qquad\qquad\qquad\qquad\qquad k = \nu+1, \dots, n-1,
\end{aligned}$$

for $0 \le t \le 1$. These functions also form a basis for the solution space of the differential equation (2.1), and together with their derivatives they can be expressed as

$$\begin{aligned}
u_{10}^{(\alpha)}(t,\rho) &= \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_0 (t-1)}\big[(\mathrm{i}\omega_0)^{\alpha} + F_{10\alpha}(t,\rho)\big],\\
u_{1k}^{(\alpha)}(t,\rho) &= \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{1k\alpha}(t,\rho)\big], \qquad k = 1, \dots, \nu-1,\\
u_{1\nu}^{(\alpha)}(t,\rho) &= \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_\nu t}\big[(\mathrm{i}\omega_\nu)^{\alpha} + F_{1\nu\alpha}(t,\rho)\big],\\
u_{1k}^{(\alpha)}(t,\rho) &= \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{1k\alpha}(t,\rho)\big], \quad k = \nu+1, \dots, n-1,
\end{aligned} \tag{6.40}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, and for $\alpha = 0, 1, \dots, n-1$. From equation (5.48) we have the bounds

$$|u_{1k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 0, 1, \dots, n-1, \tag{6.41}$$

for $\rho = a + \mathrm{i}b$ in the sector S_1 (where $b \le 0$) with $|\rho| > R_1$.

For $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in T_1$ satisfying $|\rho| > R_0$, we can again compute the Green's function $g(t,s;\lambda)$ of the differential operator $\lambda I - L_0$, but now we use the basis $v_{1k}(\,\cdot\,,\rho)$, $k = 0, 1, \dots, n-1$:

$$\begin{aligned}
g(t,s;\lambda) &= \sum_{k=0}^{n-1} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le s < t \le 1,\\
g(t,s;\lambda) &= 0, \qquad\qquad\qquad\qquad\qquad 0 \le t < s \le 1,
\end{aligned} \tag{6.42}$$

where the functions $\eta_{1k}(\,\cdot\,,\rho)$, $k=0,1,\dots,n-1$, belong to $H^n[0,1]$ and are determined by the linear system

$$\sum_{k=0}^{n-1} v_{1k}^{(\alpha)}(s,\rho)\eta_{1k}(s,\rho) = -\delta_{\alpha\,n-1}\mathrm{i}^n, \qquad \alpha = 0,1,\dots,n-1, \tag{6.43}$$

for $0 \le s \le 1$. Using (6.38), we can write this system in the form

$$\sum_{k=0}^{n-1}\big[(\mathrm{i}\omega_k)^\alpha + F_{1k\alpha}(s,\rho)\big]\mathrm{e}^{\mathrm{i}\rho\omega_k s}\eta_{1k}(s,\rho) = -\delta_{\alpha\,n-1}\mathrm{i}^n\rho^{-(n-1)}, \qquad \alpha = 0,1,\dots,n-1, \tag{6.44}$$

for $0 \le s \le 1$. We will consider this as a linear system for the unknowns $\mathrm{e}^{\mathrm{i}\rho\omega_k s}\eta_{1k}(s,\rho)$, $k=0,1,\dots,n-1$, with coefficient matrix

$$A_1(s,\rho) := \big((\mathrm{i}\omega_k)^\alpha + F_{1k\alpha}(s,\rho)\big)$$

that is nonsingular for $0 \le s \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$.

Proceeding as above, we express the inverse $A_1(s,\rho)^{-1}$ in the form

$$A_1(s,\rho)^{-1} := \Big(\frac{1}{n\mathrm{i}^k}\,\omega_\alpha^{n-k}\,[1 + G_{1k\alpha}(s,\rho)]\Big)$$

for $0 \le s \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, and it then follows that

$$\eta_{1k}(s,\rho) = -\,\frac{\mathrm{i}\omega_k}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{1\,n-1\,k}(s,\rho)] \tag{6.45}$$

for $0 \le s \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k = 0,1,\dots,n-1$. For fixed ρ the functions $G_{1\,n-1\,k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$, and appealing to the continuity of matrix inversion once more, the $G_{1k\alpha}(s,\rho)$ go to 0 uniformly on $[0,1]\times T_1$ as $|\rho|\to\infty$ for $k,\alpha = 0,1,\dots,n-1$. Without loss of generality we can assume that the constant R_1 chosen earlier also produces the bound

$$|G_{1k\alpha}(s,\rho)| \le 1 \tag{6.46}$$

for $0 \le s \le 1$, for $\rho \in T_1$ with $|\rho| > R_1$, and for $k,\alpha = 0,1,\dots,n-1$.

Thus, for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$, the Green's function for the differential operator $\lambda I - L_0$ is given by

$$\begin{aligned} g(t,s;\lambda) &= -\,\frac{1}{n\rho^{n-1}}\sum_{k=0}^{n-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{1k0}(t,\rho)][1 + G_{1\,n-1\,k}(s,\rho)], && 0 \le s < t \le 1,\\ g(t,s;\lambda) &= 0, && 0 \le t < s \le 1, \end{aligned} \tag{6.47}$$

where the functions $F_{1k0}(\,\cdot\,,\rho)$ and $G_{1\,n-1\,k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$ and satisfy the bounds given in equations (6.39) and (6.46).

Next, recall that $\omega_n = \omega_0 = 1$ and $\omega_\nu = -1$. To simplify the discussion, we set

$$\begin{aligned} &v_{1n}(t,\rho) := v_{10}(t,\rho), \\ &F_{1n\alpha}(t,\rho) := F_{10\alpha}(t,\rho), \qquad \alpha = 0,1,\ldots,n-1, \\ &\eta_{1n}(s,\rho) := \eta_{10}(s,\rho), \qquad G_{1\,n-1\,n}(s,\rho) := G_{1\,n-1\,0}(s,\rho), \end{aligned}$$

for $0 \le t,s \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$. With this change of notation, we can then rewrite (6.47) in the form

$$\begin{aligned} g(t,s;\lambda) &= -\frac{1}{n\rho^{n-1}}\sum_{k=1}^{n}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{1k0}(t,\rho)][1+G_{1\,n-1\,k}(s,\rho)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s < t \le 1, \\ g(t,s;\lambda) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 0 \le t < s \le 1, \end{aligned} \tag{6.48}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$. The representations (6.42), (6.47), and (6.48) can be expressed in the alternate form

$$g(t,s;\lambda) = k_1(t,s;\rho) + \ell_1(t,s;\rho)$$

for $t \ne s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$, where

$$\begin{aligned} k_1(t,s;\rho) &:= \sum_{k=1}^{\nu} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le s < t \le 1, \\ k_1(t,s;\rho) &:= -\sum_{k=\nu+1}^{n} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le t < s \le 1, \end{aligned} \tag{6.49}$$

and

$$\ell_1(t,s;\rho) := \sum_{k=\nu+1}^{n} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le t,s \le 1.$$

For each $\rho \in T_1$ with $|\rho| > R_0$, let $K_{1\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$K_{1\rho}u(t) := \int_0^1 k_1(t,s;\rho)u(s)\,ds, \qquad 0 \le t \le 1,$$

for $u \in L^2[0,1]$. If $u \in L^2[0,1]$ and $v = K_{1\rho}u$, then it follows that v belongs to $H^n[0,1]$ and $(\rho^n I - \ell)v = u$, and by direct calculation

$$\begin{aligned} v(t) &= \int_0^1 k_1(t,s;\rho)u(s)\,ds \\ &= \sum_{k=1}^{\nu} v_{1k}(t,\rho)\int_0^t \eta_{1k}(s,\rho)u(s)\,ds - \sum_{k=\nu+1}^{n} v_{1k}(t,\rho)\int_t^1 \eta_{1k}(s,\rho)u(s)\,ds \end{aligned}$$

for $0 \le t \le 1$. Using equation (6.43) with $\alpha = 0$, we get

$$\begin{aligned} v'(t) &= \sum_{k=1}^{\nu} v'_{1k}(t,\rho)\int_0^t \eta_{1k}(s,\rho)u(s)\,ds - \sum_{k=\nu+1}^{n} v'_{1k}(t,\rho)\int_t^1 \eta_{1k}(s,\rho)u(s)\,ds \\ &= \int_0^1 \frac{\partial k_1}{\partial t}(t,s;\rho)u(s)\,ds \end{aligned}$$

for $0 \le t \le 1$. Proceeding by induction, we see immediately that the derivatives of $v = K_{1\rho}u$ satisfy the equations

$$\begin{aligned} v^{(j)}(t) &= \int_0^1 \frac{\partial^j k_1}{\partial t^j}(t,s;\rho)u(s)\,ds \\ &= \sum_{k=1}^{\nu} v^{(j)}_{1k}(t,\rho)\int_0^t \eta_{1k}(s,\rho)u(s)\,ds - \sum_{k=\nu+1}^{n} v^{(j)}_{1k}(t,\rho)\int_t^1 \eta_{1k}(s,\rho)u(s)\,ds \end{aligned} \tag{6.50}$$

for $0 \le t \le 1$ and for $j = 0, 1, \ldots, n-1$, valid for each $\rho \in T_1$ with $|\rho| > R_0$.

Using (6.38) and (6.45), for $\rho \in T_1$ with $|\rho| > R_0$ the kernel $k_1(t,s;\rho)$ can be expressed as

$$\begin{aligned} k_1(t,s;\rho) &= -\frac{1}{n\rho^{n-1}}\sum_{k=1}^{\nu}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{1k0}(t,\rho)][1+G_{1\,n-1\,k}(s,\rho)], \\ &\qquad\qquad 0 \le s < t \le 1, \\ k_1(t,s;\rho) &= \frac{1}{n\rho^{n-1}}\sum_{k=\nu+1}^{n}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{1k0}(t,\rho)][1+G_{1\,n-1\,k}(s,\rho)], \\ &\qquad\qquad 0 \le t < s \le 1. \end{aligned} \tag{6.51}$$

Now take any point $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_0$. If t, s are real numbers with $0 \le s < t \le 1$, then $0 < t - s \le 1$, and by (5.8) we obtain the estimates

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le \mathrm{e}^{-(t-s)\alpha|\rho|} \le 1, \qquad k = 1,\ldots,\nu-1, \tag{6.52}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_\nu(t-s)}\right| = \mathrm{e}^{b(t-s)}. \tag{6.53}$$

On the other hand, for real numbers t, s with $0 \le t < s \le 1$, we have

$$0 < s-t \le 1, \qquad -1 \le t-s < 0, \qquad 0 \le 1+t-s < 1,$$

and hence, by (5.9)

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(1+t-s-1)}\right| \le \mathrm{e}^{-(s-t)\alpha|\rho|} \le 1, \qquad k = \nu+1,\ldots,n-1, \tag{6.54}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_n(t-s)}\right| = \mathrm{e}^{b(s-t)}. \tag{6.55}$$

In particular, if $\rho = a + \mathrm{i}b \in S_1$ (where $b \le 0$) with $|\rho| > R_1$, then these estimates give

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le s < t \le 1, \quad k = 1, \dots, \nu, \tag{6.56}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le t < s \le 1, \quad k = \nu + 1, \dots, n. \tag{6.57}$$

Applying (6.56), (6.57) and (6.39), (6.46) to the representation (6.51), it follows that

$$|k_1(t, s; \rho)| \le \frac{2}{|\rho|^{n-1}} \tag{6.58}$$

for $t \ne s$ in $[0, 1]$ and for $\rho \in S_1$ with $|\rho| > R_1$.

Finally, fix any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$, and assume that $\Delta_1(\rho) \ne 0$, so the point λ belongs to the resolvent set $\rho(L)$. We are going to establish representations of the resolvent $R_\lambda(L)$ and the associated Green's function $G(t, s; \lambda)$. Indeed, take any $u \in L^2[0, 1]$, and set

$$v = K_{1\rho} u \quad \text{and} \quad w = R_\lambda(L)u.$$

Clearly the functions v and w belong to $H^n[0, 1]$, and

$$(\lambda I - \ell)v = u = (\lambda I - \ell)w.$$

Thus, there exist constants $c_0, c_1, \dots, c_{n-1}$ (depending on ρ) such that

$$R_\lambda(L)u(t) = w(t) = v(t) + \sum_{k=0}^{n-1} c_k u_{1k}(t, \rho), \qquad 0 \le t \le 1.$$

The functions $u_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \dots, n-1$, were introduced earlier. They form a basis for the solution space of the differential equation (2.1), and the characteristic determinant $\Delta_1(\rho)$ is defined in terms of them.

Applying the boundary value B_i to the last equation, we obtain the linear system

$$\sum_{k=0}^{n-1} M_{1ik}(\rho) c_k = -B_i(v), \qquad i = 1, \dots, n, \tag{6.59}$$

for the constants $c_0, c_1, \dots, c_{n-1}$, where as in Chapter 5

$$M_{1ik}(\rho) = B_i(u_{1k}(\,\cdot\,, \rho)) = \sum_{j=0}^{m_i} \alpha_{ij} u_{1k}^{(j)}(0, \rho) + \sum_{j=0}^{m_i} \beta_{ij} u_{1k}^{(j)}(1, \rho)$$

for $i = 1, \dots, n$, $k = 0, 1, \dots, n-1$. Note that $\det(M_{1ik}(\rho)) = \Delta_1(\rho) \ne 0$.

Fix an index i with $1 \le i \le n$, and consider the quantity $B_i(v) = B_i(K_{1\rho}u)$. From equation (6.50) and equations (6.38), (6.45), we have

$$
\begin{aligned}
v^{(j)}(0) &= -\sum_{k=\nu+1}^{n} v_{1k}^{(j)}(0,\rho)\int_0^1 \eta_{1k}(s,\rho)u(s)\,ds \\
&= \frac{1}{n\rho^{n-1}}\sum_{k=\nu+1}^{n}(\mathrm{i}\omega_k)\rho^j[(\mathrm{i}\omega_k)^j \\
&\qquad + F_{1kj}(0,\rho)]\int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1+G_{1\,n-1\,k}(s,\rho)]u(s)\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
v^{(j)}(1) &= \sum_{k=1}^{\nu} v_{1k}^{(j)}(1,\rho)\int_0^1 \eta_{1k}(s,\rho)u(s)\,ds \\
&= -\frac{1}{n\rho^{n-1}}\sum_{k=1}^{\nu}(\mathrm{i}\omega_k)\rho^j[(\mathrm{i}\omega_k)^j \\
&\qquad + F_{1kj}(1,\rho)]\int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)}[1+G_{1\,n-1\,k}(s,\rho)]u(s)\,ds
\end{aligned}
$$

for $j = 0, 1, \ldots, n-1$, and hence,

$$
\begin{aligned}
B_i(v) &= \sum_{j=0}^{m_i}\alpha_{ij}v^{(j)}(0) + \sum_{j=0}^{m_i}\beta_{ij}v^{(j)}(1) \\
&= \frac{1}{n\rho^{n-1}}\sum_{k=1}^{\nu}\Big\{\sum_{j=0}^{m_i}(-1)\beta_{ij}(\mathrm{i}\omega_k)\rho^j[(\mathrm{i}\omega_k)^j + F_{1kj}(1,\rho)]\Big\} \\
&\qquad \times\int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)}[1+G_{1\,n-1\,k}(s,\rho)]u(s)\,ds \\
&\quad + \frac{1}{n\rho^{n-1}}\sum_{k=\nu+1}^{n}\Big\{\sum_{j=0}^{m_i}\alpha_{ij}(\mathrm{i}\omega_k)\rho^j[(\mathrm{i}\omega_k)^j + F_{1kj}(0,\rho)]\Big\} \\
&\qquad \times\int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1+G_{1\,n-1\,k}(s,\rho)]u(s)\,ds
\end{aligned}
\tag{6.60}
$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $k = 1, \ldots, \nu$ define

$$
\mathcal{T}_{1ik}(\rho) := \sum_{j=0}^{m_i}(-1)\beta_{ij}(\mathrm{i}\omega_k)\rho^j[(\mathrm{i}\omega_k)^j + F_{1kj}(1,\rho)]
$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu+1, \ldots, n$ define

$$
\mathcal{T}_{1ik}(\rho) := \sum_{j=0}^{m_i}\alpha_{ij}(\mathrm{i}\omega_k)\rho^j[(\mathrm{i}\omega_k)^j + F_{1kj}(0,\rho)]
$$

for $\rho \in T_1$ with $|\rho| > R_0$; and for $k = 1, \ldots, \nu$ define

$$U_{1k}(s,\rho) := \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)}[1 + G_{1\,n-1\,k}(s,\rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $k = \nu + 1, \ldots, n$ define

$$U_{1k}(s,\rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{1\,n-1\,k}(s,\rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_1$ with $|\rho| > R_0$. Then we can express (6.60) in the simpler form

$$B_i(v) = \frac{1}{n\rho^{n-1}} \sum_{k=1}^{n} \mathcal{T}_{1ik}(\rho) \int_0^1 U_{1k}(s,\rho) u(s)\, ds \tag{6.61}$$

for $i = 1, \ldots, n$, where the functions $\mathcal{T}_{1ik}(\rho)$ are analytic functions of ρ on G_1, and for fixed ρ in T_1 with $|\rho| > R_0$ the functions $U_{1k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$.

In terms of the matrix $(M_{1ik}(\rho))$, let $\widetilde{M}_{1ik}(\rho)$ denote the cofactor of the entry $M_{1ik}(\rho)$:

$\widetilde{M}_{1ik}(\rho) := (-1)^{i+k+1}$ times the determinant of the $(n-1) \times (n-1)$
submatrix of $(M_{1j\ell}(\rho))$ obtained by deleting
the ith row and kth column.

Clearly the cofactors $\widetilde{M}_{1ik}(\rho)$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$, are analytic functions in the ρ variable on the open set G_1. Using these cofactors, we proceed to solve for the constants $c_0, c_1, \ldots, c_{n-1}$ in the linear system (6.59) by means of Cramer's rule:

$$\begin{aligned} c_k &= \frac{-1}{\Delta_1(\rho)} \det \begin{array}{c} \\ \left(\begin{array}{ccccc} & & k\text{th column} & & \\ M_{110}(\rho) & \cdots & B_1(v) & \cdots & M_{11n-1}(\rho) \\ \vdots & & \vdots & & \vdots \\ M_{1n0}(\rho) & \cdots & B_n(v) & \cdots & M_{1nn-1}(\rho) \end{array} \right) \end{array} \\ &= \frac{-1}{\Delta_1(\rho)} \sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho) B_j(v), \end{aligned}$$

or

$$c_k = \frac{-1}{n\rho^{n-1}\Delta_1(\rho)} \sum_{l=1}^{n} \sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho) \mathcal{T}_{1jl}(\rho) \int_0^1 U_{1l}(s,\rho) u(s)\, ds \tag{6.62}$$

for $k = 0, 1, \ldots, n-1$.

Combining the above results, we conclude that

$$\begin{aligned} R_\lambda(L)u(t) &= K_{1\rho}u(t) \\ &\quad - \frac{1}{n\rho^{n-1}\Delta_1(\rho)} \sum_{k=0}^{n-1} \sum_{j,l=1}^{n} \widetilde{M}_{1jk}(\rho) \mathcal{T}_{1jl}(\rho) u_{1k}(t,\rho) \int_0^1 U_{1l}(s,\rho) u(s)\, ds \end{aligned} \tag{6.63}$$

for $0 \leq t \leq 1$ and for $u \in L^2[0,1]$, where (6.63) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\varDelta_1(\rho) \neq 0$. Also, from (6.63) the associated Green's function is given by

$$\begin{aligned} G(t,s;\lambda) &= k_1(t,s;\rho) \\ &\quad - \frac{1}{n\rho^{n-1}\varDelta_1(\rho)} \sum_{k=0}^{n-1} \sum_{j,l=1}^{n} \widetilde{M}_{1jk}(\rho)\mathcal{T}_{1jl}(\rho)u_{1k}(t,\rho)U_{1l}(s,\rho) \end{aligned} \tag{6.64}$$

for $t \neq s$ in $[0,1]$, where (6.64) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\varDelta_1(\rho) \neq 0$.

Let us proceed to determine bounds and growth rates for the various functions appearing in equations (6.63) and (6.64). For the basis functions $u_{1k}(t,\rho)$ we have already established the necessary boundedness in equation (6.41). For the kernel $k_1(t,s;\rho)$ the required growth rate is given in equation (6.58). Consider the functions $\mathcal{T}_{1ik}(\rho)$, $U_{1k}(s,\rho)$, and $\widetilde{M}_{1ik}(\rho)$.

First, for the functions $\mathcal{T}_{1ik}(\rho)$, from their definitions and the estimate (6.39) it is immediate that

$$|\mathcal{T}_{1ik}(\rho)| \leq \gamma_1 |\rho|^{m_i} \tag{6.65}$$

for $\rho \in T_1$ with $|\rho| > R_1$ and for $i = 1,\ldots,n$, $k = 1,\ldots,n$. Second, for the functions $U_{1k}(s,\rho)$, from their definitions and the earlier estimates (5.8), (5.9), together with (6.46), we obtain the estimates (replace t by $1-s$)

$$|U_{1k}(s,\rho)| \leq 2\mathrm{e}^{-(1-s)\alpha|\rho|} \leq 2, \qquad k = 1,\ldots,\nu-1, \tag{6.66}$$

$$|U_{1\nu}(s,\rho)| \leq 2\mathrm{e}^{b(1-s)}, \tag{6.67}$$

$$|U_{1k}(s,\rho)| \leq 2\mathrm{e}^{-s\alpha|\rho|} \leq 2, \qquad k = \nu+1,\ldots,n-1, \tag{6.68}$$

$$|U_{1n}(s,\rho)| \leq 2\mathrm{e}^{bs} \tag{6.69}$$

for $0 \leq s \leq 1$ and for $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_1 (where $b \leq 0$) with $|\rho| > R_1$, we have

$$|U_{1k}(s,\rho)| \leq 2, \qquad 0 \leq s \leq 1, \quad k = 1,\ldots,n. \tag{6.70}$$

Third, fix indices i and k with $1 \leq i \leq n$ and $0 \leq k \leq n-1$, and let us consider the cofactor $\widetilde{M}_{1ik}(\rho)$ for $\rho \in G_1$. It is formed by taking $(-1)^{i+k+1}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the kth column of the matrix

$$\begin{array}{cccc} & {\scriptstyle 1\leq k\leq \nu-1} & & {\scriptstyle \nu+1\leq k\leq n-1} \\ \left(\begin{matrix} P_{110}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{110}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) & P_{11\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{11\nu}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{11k}(\rho) \\ \vdots & \vdots & \vdots & \vdots \\ P_{1n0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{1n0}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) & P_{1n\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{1n\nu}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{1nk}(\rho) \end{matrix}\right) \end{array}.$$

Suppose we proceed to expand the determinant for $\widetilde{M}_{1ik}(\rho)$ in the same manner as $\Delta_1(\rho)$ was expanded in Chapter 5. See equation (5.56). If $1 \le k \le \nu-1$ or $\nu+1 \le k \le n-1$, then in the determinant of the $(n-1)\times(n-1)$ submatrix leading to $\widetilde{M}_{1ik}(\rho)$, we first factor out the term $\mathrm{e}^{-\mathrm{i}\rho}$ from the original 0th and νth columns, and then we apply linearity to these two columns; this yields the terms 1, $\mathrm{e}^{-\mathrm{i}\rho}$, $\mathrm{e}^{-2\mathrm{i}\rho}$. On the other hand, if $k = 0$, then $\mathrm{e}^{-\mathrm{i}\rho}$ is factored out of the original νth column, and then linearity is applied to this column; this yields the terms 1, $\mathrm{e}^{-\mathrm{i}\rho}$. And if $k = \nu$, then initially $\mathrm{e}^{-\mathrm{i}\rho}$ is factored out of the original 0th column, and then linearity is applied to this column; this also yields the terms 1, $\mathrm{e}^{-\mathrm{i}\rho}$.

Next, all the $(n-1)\times(n-1)$ determinants are expanded using linearity in all $n-1$ columns. This expansion produces the following representations: for $1 \le k \le \nu-1$ or for $\nu+1 \le k \le n-1$,

$$\begin{aligned}\widetilde{M}_{1ik}(\rho) &= \mathrm{e}^{-2\mathrm{i}\rho}[\tilde{\pi}''_{ik}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \tilde{\pi}'_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\pi}_{ik}(\rho) \\ &\qquad + \tilde{\phi}''_{1ik}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \tilde{\phi}'_{1ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\phi}_{1ik}(\rho)] \\ &= \tilde{\pi}''_{ik}(\rho) + \tilde{\pi}'_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \tilde{\pi}_{ik}(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \\ &\qquad + \tilde{\phi}''_{1ik}(\rho) + \tilde{\phi}'_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \tilde{\phi}_{1ik}(\rho)\mathrm{e}^{-2\mathrm{i}\rho}\end{aligned} \tag{6.71}$$

for $\rho \in G_1$; for $k = 0$

$$\begin{aligned}\widetilde{M}_{1i0}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}[\tilde{\pi}'_{i0}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\pi}_{i0}(\rho) + \tilde{\phi}'_{1i0}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\phi}_{1i0}(\rho)] \\ &= \tilde{\pi}'_{i0}(\rho) + \tilde{\pi}_{i0}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \tilde{\phi}'_{1i0}(\rho) + \tilde{\phi}_{1i0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}\end{aligned} \tag{6.72}$$

for $\rho \in G_1$; and for $k = \nu$

$$\begin{aligned}\widetilde{M}_{1i\nu}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}[\tilde{\pi}'_{i\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\pi}_{i\nu}(\rho) + \tilde{\phi}'_{1i\nu}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\phi}_{1i\nu}(\rho)] \\ &= \tilde{\pi}'_{i\nu}(\rho) + \tilde{\pi}_{i\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \tilde{\phi}'_{1i\nu}(\rho) + \tilde{\phi}_{1i\nu}(\rho)\mathrm{e}^{-\mathrm{i}\rho}\end{aligned} \tag{6.73}$$

for $\rho \in G_1$. In these equations the functions $\tilde{\pi}''_{ik}$, $\tilde{\pi}'_{ik}$, $\tilde{\pi}_{ik}$ are formed from the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$ introduced in Chapter 3. They are the same functions that appear in (6.33); they are analytic for $\rho \neq 0$ in $\mathbb{C}$, and each one has the simple form

$$\sum_{j=-(n-1)(m-1)}^{p_0-m_i} A_{ikj}\rho^j, \qquad \rho \neq 0 \text{ in } \mathbb{C}.$$

All the perturbation terms are contained in the functions $\tilde{\phi}''_{1ik}$, $\tilde{\phi}'_{1ik}$, $\tilde{\phi}_{1ik}$; they are analytic for ρ in the open set G_1, and satisfy the growth rates

$$\begin{gathered}|\tilde{\phi}''_{1ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)}, \qquad |\tilde{\phi}'_{1ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)}, \\ |\tilde{\phi}_{1ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)}\end{gathered} \tag{6.74}$$

for $\rho \in G_1$.

Fourth, consider any point $\rho = a + \mathrm{i}b$ belonging to the sector S_1 with $|\rho| > R_1$. Clearly $b \leq 0$, $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^b \leq 1$, and $|\mathrm{e}^{-2\mathrm{i}\rho}| \leq 1$, and hence, by the representations (6.71)–(6.73) and the bounds in (6.74),

$$|\widetilde{M}_{1ik}(\rho)| \leq 3\gamma_3|\rho|^{p_0-m_i} + 3\gamma_2|\rho|^{-(m-p_0+m_i)} \leq \gamma_4|\rho|^{p_0-m_i} \tag{6.75}$$

for $i = 1, \dots, n$, $k = 0, 1, \dots, n-1$. Combining this result with (6.65), we obtain the estimate

$$\Big|\sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho)\mathcal{T}_{1jl}(\rho)\Big| \leq \sum_{j=1}^{n} \gamma_4|\rho|^{p_0-m_j} \cdot \gamma_1|\rho|^{m_j} \leq \gamma_5|\rho|^{p_0} \tag{6.76}$$

for $\rho \in S_1$ with $|\rho| > R_1$ and for $k = 0, 1, \dots, n-1$, $l = 1, \dots, n$.

Finally, take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$, with $|\rho| > R_1$, and with $\Delta_1(\rho) \neq 0$. Clearly $\rho \in T_1$ with $|\rho| > R_1 \geq R_0$, $\rho \in G_1$, and $b \leq 0$, and λ belongs to the resolvent set $\rho(L)$ with the Green's function $G(t, s; \lambda)$ given by (6.64). Applying the estimates (6.58), (6.76), (6.41), and (6.70) to (6.64), we see that

$$|G(t, s; \lambda)| \leq \frac{2}{|\rho|^{n-1}} + \frac{1}{n|\rho|^{n-1}|\Delta_1(\rho)|} \sum_{k=0}^{n-1}\sum_{l=1}^{n} \gamma_5|\rho|^{p_0} \cdot 2 \cdot 2,$$

or

$$|G(t, s; \lambda)| \leq \frac{2}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_1(\rho)|} \tag{6.77}$$

for $t \neq s$ in $[0, 1]$, where (6.77) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in S_1$ and $|\rho| > R_1$ and with $\Delta_1(\rho) \neq 0$. Cf. equation (6.37) and equation (9.8) in Chapter 4 of [34]. In Chapter 9 we will use the estimates (6.37) and (6.77) of the Green's function $G(t, s; \lambda)$ to establish the completeness of the generalized eigenfunctions of the differential operator L.

6.2 The Green's Function for n Odd

Assume that n is odd: $n = 2\nu - 1 \geq 3$. For the odd order case the sectors S_0 and S_1 are given by

$$S_0\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{2n} \leq \theta \leq \frac{\pi}{2n},$$
$$S_1\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \pi - \frac{\pi}{2n} \leq \theta \leq \pi + \frac{\pi}{2n},$$

with T_0 and T_1 the corresponding translated sectors. In the last half of this chapter we again construct representations of the resolvent and the Green's function, and then use these representation to derive their growth rates for ρ

belonging to the sectors S_0 and S_1. The treatment closely follows the development in the first half for the case n even. Consequently, we will only indicate the main features of the theory for the odd order case.

For each $\rho \in T_0$ with $|\rho| > R_0$, let $v_{00}(\,\cdot\,,\rho), v_{01}(\,\cdot\,,\rho), \ldots, v_{0\,n-1}(\,\cdot\,,\rho)$ be the basis for the solution space of the differential equation (2.1) determined in Theorem 4.6. In Chapter 5 we showed that (see equations (5.80) and (5.81))

$$v_{0k}^{(\alpha)}(t,\rho) = \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{0k\alpha}(t,\rho)\big] \tag{6.78}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$, where the function $F_{0k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$ with

$$|F_{0k\alpha}(t,\rho)| \le 1 \tag{6.79}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_1 \ge R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. Also in Chapter 5, for $\rho \in T_0$ with $|\rho| > R_0$ we formed the modified solutions $u_{00}(\,\cdot\,,\rho), u_{01}(\,\cdot\,,\rho), \ldots, u_{0\,n-1}(\,\cdot\,,\rho)$ of the differential equation (2.1). These functions have the representations (5.82):

$$\begin{aligned} u_{0k}^{(\alpha)}(t,\rho) &= \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{0k\alpha}(t,\rho)\big], \qquad k = 0, 1, \ldots, \nu - 1, \\ u_{0k}^{(\alpha)}(t,\rho) &= \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{0k\alpha}(t,\rho)\big], \quad k = \nu, \ldots, n - 1, \end{aligned} \tag{6.80}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $\alpha = 0, 1, \ldots, n-1$. From equations (5.89)–(5.91) we have the bounds

$$|u_{00}(t,\rho)| \le 2\mathrm{e}^{-bt}, \tag{6.81}$$

$$|u_{0k}(t,\rho)| \le 2\mathrm{e}^{-t\alpha|\rho|} \le 2, \qquad k = 1, \ldots, \nu - 1, \tag{6.82}$$

$$|u_{0k}(t,\rho)| \le 2\mathrm{e}^{-(1-t)\alpha|\rho|} \le 2, \quad k = \nu, \ldots, n - 1, \tag{6.83}$$

for $0 \le t \le 1$ and for $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_0 with $|\rho| > R_1$ and $b \ge 0$, we have

$$|u_{0k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 0, 1, \ldots, n - 1. \tag{6.84}$$

We must form another set of modified solutions in order to treat the case of $\rho = a + \mathrm{i}b$ in S_0 with $|\rho| > R_1$ and $b \le 0$.

Indeed, set $v_{0n}(t,\rho) := v_{00}(t,\rho)$ and

$$F_{0n\alpha}(t,\rho) := F_{00\alpha}(t,\rho), \qquad \alpha = 0, 1, \ldots, n - 1,$$

for $0 \le t \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$, and then set

$$\begin{aligned} u_{0n}(t,\rho) :&= \mathrm{e}^{-\mathrm{i}\rho\omega_0} u_{00}(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho\omega_n} v_{0n}(t,\rho) \\ &= \mathrm{e}^{\mathrm{i}\rho\omega_n(t-1)}[1 + F_{0n0}(t,\rho)] \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. Clearly

$$u_{0n}^{(\alpha)}(t,\rho) = \rho^{\alpha} \mathrm{e}^{\mathrm{i}\rho\omega_n(t-1)} \big[(\mathrm{i}\omega_n)^{\alpha} + F_{0n\alpha}(t,\rho)\big] \tag{6.85}$$

for $0 \le t \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $\alpha = 0, 1, \ldots, n-1$, and clearly the functions $u_{0k}(\,\cdot\,,\rho)$, $k = 1, \ldots, n$, also form a basis for the solution space of the differential equation (2.1). Note that

$$|u_{0n}(t,\rho)| \le 2\mathrm{e}^{-b(t-1)} \tag{6.86}$$

for $0 \le t \le 1$ and for $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_1$, and hence, for $\rho = a + \mathrm{i}b$ in S_0 with $|\rho| > R_1$ and $b \le 0$, we have

$$|u_{0k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 1, \ldots, n. \tag{6.87}$$

In terms of this new basis, we can form the new characteristic determinant

$$\Delta_0^*(\rho) := \det \begin{pmatrix} B_1(u_{01}(\,\cdot\,,\rho)) & \cdots & B_1(u_{0n}(\,\cdot\,,\rho)) \\ \vdots & & \vdots \\ B_n(u_{01}(\,\cdot\,,\rho)) & \cdots & B_n(u_{0n}(\,\cdot\,,\rho)) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\rho} \Delta_0(\rho)$$

for $\rho \in G_0$.

Next, as earlier in this chapter let L_0 be the nth order differential operator in $L^2[0,1]$ defined by

$$\mathcal{D}(L_0) = \{u \in H^n[0,1] \mid u^{(n-i)}(0) = 0,\ i = 1, \ldots, n\}, \qquad L_0 u = \ell u.$$

Then the resolvent set $\rho(L_0)$ is equal to $\mathbb{C}$, and the Green's function $g(t,s;\lambda)$ of the differential operator $\lambda I - L_0$ is given by

$$\begin{aligned} g(t,s;\lambda) &= \sum_{k=0}^{n-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), && 0 \le s < t \le 1, \\ g(t,s;\lambda) &= 0, && 0 \le t < s \le 1, \end{aligned} \tag{6.88}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$. The functions $\eta_{0k}(\,\cdot\,,\rho)$, $k = 0, 1, \ldots, n-1$, belong to $H^n[0,1]$ and are determined by the linear system

$$\sum_{k=0}^{n-1} v_{0k}^{(\alpha)}(s,\rho)\eta_{0k}(s,\rho) = -\delta_{\alpha\, n-1}\mathrm{i}^{\,n}, \qquad \alpha = 0, 1, \ldots, n-1, \tag{6.89}$$

for $0 \le s \le 1$. This system can be rewritten in the form

$$\begin{gathered} \sum_{k=0}^{n-1} \big[(\mathrm{i}\omega_k)^{\alpha} + F_{0k\alpha}(s,\rho)\big] \mathrm{e}^{\mathrm{i}\rho\omega_k s} \eta_{0k}(s,\rho) = -\delta_{\alpha\, n-1}\mathrm{i}^{\,n}\rho^{-(n-1)}, \\ \alpha = 0, 1, \ldots, n-1, \end{gathered} \tag{6.90}$$

for $0 \le s \le 1$. We know that the $n \times n$ matrix $A_0(s,\rho) := \big((\mathrm{i}\omega_k)^{\alpha} + F_{0k\alpha}(s,\rho)\big)$ is nonsingular for $0 \le s \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$, and its inverse can be expressed in the form

$$A_0(s,\rho)^{-1} := \Big(\frac{1}{n\mathrm{i}^k}\,\omega_\alpha^{n-k}\,[1+G_{0k\alpha}(s,\rho)]\Big)$$

for $0 \le s \le 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. It follows that

$$\eta_{0k}(s,\rho) = -\,\frac{\mathrm{i}\omega_k}{n\rho^{n-1}}\,\mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1+G_{0\,n-1\,k}(s,\rho)] \tag{6.91}$$

for $0 \le s \le 1$, for $\rho \in T_0$ with $|\rho| > R_0$, and for $k = 0,1,\ldots,n-1$, with the functions $G_{0\,n-1\,k}(\,\cdot\,,\rho)$ belonging to $H^n[0,1]$. Since the functions $G_{0k\alpha}(s,\rho) \to 0$ uniformly on $[0,1]\times T_0$ as $|\rho|\to\infty$ for $k,\alpha = 0,1,\ldots,n-1$, we can assume that the constant R_1 chosen earlier also yields the bound

$$|G_{0k\alpha}(s,\rho)| \le 1 \tag{6.92}$$

for $0 \le s \le 1$, for $\rho \in T_0$ with $|\rho| > R_1$, and for $k,\alpha = 0,1,\ldots,n-1$.

Summarizing, for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$, the Green's function for the differential operator $\lambda I - L_0$ is given by

$$\begin{aligned} g(t,s;\lambda) &= -\,\frac{1}{n\rho^{n-1}}\sum_{k=0}^{n-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{0k0}(t,\rho)][1+G_{0\,n-1\,k}(s,\rho)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s < t \le 1, \\ g(t,s;\lambda) &= 0, \qquad\qquad\qquad\qquad\qquad\quad 0 \le t < s \le 1, \end{aligned} \tag{6.93}$$

where the functions $F_{0k0}(\,\cdot\,,\rho)$ and $G_{0\,n-1\,k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$ and satisfy the bounds given in equations (6.79) and (6.92).

Next, we rewrite (6.88) or (6.93) in the form

$$g(t,s;\lambda) = k_0(t,s;\rho) + \ell_0(t,s;\rho)$$

for $t \ne s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$, where

$$\begin{aligned} k_0(t,s;\rho) &:= \sum_{k=0}^{\nu-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), && 0 \le s < t \le 1, \\ k_0(t,s;\rho) &:= -\sum_{k=\nu}^{n-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), && 0 \le t < s \le 1, \end{aligned} \tag{6.94}$$

and

$$\ell_0(t,s;\rho) := \sum_{k=\nu}^{n-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), \qquad 0 \le t,s \le 1.$$

The Green's function $g(t,s;\lambda)$ is the kernel of the integral operator $R_\lambda(L_0)$, which assigns to each u in $L^2[0,1]$ the unique solution $z \in \mathcal{D}(L_0)$ of the initial value problem $(\lambda I - L_0)z = u$. On the other hand, the function $\ell_0(t,s;\rho)$ is the kernel of an integral operator which maps $L^2[0,1]$ into the solution space of the differential equation $(\rho^n I - \ell)u = 0$.

For each $\rho \in T_0$ with $|\rho| > R_0$, let $K_{0\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$K_{0\rho}u(t) := \int_0^1 k_0(t,s;\rho)u(s)\,ds, \qquad 0 \le t \le 1,$$

for $u \in L^2[0,1]$. From the above remarks it follows that if $u \in L^2[0,1]$ and $v = K_{0\rho}u$, then v belongs to $H^n[0,1]$ and $(\rho^n I - \ell)v = u$, and using equation (6.89) and induction, the derivatives of $v = K_{0\rho}u$ are given by

$$\begin{aligned} v^{(j)}(t) &= \int_0^1 \frac{\partial^j k_0}{\partial t^j}(t,s;\rho)u(s)\,ds \\ &= \sum_{k=0}^{\nu-1} v_{0k}^{(j)}(t,\rho)\int_0^t \eta_{0k}(s,\rho)u(s)\,ds - \sum_{k=\nu}^{n-1} v_{0k}^{(j)}(t,\rho)\int_t^1 \eta_{0k}(s,\rho)u(s)\,ds \end{aligned} \tag{6.95}$$

for $0 \le t \le 1$ and for $j = 0, 1, \ldots, n-1$, valid for each $\rho \in T_0$ with $|\rho| > R_0$.

Using (6.78) and (6.91), the kernel $k_0(t,s;\rho)$ can be expressed as

$$\begin{aligned} k_0(t,s;\rho) &= -\frac{1}{n\rho^{n-1}}\sum_{k=0}^{\nu-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{0k0}(t,\rho)][1 + G_{0\,n-1\,k}(s,\rho)], \\ &\qquad\qquad 0 \le s < t \le 1, \\ k_0(t,s;\rho) &= \frac{1}{n\rho^{n-1}}\sum_{k=\nu}^{n-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{0k0}(t,\rho)][1 + G_{0\,n-1\,k}(s,\rho)], \\ &\qquad\qquad 0 \le t < s \le 1, \end{aligned} \tag{6.96}$$

for $\rho \in T_0$ with $|\rho| > R_0$. Now take any point $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_0$. If t, s are real numbers with $0 \le s < t \le 1$, then $0 < t - s \le 1$, and by (5.83) and (5.84) we obtain the estimates

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0(t-s)}\right| = \mathrm{e}^{-b(t-s)}, \tag{6.97}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le \mathrm{e}^{-(t-s)\alpha|\rho|} \le 1, \qquad k = 1, \ldots, \nu - 1. \tag{6.98}$$

On the other hand, for real numbers t, s with $0 \le t < s \le 1$, we have $0 \le 1 + t - s < 1$, and hence, by (5.85)

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(1+t-s-1)}\right| \le \mathrm{e}^{-(s-t)\alpha|\rho|} \le 1, \qquad k = \nu, \ldots, n-1. \tag{6.99}$$

In particular, if $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| > R_1$ and $b \ge 0$, then these estimates give

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le s < t \le 1, \quad k = 0, 1, \ldots, \nu - 1, \tag{6.100}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le t < s \le 1, \quad k = \nu, \ldots, n - 1. \tag{6.101}$$

Applying (6.100), (6.101) and (6.79), (6.92) to the representation (6.96), it follows that

$$|k_0(t, s; \rho)| \le \frac{4}{|\rho|^{n-1}} \tag{6.102}$$

for $t \ne s$ in $[0, 1]$ and for $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| > R_1$ and $b \ge 0$.

Now fix any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\Delta_0(\rho) \ne 0$, so the point λ belongs to the resolvent set $\rho(L)$. Take any function $u \in L^2[0, 1]$, and set

$$v = K_{0\rho}u \quad \text{and} \quad w = R_\lambda(L)u.$$

Then v and w belong to $H^n[0, 1]$, and $(\lambda I - \ell)v = u = (\lambda I - \ell)w$. Thus, there exist constants $c_0, c_1, \ldots, c_{n-1}$ (depending on ρ) such that

$$R_\lambda(L)u(t) = w(t) = v(t) + \sum_{k=0}^{n-1} c_k u_{0k}(t, \rho), \qquad 0 \le t \le 1.$$

The functions $u_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n - 1$, are the modified solutions of the differential equation (2.1) introduced earlier; the characteristic determinant $\Delta_0(\rho)$ is defined in terms of them.

Applying the boundary value B_i to the last equation, we obtain the linear system

$$\sum_{k=0}^{n-1} M_{0ik}(\rho)c_k = -B_i(v), \qquad i = 1, \ldots, n, \tag{6.103}$$

for the constants $c_0, c_1, \ldots, c_{n-1}$, where as in Chapter 5

$$M_{0ik}(\rho) = B_i(u_{0k}(\,\cdot\,, \rho)) = \sum_{j=0}^{m_i} \alpha_{ij} u_{0k}^{(j)}(0, \rho) + \sum_{j=0}^{m_i} \beta_{ij} u_{0k}^{(j)}(1, \rho)$$

for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n - 1$. The $n \times n$ coefficient matrix $(M_{0ik}(\rho))$ in (6.103) is nonsingular because $\det(M_{0ik}(\rho)) = \Delta_0(\rho) \ne 0$.

Fix an index i with $1 \le i \le n$, and consider the quantity $B_i(v) = B_i(K_{0\rho}u)$. From equation (6.95) and equations (6.78), (6.91), we have

$$\begin{aligned}
B_i(v) &= \sum_{j=0}^{m_i} \alpha_{ij} v^{(j)}(0) + \sum_{j=0}^{m_i} \beta_{ij} v^{(j)}(1) \\
&= \frac{1}{n\rho^{n-1}} \sum_{k=0}^{\nu-1} \Big\{ \sum_{j=0}^{m_i} (-1)\beta_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(1,\rho)] \Big\} \\
&\qquad \times \int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)} [1 + G_{0\,n-1\,k}(s,\rho)] u(s)\, ds \qquad (6.104) \\
&\quad + \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n-1} \Big\{ \sum_{j=0}^{m_i} \alpha_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(0,\rho)] \Big\} \\
&\qquad \times \int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{0\,n-1\,k}(s,\rho)] u(s)\, ds
\end{aligned}$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$ define

$$\mathcal{T}_{0ik}(\rho) := \sum_{j=0}^{m_i} (-1)\beta_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(1,\rho)]$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ define

$$\mathcal{T}_{0ik}(\rho) := \sum_{j=0}^{m_i} \alpha_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(0,\rho)]$$

for $\rho \in T_0$ with $|\rho| > R_0$; and for $k = 0, 1, \ldots, \nu - 1$ define

$$U_{0k}(s,\rho) := \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)} [1 + G_{0\,n-1\,k}(s,\rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $k = \nu, \ldots, n-1$ define

$$U_{0k}(s,\rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{0\,n-1\,k}(s,\rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_0$ with $|\rho| > R_0$. Then (6.104) can be expressed in the simpler form

$$B_i(v) = \frac{1}{n\rho^{n-1}} \sum_{k=0}^{n-1} \mathcal{T}_{0ik}(\rho) \int_0^1 U_{0k}(s,\rho) u(s)\, ds \qquad (6.105)$$

for $i = 1, \ldots, n$, where the functions $\mathcal{T}_{0ik}(\rho)$ are analytic functions of ρ on G_0, and for fixed ρ in T_0 with $|\rho| > R_0$ the functions $U_{0k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$.

In terms of the matrix $(M_{0ik}(\rho))$, let $\widetilde{M}_{0ik}(\rho)$ denote the cofactor of the entry $M_{0ik}(\rho)$. Clearly the cofactors $\widetilde{M}_{0ik}(\rho)$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$, are analytic functions in the ρ variable on the open set G_0, and in terms of them we can solve for the constants $c_0, c_1, \ldots, c_{n-1}$ that appear in the linear system (6.103):

$$\begin{aligned} c_k &= \frac{-1}{\varDelta_0(\rho)} \det \begin{array}{c} \qquad\qquad\qquad k\text{th column} \\ \begin{pmatrix} M_{010}(\rho) & \cdots & B_1(v) & \cdots & M_{01n-1}(\rho) \\ \vdots & & \vdots & & \vdots \\ M_{0n0}(\rho) & \cdots & B_n(v) & \cdots & M_{0nn-1}(\rho) \end{pmatrix} \end{array} \\ &= \frac{-1}{\varDelta_0(\rho)} \sum_{j=1}^{n} \widetilde{M}_{0jk}(\rho) B_j(v), \end{aligned}$$

or

$$c_k = \frac{-1}{n\rho^{n-1}\varDelta_0(\rho)} \sum_{l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M}_{0jk}(\rho) \mathcal{T}_{0jl}(\rho) \int_0^1 U_{0l}(s,\rho) u(s)\, ds \tag{6.106}$$

for $k = 0, 1, \ldots, n-1$.

Combining the above results, we conclude that

$$\begin{aligned} R_\lambda(L)u(t) &= K_{0\rho}u(t) \\ &\quad - \frac{1}{n\rho^{n-1}\varDelta_0(\rho)} \sum_{k,l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M}_{0jk}(\rho) \mathcal{T}_{0jl}(\rho) u_{0k}(t,\rho) \int_0^1 U_{0l}(s,\rho) u(s)\, ds \end{aligned} \tag{6.107}$$

for $0 \le t \le 1$ and for $u \in L^2[0,1]$, where (6.107) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\varDelta_0(\rho) \neq 0$. The associated Green's function is then given by

$$\begin{aligned} G(t,s;\lambda) &= k_0(t,s;\rho) \\ &\quad - \frac{1}{n\rho^{n-1}\varDelta_0(\rho)} \sum_{k,l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M}_{0jk}(\rho) \mathcal{T}_{0jl}(\rho) u_{0k}(t,\rho) U_{0l}(s,\rho) \end{aligned} \tag{6.108}$$

for $t \neq s$ in $[0,1]$, where (6.108) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\varDelta_0(\rho) \neq 0$.

Let us determine bounds and growth rates for the various functions appearing in equations (6.107) and (6.108). For the basis functions $u_{0k}(t,\rho)$, $k = 0, 1, \ldots, n-1$, we have the bounds given previously in equation (6.84). For the kernel $k_0(t,s;\rho)$ the required growth rate is determined by equation (6.102). Note the restriction $\operatorname{Im}\rho \ge 0$ that applies to these estimates.

Consider the functions $\mathcal{T}_{0ik}(\rho)$, $U_{0k}(s,\rho)$, and $\widetilde{M}_{0ik}(\rho)$. First, from the definitions of the $\mathcal{T}_{0ik}(\rho)$ and the estimate (6.79), it is immediate that

$$|\mathcal{T}_{0ik}(\rho)| \le \gamma_1 |\rho|^{m_i} \tag{6.109}$$

for $\rho \in T_0$ with $|\rho| > R_1$ and for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$. Second, from the definitions of the $U_{0k}(s,\rho)$ and the estimates (5.83)–(5.85), together with (6.92), we obtain the estimates (replace t by $1-s$)

$$|U_{00}(s,\rho)| \leq 2\mathrm{e}^{-b(1-s)}, \tag{6.110}$$

$$|U_{0k}(s,\rho)| \leq 2\mathrm{e}^{-(1-s)\alpha|\rho|} \leq 2, \qquad k = 1, \ldots, \nu - 1, \tag{6.111}$$

$$|U_{0k}(s,\rho)| \leq 2\mathrm{e}^{-s\alpha|\rho|} \leq 2, \qquad k = \nu, \ldots, n - 1, \tag{6.112}$$

for $0 \leq s \leq 1$ and for $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_0 with $|\rho| > R_1$ and $b \geq 0$, we have

$$|U_{0k}(s,\rho)| \leq 2, \qquad 0 \leq s \leq 1, \quad k = 0, 1, \ldots, n-1. \tag{6.113}$$

Third, fix indices i and k with $1 \leq i \leq n$ and $0 \leq k \leq n-1$. The cofactor $\widetilde{M}_{0ik}(\rho)$ is formed by taking $(-1)^{i+k+1}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the kth column of the matrix

$$\begin{array}{c} \begin{array}{ccc} \phantom{P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho}} & {\scriptstyle 1 \leq k \leq \nu-1} & {\scriptstyle \nu \leq k \leq n-1} \end{array} \\ \begin{pmatrix} P_{010}(\rho)+Q_{010}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) \\ \vdots & \vdots & \vdots \\ P_{0n0}(\rho)+Q_{0n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) \end{pmatrix}. \end{array}$$

Suppose we expand the determinant for $\widetilde{M}_{0ik}(\rho)$ in the same manner as $\Delta_0(\rho)$ was expanded earlier. See equation (5.99). If $1 \leq k \leq \nu - 1$ or $\nu \leq k \leq n-1$, then the determinant of the $(n-1) \times (n-1)$ submatrix that leads to $\widetilde{M}_{0ik}(\rho)$ is first expanded using linearity in the 0th column; this yields $\mathrm{e}^{\mathrm{i}\rho}$, 1 terms. On the other hand, if $k = 0$, then a 1 term is produced, but no $\mathrm{e}^{\mathrm{i}\rho}$ term appears.

Next, all the $(n-1) \times (n-1)$ determinants are expanded using linearity in all $n-1$ columns. This expansion produces the representation

$$\widetilde{M}_{0ik}(\rho) = \tilde{\pi}'_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\pi}_{ik}(\rho) + \tilde{\phi}'_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\phi}_{0ik}(\rho) \tag{6.114}$$

for $\rho \in G_0$. In this equation the functions $\tilde{\pi}'_{ik}$, $\tilde{\pi}_{ik}$ are formed from the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$ introduced in Chapter 3; they are analytic for $\rho \neq 0$ in $\mathbb{C}$, and each one has the simple form

$$\sum_{j=-(n-1)(m-1)}^{p_0 - m_i} A_{ikj}\rho^j, \qquad \rho \neq 0 \text{ in } \mathbb{C}.$$

The functions $\tilde{\phi}'_{0ik}$, $\tilde{\phi}_{0ik}$ are analytic for ρ in the open set G_0, and satisfy the growth rates

$$|\tilde{\phi}'_{0ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)}, \qquad |\tilde{\phi}_{0ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)} \tag{6.115}$$

for $\rho \in G_0$. For the special case $k = 0$, $\tilde{\pi}'_{ik}(\rho) \equiv 0$ and $\tilde{\phi}'_{0ik}(\rho) \equiv 0$.

Take any point $\rho = a + \mathrm{i}b$ in the sector S_0 with $|\rho| > R_1$ and $b \geq 0$. Clearly $|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \leq 1$, and hence, by (6.114) and (6.115)

$$|\widetilde{M}_{0ik}(\rho)| \leq 2\gamma_3|\rho|^{p_0-m_i} + 2\gamma_2|\rho|^{-(m-p_0+m_i)} \leq \gamma_4|\rho|^{p_0-m_i} \tag{6.116}$$

for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$. Upon combining (6.116) with (6.109), we obtain the estimate

$$\Big|\sum_{j=1}^{n} \widetilde{M}_{0jk}(\rho)\mathcal{T}_{0jl}(\rho)\Big| \leq \sum_{j=1}^{n} \gamma_4|\rho|^{p_0-m_j} \cdot \gamma_1|\rho|^{m_j} \leq \gamma_5|\rho|^{p_0} \tag{6.117}$$

for $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| > R_1$ and $b \geq 0$ and for $k, l = 0, 1, \ldots, n-1$.

Finally, take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$, with $|\rho| > R_1$ and $b \geq 0$, and with $\Delta_0(\rho) \neq 0$. Clearly $\rho \in \operatorname{Int} T_0$ with $|\rho| > R_1 \geq R_0$, $\rho \in G_0$, and λ belongs to the resolvent set $\rho(L)$ with the Green's function $G(t, s; \lambda)$ given by (6.108). Applying the estimates (6.102), (6.117), (6.84), and (6.113) to (6.108), we see that

$$|G(t, s; \lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{1}{n|\rho|^{n-1}|\Delta_0(\rho)|} \sum_{k,l=0}^{n-1} \gamma_5|\rho|^{p_0} \cdot 2 \cdot 2,$$

or

$$|G(t, s; \lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_0(\rho)|} \tag{6.118}$$

for $t \neq s$ in $[0, 1]$, where (6.118) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$, with $|\rho| > R_1$ and $b \geq 0$, and with $\Delta_0(\rho) \neq 0$.

The representation (6.108) of the Green's function $G(t, s; \lambda)$ is still valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and with $|\rho| > R_1$ and $b \leq 0$, but in this form it is difficult to obtain the necessary bounds and growth rates. To remedy this situation, we make simple modifications in the above work: we first alter the integral operator $K_{0\rho}$, and then replace the basis $u_{0k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, with the basis $u_{0k}(\,\cdot\,, \rho)$, $k = 1, \ldots, n$.

Recall that $\omega_n = \omega_0 = 1$, and earlier we introduced the notation $v_{0n}(t, \rho) := v_{00}(t, \rho)$ and

$$F_{0n\alpha}(t, \rho) := F_{00\alpha}(t, \rho), \qquad \alpha = 0, 1, \ldots, n-1,$$

for $0 \leq t \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. Now set $\eta_{0n}(s, \rho) := \eta_{00}(s, \rho)$ and

$$G_{0kn}(s, \rho) := G_{0k0}(s, \rho), \qquad k = 0, 1, \ldots, n-1,$$

for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_0$. Clearly

$$|F_{0n0}(t, \rho)| = |F_{000}(t, \rho)| \leq 1 \tag{6.119}$$

for $0 \leq t \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_1$, and

$$|G_{0\,n-1\,n}(s, \rho)| = |G_{0\,n-1\,0}(s, \rho)| \leq 1 \tag{6.120}$$

for $0 \leq s \leq 1$ and for $\rho \in T_0$ with $|\rho| > R_1$. With this change in notation, we can rewrite the representation (6.93) of the Green's function $g(t, s; \lambda)$ as

$$
\begin{aligned}
g(t,s;\lambda) &= -\frac{1}{n\rho^{n-1}}\sum_{k=1}^{n}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{0k0}(t,\rho)][1+G_{0\,n-1\,k}(s,\rho)], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s < t \le 1, \\
g(t,s;\lambda) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < s \le 1,
\end{aligned}
\tag{6.121}
$$

for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$. Here the functions $F_{0k0}(\,\cdot\,,\rho)$ and $G_{0\,n-1\,k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$ and satisfy the bounds given in (6.79), (6.92), and (6.119), (6.120).

Next, we rewrite (6.88) or (6.121) in the alternate form

$$
g(t,s;\lambda) = k_0^*(t,s;\rho) + \ell_0^*(t,s;\rho)
$$

for $t \ne s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_0$ with $|\rho| > R_0$, where

$$
\begin{aligned}
k_0^*(t,s;\rho) &:= \sum_{k=1}^{\nu-1} v_{0k}(t,\rho)\eta_{0k}(s,\rho), \qquad 0 \le s < t \le 1, \\
k_0^*(t,s;\rho) &:= -\sum_{k=\nu}^{n} v_{0k}(t,\rho)\eta_{0k}(s,\rho), \qquad 0 \le t < s \le 1,
\end{aligned}
\tag{6.122}
$$

and

$$
\ell_0^*(t,s;\rho) := \sum_{k=\nu}^{n} v_{0k}(t,\rho)\eta_{0k}(s,\rho), \qquad 0 \le t, s \le 1.
$$

Observe that the function $\ell_0^*(t,s;\rho)$ is the kernel of an integral operator which maps $L^2[0,1]$ into the solution space of the differential equation $(\rho^n I - \ell)u = 0$. For each $\rho \in T_0$ with $|\rho| > R_0$, let $K_{0\rho}^*$ be the integral operator on $L^2[0,1]$ defined by

$$
K_{0\rho}^* u(t) := \int_0^1 k_0^*(t,s;\rho)u(s)\,ds, \qquad 0 \le t \le 1,
$$

for $u \in L^2[0,1]$. If $u \in L^2[0,1]$ and $v = K_{0\rho}^* u$, then it follows that v belongs to $H^n[0,1]$ and $(\rho^n I - \ell)v = u$. Using equation (6.89) and induction, we see immediately that the derivatives of $v = K_{0\rho}^* u$ are given by

$$
\begin{aligned}
v^{(j)}(t) &= \int_0^1 \frac{\partial^j k_0^*}{\partial t^j}(t,s;\rho)u(s)\,ds \\
&= \sum_{k=1}^{\nu-1} v_{0k}^{(j)}(t,\rho)\int_0^t \eta_{0k}(s,\rho)u(s)\,ds - \sum_{k=\nu}^{n} v_{0k}^{(j)}(t,\rho)\int_t^1 \eta_{0k}(s,\rho)u(s)\,ds
\end{aligned}
\tag{6.123}
$$

for $0 \le t \le 1$ and for $j = 0, 1, \ldots, n-1$, valid for each $\rho \in T_0$ with $|\rho| > R_0$.

From (6.78) and (6.91) the kernel $k_0^*(t,s;\rho)$ can be expressed in the form

$$\begin{aligned}
k_0^*(t,s;\rho) &= -\frac{1}{n\rho^{n-1}}\sum_{k=1}^{\nu-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{0k0}(t,\rho)][1+G_{0\,n-1\,k}(s,\rho)], \\
&\qquad\qquad 0 \le s < t \le 1, \\
k_0^*(t,s;\rho) &= \frac{1}{n\rho^{n-1}}\sum_{k=\nu}^{n}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{0k0}(t,\rho)][1+G_{0\,n-1\,k}(s,\rho)], \\
&\qquad\qquad 0 \le t < s \le 1,
\end{aligned} \tag{6.124}$$

for $\rho \in T_0$ with $|\rho| > R_0$. Now take any point $\rho = a + \mathrm{i}b \in T_0$ with $|\rho| > R_0$. If t, s are real numbers with $0 \le s < t \le 1$, then $0 < t - s \le 1$, and by (5.84) we obtain the estimate

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le \mathrm{e}^{-(t-s)\alpha|\rho|} \le 1, \qquad k = 1, \ldots, \nu - 1. \tag{6.125}$$

On the other hand, for real numbers t, s with $0 \le t < s \le 1$, we have $0 \le 1 + t - s < 1$, and by (5.85)

$$\begin{aligned}
\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(1+t-s-1)}\right| &\le \mathrm{e}^{-(s-t)\alpha|\rho|} \le 1, \\
k &= \nu, \ldots, n-1,
\end{aligned} \tag{6.126}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_n(t-s)}\right| = \mathrm{e}^{b(s-t)}. \tag{6.127}$$

In particular, if $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| > R_1$ and $b \le 0$, then these estimates give

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le s < t \le 1, \quad k = 1, \ldots, \nu - 1, \tag{6.128}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le t < s \le 1, \quad k = \nu, \ldots, n. \tag{6.129}$$

Applying (6.128), (6.129) and (6.79), (6.119) and (6.92), (6.120) to the representation (6.124), it follows that

$$|k_0^*(t,s;\rho)| \le \frac{4}{|\rho|^{n-1}} \tag{6.130}$$

for $t \ne s$ in $[0,1]$ and for $\rho \in S_0$ with $|\rho| > R_1$ and $b \le 0$.

Using the integral operator $K_{0\rho}^*$, the alternate basis $u_{0k}(\,\cdot\,,\rho)$, $k = 1, \ldots, n$, and the alternate characteristic determinant

$$\Delta_0^*(\rho) = \det\begin{pmatrix} B_1(u_{01}(\,\cdot\,,\rho)) & \cdots & B_1(u_{0n}(\,\cdot\,,\rho)) \\ \vdots & & \vdots \\ B_n(u_{01}(\,\cdot\,,\rho)) & \cdots & B_n(u_{0n}(\,\cdot\,,\rho)) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}\rho}\Delta_0(\rho), \quad \rho \in G_0, \tag{6.131}$$

we can establish alternate representations for the resolvent $R_\lambda(L)$ and the associated Green's function $G(t,s;\lambda)$. Fix any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$, and assume that $\Delta_0(\rho) \ne 0$. Clearly $\Delta_0^*(\rho) \ne 0$, and the point λ belongs to the resolvent set $\rho(L)$. Take any function $u \in L^2[0,1]$, and set $v = K_{0\rho}^* u$ and

$w = R_\lambda(L)u$. The functions v and w belong to $H^n[0,1]$, and $(\lambda I - \ell)v = u = (\lambda I - \ell)w$. Thus, there exist constants $c_1, \dots, c_n$ such that

$$R_\lambda(L)u(t) = w(t) = v(t) + \sum_{k=1}^{n} c_k u_{0k}(t,\rho), \qquad 0 \le t \le 1.$$

Applying the boundary value B_i to the last equation, we obtain the linear system

$$\sum_{k=1}^{n} M^*_{0ik}(\rho) c_k = -B_i(v), \qquad i = 1, \dots, n, \tag{6.132}$$

for the constants $c_1, \dots, c_n$, where

$$M^*_{0ik}(\rho) = B_i(u_{0k}(\,\cdot\,,\rho)) = \sum_{j=0}^{m_i} \alpha_{ij} u^{(j)}_{0k}(0,\rho) + \sum_{j=0}^{m_i} \beta_{ij} u^{(j)}_{0k}(1,\rho)$$

for $i = 1, \dots, n$, $k = 1, \dots, n$. The $n \times n$ coefficient matrix $(M^*_{0ik}(\rho))$ in (6.132) is nonsingular because $\det(M^*_{0ik}(\rho)) = \Delta^*_0(\rho) \neq 0$.

Fix an index i with $1 \le i \le n$, and consider the quantity $B_i(v) = B_i(K^*_{0\rho}u)$. From equation (6.123) and equations (6.78), (6.91), we have

$$\begin{aligned} B_i(v) &= \sum_{j=0}^{m_i} \alpha_{ij} v^{(j)}(0) + \sum_{j=0}^{m_i} \beta_{ij} v^{(j)}(1) \\ &= \frac{1}{n\rho^{n-1}} \sum_{k=1}^{\nu-1} \Big\{ \sum_{j=0}^{m_i} (-1)\beta_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(1,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)} [1 + G_{0\,n-1\,k}(s,\rho)] u(s)\, ds \\ &\quad + \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n} \Big\{ \sum_{j=0}^{m_i} \alpha_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(0,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{0\,n-1\,k}(s,\rho)] u(s)\, ds \end{aligned} \tag{6.133}$$

for $i = 1, \dots, n$. For $i = 1, \dots, n$ and $k = 1, \dots, \nu - 1$ define

$$\mathcal{T}^*_{0ik}(\rho) := \sum_{j=0}^{m_i} (-1)\beta_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(1,\rho)]$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $i = 1, \dots, n$ and $k = \nu, \dots, n$ define

$$\mathcal{T}^*_{0ik}(\rho) := \sum_{j=0}^{m_i} \alpha_{ij}(\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{0kj}(0,\rho)]$$

for $\rho \in T_0$ with $|\rho| > R_0$; and for $k = 1, \ldots, \nu - 1$ define

$$U^*_{0k}(s, \rho) := \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)}[1 + G_{0\,n-1\,k}(s, \rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_0$ with $|\rho| > R_0$, and for $k = \nu, \ldots, n$ define

$$U^*_{0k}(s, \rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{0\,n-1\,k}(s, \rho)], \qquad 0 \le s \le 1,$$

for $\rho \in T_0$ with $|\rho| > R_0$. Then (6.133) can be expressed in the simpler form

$$B_i(v) = \frac{1}{n\rho^{n-1}} \sum_{k=1}^{n} \mathcal{T}^*_{0ik}(\rho) \int_0^1 U^*_{0k}(s, \rho) u(s)\, ds \tag{6.134}$$

for $i = 1, \ldots, n$, where the functions $\mathcal{T}^*_{0ik}(\rho)$ are analytic functions of ρ on G_0, and for fixed ρ in T_0 with $|\rho| > R_0$ the functions $U^*_{0k}(\,\cdot\,, \rho)$ belong to $H^n[0, 1]$.

For the matrix $(M^*_{0ik}(\rho))$, let $\widetilde{M}^*_{0ik}(\rho)$ denote the cofactor of the entry $M^*_{0ik}(\rho)$. These cofactors are analytic functions in the ρ variable on the open set G_0. They appear naturally when we solve for the constants $c_1, \ldots, c_n$ in the linear system (6.132):

$$\begin{aligned} c_k &= \frac{-1}{\Delta^*_0(\rho)} \det \overset{k\text{th column}}{\begin{pmatrix} M^*_{011}(\rho) & \cdots & B_1(v) & \cdots & M^*_{01n}(\rho) \\ \vdots & & \vdots & & \vdots \\ M^*_{0n1}(\rho) & \cdots & B_n(v) & \cdots & M^*_{0nn}(\rho) \end{pmatrix}} \\ &= \frac{-1}{\Delta^*_0(\rho)} \sum_{j=1}^{n} \widetilde{M}^*_{0jk}(\rho) B_j(v), \end{aligned}$$

or

$$c_k = \frac{-1}{n\rho^{n-1}\Delta^*_0(\rho)} \sum_{l=1}^{n} \sum_{j=1}^{n} \widetilde{M}^*_{0jk}(\rho) \mathcal{T}^*_{0jl}(\rho) \int_0^1 U^*_{0l}(s, \rho) u(s)\, ds \tag{6.135}$$

for $k = 1, \ldots, n$.

From the above we conclude that

$$\begin{aligned} R_\lambda(L)u(t) &= K^*_{0\rho}u(t) \\ &\quad - \frac{1}{n\rho^{n-1}\Delta^*_0(\rho)} \sum_{k,l=1}^{n} \sum_{j=1}^{n} \widetilde{M}^*_{0jk}(\rho) \mathcal{T}^*_{0jl}(\rho) u_{0k}(t, \rho) \int_0^1 U^*_{0l}(s, \rho) u(s)\, ds \end{aligned} \tag{6.136}$$

for $0 \le t \le 1$ and for $u \in L^2[0, 1]$, where (6.136) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\Delta^*_0(\rho) \ne 0$ (equivalently $\Delta_0(\rho) \ne 0$). From (6.136) the associated Green's function is given by

$$G(t,s;\lambda) = k_0^*(t,s;\rho) \\ - \frac{1}{n\rho^{n-1}\varDelta_0^*(\rho)} \sum_{k,l=1}^{n} \sum_{j=1}^{n} \widetilde{M}_{0jk}^*(\rho)\mathcal{T}_{0jl}^*(\rho)u_{0k}(t,\rho)U_{0l}^*(s,\rho) \tag{6.137}$$

for $t \neq s$ in $[0,1]$, valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_0$ and with $\varDelta_0^*(\rho) \neq 0$.

Next, we determine bounds and growth rates for the functions appearing in (6.136) and (6.137). For the basis functions $u_{0k}(t,\rho)$, $k = 1,\dots,n$, we have already established the necessary bounds in equation (6.87). For the kernel $k_0^*(t,s;\rho)$ the required growth rate is given in equation (6.130). Note the restriction $\operatorname{Im}\rho \leq 0$ that applies to these estimates. Let us calculate bounds and growth rates for the functions $\mathcal{T}_{0ik}^*(\rho)$, $U_{0k}^*(s,\rho)$, and $\widetilde{M}_{0ik}^*(\rho)$.

First, consider the functions $\mathcal{T}_{0ik}^*(\rho)$. From their definitions and the estimates (6.79) and (6.119), it is immediate that

$$|\mathcal{T}_{0ik}^*(\rho)| \leq \gamma_1|\rho|^{m_i} \tag{6.138}$$

for $\rho \in T_0$ with $|\rho| > R_1$ and for $i = 1,\dots,n$, $k = 1,\dots,n$. Second, for the functions $U_{0k}^*(s,\rho)$, from their definitions and the estimates (5.84) and (5.85), together with (6.92) and (6.120), we obtain the estimates (replace t by $1-s$)

$$|U_{0k}^*(s,\rho)| \leq 2\mathrm{e}^{-(1-s)\alpha|\rho|} \leq 2, \qquad k = 1,\dots,\nu-1, \tag{6.139}$$

$$|U_{0k}^*(s,\rho)| \leq 2\mathrm{e}^{-s\alpha|\rho|} \leq 2, \qquad k = \nu,\dots,n-1, \tag{6.140}$$

$$|U_{0n}^*(s,\rho)| \leq 2\mathrm{e}^{bs} \tag{6.141}$$

for $0 \leq s \leq 1$ and for $\rho = a+\mathrm{i}b \in T_0$ with $|\rho| > R_1$. In particular, for $\rho = a+\mathrm{i}b$ belonging to the sector S_0 with $|\rho| > R_1$ and $b \leq 0$, we have

$$|U_{0k}^*(s,\rho)| \leq 2, \qquad 0 \leq s \leq 1, \quad k = 1,\dots,n. \tag{6.142}$$

Third, fix indices i and k with $1 \leq i \leq n$ and $1 \leq k \leq n$. Then the cofactor $\widetilde{M}_{0ik}^*(\rho)$ is formed by taking $(-1)^{i+k}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and the kth column of the matrix

$$\begin{array}{c} \begin{array}{ccc} 1\leq k\leq\nu-1 \qquad & \nu\leq k\leq n-1 \qquad & \qquad\qquad\qquad \end{array} \\ \begin{pmatrix} \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & \widehat{P}_{1k}(\rho)+\widetilde{F}_{01k}(\rho) & P_{010}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{010}(\rho) \\ \vdots & \vdots & \vdots \\ \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & \widehat{P}_{nk}(\rho)+\widetilde{F}_{0nk}(\rho) & P_{0n0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{0n0}(\rho) \end{pmatrix} \end{array}.$$

Suppose we expand the determinant for $\widetilde{M}_{0ik}^*(\rho)$ in the same manner as $\varDelta_0(\rho)$ was expanded earlier. See equation (5.99). Now we emphasize the nth column instead of the 0th column. If $1 \leq k \leq \nu-1$ or $\nu \leq k \leq n-1$, then the determinant of the $(n-1)\times(n-1)$ submatrix that leads to $\widetilde{M}_{0ik}^*(\rho)$ is first expanded using linearity in the nth column; this yields $\mathrm{e}^{-\mathrm{i}\rho}$, 1 terms. On the other hand, if $k = n$, then a 1 term is produced, but no $\mathrm{e}^{-\mathrm{i}\rho}$ term appears.

Next, all the $(n-1)\times(n-1)$ determinants are expanded using linearity in all $n-1$ columns. This expansion produces the following representations: for $1 \le k \le \nu - 1$ or for $\nu \le k \le n-1$,

$$\begin{aligned}\widetilde{M}^*_{0ik}(\rho) &= \mathrm{e}^{-\mathrm{i}\rho}[\tilde{\pi}'_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\pi}_{ik}(\rho) + \tilde{\phi}'_{0ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \tilde{\phi}_{0ik}(\rho)] \\ &= \tilde{\pi}'_{ik}(\rho) + \tilde{\pi}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \tilde{\phi}'_{0ik}(\rho) + \tilde{\phi}_{0ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho}\end{aligned} \qquad (6.143)$$

for $\rho \in G_0$; and for $k = n$,

$$\widetilde{M}^*_{0in}(\rho) = \tilde{\pi}_{i0}(\rho) + \tilde{\phi}_{0i0}(\rho) \qquad (6.144)$$

for $\rho \in G_0$. In these equations the functions $\tilde{\pi}'_{ik}$, $\tilde{\pi}_{ik}$ and $\tilde{\phi}'_{0ik}$, $\tilde{\phi}_{0ik}$ are the same functions that appear in equation (6.114). The functions $\tilde{\pi}'_{ik}$, $\tilde{\pi}_{ik}$ are formed from the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$ introduced in Chapter 3. They are analytic for $\rho \ne 0$ in $\mathbb{C}$, and each one has the simple form

$$\sum_{j=-(n-1)(m-1)}^{p_0-m_i} A_{ikj}\rho^j, \qquad \rho \ne 0 \text{ in } \mathbb{C}.$$

The functions $\tilde{\phi}'_{0ik}$, $\tilde{\phi}_{0ik}$ are analytic for ρ in the open set G_0, and satisfy the growth rates

$$|\tilde{\phi}'_{0ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)}, \qquad |\tilde{\phi}_{0ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)} \qquad (6.145)$$

for $\rho \in G_0$.

Take any point $\rho = a + \mathrm{i}b$ in the sector S_0 with $|\rho| > R_1$ and $b \le 0$. Clearly $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^b \le 1$, and hence, by (6.143)–(6.145)

$$|\widetilde{M}^*_{0ik}(\rho)| \le 2\gamma_3|\rho|^{p_0-m_i} + 2\gamma_2|\rho|^{-(m-p_0+m_i)} \le \gamma_4|\rho|^{p_0-m_i} \qquad (6.146)$$

for $i = 1,\dots,n$, $k = 1,\dots,n$. Combining this result with (6.138), we obtain the estimate

$$\Big|\sum_{j=1}^{n} \widetilde{M}^*_{0jk}(\rho)\mathcal{T}^*_{0jl}(\rho)\Big| \le \sum_{j=1}^{n} \gamma_4|\rho|^{p_0-m_j}\cdot\gamma_1|\rho|^{m_j} \le \gamma_5|\rho|^{p_0} \qquad (6.147)$$

for $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| > R_1$ and $b \le 0$ and for $k, l = 1,\dots,n$.

Finally, take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$, with $|\rho| > R_1$ and $b \le 0$, and with $\Delta_0(\rho) \ne 0$, so $\Delta^*_0(\rho) \ne 0$. Clearly $\rho \in \operatorname{Int} T_0$ with $|\rho| > R_1 \ge R_0$, $\rho \in G_0$, and λ belongs to the resolvent set $\rho(L)$ with the Green's function $G(t,s;\lambda)$ given by equation (6.137). Applying the estimates (6.130), (6.147), (6.87), and (6.143) and the relation (6.131) to the representation (6.137), we see that

$$|G(t,s;\lambda)| \le \frac{4}{|\rho|^{n-1}} + \frac{1}{n|\rho|^{n-1}|\Delta^*_0(\rho)|}\sum_{k,l=1}^{n}\gamma_5|\rho|^{p_0}\cdot 2\cdot 2,$$

or

$$|G(t,s;\lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}\mathrm{e}^{b}|\Delta_0(\rho)|} \tag{6.148}$$

for $t \neq s$ in $[0,1]$, where (6.148) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$, with $|\rho| > R_1$ and $b \leq 0$, and with $\Delta_0(\rho) \neq 0$. Compare this growth rate to the growth rate given previously in equation (6.118). Note the extra factor e^b that appears in the denominator of (6.148).

To develop the growth rate of the Green's function $G(t,s;\lambda)$ for $\lambda = \rho^n$ with ρ belonging to the sector S_1, we simply recopy the above material, and then make the few changes needed for the new material. Let us proceed with these extensions of the theory.

For each $\rho \in T_1$ with $|\rho| > R_0$, let $v_{10}(\,\cdot\,,\rho), v_{11}(\,\cdot\,,\rho), \ldots, v_{1\,n-1}(\,\cdot\,,\rho)$ be the basis for the solution space of the differential equation (2.1) determined in Theorem 4.7. In Chapter 5 we showed that (see (5.107) and (5.108))

$$v_{1k}^{(\alpha)}(t,\rho) = \rho^{\alpha}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{1k\alpha}(t,\rho)\big] \tag{6.149}$$

for $0 \leq t \leq 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$, where the function $F_{1k\alpha}(\,\cdot\,,\rho)$ belongs to $H^{n-\alpha}[0,1]$ with

$$|F_{1k\alpha}(t,\rho)| \leq 1 \tag{6.150}$$

for $0 \leq t \leq 1$, for $\rho \in T_1$ with $|\rho| > R_1 \geq R_0$, and for $k, \alpha = 0, 1, \ldots, n-1$. Also in Chapter 5, for $\rho \in T_1$ with $|\rho| > R_0$ we formed the modified solutions $u_{10}(\,\cdot\,,\rho), u_{11}(\,\cdot\,,\rho), \ldots, u_{1\,n-1}(\,\cdot\,,\rho)$ of the differential equation (2.1). These functions have the representations (5.109):

$$\begin{aligned} u_{1k}^{(\alpha)}(t,\rho) &= \rho^{\alpha}\mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{1k\alpha}(t,\rho)\big], & k &= 0,1,\ldots,\nu-1, \\ u_{1k}^{(\alpha)}(t,\rho) &= \rho^{\alpha}\mathrm{e}^{\mathrm{i}\rho\omega_k t}\big[(\mathrm{i}\omega_k)^{\alpha} + F_{1k\alpha}(t,\rho)\big], & k &= \nu,\ldots,n-1, \end{aligned} \tag{6.151}$$

for $0 \leq t \leq 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $\alpha = 0, 1, \ldots, n-1$. From equations (5.116)–(5.118) we have the bounds

$$|u_{10}(t,\rho)| \leq 2\mathrm{e}^{-b(t-1)}, \tag{6.152}$$

$$|u_{1k}(t,\rho)| \leq 2\mathrm{e}^{-(1-t)\alpha|\rho|} \leq 2, \qquad k = 1,\ldots,\nu-1, \tag{6.153}$$

$$|u_{1k}(t,\rho)| \leq 2\mathrm{e}^{-t\alpha|\rho|} \leq 2, \qquad k = \nu,\ldots,n-1, \tag{6.154}$$

for $0 \leq t \leq 1$ and for $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_1$. In particular, for $\rho = a + \mathrm{i}b$ belonging to the sector S_1 with $|\rho| > R_1$ and $b \leq 0$, we have

$$|u_{1k}(t,\rho)| \leq 2, \qquad 0 \leq t \leq 1, \quad k = 0,1,\ldots,n-1. \tag{6.155}$$

We must form another set of modified solutions in order to treat the case of $\rho = a + \mathrm{i}b$ in S_1 with $|\rho| > R_1$ and $b \geq 0$.

Indeed, set $v_{1n}(t,\rho) := v_{10}(t,\rho)$ and

$$F_{1n\alpha}(t,\rho) := F_{10\alpha}(t,\rho), \qquad \alpha = 0,1,\dots,n-1,$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, and then set

$$\begin{aligned} u_{1n}(t,\rho) :&= \mathrm{e}^{\mathrm{i}\rho\omega_0} u_{10}(t,\rho) = v_{1n}(t,\rho) \\ &= \mathrm{e}^{\mathrm{i}\rho\omega_n t}[1 + F_{1n0}(t,\rho)] \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$. Clearly we have the representation

$$u_{1n}^{(\alpha)}(t,\rho) = \rho^\alpha \mathrm{e}^{\mathrm{i}\rho\omega_n t}\big[(\mathrm{i}\omega_n)^\alpha + F_{1n\alpha}(t,\rho)\big] \tag{6.156}$$

for $0 \le t \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $\alpha = 0,1,\dots,n-1$, and clearly the functions $u_{1k}(\,\cdot\,,\rho)$, $k = 1,\dots,n$, also form a basis for the solution space of the differential equation (2.1). Observe that

$$|u_{1n}(t,\rho)| \le 2\mathrm{e}^{-bt} \tag{6.157}$$

for $0 \le t \le 1$ and for $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_1$, and hence, for $\rho = a + \mathrm{i}b$ in S_1 with $|\rho| > R_1$ and $b \ge 0$, we have

$$|u_{1k}(t,\rho)| \le 2, \qquad 0 \le t \le 1, \quad k = 1,\dots,n. \tag{6.158}$$

In terms of this new basis, we can form the new characteristic determinant

$$\varDelta_1^*(\rho) := \det\begin{pmatrix} B_1(u_{11}(\,\cdot\,,\rho)) & \cdots & B_1(u_{1n}(\,\cdot\,,\rho)) \\ \vdots & & \vdots \\ B_n(u_{11}(\,\cdot\,,\rho)) & \cdots & B_n(u_{1n}(\,\cdot\,,\rho)) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\rho}\varDelta_1(\rho)$$

for $\rho \in G_1$.

Next, let L_0 be the nth order differential operator in $L^2[0,1]$ introduced earlier:

$$\mathcal{D}(L_0) = \{u \in H^n[0,1] \mid u^{(n-i)}(0) = 0,\, i = 1,\dots,n\}, \qquad L_0 u = \ell u.$$

Clearly the resolvent set $\rho(L_0)$ is equal to $\mathbb{C}$, and the Green's function $g(t,s;\lambda)$ of the differential operator $\lambda I - L_0$ is given by

$$\begin{aligned} g(t,s;\lambda) &= \sum_{k=0}^{n-1} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad && 0 \le s < t \le 1, \\ g(t,s;\lambda) &= 0, && 0 \le t < s \le 1, \end{aligned} \tag{6.159}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$. The functions $\eta_{1k}(\,\cdot\,,\rho)$, $k = 0,1,\dots,n-1$, belong to $H^n[0,1]$ and are determined by the linear system

$$\sum_{k=0}^{n-1} v_{1k}^{(\alpha)}(s,\rho)\eta_{1k}(s,\rho) = -\delta_{\alpha\, n-1}\mathrm{i}^{\,n}, \qquad \alpha = 0,1,\dots,n-1, \tag{6.160}$$

for $0 \le s \le 1$. This system can be rewritten in the form

$$\sum_{k=0}^{n-1} \big[(\mathrm{i}\omega_k)^\alpha + F_{1k\alpha}(s,\rho)\big]\mathrm{e}^{\mathrm{i}\rho\omega_k s}\eta_{1k}(s,\rho) = -\delta_{\alpha\, n-1}\mathrm{i}^{\,n}\rho^{-(n-1)}, \qquad \alpha = 0,1,\dots,n-1, \tag{6.161}$$

for $0 \le s \le 1$. We know that the $n \times n$ matrix $A_1(s,\rho) := \big((\mathrm{i}\omega_k)^\alpha + F_{1k\alpha}(s,\rho)\big)$ is nonsingular for $0 \le s \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$, and its inverse can be expressed in the form

$$A_1(s,\rho)^{-1} := \Big(\frac{1}{n\mathrm{i}^{\,k}}\, \omega_\alpha^{n-k}\, [1 + G_{1k\alpha}(s,\rho)]\Big)$$

for $0 \le s \le 1$ and for $\rho \in T_1$ with $|\rho| > R_0$. It follows that

$$\eta_{1k}(s,\rho) = -\frac{\mathrm{i}\omega_k}{n\rho^{n-1}}\, \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{1\, n-1\, k}(s,\rho)] \tag{6.162}$$

for $0 \le s \le 1$, for $\rho \in T_1$ with $|\rho| > R_0$, and for $k = 0,1,\dots,n-1$, with the functions $G_{1\, n-1\, k}(\,\cdot\,,\rho)$ belonging to $H^n[0,1]$. Since the functions $G_{1k\alpha}(s,\rho) \to 0$ uniformly on $[0,1] \times T_1$ as $|\rho| \to \infty$ for $k, \alpha = 0,1,\dots,n-1$, we can assume that the constant R_1 chosen earlier also produces the bound

$$|G_{1k\alpha}(s,\rho)| \le 1 \tag{6.163}$$

for $0 \le s \le 1$, for $\rho \in T_1$ with $|\rho| > R_1$, and for $k, \alpha = 0,1,\dots,n-1$.

Summarizing, for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$, the Green's function for the differential operator $\lambda I - L_0$ is given by

$$\begin{aligned} g(t,s;\lambda) &= -\frac{1}{n\rho^{n-1}} \sum_{k=0}^{n-1} (\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{1k0}(t,\rho)][1 + G_{1\, n-1\, k}(s,\rho)], \\ &\qquad\qquad 0 \le s < t \le 1, \\ g(t,s;\lambda) &= 0, \qquad\qquad 0 \le t < s \le 1, \end{aligned} \tag{6.164}$$

where the functions $F_{1k0}(\,\cdot\,,\rho)$ and $G_{1\, n-1\, k}(\,\cdot\,,\rho)$ belong to $H^n[0,1]$ and satisfy the bounds given in equations (6.150) and (6.163).

Next, we rewrite (6.159) or (6.164) in the form

$$g(t,s;\lambda) = k_1(t,s;\rho) + \ell_1(t,s;\rho)$$

for $t \ne s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$, where

$$\begin{aligned} k_1(t,s;\rho) &:= \sum_{k=\nu}^{n-1} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le s < t \le 1, \\ k_1(t,s;\rho) &:= -\sum_{k=0}^{\nu-1} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le t < s \le 1, \end{aligned} \tag{6.165}$$

and

$$\ell_1(t,s;\rho) := \sum_{k=0}^{\nu-1} v_{1k}(t,\rho)\eta_{1k}(s,\rho), \qquad 0 \le t, s \le 1.$$

For each $\rho \in T_1$ with $|\rho| > R_0$, let $K_{1\rho}$ be the integral operator on $L^2[0,1]$ defined by

$$K_{1\rho}u(t) := \int_0^1 k_1(t,s;\rho)u(s)\,ds, \qquad 0 \le t \le 1,$$

for $u \in L^2[0,1]$. If $u \in L^2[0,1]$ and $v = K_{1\rho}u$, then it follows that v belongs to $H^n[0,1]$ and $(\rho^n I - \ell)v = u$, and by direct calculation

$$\begin{aligned} v(t) &= \int_0^1 k_1(t,s;\rho)u(s)\,ds \\ &= \sum_{k=\nu}^{n-1} v_{1k}(t,\rho)\int_0^t \eta_{1k}(s,\rho)u(s)\,ds - \sum_{k=0}^{\nu-1} v_{1k}(t,\rho)\int_t^1 \eta_{1k}(s,\rho)u(s)\,ds \end{aligned}$$

for $0 \le t \le 1$. Proceeding by induction, we see that the derivatives of $v = K_{1\rho}u$ satisfy the equations

$$\begin{aligned} v^{(j)}(t) &= \int_0^1 \frac{\partial^j k_1}{\partial t^j}(t,s;\rho)u(s)\,ds \\ &= \sum_{k=\nu}^{n-1} v_{1k}^{(j)}(t,\rho)\int_0^t \eta_{1k}(s,\rho)u(s)\,ds - \sum_{k=0}^{\nu-1} v_{1k}^{(j)}(t,\rho)\int_t^1 \eta_{1k}(s,\rho)u(s)\,ds \end{aligned} \tag{6.166}$$

for $0 \le t \le 1$ and for $j = 0, 1, \ldots, n-1$, valid for each $\rho \in T_1$ with $|\rho| > R_0$.

Using (6.149) and (6.162), the kernel $k_1(t,s;\rho)$ can be expressed as

$$\begin{aligned} k_1(t,s;\rho) &= -\frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n-1} (\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{1k0}(t,\rho)][1 + G_{1\,n-1\,k}(s,\rho)], \\ &\qquad 0 \le s < t \le 1, \\ k_1(t,s;\rho) &= \frac{1}{n\rho^{n-1}} \sum_{k=0}^{\nu-1} (\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1 + F_{1k0}(t,\rho)][1 + G_{1\,n-1\,k}(s,\rho)], \\ &\qquad 0 \le t < s \le 1, \end{aligned} \tag{6.167}$$

for $\rho \in T_1$ with $|\rho| > R_0$. Now take any point $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_0$. If t, s are real numbers with $0 \leq s < t \leq 1$, then $0 < t - s \leq 1$, and by (5.112) we obtain the estimates

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \leq \mathrm{e}^{-(t-s)\alpha|\rho|} \leq 1, \quad k = \nu, \ldots, n-1. \tag{6.168}$$

On the other hand, for real numbers t, s with $0 \leq t < s \leq 1$, we have $0 \leq 1 + t - s < 1$, and hence, by (5.111)

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_0(t-s)}\right| = \mathrm{e}^{-b(t-s)}, \tag{6.169}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| = \left|\mathrm{e}^{\mathrm{i}\rho\omega_k(1+t-s-1)}\right| \leq \mathrm{e}^{-(s-t)\alpha|\rho|} \leq 1, \quad k = 1, \ldots, \nu - 1. \tag{6.170}$$

In particular, if $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| > R_1$ and $b \leq 0$, then these estimates give

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \leq 1, \qquad 0 \leq s < t \leq 1, \quad k = \nu, \ldots, n-1, \tag{6.171}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \leq 1, \qquad 0 \leq t < s \leq 1, \quad k = 0, 1, \ldots, \nu - 1. \tag{6.172}$$

Applying (6.171), (6.172) and (6.150), (6.163) to the representation (6.167), it follows that

$$|k_1(t, s; \rho)| \leq \frac{4}{|\rho|^{n-1}} \tag{6.173}$$

for $t \neq s$ in $[0, 1]$ and for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| > R_1$ and $b \leq 0$.

Fix any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$, and assume that $\Delta_1(\rho) \neq 0$, so the point λ belongs to the resolvent set $\rho(L)$. Using the integral operator $K_{1\rho}$, we can establish our representations of the resolvent $R_\lambda(L)$ and the associated Green's function $G(t, s; \lambda)$. Indeed, take any function $u \in L^2[0, 1]$, and set

$$v = K_{1\rho}u \quad \text{and} \quad w = R_\lambda(L)u.$$

Clearly the functions v and w belong to $H^n[0, 1]$, and

$$(\lambda I - \ell)v = u = (\lambda I - \ell)w.$$

Thus, there exist constants $c_0, c_1, \ldots, c_{n-1}$ (depending on ρ) such that

$$R_\lambda(L)u(t) = w(t) = v(t) + \sum_{k=0}^{n-1} c_k u_{1k}(t, \rho), \qquad 0 \leq t \leq 1.$$

The functions $u_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, are the modified solutions of the differential equation (2.1) introduced earlier. They form a basis for the solution space of the differential equation (2.1), and the characteristic determinant $\Delta_1(\rho)$ is defined in terms of them.

Applying the boundary value B_i to both sides of the last equation, we obtain the linear system

$$\sum_{k=0}^{n-1} M_{1ik}(\rho)c_k = -B_i(v), \qquad i = 1, \ldots, n, \tag{6.174}$$

for the constants $c_0, c_1, \ldots, c_{n-1}$, where as in Chapter 5

$$M_{1ik}(\rho) = B_i(u_{1k}(\,\cdot\,,\rho)) = \sum_{j=0}^{m_i} \alpha_{ij} u_{1k}^{(j)}(0,\rho) + \sum_{j=0}^{m_i} \beta_{ij} u_{1k}^{(j)}(1,\rho)$$

for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$. Note that $\det(M_{1ik}(\rho)) = \mathit{\Delta}_1(\rho) \neq 0$.

Fix an index i with $1 \leq i \leq n$, and consider the quantity $B_i(v) = B_i(K_{1\rho}u)$. From equation (6.166) and equations (6.149), (6.162), we have

$$\begin{aligned} B_i(v) &= \sum_{j=0}^{m_i} \alpha_{ij} v^{(j)}(0) + \sum_{j=0}^{m_i} \beta_{ij} v^{(j)}(1) \\ &= \frac{1}{n\rho^{n-1}} \sum_{k=0}^{\nu-1} \Big\{ \sum_{j=0}^{m_i} \alpha_{ij} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(0,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{1\,n-1\,k}(s,\rho)] u(s)\, ds \\ &\quad + \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n-1} \Big\{ \sum_{j=0}^{m_i} (-1)\beta_{ij} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(1,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k (1-s)} [1 + G_{1\,n-1\,k}(s,\rho)] u(s)\, ds \end{aligned} \tag{6.175}$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu-1$ define

$$\mathcal{T}_{1ik}(\rho) := \sum_{j=0}^{m_i} \alpha_{ij} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(0,\rho)]$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$ define

$$\mathcal{T}_{1ik}(\rho) := \sum_{j=0}^{m_i} (-1)\beta_{ij} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(1,\rho)]$$

for $\rho \in T_1$ with $|\rho| > R_0$; and for $k = 0, 1, \ldots, \nu-1$ define

$$U_{1k}(s,\rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{1\,n-1\,k}(s,\rho)], \qquad 0 \leq s \leq 1,$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $k = \nu, \ldots, n-1$ define

$$U_{1k}(s,\rho) := \mathrm{e}^{\mathrm{i}\rho\omega_k (1-s)} [1 + G_{1\,n-1\,k}(s,\rho)], \qquad 0 \leq s \leq 1,$$

for $\rho \in T_1$ with $|\rho| > R_0$. Then we can express (6.175) in the simpler form

$$B_i(v) = \frac{1}{n\rho^{n-1}} \sum_{k=0}^{n-1} \mathcal{T}_{1ik}(\rho) \int_0^1 U_{1k}(s,\rho)u(s)\, ds \tag{6.176}$$

for $i = 1, \ldots, n$, where the functions $\mathcal{T}_{1ik}(\rho)$ are analytic functions of ρ on G_1, and for fixed ρ in T_1 with $|\rho| > R_0$ the functions $U_{1k}(\,\cdot\,, \rho)$ belong to $H^n[0,1]$.

In terms of the matrix $(M_{1ik}(\rho))$, let $\widetilde{M}_{1ik}(\rho)$ denote the cofactor of the entry $M_{1ik}(\rho)$. Clearly the cofactors $\widetilde{M}_{1ik}(\rho)$, $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$, are analytic functions in the ρ variable on the open set G_1. These cofactors arise naturally when we solve for the constants $c_0, c_1, \ldots, c_{n-1}$ in the linear system (6.174) by means of Cramer's rule:

$$\begin{aligned} c_k &= \frac{-1}{\Delta_1(\rho)} \det \begin{array}{c} \\ \end{array} \begin{pmatrix} & & k\text{th column} & & \\ M_{110}(\rho) & \cdots & B_1(v) & \cdots & M_{11n-1}(\rho) \\ \vdots & & \vdots & & \vdots \\ M_{1n0}(\rho) & \cdots & B_n(v) & \cdots & M_{1nn-1}(\rho) \end{pmatrix} \\ &= \frac{-1}{\Delta_1(\rho)} \sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho) B_j(v), \end{aligned}$$

or

$$c_k = \frac{-1}{n\rho^{n-1}\Delta_1(\rho)} \sum_{l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho) \mathcal{T}_{1jl}(\rho) \int_0^1 U_{1l}(s,\rho)u(s)\, ds \tag{6.177}$$

for $k = 0, 1, \ldots, n-1$.

Combining these results, we conclude that

$$\begin{aligned} R_\lambda(L)u(t) &= K_{1\rho}u(t) \\ &\quad - \frac{1}{n\rho^{n-1}\Delta_1(\rho)} \sum_{k,l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho) \mathcal{T}_{1jl}(\rho) u_{1k}(t,\rho) \int_0^1 U_{1l}(s,\rho)u(s)\, ds \end{aligned} \tag{6.178}$$

for $0 \le t \le 1$ and for $u \in L^2[0,1]$, where (6.178) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\Delta_1(\rho) \neq 0$. The associated Green's function is then given by

$$\begin{aligned} G(t,s;\lambda) &= k_1(t,s;\rho) \\ &\quad - \frac{1}{n\rho^{n-1}\Delta_1(\rho)} \sum_{k,l=0}^{n-1} \sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho) \mathcal{T}_{1jl}(\rho) u_{1k}(t,\rho) U_{1l}(s,\rho) \end{aligned} \tag{6.179}$$

for $t \neq s$ in $[0,1]$, where (6.179) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\Delta_1(\rho) \neq 0$.

Let us determine bounds and growth rates for the functions appearing in (6.178) and (6.179). For the basis functions $u_{1k}(t,\rho)$, $k = 0, 1, \ldots, n-1$, we have already established the necessary bounds in equation (6.155). For the kernel $k_1(t,s;\rho)$ the required growth rate is given in equation (6.173). Note the restriction $\operatorname{Im}\rho \le 0$ that applies to these estimates.

Consider the functions $\mathcal{T}_{1ik}(\rho)$, $U_{1k}(s,\rho)$, and $\widetilde{M}_{1ik}(\rho)$. First, from the definitions of the $\mathcal{T}_{1ik}(\rho)$ and the estimate (6.150), it is immediate that

$$|\mathcal{T}_{1ik}(\rho)| \le \gamma_1|\rho|^{m_i} \tag{6.180}$$

for $\rho \in T_1$ with $|\rho| > R_1$ and for $i = 1,\dots,n$, $k = 0,1,\dots,n-1$. Second, from the definitions of the $U_{1k}(s,\rho)$ and the estimates (5.111)–(5.112), together with (6.163), we obtain the estimates (replace t by $1-s$)

$$|U_{10}(s,\rho)| \le 2\mathrm{e}^{bs}, \tag{6.181}$$

$$|U_{1k}(s,\rho)| \le 2\mathrm{e}^{-s\alpha|\rho|} \le 2, \qquad k = 1,\dots,\nu-1, \tag{6.182}$$

$$|U_{1k}(s,\rho)| \le 2\mathrm{e}^{-(1-s)\alpha|\rho|} \le 2, \qquad k = \nu,\dots,n-1, \tag{6.183}$$

for $0 \le s \le 1$ and for $\rho = a+\mathrm{i}b \in T_1$ with $|\rho| > R_1$. In particular, for $\rho = a+\mathrm{i}b$ belonging to the sector S_1 with $|\rho| > R_1$ and $b \le 0$, we have

$$|U_{1k}(s,\rho)| \le 2, \qquad 0 \le s \le 1, \quad k = 0,1,\dots,n-1. \tag{6.184}$$

Third, fix indices i and k with $1 \le i \le n$ and $0 \le k \le n-1$. The cofactor $\widetilde{M}_{1ik}(\rho)$ is formed by taking $(-1)^{i+k+1}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and the kth column of the matrix

$$\begin{array}{ccc} & {\scriptstyle 1\le k\le \nu-1} & {\scriptstyle \nu\le k\le n-1} \\ \left(\begin{matrix} P_{110}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{110}(\rho) \\ \vdots \\ P_{1n0}(\rho)\mathrm{e}^{-\mathrm{i}\rho}+Q_{1n0}(\rho) \end{matrix}\right. & \begin{matrix} \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) \\ \vdots \\ \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) \end{matrix} & \left.\begin{matrix} \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) \\ \vdots \\ \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) \end{matrix}\right) \end{array}.$$

Suppose we proceed to expand the determinant for $\widetilde{M}_{1ik}(\rho)$ in the same manner as $\varDelta_1(\rho)$ was expanded in Chapter 5. See equation (5.125). If $1 \le k \le \nu-1$ or $\nu \le k \le n-1$, then the determinant of the $(n-1)\times(n-1)$ submatrix that leads to $\widetilde{M}_{1ik}(\rho)$ is first expanded using linearity in the 0th column; this yields $\mathrm{e}^{-\mathrm{i}\rho}$, 1 terms. On the other hand, if $k = 0$, then a 1 term is produced, but no $\mathrm{e}^{-\mathrm{i}\rho}$ term appears.

Next, all the $(n-1)\times(n-1)$ determinants are expanded using linearity in all $n-1$ columns. This expansion produces the representation

$$\widetilde{M}_{1ik}(\rho) = \widetilde{\varPi}'_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\varPi}_{ik}(\rho) + \widetilde{\varPhi}'_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\varPhi}_{1ik}(\rho) \tag{6.185}$$

for $\rho \in G_1$. In this equation the functions $\widetilde{\varPi}'_{ik}$, $\widetilde{\varPi}_{ik}$ are formed from the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$ introduced in Chapter 3; they are analytic for $\rho \ne 0$ in $\mathbb{C}$, and each one has the simple form

$$\sum_{j=-(n-1)(m-1)}^{p_0-m_i} B_{ikj}\rho^j, \qquad \rho \neq 0 \text{ in } \mathbb{C}.$$

The functions $\widetilde{\Phi}'_{1ik}$, $\widetilde{\Phi}_{1ik}$ are analytic for ρ in the open set G_1, and satisfy the growth rates

$$|\widetilde{\Phi}'_{1ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)}, \qquad |\widetilde{\Phi}_{1ik}(\rho)| \leq \gamma_2|\rho|^{-(m-p_0+m_i)} \tag{6.186}$$

for $\rho \in G_1$. In the special case $k = 0$, then $\widetilde{\Pi}'_{ik}(\rho) \equiv 0$ and $\widetilde{\Phi}'_{1ik}(\rho) \equiv 0$.

Take any point $\rho = a + \mathrm{i}b$ in the sector S_1 with $|\rho| > R_1$ and $b \leq 0$. Clearly $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^b \leq 1$, and hence, by (6.185) and (6.186)

$$|\widetilde{M}_{1ik}(\rho)| \leq 2\gamma_3|\rho|^{p_0-m_i} + 2\gamma_2|\rho|^{-(m-p_0+m_i)} \leq \gamma_4|\rho|^{p_0-m_i} \tag{6.187}$$

for $i = 1, \ldots, n$, $k = 0, 1, \ldots, n-1$. Combining this result with (6.180), we obtain the estimate

$$\Big|\sum_{j=1}^{n} \widetilde{M}_{1jk}(\rho)\mathcal{T}_{1jl}(\rho)\Big| \leq \sum_{j=1}^{n} \gamma_4|\rho|^{p_0-m_j} \cdot \gamma_1|\rho|^{m_j} \leq \gamma_5|\rho|^{p_0} \tag{6.188}$$

for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| > R_1$ and $b \leq 0$ and for $k, l = 0, 1, \ldots, n-1$.

Finally, take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a+\mathrm{i}b \in S_1$, with $|\rho| > R_1$ and $b \leq 0$, and with $\Delta_1(\rho) \neq 0$. Clearly $\rho \in \operatorname{Int} T_1$ with $|\rho| > R_1 \geq R_0$, $\rho \in G_1$, and λ belongs to the resolvent set $\rho(L)$ with the Green's function $G(t, s; \lambda)$ given by (6.179). Applying the estimates (6.173), (6.188), (6.155), and (6.184) to (6.179), we see that

$$|G(t, s; \lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{1}{n|\rho|^{n-1}|\Delta_1(\rho)|} \sum_{k,l=0}^{n-1} \gamma_5|\rho|^{p_0} \cdot 2 \cdot 2,$$

or

$$|G(t, s; \lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_1(\rho)|} \tag{6.189}$$

for $t \neq s$ in $[0, 1]$, where (6.189) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$, with $|\rho| > R_1$ and $b \leq 0$, and with $\Delta_1(\rho) \neq 0$.

The representation (6.179) of the Green's function $G(t, s; \lambda)$ is also valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$ and with $|\rho| > R_1$ and $b \geq 0$, but in this form it is difficult to obtain the necessary bounds and growth rates. To remedy this situation, we proceed to make simple modifications in the above work: we first alter the integral operator $K_{1\rho}$, and then replace the basis $u_{1k}(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, with the basis $u_{1k}(\,\cdot\,, \rho)$, $k = 1, \ldots, n$.

Recall that $\omega_n = \omega_0 = 1$, and earlier we introduced the notation

$$\begin{aligned} &v_{1n}(t, \rho) := v_{10}(t, \rho), \\ &F_{1n\alpha}(t, \rho) := F_{10\alpha}(t, \rho), \qquad \alpha = 0, 1, \ldots, n-1, \end{aligned}$$

for $0 \leq t \leq 1$ and for $\rho \in T_1$ with $|\rho| > R_0$. Now set $\eta_{1n}(s, \rho) := \eta_{10}(s, \rho)$ and

$$G_{1kn}(s, \rho) := G_{1k0}(s, \rho), \qquad k = 0, 1, \ldots, n-1,$$

for $0 \leq s \leq 1$ and for $\rho \in T_1$ with $|\rho| > R_0$. Clearly

$$|F_{1n0}(t, \rho)| = |F_{100}(t, \rho)| \leq 1 \tag{6.190}$$

for $0 \leq t \leq 1$ and for $\rho \in T_1$ with $|\rho| > R_1$, and

$$|G_{1\,n-1\,n}(s, \rho)| = |G_{1\,n-1\,0}(s, \rho)| \leq 1 \tag{6.191}$$

for $0 \leq s \leq 1$ and for $\rho \in T_1$ with $|\rho| > R_1$. With this change in notation, we can rewrite the representation (6.164) of the Green's function $g(t, s; \lambda)$ as

$$\begin{aligned} g(t, s; \lambda) &= -\frac{1}{n\rho^{n-1}} \sum_{k=1}^{n} (\mathrm{i}\omega_k) \mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)} [1 + F_{1k0}(t, \rho)][1 + G_{1\,n-1\,k}(s, \rho)], \\ &\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s < t \leq 1, \\ g(t, s; \lambda) &= 0, \qquad\qquad\qquad\qquad\qquad 0 \leq t < s \leq 1, \end{aligned} \tag{6.192}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$. Here the functions $F_{1k0}(\,\cdot\,, \rho)$ and $G_{1\,n-1\,k}(\,\cdot\,, \rho)$ belong to $H^n[0, 1]$ and satisfy the bounds given in equations (6.150), (6.190) and (6.163), (6.191).

Next, we rewrite (6.159) or (6.192) in the alternate form

$$g(t, s; \lambda) = k_1^*(t, s; \rho) + \ell_1^*(t, s; \rho)$$

for $t \neq s$ in $[0, 1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ and $\rho \in T_1$ with $|\rho| > R_0$, where

$$\begin{aligned} k_1^*(t, s; \rho) &:= \sum_{k=\nu}^{n} v_{1k}(t, \rho)\eta_{1k}(s, \rho), \qquad 0 \leq s < t \leq 1, \\ k_1^*(t, s; \rho) &:= -\sum_{k=1}^{\nu-1} v_{1k}(t, \rho)\eta_{1k}(s, \rho), \qquad 0 \leq t < s \leq 1, \end{aligned} \tag{6.193}$$

and

$$\ell_1^*(t, s; \rho) := \sum_{k=1}^{\nu-1} v_{1k}(t, \rho)\eta_{1k}(s, \rho), \qquad 0 \leq t, s \leq 1.$$

Note that the function $\ell_1^*(t, s; \rho)$ is the kernel of an integral operator which maps $L^2[0, 1]$ into the solution space of the differential equation $(\rho^n I - \ell)u = 0$. For each $\rho \in T_1$ with $|\rho| > R_0$, let $K_{1\rho}^*$ be the integral operator on $L^2[0, 1]$ defined by

$$K_{1\rho}^* u(t) := \int_0^1 k_1^*(t, s; \rho)u(s)\, ds, \qquad 0 \leq t \leq 1,$$

for $u \in L^2[0,1]$. If $u \in L^2[0,1]$ and $v = K_{1\rho}^* u$, then it follows that v belongs to $H^n[0,1]$ and $(\rho^n I - \ell)v = u$. Using equation (6.160) and induction, we see immediately that the derivatives of $v = K_{1\rho}^* u$ are given by

$$\begin{aligned} v^{(j)}(t) &= \int_0^1 \frac{\partial^j k_1^*}{\partial t^j}(t,s;\rho)u(s)\,ds \\ &= \sum_{k=\nu}^{n} v_{1k}^{(j)}(t,\rho)\int_0^t \eta_{1k}(s,\rho)u(s)\,ds - \sum_{k=1}^{\nu-1} v_{1k}^{(j)}(t,\rho)\int_t^1 \eta_{1k}(s,\rho)u(s)\,ds \end{aligned} \tag{6.194}$$

for $0 \le t \le 1$ and for $j = 0,1,\ldots,n-1$, valid for each $\rho \in T_1$ with $|\rho| > R_0$.

From (6.149) and (6.162) the kernel $k_1^*(t,s;\rho)$ can be expressed in the form

$$\begin{aligned} k_1^*(t,s;\rho) &= -\frac{1}{n\rho^{n-1}}\sum_{k=\nu}^{n}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{1k0}(t,\rho)][1+G_{1\,n-1\,k}(s,\rho)], \\ &\qquad 0 \le s < t \le 1, \\ k_1^*(t,s;\rho) &= \frac{1}{n\rho^{n-1}}\sum_{k=1}^{\nu-1}(\mathrm{i}\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}[1+F_{1k0}(t,\rho)][1+G_{1\,n-1\,k}(s,\rho)], \\ &\qquad 0 \le t < s \le 1, \end{aligned} \tag{6.195}$$

for $\rho \in T_1$ with $|\rho| > R_0$. Now take any point $\rho = a + \mathrm{i}b \in T_1$ with $|\rho| > R_0$. If t, s are real numbers with $0 \le s < t \le 1$, then from (6.168) we obtain the estimates

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le \mathrm{e}^{-(t-s)\alpha|\rho|} \le 1, \qquad k = \nu,\ldots,n-1, \tag{6.196}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_n(t-s)}\right| = \mathrm{e}^{-b(t-s)}. \tag{6.197}$$

On the other hand, for real numbers t, s with $0 \le t < s \le 1$, from (6.170) we get

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le \mathrm{e}^{-(s-t)\alpha|\rho|} \le 1, \qquad k = 1,\ldots,\nu-1. \tag{6.198}$$

In particular, if $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| > R_1$ and $b \ge 0$, then these estimates give

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le s < t \le 1, \quad k = \nu,\ldots,n, \tag{6.199}$$

$$\left|\mathrm{e}^{\mathrm{i}\rho\omega_k(t-s)}\right| \le 1, \qquad 0 \le t < s \le 1, \quad k = 1,\ldots,\nu-1. \tag{6.200}$$

Applying (6.199), (6.200) and (6.150), (6.190) and (6.163), (6.191) to the representation (6.195), it follows that

$$|k_1^*(t,s;\rho)| \le \frac{4}{|\rho|^{n-1}} \tag{6.201}$$

for $t \neq s$ in $[0,1]$ and for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| > R_1$ and $b \geq 0$.

Using the integral operator $K^*_{1\rho}$, the alternate basis $u_{1k}(\,\cdot\,,\rho)$, $k = 1,\dots,n$, and the alternate characteristic determinant

$$\varDelta_1^*(\rho) = \det\begin{pmatrix} B_1(u_{11}(\,\cdot\,,\rho)) & \cdots & B_1(u_{1n}(\,\cdot\,,\rho)) \\ \vdots & & \vdots \\ B_n(u_{11}(\,\cdot\,,\rho)) & \cdots & B_n(u_{1n}(\,\cdot\,,\rho)) \end{pmatrix} = \mathrm{e}^{\mathrm{i}\rho}\varDelta_1(\rho), \qquad \rho \in G_1, \tag{6.202}$$

we now proceed to establish alternate representations for the resolvent $R_\lambda(L)$ and the associated Green's function $G(t,s;\lambda)$. Fix any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$, and assume that $\varDelta_1(\rho) \neq 0$. Clearly $\varDelta_1^*(\rho) \neq 0$, and the point λ belongs to the resolvent set $\rho(L)$. Take any function $u \in L^2[0,1]$, and set $v = K^*_{1\rho}u$ and $w = R_\lambda(L)u$. The functions v and w belong to $H^n[0,1]$, and $(\lambda I - \ell)v = u = (\lambda I - \ell)w$. Thus, there exist constants $c_1,\dots,c_n$ such that

$$R_\lambda(L)u(t) = w(t) = v(t) + \sum_{k=1}^{n} c_k u_{1k}(t,\rho), \qquad 0 \leq t \leq 1.$$

Applying the boundary value B_i to the last equation, we obtain the linear system

$$\sum_{k=1}^{n} M^*_{1ik}(\rho)c_k = -B_i(v), \qquad i = 1,\dots,n, \tag{6.203}$$

for the constants $c_1,\dots,c_n$, where

$$M^*_{1ik}(\rho) = B_i(u_{1k}(\,\cdot\,,\rho)) = \sum_{j=0}^{m_i} \alpha_{ij} u_{1k}^{(j)}(0,\rho) + \sum_{j=0}^{m_i} \beta_{ij} u_{1k}^{(j)}(1,\rho)$$

for $i = 1,\dots,n$, $k = 1,\dots,n$. The $n \times n$ coefficient matrix $(M^*_{1ik}(\rho))$ in (6.203) is nonsingular because $\det(M^*_{1ik}(\rho)) = \varDelta_1^*(\rho) \neq 0$.

Fix an index i with $1 \leq i \leq n$, and consider the quantity $B_i(v) = B_i(K^*_{1\rho}u)$. From equation (6.194) and equations (6.149), (6.162), we have

$$\begin{aligned} B_i(v) &= \sum_{j=0}^{m_i} \alpha_{ij} v^{(j)}(0) + \sum_{j=0}^{m_i} \beta_{ij} v^{(j)}(1) \\ &= \frac{1}{n\rho^{n-1}} \sum_{k=1}^{\nu-1} \Big\{ \sum_{j=0}^{m_i} \alpha_{ij} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(0,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{-\mathrm{i}\rho\omega_k s}[1 + G_{1\,n-1\,k}(s,\rho)]u(s)\,ds \\ &\quad + \frac{1}{n\rho^{n-1}} \sum_{k=\nu}^{n} \Big\{ \sum_{j=0}^{m_i} (-1)\beta_{ij} (\mathrm{i}\omega_k)\rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(1,\rho)] \Big\} \\ &\qquad \times \int_0^1 \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)}[1 + G_{1\,n-1\,k}(s,\rho)]u(s)\,ds \end{aligned} \tag{6.204}$$

for $i = 1, \ldots, n$. For $i = 1, \ldots, n$ and $k = 1, \ldots, \nu - 1$ define

$$\mathcal{T}^*_{1ik}(\rho) := \sum_{j=0}^{m_i} \alpha_{ij} (\mathrm{i}\omega_k) \rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(0, \rho)]$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $i = 1, \ldots, n$ and $k = \nu, \ldots, n$ define

$$\mathcal{T}^*_{1ik}(\rho) := \sum_{j=0}^{m_i} (-1) \beta_{ij} (\mathrm{i}\omega_k) \rho^j [(\mathrm{i}\omega_k)^j + F_{1kj}(1, \rho)]$$

for $\rho \in T_1$ with $|\rho| > R_0$; and for $k = 1, \ldots, \nu - 1$ define

$$U^*_{1k}(s, \rho) := \mathrm{e}^{-\mathrm{i}\rho\omega_k s} [1 + G_{1\,n-1\,k}(s, \rho)], \qquad 0 \leq s \leq 1,$$

for $\rho \in T_1$ with $|\rho| > R_0$, and for $k = \nu, \ldots, n$ define

$$U^*_{1k}(s, \rho) := \mathrm{e}^{\mathrm{i}\rho\omega_k(1-s)} [1 + G_{1\,n-1\,k}(s, \rho)], \qquad 0 \leq s \leq 1,$$

for $\rho \in T_1$ with $|\rho| > R_0$. Then (6.204) takes on the simpler form

$$B_i(v) = \frac{1}{n\rho^{n-1}} \sum_{k=1}^{n} \mathcal{T}^*_{1ik}(\rho) \int_0^1 U^*_{1k}(s, \rho) u(s)\, ds \tag{6.205}$$

for $i = 1, \ldots, n$, where the functions $\mathcal{T}^*_{1ik}(\rho)$ are analytic functions of ρ on G_1, and for fixed ρ in T_1 with $|\rho| > R_0$ the functions $U^*_{1k}(\,\cdot\,, \rho)$ belong to $H^n[0, 1]$.

In terms of the matrix $(M^*_{1ik}(\rho))$, let $\widetilde{M}^*_{1ik}(\rho)$ denote the cofactor of the entry $M^*_{1ik}(\rho)$. These cofactors are analytic functions in the ρ variable on the open set G_1; they appear naturally when we solve for the constants $c_1, \ldots, c_n$ in the linear system (6.203):

$$\begin{aligned} c_k &= \frac{-1}{\Delta^*_1(\rho)} \det \begin{array}{c} \qquad\qquad k\text{th column} \\ \begin{pmatrix} M^*_{111}(\rho) & \cdots & B_1(v) & \cdots & M^*_{11n}(\rho) \\ \vdots & & \vdots & & \vdots \\ M^*_{1n1}(\rho) & \cdots & B_n(v) & \cdots & M^*_{1nn}(\rho) \end{pmatrix} \end{array} \\ &= \frac{-1}{\Delta^*_1(\rho)} \sum_{j=1}^{n} \widetilde{M}^*_{1jk}(\rho) B_j(v), \end{aligned}$$

or

$$c_k = \frac{-1}{n\rho^{n-1} \Delta^*_1(\rho)} \sum_{l=1}^{n} \sum_{j=1}^{n} \widetilde{M}^*_{1jk}(\rho) \mathcal{T}^*_{1jl}(\rho) \int_0^1 U^*_{1l}(s, \rho) u(s)\, ds \tag{6.206}$$

for $k = 1, \ldots, n$.

Combining the above results, we conclude that

$$R_\lambda(L)u(t) = K^*_{1\rho}u(t) - \frac{1}{n\rho^{n-1}\varDelta^*_1(\rho)} \sum_{k,l=1}^{n} \sum_{j=1}^{n} \widetilde{M}^*_{1jk}(\rho)\mathcal{T}^*_{1jl}(\rho)u_{1k}(t,\rho)\int_0^1 U^*_{1l}(s,\rho)u(s)\,ds \tag{6.207}$$

for $0 \le t \le 1$ and for $u \in L^2[0,1]$, where (6.207) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\varDelta^*_1(\rho) \neq 0$ (equivalently $\varDelta_1(\rho) \neq 0$). Also, from (6.207) the associated Green's function is given by

$$G(t,s;\lambda) = k^*_1(t,s;\rho) - \frac{1}{n\rho^{n-1}\varDelta^*_1(\rho)} \sum_{k,l=1}^{n} \sum_{j=1}^{n} \widetilde{M}^*_{1jk}(\rho)\mathcal{T}^*_{1jl}(\rho)u_{1k}(t,\rho)U^*_{1l}(s,\rho) \tag{6.208}$$

for $t \neq s$ in $[0,1]$, valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in G_1$ and with $\varDelta^*_1(\rho) \neq 0$.

Next, we determine bounds and growth rates for the functions appearing in (6.207) and (6.208). For the basis functions $u_{1k}(t,\rho)$, $k = 1,\ldots,n$, we have already established the necessary bounds in equation (6.158). For the kernel $k^*_1(t,s;\rho)$ the required growth rate is given in equation (6.201). Note the restriction $\operatorname{Im}\rho \ge 0$ that applies to these inequalities. Let us calculate bounds and growth rates for the functions $\mathcal{T}^*_{1ik}(\rho)$, $U^*_{1k}(s,\rho)$, and $\widetilde{M}^*_{1ik}(\rho)$.

First, consider the functions $\mathcal{T}^*_{1ik}(\rho)$. From their definitions and the estimates (6.150) and (6.190), it is immediate that

$$|\mathcal{T}^*_{1ik}(\rho)| \le \gamma_1|\rho|^{m_i} \tag{6.209}$$

for $\rho \in T_1$ with $|\rho| > R_1$ and for $i = 1,\ldots,n$, $k = 1,\ldots,n$. Second, for the functions $U^*_{1k}(s,\rho)$, from their definitions and the estimates (5.111)–(5.112), together with (6.163) and (6.191), we obtain the estimates (replace t by $1-s$)

$$|U^*_{1k}(s,\rho)| \le 2\mathrm{e}^{-s\alpha|\rho|} \le 2, \qquad k = 1,\ldots,\nu-1, \tag{6.210}$$

$$|U^*_{1k}(s,\rho)| \le 2\mathrm{e}^{-(1-s)\alpha|\rho|} \le 2, \qquad k = \nu,\ldots,n-1, \tag{6.211}$$

$$|U^*_{1n}(s,\rho)| \le 2\mathrm{e}^{-b(1-s)} \tag{6.212}$$

for $0 \le s \le 1$ and for $\rho = a+\mathrm{i}b \in T_1$ with $|\rho| > R_1$. In particular, for $\rho = a+\mathrm{i}b$ belonging to the sector S_1 with $|\rho| > R_1$ and $b \ge 0$, we have

$$|U^*_{1k}(s,\rho)| \le 2, \qquad 0 \le s \le 1, \quad k = 1,\ldots,n. \tag{6.213}$$

Third, fix indices i and k with $1 \le i \le n$ and $1 \le k \le n$. Then the cofactor $\widetilde{M}^*_{1ik}(\rho)$ is formed by taking $(-1)^{i+k}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and the kth column of the matrix

$$\begin{array}{c} \begin{array}{ccc} \scriptstyle 1\le k\le \nu-1 & \scriptstyle \nu\le k\le n-1 & \end{array} \\ \begin{pmatrix} \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) & \widetilde{F}_{11k}(\rho)+\widehat{Q}_{1k}(\rho) & P_{110}(\rho)+Q_{110}(\rho)\mathrm{e}^{\mathrm{i}\rho} \\ \vdots & \vdots & \vdots \\ \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) & \widetilde{F}_{1nk}(\rho)+\widehat{Q}_{nk}(\rho) & P_{1n0}(\rho)+Q_{1n0}(\rho)\mathrm{e}^{\mathrm{i}\rho} \end{pmatrix}. \end{array}$$

Suppose we expand the determinant for $\widetilde{M}^*_{1ik}(\rho)$ in the same manner as $\varDelta_1(\rho)$ was expanded in Chapter 5. See equation (5.125). Now we emphasize the nth column instead of the 0th column. If $1 \le k \le \nu - 1$ or $\nu \le k \le n - 1$, then the determinant of the $(n-1) \times (n-1)$ submatrix that leads to $\widetilde{M}^*_{1ik}(\rho)$ is first expanded using linearity in the nth column; this yields $\mathrm{e}^{\mathrm{i}\rho}$, 1 terms. On the other hand, if $k = n$, then a 1 term is produced, but no $\mathrm{e}^{\mathrm{i}\rho}$ term appears.

Next, all the $(n-1) \times (n-1)$ determinants are expanded using linearity in all $n-1$ columns. This expansion produces the following representations: for $1 \le k \le \nu - 1$ or for $\nu \le k \le n - 1$,

$$\begin{aligned}\widetilde{M}^*_{1ik}(\rho) &= \mathrm{e}^{\mathrm{i}\rho}[\widetilde{\varPi}'_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\varPi}_{ik}(\rho) + \widetilde{\varPhi}'_{1ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \widetilde{\varPhi}_{1ik}(\rho)] \\ &= \widetilde{\varPi}'_{ik}(\rho) + \widetilde{\varPi}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \widetilde{\varPhi}'_{1ik}(\rho) + \widetilde{\varPhi}_{1ik}(\rho)\mathrm{e}^{\mathrm{i}\rho}\end{aligned} \tag{6.214}$$

for $\rho \in G_1$; and for $k = n$,

$$\widetilde{M}^*_{1in}(\rho) = \widetilde{\varPi}_{i0}(\rho) + \widetilde{\varPhi}_{1i0}(\rho) \tag{6.215}$$

for $\rho \in G_1$. In these representations the functions $\widetilde{\varPi}'_{ik}$, $\widetilde{\varPi}_{ik}$ and $\widetilde{\varPhi}'_{1ik}$, $\widetilde{\varPhi}_{1ik}$ are the same functions that appear in equation (6.185). The functions $\widetilde{\varPi}'_{ik}$, $\widetilde{\varPi}_{ik}$ are formed from the functions $\widehat{P}_{ik}$, $\widehat{Q}_{ik}$ introduced in Chapter 3. They are analytic for $\rho \ne 0$ in $\mathbb{C}$, and each one has the simple form

$$\sum_{j=-(n-1)(m-1)}^{p_0-m_i} B_{ikj}\rho^j, \qquad \rho \ne 0 \text{ in } \mathbb{C}.$$

The functions $\widetilde{\varPhi}'_{1ik}$, $\widetilde{\varPhi}_{1ik}$ are analytic for ρ in the open set G_1, and satisfy the growth rates

$$|\widetilde{\varPhi}'_{1ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)}, \qquad |\widetilde{\varPhi}_{1ik}(\rho)| \le \gamma_2|\rho|^{-(m-p_0+m_i)} \tag{6.216}$$

for $\rho \in G_1$.

Take any point $\rho = a + \mathrm{i}b$ in the sector S_1 with $|\rho| > R_1$ and $b \ge 0$. Clearly $|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \le 1$, and hence, by (6.214)–(6.216)

$$|\widetilde{M}^*_{1ik}(\rho)| \le 2\gamma_3|\rho|^{p_0-m_i} + 2\gamma_2|\rho|^{-(m-p_0+m_i)} \le \gamma_4|\rho|^{p_0-m_i} \tag{6.217}$$

for $i = 1, \dots, n$, $k = 1, \dots, n$. Combining this result with (6.209), we obtain the estimate

$$\Big|\sum_{j=1}^{n} \widetilde{M}^*_{1jk}(\rho)\mathcal{T}^*_{1jl}(\rho)\Big| \le \sum_{j=1}^{n} \gamma_4|\rho|^{p_0-m_j} \cdot \gamma_1|\rho|^{m_j} \le \gamma_5|\rho|^{p_0} \tag{6.218}$$

for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| > R_1$ and $b \ge 0$ and for $k, l = 1, \dots, n$.

Finally, take any point $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$, with $|\rho| > R_1$ and $b \ge 0$, and with $\varDelta_1(\rho) \ne 0$, so $\varDelta^*_1(\rho) \ne 0$. Clearly $\rho \in \operatorname{Int} T_1$ with

$|\rho| > R_1 \geq R_0$, $\rho \in G_1$, and λ belongs to the resolvent set $\rho(L)$ with the Green's function $G(t, s; \lambda)$ given by equation (6.208). Applying the estimates (6.201), (6.218), (6.158), and (6.213) and the relation (6.202) to (6.208), we see that

$$|G(t, s; \lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{1}{n|\rho|^{n-1}|\Delta_1^*(\rho)|} \sum_{k,l=1}^{n} \gamma_5 |\rho|^{p_0} \cdot 2 \cdot 2,$$

or

$$|G(t, s; \lambda)| \leq \frac{4}{|\rho|^{n-1}} + \frac{\gamma |\rho|^{p_0}}{n|\rho|^{n-1} \mathrm{e}^{-b} |\Delta_1(\rho)|} \tag{6.219}$$

for $t \neq s$ in $[0, 1]$, where equation (6.219) is valid for $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$, with $|\rho| > R_1$ and $b \geq 0$, and with $\Delta_1(\rho) \neq 0$. Compare this growth rate to the growth rate given in equation (6.189). Note the additional factor e^{-b} that appears in the denominator of (6.219). In Chapter 9 we will use the estimates (6.118), (6.148) and (6.189), (6.219) of the Green's function $G(t, s; \lambda)$ to establish the completeness of the generalized eigenfunctions of the differential operator L.

7

The Eigenvalues for n Even

In this chapter we compute the eigenvalues of the differential operator L for the case n even. We assume the hypotheses of Chapters 3–5: (i) $n = 2\nu \geq 2$; (ii) the differential operator L is either regular or simply irregular; (iii) the integers p and q have been determined with $-\infty < p \leq q \leq p_0$ and with $a_p \neq 0$, $c_p \neq 0$, and $a_\kappa = c_\kappa = 0$ for $\kappa = p+1, \ldots, p_0$ and $b_\kappa = 0$ for $\kappa = q+1, \ldots, p_0$; (iv) the translated sectors T_0 and T_1 have been chosen with condition (3.31) being satisfied for the case $p = q$; (v) the integer m has been fixed with $m > n$, $m > p_0$, and $-(m-p_0-1) \leq p \leq p_0$; and (vi) the functions π_i, $i = 0, 1, 2$, have been determined as per Chapter 3 or equation (5.29). The functions π_i are given explicitly by the formulas

$$\pi_2(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \quad \pi_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa, \quad \pi_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} c_\kappa \rho^\kappa \tag{7.1}$$

for $\rho \neq 0$ in $\mathbb{C}$. By our choice of the integer q, if $p = q$, then the constant $b_p = b_q$ can be either zero or nonzero, while if $p < q$, then the constant b_q is nonzero.

Let Δ_0 and Δ_1 be the characteristic determinants of L determined in Theorem 5.1 and Theorem 5.2, respectively. Δ_0 is analytic on the open set $G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$ and has the representation

$$\begin{aligned} \Delta_0(\rho) = {} & \pi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \pi_0(\rho) \\ & + \Phi_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \Phi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Phi_{00}(\rho) \end{aligned} \tag{7.2}$$

for $\rho \in G_0$, where the functions Φ_{0i}, $i = 0, 1, 2$, are analytic on G_0 with

$$|\Phi_{0i}(\rho)| \leq \gamma |\rho|^{-(m-p_0)} \tag{7.3}$$

for $\rho \in G_0$ and for $i = 0, 1, 2$. Similarly, the characteristic determinant Δ_1 is analytic on the open set $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$ and has the representation

$$\begin{aligned}\Delta_1(\rho) &= \pi_2(\rho) + \pi_1(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \pi_0(\rho)\mathrm{e}^{-2\mathrm{i}\rho} \\ &\quad + \Phi_{12}(\rho) + \Phi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \Phi_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho}\end{aligned} \tag{7.4}$$

for $\rho \in G_1$, where the functions Φ_{1i}, $i = 0, 1, 2$, are analytic on G_1 with

$$|\Phi_{1i}(\rho)| \leq \gamma|\rho|^{-(m-p_0)} \tag{7.5}$$

for $\rho \in G_1$ and for $i = 0, 1, 2$.

To determine the eigenvalues of L, we calculate the zeros of the characteristic determinants Δ_0 and Δ_1. The analysis divides quite naturally into several cases determined by the relative size of the integers p and q.

Assume that $p = q$. This case divides below into Case 1 and Case 2. Case 1 corresponds to simple eigenvalues, while Case 2 allows the possibility of multiple eigenvalues. These two cases include all the cases in which L is regular, as well as many cases in which L is simply irregular. Consider first the characteristic determinant Δ_0 on the open set G_0. Let

$$f_0(\rho) := a_p\mathrm{e}^{2\mathrm{i}\rho} + b_p\mathrm{e}^{\mathrm{i}\rho} + c_p$$

for $\rho \in \mathbb{C}$, an entire function, and let

$$\begin{aligned}g_0(\rho) := &\sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a_\kappa}{\rho^{p-\kappa}}\,\mathrm{e}^{2\mathrm{i}\rho} + \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{b_\kappa}{\rho^{p-\kappa}}\,\mathrm{e}^{\mathrm{i}\rho} + \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{c_\kappa}{\rho^{p-\kappa}} \\ &+ \frac{1}{\rho^p}\left[\Phi_{02}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \Phi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Phi_{00}(\rho)\right]\end{aligned}$$

for $\rho \in G_0$, where the function g_0 is analytic on the open set G_0. From (7.2) we obtain the representation

$$\Delta_0(\rho) = \rho^p[f_0(\rho) + g_0(\rho)] \tag{7.6}$$

for $\rho \in G_0$.

Next, consider the characteristic determinant Δ_1 on the open set G_1. Here we let

$$f_1(\rho) := a_p + b_p\mathrm{e}^{-\mathrm{i}\rho} + c_p\mathrm{e}^{-2\mathrm{i}\rho} = \mathrm{e}^{-2\mathrm{i}\rho}f_0(\rho)$$

for $\rho \in \mathbb{C}$, an entire function, and let

$$\begin{aligned}g_1(\rho) := &\sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a_\kappa}{\rho^{p-\kappa}} + \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{b_\kappa}{\rho^{p-\kappa}}\,\mathrm{e}^{-\mathrm{i}\rho} + \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{c_\kappa}{\rho^{p-\kappa}}\,\mathrm{e}^{-2\mathrm{i}\rho} \\ &+ \frac{1}{\rho^p}\left[\Phi_{12}(\rho) + \Phi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \Phi_{10}(\rho)\mathrm{e}^{-2\mathrm{i}\rho}\right]\end{aligned}$$

for $\rho \in G_1$, where the function g_1 is analytic on the open set G_1. From (7.4) we have

$$\Delta_1(\rho) = \rho^p[f_1(\rho) + g_1(\rho)] \tag{7.7}$$

for $\rho \in G_1$.

Recall that the constant $d > 0$ has been chosen in Chapter 3 to satisfy condition (3.31), namely

$$|a_p|\mathrm{e}^{-2d} + |b_p|\mathrm{e}^{-d} + |c_p|\mathrm{e}^{-2d} \leq \frac{1}{4}\,|a_p| = \frac{1}{4}\,|c_p|, \tag{7.8}$$

and that the translated sectors T_0, T_1 have been selected such that the sectors S_0, S_1 lie in the interiors of T_0, T_1, respectively, and such that the horizontal strip

$$\Gamma = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \geq -\pi \text{ and } |b| \leq d\}$$

lies in the interiors of both T_0 and T_1.

Let us examine the functions f_0 and g_0 which make up the characteristic determinant Δ_0. First, if $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \geq d$, then $|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \leq \mathrm{e}^{-d}$, $|\mathrm{e}^{2\mathrm{i}\rho}| \leq \mathrm{e}^{-2d}$, and by inequality (7.8)

$$\begin{aligned} |f_0(\rho)| &\geq |c_p| - |a_p||\mathrm{e}^{2\mathrm{i}\rho}| - |b_p||\mathrm{e}^{\mathrm{i}\rho}| \\ &\geq |c_p| - |a_p|\mathrm{e}^{-2d} - |b_p|\mathrm{e}^{-d} \geq |c_p| - \frac{1}{4}\,|c_p|. \end{aligned}$$

Thus,

$$|f_0(\rho)| \geq \frac{3}{4}\,|c_p| = \frac{3}{4}\,|a_p| \quad \text{for } \rho = a + \mathrm{i}b \in \mathbb{C} \text{ with } b \geq d.$$

Second, take any $\rho = a + \mathrm{i}b \in G_0$ with $b \geq -d$. Clearly $|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \leq \mathrm{e}^{d}$ and $|\mathrm{e}^{2\mathrm{i}\rho}| \leq \mathrm{e}^{2d}$. Also, since $-(m - p_0 - 1) \leq p \leq p_0$, we have $m - p_0 + p \geq m - p_0 - (m - p_0 - 1) = 1$, and hence, by (7.3)

$$\begin{aligned} |g_0(\rho)| &\leq \frac{\gamma_1}{|\rho|} + \frac{1}{|\rho|^p} \cdot \frac{\gamma_2}{|\rho|^{m-p_0}} \\ &= \frac{\gamma_1}{|\rho|} + \frac{\gamma_2}{|\rho|^{m-p_0+p}} \leq \frac{\gamma_1 + \gamma_2}{|\rho|}\,. \end{aligned}$$

Therefore,

$$|g_0(\rho)| \leq \frac{\gamma_0}{|\rho|} \quad \text{for } \rho = a + \mathrm{i}b \in G_0 \text{ with } b \geq -d.$$

Third, if $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \leq -d$, then $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^{b} \leq \mathrm{e}^{-d}$, $|\mathrm{e}^{-2\mathrm{i}\rho}| \leq \mathrm{e}^{-2d}$, and by inequality (7.8) again

$$\begin{aligned} |f_1(\rho)| &= |a_p + b_p\mathrm{e}^{-\mathrm{i}\rho} + c_p\mathrm{e}^{-2\mathrm{i}\rho}| \\ &\geq |a_p| - |b_p|\mathrm{e}^{-d} - |c_p|\mathrm{e}^{-2d} \geq |a_p| - \frac{1}{4}\,|a_p|. \end{aligned}$$

Hence,

$$|f_1(\rho)| \geq \frac{3}{4}\,|a_p| \quad \text{for } \rho = a + \mathrm{i}b \in \mathbb{C} \text{ with } b \leq -d.$$

Fourth, consider the function g_1 appearing in the representation (7.7) for $\Delta_1(\rho)$. Take any point $\rho = a + \mathrm{i}b \in G_1$ with $b \leq d$. Then $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^b \leq \mathrm{e}^d$, $|\mathrm{e}^{-2\mathrm{i}\rho}| \leq \mathrm{e}^{2d}$, and by (7.5)

$$|g_1(\rho)| \leq \frac{\gamma_3}{|\rho|} + \frac{1}{|\rho|^p} \cdot \frac{\gamma_4}{|\rho|^{m-p_0}} \leq \frac{\gamma_3 + \gamma_4}{|\rho|},$$

and hence,

$$|g_1(\rho)| \leq \frac{\gamma_0}{|\rho|} \quad \text{for } \rho = a + \mathrm{i}b \in G_1 \text{ with } b \leq d.$$

In terms of the constant γ_0 appearing in the estimates for g_0 and g_1, choose a constant $r_1 > R_1 \geq R_0$ such that

$$\frac{\gamma_0}{|\rho|} \leq \frac{1}{4}\,|a_p| \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \geq r_1.$$

It follows from the above that if $\rho = a + \mathrm{i}b \in G_0$ with $|\rho| \geq r_1$ and $b \geq d$, then

$$\begin{aligned} |\Delta_0(\rho)| &\geq |\rho|^p\{|f_0(\rho)| - |g_0(\rho)|\} \\ &\geq |\rho|^p\left\{\frac{3}{4}\,|a_p| - \frac{1}{4}\,|a_p|\right\} = \frac{1}{2}\,|a_p||\rho|^p > 0. \end{aligned} \tag{7.9}$$

On the other hand, if $\rho = a + \mathrm{i}b \in G_1$ with $|\rho| \geq r_1$ and $b \leq -d$, then

$$\begin{aligned} |\Delta_1(\rho)| &\geq |\rho|^p\{|f_1(\rho)| - |g_1(\rho)|\} \\ &\geq |\rho|^p\left\{\frac{3}{4}\,|a_p| - \frac{1}{4}\,|a_p|\right\} = \frac{1}{2}\,|a_p||\rho|^p > 0. \end{aligned} \tag{7.10}$$

The estimates (7.9) and (7.10) are our initial growth rates for the characteristic determinants Δ_0 and Δ_1 relative to the open sets G_0 and G_1, respectively. They will play a crucial role in Chapter 9 where we show that the generalized eigenfunctions of the differential operator L are complete in the Hilbert space $L^2[0, 1]$. As an immediate application of (7.9) and (7.10), we have the following theorem which establishes *apriori estimates* for the eigenvalues of L relative to the sets G_0 and G_1.

Theorem 7.1. *Assume that* $p = q$. *Let* $\lambda = \rho^n \in \mathbb{C}$ *with* $\rho = a + \mathrm{i}b \in G_0$ *and with* $|\rho| \geq r_1$.

(a) *If* $b \geq d$, *then* $\Delta_0(\rho) \neq 0$ *and* $\lambda \in \rho(L)$.

(b) *If* λ *is an eigenvalue of* L, *then* $\Delta_0(\rho) = 0$ *and* $b < d$.

In addition, let $\lambda = \rho^n \in \mathbb{C}$ *with* $\rho = a + \mathrm{i}b \in G_1$ *and with* $|\rho| \geq r_1$.

(c) *If* $b \leq -d$, *then* $\Delta_1(\rho) \neq 0$ *and* $\lambda \in \rho(L)$.

(d) *If* λ *is an eigenvalue of* L, *then* $\Delta_1(\rho) = 0$ *and* $b > -d$.

It follows from the theorem that the resolvent set $\rho(L)$ is nonempty. Thus, the differential operator L is a Fredholm operator with Fredholm set $\Phi(L) = \mathbb{C}$

and with resolvent set $\rho(L) \neq \emptyset$: this implies that the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. See [34, p. 58 or p.60].

We next focus our search for the zeros of Δ_0 on the horizontal strip Γ. Let ξ_0 and η_0 be the roots of the quadratic polynomial $Q(z) := a_p z^2 + b_p z + c_p$, so

$$Q(z) = a_p[z - \xi_0][z - \eta_0]$$

with $a_p \xi_0 \eta_0 = c_p$, $\xi_0 \neq 0$, $\eta_0 \neq 0$ and with $\eta_0 = -1/(\omega_p \xi_0)$ by (3.30). Then the function f_0 can be written in the form

$$f_0(\rho) = a_p[\mathrm{e}^{\mathrm{i}\rho} - \xi_0][\mathrm{e}^{\mathrm{i}\rho} - \eta_0]$$

for $\rho \in \mathbb{C}$, and we see immediately that the zeros of f_0 are given by

$$\begin{aligned} \mu'_k &:= (2\pi k + \operatorname{Arg} \xi_0) - \mathrm{i} \ln |\xi_0|, \qquad k = 0, \pm 1, \pm 2, \dots, \\ \mu''_k &:= (2\pi k + \operatorname{Arg} \eta_0) - \mathrm{i} \ln |\eta_0|, \qquad k = 0, \pm 1, \pm 2, \dots, \end{aligned}$$

with $|\eta_0| = 1/|\xi_0|$ and $\arg \eta_0 = -\arg \xi_0 - 2\pi p/n + \pi$. From the above estimates for f_0, the zeros μ'_k, μ''_k must all lie in the interior of the horizontal strip $|\operatorname{Im} \rho| \leq d$, i.e., $-d < \ln |\xi_0| < d$. In case $\xi_0 \neq \eta_0$, we have $\mu'_k \neq \mu''_l$ for all k, l, with each μ'_k and each μ''_k a zero of order 1 of f_0. In the special case $\xi_0 = \eta_0$, where Q has a double root, we have $b_p^2 = -4a_p^2/\omega_p$,

$$\xi_0 = \eta_0 = -\frac{b_p}{2a_p} = \pm \frac{\mathrm{i}}{\sqrt{\omega_p}} \quad \text{and} \quad |\xi_0| = |\eta_0| = 1,$$

and

$$\mu'_k = \mu''_k = 2\pi k + \operatorname{Arg} \xi_0 := \mu_k \quad \text{for } k = 0, \pm 1, \pm 2, \dots;$$

in this case each μ_k is a real zero of order 2 of f_0, and the relation $b_p^2 = -4a_p^2/\omega_p$ implies that $b_p \neq 0$. In both cases we will show that the zeros of Δ_0 and $f_0 + g_0$ in Γ appear as perturbations of the μ'_k, μ''_k.

Since $-\pi < \operatorname{Arg} \xi_0 \leq \pi$ and $-\pi < \operatorname{Arg} \eta_0 \leq \pi$, we can select a constant $\omega \geq \pi$ such that $\omega - 2\pi < \operatorname{Arg} \xi_0 < \omega$ and $\omega - 2\pi < \operatorname{Arg} \eta_0 < \omega$. Then for $k = 1, 2, \dots$ we introduce the rectangles

$$R'_k := \{\rho \in \mathbb{C} \mid \omega - 2\pi \leq \operatorname{Re} \rho \leq \omega + 2\pi(k-1) \text{ and } |\operatorname{Im} \rho| \leq d\}.$$

Clearly these rectangles lie in the horizontal strip Γ, and hence, they lie in the interior of the sector T_0, and the zeros μ'_0, μ''_0 lie in the interior of the rectangle R'_1. Choose a constant δ with $0 < \delta \leq \pi/4$ such that the two closed disks $|\rho - \mu'_0| \leq \delta$ and $|\rho - \mu''_0| \leq \delta$ both lie in the interior of R'_1 and such that these two disks are disjoint in the case $\xi_0 \neq \eta_0$. For $k = 0, \pm 1, \pm 2, \dots$ form the circles

$$\Gamma'_k := \{\rho \in \mathbb{C} \mid |\rho - \mu'_k| = \delta\}, \qquad \Gamma''_k := \{\rho \in \mathbb{C} \mid |\rho - \mu''_k| = \delta\},$$

where in the case $\xi_0 = \eta_0$ we set

$$\varGamma'_k = \varGamma''_k := \varGamma_k \quad \text{for } k = 0, \pm 1, \pm 2, \ldots .$$

The following properties are obvious from these definitions: (i) the circles $\varGamma'_k$, $\varGamma''_k$, $k \geq 0$, lie in the interior of the horizontal strip $\varGamma$; (ii) the $\varGamma'_k$, $\varGamma''_l$ and the points inside them do not overlap each other in the case $\xi_0 \neq \eta_0$; (iii) the $\varGamma_k$ and the points inside them do not overlap each other in the case $\xi_0 = \eta_0$; and (iv) for each positive integer k_0 the circles $\varGamma'_k$, $\varGamma''_k$, $0 \leq k < k_0$, lie in the interior of the rectangle R'_{k_0}, the circles $\varGamma'_k$, $\varGamma''_k$, $k \geq k_0$, lie in the exterior of R'_{k_0} and to the right of R'_{k_0}, and the circles $\varGamma'_k$, $\varGamma''_k$, $k < 0$, lie in the exterior of R'_{k_0} and to the left of R'_{k_0}.

To complete the geometry, let $\varGamma_*$ be the subset of T_0 defined by

$$\varGamma_* := \{\rho = a + \mathrm{i}b \in \varGamma \mid \rho \text{ is not inside any of the circles } \varGamma'_k,\, \varGamma''_k\}.$$

In the sequel we refer to $\varGamma_*$ as a *punctured horizontal strip.*

It is clear that $f_0(\rho) \neq 0$ for all $\rho \in R'_1$ not in the circles $\varGamma'_0$, $\varGamma''_0$. Set

$$m_* := \min\{\, |f_0(\rho)| \mid \rho \in R'_1 \text{ with } \rho \text{ not in } \varGamma'_0,\, \varGamma''_0 \,\} > 0.$$

Since $f_0(\rho + 2\pi) = f_0(\rho)$ for all $\rho \in \mathbb{C}$, it follows that $|f_0(\rho)| \geq m_*$ for all points $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|b| \leq d$ and with ρ not in any of the circles $\varGamma'_k$, $\varGamma''_k$. If we set $m_0 := \min\{3|a_p|/4,\, m_*\} > 0$, then our estimates for f_0 combine to yield the result

$$|f_0(\rho)| \geq m_0 > 0 \tag{7.11}$$

for all $\rho \in \mathbb{C}$ with ρ not in any of the circles $\varGamma'_k$, $\varGamma''_k$.

Select a positive integer k_0 such that the constant $y_0 := \omega + 2\pi(k_0 - 1)$ has the following properties: $y_0 \geq r_1$, $y_0 \geq \omega$, and

$$\frac{\gamma_0}{|\rho|} \leq \frac{m_0}{2} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \geq y_0,$$

where γ_0 is the constant introduced above in the estimates for g_0 and g_1. Clearly $y_0 > 1$. Then for any point $\rho = a + \mathrm{i}b \in \varGamma$ with $|\rho| \geq y_0$, we have $|\rho| \geq r_1 > R_0$, so $\rho \in G_0$, and by our previous estimate for g_0,

$$|g_0(\rho)| \leq \frac{\gamma_0}{|\rho|} \leq \frac{m_0}{2}. \tag{7.12}$$

Combining (7.11) and (7.12), we conclude that

$$|g_0(\rho)| \leq \frac{m_0}{2} < m_0 \leq |f_0(\rho)| \tag{7.13}$$

and

$$|f_0(\rho) + g_0(\rho)| \geq \frac{m_0}{2} \tag{7.14}$$

for all $\rho = a + \mathrm{i}b \in \varGamma_*$ with $|\rho| \geq y_0$, and hence, by (7.6)

$$|\varDelta_0(\rho)| \geq \frac{m_0}{2}\, |\rho|^p > 0 \tag{7.15}$$

for all $\rho = a + \mathrm{i}b \in \varGamma_*$ with $|\rho| \geq y_0$.

The estimate (7.15) is our principal result for the growth rate of the characteristic determinant $\varDelta_0$ on the punctured horizontal strip $\varGamma_*$ for the case $p = q$. At this point we divide the discussion into the two cases where $\xi_0 \neq \eta_0$ and $\xi_0 = \eta_0$.

7.1 Case 1. $p = q$, $\xi_0 \neq \eta_0$

Assume that $p = q$ and $\xi_0 \neq \eta_0$. Suppose $\rho = a + ib$ is a zero of $\varDelta_0$ in G_0 with $a \geq -\pi$, $b \geq -d$, and $|\rho| \geq y_0$. What can we say about the location of ρ? By Theorem 7.1 and the inequality (7.15), ρ must lie in one of the circles $\varGamma'_k$, $\varGamma''_k$ for some integer k.

Let us consider the circles $\varGamma'_k$, $\varGamma''_k$ for $k \geq k_0$, which lie in the interior of T_0 and in the interior of the horizontal strip $\varGamma$ and to the right of the rectangle $R'_{k_0} = [\omega - 2\pi, y_0] \times [-d, d]$. From (7.13) we have $|g_0(\rho)| < |f_0(\rho)|$ for all points ρ on $\varGamma'_k$, $\varGamma''_k$ for $k \geq k_0$, and hence, by Rouché's Theorem $\varDelta_0$ and $f_0 + g_0$ have precisely the same number of zeros as f_0 inside $\varGamma'_k$ and $\varGamma''_k$ for all $k \geq k_0$. But f_0 has only the single zero μ'_k of order 1 inside $\varGamma'_k$ and only the single zero μ''_k of order 1 inside $\varGamma''_k$, implying that $\varDelta_0$ has exactly one zero ρ'_k inside $\varGamma'_k$ with ρ'_k having order 1 for $k \geq k_0$, and $\varDelta_0$ has exactly one zero ρ''_k inside $\varGamma''_k$ with ρ''_k having order 1 for $k \geq k_0$.

Setting

$$\begin{aligned} \lambda'_k &:= (\rho'_k)^n, \qquad k = k_0, k_0 + 1, \ldots, \\ \lambda''_k &:= (\rho''_k)^n, \qquad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

it follows that the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$, are eigenvalues of L. Since the points ρ'_k, ρ''_k, $k = k_0, k_0 + 1, \ldots$, are zeros of order 1 of $\varDelta_0$, applying our earlier work the corresponding algebraic multiplicities and ascents are

$$\begin{aligned} \nu(\lambda'_k) &= m(\lambda'_k) = 1, \qquad k = k_0, k_0 + 1, \ldots, \\ \nu(\lambda''_k) &= m(\lambda''_k) = 1, \qquad k = k_0, k_0 + 1, \ldots. \end{aligned} \tag{7.16}$$

See Theorem 2.1 of Chapter 4 in [34]; the proof given for the principal part T also works for the differential operator L.

Now suppose that λ_0 is any eigenvalue of L distinct from the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$. Choose $\rho_0 \in S_0 \cup S_1$ such that $\lambda_0 = (\rho_0)^n$. Clearly $\rho_0 \in \operatorname{Int} T_0 \cup \operatorname{Int} T_1$. There are two possible locations for the point ρ_0: either ρ_0 lies in the disk $|\rho| < y_0$, or $|\rho_0| \geq y_0$. In the former case only a finite number of such ρ_0 are possible because the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. Assume that ρ_0 belongs to the latter case, so $|\rho_0| \geq y_0$. Let $\rho_0 = a_0 + ib_0$. First, if $b_0 \geq d$, then we would have $\rho_0 \in S_0$ with $|\rho_0| \geq r_1$, and hence, ρ_0 would belong to the open set G_0 with $|\rho_0| \geq r_1$; but Theorem 7.1 would then place λ_0 in $\rho(L)$ — a contradiction. Thus, we must have $b_0 < d$.

Second, if $b_0 \le -d$, then we would have $\rho_0 \in S_1$ with $|\rho_0| \ge r_1$, so ρ_0 would belong to the open set G_1 with $|\rho_0| \ge r_1$, and Theorem 7.1 would again place λ_0 in $\rho(L)$ — a contradiction. Therefore, for the imaginary part of ρ_0 we must have $-d < b_0 < d$.

Now for the real part, clearly $a_0 \ge 0$ from the simple geometry of the sector $S_0 \cup S_1$, and hence, ρ_0 lies in the interior of the horizontal strip Γ and ρ_0 lies in $\operatorname{Int} T_0$. Suppose $a_0 > y_0$. We know that ρ_0 does not lie in any of the circles Γ'_k, Γ''_k for $k \ge k_0$ because λ_0 is distinct from the λ'_k, λ''_k. The circles Γ'_k, Γ''_k, $-\infty < k < k_0$, either lie in the interior of the rectangle R'_{k_0} or lie to the left of R'_{k_0}, and hence, ρ_0 can not be in any of these circles. This implies that ρ_0 belongs to the punctured horizontal strip Γ_* with $|\rho_0| \ge a_0 > y_0$, and by inequality (7.15) we have $\Delta_0(\rho_0) \ne 0$ — again putting λ_0 in $\rho(L)$. This contradiction shows that we must have $a_0 \le y_0$, and hence, ρ_0 must lie in the rectangle $[0, y_0] \times [-d, d]$. Only a finite number of such ρ_0 are possible. Thus, the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$, account for all but a finite number of the eigenvalues of L.

To complete this case, we derive asymptotic formulas for the zeros ρ'_k, ρ''_k of Δ_0. Indeed, let G be the entire function defined by $G(\rho) := a_p[\mathrm{e}^{\mathrm{i}\rho} - \eta_0]$, and set

$$M_0 := \min\{\, |G(\rho)| \mid \rho \in \mathbb{C} \text{ with } |\rho - \mu'_0| \le \delta\} > 0.$$

Because G has period 2π, it follows that $|G(\rho'_k)| \ge M_0$ for $k = k_0, k_0 + 1, \ldots$. If we set $\zeta'_k := -g_0(\rho'_k)/\xi_0 G(\rho'_k)$, $k = k_0, k_0 + 1, \ldots$, then we can write the equation $f_0(\rho'_k) + g_0(\rho'_k) = 0$ as $\mathrm{e}^{\mathrm{i}\rho'_k} = \xi_0 + \xi_0 \zeta'_k$, and upon dividing by $\mathrm{e}^{\mathrm{i}\mu'_k} = \xi_0$, it becomes

$$\mathrm{e}^{\mathrm{i}(\rho'_k - \mu'_k)} = 1 + \zeta'_k.$$

But $|\operatorname{Re}(\rho'_k - \mu'_k)| \le |\rho'_k - \mu'_k| < \delta \le \pi/4$, so

$$\rho'_k - \mu'_k = -\mathrm{i}\operatorname{Log}[1 + \zeta'_k], \qquad k = k_0, k_0 + 1, \ldots. \tag{7.17}$$

For each integer $k \ge k_0$ we have

$$\begin{aligned} |\rho'_k| &\ge |\mu'_k| - |\rho'_k - \mu'_k| \ge 2\pi k + \operatorname{Arg} \xi_0 - \delta \\ &\ge 2\pi k - \pi - \frac{\pi}{4} \ge 6k - 5 \ge k \end{aligned}$$

and

$$|\zeta'_k| = \frac{|g_0(\rho'_k)|}{|\xi_0||G(\rho'_k)|} \le \frac{\gamma_0}{|\xi_0| M_0 |\rho'_k|} \le \frac{\gamma}{k}. \tag{7.18}$$

Since

$$-\mathrm{i}\operatorname{Log}[1 + z] = zH(z) \quad \text{for } |z| < 1,$$

with H analytic on the disk $|z| < 1$, from (7.17) and (7.18) we obtain the estimate

$$|\rho'_k - \mu'_k| \le \frac{\gamma}{k}, \qquad k = k_0, k_0 + 1, \ldots, \tag{7.19}$$

for an appropriate constant $\gamma > 0$. A similar argument shows that

$$|\rho_k'' - \mu_k''| \leq \frac{\gamma}{k}, \qquad k = k_0, k_0 + 1, \ldots . \tag{7.20}$$

The estimates (7.19) and (7.20) are the desired asymptotic formulas for Case 1.

We summarize these results for the eigenvalues as a theorem.

Theorem 7.2. *Let the differential operator L belong to Case 1, where the integers p and q satisfy the conditions $-\infty < p = q \leq p_0$ and where ξ_0 and η_0 are the roots of the quadratic polynomial $Q(z) = a_p z^2 + b_p z + c_p$ with $\xi_0 \neq \eta_0$ (so $|\eta_0| = 1/|\xi_0|$ and $\arg \eta_0 = -\arg \xi_0 - 2\pi p/n + \pi$). Then the elements of the spectrum $\sigma(L)$ can be listed as two distinct sequences*

$$\lambda_k' = (\rho_k')^n, \quad k = k_0, k_0 + 1, \ldots, \qquad \lambda_k'' = (\rho_k'')^n, \quad k = k_0, k_0 + 1, \ldots,$$

plus a finite number of additional points, where

$$\begin{aligned} \rho_k' &= (2\pi k + \operatorname{Arg} \xi_0) - \mathrm{i} \ln |\xi_0| + \epsilon_k', \quad k = k_0, k_0 + 1, \ldots, \\ \rho_k'' &= (2\pi k + \operatorname{Arg} \eta_0) + \mathrm{i} \ln |\xi_0| + \epsilon_k'', \quad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

with $|\epsilon_k'| \leq \gamma/k$ and $|\epsilon_k''| \leq \gamma/k$ for $k = k_0, k_0 + 1, \ldots$. Moreover, the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda_k') = m(\lambda_k') = 1, \quad k = k_0, k_0 + 1, \ldots,$$

$$\nu(\lambda_k'') = m(\lambda_k'') = 1, \quad k = k_0, k_0 + 1, \ldots .$$

7.2 Case 2. $p = q$, $\xi_0 = \eta_0$

Assume that $p = q$ and $\xi_0 = \eta_0$. In this case the polynomial

$$Q(z) = a_p z^2 + b_p z + c_p$$

has the double root $\xi_0 = \eta_0$, and multiple eigenvalues are possible. If $\rho = a + \mathrm{i}b \in G_0$ is a zero of Δ_0 with $a \geq -\pi$, $b \geq -d$, and $|\rho| \geq y_0$, then by Theorem 7.1 and (7.15) ρ must lie in one of the circles $\Gamma_k = \Gamma_k' = \Gamma_k''$ for some integer k. Let us consider the circles Γ_k, $k \geq k_0$, which lie in the interior of T_0 and in the interior of the horizontal strip Γ and to the right of the rectangle $R_{k_0}' = [\omega - 2\pi, y_0] \times [-d, d]$. From (7.13) we have $|g_0(\rho)| < |f_0(\rho)|$ for all points ρ on Γ_k for $k \geq k_0$, and hence, by Rouché's Theorem Δ_0 and $f_0 + g_0$ have precisely the same number of zeros as f_0 inside Γ_k for all $k \geq k_0$. But f_0 has only the single zero μ_k of order 2 inside Γ_k, implying that Δ_0 has two zeros ρ_k' and ρ_k'' inside Γ_k for $k \geq k_0$, where either $\rho_k' \neq \rho_k''$ with ρ_k' and ρ_k'' both being zeros of order 1 or $\rho_k' = \rho_k''$ with ρ_k' being a zero of order 2.

Setting

$$\begin{aligned} \lambda_k' &:= (\rho_k')^n, \qquad k = k_0, k_0 + 1, \ldots, \\ \lambda_k'' &:= (\rho_k'')^n, \qquad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

it follows that the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$, are eigenvalues of L. In addition, if $\rho'_k \neq \rho''_k$, then $\lambda'_k \neq \lambda''_k$ and by our earlier work the algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \qquad \nu(\lambda''_k) = m(\lambda''_k) = 1; \tag{7.21}$$

if $\rho'_k = \rho''_k$, then $\lambda'_k = \lambda''_k$, and again by our previous work the algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = 2, \qquad m(\lambda'_k) = 1 \quad \text{or} \quad m(\lambda'_k) = 2. \tag{7.22}$$

Let λ_0 be any eigenvalue of L distinct from the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$. Choose $\rho_0 \in S_0 \cup S_1$ such that $\lambda_0 = (\rho_0)^n$. Then either ρ_0 lies in the disk $|\rho| < y_0$, or the modulus satisfies the condition $|\rho_0| \geq y_0$ with $|\operatorname{Im} \rho_0| < d$ by Theorem 7.1. In the latter case (7.15) implies that ρ_0 belongs to the rectangle $[0, y_0] \times [-d, d]$. In either case only a finite number of such ρ_0 are possible because the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. Thus, we conclude that the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$, account for all but a finite number of the eigenvalues of L.

Next, let us derive asymptotic formulas for the zeros ρ'_k, ρ''_k of Δ_0. Fix any index $k \geq k_0$. We know that $f_0(\rho'_k) + g_0(\rho'_k) = 0$, so

$$\left[\mathrm{e}^{\mathrm{i}\rho'_k} - \xi_0\right]^2 = -\frac{1}{a_p}\, g_0(\rho'_k).$$

By constructing an appropriate analytic branch of the square root $\sqrt{}$ (depending on k), the last equation can be rewritten as

$$\mathrm{e}^{\mathrm{i}\rho'_k} = \xi_0 + \sqrt{-\frac{1}{a_p}\, g_0(\rho'_k)}\,,$$

and upon dividing by $\mathrm{e}^{\mathrm{i}\mu_k} = \xi_0$, it becomes

$$\mathrm{e}^{\mathrm{i}(\rho'_k - \mu_k)} = 1 + \underbrace{\frac{1}{\xi_0}\sqrt{-\frac{1}{a_p}\, g_0(\rho'_k)}}_{\zeta'_k}\,.$$

Since $|\operatorname{Re}(\rho'_k - \mu_k)| \leq |\rho'_k - \mu_k| < \delta \leq \pi/4$, we get

$$\rho'_k - \mu_k = -\mathrm{i}\operatorname{Log}\left[1 + \zeta'_k\right].$$

Now

$$|\rho'_k| \geq |\mu_k| - |\rho'_k - \mu_k| \geq 2\pi k + \operatorname{Arg}\xi_0 - \frac{\pi}{4} \geq 6k - 5 \geq k$$

and

$$|\zeta_k'| = \frac{1}{|\xi_0|}\sqrt{\frac{1}{|a_p|}\,|g_0(\rho_k')|} \le \frac{1}{|\xi_0|}\sqrt{\frac{\gamma_0}{|a_p||\rho_k'|}} \le \frac{\gamma}{\sqrt{k}}\,.$$

As in Case 1 this leads to the estimate

$$|\rho_k' - \mu_k| \le \frac{\gamma}{\sqrt{k}}\,, \qquad k = k_0, k_0+1, \ldots, \tag{7.23}$$

for an appropriate constant $\gamma > 0$. The same argument shows that

$$|\rho_k'' - \mu_k| \le \frac{\gamma}{\sqrt{k}}\,, \qquad k = k_0, k_0+1, \ldots. \tag{7.24}$$

These last two results are our asymptotic formulas for the zeros in Case 2.

We now summarize the above results as a theorem.

Theorem 7.3. *Let the differential operator L belong to Case 2, where the integers p and q satisfy* $-\infty < p = q \le p_0$ *and where* $\xi_0 = \eta_0$ *is the double root of the quadratic polynomial* $Q(z) = a_p z^2 + b_p z + c_p$ *(so* $\xi_0 = \eta_0 = \pm\mathrm{i}/\sqrt{\omega_p}$*). Then the elements of the spectrum* $\sigma(L)$ *can be listed as two sequences*

$$\lambda_k' = (\rho_k')^n, \quad k = k_0, k_0+1, \ldots, \qquad \lambda_k'' = (\rho_k'')^n, \quad k = k_0, k_0+1, \ldots,$$

plus a finite number of additional points, where

$$\rho_k' = 2\pi k + \operatorname{Arg}\xi_0 + \epsilon_k', \quad k = k_0, k_0+1, \ldots,$$
$$\rho_k'' = 2\pi k + \operatorname{Arg}\xi_0 + \epsilon_k'', \quad k = k_0, k_0+1, \ldots,$$

with $|\epsilon_k'| \le \gamma/\sqrt{k}$ *and* $|\epsilon_k''| \le \gamma/\sqrt{k}$ *for* $k = k_0, k_0+1, \ldots$. *For each* $k \ge k_0$ *if* $\rho_k' \ne \rho_k''$*, then* $\lambda_k' \ne \lambda_k''$ *and the algebraic multiplicities and ascents are*

$$\nu(\lambda_k') = m(\lambda_k') = 1 \quad \text{and} \quad \nu(\lambda_k'') = m(\lambda_k'') = 1,$$

while if $\rho_k' = \rho_k''$*, then* $\lambda_k' = \lambda_k''$ *and the algebraic multiplicities and ascents are*

$$\nu(\lambda_k') = 2 \quad \text{and} \quad m(\lambda_k') = 1 \ \text{or} \ m(\lambda_k') = 2.$$

Question. There are examples of differential operators L belonging to Case 2 that are regular, where the integers p and q satisfy the conditions $p = q = p_0$. For example, consider a second order differential operator L determined by periodic boundary conditions. Are there examples of differential operators L belonging to Case 2 that are simply irregular, i.e., where the integers p and q satisfy the conditions $p = q < p_0$?

7.3 Case 3. $p < q$

Assume that $p < q$. This case becomes Case 3, the so-called *logarithmic case.* For this case the differential operator L is always simply irregular. For the functions in (7.1) we know that the coefficients satisfy $a_p \neq 0$, $b_q \neq 0$, and $c_p \neq 0$. Set $n_0 := q - p > 0$. By (7.1) and (7.2) we can write the characteristic determinant Δ_0 in the form

$$\Delta_0(\rho) = \rho^p \big\{ a_p \mathrm{e}^{2\mathrm{i}\rho}[1 + \phi_{02}(\rho)] + b_q \rho^{n_0} \mathrm{e}^{\mathrm{i}\rho}[1 + \phi_{01}(\rho)] + c_p[1 + \phi_{00}(\rho)] \big\} \quad (7.25)$$

for $\rho \in G_0$, where

$$\phi_{02}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a_\kappa}{a_p \rho^{p-\kappa}} + \frac{1}{a_p \rho^p} \Phi_{02}(\rho),$$

$$\phi_{01}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{q-1} \frac{b_\kappa}{b_q \rho^{q-\kappa}} + \frac{1}{b_q \rho^q} \Phi_{01}(\rho),$$

$$\phi_{00}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{c_\kappa}{c_p \rho^{p-\kappa}} + \frac{1}{c_p \rho^p} \Phi_{00}(\rho)$$

for $\rho \in G_0$. The functions ϕ_{0i}, $i = 0, 1, 2$, are analytic on the open set G_0, and recalling that $m - p_0 + p \geq 1$ and $m - p_0 + q \geq 1$, by (7.3) we obtain the growth rates

$$|\phi_{02}(\rho)| \leq \frac{\gamma_0}{|\rho|}, \qquad |\phi_{01}(\rho)| \leq \frac{\gamma_0}{|\rho|}, \qquad |\phi_{00}(\rho)| \leq \frac{\gamma_0}{|\rho|} \quad (7.26)$$

for $\rho \in G_0$. Choose a constant $r_1 > R_1 \geq R_0$ such that

$$1/2 \leq |1 + \phi_{02}(\rho)| \leq 2, \quad 1/2 \leq |1 + \phi_{01}(\rho)| \leq 2, \quad 1/2 \leq |1 + \phi_{00}(\rho)| \leq 2$$

for $\rho \in G_0$ with $|\rho| \geq r_1$.

Similarly, by (7.1) and (7.4) the characteristic determinant Δ_1 can be expressed as

$$\Delta_1(\rho) = \rho^p \big\{ a_p[1+\phi_{12}(\rho)] + b_q \rho^{n_0} \mathrm{e}^{-\mathrm{i}\rho}[1+\phi_{11}(\rho)] + c_p \mathrm{e}^{-2\mathrm{i}\rho}[1+\phi_{10}(\rho)] \big\} \quad (7.27)$$

for $\rho \in G_1$, where

$$\phi_{12}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a_\kappa}{a_p \rho^{p-\kappa}} + \frac{1}{a_p \rho^p} \Phi_{12}(\rho),$$

$$\phi_{11}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{q-1} \frac{b_\kappa}{b_q \rho^{q-\kappa}} + \frac{1}{b_q \rho^q} \Phi_{11}(\rho),$$

$$\phi_{10}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{c_\kappa}{c_p \rho^{p-\kappa}} + \frac{1}{c_p \rho^p} \Phi_{10}(\rho)$$

for $\rho \in G_1$. The functions ϕ_{1i}, $i = 0, 1, 2$, are analytic on the open set G_1, and by (7.5)

$$|\phi_{12}(\rho)| \leq \frac{\gamma_0}{|\rho|}, \qquad |\phi_{11}(\rho)| \leq \frac{\gamma_0}{|\rho|}, \qquad |\phi_{10}(\rho)| \leq \frac{\gamma_0}{|\rho|} \tag{7.28}$$

for $\rho \in G_1$. Without loss of generality we can assume that the constant r_1 chosen above also produces the inequalities

$$1/2 \leq |1 + \phi_{12}(\rho)| \leq 2, \quad 1/2 \leq |1 + \phi_{11}(\rho)| \leq 2, \quad 1/2 \leq |1 + \phi_{10}(\rho)| \leq 2$$

for $\rho \in G_1$ with $|\rho| \geq r_1$.

Set $\mu_0 := -b_q/c_p$, a nonzero complex constant, and let ω be the positive real number defined by the equation $1/\omega := 1/|\mu_0|^{1/n_0} + n_0$. Choose real numbers α and β with $0 < \alpha < [1/(2|\mu_0|)]^{1/n_0}$, $\beta > [2/|\mu_0|]^{1/n_0}$, and

$$2|\mu_0|(2\alpha)^{n_0} \leq \frac{1}{8}, \qquad \frac{4}{|\mu_0|\beta^{n_0}} \leq \frac{1}{4}.$$

Clearly $1/\beta < |\mu_0|^{1/n_0} < 1/\alpha$. We will first study the characteristic determinant $\varDelta_0$ on the sector S_0, which lies in Quadrant I. Note that if ρ is any point in S_0 with $|\rho| > R_0$, then ρ belongs to the open set G_0, and hence, we will be working in a region where $\varDelta_0$ is analytic. In terms of the constants α and β, form the region

$$\varOmega_0 := \bigl\{\rho = a + \mathrm{i}b \in S_0 \bigm| \alpha \mathrm{e}^{b/n_0} \leq a \leq \beta \mathrm{e}^{b/n_0}\bigr\}$$

and the two complementary regions

$$\varOmega_{0\infty} := \bigl\{\rho = a + \mathrm{i}b \in S_0 \bigm| a \leq \alpha \mathrm{e}^{b/n_0}\bigr\},$$

$$\varOmega_{0\square} := \bigl\{\rho = a + \mathrm{i}b \in S_0 \bigm| \beta \mathrm{e}^{b/n_0} \leq a\bigr\}.$$

Clearly these three regions lie in Quadrant I. An equivalent definition of $\varOmega_0$ is

$$\varOmega_0 = \bigl\{\rho = a + \mathrm{i}b \in S_0 \bigm| a \geq \alpha \text{ and } n_0 \ln[a/\beta] \leq b \leq n_0 \ln[a/\alpha]\bigr\},$$

so in the sequel $\varOmega_0$ is referred to as a *logarithmic strip*. Also, observe that

$$n_0 \ln[a/\beta] < n_0 \ln[|\mu_0|^{1/n_0} a] < n_0 \ln[a/\alpha]$$

for $a \geq \beta$, so the logarithmic curve $b = n_0 \ln[|\mu_0|^{1/n_0} a]$ runs down the 'middle' of $\varOmega_0$.

Let us begin by calculating the growth rate of $\varDelta_0$ on the region $\varOmega_{0\infty}$. For any point $\rho = a + \mathrm{i}b \in \varOmega_{0\infty}$ we have

$$\begin{aligned} |\rho| \leq a + b &\leq \alpha \mathrm{e}^{b/n_0} + n_0 \mathrm{e}^{b/n_0} \\ &\leq [1/|\mu_0|^{1/n_0} + n_0]\mathrm{e}^{b/n_0} = \frac{1}{\omega}\,\mathrm{e}^{b/n_0}, \end{aligned}$$

and hence,

$$n_0 \ln[\omega|\rho|] \leq b \quad \text{for all } \rho = a + \mathrm{i}b \in \varOmega_{0\infty}. \tag{7.29}$$

Choose a constant $x_0 > 0$ such that $x \leq \alpha e^{x/n_0}$ and $e^{-2x} \leq 1/16$ for all $x \in \mathbb{R}$ with $x \geq x_0$, and then choose a second constant r_2 with $r_2 \geq r_1$ and $r_2 \geq \beta$ and with $x_0 \leq n_0 \ln[\omega|\rho|]$ for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_2$. If $\rho = a + ib \in \Omega_{0\infty}$ with $|\rho| \geq r_2$, then by (7.29) we have $b \geq x_0$, $b \leq \alpha e^{b/n_0}$ and $e^{-2b} \leq 1/16$, and

$$|\rho| \leq a + b \leq \alpha e^{b/n_0} + \alpha e^{b/n_0} = 2\alpha e^{b/n_0}. \tag{7.30}$$

Take any point $\rho = a + ib \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. Clearly $b \geq x_0 > 0$. Combining (7.30) with (7.25), we have

$$\Delta_0(\rho) = c_p\rho^p\Big\{[1+\phi_{00}(\rho)] - \mu_0\rho^{n_0}[1+\phi_{01}(\rho)]e^{i\rho} + \frac{a_p}{c_p}[1+\phi_{02}(\rho)]e^{2i\rho}\Big\}$$

and (recall $|a_p| = |c_p|$ by (3.30))

$$\begin{aligned}|\Delta_0(\rho)| &\geq |c_p||\rho|^p\left\{\frac{1}{2} - 2|\mu_0||\rho|^{n_0}e^{-b} - 2e^{-2b}\right\}\\ &\geq |c_p||\rho|^p\left\{\frac{1}{2} - 2|\mu_0|\cdot(2\alpha)^{n_0}e^{b}\cdot e^{-b} - 2\cdot\frac{1}{16}\right\} \geq \frac{1}{4}|c_p||\rho|^p,\end{aligned}$$

or

$$|\Delta_0(\rho)| \geq \frac{1}{4}|a_p||\rho|^p > 0 \tag{7.31}$$

for all $\rho = a + ib \in \Omega_{0\infty}$ with $|\rho| \geq r_2$.

To determine the growth rate of Δ_0 on the region $\Omega_{0\square}$, we first use (7.25) to express Δ_0 in the form

$$\begin{aligned}\Delta_0(\rho) = b_q\rho^q e^{i\rho}\Big\{&[1+\phi_{01}(\rho)] + \frac{a_p}{b_q\rho^{n_0}}[1+\phi_{02}(\rho)]e^{i\rho}\\ &+ \frac{c_p}{b_q\rho^{n_0}}[1+\phi_{00}(\rho)]e^{-i\rho}\Big\}\end{aligned}$$

for $\rho \in G_0$. Then for any point $\rho = a + ib \in \Omega_{0\square}$ with $|\rho| \geq r_2$ we have

$$|\rho| \geq a \geq \beta e^{b/n_0}, \tag{7.32}$$

and hence,

$$\begin{aligned}|\Delta_0(\rho)| &\geq |b_q||\rho|^q e^{-b}\Big\{\frac{1}{2} - \frac{2}{|\mu_0||\rho|^{n_0}}\left[e^{-b} + e^{b}\right]\Big\}\\ &\geq |b_q||\rho|^q e^{-b}\Big\{\frac{1}{2} - \frac{4e^{b}}{|\mu_0|\cdot\beta^{n_0}e^{b}}\Big\} \geq \frac{1}{4}|b_q||\rho|^q e^{-b}\\ &= \frac{1}{4}|b_q||\rho|^p\cdot|\rho|^{n_0}e^{-b} \geq \frac{\beta^{n_0}}{4}|b_q||\rho|^p \geq \frac{4}{|\mu_0|}|b_q||\rho|^p > 0,\end{aligned}$$

or

$$|\Delta_0(\rho)| \geq \frac{1}{4}|b_q||\rho|^q e^{-b} \geq \frac{4}{|\mu_0|}|b_q||\rho|^p > 0 \tag{7.33}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{0\square}$ with $|\rho| \geq r_2$.

Next, we rework the above material for the characteristic determinant Δ_1 on the sector S_1. For $\rho \in S_1$ with $|\rho| > R_0$, we have $\rho \in G_1$, and we are working in a region of analyticity for Δ_1. Now let us form the region

$$\Omega_1 := \left\{\rho = a + \mathrm{i}b \in S_1 \;\middle|\; \alpha\mathrm{e}^{-b/n_0} \leq a \leq \beta\mathrm{e}^{-b/n_0}\right\}$$

and the two complementary regions

$$\Omega_{1\infty} := \left\{\rho = a + \mathrm{i}b \in S_1 \;\middle|\; a \leq \alpha\mathrm{e}^{-b/n_0}\right\},$$

$$\Omega_{1\square} := \left\{\rho = a + \mathrm{i}b \in S_1 \;\middle|\; \beta\mathrm{e}^{-b/n_0} \leq a\right\}.$$

These three regions lie in Quadrant IV, and Ω_1 can be expressed as

$$\Omega_1 = \left\{\rho = a + \mathrm{i}b \in S_1 \;\middle|\; a \geq \alpha \text{ and } -n_0 \ln[a/\alpha] \leq b \leq -n_0 \ln[a/\beta]\right\}.$$

The logarithmic curve $b = -n_0 \ln[|\mu_0|^{1/n_0} a]$ runs down the 'middle' of Ω_1.

Let us first calculate the growth rate of Δ_1 on the region $\Omega_{1\infty}$. For any point $\rho = a + \mathrm{i}b$ in $\Omega_{1\infty}$ we have

$$\begin{aligned}|\rho| &\leq a + |b| \leq \alpha\mathrm{e}^{-b/n_0} + n_0\mathrm{e}^{|b|/n_0} \\ &\leq [1/|\mu_0|^{1/n_0} + n_0]\mathrm{e}^{|b|/n_0} = \frac{1}{\omega}\,\mathrm{e}^{|b|/n_0},\end{aligned}$$

and hence,

$$n_0 \ln[\omega|\rho|] \leq |b| \quad \text{for all } \rho = a + \mathrm{i}b \in \Omega_{1\infty}. \tag{7.34}$$

Note that if $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2$, then by (7.34) we have $|b| \geq x_0$, $|b| \leq \alpha\mathrm{e}^{|b|/n_0}$ and $\mathrm{e}^{-2|b|} \leq 1/16$, and

$$|\rho| \leq a + |b| \leq \alpha\mathrm{e}^{-b/n_0} + \alpha\mathrm{e}^{|b|/n_0} = 2\alpha\mathrm{e}^{|b|/n_0}. \tag{7.35}$$

Now take any point $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2$. Clearly $|b| \geq x_0 > 0$. Combining (7.35) with (7.27), we have

$$\Delta_1(\rho) = a_p\rho^p\Big\{[1 + \phi_{12}(\rho)] + \frac{b_q}{a_p}\rho^{n_0}[1 + \phi_{11}(\rho)]\mathrm{e}^{-\mathrm{i}\rho} + \frac{c_p}{a_p}[1 + \phi_{10}(\rho)]\mathrm{e}^{-2\mathrm{i}\rho}\Big\}$$

and

$$\begin{aligned}|\Delta_1(\rho)| &\geq |a_p||\rho|^p\left\{\frac{1}{2} - 2|\mu_0||\rho|^{n_0}\mathrm{e}^{b} - 2\mathrm{e}^{2b}\right\} \\ &\geq |a_p||\rho|^p\left\{\frac{1}{2} - 2|\mu_0|\cdot(2\alpha)^{n_0}\mathrm{e}^{-b}\cdot\mathrm{e}^{b} - 2\cdot\frac{1}{16}\right\} \geq \frac{1}{4}\,|a_p||\rho|^p,\end{aligned}$$

or

$$|\Delta_1(\rho)| \geq \frac{1}{4}\,|a_p||\rho|^p > 0 \tag{7.36}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ with $|\rho| \geq r_2$. Compare this growth rate to (7.31).

To determine the growth rate of $\varDelta_1$ on the region $\varOmega_{1\Box}$, we use (7.27) to express $\varDelta_1$ in the form

$$\varDelta_1(\rho) = b_q \rho^q \mathrm{e}^{-\mathrm{i}\rho} \left\{ [1 + \phi_{11}(\rho)] + \frac{a_p}{b_q \rho^{n_0}} [1 + \phi_{12}(\rho)] \mathrm{e}^{\mathrm{i}\rho} + \frac{c_p}{b_q \rho^{n_0}} [1 + \phi_{10}(\rho)] \mathrm{e}^{-\mathrm{i}\rho} \right\}$$

for $\rho \in G_1$. Then for any point $\rho = a + \mathrm{i}b \in \varOmega_{1\Box}$ with $|\rho| \geq r_2$ we have

$$|\rho| \geq a \geq \beta \mathrm{e}^{-b/n_0} = \beta \mathrm{e}^{|b|/n_0}, \tag{7.37}$$

and hence,

$$\begin{aligned} |\varDelta_1(\rho)| &\geq |b_q||\rho|^q \mathrm{e}^b \left\{ \frac{1}{2} - \frac{2}{|\mu_0||\rho|^{n_0}} \left[\mathrm{e}^{-b} + \mathrm{e}^b \right] \right\} \\ &\geq |b_q||\rho|^q \mathrm{e}^b \left\{ \frac{1}{2} - \frac{4\mathrm{e}^{|b|}}{|\mu_0| \cdot \beta^{n_0} \mathrm{e}^{|b|}} \right\} \geq \frac{1}{4} |b_q||\rho|^q \mathrm{e}^b \\ &\geq \frac{1}{4} |b_q||\rho|^p \cdot |\rho|^{n_0} \mathrm{e}^{-|b|} \geq \frac{\beta^{n_0}}{4} |b_q||\rho|^p \geq \frac{4}{|\mu_0|} |b_q||\rho|^p > 0, \end{aligned}$$

or

$$|\varDelta_1(\rho)| \geq \frac{1}{4} |b_q||\rho|^q \mathrm{e}^b \geq \frac{4}{|\mu_0|} |b_q||\rho|^p > 0 \tag{7.38}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{1\Box}$ with $|\rho| \geq r_2$. Cf. (7.33).

As an application of these growth rates on the regions $\varOmega_{0\infty}$, $\varOmega_{0\Box}$ and $\varOmega_{1\infty}$, $\varOmega_{1\Box}$, we obtain the following *apriori estimates* for the eigenvalues of L.

Theorem 7.4. *Assume $p < q$. For any point $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and $|\rho| \geq r_2$:*

(a) *If $\rho \in \varOmega_{0\infty}$ or $\rho \in \varOmega_{0\Box}$, then $\varDelta_0(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(b) *If λ is an eigenvalue of L, then $\varDelta_0(\rho) = 0$ and ρ lies in the interior of the logarithmic strip $\varOmega_0$.*

Also, for any point $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$ and $|\rho| \geq r_2$:

(c) *If $\rho \in \varOmega_{1\infty}$ or $\rho \in \varOmega_{1\Box}$, then $\varDelta_1(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(d) *If λ is an eigenvalue of L, then $\varDelta_1(\rho) = 0$ and ρ lies in the interior of the logarithmic strip $\varOmega_1$.*

The theorem implies that the resolvent set $\rho(L)$ is nonempty, and hence, by our earlier remarks the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$.

Next, we calculate the eigenvalues of L that correspond to the zeros of $\varDelta_0$ in the logarithmic strip $\varOmega_0$. Following this we compute the eigenvalues corresponding to the zeros of $\varDelta_1$ in the logarithmic strip $\varOmega_1$. Set $\xi := 1 + n_0/\alpha$ and $\eta := \beta + n_0$. Then for $\rho = a + \mathrm{i}b \in \varOmega_0$ we have

$$|\rho| \leq a + b \leq a + n_0 \ln[a/\alpha] \leq a + (n_0/\alpha)a = \xi a \tag{7.39}$$

and

$$|\rho| \le a + b \le \beta \mathrm{e}^{b/n_0} + b \le \beta \mathrm{e}^{b/n_0} + n_0 \mathrm{e}^{b/n_0} = \eta \mathrm{e}^{b/n_0}. \tag{7.40}$$

In studying the characteristic determinant on the logarithmic strip Ω_0, we use (7.25) to write Δ_0 in the form

$$\Delta_0(\rho) = c_p \rho^p \mathrm{e}^{\mathrm{i}\rho} \big\{ \mathrm{e}^{-\mathrm{i}\rho} - \mu_0 \rho^{n_0} [1 + h_0(\rho)] \big\} \tag{7.41}$$

for $\rho \in G_0$, where h_0 is the function given by

$$h_0(\rho) := -\frac{\mathrm{e}^{-\mathrm{i}\rho}}{\mu_0 \rho^{n_0}} \, \phi_{00}(\rho) + \phi_{01}(\rho) + \frac{a_p}{b_q \rho^{n_0}} \, [1 + \phi_{02}(\rho)] \mathrm{e}^{\mathrm{i}\rho}$$

for $\rho \in G_0$. The function h_0 is analytic on the open set G_0. Observe that for any point $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \ge r_2$, the inequalities defining Ω_0 yield

$$\left| \frac{\mathrm{e}^{-\mathrm{i}\rho}}{\mu_0 \rho^{n_0}} \right| = \frac{\mathrm{e}^b}{|\mu_0||\rho|^{n_0}} \le \frac{1}{|\mu_0||\rho|^{n_0}} \cdot \frac{a^{n_0}}{\alpha^{n_0}} \le \frac{1}{|\mu_0| \alpha^{n_0}},$$

and clearly $|1 + \phi_{02}(\rho)||\mathrm{e}^{\mathrm{i}\rho}| = |1 + \phi_{02}(\rho)| \mathrm{e}^{-b} \le 2$ because $b \ge 0$ for $\rho \in S_0$, and hence, by the inequalities in (7.26)

$$|h_0(\rho)| \le \frac{\gamma_1}{|\rho|} \tag{7.42}$$

for $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \ge r_2$.

Set

$$\Omega' := \big\{ \rho = a + \mathrm{i}b \in S_0 \;\big|\; a \ge r_2, \; n_0 \ln[a/\beta] \le b \le n_0 \ln[a/\alpha] \big\},$$

which is a subset of both G_0 and Ω_0 (recall that $r_2 \ge \beta > \alpha$). Fix a real number δ with $0 < \delta \le \pi/4$ and $0 < \delta < (\ln 2)/(1 + n_0)$, and then for $k = 1, 2, \ldots$ define

$$\begin{cases} \alpha'_k := 2\pi k - \operatorname{Arg} \mu_0, & \beta'_k := n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k], \\ \mu'_k := \alpha'_k + \mathrm{i}\beta'_k, \end{cases}$$

and introduce the circles

$$\Gamma'_k := \{ \rho \in \mathbb{C} \mid |\rho - \mu'_k| = \delta \}.$$

Choose an integer $k_1 \ge 2$ such that $y'_1 := \alpha'_{k_1} - \pi \ge r_2$. Note that $\alpha'_k - \pi \ge r_2 \ge \beta$ and $\alpha'_k \ge 3\pi$ for $k = k_1, k_1 + 1, \ldots$. Also, introduce the *logarithmic rectangles*

$$R'_k := \big\{ \rho = a + \mathrm{i}b \in \mathbb{C} \mid \alpha'_k - \pi \le a \le \alpha'_k + \pi, \; n_0 \ln[a/\beta] \le b \le n_0 \ln[a/\alpha] \big\}$$

for $k = k_1, k_1 + 1, \ldots$. Without loss of generality we can assume that k_1 is sufficiently large to guarantee that each R'_k is contained in S_0, and hence, for

$k = k_1, k_1 + 1, \dots$ the point μ'_k lies in the interior of R'_k with R'_k a subset of Ω'.

Fix any index $k \geq k_1$, and take any point $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|\rho - \mu'_k| \leq \delta$. We assert that ρ lies in the interior of R'_k. Indeed, we clearly have $|a - \alpha'_k| \leq \delta < \pi$ and $|b - n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k]| \leq \delta$, so

$$\begin{aligned}|b - n_0 \ln[|\mu_0|^{1/n_0} a]| &\leq |b - n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k]| \\ &\qquad + |n_0 \ln[|\mu_0|^{1/n_0} a] - n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k]| \\ &\leq \delta + n_0|a - \alpha'_k| \leq \delta(1 + n_0) < \ln 2.\end{aligned}$$

It follows that $n_0 \ln[(|\mu_0|/2)^{1/n_0} a] < b < n_0 \ln[(2|\mu_0|)^{1/n_0} a]$ and

$$n_0 \ln[a/\beta] < b < n_0 \ln[a/\alpha].$$

This establishes the assertion, and it is immediate that the circle Γ'_k lies in the interior of the logarithmic rectangle R'_k for $k = k_1, k_1 + 1, \dots$. To complete the setup of the geometry, for $k = k_1, k_1 + 1, \dots$ let Ω'_k be the *punctured logarithmic rectangle* formed from R'_k by removing all the points inside Γ'_k.

The next step is to establish the growth rate of Δ_0 on each of the regions Ω'_k. Note that

$$\mathrm{e}^{-\mathrm{i}\mu'_k} = \mu_0(\alpha'_k)^{n_0}$$

for $k = k_1, k_1 + 1, \dots$. Let f_k, $k = k_1, k_1 + 1, \dots$, and g_k, $k = k_1, k_1 + 1, \dots$, be the sequences of functions defined by

$$f_k(\rho) := \mathrm{e}^{-\mathrm{i}(\rho - \mu'_k)} - 1 \quad \text{for } \rho \in \mathbb{C},$$

$$g_k(\rho) := -h_0(\rho) - \sum_{j=1}^{n_0} \binom{n_0}{j} \Big[\frac{1}{\alpha'_k}(\rho - \mu'_k) + \frac{\mathrm{i}\beta'_k}{\alpha'_k}\Big]^j \big[1 + h_0(\rho)\big] \quad \text{for } \rho \in G_0.$$

The functions f_k are entire functions, while the functions g_k are analytic on the open set G_0. We can use (7.41) to write Δ_0 in its final form

$$\begin{aligned}\Delta_0(\rho) &= c_p \mu_0(\alpha'_k)^{n_0} \rho^p \mathrm{e}^{\mathrm{i}\rho} \Big\{\mathrm{e}^{-\mathrm{i}\rho} \cdot \mathrm{e}^{\mathrm{i}\mu'_k} - \Big[\frac{\rho}{\alpha'_k}\Big]^{n_0} \big[1 + h_0(\rho)\big]\Big\} \\ &= c_p \mu_0(\alpha'_k)^{n_0} \rho^p \mathrm{e}^{\mathrm{i}\rho} \Big\{\mathrm{e}^{-\mathrm{i}(\rho - \mu'_k)} - \Big[\frac{1}{\alpha'_k}(\rho - \mu'_k) + \frac{\mathrm{i}\beta'_k}{\alpha'_k} + 1\Big]^{n_0} \big[1 + h_0(\rho)\big]\Big\},\end{aligned}$$

or

$$\Delta_0(\rho) = c_p \mu_0(\alpha'_k)^{n_0} \rho^p \mathrm{e}^{\mathrm{i}\rho} [f_k(\rho) + g_k(\rho)] \tag{7.43}$$

for $\rho \in G_0$ and for $k = k_1, k_1 + 1, \dots$. Here we have a family of representations for Δ_0 depending on the integer k. We will use the kth representation to determine the growth rate of Δ_0 on the kth region Ω'_k.

In terms of the constants α, β, δ, choose $d_0 > 0$ such that

$$n_0 \ln[2/(|\mu_0|^{1/n_0}\alpha)] \leq d_0, \qquad n_0 \ln[2|\mu_0|^{1/n_0}\beta] \leq d_0,$$

and $\delta < d_0$, and then form the punctured rectangle

$$R_* := \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid -\pi \le a \le \pi,\ -d_0 \le b \le d_0,\ \text{and } |\rho| \ge \delta\}.$$

Set $m_0 := \min\{\,|\mathrm{e}^{-\mathrm{i}\rho} - 1| \mid \rho \in R_*\} > 0$.

Fix any index $k \ge k_1$ and any point $\rho = a + \mathrm{i}b \in \Omega'_k$. It follows that $-\pi \le a - \alpha'_k \le \pi$, $\pi/\alpha'_k \le 1/2$ because $\alpha'_k \ge 3\pi$,

$$\frac{a}{\alpha'_k} \le \frac{\alpha'_k + \pi}{\alpha'_k} \le 2, \qquad \frac{a}{\alpha'_k} \ge \frac{\alpha'_k - \pi}{\alpha'_k} \ge \frac{1}{2},$$

and

$$\begin{aligned} b - \beta'_k &\le n_0 \ln[a/\alpha] - n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k] \\ &= n_0 \ln[a/(|\mu_0|^{1/n_0}\alpha\alpha'_k)] \le n_0 \ln[2/(|\mu_0|^{1/n_0}\alpha)] \le d_0, \end{aligned}$$

$$\begin{aligned} b - \beta'_k &\ge n_0 \ln[a/\beta] - n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k] \\ &= n_0 \ln[a/(|\mu_0|^{1/n_0}\beta\alpha'_k)] \ge n_0 \ln[1/(2|\mu_0|^{1/n_0}\beta)] \ge -d_0. \end{aligned}$$

Thus, the point $\rho - \mu'_k$ belongs to the punctured rectangle R_* with $|\rho - \mu'_k| \le \pi + d_0$. We conclude that

$$|f_k(\rho)| \ge m_0 \tag{7.44}$$

for $k \ge k_1$ and for $\rho \in \Omega'_k$. Note that the constant m_0 is independent of the index k.

Clearly $\lim_{k\to\infty} 1/\alpha'_k = \lim_{k\to\infty} \beta'_k/\alpha'_k = 0$. In terms of the constant γ_1 that appears in the inequality (7.42), select an integer $k_0 \ge k_1$ such that $y'_0 := \alpha'_{k_0} - \pi \ge r_2$, such that

$$\frac{\gamma_1}{|\rho|} \le \frac{m_0}{4} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \ge y'_0,$$

and such that

$$\sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{\pi + d_0}{\alpha'_k} + \frac{\beta'_k}{\alpha'_k}\right]^j \left[1 + \frac{\gamma_1}{r_2}\right] \le \frac{m_0}{4} \quad \text{for all } k \ge k_0.$$

Then for $k \ge k_0$ and for $\rho = a + \mathrm{i}b \in \Omega'_k$, we have $|\rho| \ge a \ge \alpha'_k - \pi \ge y'_0 \ge r_2$, and hence, by (7.42) and the definition of the integer k_0:

$$|g_k(\rho)| \le \frac{m_0}{2} < m_0 \le |f_k(\rho)|. \tag{7.45}$$

Also, since $\alpha'_k \ge 3\pi$, we have $a \ge \alpha'_k - \pi \ge 2\pi$ or $a/2 \ge \pi$, and $\alpha'_k \ge a - \pi \ge a/2 \ge |\rho|/(2\xi)$ by (7.39). Therefore, from (7.43) and (7.45) we conclude that

$$|\Delta_0(\rho)| \ge |c_p||\mu_0|(\alpha'_k)^{n_0}|\rho|^p \mathrm{e}^{-b} \cdot \frac{m_0}{2} \ge \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\,|\rho|^q \mathrm{e}^{-b},$$

or

$$|\Delta_0(\rho)| \geq \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\,|\rho|^q\mathrm{e}^{-b} > 0 \tag{7.46}$$

for $k \geq k_0$ and for $\rho = a + \mathrm{i}b \in \Omega'_k$.

The estimate (7.45) is local in character in that it depends on k: it is valid only on the region Ω'_k. In contrast, the estimate (7.46) is global because the constant on the right is independent of k. If we introduce the *punctured logarithmic strip*

$$\Omega'_* := \bigcup_{k=k_0}^{\infty} \Omega'_k,$$

then we see that Ω'_* consists of all points $\rho = a + \mathrm{i}b \in \Omega_0$ which satisfy $a \geq y'_0$ and which do not lie inside any of the circles Γ'_k for $k \geq k_0$, and from (7.46) and the definition of Ω_0:

$$|\Delta_0(\rho)| \geq \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\,|\rho|^q\mathrm{e}^{-b} \geq \frac{m_0|a_p||\mu_0|\alpha^{n_0}}{2(2\xi)^{n_0}}\,|\rho|^p > 0 \tag{7.47}$$

for all $\rho = a + \mathrm{i}b \in \Omega'_*$.

With the basic estimates (7.45) and (7.47) in place, consider one of the circles Γ'_k for $k \geq k_0$. Since (7.45) is valid for each point ρ on Γ'_k, it follows by Rouché's Theorem that Δ_0 and $f_k + g_k$ have the same number of zeros as f_k inside Γ'_k. But μ'_k is the only zero of f_k inside Γ'_k, μ'_k being a zero of order 1. Consequently, Δ_0 has a unique zero ρ'_k inside Γ'_k with ρ'_k having order 1 for $k \geq k_0$. Setting

$$\lambda'_k := (\rho'_k)^n, \qquad k = k_0, k_0+1, \ldots,$$

the complex numbers λ'_k are eigenvalues of L, and by our earlier work the corresponding algebraic multiplicities and ascents are given by

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \qquad k = k_0, k_0+1, \ldots. \tag{7.48}$$

It is also easy to derive asymptotic formulas for the zeros ρ'_k of Δ_0. Set $\zeta'_k := -g_k(\rho'_k)$ for $k = k_0, k_0+1, \ldots$. Then $\mathrm{e}^{-\mathrm{i}(\rho'_k-\mu'_k)} = 1 + \zeta'_k$ and

$$\rho'_k - \mu'_k = \mathrm{i}\,\mathrm{Log}\,[1+\zeta'_k] \tag{7.49}$$

for $k = k_0, k_0+1, \ldots$, where

$$|\zeta'_k| \leq \frac{\gamma_1}{|\rho'_k|} + \sum_{j=1}^{n_0}\binom{n_0}{j}\left[\frac{\delta}{\alpha'_k} + \frac{\beta'_k}{\alpha'_k}\right]^j\left[1 + \frac{\gamma_1}{r_2}\right].$$

Now for each $k \geq k_0$, $\alpha'_k \geq 2\pi k - \pi \geq k$, $\beta'_k \leq n_0 \ln[|\mu_0|^{1/n_0}(2\pi k+\pi)] \leq \gamma_2 \ln k$, and $|\rho'_k| \geq |\mu'_k| - |\rho'_k - \mu'_k| \geq \alpha'_k - \delta \geq 6k - 5 \geq k$, which yields $|\zeta'_k| \leq \gamma_3 \ln k/k$ and

$$|\rho'_k - \mu'_k| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0+1, \ldots. \tag{7.50}$$

To compute the zeros of $\mathit{\Delta}_1$ in the logarithmic strip $\mathit{\Omega}_1$, we begin by using (7.27) to express $\mathit{\Delta}_1$ in the form

$$\mathit{\Delta}_1(\rho) = a_p \rho^p \mathrm{e}^{-\mathrm{i}\rho} \big\{ \mathrm{e}^{\mathrm{i}\rho} - \mu_1 \rho^{n_0} [1 + h_1(\rho)] \big\} \tag{7.51}$$

for $\rho \in G_1$, where $\mu_1 := -b_q/a_p = -\mu_0/\omega_p$, $|\mu_1| = |\mu_0|$, and h_1 is the analytic function given by

$$h_1(\rho) := -\frac{\mathrm{e}^{\mathrm{i}\rho}}{\mu_1 \rho^{n_0}}\, \phi_{12}(\rho) + \phi_{11}(\rho) + \frac{c_p}{b_q \rho^{n_0}}\, [1 + \phi_{10}(\rho)] \mathrm{e}^{-\mathrm{i}\rho}$$

for $\rho \in G_1$. For any point $\rho = a + \mathrm{i}b \in \mathit{\Omega}_1$ with $|\rho| \geq r_2$, the inequalities defining $\mathit{\Omega}_1$ give

$$\left| \frac{\mathrm{e}^{\mathrm{i}\rho}}{\mu_1 \rho^{n_0}} \right| = \frac{\mathrm{e}^{-b}}{|\mu_1||\rho|^{n_0}} \leq \frac{1}{|\mu_1||\rho|^{n_0}} \cdot \frac{a^{n_0}}{\alpha^{n_0}} \leq \frac{1}{|\mu_0|\alpha^{n_0}} \,,$$

and clearly $|1+\phi_{10}(\rho)||\mathrm{e}^{-\mathrm{i}\rho}| = |1+\phi_{10}(\rho)|\mathrm{e}^{b} \leq 2$, and hence, by the inequalities in (7.28)

$$|h_1(\rho)| \leq \frac{\gamma_1}{|\rho|} \tag{7.52}$$

for $\rho = a + \mathrm{i}b \in \mathit{\Omega}_1$ with $|\rho| \geq r_2$.

We concentrate the search for zeros in the region

$$\mathit{\Omega}'' := \big\{ \rho = a + \mathrm{i}b \in S_1 \;\big|\; a \geq r_2,\ -n_0 \ln[a/\alpha] \leq b \leq -n_0 \ln[a/\beta] \big\},$$

which is a subset of both G_1 and $\mathit{\Omega}_1$. Using the real number δ defined above, for $k = 1, 2, \ldots$ define

$$\begin{cases} \alpha_k'' := 2\pi k + \operatorname{Arg} \mu_1, & \beta_k'' := -n_0 \ln[|\mu_1|^{1/n_0} \alpha_k''], \\ \mu_k'' := \alpha_k'' + \mathrm{i}\beta_k'', \end{cases}$$

and introduce the circles

$$\mathit{\Gamma}_k'' := \{ \rho \in \mathbb{C} \mid |\rho - \mu_k''| = \delta \}.$$

Assume the positive integer k_1 also satisfies the condition $y_1'' := \alpha_{k_1}'' - \pi \geq r_2$. Clearly we have $\alpha_k'' - \pi \geq r_2 \geq \beta$ and $\alpha_k'' \geq 3\pi$ for $k = k_1, k_1 + 1, \ldots$. Let us introduce the *logarithmic rectangles*

$$R_k'' := \big\{ \rho = a + \mathrm{i}b \in \mathbb{C} \mid \alpha_k'' - \pi \leq a \leq \alpha_k'' + \pi,\ -n_0 \ln[a/\alpha] \leq b \leq -n_0 \ln[a/\beta] \big\}$$

for $k = k_1, k_1 + 1, \ldots$. Without loss of generality we can assume that k_1 is sufficiently large to guarantee that each R_k'' is contained in S_1, and hence, for $k = k_1, k_1 + 1, \ldots$ the point μ_k'' lies in the interior of R_k'' with R_k'' a subset of $\mathit{\Omega}''$. As above the circle $\mathit{\Gamma}_k''$ lies in the interior of the logarithmic rectangle R_k'' for $k = k_1, k_1 + 1, \ldots$. Finally, for $k = k_1, k_1 + 1, \ldots$ let $\mathit{\Omega}_k''$ be the *punctured logarithmic rectangle* formed from R_k'' by removing all the points inside $\mathit{\Gamma}_k''$.

We next determine the growth rate of $\varDelta_1$ on each of the regions $\varOmega''_k$. Observe that the points μ''_k satisfy the equation

$$\mathrm{e}^{\mathrm{i}\mu''_k} = \mu_1(\alpha''_k)^{n_0}$$

for $k = k_1, k_1+1, \ldots$. If F_k, $k = k_1, k_1+1, \ldots$, and G_k, $k = k_1, k_1+1, \ldots$, are the sequences of analytic functions defined by

$$F_k(\rho) = \mathrm{e}^{\mathrm{i}(\rho-\mu''_k)} - 1 \quad \text{for } \rho \in \mathbb{C},$$

$$G_k(\rho) = -h_1(\rho) - \sum_{j=1}^{n_0} \binom{n_0}{j} \Big[\frac{1}{\alpha''_k}(\rho - \mu''_k) + \frac{\mathrm{i}\beta''_k}{\alpha''_k} \Big]^j \big[1 + h_1(\rho)\big] \quad \text{for } \rho \in G_1,$$

then we can rewrite $\varDelta_1$ in the form

$$\begin{aligned} \varDelta_1(\rho) &= a_p\mu_1(\alpha''_k)^{n_0}\rho^p\mathrm{e}^{-\mathrm{i}\rho}\Big\{\mathrm{e}^{\mathrm{i}\rho}\cdot\mathrm{e}^{-\mathrm{i}\mu''_k} - \Big[\frac{\rho}{\alpha''_k}\Big]^{n_0}\big[1+h_1(\rho)\big]\Big\} \\ &= a_p\mu_1(\alpha''_k)^{n_0}\rho^p\mathrm{e}^{-\mathrm{i}\rho}\Big\{\mathrm{e}^{\mathrm{i}(\rho-\mu''_k)} \\ &\quad - \Big[\frac{1}{\alpha''_k}(\rho-\mu''_k) + \frac{\mathrm{i}\beta''_k}{\alpha''_k} + 1\Big]^{n_0}\big[1+h_1(\rho)\big]\Big\}, \end{aligned}$$

or

$$\varDelta_1(\rho) = a_p\mu_1(\alpha''_k)^{n_0}\rho^p\mathrm{e}^{-\mathrm{i}\rho}[F_k(\rho) + G_k(\rho)] \tag{7.53}$$

for $\rho \in G_1$ and for $k = k_1, k_1+1, \ldots$. In the last equation we actually have a family of representations for $\varDelta_1$ depending on the integer k.

Using the constant d_0 defined above, let us form the punctured rectangle

$$R_* = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid -\pi \le a \le \pi,\ -d_0 \le b \le d_0,\ \text{and } |\rho| \ge \delta\}.$$

Clearly $\min\{\,|\mathrm{e}^{\mathrm{i}\rho} - 1| \mid \rho \in R_*\} = m_0 > 0$. For any index $k \ge k_1$ and any point $\rho = a + \mathrm{i}b \in \varOmega''_k$, we again get $-\pi \le a - \alpha''_k \le \pi$, $1/2 \le a/\alpha''_k \le 2$, and $-d_0 \le b - \beta''_k \le d_0$, and hence, the point $\rho - \mu''_k$ belongs to the punctured rectangle R_* with $|\rho - \mu''_k| \le \pi + d_0$, and

$$|F_k(\rho)| \ge m_0. \tag{7.54}$$

Since $\lim_{k\to\infty} 1/\alpha''_k = \lim_{k\to\infty} \beta''_k/\alpha''_k = 0$, in terms of (7.52) it can be assumed that the previous integer k_0 also guarantees that $y''_0 := \alpha''_{k_0} - \pi \ge r_2$,

$$\frac{\gamma_1}{|\rho|} \le \frac{m_0}{4} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \ge y''_0,$$

and

$$\sum_{j=1}^{n_0} \binom{n_0}{j} \Big[\frac{\pi + d_0}{\alpha''_k} + \frac{|\beta''_k|}{\alpha''_k}\Big]^j \Big[1 + \frac{\gamma_1}{r_2}\Big] \le \frac{m_0}{4} \quad \text{for all } k \ge k_0.$$

Then for each $k \geq k_0$ and each $\rho = a + \mathrm{i}b \in \Omega_k''$, we have $|\rho| \geq a \geq \alpha_k'' - \pi \geq y_0'' \geq r_2$, and by (7.52) and the defining properties of k_0:

$$|G_k(\rho)| \leq \frac{m_0}{2} < m_0 \leq |F_k(\rho)|. \tag{7.55}$$

Also, $\alpha_k'' \geq a - \pi \geq a/2 \geq |\rho|/(2\xi)$, and from (7.53) and (7.55) it follows that

$$|\Delta_1(\rho)| \geq |a_p||\mu_1|(\alpha_k'')^{n_0}|\rho|^p \mathrm{e}^b \cdot \frac{m_0}{2} \geq \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\, |\rho|^q \mathrm{e}^b > 0 \tag{7.56}$$

for $k \geq k_0$ and for $\rho = a + \mathrm{i}b \in \Omega_k''$. Cf. the estimate (7.46).

The estimate (7.55) is local as it depends on k, while the estimate (7.56) is global being independent of k. Introducing the *punctured logarithmic strip*

$$\Omega_*'' := \bigcup_{k=k_0}^{\infty} \Omega_k'',$$

we see that Ω_*'' consists of all points $\rho = a + \mathrm{i}b \in \Omega_1$ which satisfy $a \geq y_0''$ and which do not lie inside any of the circles Γ_k'' for $k \geq k_0$, and from (7.56) and the definition of Ω_1:

$$|\Delta_1(\rho)| \geq \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\, |\rho|^q \mathrm{e}^b \geq \frac{m_0|a_p||\mu_0|\alpha^{n_0}}{2(2\xi)^{n_0}}\, |\rho|^p > 0 \tag{7.57}$$

for all $\rho = a + \mathrm{i}b \in \Omega_*''$. Cf. (7.47).

Consider one of the circles Γ_k'' for $k \geq k_0$. Inequality (7.55) is valid for each point ρ on the circle Γ_k'', so by Rouché's Theorem Δ_1 and $F_k + G_k$ have the same number of zeros as F_k inside Γ_k''. The point μ_k'' is the only zero of F_k inside Γ_k'' with μ_k'' being a zero of order 1. We conclude that Δ_1 has a unique zero ρ_k'' inside Γ_k'' of order 1 for $k \geq k_0$. Setting

$$\lambda_k'' := (\rho_k'')^n, \qquad k = k_0, k_0 + 1, \ldots,$$

the λ_k'' are all eigenvalues of L with algebraic multiplicities and ascents

$$\nu(\lambda_k'') = m(\lambda_k'') = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{7.58}$$

To derive asymptotic formulas, set $\zeta_k'' := -G_k(\rho_k'')$ for $k = k_0, k_0 + 1, \ldots$. It is immediate that $\mathrm{e}^{\mathrm{i}(\rho_k'' - \mu_k'')} = 1 + \zeta_k''$ and

$$\rho_k'' - \mu_k'' = -\mathrm{i}\,\mathrm{Log}\,[1 + \zeta_k''] \tag{7.59}$$

for $k = k_0, k_0 + 1, \ldots$, where

$$|\zeta_k''| \leq \frac{\gamma_1}{|\rho_k''|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{\delta}{\alpha_k''} + \frac{|\beta_k''|}{\alpha_k''}\right]^j \left[1 + \frac{\gamma_1}{r_2}\right].$$

For each $k \geq k_0$ we have $\alpha_k'' \geq 2\pi k - \pi \geq k$, $|\beta_k''| \leq n_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \pi)] \leq \gamma_2 \ln k$, and $|\rho_k''| \geq |\mu_k''| - |\rho_k'' - \mu_k''| \geq \alpha_k'' - \delta \geq 6k - 5 \geq k$, which yields the estimates $|\zeta_k''| \leq \gamma_3 \ln k/k$ and

$$|\rho_k'' - \mu_k''| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0 + 1, \ldots. \tag{7.60}$$

Finally, we assert that the λ_k', λ_k'', $k = k_0, k_0 + 1, \ldots$, account for all but a finite number of the eigenvalues of L. Indeed, suppose $\lambda_0 = (\rho_0)^n$ is any eigenvalue of L distinct from the λ_k', λ_k'', with $\rho_0 = a_0 + ib_0$ belonging to the sector $S_0 \cup S_1$. Only a finite number of such ρ_0 are possible in the disk $|\rho| < r_2$. Assume that $|\rho_0| \geq r_2$.

First, consider the case where $\rho_0 \in S_0$. By Theorem 7.4 we know that $\Delta_0(\rho_0) = 0$ and ρ_0 lies in the interior of the logarithmic strip Ω_0. Now ρ_0 does not lie in any of the circles Γ_k' for $k \geq k_0$ since λ_0 is distinct from the λ_k', and ρ_0 does not lie in the punctured logarithmic strip Ω_*' by virtue of (7.47). Thus, we must have $\rho_0 \in \Omega_0$ with $\alpha \leq a_0 < y_0'$. But this implies that $0 \leq b_0 \leq n_0 \ln[a_0/\alpha] < n_0 \ln[y_0'/\alpha]$, so these ρ_0 lie in a bounded region of the ρ plane, and again only a finite number of such ρ_0 are possible.

Second, consider the other possible case where $\rho_0 \in S_1$. By Theorem 7.4 we have $\Delta_1(\rho_0) = 0$ with ρ_0 lying in the interior of the logarithmic strip Ω_1. Following the same argument as in the previous case, only a finite number of such ρ_0 are possible. This establishes the assertion.

We summarize the above results for this logarithmic case in the following theorem.

Theorem 7.5. *Let the differential operator L belong to Case 3, a logarithmic case, where the integers p and q satisfy the conditions $-\infty < p < q \leq p_0$, and let $n_0 = q - p$, $\mu_0 = -b_q/c_p \neq 0$, and $\mu_1 = -b_q/a_p \neq 0$ (so $|\mu_1| = |\mu_0|$ and $\arg \mu_1 = \arg \mu_0 - 2\pi p/n + \pi$). Then the elements of the spectrum $\sigma(L)$ can be listed as two distinct sequences*

$$\lambda_k' = (\rho_k')^n, \quad k = k_0, k_0 + 1, \ldots, \qquad \lambda_k'' = (\rho_k'')^n, \quad k = k_0, k_0 + 1, \ldots,$$

plus a finite number of additional points, where

$$\rho_k' = (2\pi k - \operatorname{Arg} \mu_0) + in_0 \ln[|\mu_0|^{1/n_0}(2\pi k - \operatorname{Arg} \mu_0)] + \epsilon_k', \\ k = k_0, k_0 + 1, \ldots,$$

$$\rho_k'' = (2\pi k + \operatorname{Arg} \mu_1) - in_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \operatorname{Arg} \mu_1)] + \epsilon_k'', \\ k = k_0, k_0 + 1, \ldots,$$

with $|\epsilon_k'| \leq \gamma \ln k/k$ and $|\epsilon_k''| \leq \gamma \ln k/k$ for $k = k_0, k_0 + 1, \ldots$. Moreover, the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda_k') = m(\lambda_k') = 1, \quad k = k_0, k_0 + 1, \ldots,$$

$$\nu(\lambda_k'') = m(\lambda_k'') = 1, \quad k = k_0, k_0 + 1, \ldots.$$

8

The Eigenvalues for n Odd

In this chapter we compute the eigenvalues of the differential operator L for the case n odd. Throughout the hypotheses of Chapters 3–5 are assumed: (i) $n = 2\nu - 1 \geq 3$; (ii) the differential operator L is either regular or simply irregular; (iii) the integers p and q have been determined with $-\infty < p, q \leq p_0$ and with $a_p \neq 0$, $b_q \neq 0$, and $a_\kappa = 0$ for $\kappa = p+1, \ldots, p_0$ and $b_\kappa = 0$ for $\kappa = q+1, \ldots, p_0$; (iv) the translated sectors T_0 and T_1 have been chosen with condition (3.51) being satisfied for the case $p = q$; (v) the integer m has been fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p, q \leq p_0$; and (vi) the functions π_i, $i = 0, 1$, and the functions π_i', $i = 0, 1$, have been determined as per Chapter 3 or equations (5.103) and (5.129). Specifically, the functions π_1 and π_0 are given by

$$\pi_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \qquad \pi_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa \tag{8.1}$$

for $\rho \neq 0$ in $\mathbb{C}$, while the functions π_1' and π_0' are given by

$$\pi_1'(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} a_\kappa' \rho^\kappa, \qquad \pi_0'(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} b_\kappa' \rho^\kappa \tag{8.2}$$

for $\rho \neq 0$ in $\mathbb{C}$. The leading coefficients in these representations are related by equation (5.133), viz. $|a_q'| = |b_q|$ and $|b_p'| = |a_p|$.

To determine the eigenvalues of L, we calculate the zeros of the characteristic determinant Δ_0 in the open set G_0 and the zeros of the characteristic determinant Δ_1 in the open set G_1. The basic properties of Δ_0 and Δ_1 have been developed previously in Theorem 5.4 and Theorem 5.5. Our analysis divides naturally into the three cases where $p = q$, $p < q$, and $p > q$. The latter two cases are logarithmic cases.

8.1 Case 1. $p = q$

Assume that $p = q$. This first case is a case with simple eigenvalues; it includes all of the regular differential operators and many of the simply irregular ones. We begin by working on the sector T_0 and the corresponding open set $G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}$. Let

$$f_0(\rho) := a_p \mathrm{e}^{\mathrm{i}\rho} + b_p = a_p \big[\mathrm{e}^{\mathrm{i}\rho} - \xi_0\big]$$

for $\rho \in \mathbb{C}$, where $\xi_0 := -b_p/a_p \neq 0$, and let

$$g_0(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a_\kappa}{\rho^{p-\kappa}} \mathrm{e}^{\mathrm{i}\rho} + \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{b_\kappa}{\rho^{p-\kappa}} + \frac{1}{\rho^p} \big[\Phi_{01}(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Phi_{00}(\rho)\big]$$

for $\rho \in G_0$. From the representation (5.105) or Theorem 5.4 we have

$$\Delta_0(\rho) = \rho^p[f_0(\rho) + g_0(\rho)] \tag{8.3}$$

for $\rho \in G_0$, where the function f_0 is an entire function and the function g_0 is analytic on the open set G_0. In Chapter 3 the constant $d > 0$ was selected to satisfy condition (3.51):

$$|a_p|\mathrm{e}^{-d} + |b_p|\mathrm{e}^{-d} \leq \frac{1}{4} \min\{|a_p|, |b_p|\}. \tag{8.4}$$

From equation (3.53) we also have

$$|a'_p|\mathrm{e}^{-d} + |b'_p|\mathrm{e}^{-d} \leq \frac{1}{4} \min\{|a'_p|, |b'_p|\} = \frac{1}{4} \min\{|a_p|, |b_p|\}. \tag{8.5}$$

Let us examine the functions f_0 and g_0 which make up Δ_0. First, if $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \geq d$, then $|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \leq \mathrm{e}^{-d}$, and by (8.4)

$$|f_0(\rho)| \geq |b_p| - |a_p||\mathrm{e}^{\mathrm{i}\rho}| \geq |b_p| - |a_p|\mathrm{e}^{-d} \geq |b_p| - \frac{1}{4}|b_p|.$$

Thus, we obtain the inequality

$$|f_0(\rho)| \geq \frac{3}{4}|b_p|$$

for $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \geq d$.

Second, take any point $\rho = a + \mathrm{i}b \in G_0$ with $b \geq -d$. Clearly $|\rho| > R_0 \geq 1$, $|\mathrm{e}^{\mathrm{i}\rho}| \leq \mathrm{e}^d$, and $-(m-p_0-1) \leq p \leq p_0$, so $m-p_0+p \geq m-p_0-(m-p_0-1) = 1$. Applying (5.106), it follows that

$$\begin{aligned} |g_0(\rho)| &\leq \frac{\gamma_1}{|\rho|} + \frac{1}{|\rho|^p} \cdot \frac{\gamma_2}{|\rho|^{m-p_0}} \\ &= \frac{\gamma_1}{|\rho|} + \frac{\gamma_2}{|\rho|^{m-p_0+p}} \leq \frac{\gamma_1 + \gamma_2}{|\rho|}. \end{aligned}$$

Therefore,

$$|g_0(\rho)| \leq \frac{\gamma_0}{|\rho|}$$

for $\rho = a + \mathrm{i}b \in G_0$ with $b \geq -d$.

Third, if $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \leq -d$, then $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^{b} \leq \mathrm{e}^{-d}$ and

$$|f_0(\rho)| = |\mathrm{e}^{\mathrm{i}\rho}||a_p + b_p\mathrm{e}^{-\mathrm{i}\rho}| \geq \mathrm{e}^{-b}\left\{|a_p| - |b_p|\mathrm{e}^{-d}\right\} \geq \mathrm{e}^{-b}\Big\{|a_p| - \frac{1}{4}\,|a_p|\Big\}.$$

Thus,

$$|f_0(\rho)| \geq \frac{3}{4}\,\mathrm{e}^{-b}|a_p| \geq \frac{3}{4}\,|a_p|$$

for $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \leq -d$.

Fourth, for any $\rho = a + \mathrm{i}b \in G_0$ with $b \leq -d$, by (5.106) once more

$$|g_0(\rho)| \leq \frac{\gamma_3}{|\rho|}\,\mathrm{e}^{-b} + \frac{1}{|\rho|^p}\cdot\frac{\gamma_4}{|\rho|^{m-p_0}}\,\mathrm{e}^{-b} \leq \frac{\gamma_3 + \gamma_4}{|\rho|}\,\mathrm{e}^{-b},$$

and hence,

$$|g_0(\rho)| \leq \frac{\gamma_0\mathrm{e}^{-b}}{|\rho|}$$

for $\rho = a + \mathrm{i}b \in G_0$ with $b \leq -d$.

In terms of the constant γ_0 that appears in the estimates for g_0, choose a constant $r_1 > R_1 \geq R_0$ such that

$$\frac{\gamma_0}{|\rho|} \leq \frac{1}{4}\min\{|a_p|, |b_p|\} = \frac{1}{4}\min\{|a'_p|, |b'_p|\}$$

for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_1$. It follows from the above that if $\rho = a + \mathrm{i}b \in G_0$ with $|\rho| \geq r_1$ and $b \geq d$, then

$$\begin{aligned}|\Delta_0(\rho)| &\geq |\rho|^p\{|f_0(\rho)| - |g_0(\rho)|\}\\ &\geq |\rho|^p\left\{\frac{3}{4}\,|b_p| - \frac{1}{4}\,|b_p|\right\} = \frac{1}{2}\,|b_p||\rho|^p > 0.\end{aligned} \tag{8.6}$$

On the other hand, if $\rho = a + \mathrm{i}b \in G_0$ with $|\rho| \geq r_1$ and $b \leq -d$, then

$$\begin{aligned}|\Delta_0(\rho)| &\geq |\rho|^p\left\{\frac{3}{4}\,\mathrm{e}^{-b}|a_p| - \frac{1}{4}\,\mathrm{e}^{-b}|a_p|\right\}\\ &= \frac{1}{2}\,\mathrm{e}^{-b}|a_p||\rho|^p \geq \frac{1}{2}\,|a_p||\rho|^p > 0.\end{aligned} \tag{8.7}$$

The estimates (8.6) and (8.7) are our initial growth rates for the characteristic determinant Δ_0 relative to the open set G_0.

Next, we consider the characteristic determinant Δ_1, and proceed to examine its behavior on the open set $G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\}$. Let

$$f_1(\rho) := a'_p \mathrm{e}^{-\mathrm{i}\rho} + b'_p = a'_p\big[\mathrm{e}^{-\mathrm{i}\rho} - \eta_0\big]$$

for $\rho \in \mathbb{C}$, where by (5.133) $\eta_0 := -b'_p/a'_p = 1/(\omega_p\xi_0)$ (cf. [36, p. 59]), and let

$$g_1(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a'_\kappa}{\rho^{p-\kappa}}\,\mathrm{e}^{-\mathrm{i}\rho} + \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{b'_\kappa}{\rho^{p-\kappa}} + \frac{1}{\rho^p}\,\big[\varPhi_{11}(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \varPhi_{10}(\rho)\big]$$

for $\rho \in G_1$. Then from (5.131) or Theorem 5.5 we have

$$\varDelta_1(\rho) = \rho^p[f_1(\rho) + g_1(\rho)] \tag{8.8}$$

for $\rho \in G_1$, where the function f_1 is an entire function and the function g_1 is analytic on the open set G_1.

Consider the functions f_1 and g_1. First, if $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \le -d$, then we have $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^b \le \mathrm{e}^{-d}$, and by (8.5)

$$|f_1(\rho)| \ge |b'_p| - |a'_p|\mathrm{e}^{-d} \ge |b'_p| - \frac{1}{4}\,|b'_p|.$$

Hence,

$$|f_1(\rho)| \ge \frac{3}{4}\,|b'_p|$$

for $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \le -d$.

Second, take any point $\rho = a + \mathrm{i}b \in G_1$ with $b \le d$. Then $|\rho| \ge 1$, $|\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^b \le \mathrm{e}^d$, and by equation (5.132)

$$|g_1(\rho)| \le \frac{\gamma'_1}{|\rho|} + \frac{1}{|\rho|^p}\cdot\frac{\gamma'_2}{|\rho|^{m-p_0}} \le \frac{\gamma'_1 + \gamma'_2}{|\rho|}\,.$$

Thus,

$$|g_1(\rho)| \le \frac{\gamma'_0}{|\rho|}$$

for $\rho = a + \mathrm{i}b \in G_1$ with $b \le d$.

Third, if $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \ge d$, then $|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \le \mathrm{e}^{-d}$ and

$$|f_1(\rho)| = |\mathrm{e}^{-\mathrm{i}\rho}||a'_p + b'_p\mathrm{e}^{\mathrm{i}\rho}| \ge \mathrm{e}^b\{|a'_p| - |b'_p|\mathrm{e}^{-d}\} \ge \mathrm{e}^b\Big\{|a'_p| - \frac{1}{4}\,|a'_p|\Big\}.$$

Therefore,

$$|f_1(\rho)| \ge \frac{3}{4}\,\mathrm{e}^b|a'_p| \ge \frac{3}{4}\,|a'_p|$$

for $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $b \ge d$.

Fourth, for any $\rho = a + \mathrm{i}b \in G_1$ with $b \ge d$, using (5.132) once more, we have

$$|g_1(\rho)| \le \mathrm{e}^b\Big\{\frac{\gamma'_3}{|\rho|} + \frac{1}{|\rho|^p}\cdot\frac{\gamma'_4}{|\rho|^{m-p_0}}\Big\},$$

and hence,

$$|g_1(\rho)| \leq \frac{\gamma_0' \mathrm{e}^b}{|\rho|}$$

for $\rho = a + \mathrm{i}b \in G_1$ with $b \geq d$.

Without loss of generality we can assume that $\gamma_0 = \gamma_0'$, so

$$\frac{\gamma_0'}{|\rho|} \leq \frac{1}{4} \min\{|a_p|, |b_p|\} = \frac{1}{4} \min\{|a_p'|, |b_p'|\}$$

for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_1$. It follows from the above that if $\rho = a + \mathrm{i}b \in G_1$ with $|\rho| \geq r_1$ and $b \leq -d$, then

$$\begin{aligned} |\Delta_1(\rho)| &\geq |\rho|^p \{|f_1(\rho)| - |g_1(\rho)|\} \\ &\geq |\rho|^p \left\{ \frac{3}{4} |b_p'| - \frac{1}{4} |b_p'| \right\} = \frac{1}{2} |b_p'| |\rho|^p > 0, \end{aligned} \tag{8.9}$$

while if $\rho = a + \mathrm{i}b \in G_1$ with $|\rho| \geq r_1$ and $b \geq d$, then

$$\begin{aligned} |\Delta_1(\rho)| &\geq |\rho|^p \left\{ \frac{3}{4} \mathrm{e}^b |a_p'| - \frac{1}{4} \mathrm{e}^b |a_p'| \right\} \\ &= \frac{1}{2} \mathrm{e}^b |a_p'| |\rho|^p \geq \frac{1}{2} |a_p'| |\rho|^p > 0. \end{aligned} \tag{8.10}$$

The estimates (8.9) and (8.10) are our initial growth rates for Δ_1 relative to the open set G_1.

As an immediate application of (8.6), (8.7) and (8.9), (8.10), we have the following theorem which establishes *apriori estimates* for the eigenvalues of L.

Theorem 8.1. *Assume that $p = q$. Let $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in G_0$ and with $|\rho| \geq r_1$.*

(a) *If $|b| \geq d$, then $\Delta_0(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(b) *If λ is an eigenvalue of L, then $\Delta_0(\rho) = 0$ and $|b| < d$.*

In addition, let $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in G_1$ and with $|\rho| \geq r_1$.

(c) *If $|b| \geq d$, then $\Delta_1(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(d) *If λ is an eigenvalue of L, then $\Delta_1(\rho) = 0$ and $|b| < d$.*

From the theorem the resolvent set $\rho(L)$ is nonempty. Consequently, the differential operator L is a Fredholm operator with Fredholm set $\Phi(L) = \mathbb{C}$ and with resolvent set $\rho(L) \neq \emptyset$; this implies that the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. See [34, p. 58 or p. 60].

We next focus our search for the zeros of Δ_0 on the horizontal strip

$$\Gamma_0 = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \geq -\pi \text{ and } |b| \leq d\}.$$

Recall that the sector T_0 was selected in Chapter 3 so that the horizontal strip Γ_0 lies in the interior of T_0. Clearly the zeros of f_0 are given by the sequence

$$\mu_k' := (2\pi k + \operatorname{Arg} \xi_0) - \mathrm{i} \ln |\xi_0|, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

where each μ'_k is a zero of order 1 of f_0. From the above estimates for f_0, it follows that the μ'_k all lie in the interior of the horizontal strip $|\operatorname{Im}\rho| \leq d$, i.e., $-d < \ln|\xi_0| < d$. We will show that the zeros of Δ_0 and $f_0 + g_0$ that lie in the horizontal strip Γ_0 appear as perturbations of the zeros μ'_k.

Since $-\pi < \operatorname{Arg}\xi_0 \leq \pi$, we can choose a real number $\omega \geq \pi$ such that $\omega - 2\pi < \operatorname{Arg}\xi_0 < \omega$. Then for $k = 1, 2, \ldots$ introduce the rectangles

$$R'_k := \{\rho \in \mathbb{C} \mid \omega - 2\pi \leq \operatorname{Re}\rho \leq \omega + 2\pi(k-1) \text{ and } |\operatorname{Im}\rho| \leq d\}.$$

Clearly these rectangles lie in the horizontal strip Γ_0, and hence, they lie in the interior of the sector T_0, and the zero μ'_0 lies in the interior of the rectangle R'_1. Choose a real number δ with $0 < \delta \leq \pi/4$ such that the disk $|\rho - \mu'_0| \leq \delta$ lies in the interior of R'_1. For $k = 0, \pm 1, \pm 2, \ldots$ form the circles

$$\Gamma'_k := \{\rho \in \mathbb{C} \mid |\rho - \mu'_k| = \delta\}.$$

The following properties are obvious from these definitions: (i) the circles Γ'_k, $k \geq 0$, lie in the interior of the horizontal strip Γ_0; (ii) the Γ'_k and the points inside them do not overlap each other; and (iii) for each positive integer k_0 the circles Γ'_k, $0 \leq k < k_0$, lie in the interior of the rectangle R'_{k_0}, the circles Γ'_k, $k \geq k_0$, lie in the exterior of R'_{k_0} and to the right of R'_{k_0}, and the circles Γ'_k, $k < 0$, lie in the exterior of R'_{k_0} and to the left of R'_{k_0}.

To complete the geometry, let Ω_0 be the subset of the sector T_0 defined by

$$\Omega_0 := \{\rho = a + ib \in \Gamma_0 \mid \rho \text{ is not inside any of the circles } \Gamma'_k\}.$$

In the sequel we refer to Ω_0 as a *punctured horizontal strip.*

Clearly $f_0(\rho) \neq 0$ for all $\rho \in R'_1$ which do not lie in the circle Γ'_0. Let

$$m_* := \min\{|f_0(\rho)| \mid \rho \in R'_1 \text{ with } \rho \text{ not in } \Gamma'_0\} > 0.$$

Then $f_0(\rho + 2\pi) = f_0(\rho)$ for all $\rho \in \mathbb{C}$, and hence, $|f_0(\rho)| \geq m_*$ for all $\rho = a + ib \in \mathbb{C}$ with $|b| \leq d$ and with ρ not in any of the circles Γ'_k. Setting

$$m_0 := \min\left\{\frac{3}{4}|a_p|, \frac{3}{4}|b_p|, m_*\right\} > 0,$$

the estimates for f_0 combine to yield

$$|f_0(\rho)| \geq m_0 > 0 \tag{8.11}$$

for all $\rho \in \mathbb{C}$ with ρ not in any of the circles Γ'_k.

Choose a positive integer k_0 such that the constant $y_0 := \omega + 2\pi(k_0 - 1)$ has the following properties: $y_0 \geq r_1$ and

$$\frac{\gamma_0}{|\rho|} \leq \frac{m_0}{2} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \geq y_0,$$

where $\gamma_0 = \gamma_0'$ is the constant introduced above in the estimates for g_0 and g_1. Clearly $y_0 > 1$ and $y_0 \geq \omega$. Then for any point $\rho = a + \mathrm{i}b \in \Gamma_0$ with $|\rho| \geq y_0$, we have $|\rho| \geq r_1 > R_0$, so $\rho \in G_0$, and by our previous estimates for g_0,

$$|g_0(\rho)| \leq \frac{\gamma_0}{|\rho|} \leq \frac{m_0}{2}. \tag{8.12}$$

Combining (8.11) and (8.12), we conclude that

$$|g_0(\rho)| \leq \frac{m_0}{2} < m_0 \leq |f_0(\rho)| \tag{8.13}$$

and

$$|f_0(\rho) + g_0(\rho)| \geq \frac{m_0}{2} \tag{8.14}$$

for all $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \geq y_0$, and hence, by the representation (8.3)

$$|\Delta_0(\rho)| \geq \frac{m_0}{2}\,|\rho|^p > 0 \tag{8.15}$$

for all $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \geq y_0$. The estimate (8.15) is our main result for the growth rate of the characteristic determinant Δ_0 on the punctured horizontal strip Ω_0. It immediately implies that Δ_0 and $f_0 + g_0$ have no zeros in Ω_0 when $|\rho| \geq y_0$.

Now let us consider the circles Γ_k' for $k \geq k_0$, which lie in the interior of T_0 and in the interior of the horizontal strip Γ_0 and to the right of the rectangle $R_{k_0}' = [\omega - 2\pi, y_0] \times [-d, d]$. From (8.13) we have $|g_0(\rho)| < |f_0(\rho)|$ for all points ρ on Γ_k' for $k \geq k_0$, and hence, by Rouché's Theorem Δ_0 and $f_0 + g_0$ have precisely the same number of zeros as f_0 inside Γ_k' for all $k \geq k_0$. But f_0 has only the single zero μ_k' of order 1 inside Γ_k', implying that Δ_0 has exactly one zero ρ_k' inside Γ_k' with ρ_k' having order 1 for $k \geq k_0$. Setting

$$\lambda_k' := (\rho_k')^n, \qquad k = k_0, k_0 + 1, \ldots,$$

it follows that the λ_k', $k = k_0, k_0 + 1, \ldots$, are eigenvalues of L, and by our earlier work the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda_k') = m(\lambda_k') = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.16}$$

Let us derive asymptotic formulas for the zeros ρ_k', $k = k_0, k_0 + 1, \ldots$, of Δ_0. If we set $\zeta_k' = -g_0(\rho_k')/(a_p \xi_0)$, $k = k_0, k_0 + 1, \ldots$, then we can rewrite the equation $f_0(\rho_k') + g_0(\rho_k') = 0$ as $\mathrm{e}^{\mathrm{i}\rho_k'} = \xi_0 + \xi_0 \zeta_k'$, and upon dividing by $\mathrm{e}^{\mathrm{i}\mu_k'} = \xi_0$, it becomes

$$\mathrm{e}^{\mathrm{i}(\rho_k' - \mu_k')} = 1 + \zeta_k'.$$

But $|\mathrm{Re}(\rho_k' - \mu_k')| \leq |\rho_k' - \mu_k'| < \delta \leq \pi/4$, so

$$\rho_k' - \mu_k' = -\mathrm{i}\,\mathrm{Log}\,[1 + \zeta_k'], \qquad k = k_0, k_0 + 1, \ldots. \tag{8.17}$$

Now fix an integer $k \geq k_0$, and consider the zero ρ_k'. Clearly

$$\begin{aligned}|\rho_k'| &\geq |\mu_k'| - |\rho_k' - \mu_k'| \geq 2\pi k + \operatorname{Arg} \xi_0 - \delta \\ &\geq 2\pi k - \pi - \pi/4 \geq 6k - 5 \geq k\end{aligned}$$

and

$$|\zeta_k'| = \frac{|g_0(\rho_k')|}{|\xi_0||a_p|} \leq \frac{\gamma_0}{|\xi_0||a_p||\rho_k'|} \leq \frac{\gamma'}{k}\,. \tag{8.18}$$

Since

$$-\mathrm{i}\operatorname{Log}[1+z] = zH(z) \quad \text{for } |z| < 1,$$

with H analytic on the disk $|z| < 1$, from (8.17) and (8.18) we obtain the estimate

$$|\rho_k' - \mu_k'| \leq \frac{\gamma}{k}\,, \qquad k = k_0, k_0+1, \ldots, \tag{8.19}$$

for an appropriate constant $\gamma > 0$. This is the desired asymptotic formula.

Next, we compute the zeros of Δ_1 in the horizontal strip

$$\Gamma_1 = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid a \leq \pi \text{ and } |b| \leq d\}.$$

Recall that the sector T_1 was selected in Chapter 3 so that the horizontal strip Γ_1 lies in the interior of T_1. The zeros of the entire function f_1 are given by

$$\mu_k'' := -(2\pi k + \operatorname{Arg} \eta_0) + \mathrm{i} \ln |\eta_0|, \qquad k = 0, \pm 1, \pm 2, \ldots,$$

where $|\eta_0| = 1/|\xi_0|$ and $\arg \eta_0 = -\arg \xi_0 - 2\pi p/n$ and where each μ_k'' is a zero of order 1 of f_1. Clearly the μ_k'' all lie in the interior of the horizontal strip $|\operatorname{Im} \rho| \leq d$, i.e., $-d < \ln |\eta_0| < d$. We will show that the zeros of Δ_1 and $f_1 + g_1$ in Γ_1 appear as perturbations of the μ_k''.

Choose a constant $\omega' \leq \pi$ such that $\omega' - 2\pi < -\operatorname{Arg} \eta_0 < \omega'$, and then for $k = 1, 2, \ldots$ introduce the rectangles

$$R_k'' := \{\rho \in \mathbb{C} \mid \omega' - 2\pi k \leq \operatorname{Re} \rho \leq \omega' \text{ and } |\operatorname{Im} \rho| \leq d\}.$$

Clearly the zero μ_0'' lies in the interior of the rectangle R_1'', so we can choose a constant δ' with $0 < \delta' \leq \pi/4$ such that the disk $|\rho - \mu_0''| \leq \delta'$ lies in the interior of R_1''. Without loss of generality we can assume that $\delta = \delta'$. For $k = 0, \pm 1, \pm 2, \ldots$ form the circles

$$\Gamma_k'' := \{\rho \in \mathbb{C} \mid |\rho - \mu_k''| = \delta\}.$$

The following properties are obvious from these definitions: (i) the circles Γ_k'', $k \geq 0$, lie in the interior of the horizontal strip Γ_1; (ii) the Γ_k'' and the points inside them do not overlap each other; and (iii) for each positive integer k_0 the circles Γ_k'', $0 \leq k < k_0$, lie in the interior of the rectangle R_{k_0}'', the circles Γ_k'', $k \geq k_0$, lie in the exterior of R_{k_0}'' and to the left of R_{k_0}'', and the circles Γ_k'', $k < 0$, lie in the exterior of R_{k_0}'' and to the right of R_{k_0}''. Finally, let Ω_1 be the *punctured horizontal strip* in the ρ plane defined by

$$\Omega_1 := \{\rho = a + \mathrm{i}b \in \Gamma_1 \mid \rho \text{ is not inside any of the circles } \Gamma_k''\}.$$

It is clear that $f_1(\rho) \neq 0$ for all $\rho \in R_1''$ which do not lie in the circle Γ_0''. Set

$$m^* := \min\{|f_1(\rho)| \mid \rho \in R_1'' \text{ with } \rho \text{ not in } \Gamma_0''\} > 0.$$

Since $f_1(\rho + 2\pi) = f_1(\rho)$ for all points $\rho \in \mathbb{C}$, it follows that $|f_1(\rho)| \geq m^*$ for all $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|b| \leq d$ and with ρ not in any of the circles Γ_k''. If we set

$$m_\diamond := \min\left\{\frac{3}{4}\,|a_p'|, \frac{3}{4}\,|b_p'|, m^*\right\} > 0,$$

then the estimates for f_1 combine to yield

$$|f_1(\rho)| \geq m_\diamond > 0 \tag{8.20}$$

for all $\rho \in \mathbb{C}$ with ρ not in any of the circles Γ_k''.

Select a positive integer k_1 such that the real number $x_0 := \omega' - 2\pi k_1$ has the properties: $x_0 \leq -r_1$ and

$$\frac{\gamma_0}{|\rho|} \leq \frac{m_\diamond}{2} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \geq |x_0|,$$

where $\gamma_0 = \gamma_0'$ is again the constant introduced above in the estimates for g_0 and g_1. Without loss of generality we can assume that $k_1 = k_0$. Clearly $|x_0| \geq r_1 > 1$ and $x_0 \leq \omega' - 2\pi$. Then for any point $\rho = a + \mathrm{i}b \in \Gamma_1$ with $|\rho| \geq |x_0|$, we have $|\rho| \geq r_1 > R_0$, so $\rho \in G_1$, and by our previous estimates for g_1,

$$|g_1(\rho)| \leq \frac{\gamma_0}{|\rho|} \leq \frac{m_\diamond}{2}\,. \tag{8.21}$$

Combining (8.20) and (8.21), we conclude that

$$|g_1(\rho)| \leq \frac{m_\diamond}{2} < m_\diamond \leq |f_1(\rho)| \tag{8.22}$$

and

$$|f_1(\rho) + g_1(\rho)| \geq \frac{m_\diamond}{2} \tag{8.23}$$

for $\rho = a + \mathrm{i}b \in \Omega_1$ with $|\rho| \geq |x_0|$, and hence, by the representation (8.8)

$$|\Delta_1(\rho)| \geq \frac{m_\diamond}{2}\,|\rho|^p > 0 \tag{8.24}$$

for $\rho = a + \mathrm{i}b \in \Omega_1$ with $|\rho| \geq |x_0|$. The estimate (8.24) is our main result for the growth rate of Δ_1 on the punctured horizontal strip Ω_1. It immediately implies that Δ_1 and $f_1 + g_1$ have no zeros in Ω_1 when $|\rho| \geq |x_0|$.

Now let us consider the circles Γ_k'' for $k \geq k_0$, which lie in Γ_1 and to the left of the rectangle $R_{k_0}'' = [x_0, \omega'] \times [-d, d]$. From (8.22) we have $|g_1(\rho)| < |f_1(\rho)|$ for all points ρ on Γ_k'' for $k \geq k_0$, and hence, by Rouché's Theorem Δ_1 and $f_1 + g_1$ have precisely the same number of zeros as f_1 inside Γ_k'' for all $k \geq k_0$.

But f_1 has only the single zero μ_k'' of order 1 inside Γ_k'', implying that Δ_1 has exactly one zero ρ_k'' inside Γ_k'' with ρ_k'' having order 1 for each $k \geq k_0$. Setting

$$\lambda_k'' := (\rho_k'')^n, \qquad k = k_0, k_0 + 1, \ldots,$$

the λ_k'' are eigenvalues of L with algebraic multiplicities and ascents

$$\nu(\lambda_k'') = m(\lambda_k'') = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.25}$$

Let us derive asymptotic formulas for the zeros ρ_k'', $k = k_0, k_0 + 1, \ldots$. Indeed, set $\zeta_k'' = -g_1(\rho_k'')/(a_p'\eta_0)$, $k = k_0, k_0 + 1, \ldots$. Then the equation $f_1(\rho_k'') + g_1(\rho_k'') = 0$ can be rewritten as

$$\mathrm{e}^{-\mathrm{i}\rho_k''} = \eta_0 + \eta_0\zeta_k'',$$

and dividing by $\mathrm{e}^{-\mathrm{i}\mu_k''} = \eta_0$, it becomes $\mathrm{e}^{-\mathrm{i}(\rho_k'' - \mu_k'')} = 1 + \zeta_k''$ or

$$\rho_k'' - \mu_k'' = \mathrm{i}\,\mathrm{Log}\,[1 + \zeta_k''], \qquad k = k_0, k_0 + 1, \ldots. \tag{8.26}$$

Now fix an integer $k \geq k_0$, and consider the zero ρ_k''. Clearly

$$\begin{aligned} |\rho_k''| &\geq |\mu_k''| - |\rho_k'' - \mu_k''| \geq 2\pi k + \mathrm{Arg}\,\eta_0 - \delta \\ &\geq 2\pi k - \pi - \pi/4 \geq k \quad \text{for } k \text{ sufficiently large,} \end{aligned}$$

and hence,

$$|\zeta_k''| = \frac{|g_1(\rho_k'')|}{|\eta_0||a_p'|} \leq \frac{\gamma_0}{|\eta_0||a_p'||\rho_k''|} \leq \frac{\gamma''}{k}\,. \tag{8.27}$$

Since $\mathrm{i}\,\mathrm{Log}\,[1 + z] = zH(z)$ for $|z| < 1$, with H analytic on the disk $|z| < 1$, from (8.26) and (8.27) we obtain the estimate

$$|\rho_k'' - \mu_k''| \leq \frac{\gamma}{k}\,, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.28}$$

Finally, suppose that $\lambda_0 \neq 0$ is any eigenvalue of L which is distinct from the λ_k', λ_k'', $k = k_0, k_0 + 1, \ldots$. Then we can express λ_0 in the form $\lambda_0 = (\rho_0)^n$ where $\rho_0 = a_0 + \mathrm{i}b_0 \neq 0$ belongs to either the sector S_0 or the sector S_1. Let us consider these two cases separately.

First, assume $\rho_0 \in S_0$. Clearly $\rho_0 \in \mathrm{Int}\,T_0$. There are two possible locations for the point ρ_0: either ρ_0 lies in the disk $|\rho| < y_0$, or $|\rho_0| \geq y_0$. In the former case only a finite number of such ρ_0 are possible because the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. Assume that ρ_0 belongs to the latter case, so $|\rho_0| \geq y_0$. Then $|\rho_0| \geq y_0 \geq r_1 > R_0$, and hence, $\rho_0 \in G_0$. By Theorem 8.1we must have $\Delta_0(\rho_0) = 0$ and $|b_0| < d$. From the simple geometry of the sector S_0 it is immediate that $a_0 \geq 0$, and hence, ρ_0 lies in the interior of the horizontal strip Γ_0. Suppose $a_0 > y_0$. We know that ρ_0 does not lie in any of the circles Γ_k' for $k \geq k_0$ because λ_0 is distinct from the λ_k'. The circles Γ_k',

$-\infty < k < k_0$, either lie in the interior of the rectangle R'_{k_0} or lie to the left of R'_{k_0}, and hence, ρ_0 can not be in any of these circles. This implies that ρ_0 belongs to the punctured horizontal strip Ω_0 with $|\rho_0| \geq a_0 > y_0$, and by the inequality (8.15) we have $\Delta_0(\rho_0) \neq 0$ — a contradiction. This contradiction shows that we must have $a_0 \leq y_0$, and hence, ρ_0 must lie in the rectangle $[0, y_0] \times [-d, d]$. Only a finite number of such ρ_0 are possible.

Second, assume $\rho_0 \in S_1$, so $\rho_0 \in \operatorname{Int} T_1$. Now either ρ_0 lies in the disk $|\rho| < |x_0|$, or $|\rho_0| \geq |x_0|$. In the former case only a finite number of such ρ_0 are possible. Assume that $|\rho_0| \geq |x_0|$, so $|\rho_0| \geq |x_0| \geq r_1 > R_0$, and hence, $\rho_0 \in G_1$. Again by Theorem 8.1 we must have $\Delta_1(\rho_0) = 0$ and $|b_0| < d$. From the simple geometry of the sector S_1 we must also have $a_0 \leq 0$. It follows that ρ_0 lies in the interior of the horizontal strip Γ_1. Suppose $a_0 < x_0$. We know that ρ_0 does not lie in any of the circles Γ''_k for $k \geq k_0$ because λ_0 is distinct from the λ''_k. The circles Γ''_k, $-\infty < k < k_0$, either lie in the interior of the rectangle R''_{k_0} or lie to the right of R''_{k_0}, and hence, ρ_0 can not be in any of these circles. This implies that ρ_0 belongs to the punctured horizontal strip Ω_1 with $|\rho_0| \geq |a_0| > |x_0|$, and by the estimate (8.24) we have $\Delta_1(\rho_0) \neq 0$ — a contradiction. We conclude that $x_0 \leq a_0 \leq 0$, and hence, ρ_0 must lie in the rectangle $[x_0, 0] \times [-d, d]$. Again only a finite number of such ρ_0 are possible.

Combining the two cases, we conclude that the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$, account for all but a finite number of the eigenvalues of L.

These results for the eigenvalues are summarized below in a theorem.

Theorem 8.2. *Let the differential operator L belong to Case 1, where the integers p and q satisfy the conditions $-\infty < p = q \leq p_0$ and where $\xi_0 = -b_p/a_p \neq 0$ and $\eta_0 = -b'_p/a'_p \neq 0$ (so $|\eta_0| = 1/|\xi_0|$, $\arg \eta_0 = -\arg \xi_0 - 2\pi p/n$). Then the elements of the spectrum $\sigma(L)$ can be listed as two sequences*

$$\lambda'_k = (\rho'_k)^n, \quad k = k_0, k_0 + 1, \ldots, \qquad \lambda''_k = (\rho''_k)^n, \quad k = k_0, k_0 + 1, \ldots,$$

plus a finite number of additional points, where

$$\begin{aligned} \rho'_k &= (2\pi k + \operatorname{Arg} \xi_0) - \mathrm{i} \ln |\xi_0| + \epsilon'_k, \qquad k = k_0, k_0 + 1, \ldots, \\ \rho''_k &= -(2\pi k + \operatorname{Arg} \eta_0) - \mathrm{i} \ln |\xi_0| + \epsilon''_k, \quad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

with $|\epsilon'_k| \leq \gamma/k$ and $|\epsilon''_k| \leq \gamma/k$ for $k = k_0, k_0 + 1, \ldots$. Moreover, the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \quad k = k_0, k_0 + 1, \ldots,$$

$$\nu(\lambda''_k) = m(\lambda''_k) = 1, \quad k = k_0, k_0 + 1, \ldots.$$

8.2 Case 2. $p < q$

Assume that $p < q$. This second case is a logarithmic case; all differential operators belonging to it are simply irregular. We begin by making some simple observations about the translated sectors $T_0 = \{\rho - \tau_0 \mid \rho \in S_0\}$ and $T_1 = \{\rho - \tau_1 \mid \rho \in S_1\}$. Take any point ρ in the sector T_0. Then the point $\rho + \tau_0$ belongs to the sector S_0, and it can be expressed in the form $\rho + \tau_0 = |\rho + \tau_0| \mathrm{e}^{\mathrm{i}\theta}$ with $-\pi/(2n) \le \theta \le \pi/(2n)$ (recall $n \ge 3$). Hence,

$$\begin{aligned} \operatorname{Re} \rho &= \operatorname{Re}(\rho + \tau_0) - \operatorname{Re} \tau_0 = |\rho + \tau_0| \cos \theta - \operatorname{Re} \tau_0 \\ &\ge |\rho| \cos \frac{\pi}{2n} - |\tau_0| \cos \frac{\pi}{2n} - \operatorname{Re} \tau_0. \end{aligned}$$

It follows that $\operatorname{Re} \rho \ge 0$ for all points $\rho \in T_0$ with $|\rho|$ sufficiently large. A similar argument show that $\operatorname{Re} \rho \le 0$ for all $\rho \in T_1$ with $|\rho|$ sufficiently large. Without loss of generality we can assume that in forming the open sets

$$G_0 = \{\rho \in \operatorname{Int} T_0 \mid |\rho| > R_0\}, \qquad G_1 = \{\rho \in \operatorname{Int} T_1 \mid |\rho| > R_0\},$$

the constant R_0 has been chosen sufficiently large to guarantee that $\operatorname{Re} \rho \ge 0$ for all $\rho \in G_0$ and $\operatorname{Re} \rho \le 0$ for all $\rho \in G_1$. We will assume this property is also valid for Case 3 which follows later in the chapter.

Set $n_0 := q - p > 0$, and consider the sector T_0 and the corresponding open set G_0. By (5.105) or Theorem 5.4 and (8.1) we can write the characteristic determinant Δ_0 in the form

$$\Delta_0(\rho) = \rho^p \{ a_p \mathrm{e}^{\mathrm{i}\rho} [1 + \phi_{01}(\rho)] + b_q \rho^{n_0} [1 + \phi_{00}(\rho)] \} \tag{8.29}$$

for $\rho \in G_0$, where

$$\begin{aligned} \phi_{01}(\rho) &:= \sum_{\kappa = -(m - p_0 - 1)}^{p-1} \frac{a_\kappa}{a_p \rho^{p-\kappa}} + \frac{1}{a_p \rho^p} \Phi_{01}(\rho), \\ \phi_{00}(\rho) &:= \sum_{\kappa = -(m - p_0 - 1)}^{q-1} \frac{b_\kappa}{b_q \rho^{q-\kappa}} + \frac{1}{b_q \rho^q} \Phi_{00}(\rho) \end{aligned}$$

for $\rho \in G_0$. The functions ϕ_{01}, ϕ_{00} are analytic on the open set G_0, and recalling that $m - p_0 + p \ge 1$ and $m - p_0 + q \ge 1$, by (5.106) we obtain the growth rates

$$|\phi_{01}(\rho)| \le \frac{\gamma_0}{|\rho|}, \qquad |\phi_{00}(\rho)| \le \frac{\gamma_0}{|\rho|} \tag{8.30}$$

for $\rho \in G_0$.

Set $\mu_0 := -b_q / a_p \ne 0$. Choose a constant $r_1 > R_1 \ge R_0$ such that

$$1/2 \le |1 + \phi_{01}(\rho)| \le 2, \qquad 1/2 \le |1 + \phi_{00}(\rho)| \le 2$$

for $\rho \in G_0$ with $|\rho| \ge r_1$ and such that

$$\frac{2}{|\mu_0| |\rho|^{n_0}} \le \frac{1}{4}$$

for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_1$. Then (8.29) can be rewritten in the form

$$\varDelta_0(\rho) = b_q \rho^q \Big\{ [1 + \phi_{00}(\rho)] + \frac{a_p}{b_q \rho^{n_0}} [1 + \phi_{01}(\rho)] \mathrm{e}^{\mathrm{i}\rho} \Big\}$$

for $\rho \in G_0$, and for any point $\rho = a + \mathrm{i}b \in G_0$ with $|\rho| \geq r_1$ and $b \geq 0$, we have

$$\begin{aligned} |\varDelta_0(\rho)| &\geq |b_q||\rho|^q \Big\{ \frac{1}{2} - \frac{2}{|\mu_0||\rho|^{n_0}} \mathrm{e}^{-b} \Big\} \\ &\geq |b_q||\rho|^q \Big\{ \frac{1}{2} - \frac{1}{4} \cdot 1 \Big\} = \frac{1}{4} |b_q||\rho|^q > 0. \end{aligned} \tag{8.31}$$

Consequently, as we search for the zeros of $\varDelta_0$ in the open set G_0, we will concentrate our search in Quadrant IV.

Let ω be the positive real number defined by the equation $1/\omega := 1/|\mu_0|^{1/n_0} + n_0$. Choose real numbers α and β with $0 < \alpha < [1/(2|\mu_0|)]^{1/n_0}$, $\beta > [2/|\mu_0|]^{1/n_0}$, and

$$2|\mu_0|(2\alpha)^{n_0} \leq \frac{1}{4}, \qquad \frac{2}{|\mu_0|\beta^{n_0}} \leq \frac{1}{4}.$$

Clearly $1/\beta < |\mu_0|^{1/n_0} < 1/\alpha$. We will first study the characteristic determinant $\varDelta_0$ on the sector S_0. Note that if ρ is any point in S_0 with $|\rho| > R_0$, then ρ belongs to the open set G_0, and hence, we will be working in a region where $\varDelta_0$ is analytic. In terms of the constants μ_0, α, and β, we form the *logarithmic strip*

$$\begin{aligned} \varOmega_0 &:= \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| b \leq 0,\ \alpha \mathrm{e}^{-b/n_0} \leq a \leq \beta \mathrm{e}^{-b/n_0} \big\} \\ &= \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| a \geq \alpha,\ b \leq 0,\ -n_0 \ln[a/\alpha] \leq b \leq -n_0 \ln[a/\beta] \big\} \end{aligned}$$

and the two complementary regions

$$\varOmega_{0\infty} := \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| b \leq 0,\ a \leq \alpha \mathrm{e}^{-b/n_0} \big\},$$

$$\varOmega_{0\square} := \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| b \leq 0,\ \beta \mathrm{e}^{-b/n_0} \leq a \big\}.$$

These three regions are contained in Quadrant IV. Note that

$$-n_0 \ln[a/\alpha] < -n_0 \ln[|\mu_0|^{1/n_0} a] < -n_0 \ln[a/\beta]$$

for $a \geq \beta$, so the logarithmic curve $b = -n_0 \ln[|\mu_0|^{1/n_0} a]$ runs down the 'middle' of the logarithmic strip $\varOmega_0$.

Let us begin by calculating the growth rate of $\varDelta_0$ on the region $\varOmega_{0\infty}$. For any point $\rho = a + ib$ belonging to $\varOmega_{0\infty}$, we have $a \geq 0$, $b \leq 0$, and

$$\begin{aligned} |\rho| &\leq |a| + |b| \leq \alpha \mathrm{e}^{|b|/n_0} + n_0 \mathrm{e}^{|b|/n_0} \\ &\qquad \leq [1/|\mu_0|^{1/n_0} + n_0] \mathrm{e}^{|b|/n_0} = \frac{1}{\omega} \mathrm{e}^{|b|/n_0}, \end{aligned}$$

and hence,

$$n_0 \ln[\omega|\rho|] \leq |b| \quad \text{for all } \rho = a + \mathrm{i}b \in \Omega_{0\infty}. \tag{8.32}$$

Choose a real number $x_0 > 0$ such that $x \leq \alpha \mathrm{e}^{x/n_0}$ for all $x \in \mathbb{R}$ with $x \geq x_0$, and then choose a real number r_2 such that $r_2 \geq r_1$ and $r_2 \geq \beta$ and such that $x_0 \leq n_0 \ln[\omega|\rho|]$ for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_2$. If $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$, then by (8.32) we have $|b| \geq x_0$, $|b| \leq \alpha \mathrm{e}^{|b|/n_0}$, and

$$|\rho| \leq |a| + |b| \leq \alpha \mathrm{e}^{|b|/n_0} + \alpha \mathrm{e}^{|b|/n_0} = 2\alpha \mathrm{e}^{|b|/n_0}. \tag{8.33}$$

Combining (8.33) with (8.29) gives

$$\Delta_0(\rho) = a_p \rho^p \mathrm{e}^{\mathrm{i}\rho} \Big\{ [1 + \phi_{01}(\rho)] + \frac{b_q}{a_p}\, \rho^{n_0} [1 + \phi_{00}(\rho)] \mathrm{e}^{-\mathrm{i}\rho} \Big\}$$

and

$$\begin{aligned} |\Delta_0(\rho)| &\geq |a_p||\rho|^p \mathrm{e}^{-b} \left\{ \frac{1}{2} - 2|\mu_0||\rho|^{n_0} \mathrm{e}^{b} \right\} \\ &\geq |a_p||\rho|^p \mathrm{e}^{-b} \left\{ \frac{1}{2} - 2|\mu_0| \cdot (2\alpha)^{n_0} \mathrm{e}^{-b} \cdot \mathrm{e}^{b} \right\} \geq \frac{1}{4}\, |a_p||\rho|^p \mathrm{e}^{-b}, \end{aligned}$$

or

$$|\Delta_0(\rho)| \geq \frac{1}{4}\, |a_p||\rho|^p \mathrm{e}^{-b} \geq \frac{|a_p|}{4(2\alpha)^{n_0}}\, |\rho|^q > 0 \tag{8.34}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. The inequality (8.34) establishes the growth rate for Δ_0 on the region $\Omega_{0\infty}$.

To determine the growth rate on the region $\Omega_{0\square}$, we again express the characteristic determinant in the form

$$\Delta_0(\rho) = b_q \rho^q \Big\{ [1 + \phi_{00}(\rho)] + \frac{a_p}{b_q \rho^{n_0}}\, [1 + \phi_{01}(\rho)] \mathrm{e}^{\mathrm{i}\rho} \Big\}$$

for $\rho \in G_0$. Then for any point $\rho = a + \mathrm{i}b \in \Omega_{0\square}$ we have $|\rho| \geq |a| \geq \beta \mathrm{e}^{|b|/n_0}$, and hence,

$$\begin{aligned} |\Delta_0(\rho)| &\geq |b_q||\rho|^q \Big\{ \frac{1}{2} - \frac{2\mathrm{e}^{-b}}{|\mu_0||\rho|^{n_0}} \Big\} \\ &\geq |b_q||\rho|^q \Big\{ \frac{1}{2} - \frac{2\mathrm{e}^{|b|}}{|\mu_0| \cdot \beta^{n_0} \mathrm{e}^{|b|}} \Big\} \geq \frac{1}{4}\, |b_q||\rho|^q, \end{aligned}$$

or

$$|\Delta_0(\rho)| \geq \frac{1}{4}\, |b_q||\rho|^q > 0 \tag{8.35}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{0\square}$. This is our growth rate for Δ_0 on the region $\Omega_{0\square}$.

Relative to the sector T_1 and the corresponding open set G_1, we can use (5.131) or Theorem 5.5 and (8.2) to write the characteristic determinant Δ_1 in the form

$$\Delta_1(\rho) = \rho^q \Big\{ a'_q \mathrm{e}^{-\mathrm{i}\rho}\big[1+\phi_{11}(\rho)\big] + \frac{b'_p}{\rho^{n_0}}\big[1+\phi_{10}(\rho)\big] \Big\} \tag{8.36}$$

for $\rho \in G_1$, where

$$\phi_{11}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{q-1} \frac{a'_\kappa}{a'_q \rho^{q-\kappa}} + \frac{1}{a'_q \rho^q} \Phi_{11}(\rho),$$
$$\phi_{10}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{b'_\kappa}{b'_p \rho^{p-\kappa}} + \frac{1}{b'_p \rho^p} \Phi_{10}(\rho)$$

for $\rho \in G_1$. The functions ϕ_{11}, ϕ_{10} are analytic on the open set G_1 with

$$|\phi_{11}(\rho)| \le \frac{\gamma'_0}{|\rho|}, \qquad |\phi_{10}(\rho)| \le \frac{\gamma'_0}{|\rho|} \tag{8.37}$$

for $\rho \in G_1$ (see equation (5.132)).

Set $\mu_1 := -a'_q/b'_p \neq 0$. From (5.133) we have $b'_p = \omega^p_{\nu-1} a_p$ and $a'_q = \omega^q_\nu b_q$, and hence,

$$\mu_1 = -\omega^q_\nu b_q / (\omega^p_{\nu-1} a_p) = \omega_{n_0\nu+p}\mu_0. \tag{8.38}$$

Clearly $|\mu_1| = |\mu_0|$ and $\arg\mu_1 = \arg\mu_0 + 2\pi(n_0\nu + p)/n$. Choose a constant $r'_1 > R_1 \ge R_0$ such that

$$1/2 \le |1+\phi_{11}(\rho)| \le 2, \qquad 1/2 \le |1+\phi_{10}(\rho)| \le 2$$

for $\rho \in G_1$ with $|\rho| \ge r'_1$ and such that

$$\frac{2}{|\mu_1||\rho|^{n_0}} \le \frac{1}{4}$$

for all $\rho \in \mathbb{C}$ with $|\rho| \ge r'_1$. Then (8.36) can be rewritten in the form

$$\Delta_1(\rho) = a'_q \rho^q \mathrm{e}^{-\mathrm{i}\rho} \Big\{ [1+\phi_{11}(\rho)] + \frac{b'_p}{a'_q \rho^{n_0}}\,[1+\phi_{10}(\rho)]\mathrm{e}^{\mathrm{i}\rho} \Big\}$$

for $\rho \in G_1$, and for any point $\rho = a + \mathrm{i}b \in G_1$ with $|\rho| \ge r'_1$ and $b \ge 0$, we have

$$\begin{aligned} |\Delta_1(\rho)| &\ge |a'_q||\rho|^q \mathrm{e}^b \Big\{ \frac{1}{2} - \frac{2}{|\mu_1||\rho|^{n_0}}\,\mathrm{e}^{-b} \Big\} \\ &\ge |a'_q||\rho|^q \mathrm{e}^b \Big\{ \frac{1}{2} - \frac{1}{4} \cdot 1 \Big\} = \frac{1}{4}\,|a'_q||\rho|^q \mathrm{e}^b > 0. \end{aligned} \tag{8.39}$$

In view of this result, our search for the zeros of the characteristic determinant Δ_1 in the open set G_1 will be concentrated in Quadrant III.

Set $\omega_1 := \omega$, $\alpha_1 := \alpha$, and $\beta_1 := \beta$. Clearly $1/\omega_1 = 1/|\mu_1|^{1/n_0} + n_0$, $0 < \alpha_1 < [1/(2|\mu_1|)]^{1/n_0}$, $\beta_1 > [2/|\mu_1|]^{1/n_0}$, and

$$2|\mu_1|(2\alpha_1)^{n_0} \le \frac{1}{4}, \qquad \frac{2}{|\mu_1|(\beta_1)^{n_0}} \le \frac{1}{4}.$$

Note that for $\rho \in S_1$ with $|\rho| > R_0$, we have $\rho \in G_1$, and we are working in a region of analyticity for $\varDelta_1$. In terms of the constants μ_1, α_1, and β_1, we form the *logarithmic strip*

$$\begin{aligned}\varOmega_1 &:= \bigl\{\rho = a + \mathrm{i}b \in S_1 \bigm| b \le 0,\ -\beta_1 \mathrm{e}^{-b/n_0} \le a \le -\alpha_1 \mathrm{e}^{-b/n_0}\bigr\}\\ &= \bigl\{\rho = a + \mathrm{i}b \in S_1 \bigm| a \le -\alpha_1,\ b \le 0,\ -n_0 \ln[-a/\alpha_1] \le b \le -n_0 \ln[-a/\beta_1]\bigr\}\end{aligned}$$

and the two complementary regions

$$\varOmega_{1\infty} := \bigl\{\rho = a + \mathrm{i}b \in S_1 \bigm| b \le 0,\ -\alpha_1 \mathrm{e}^{-b/n_0} \le a\bigr\},$$

$$\varOmega_{1\square} := \bigl\{\rho = a + \mathrm{i}b \in S_1 \bigm| b \le 0,\ a \le -\beta_1 \mathrm{e}^{-b/n_0}\bigr\}.$$

These three regions are contained in Quadrant III. Also, observe that

$$-n_0 \ln[-a/\alpha_1] < -n_0 \ln[-|\mu_1|^{1/n_0} a] < -n_0 \ln[-a/\beta_1]$$

for $a \le -\beta_1$, so the logarithmic curve $b = -n_0 \ln[-|\mu_1|^{1/n_0} a]$ runs down the 'middle' of $\varOmega_1$.

To calculate the growth rate of $\varDelta_1$ on $\varOmega_{1\infty}$, take any point $\rho = a + \mathrm{i}b$ in $\varOmega_{1\infty}$. Clearly $a \le 0$, $b \le 0$, $|a| = -a \le \alpha_1 \mathrm{e}^{|b|/n_0}$, and

$$|\rho| \le |a| + |b| \le \alpha_1 \mathrm{e}^{|b|/n_0} + n_0 \mathrm{e}^{|b|/n_0} \le [1/|\mu_1|^{1/n_0} + n_0] \mathrm{e}^{|b|/n_0} = \frac{1}{\omega_1} \mathrm{e}^{|b|/n_0}$$

or

$$n_0 \ln[\omega_1 |\rho|] \le |b|. \tag{8.40}$$

For the constant $x_1 := x_0$, we know that $x \le \alpha_1 \mathrm{e}^{x/n_0}$ for all $x \in \mathbb{R}$ with $x \ge x_1$. Choose a real number r_2' such that $r_2' \ge r_1'$ and $r_2' \ge \beta_1$ and such that

$$x_1 \le n_0 \ln[\omega_1 |\rho|]$$

for all $\rho \in \mathbb{C}$ with $|\rho| \ge r_2'$. By working with the maximum of the two constants r_2 and r_2', we can assume without loss of generality that $r_2 = r_2'$.

Now consider any point $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ with $|\rho| \ge r_2'$. Then by (8.40) we have $|b| \ge x_1$ and $|b| \le \alpha_1 \mathrm{e}^{|b|/n_0}$, and hence,

$$|\rho| \le |a| + |b| \le \alpha_1 \mathrm{e}^{|b|/n_0} + \alpha_1 \mathrm{e}^{|b|/n_0} = 2\alpha_1 \mathrm{e}^{|b|/n_0}.$$

We conclude that

$$|\rho| \le 2\alpha_1 \mathrm{e}^{|b|/n_0} \tag{8.41}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ with $|\rho| \ge r_2'$.

Combining (8.41) with (8.36), we have

$$\varDelta_1(\rho) = b_p' \rho^p \bigl\{[1 + \phi_{10}(\rho)] + (a_q'/b_p') \rho^{n_0} [1 + \phi_{11}(\rho)] \mathrm{e}^{-\mathrm{i}\rho}\bigr\}$$

and

$$|\varDelta_1(\rho)| \geq |b'_p||\rho|^p \left\{ \frac{1}{2} - 2|\mu_1||\rho|^{n_0}\mathrm{e}^b \right\}$$
$$\geq |b'_p||\rho|^p \left\{ \frac{1}{2} - 2|\mu_1| \cdot (2\alpha_1)^{n_0}\mathrm{e}^{-b} \cdot \mathrm{e}^b \right\} \geq \frac{1}{4}\,|b'_p||\rho|^p,$$

or

$$|\varDelta_1(\rho)| \geq \frac{1}{4}\,|b'_p||\rho|^p > 0 \tag{8.42}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ with $|\rho| \geq r'_2$. The inequality (8.42) establishes the growth rate for $\varDelta_1$ on the region $\varOmega_{1\infty}$.

For the growth rate on $\varOmega_{1\Box}$, we again express the characteristic determinant in the form

$$\varDelta_1(\rho) = a'_q\rho^q\mathrm{e}^{-\mathrm{i}\rho}\Big\{[1 + \phi_{11}(\rho)] + \frac{b'_p}{a'_q\rho^{n_0}}\,[1 + \phi_{10}(\rho)]\mathrm{e}^{\mathrm{i}\rho}\Big\}$$

for $\rho \in G_1$. Then for any point $\rho = a + \mathrm{i}b \in \varOmega_{1\Box}$ we have $|\rho| \geq |a| \geq \beta_1\mathrm{e}^{|b|/n_0}$, and hence,

$$|\varDelta_1(\rho)| \geq |a'_q||\rho|^q\mathrm{e}^b\left\{\frac{1}{2} - \frac{2\mathrm{e}^{-b}}{|\mu_1||\rho|^{n_0}}\right\}$$
$$\geq |a'_q||\rho|^q\mathrm{e}^b\left\{\frac{1}{2} - \frac{2\mathrm{e}^{|b|}}{|\mu_1| \cdot (\beta_1)^{n_0}\mathrm{e}^{|b|}}\right\} \geq \frac{1}{4}\,|a'_q||\rho|^q\mathrm{e}^b,$$

or

$$|\varDelta_1(\rho)| \geq \frac{1}{4}\,|a'_q||\rho|^q\mathrm{e}^b \geq \frac{1}{4}\,|a'_q|(\beta_1)^{n_0}|\rho|^p > 0 \tag{8.43}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{1\Box}$. This is our growth rate for $\varDelta_1$ on the region $\varOmega_{1\Box}$.

As an application of these growth rates on the regions $\varOmega_{0\infty}$, $\varOmega_{0\Box}$ and $\varOmega_{1\infty}$, $\varOmega_{1\Box}$, we obtain the following *apriori estimates* for the eigenvalues of L. Recall that we are assuming that $r_2 = r'_2$.

Theorem 8.3. *Assume that $p < q$. Let $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and with $|\rho| \geq r_2$.*

(a) *If $\rho \in \varOmega_{0\infty}$ or $\rho \in \varOmega_{0\Box}$, then $\varDelta_0(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(b) *If λ is an eigenvalue of L, then $\varDelta_0(\rho) = 0$ and ρ lies in the interior of the logarithmic strip $\varOmega_0$.*

In addition, let $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$ and with $|\rho| \geq r_2$.

(c) *If $\rho \in \varOmega_{1\infty}$ or $\rho \in \varOmega_{1\Box}$, then $\varDelta_1(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(d) *If λ is an eigenvalue of L, then $\varDelta_1(\rho) = 0$ and ρ lies in the interior of the logarithmic strip $\varOmega_1$.*

By the theorem the resolvent set $\rho(L)$ is nonempty, and hence, by our earlier remarks the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$.

Next, let us consider the behavior of Δ_0 on the logarithmic strip Ω_0. Setting $\xi := 1 + n_0/\alpha$, for each point $\rho = a + \mathrm{i}b \in \Omega_0$ we have

$$|\rho| \leq |a| + |b| \leq |a| + n_0 \ln[|a|/\alpha] \leq |a| + (n_0/\alpha)|a| = \xi|a|. \tag{8.44}$$

Relative to the strip Ω_0 the characteristic determinant can be written in the form

$$\Delta_0(\rho) = a_p\rho^p \big\{\mathrm{e}^{\mathrm{i}\rho} - \mu_0\rho^{n_0}[1 + h_0(\rho)]\big\} \tag{8.45}$$

for $\rho \in G_0$, where h_0 is the analytic function given by

$$h_0(\rho) := -\frac{\mathrm{e}^{\mathrm{i}\rho}}{\mu_0\rho^{n_0}}\,\phi_{01}(\rho) + \phi_{00}(\rho)$$

for $\rho \in G_0$. If $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \geq r_2$, then

$$\left|\frac{\mathrm{e}^{\mathrm{i}\rho}}{\mu_0\rho^{n_0}}\right| = \frac{\mathrm{e}^{-b}}{|\mu_0||\rho|^{n_0}} \leq \frac{1}{|\mu_0||\rho|^{n_0}} \cdot \frac{|a|^{n_0}}{\alpha^{n_0}} \leq \frac{1}{|\mu_0|\alpha^{n_0}},$$

and hence, by (8.30)

$$|h_0(\rho)| \leq \frac{\gamma_1}{|\rho|} \tag{8.46}$$

for all $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \geq r_2$.

Fix a real number δ with $0 < \delta \leq \pi/4$ and $0 < \delta < (\ln 2)/(1 + n_0)$, and then for the integers $k = 1, 2, \ldots$ define

$$\begin{cases} \alpha'_k := 2\pi k + \operatorname{Arg}\mu_0, \qquad \beta'_k := -n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k], \\ \mu'_k := \alpha'_k + \mathrm{i}\beta'_k, \end{cases}$$

and introduce the circles

$$\Gamma'_k := \{\rho \in \mathbb{C} \mid |\rho - \mu'_k| = \delta\}.$$

Choose a positive integer $k_1 \geq 2$ such that $y'_1 := \alpha'_{k_1} - \pi \geq r_2$. Note that $\alpha'_k - \pi \geq r_2 \geq \beta$ and $\alpha'_k \geq 3\pi$ for $k = k_1, k_1 + 1, \ldots$. Also, we introduce the *logarithmic rectangles*

$$R'_k := \big\{\rho = a + \mathrm{i}b \in \mathbb{C} \;\big|\; \alpha'_k - \pi \leq a \leq \alpha'_k + \pi,\ -n_0 \ln[a/\alpha] \leq b \leq -n_0 \ln[a/\beta]\big\}$$

for $k = k_1, k_1 + 1, \ldots$. Without loss of generality we can assume that k_1 is sufficiently large to guarantee that each R'_k is contained in the sector S_0, and hence, for $k = k_1, k_1 + 1, \ldots$ the point μ'_k lies in the interior of R'_k with R'_k a subset of Ω_0.

Fix any index $k \geq k_1$, and take any point $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|\rho - \mu'_k| \leq \delta$. We assert that ρ lies in the interior of R'_k. Indeed, we clearly have $|a - \alpha'_k| \leq \delta < \pi$ and $|b + n_0 \ln[|\mu_0|^{1/n_0}\alpha'_k]| \leq \delta$, so

$$\begin{aligned}|b + n_0 \ln[|\mu_0|^{1/n_0} a]| &\le |b + n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k]| \\ &\quad + |n_0 \ln[|\mu_0|^{1/n_0} a] - n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k]| \\ &\le \delta + n_0|a - \alpha'_k| \le \delta(1 + n_0) < \ln 2.\end{aligned}$$

Thus, $-n_0 \ln[(2|\mu_0|)^{1/n_0} a] < b < -n_0 \ln[(|\mu_0|/2)^{1/n_0} a]$ and

$$-n_0 \ln[a/\alpha] < b < -n_0 \ln[a/\beta].$$

This establishes the assertion, and it is immediate that the circle Γ'_k lies in the interior of the logarithmic rectangle R'_k for $k = k_1, k_1 + 1, \ldots$. To complete the setup of the geometry, for $k = k_1, k_1 + 1, \ldots$ let Ω'_k be the *punctured logarithmic rectangle* formed from R'_k by removing all the points inside Γ'_k.

The next step is to establish the growth rate of Δ_0 on each of the regions Ω'_k. Note that

$$e^{i\mu'_k} = \mu_0(\alpha'_k)^{n_0}$$

for $k = k_1, k_1 + 1, \ldots$. Let f_k, $k = k_1, k_1 + 1, \ldots$, and g_k, $k = k_1, k_1 + 1, \ldots$, be the sequences of analytic functions defined by

$$f_k(\rho) := e^{i(\rho - \mu'_k)} - 1 \quad \text{for } \rho \in \mathbb{C},$$

$$g_k(\rho) := -h_0(\rho) - \sum_{j=1}^{n_0} \binom{n_0}{j} \Big[\frac{1}{\alpha'_k}(\rho - \mu'_k) + \frac{i\beta'_k}{\alpha'_k}\Big]^j \big[1 + h_0(\rho)\big] \quad \text{for } \rho \in G_0.$$

Then we can use (8.45) to write Δ_0 in its final form:

$$\begin{aligned}\Delta_0(\rho) &= a_p\mu_0(\alpha'_k)^{n_0}\rho^p\Big\{e^{i\rho} \cdot e^{-i\mu'_k} - \Big[\frac{\rho}{\alpha'_k}\Big]^{n_0}\big[1 + h_0(\rho)\big]\Big\} \\ &= a_p\mu_0(\alpha'_k)^{n_0}\rho^p\Big\{e^{i(\rho-\mu'_k)} - \Big[\frac{1}{\alpha'_k}(\rho - \mu'_k) + \frac{i\beta'_k}{\alpha'_k} + 1\Big]^{n_0}\big[1 + h_0(\rho)\big]\Big\},\end{aligned}$$

or

$$\Delta_0(\rho) = a_p\mu_0(\alpha'_k)^{n_0}\rho^p[f_k(\rho) + g_k(\rho)] \tag{8.47}$$

for $\rho \in G_0$ and for $k = k_1, k_1 + 1, \ldots$. Here we have a family of representations for Δ_0 depending on the integer k. We will use the kth representation to determine the growth rate of Δ_0 on the kth region Ω'_k.

In terms of the constants α, β, δ, choose $d_0 > 0$ such that

$$n_0 \ln[2/(|\mu_0|^{1/n_0}\alpha)] \le d_0, \qquad n_0 \ln[2|\mu_0|^{1/n_0}\beta] \le d_0,$$

and $\delta < d_0$, and then form the punctured rectangle

$$R_* := \{\rho = a + ib \in \mathbb{C} \mid -\pi \le a \le \pi,\ -d_0 \le b \le d_0, \text{ and } |\rho| \ge \delta\}.$$

Set $m_0 := \min\{\, |e^{i\rho} - 1| \mid \rho \in R_*\} > 0$.

Fix any index $k \ge k_1$, and take any point $\rho = a + ib \in \Omega'_k$. Clearly we have $-\pi \le a - \alpha'_k \le \pi$, $\pi/\alpha'_k \le 1/2$ because $\alpha'_k \ge 3\pi$,

$$\frac{a}{\alpha'_k} \le \frac{\alpha'_k + \pi}{\alpha'_k} \le 2, \qquad \frac{a}{\alpha'_k} \ge \frac{\alpha'_k - \pi}{\alpha'_k} \ge \frac{1}{2},$$

and

$$\begin{aligned} b - \beta'_k &\ge -n_0 \ln[a/\alpha] + n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k] \\ &= -n_0 \ln[a/(|\mu_0|^{1/n_0} \alpha \alpha'_k)] \ge -n_0 \ln[2/(|\mu_0|^{1/n_0} \alpha)] \ge -d_0, \end{aligned}$$

$$\begin{aligned} b - \beta'_k &\le -n_0 \ln[a/\beta] + n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k] \\ &= -n_0 \ln[a/(|\mu_0|^{1/n_0} \beta \alpha'_k)] \le -n_0 \ln[1/(2|\mu_0|^{1/n_0} \beta)] \le d_0. \end{aligned}$$

Thus, the translate $\rho - \mu'_k$ belongs to R_* with $|\rho - \mu'_k| \le \pi + d_0$. It follows that

$$|f_k(\rho)| \ge m_0 \tag{8.48}$$

for $k \ge k_1$ and for $\rho \in \Omega'_k$. Note that the constant m_0 is independent of the index k.

Clearly $\lim_{k\to\infty} 1/\alpha'_k = \lim_{k\to\infty} \beta'_k/\alpha'_k = 0$. In terms of the constant γ_1 that appears in the inequality (8.46), select an integer $k_0 \ge k_1$ such that

$$\frac{\gamma_1}{|\rho|} \le \frac{m_0}{4} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \ge y'_0,$$

where $y'_0 := \alpha'_{k_0} - \pi \ge r_2$, and such that

$$\sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{\pi + d_0}{\alpha'_k} + \frac{|\beta'_k|}{\alpha'_k} \right]^j \left[1 + \frac{\gamma_1}{r_2} \right] \le \frac{m_0}{4} \quad \text{for all } k \ge k_0.$$

Then for $k \ge k_0$ and for $\rho = a + \mathrm{i}b \in \Omega'_k$, we have $|\rho| \ge a \ge \alpha'_k - \pi \ge y'_0 \ge r_2$, and hence, by (8.46) and the definition of the integer k_0,

$$|g_k(\rho)| \le \frac{m_0}{2} < m_0 \le |f_k(\rho)|. \tag{8.49}$$

Also, since $\alpha'_k \ge 3\pi$, we have $a \ge \alpha'_k - \pi \ge 2\pi$ or $a/2 \ge \pi$, and $\alpha'_k \ge a - \pi \ge a/2 \ge |\rho|/(2\xi)$ by (8.44). Therefore, from (8.47) and (8.49) we conclude that

$$|\Delta_0(\rho)| \ge |a_p||\mu_0|(\alpha'_k)^{n_0}|\rho|^p \cdot \frac{m_0}{2} \ge \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\,|\rho|^q,$$

or

$$|\Delta_0(\rho)| \ge \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\,|\rho|^q > 0 \tag{8.50}$$

for $k \ge k_0$ and for $\rho \in \Omega'_k$.

The estimate (8.49) is local in character in that it depends on k: it is valid only on the region Ω'_k. In contrast, the estimate (8.50) is global because the constant on the right is independent of k. If we introduce the *punctured logarithmic strip*

$$\Omega'_* := \bigcup_{k=k_0}^{\infty} \Omega'_k,$$

then we see that Ω'_* consists of all points $\rho = a + \mathrm{i}b \in \Omega_0$ with $a \geq y'_0$ which do not lie inside any of the circles Γ'_k for $k \geq k_0$, and from the above

$$|\Delta_0(\rho)| \geq \frac{m_0|a_p||\mu_0|}{2(2\xi)^{n_0}}\,|\rho|^q > 0 \tag{8.51}$$

for all $\rho = a + \mathrm{i}b \in \Omega'_*$.

With the basic estimates (8.49) and (8.51) in place, consider one of the circles Γ'_k for $k \geq k_0$. Since (8.49) is valid for each point ρ on Γ'_k, it follows by Rouché's Theorem that Δ_0 and $f_k + g_k$ have the same number of zeros as f_k inside Γ'_k. But μ'_k is the only zero of f_k inside Γ'_k, μ'_k being a zero of order 1. Consequently, the characteristic determinant Δ_0 has a unique zero ρ'_k inside Γ'_k with ρ'_k having order 1 for $k \geq k_0$. Setting

$$\lambda'_k := (\rho'_k)^n, \qquad k = k_0, k_0 + 1, \ldots,$$

the complex numbers λ'_k are eigenvalues of L, and by our earlier work the corresponding algebraic multiplicities and ascents are given by

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.52}$$

It is also easy to derive asymptotic formulas for the zeros ρ'_k, $k = k_0$, $k_0 + 1, \ldots$. Set $\zeta'_k := -g_k(\rho'_k)$ for $k = k_0, k_0 + 1, \ldots$. Then we know that $\mathrm{e}^{\mathrm{i}(\rho'_k - \mu'_k)} = 1 + \zeta'_k$ and

$$\rho'_k - \mu'_k = -\mathrm{i}\,\mathrm{Log}\,[1 + \zeta'_k] \tag{8.53}$$

for $k = k_0, k_0 + 1, \ldots$, where

$$|\zeta'_k| \leq \frac{\gamma_1}{|\rho'_k|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{\delta}{\alpha'_k} + \frac{|\beta'_k|}{\alpha'_k}\right]^j \left[1 + \frac{\gamma_1}{r_2}\right].$$

Now for each $k \geq k_0$, $\alpha'_k \geq 2\pi k - \pi \geq k$, $|\beta'_k| \leq n_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \pi)] \leq \gamma_2 \ln k$, and

$$|\rho'_k| \geq |\mu'_k| - |\rho'_k - \mu'_k| \geq \alpha'_k - \delta \geq 6k - 5 \geq k,$$

which yields $|\zeta'_k| \leq \gamma_3 \ln k/k$ and

$$|\rho'_k - \mu'_k| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.54}$$

To determine the zeros of Δ_1 in the logarithmic strip Ω_1, we adopt an approach similar to the above. Set $\eta := 1 + n_0/\alpha_1$. Then for each point $\rho = a + \mathrm{i}b \in \Omega_1$ we have

$$|\rho| \le |a| + |b| \le |a| + n_0 \ln[|a|/\alpha_1] \le |a| + (n_0/\alpha_1)|a| = \eta|a|. \tag{8.55}$$

Relative to the strip Ω_1 we can write Δ_1 in the form

$$\Delta_1(\rho) = a'_q \rho^q \Big\{ \mathrm{e}^{-\mathrm{i}\rho} - \frac{1}{\mu_1 \rho^{n_0}} \left[1 + h_1(\rho) \right] \Big\} \tag{8.56}$$

for $\rho \in G_1$, where h_1 is the analytic function given by

$$h_1(\rho) := -\mu_1 \rho^{n_0} \mathrm{e}^{-\mathrm{i}\rho} \phi_{11}(\rho) + \phi_{10}(\rho)$$

for $\rho \in G_1$. If $\rho = a + \mathrm{i}b \in \Omega_1$ with $|\rho| \ge r'_2$, then

$$|\mu_1 \rho^{n_0} \mathrm{e}^{-\mathrm{i}\rho}| = |\mu_1||\rho|^{n_0} \mathrm{e}^{b} \le |\mu_1| \cdot \eta^{n_0} |a|^{n_0} \cdot \frac{(\beta_1)^{n_0}}{|a|^{n_0}} = |\mu_1|(\eta\beta_1)^{n_0},$$

and hence, by (8.37)

$$|h_1(\rho)| \le \frac{\gamma'_1}{|\rho|} \tag{8.57}$$

for all $\rho = a + \mathrm{i}b \in \Omega_1$ with $|\rho| \ge r'_2$.

Let δ be defined as above, so $0 < \delta \le \pi/4$ and $0 < \delta < (\ln 2)/(1 + n_0)$. For $k = 1, 2, \ldots$ define

$$\begin{cases} \alpha''_k := -(2\pi k - \operatorname{Arg} \mu_1 + \pi n_0), & \beta''_k := -n_0 \ln[-|\mu_1|^{1/n_0} \alpha''_k], \\ \mu''_k := \alpha''_k + \mathrm{i}\beta''_k, \end{cases}$$

and introduce the circles

$$\Gamma''_k := \{\rho \in \mathbb{C} \mid |\rho - \mu''_k| = \delta\}.$$

Choose an integer $k_1 \ge 2$ such that $y''_1 := \alpha''_{k_1} + \pi \le -r'_2$. Note that the α''_k satisfy $\alpha''_k + \pi \le -r'_2 \le -\beta_1$ and $|\alpha''_k| \ge 3\pi$ for $k = k_1, k_1 + 1, \ldots$. Also, introduce the *logarithmic rectangles*

$$\begin{aligned} R''_k := \big\{ \rho = a + \mathrm{i}b \in \mathbb{C} \bigm| \alpha''_k - \pi \le a \le \alpha''_k + \pi, \\ - n_0 \ln[-a/\alpha_1] \le b \le -n_0 \ln[-a/\beta_1] \big\} \end{aligned}$$

for $k = k_1, k_1 + 1, \ldots$. Without loss of generality we can assume that this new k_1 is identical to the k_1 introduced earlier, and that k_1 is sufficiently large to guarantee that each R''_k is contained in the sector S_1, and hence, for $k = k_1, k_1 + 1, \ldots$ the point μ''_k lies in the interior of the logarithmic rectangle R''_k with R''_k a subset of the logarithmic strip Ω_1.

Fix any index $k \geq k_1$, and take any point $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|\rho - \mu_k''| \leq \delta$. We assert that ρ lies in the interior of R_k''. Indeed, we clearly have $|a - \alpha_k''| \leq \delta < \pi$ and $|b + n_0 \ln[-|\mu_1|^{1/n_0}\alpha_k'']| \leq \delta$, so

$$\begin{aligned} |b + n_0 \ln[-|\mu_1|^{1/n_0} a]| &\leq |b + n_0 \ln[-|\mu_1|^{1/n_0}\alpha_k'']| \\ &\qquad + |n_0 \ln[-|\mu_1|^{1/n_0} a] - n_0 \ln[-|\mu_1|^{1/n_0}\alpha_k'']| \\ &\leq \delta + n_0|a - \alpha_k''| \leq \delta(1 + n_0) < \ln 2. \end{aligned}$$

Hence, $-n_0 \ln[-(2|\mu_1|)^{1/n_0} a] < b < -n_0 \ln[-(|\mu_1|/2)^{1/n_0} a]$, and from the definitions of α_1 and β_1 it follows that $-n_0 \ln[-a/\alpha_1] < b < -n_0 \ln[-a/\beta_1]$. We have established the assertion. It is immediate that the circle Γ_k'' lies in the interior of the logarithmic rectangle R_k'' for $k = k_1, k_1 + 1, \ldots$. To complete the setup of the geometry, for $k = k_1, k_1 + 1, \ldots$ let Ω_k'' be the *punctured logarithmic rectangle* formed from R_k'' by removing all the points inside Γ_k''.

Next, we establish the growth rate of Δ_1 on each of the regions Ω_k''. Observe that

$$\mathrm{e}^{\mathrm{i}\mu_k''} = \mu_1(\alpha_k'')^{n_0}$$

for $k = k_1, k_1+1, \ldots$. Introducing the analytic functions F_k, $k = k_1, k_1+1, \ldots$, and G_k, $k = k_1, k_1 + 1, \ldots$, defined by

$$F_k(\rho) := \mathrm{e}^{\mathrm{i}(\mu_k'' - \rho)} - 1 \quad \text{for } \rho \in \mathbb{C},$$

$$G_k(\rho) := -h_1(\rho) - \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{1}{\rho}\,(\mu_k'' - \rho) - \frac{\mathrm{i}\beta_k''}{\rho}\right]^j \left[1 + h_1(\rho)\right] \quad \text{for } \rho \in G_1,$$

we can use (8.56) to write Δ_1 in its final form:

$$\begin{aligned} \Delta_1(\rho) &= a_q'\rho^q \left\{\mathrm{e}^{-\mathrm{i}\rho} \cdot \frac{\mathrm{e}^{\mathrm{i}\mu_k''}}{\mu_1(\alpha_k'')^{n_0}} - \frac{1}{\mu_1\rho^{n_0}}[1 + h_1(\rho)]\right\} \\ &= \frac{a_q'\rho^q}{\mu_1(\alpha_k'')^{n_0}} \left\{\mathrm{e}^{\mathrm{i}(\mu_k'' - \rho)} - \left[\frac{\alpha_k''}{\rho}\right]^{n_0} [1 + h_1(\rho)]\right\} \\ &= \frac{a_q'\rho^q}{\mu_1(\alpha_k'')^{n_0}} \left\{\mathrm{e}^{\mathrm{i}(\mu_k'' - \rho)} - \left[\frac{1}{\rho}\,(\mu_k'' - \rho) - \frac{\mathrm{i}\beta_k''}{\rho} + 1\right]^{n_0} [1 + h_1(\rho)]\right\}, \end{aligned}$$

or

$$\Delta_1(\rho) = \frac{a_q'\rho^q}{\mu_1(\alpha_k'')^{n_0}}\,[F_k(\rho) + G_k(\rho)] \tag{8.58}$$

for $\rho \in G_1$ and for $k = k_1, k_1+1, \ldots$. Here we have a family of representations for Δ_1 which depends on the integer k. We use the kth representation to determine the growth rate of Δ_1 on the kth region Ω_k''.

In terms of the constant d_0 introduced earlier, we clearly have

$$n_0 \ln[2/(|\mu_1|^{1/n_0}\alpha_1)] \leq d_0, \qquad n_0 \ln[2|\mu_1|^{1/n_0}\beta_1] \leq d_0,$$

and $\delta < d_0$. As previously we can form the punctured rectangle

$$R_* = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid -\pi \le a \le \pi,\ -d_0 \le b \le d_0, \text{ and } |\rho| \ge \delta\},$$

and determine the constant $m_0 = \min\{\,|\mathrm{e}^{\mathrm{i}\rho} - 1| \mid \rho \in R_*\} > 0$.

Fix any index $k \ge k_1$, and take any point $\rho = a + \mathrm{i}b \in \Omega''_k$. Clearly $-\pi \le \alpha''_k - a \le \pi$, $\pi/|\alpha''_k| \le 1/2$ because $|\alpha''_k| \ge 3\pi$,

$$\frac{-a}{-\alpha''_k} \le \frac{-\alpha''_k + \pi}{-\alpha''_k} = 1 + \frac{\pi}{|\alpha''_k|} \le 2, \qquad \frac{-a}{-\alpha''_k} \ge \frac{-\alpha''_k - \pi}{-\alpha''_k} = 1 - \frac{\pi}{|\alpha''_k|} \ge \frac{1}{2},$$

and

$$\begin{aligned}\beta''_k - b &\le -n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k] + n_0 \ln[-a/\alpha_1]\\ &= n_0 \ln[-a/(|\mu_1|^{1/n_0}\alpha_1(-\alpha''_k))] \le n_0 \ln[2/(|\mu_1|^{1/n_0}\alpha_1)] \le d_0,\end{aligned}$$

$$\begin{aligned}\beta''_k - b &\ge -n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k] + n_0 \ln[-a/\beta_1]\\ &= n_0 \ln[-a/(|\mu_1|^{1/n_0}\beta_1(-\alpha''_k))]\\ &\ge n_0 \ln[1/(2|\mu_1|^{1/n_0}\beta_1)] \ge -d_0.\end{aligned}$$

Thus, the point $\mu''_k - \rho$ belongs to the punctured rectangle R_* with $|\mu''_k - \rho| \le \pi + d_0$. It follows that

$$|F_k(\rho)| \ge m_0 \tag{8.59}$$

for $k \ge k_1$ and for $\rho \in \Omega''_k$. Clearly the constant m_0 is independent of the index k.

On the other hand, for any index $k \ge k_1$ and for any point $\rho = a + \mathrm{i}b \in \Omega''_k$, we have $-\alpha''_k/2 = |\alpha''_k|/2 \ge \pi$, $-\alpha''_k - \pi \ge -\alpha''_k/2$, and

$$|\rho| \ge |a| = -a \ge -\alpha''_k - \pi \ge \frac{-\alpha''_k}{2} = \frac{|\alpha''_k|}{2}\,.$$

It follows that

$$|G_k(\rho)| \le \frac{\gamma'_1}{|\rho|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{2}{|\alpha''_k|}(\pi + d_0) + \frac{2|\beta''_k|}{|\alpha''_k|}\right]^j \left[1 + \frac{\gamma'_1}{r'_2}\right].$$

Since $\lim_{k\to\infty} 1/\alpha''_k = \lim_{k\to\infty} \beta''_k/\alpha''_k = 0$, in terms of the constant γ'_1 that appears in (8.57), we can select an integer $k_0 \ge k_1$ such that

$$\frac{\gamma'_1}{|\rho|} \le \frac{m_0}{4} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \ge -y''_0,$$

where $y''_0 := \alpha''_{k_0} + \pi \le -r'_2$, and such that

$$\sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{2}{|\alpha''_k|}(\pi + d_0) + \frac{2|\beta''_k|}{|\alpha''_k|}\right]^j \left[1 + \frac{\gamma'_1}{r'_2}\right] \le \frac{m_0}{4} \quad \text{for all } k \ge k_0.$$

Then for $k \geq k_0$ and for $\rho = a + \mathrm{i}b \in \Omega_k''$, we have

$$|\rho| \geq |a| \geq -\alpha_k'' - \pi \geq -y_0'' \geq r_2',$$

and hence, by (8.57), the definition of the integer k_0, and (8.59):

$$|G_k(\rho)| \leq \frac{m_0}{2} < m_0 \leq |F_k(\rho)|. \tag{8.60}$$

Also, for $k \geq k_0$ and for $\rho = a + \mathrm{i}b \in \Omega_k''$, we have $|\alpha_k''| \geq 3\pi$, so $\alpha_k'' \leq -2\pi$ and $\alpha_k'' + \pi \leq -\pi$, and hence, $a \leq \alpha_k'' + \pi \leq -\pi$ and

$$|\alpha_k''| = -\alpha_k'' \leq -a + \pi \leq -2a = 2|a| \leq 2|\rho|.$$

Therefore, from (8.58) and (8.60) we conclude that

$$|\Delta_1(\rho)| \geq \frac{|a_q'||\rho|^q}{|\mu_1||\alpha_k''|^{n_0}} \cdot \frac{m_0}{2} \geq \frac{m_0|a_q'|}{2|\mu_1|(2^{n_0})}\,|\rho|^p,$$

or

$$|\Delta_1(\rho)| \geq \frac{m_0|a_q'|}{2|\mu_1|(2^{n_0})}\,|\rho|^p > 0 \tag{8.61}$$

for $k \geq k_0$ and for $\rho \in \Omega_k''$. If we introduce the *punctured logarithmic strip*

$$\Omega_*'' := \bigcup_{k=k_0}^{\infty} \Omega_k'',$$

then we see that Ω_*'' consists of all points $\rho = a + \mathrm{i}b \in \Omega_1$ with $a \leq y_0''$ and with ρ not inside any of the circles Γ_k'' for $k \geq k_0$, and from the last equation

$$|\Delta_1(\rho)| \geq \frac{m_0|a_q'|}{2|\mu_1|(2^{n_0})}\,|\rho|^p > 0 \tag{8.62}$$

for all $\rho \in \Omega_*''$.

Consider one of the circles Γ_k'' for $k \geq k_0$. Since (8.60) is valid for each point ρ on Γ_k'', it follows by Rouché's Theorem that Δ_1 and $F_k + G_k$ have the same number of zeros as F_k inside Γ_k''. But μ_k'' is the only zero of F_k inside Γ_k'', μ_k'' being a zero of order 1. Consequently, the characteristic determinant Δ_1 has a unique zero ρ_k'' inside Γ_k'' with ρ_k'' having order 1 for $k \geq k_0$. Setting

$$\lambda_k'' := (\rho_k'')^n, \qquad k = k_0, k_0 + 1, \ldots,$$

the λ_k'' are eigenvalues of L with corresponding algebraic multiplicities and ascents

$$\nu(\lambda_k'') = m(\lambda_k'') = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.63}$$

To derive asymptotic formulas for the zeros ρ_k'', define $\zeta_k'' := -G_k(\rho_k'')$ for $k = k_0, k_0 + 1, \ldots$. Then $\mathrm{e}^{\mathrm{i}(\mu_k'' - \rho_k'')} = 1 + \zeta_k''$ and

$$\rho_k'' - \mu_k'' = \mathrm{i}\,\mathrm{Log}\,[1 + \zeta_k''] \tag{8.64}$$

for $k = k_0, k_0 + 1, \ldots$, where

$$|\zeta_k''| \le \frac{\gamma_1'}{|\rho_k''|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{2\delta}{|\alpha_k''|} + \frac{2|\beta_k''|}{|\alpha_k''|} \right]^j \left[1 + \frac{\gamma_1'}{r_2'} \right].$$

Now for each $k \ge k_0$, $|\alpha_k''| \ge 2\pi k - \pi \ge k$,

$$|\beta_k''| \le n_0 \ln[|\mu_1|^{1/n_0}(2\pi k + \pi + \pi n_0)] \le \gamma_2' \ln k,$$

and

$$|\rho_k''| \ge |\mu_k''| - |\rho_k'' - \mu_k''| \ge |\alpha_k''| - \delta \ge 6k - 5 \ge k,$$

which yields $|\zeta_k''| \le \gamma_3' \ln k / k$ and

$$|\rho_k'' - \mu_k''| \le \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.65}$$

Finally, we claim that L can have only a finite number of eigenvalues beyond the λ_k', λ_k'', $k = k_0, k_0 + 1, \ldots$. Indeed, suppose $\lambda_0 \ne 0$ is any eigenvalue of L distinct from the λ_k', λ_k''. Then λ_0 can be expressed in the form $\lambda_0 = (\rho_0)^n$ where the point $\rho_0 \ne 0$ belongs to either the sector S_0 or to the sector S_1. We look at these two cases separately.

First, assume $\rho_0 = a_0 + \mathrm{i}b_0 \in S_0$. If $|\rho_0| < r_2$, then only a finite number of such ρ_0 are possible because the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. Assume that $|\rho_0| \ge r_2$. Then by Theorem 8.3 we must have $\Delta_0(\rho_0) = 0$ and ρ_0 must lie in the interior of the logarithmic strip Ω_0. If $a_0 > y_0'$, then either $\rho_0 \in \Omega_*'$ or ρ_0 lies inside one of the circles Γ_k', $k \ge k_0$. But these two possibilities can not occur because of (8.51) and the fact that ρ_0 must be distinct from the ρ_k', $k \ge k_0$. Thus, $\alpha \le a_0 \le y_0'$ and $-n_0 \ln[y_0'/\alpha] \le b_0 \le 0$. Since these ρ_0 come from a bounded region in the ρ plane, only a finite number of such ρ_0 are possible.

Second, assume $\rho_0 = a_0 + \mathrm{i}b_0 \in S_1$. Again if $|\rho_0| < r_2$, then only a finite number of such ρ_0 are possible. Assume that $|\rho_0| \ge r_2$. Then by Theorem 8.3 we must have $\Delta_1(\rho_0) = 0$ and ρ_0 must lie in the interior of the logarithmic strip Ω_1. It is impossible to have $a_0 < y_0''$ because of (8.62) and the fact that ρ_0 must be distinct from the ρ_k'', $k \ge k_0$. Therefore, $y_0'' \le a_0 \le -\alpha_1$ and $-n_0 \ln[-y_0''/\alpha_1] \le b_0 \le 0$. Again only a finite number of such ρ_0 are possible.

Combining the two cases, we conclude that the λ_k', λ_k'', $k = k_0, k_0 + 1, \ldots$, account for all but a finite number of the eigenvalues of L.

We summarize the results for this logarithmic case as a theorem.

Theorem 8.4. *Let the differential operator L belong to Case 2, a logarithmic case, where the integers p and q satisfy the conditions $-\infty < p < q \le p_0$, and let $n_0 = q - p$, $\mu_0 = -b_q/a_p \ne 0$, and $\mu_1 = -a_q'/b_p' \ne 0$ (so $|\mu_1| = |\mu_0|$ and*

$\arg\mu_1 = \arg\mu_0 + 2\pi(n_0\nu + p)/n$). *Then the elements of the spectrum* $\sigma(L)$ *can be listed as two sequences*

$$\lambda'_k = (\rho'_k)^n, \quad k = k_0, k_0+1, \dots, \qquad \lambda''_k = (\rho''_k)^n, \quad k = k_0, k_0+1, \dots,$$

plus a finite number of additional points, where

$$\begin{aligned} \rho'_k &= (2\pi k + \operatorname{Arg}\mu_0) - \mathrm{i}n_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \operatorname{Arg}\mu_0)] + \epsilon'_k, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = k_0, k_0+1, \dots, \\ \rho''_k &= -(2\pi k - \operatorname{Arg}\mu_1 + \pi n_0) - \mathrm{i}n_0 \ln[|\mu_0|^{1/n_0}(2\pi k - \operatorname{Arg}\mu_1 + \pi n_0)] + \epsilon''_k, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = k_0, k_0+1, \dots, \end{aligned}$$

with $|\epsilon'_k| \le \gamma \ln k/k$ *and* $|\epsilon''_k| \le \gamma \ln k/k$ *for* $k = k_0, k_0+1, \dots$. *In addition, the corresponding algebraic multiplicities and ascents are*

$$\begin{aligned} \nu(\lambda'_k) = m(\lambda'_k) = 1, \quad k = k_0, k_0+1, \dots, \\ \nu(\lambda''_k) = m(\lambda''_k) = 1, \quad k = k_0, k_0+1, \dots. \end{aligned}$$

8.3 Case 3. $p > q$

Assume that $p > q$. The differential operators belonging to this case are always simply irregular. It is also a logarithmic case. As in Case 2 we will assume that the constant R_0 has been chosen sufficiently large to guarantee that $\operatorname{Re}\rho \ge 0$ for all $\rho \in G_0$ and $\operatorname{Re}\rho \le 0$ for all $\rho \in G_1$. Set $n_0 := p - q > 0$, and let us begin work in the sector T_0 and the open set G_0. By (5.105) or Theorem 5.4 and (8.1) we can write the characteristic determinant Δ_0 in the form

$$\Delta_0(\rho) = \rho^p \Big\{ a_p \mathrm{e}^{\mathrm{i}\rho}[1 + \phi_{01}(\rho)] + \frac{b_q}{\rho^{n_0}} \big[1 + \phi_{00}(\rho)\big] \Big\} \tag{8.66}$$

for $\rho \in G_0$, where

$$\begin{aligned} \phi_{01}(\rho) &:= \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{a_\kappa}{a_p \rho^{p-\kappa}} + \frac{1}{a_p \rho^p} \Phi_{01}(\rho), \\ \phi_{00}(\rho) &:= \sum_{\kappa=-(m-p_0-1)}^{q-1} \frac{b_\kappa}{b_q \rho^{q-\kappa}} + \frac{1}{b_q \rho^q} \Phi_{00}(\rho) \end{aligned}$$

for $\rho \in G_0$. The functions ϕ_{01}, ϕ_{00} are analytic on the open set G_0, and recalling that $m - p_0 + p \ge 1$ and $m - p_0 + q \ge 1$, by (5.106) we obtain the growth rates

$$|\phi_{01}(\rho)| \le \frac{\gamma_0}{|\rho|}, \qquad |\phi_{00}(\rho)| \le \frac{\gamma_0}{|\rho|} \tag{8.67}$$

for $\rho \in G_0$.

Set $\mu_0 := -a_p/b_q \neq 0$. Choose a constant $r_1 > R_1 \geq R_0$ such that

$$1/2 \leq |1 + \phi_{01}(\rho)| \leq 2, \qquad 1/2 \leq |1 + \phi_{00}(\rho)| \leq 2$$

for $\rho \in G_0$ with $|\rho| \geq r_1$ and such that

$$\frac{2}{|\mu_0||\rho|^{n_0}} \leq \frac{1}{4}$$

for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_1$. Then (8.66) can be rewritten in the form

$$\varDelta_0(\rho) = a_p \rho^p \mathrm{e}^{\mathrm{i}\rho} \Big\{ [1 + \phi_{01}(\rho)] + \frac{b_q}{a_p \rho^{n_0}} \, [1 + \phi_{00}(\rho)] \mathrm{e}^{-\mathrm{i}\rho} \Big\}$$

for $\rho \in G_0$, and for any point $\rho = a + \mathrm{i}b \in G_0$ with $|\rho| \geq r_1$ and $b \leq 0$, we have

$$\begin{aligned} |\varDelta_0(\rho)| &\geq |a_p||\rho|^p \mathrm{e}^{-b} \Big\{ \frac{1}{2} - \frac{2}{|\mu_0||\rho|^{n_0}} \, \mathrm{e}^{b} \Big\} \\ &\geq |a_p||\rho|^p \mathrm{e}^{-b} \Big\{ \frac{1}{2} - \frac{1}{4} \cdot 1 \Big\} = \frac{1}{4} \, |a_p||\rho|^p \mathrm{e}^{-b} > 0. \end{aligned} \tag{8.68}$$

Consequently, as we search for the zeros of $\varDelta_0$ in the open set G_0, we will concentrate our search in Quadrant I.

Let ω be the number defined by the equation $1/\omega := 1/|\mu_0|^{1/n_0} + n_0$. Choose real numbers α and β with $0 < \alpha < [1/(2|\mu_0|)]^{1/n_0}$, $\beta > [2/|\mu_0|]^{1/n_0}$, and

$$2|\mu_0|(2\alpha)^{n_0} \leq \frac{1}{4}, \qquad \frac{2}{|\mu_0|\beta^{n_0}} \leq \frac{1}{4}.$$

Clearly $1/\beta < |\mu_0|^{1/n_0} < 1/\alpha$. Note that if ρ is any point in S_0 with $|\rho| > R_0$, then ρ belongs to the open set G_0, and hence, we will be working in a region where $\varDelta_0$ is analytic. In terms of the constants μ_0, α, and β, we form the *logarithmic strip*

$$\begin{aligned} \varOmega_0 &:= \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| b \geq 0, \ \alpha \mathrm{e}^{b/n_0} \leq a \leq \beta \mathrm{e}^{b/n_0} \big\} \\ &= \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| a \geq \alpha, \ b \geq 0, \ n_0 \ln[a/\beta] \leq b \leq n_0 \ln[a/\alpha] \big\} \end{aligned}$$

and the two complementary regions

$$\varOmega_{0\infty} := \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| b \geq 0, \ a \leq \alpha \mathrm{e}^{b/n_0} \big\},$$

$$\varOmega_{0\Box} := \big\{ \rho = a + \mathrm{i}b \in S_0 \bigm| b \geq 0, \ \beta \mathrm{e}^{b/n_0} \leq a \big\}.$$

These three regions are contained in Quadrant I. Also, we have

$$n_0 \ln[a/\beta] < n_0 \ln[|\mu_0|^{1/n_0} a] < n_0 \ln[a/\alpha]$$

for $a \geq \beta$, and hence, the logarithmic curve $b = n_0 \ln[|\mu_0|^{1/n_0} a]$ runs down the 'middle' of the logarithmic strip Ω_0.

Let us begin by calculating the growth rate of Δ_0 on the region $\Omega_{0\infty}$. Take any point $\rho = a + \mathrm{i}b$ in $\Omega_{0\infty}$. Clearly $a \geq 0$, $b \geq 0$, and $a \leq \alpha \mathrm{e}^{b/n_0}$ and

$$|\rho| \leq |a| + |b| \leq \alpha \mathrm{e}^{|b|/n_0} + n_0 \mathrm{e}^{|b|/n_0} \leq [1/|\mu_0|^{1/n_0} + n_0] \mathrm{e}^{|b|/n_0} = \frac{1}{\omega}\, \mathrm{e}^{|b|/n_0},$$

or

$$n_0 \ln[\omega|\rho|] \leq |b| \quad \text{for all } \rho = a + \mathrm{i}b \in \Omega_{0\infty}. \tag{8.69}$$

Choose a real number $x_0 > 0$ such that $x \leq \alpha \mathrm{e}^{x/n_0}$ for all $x \in \mathbb{R}$ with $x \geq x_0$, and then choose a real number r_2 such that $r_2 \geq r_1$ and $r_2 \geq \beta$ and such that $x_0 \leq n_0 \ln[\omega|\rho|]$ for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_2$.

Now consider any point $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. Then by (8.69) we have $|b| \geq x_0$ and $|b| \leq \alpha \mathrm{e}^{|b|/n_0}$, and hence,

$$|\rho| \leq |a| + |b| \leq \alpha \mathrm{e}^{|b|/n_0} + \alpha \mathrm{e}^{|b|/n_0} = 2\alpha \mathrm{e}^{|b|/n_0}. \tag{8.70}$$

Combining (8.70) with (8.66), we have

$$\Delta_0(\rho) = b_q \rho^q \big\{[1 + \phi_{00}(\rho)] + (a_p/b_q)\rho^{n_0}[1 + \phi_{01}(\rho)]\mathrm{e}^{\mathrm{i}\rho}\big\}$$

and

$$\begin{aligned} |\Delta_0(\rho)| &\geq |b_q||\rho|^q \left\{\frac{1}{2} - 2|\mu_0||\rho|^{n_0}\mathrm{e}^{-b}\right\} \\ &\geq |b_q||\rho|^q \left\{\frac{1}{2} - 2|\mu_0| \cdot (2\alpha)^{n_0}\mathrm{e}^{b} \cdot \mathrm{e}^{-b}\right\} \geq \frac{1}{4}\,|b_q||\rho|^q, \end{aligned}$$

or

$$|\Delta_0(\rho)| \geq \frac{1}{4}\,|b_q||\rho|^q > 0 \tag{8.71}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. The inequality (8.71) establishes the growth rate for Δ_0 on the region $\Omega_{0\infty}$.

To determine the growth rate on $\Omega_{0\Box}$, we again express the characteristic determinant in the form

$$\Delta_0(\rho) = a_p \rho^p \mathrm{e}^{\mathrm{i}\rho}\Big\{[1 + \phi_{01}(\rho)] + \frac{b_q}{a_p \rho^{n_0}}\,[1 + \phi_{00}(\rho)]\mathrm{e}^{-\mathrm{i}\rho}\Big\}$$

for $\rho \in G_0$. Then for any point $\rho = a + \mathrm{i}b \in \Omega_{0\Box}$ we have $|\rho| \geq |a| \geq \beta \mathrm{e}^{|b|/n_0}$, and hence,

$$\begin{aligned} |\Delta_0(\rho)| &\geq |a_p||\rho|^p \mathrm{e}^{-b}\left\{\frac{1}{2} - \frac{2\mathrm{e}^{b}}{|\mu_0||\rho|^{n_0}}\right\} \\ &\geq |a_p||\rho|^p \mathrm{e}^{-b}\left\{\frac{1}{2} - \frac{2\mathrm{e}^{|b|}}{|\mu_0| \cdot \beta^{n_0}\mathrm{e}^{|b|}}\right\} \geq \frac{1}{4}\,|a_p||\rho|^p \mathrm{e}^{-b}, \end{aligned}$$

or

$$|\Delta_0(\rho)| \geq \frac{1}{4}\,|a_p||\rho|^p \mathrm{e}^{-b} \geq \frac{1}{4}\,|a_p|(\beta)^{n_0}|\rho|^q > 0 \tag{8.72}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{0\square}$. This is our growth rate for Δ_0 on the region $\Omega_{0\square}$.

Relative to the sector T_1 and the corresponding open set G_1, we can use (5.131) or Theorem 5.5 and (8.2) to write the characteristic determinant Δ_1 in the form

$$\Delta_1(\rho) = \rho^q \Big\{ a'_q \mathrm{e}^{-\mathrm{i}\rho}\big[1 + \phi_{11}(\rho)\big] + b'_p \rho^{n_0}\big[1 + \phi_{10}(\rho)\big]\Big\} \tag{8.73}$$

for $\rho \in G_1$, where

$$\phi_{11}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{q-1} \frac{a'_\kappa}{a'_q \rho^{q-\kappa}} + \frac{1}{a'_q \rho^q}\,\Phi_{11}(\rho),$$

$$\phi_{10}(\rho) := \sum_{\kappa=-(m-p_0-1)}^{p-1} \frac{b'_\kappa}{b'_p \rho^{p-\kappa}} + \frac{1}{b'_p \rho^p}\,\Phi_{10}(\rho)$$

for $\rho \in G_1$. The functions ϕ_{11}, ϕ_{10} are analytic on the open set G_1 with

$$|\phi_{11}(\rho)| \leq \frac{\gamma'_0}{|\rho|}, \qquad |\phi_{10}(\rho)| \leq \frac{\gamma'_0}{|\rho|} \tag{8.74}$$

for $\rho \in G_1$ (see equation (5.132)).

Set $\mu_1 := -b'_p/a'_q \neq 0$. From (5.133) we have $b'_p = \omega^p_{\nu-1} a_p$ and $a'_q = \omega^q_\nu b_q$, and hence,

$$\mu_1 = -\omega^p_{\nu-1} a_p/(\omega^q_\nu b_q) = \omega_{n_0\nu-p}\mu_0. \tag{8.75}$$

Clearly $|\mu_1| = |\mu_0|$ and $\arg\mu_1 = \arg\mu_0 + 2\pi(n_0\nu - p)/n$. Choose a constant $r'_1 > R_1 \geq R_0$ such that

$$1/2 \leq |1 + \phi_{11}(\rho)| \leq 2, \qquad 1/2 \leq |1 + \phi_{10}(\rho)| \leq 2$$

for $\rho \in G_1$ with $|\rho| \geq r'_1$ and such that

$$\frac{2}{|\mu_1||\rho|^{n_0}} \leq \frac{1}{4}$$

for all $\rho \in \mathbb{C}$ with $|\rho| \geq r'_1$. Then (8.73) can be rewritten in the form

$$\Delta_1(\rho) = b'_p \rho^p \Big\{ [1 + \phi_{10}(\rho)] + \frac{a'_q}{b'_p \rho^{n_0}}\,[1 + \phi_{11}(\rho)]\mathrm{e}^{-\mathrm{i}\rho}\Big\}$$

for $\rho \in G_1$, and for any point $\rho = a + \mathrm{i}b \in G_1$ with $|\rho| \geq r'_1$ and $b \leq 0$, we have

$$\begin{aligned} |\Delta_1(\rho)| &\geq |b'_p||\rho|^p \Big\{ \frac{1}{2} - \frac{2}{|\mu_1||\rho|^{n_0}}\,\mathrm{e}^b \Big\} \\ &\geq |b'_p||\rho|^p \Big\{ \frac{1}{2} - \frac{1}{4}\cdot 1 \Big\} = \frac{1}{4}\,|b'_p||\rho|^p > 0. \end{aligned} \tag{8.76}$$

In view of this result, our search for the zeros of the characteristic determinant $\mathit{\Delta}_1$ in the open set G_1 will be concentrated in Quadrant II.

Set $\omega_1 := \omega$, $\alpha_1 := \alpha$, and $\beta_1 := \beta$. Clearly $1/\omega_1 = 1/|\mu_1|^{1/n_0} + n_0$, $0 < \alpha_1 < [1/(2|\mu_1|)]^{1/n_0}$, $\beta_1 > [2/|\mu_1|]^{1/n_0}$, and

$$2|\mu_1|(2\alpha_1)^{n_0} \leq \frac{1}{4}\,, \qquad \frac{2}{|\mu_1|(\beta_1)^{n_0}} \leq \frac{1}{4}\,.$$

Note that for $\rho \in S_1$ with $|\rho| > R_0$, we have $\rho \in G_1$, and we are working in a region of analyticity for $\mathit{\Delta}_1$. In terms of the constants μ_1, α_1, and β_1, we form the *logarithmic strip*

$$\begin{aligned} \mathit{\Omega}_1 &:= \big\{\rho = a + \mathrm{i}b \in S_1 \bigm| b \geq 0,\ -\beta_1 \mathrm{e}^{b/n_0} \leq a \leq -\alpha_1 \mathrm{e}^{b/n_0}\big\} \\ &= \big\{\rho = a + \mathrm{i}b \in S_1 \bigm| a \leq -\alpha_1,\ b \geq 0,\ n_0 \ln[-a/\beta_1] \leq b \leq n_0 \ln[-a/\alpha_1]\big\} \end{aligned}$$

and the two complementary regions

$$\mathit{\Omega}_{1\infty} := \big\{\rho = a + \mathrm{i}b \in S_1 \bigm| b \geq 0,\ -\alpha_1 \mathrm{e}^{b/n_0} \leq a\big\},$$

$$\mathit{\Omega}_{1\square} := \big\{\rho = a + \mathrm{i}b \in S_1 \bigm| b \geq 0,\ a \leq -\beta_1 \mathrm{e}^{b/n_0}\big\}.$$

These three regions are contained in Quadrant II. Also, note that

$$n_0 \ln[-a/\beta_1] < n_0 \ln[-|\mu_1|^{1/n_0} a] < n_0 \ln[-a/\alpha_1]$$

for $a \leq -\beta_1$, so the logarithmic curve $b = n_0 \ln[-|\mu_1|^{1/n_0} a]$ runs down the 'middle' of the logarithmic strip $\mathit{\Omega}_1$.

Let us determine the growth rate of $\mathit{\Delta}_1$ on the regions $\mathit{\Omega}_{1\infty}$ and $\mathit{\Omega}_{1\square}$. For any point $\rho = a + ib$ in $\mathit{\Omega}_{1\infty}$ we have $a \leq 0$, $b \geq 0$, $|a| = -a \leq \alpha_1 \mathrm{e}^{b/n_0}$, and

$$\begin{aligned} |\rho| \leq |a| + |b| &\leq \alpha_1 \mathrm{e}^{|b|/n_0} + n_0 \mathrm{e}^{|b|/n_0} \\ &\leq [1/|\mu_1|^{1/n_0} + n_0] \mathrm{e}^{|b|/n_0} = \frac{1}{\omega_1}\, \mathrm{e}^{|b|/n_0}, \end{aligned}$$

and hence,

$$n_0 \ln[\omega_1 |\rho|] \leq |b| \quad \text{for all } \rho = a + \mathrm{i}b \in \mathit{\Omega}_{1\infty}. \tag{8.77}$$

For the constant $x_1 := x_0$, we know that $x \leq \alpha_1 \mathrm{e}^{x/n_0}$ for all $x \in \mathbb{R}$ with $x \geq x_1$. Choose a real number r_2' such that $r_2' \geq r_1'$ and $r_2' \geq \beta_1$ and such that $x_1 \leq n_0 \ln[\omega_1 |\rho|]$ for all $\rho \in \mathbb{C}$ with $|\rho| \geq r_2'$. Without loss of generality we can assume that $r_2 = r_2'$.

Now take any point $\rho = a + \mathrm{i}b \in \mathit{\Omega}_{1\infty}$ with $|\rho| \geq r_2'$. Then by (8.77) we have $|b| \geq x_1$, and hence, $|b| \leq \alpha_1 \mathrm{e}^{|b|/n_0}$ and

$$|\rho| \leq |a| + |b| \leq \alpha_1 \mathrm{e}^{|b|/n_0} + \alpha_1 \mathrm{e}^{|b|/n_0} = 2\alpha_1 \mathrm{e}^{|b|/n_0}. \tag{8.78}$$

Combining (8.78) with (8.73),

$$\mathit{\Delta}_1(\rho) = a_q' \rho^q \mathrm{e}^{-\mathrm{i}\rho} \big\{[1 + \phi_{11}(\rho)] + (b_p'/a_q') \rho^{n_0} [1 + \phi_{10}(\rho)] \mathrm{e}^{\mathrm{i}\rho}\big\}$$

and

$$\begin{aligned}|\Delta_1(\rho)| &\geq |a_q'||\rho|^q \mathrm{e}^b \left\{\frac{1}{2} - 2|\mu_1||\rho|^{n_0}\mathrm{e}^{-b}\right\} \\ &\geq |a_q'||\rho|^q \mathrm{e}^b \left\{\frac{1}{2} - 2|\mu_1| \cdot (2\alpha_1)^{n_0}\mathrm{e}^{b} \cdot \mathrm{e}^{-b}\right\} \geq \frac{1}{4}\,|a_q'||\rho|^q \mathrm{e}^b,\end{aligned}$$

or

$$|\Delta_1(\rho)| \geq \frac{1}{4}\,|a_q'||\rho|^q \mathrm{e}^b \geq \frac{|a_q'|}{4(2\alpha_1)^{n_0}}\,|\rho|^p > 0 \tag{8.79}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2'$. The inequality (8.79) establishes the growth rate for Δ_1 on the region $\Omega_{1\infty}$.

To determine the growth rate on $\Omega_{1\square}$, we again express the characteristic determinant in the form

$$\Delta_1(\rho) = b_p'\rho^p\left\{[1 + \phi_{10}(\rho)] + \frac{a_q'}{b_p'\rho^{n_0}}\,[1 + \phi_{11}(\rho)]\mathrm{e}^{-\mathrm{i}\rho}\right\}$$

for $\rho \in G_1$. Then for any point $\rho = a + \mathrm{i}b \in \Omega_{1\square}$ we have $|\rho| \geq |a| \geq \beta_1 \mathrm{e}^{b/n_0}$, and hence,

$$\begin{aligned}|\Delta_1(\rho)| &\geq |b_p'||\rho|^p \left\{\frac{1}{2} - \frac{2\mathrm{e}^b}{|\mu_1||\rho|^{n_0}}\right\} \\ &\geq |b_p'||\rho|^p \left\{\frac{1}{2} - \frac{2\mathrm{e}^b}{|\mu_1| \cdot (\beta_1)^{n_0}\mathrm{e}^b}\right\} \geq \frac{1}{4}\,|b_p'||\rho|^p,\end{aligned}$$

or

$$|\Delta_1(\rho)| \geq \frac{1}{4}\,|b_p'||\rho|^p > 0 \tag{8.80}$$

for all $\rho = a + \mathrm{i}b \in \Omega_{1\square}$. This is our growth rate for Δ_1 on the region $\Omega_{1\square}$.

These growth rates on the regions $\Omega_{0\infty}$, $\Omega_{0\square}$ and $\Omega_{1\infty}$, $\Omega_{1\square}$ combine to give the following *apriori estimates* for the eigenvalues of L. Recall that $r_2 = r_2'$.

Theorem 8.5. *Assume that $p > q$. Let $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and with $|\rho| \geq r_2$.*

(a) *If $\rho \in \Omega_{0\infty}$ or $\rho \in \Omega_{0\square}$, then $\Delta_0(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(b) *If λ is an eigenvalue of L, then $\Delta_0(\rho) = 0$ and ρ lies in the interior of the logarithmic strip Ω_0.*

In addition, let $\lambda = \rho^n \in \mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$ and with $|\rho| \geq r_2$.

(c) *If $\rho \in \Omega_{1\infty}$ or $\rho \in \Omega_{1\square}$, then $\Delta_1(\rho) \neq 0$ and $\lambda \in \rho(L)$.*

(d) *If λ is an eigenvalue of L, then $\Delta_1(\rho) = 0$ and ρ lies in the interior of the logarithmic strip Ω_1.*

The theorem shows that the resolvent set $\rho(L)$ is nonempty, and hence, the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$.

Next, let us compute the actual zeros of Δ_0 in the logarithmic strip Ω_0. Setting $\xi := 1 + n_0/\alpha$, for each point $\rho = a + \mathrm{i}b \in \Omega_0$ we have

$$|\rho| \leq |a| + |b| \leq |a| + n_0 \ln[|a|/\alpha] \leq |a| + (n_0/\alpha)|a| = \xi|a|. \tag{8.81}$$

Relative to the strip Ω_0 the characteristic determinant can be written in the form

$$\Delta_0(\rho) = a_p \rho^p \Big\{ \mathrm{e}^{\mathrm{i}\rho} - \frac{1}{\mu_0 \rho^{n_0}} \big[1 + h_0(\rho)\big] \Big\} \tag{8.82}$$

for $\rho \in G_0$, where h_0 is the analytic function given by

$$h_0(\rho) := -\mu_0 \rho^{n_0} \mathrm{e}^{\mathrm{i}\rho} \phi_{01}(\rho) + \phi_{00}(\rho)$$

for $\rho \in G_0$. If $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \geq r_2$, then

$$|\mu_0 \rho^{n_0} \mathrm{e}^{\mathrm{i}\rho}| = |\mu_0||\rho|^{n_0} \mathrm{e}^{-b} \leq |\mu_0| \cdot \xi^{n_0} |a|^{n_0} \cdot \frac{\beta^{n_0}}{|a|^{n_0}} = |\mu_0| \xi^{n_0} \beta^{n_0},$$

and hence, by (8.67)

$$|h_0(\rho)| \leq \frac{\gamma_1}{|\rho|} \tag{8.83}$$

for all $\rho = a + \mathrm{i}b \in \Omega_0$ with $|\rho| \geq r_2$.

Fix a real number δ with $0 < \delta \leq \pi/4$ and $0 < \delta < (\ln 2)/(1 + n_0)$, and then for the integers $k = 1, 2, \ldots$ define

$$\begin{cases} \alpha_k' := 2\pi k - \operatorname{Arg} \mu_0, & \beta_k' := n_0 \ln[|\mu_0|^{1/n_0} \alpha_k'], \\ \mu_k' := \alpha_k' + \mathrm{i}\beta_k', \end{cases}$$

and introduce the circles

$$\Gamma_k' := \{\rho \in \mathbb{C} \mid |\rho - \mu_k'| = \delta\}.$$

Choose a positive integer $k_1 \geq 2$ such that $y_1' := \alpha_{k_1}' - \pi \geq r_2$. Note that the α_k' satisfy $\alpha_k' - \pi \geq r_2 \geq \beta$ and $\alpha_k' \geq 3\pi$ for $k = k_1, k_1 + 1, \ldots$. Introduce the *logarithmic rectangles*

$$R_k' := \big\{\rho = a + \mathrm{i}b \in \mathbb{C} \;\big|\; \alpha_k' - \pi \leq a \leq \alpha_k' + \pi,\; n_0 \ln[a/\beta] \leq b \leq n_0 \ln[a/\alpha]\big\}$$

for $k = k_1, k_1 + 1, \ldots$. Without loss of generality we can assume that k_1 is sufficiently large to guarantee that each R_k' is contained in the sector S_0, and hence, for $k = k_1, k_1 + 1, \ldots$ the point μ_k' lies in the interior of R_k' with R_k' a subset of Ω_0.

Take any index $k \geq k_1$, and take any point $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|\rho - \mu_k'| \leq \delta$. We assert that ρ lies in the interior of R_k'. Indeed, we clearly have $|a - \alpha_k'| \leq \delta < \pi$ and $|b - n_0 \ln[|\mu_0|^{1/n_0} \alpha_k']| \leq \delta$, so

$$\begin{aligned} |b - n_0 \ln[|\mu_0|^{1/n_0} a]| &\leq |b - n_0 \ln[|\mu_0|^{1/n_0} \alpha_k']| \\ &\quad + |n_0 \ln[|\mu_0|^{1/n_0} a] - n_0 \ln[|\mu_0|^{1/n_0} \alpha_k']| \\ &\leq \delta + n_0 |a - \alpha_k'| \leq \delta(1 + n_0) < \ln 2. \end{aligned}$$

It follows that $n_0 \ln[(|\mu_0|/2)^{1/n_0} a] < b < n_0 \ln[(2|\mu_0|)^{1/n_0} a]$ and

$$n_0 \ln[a/\beta] < b < n_0 \ln[a/\alpha].$$

This establishes the assertion, and it is immediate that the circle Γ'_k lies in the interior of the logarithmic rectangle R'_k for $k = k_1, k_1+1, \ldots$. To complete the setup of the geometry, for $k = k_1, k_1+1, \ldots$ let Ω'_k be the *punctured logarithmic rectangle* formed from R'_k by removing all the points inside Γ'_k.

To establish the growth rate of Δ_0 on the region Ω'_k, first note that

$$\mathrm{e}^{\mathrm{i}\mu'_k} = \frac{1}{\mu_0 (\alpha'_k)^{n_0}}$$

for $k = k_1, k_1+1, \ldots$. Let f_k, $k = k_1, k_1+1, \ldots$, and g_k, $k = k_1, k_1+1, \ldots$, be the sequences of analytic functions defined by

$$f_k(\rho) := \mathrm{e}^{\mathrm{i}(\rho - \mu'_k)} - 1 \quad \text{for } \rho \in \mathbb{C},$$

$$g_k(\rho) := -h_0(\rho) - \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{1}{\rho}(\mu'_k - \rho) - \frac{\mathrm{i}\beta'_k}{\rho} \right]^j [1 + h_0(\rho)] \quad \text{for } \rho \in G_0.$$

Then we use (8.82) to write Δ_0 in its final form:

$$\begin{aligned}
\Delta_0(\rho) &= a_p \rho^p \Big\{ \mathrm{e}^{\mathrm{i}\rho} \cdot \frac{\mathrm{e}^{-\mathrm{i}\mu'_k}}{\mu_0 (\alpha'_k)^{n_0}} - \frac{1}{\mu_0 \rho^{n_0}} [1 + h_0(\rho)] \Big\} \\
&= \frac{a_p \rho^p}{\mu_0 (\alpha'_k)^{n_0}} \Big\{ \mathrm{e}^{\mathrm{i}(\rho - \mu'_k)} - \left[\frac{\alpha'_k}{\rho} \right]^{n_0} [1 + h_0(\rho)] \Big\} \\
&= \frac{a_p \rho^p}{\mu_0 (\alpha'_k)^{n_0}} \Big\{ \mathrm{e}^{\mathrm{i}(\rho - \mu'_k)} - \left[\frac{1}{\rho}(\mu'_k - \rho) - \frac{\mathrm{i}\beta'_k}{\rho} + 1 \right]^{n_0} [1 + h_0(\rho)] \Big\},
\end{aligned}$$

or

$$\Delta_0(\rho) = \frac{a_p \rho^p}{\mu_0 (\alpha'_k)^{n_0}} [f_k(\rho) + g_k(\rho)] \tag{8.84}$$

for $\rho \in G_0$ and for $k = k_1, k_1+1, \ldots$. We will use the kth representation in this family to compute the growth rate of Δ_0 on the kth region Ω'_k.

Choose a constant $d_0 > 0$ such that

$$n_0 \ln[2/(|\mu_0|^{1/n_0} \alpha)] \le d_0, \qquad n_0 \ln[2|\mu_0|^{1/n_0} \beta] \le d_0,$$

and $\delta < d_0$, and then form the punctured rectangle

$$R_* := \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid -\pi \le a \le \pi,\ -d_0 \le b \le d_0, \text{ and } |\rho| \ge \delta\}.$$

Set $m_0 := \min\{\, |\mathrm{e}^{\mathrm{i}\rho} - 1| \mid \rho \in R_* \} > 0$.

Fix any index $k \ge k_1$, and take any point $\rho = a + \mathrm{i}b \in \Omega'_k$. Clearly we have $-\pi \le a - \alpha'_k \le \pi$, $\pi/\alpha'_k \le 1/2$ because $\alpha'_k \ge 3\pi$,

$$\frac{a}{\alpha'_k} \le \frac{\alpha'_k + \pi}{\alpha'_k} \le 2, \qquad \frac{a}{\alpha'_k} \ge \frac{\alpha'_k - \pi}{\alpha'_k} \ge \frac{1}{2},$$

and

$$\begin{aligned} b - \beta'_k &\le n_0 \ln[a/\alpha] - n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k] \\ &= n_0 \ln[a/(|\mu_0|^{1/n_0} \alpha \alpha'_k)] \le n_0 \ln[2/(|\mu_0|^{1/n_0} \alpha)] \le d_0, \end{aligned}$$

$$\begin{aligned} b - \beta'_k &\ge n_0 \ln[a/\beta] - n_0 \ln[|\mu_0|^{1/n_0} \alpha'_k] \\ &= n_0 \ln[a/(|\mu_0|^{1/n_0} \beta \alpha'_k)] \ge n_0 \ln[1/(2|\mu_0|^{1/n_0} \beta)] \ge -d_0. \end{aligned}$$

Thus, the point $\rho - \mu'_k$ belongs to the punctured rectangle R_* with $|\rho - \mu'_k| \le \pi + d_0$. It follows that

$$|f_k(\rho)| \ge m_0 \tag{8.85}$$

for $k \ge k_1$ and for $\rho \in \Omega'_k$. Note that the constant m_0 is independent of the index k.

On the other hand, for any index $k \ge k_1$ and for any point $\rho = a + \mathrm{i}b \in \Omega'_k$, we have $\alpha'_k/2 = |\alpha'_k|/2 \ge \pi$, $\alpha'_k - \pi \ge \alpha'_k/2$, and

$$|\rho| \ge |a| = a \ge \alpha'_k - \pi \ge \frac{\alpha'_k}{2} = \frac{|\alpha'_k|}{2}\,.$$

It follows that

$$|g_k(\rho)| \le \frac{\gamma_1}{|\rho|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{2}{|\alpha'_k|} (\pi + d_0) + \frac{2|\beta'_k|}{|\alpha'_k|} \right]^j \left[1 + \frac{\gamma_1}{r_2} \right].$$

Since $\lim_{k\to\infty} 1/\alpha'_k = \lim_{k\to\infty} \beta'_k/\alpha'_k = 0$, in terms of the constant γ_1 that appears in (8.83), we can select an integer $k_0 \ge k_1$ such that

$$\frac{\gamma_1}{|\rho|} \le \frac{m_0}{4} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \ge y'_0,$$

where $y'_0 := \alpha'_{k_0} - \pi \ge r_2$, and such that

$$\sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{2}{|\alpha'_k|} (\pi + d_0) + \frac{2|\beta'_k|}{|\alpha'_k|} \right]^j \left[1 + \frac{\gamma_1}{r_2} \right] \le \frac{m_0}{4} \quad \text{for all } k \ge k_0.$$

Then for $k \ge k_0$ and for $\rho = a + \mathrm{i}b \in \Omega'_k$, we have $|\rho| \ge a \ge \alpha'_k - \pi \ge y'_0 \ge r_2$, and hence, by (8.83), the definition of the integer k_0, and (8.85):

$$|g_k(\rho)| \le \frac{m_0}{2} < m_0 \le |f_k(\rho)|. \tag{8.86}$$

Also, for $k \ge k_0$ and for $\rho = a + \mathrm{i}b \in \Omega'_k$, we have $\alpha'_k \ge 3\pi$, so $\alpha'_k - \pi \ge \pi$, and hence, $a \ge \alpha'_k - \pi \ge \pi$ and

$$|\alpha_k'| = \alpha_k' \le a + \pi \le 2a = 2|a| \le 2|\rho|.$$

From (8.84) and (8.86) we conclude that

$$|\varDelta_0(\rho)| \ge \frac{|a_p||\rho|^p}{|\mu_0||\alpha_k'|^{n_0}} \cdot \frac{m_0}{2} \ge \frac{m_0|a_p|}{2|\mu_0|(2^{n_0})}\,|\rho|^q,$$

or

$$|\varDelta_0(\rho)| \ge \frac{m_0|a_p|}{2|\mu_0|(2^{n_0})}\,|\rho|^q > 0 \tag{8.87}$$

for $k \ge k_0$ and for $\rho \in \varOmega_k'$. Introducing the *punctured logarithmic strip*

$$\varOmega_*' = \bigcup_{k=k_0}^{\infty} \varOmega_k',$$

we see that $\varOmega_*'$ consists of all points $\rho = a + \mathrm{i}b \in \varOmega_0$ with $a \ge y_0'$ which do not lie inside any of the circles $\varGamma_k'$ for $k \ge k_0$, and from the above

$$|\varDelta_0(\rho)| \ge \frac{m_0|a_p|}{2|\mu_0|(2^{n_0})}\,|\rho|^q > 0 \tag{8.88}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_*'$.

Now consider one of the circles $\varGamma_k'$ for $k \ge k_0$. The inequality (8.86) is valid for each point ρ on $\varGamma_k'$, and hence, by Rouché's Theorem $\varDelta_0$ and $f_k + g_k$ have the same number of zeros as f_k inside $\varGamma_k'$. But the only zero of f_k inside the circle $\varGamma_k'$ occurs at the center μ_k', with μ_k' being a zero of order 1. It follows that the characteristic determinant $\varDelta_0$ has a unique zero ρ_k' inside $\varGamma_k'$, and ρ_k' is a zero of order 1 for $k \ge k_0$. If we set

$$\lambda_k' := (\rho_k')^n, \qquad k = k_0, k_0 + 1, \ldots,$$

then the λ_k' are eigenvalues of L with algebraic multiplicities and ascents

$$\nu(\lambda_k') = m(\lambda_k') = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.89}$$

To establish asymptotic formulas, set $\zeta_k' = -g_k(\rho_k')$ for $k = k_0, k_0 + 1, \ldots$. Then $\mathrm{e}^{\mathrm{i}(\rho_k' - \mu_k')} = 1 + \zeta_k'$ and

$$\rho_k' - \mu_k' = -\mathrm{i}\,\mathrm{Log}\,[1 + \zeta_k'] \tag{8.90}$$

for $k = k_0, k_0 + 1, \ldots$, where

$$|\zeta_k'| \le \frac{\gamma_1}{|\rho_k'|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{2\delta}{|\alpha_k'|} + \frac{2|\beta_k'|}{|\alpha_k'|}\right]^j \left[1 + \frac{\gamma_1}{r_2}\right].$$

Now for each $k \ge k_0$, $\alpha_k' \ge 2\pi k - \pi \ge k$, $\beta_k' \le n_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \pi)] \le \gamma_2 \ln k$, and

$$|\rho'_k| \geq |\mu'_k| - |\rho'_k - \mu'_k| \geq \alpha'_k - \delta \geq 6k - 5 \geq k,$$

which gives $|\zeta'_k| \leq \gamma_3 \ln k/k$ and

$$|\rho'_k - \mu'_k| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0 + 1, \dots . \tag{8.91}$$

The calculation of the zeros of $\varDelta_1$ in the logarithmic strip $\varOmega_1$ is similar to the above. Indeed, setting $\eta := 1 + n_0/\alpha_1$, for each point $\rho = a + \mathrm{i}b \in \varOmega_1$ we have $|b| \leq n_0 \ln[|a|/\alpha_1]$ and

$$|\rho| \leq |a| + n_0 \ln[|a|/\alpha_1] \leq |a| + (n_0/\alpha_1)|a| = \eta|a|. \tag{8.92}$$

Relative to $\varOmega_1$ we can express $\varDelta_1$ in the form

$$\varDelta_1(\rho) = a'_q \rho^q \bigl\{ \mathrm{e}^{-\mathrm{i}\rho} - \mu_1 \rho^{n_0} [1 + h_1(\rho)] \bigr\} \tag{8.93}$$

for $\rho \in G_1$, where h_1 is the analytic function given by

$$h_1(\rho) := -\frac{\mathrm{e}^{-\mathrm{i}\rho}}{\mu_1 \rho^{n_0}} \, \phi_{11}(\rho) + \phi_{10}(\rho)$$

for $\rho \in G_1$. Note that for each point $\rho = a + \mathrm{i}b \in \varOmega_1$ with $|\rho| \geq r'_2$, we have

$$\left| \frac{\mathrm{e}^{-\mathrm{i}\rho}}{\mu_1 \rho^{n_0}} \right| = \frac{\mathrm{e}^b}{|\mu_1||\rho|^{n_0}} \leq \frac{1}{|\mu_1||\rho|^{n_0}} \cdot \frac{|a|^{n_0}}{(\alpha_1)^{n_0}} \leq \frac{1}{|\mu_1|(\alpha_1)^{n_0}},$$

and hence, from (8.74)

$$|h_1(\rho)| \leq \frac{\gamma'_1}{|\rho|} \tag{8.94}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_1$ with $|\rho| \geq r'_2$.

Let δ be defined as above, so $0 < \delta \leq \pi/4$ and $0 < \delta < (\ln 2)/(1 + n_0)$. For $k = 1, 2, \dots$ set

$$\begin{cases} \alpha''_k := -(2\pi k + \operatorname{Arg} \mu_1 + \pi n_0), \qquad \beta''_k := n_0 \ln[-|\mu_1|^{1/n_0} \alpha''_k], \\ \mu''_k := \alpha''_k + \mathrm{i}\beta''_k, \end{cases}$$

and then form the circles

$$\varGamma''_k := \{\rho \in \mathbb{C} \mid |\rho - \mu''_k| = \delta\}.$$

Select an integer $k_1 \geq 2$ with $y''_1 := \alpha''_{k_1} + \pi \leq -r'_2$. Clearly $\alpha''_k + \pi \leq -r'_2 \leq -\beta_1$ and $|\alpha''_k| \geq 3\pi$ for $k = k_1, k_1 + 1, \dots$. Also, let us form the *logarithmic rectangles*

$$R''_k := \bigl\{ \rho = a + \mathrm{i}b \in \mathbb{C} \bigm| \alpha''_k - \pi \leq a \leq \alpha''_k + \pi, \ n_0 \ln[-a/\beta_1] \leq b \leq n_0 \ln[-a/\alpha_1] \bigr\}$$

for $k = k_1, k_1 + 1, \dots$. We can assume that this new index k_1 is identical to the k_1 introduced earlier, and that k_1 is sufficiently large to guarantee that each R''_k is contained in the sector S_1. Consequently, for $k = k_1, k_1 + 1, \ldots$ the point μ''_k lies in the interior of the logarithmic rectangle R''_k, and R''_k is a subset of the logarithmic strip Ω_1.

Take any index $k \geq k_1$ and any point $\rho = a + \mathrm{i}b \in \mathbb{C}$ with $|\rho - \mu''_k| \leq \delta$. Then $|a - \alpha''_k| \leq \delta < \pi$ and $|b - n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k]| \leq \delta$, and hence,

$$\begin{aligned}|b - n_0 \ln[-|\mu_1|^{1/n_0} a]| &\leq |b - n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k]| \\ &\quad + |n_0 \ln[-|\mu_1|^{1/n_0} a] - n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k]| \\ &\leq \delta + n_0 |a - \alpha''_k| \leq \delta(1 + n_0) < \ln 2.\end{aligned}$$

Thus, $n_0 \ln[-(|\mu_1|/2)^{1/n_0} a] < b < n_0 \ln[-(2|\mu_1|)^{1/n_0} a]$, and from the definitions of α_1 and β_1 we conclude that $n_0 \ln[-a/\beta_1] < b < n_0 \ln[-a/\alpha_1]$. This shows that the point ρ lies in the interior of R''_k. It is immediate that the circle Γ''_k lies in the interior of the logarithmic rectangle R''_k. For $k = k_1, k_1 + 1, \ldots$ let Ω''_k be the *punctured logarithmic rectangle* formed from R''_k by removing all the points inside Γ''_k.

Next, let us calculate the growth rate of Δ_1 on each of the regions Ω''_k. From the definition of μ''_k it follows that

$$\mathrm{e}^{\mathrm{i}\mu''_k} = \frac{1}{\mu_1(\alpha''_k)^{n_0}} \quad \text{for } k = k_1, k_1 + 1, \ldots.$$

Let F_k, $k = k_1, k_1 + 1, \ldots$, and G_k, $k = k_1, k_1 + 1, \ldots$, be the sequences of analytic functions defined by

$$F_k(\rho) := \mathrm{e}^{\mathrm{i}(\mu''_k - \rho)} - 1 \quad \text{for } \rho \in \mathbb{C},$$

$$G_k(\rho) := -h_1(\rho) - \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{1}{\alpha''_k}(\rho - \mu''_k) + \frac{\mathrm{i}\beta''_k}{\alpha''_k}\right]^j \left[1 + h_1(\rho)\right] \quad \text{for } \rho \in G_1.$$

Then (8.93) can be used to express Δ_1 in its final form:

$$\begin{aligned}\Delta_1(\rho) &= a'_q \mu_1 (\alpha''_k)^{n_0} \rho^q \Big\{\mathrm{e}^{-\mathrm{i}\rho} \cdot \mathrm{e}^{\mathrm{i}\mu''_k} - \left[\frac{\rho}{\alpha''_k}\right]^{n_0} \left[1 + h_1(\rho)\right]\Big\} \\ &= a'_q \mu_1 (\alpha''_k)^{n_0} \rho^q \Big\{\mathrm{e}^{\mathrm{i}(\mu''_k - \rho)} - \left[\frac{1}{\alpha''_k}(\rho - \mu''_k) + \frac{\mathrm{i}\beta''_k}{\alpha''_k} + 1\right]^{n_0} \left[1 + h_1(\rho)\right]\Big\},\end{aligned}$$

or

$$\Delta_1(\rho) = a'_q \mu_1 (\alpha''_k)^{n_0} \rho^q [F_k(\rho) + G_k(\rho)] \tag{8.95}$$

for all $\rho \in G_1$ and for $k = k_1, k_1 + 1, \ldots$. We will use (8.95) to calculate the growth rate of Δ_1 on the region Ω''_k.

In terms of the constant d_0 introduced earlier, we clearly have

$$n_0 \ln[2/(|\mu_1|^{1/n_0}\alpha_1)] \leq d_0, \qquad n_0 \ln[2|\mu_1|^{1/n_0}\beta_1] \leq d_0,$$

and $\delta < d_0$. As above we can form the punctured rectangle

$$R_* = \{\rho = a + \mathrm{i}b \in \mathbb{C} \mid -\pi \le a \le \pi,\ -d_0 \le b \le d_0,\ \text{and } |\rho| \ge \delta\},$$

and determine the constant $m_0 = \min\{\,|\mathrm{e}^{\mathrm{i}\rho} - 1| \mid \rho \in R_*\} > 0$.

Take any index $k \ge k_1$, and take any point $\rho = a + \mathrm{i}b \in \Omega''_k$. Then the point $\mu''_k - \rho$ satisfies the following conditions: $-\pi \le \alpha''_k - a \le \pi$, $\pi/|\alpha''_k| \le 1/2$ because $|\alpha''_k| \ge 3\pi$,

$$\frac{-a}{-\alpha''_k} \le \frac{-\alpha''_k + \pi}{-\alpha''_k} = 1 + \frac{\pi}{|\alpha''_k|} \le 2, \qquad \frac{-a}{-\alpha''_k} \ge \frac{-\alpha''_k - \pi}{-\alpha''_k} = 1 - \frac{\pi}{|\alpha''_k|} \ge \frac{1}{2},$$

and

$$\begin{aligned}\beta''_k - b &\le n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k] - n_0 \ln[-a/\beta_1]\\ &= -n_0 \ln[-a/(|\mu_1|^{1/n_0}\beta_1(-\alpha''_k))] \le -n_0 \ln[1/(2|\mu_1|^{1/n_0}\beta_1)] \le d_0,\end{aligned}$$

$$\begin{aligned}\beta''_k - b &\ge n_0 \ln[-|\mu_1|^{1/n_0}\alpha''_k] - n_0 \ln[-a/\alpha_1]\\ &= -n_0 \ln[-a/(|\mu_1|^{1/n_0}\alpha_1(-\alpha''_k))] \ge -n_0 \ln[2/(|\mu_1|^{1/n_0}\alpha_1)] \ge -d_0.\end{aligned}$$

Thus, the point $\mu''_k - \rho$ belongs to the punctured rectangle R_* with $|\mu''_k - \rho| \le \pi + d_0$. We conclude that

$$|F_k(\rho)| \ge m_0 \tag{8.96}$$

for $k \ge k_1$ and for $\rho \in \Omega''_k$.

Clearly $\lim_{k\to\infty} 1/\alpha''_k = \lim_{k\to\infty} \beta''_k/\alpha''_k = 0$. Choose an integer $k_0 \ge k_1$ such that

$$\frac{\gamma'_1}{|\rho|} \le \frac{m_0}{4} \quad \text{for all } \rho \in \mathbb{C} \text{ with } |\rho| \ge -y''_0,$$

where $y''_0 := \alpha''_{k_0} + \pi \le -r'_2$, and such that

$$\sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{\pi + d_0}{|\alpha''_k|} + \frac{|\beta''_k|}{|\alpha''_k|}\right]^j \left[1 + \frac{\gamma'_1}{r'_2}\right] \le \frac{m_0}{4} \quad \text{for all } k \ge k_0.$$

Then for $k \ge k_0$ and for $\rho = a + \mathrm{i}b \in \Omega''_k$, we have $|\rho| \ge |a| \ge -\alpha''_k - \pi \ge -y''_0 \ge r'_2$, and by (8.94), the definitions of G_k and k_0, and (8.96):

$$|G_k(\rho)| \le \frac{m_0}{2} < m_0 \le |F_k(\rho)|. \tag{8.97}$$

Also, since $|\alpha''_k| \ge 3\pi$, we have $-a \ge -\alpha''_k - \pi \ge 2\pi$, $-a/2 \ge \pi$, and

$$|\alpha''_k| = -\alpha''_k \ge -a - \pi \ge -a/2 = |a|/2 \ge |\rho|/(2\eta).$$

From (8.95) we conclude that

$$|\Delta_1(\rho)| \ge |a'_q||\mu_1||\alpha''_k|^{n_0}|\rho|^q \cdot \frac{m_0}{2} \ge \frac{m_0|a'_q||\mu_1|}{2(2\eta)^{n_0}}\,|\rho|^p,$$

or

$$|\Delta_1(\rho)| \geq \frac{m_0 |a_q'||\mu_1|}{2(2\eta)^{n_0}} |\rho|^p > 0 \tag{8.98}$$

for $k \geq k_0$ and for $\rho \in \Omega_k''$. If we introduce the *punctured logarithmic strip*

$$\Omega_*'' = \bigcup_{k=k_0}^{\infty} \Omega_k'',$$

then Ω_*'' consists of all points $\rho = a + \mathrm{i}b \in \Omega_1$ with $a \leq y_0''$ and with ρ not inside any of the circles Γ_k'' for $k \geq k_0$, and from (8.98)

$$|\Delta_1(\rho)| \geq \frac{m_0 |a_q'||\mu_1|}{2(2\eta)^{n_0}} |\rho|^p > 0 \tag{8.99}$$

for all $\rho \in \Omega_*''$.

Let us examine one of the circles Γ_k'' for $k \geq k_0$. Since (8.97) is valid for each point ρ on Γ_k'', by Rouché's Theorem Δ_1 and $F_k + G_k$ must have the same number of zeros as F_k inside Γ_k''. Clearly μ_k'' is the only zero of F_k inside Γ_k'', μ_k'' being a zero of order 1. Therefore, Δ_1 has a unique zero ρ_k'' of order 1 inside the circle Γ_k'' for $k \geq k_0$. Corresponding to these zeros, the complex numbers

$$\lambda_k'' := (\rho_k'')^n, \qquad k = k_0, k_0 + 1, \ldots,$$

are eigenvalues of L with algebraic multiplicities and ascents

$$\nu(\lambda_k'') = m(\lambda_k'') = 1, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.100}$$

For the asymptotic formulas, set $\zeta_k'' = -G_k(\rho_k'')$ for $k = k_0, k_0 + 1, \ldots$. Then we know that $\mathrm{e}^{-\mathrm{i}(\rho_k'' - \mu_k'')} = 1 + \zeta_k''$ and

$$\rho_k'' - \mu_k'' = \mathrm{i}\,\mathrm{Log}\,[1 + \zeta_k''] \tag{8.101}$$

for $k = k_0, k_0 + 1, \ldots$, where

$$|\zeta_k''| \leq \frac{\gamma_1'}{|\rho_k''|} + \sum_{j=1}^{n_0} \binom{n_0}{j} \left[\frac{\delta}{|\alpha_k''|} + \frac{|\beta_k''|}{|\alpha_k''|}\right]^j \left[1 + \frac{\gamma_1'}{r_2'}\right].$$

Also, we have $|\alpha_k''| \geq 2\pi k - \pi \geq k$, $|\beta_k''| = \beta_k'' \leq n_0 \ln[|\mu_1|^{1/n_0}(2\pi k + \pi + \pi n_0)] \leq \gamma_2' \ln k$, and $|\rho_k''| \geq |\mu_k''| - |\rho_k'' - \mu_k''| \geq |\alpha_k''| - \delta \geq 6k - 5 \geq k$ for each $k \geq k_0$, which leads to the estimates $|\zeta_k''| \leq \gamma_3' \ln k / k$ and

$$|\rho_k'' - \mu_k''| \leq \frac{\gamma \ln k}{k}, \qquad k = k_0, k_0 + 1, \ldots. \tag{8.102}$$

Finally, we assert that L has only a finite number of eigenvalues other than the λ_k', λ_k'', $k = k_0, k_0 + 1, \ldots$. Indeed, suppose λ_0 is a nonzero eigenvalue of

L which is distinct from the λ'_k, λ''_k. Then λ_0 can be expressed in the form $\lambda_0 = (\rho_0)^n$, where the point $\rho_0 = a_0 + ib_0$ belongs to either the sector S_0 or to the sector S_1. We look at these two cases separately.

First, consider the case when ρ_0 belongs to S_0. We know that either ρ_0 lies in the disk $|\rho| < r_2$, or $|\rho_0| \geq r_2$. In case $|\rho_0| < r_2$, only a finite number of such ρ_0 are possible because the spectrum $\sigma(L)$ is a countable set having no limit points in $\mathbb{C}$. Assume that $|\rho_0| \geq r_2$. Then by Theorem 8.5 we must have $\Delta_0(\rho_0) = 0$ and ρ_0 must lie in the interior of the logarithmic strip Ω_0. If $a_0 > y'_0$, then either $\rho_0 \in \Omega'_*$ or ρ_0 lies inside one of the circles Γ'_k, $k \geq k_0$. But these two possibilities can not occur because of (8.88) and the fact that ρ_0 must be distinct from the ρ'_k, $k \geq k_0$. Thus, $\alpha \leq a_0 \leq y'_0$ and $0 \leq b_0 \leq n_0 \ln[y'_0/\alpha]$. Since these ρ_0 come from a bounded region in the ρ plane, only a finite number of such ρ_0 are possible.

Second, assume that ρ_0 belongs to S_1. If $|\rho_0| < r_2$, then only a finite number of such ρ_0 are possible. Assume that $|\rho_0| \geq r_2$. Again by Theorem 8.5 we must have $\Delta_1(\rho_0) = 0$ and ρ_0 must lie in the interior of the logarithmic strip Ω_1. It is impossible to have $a_0 < y''_0$ because of (8.99) and the fact that ρ_0 must be distinct from the ρ''_k, $k \geq k_0$. Therefore, $y''_0 \leq a_0 \leq -\alpha_1$ and $0 \leq b_0 \leq n_0 \ln[-y''_0/\alpha_1]$. Again only a finite number of such ρ_0 are possible.

We conclude that the λ'_k, λ''_k, $k = k_0, k_0 + 1, \ldots$, account for all but a finite number of the eigenvalues of L.

The results for this logarithmic case are summarized below in a theorem.

Theorem 8.6. *Let the differential operator L belong to Case 3, a logarithmic case, where the integers p and q satisfy the conditions $-\infty < q < p \leq p_0$, and let $n_0 = p - q$, $\mu_0 = -a_p/b_q \neq 0$, and $\mu_1 = -b'_p/a'_q \neq 0$ (so $|\mu_1| = |\mu_0|$ and $\arg \mu_1 = \arg \mu_0 + 2\pi(n_0\nu - p)/n$). Then the elements of the spectrum $\sigma(L)$ can be listed as two sequences*

$$\lambda'_k = (\rho'_k)^n, \quad k = k_0, k_0 + 1, \ldots, \qquad \lambda''_k = (\rho''_k)^n, \quad k = k_0, k_0 + 1, \ldots,$$

plus a finite number of additional points, where

$$\begin{aligned} \rho'_k &= (2\pi k - \operatorname{Arg} \mu_0) + in_0 \ln[|\mu_0|^{1/n_0}(2\pi k - \operatorname{Arg} \mu_0)] + \epsilon'_k, \\ &\qquad\qquad\qquad\qquad k = k_0, k_0 + 1, \ldots, \\ \rho''_k &= -(2\pi k + \operatorname{Arg} \mu_1 + \pi n_0) + in_0 \ln[|\mu_0|^{1/n_0}(2\pi k + \operatorname{Arg} \mu_1 + \pi n_0)] + \epsilon''_k, \\ &\qquad\qquad\qquad\qquad k = k_0, k_0 + 1, \ldots, \end{aligned}$$

with $|\epsilon'_k| \leq \gamma \ln k/k$ and $|\epsilon''_k| \leq \gamma \ln k/k$ for $k = k_0, k_0 + 1, \ldots$. In addition, the corresponding algebraic multiplicities and ascents are

$$\nu(\lambda'_k) = m(\lambda'_k) = 1, \quad k = k_0, k_0 + 1, \ldots,$$

$$\nu(\lambda''_k) = m(\lambda''_k) = 1, \quad k = k_0, k_0 + 1, \ldots.$$

9

Completeness of the Generalized Eigenfunctions

To demonstrate the completeness of the generalized eigenfunctions of L, we will determine the growth rates of the Green's function $G(t, s; \lambda)$ and the resolvent $R_\lambda(L)$ along various rays in the λ plane, and then appeal to Theorem 6.2 in Chapter 2 of [34].

From the results of Chapters 7 and 8, we know that the resolvent set $\rho(L)$ is nonempty, and hence, the differential operator L is a Hilbert-Schmidt discrete linear operator in $L^2[0,1]$. The spectrum $\sigma(L)$ is a countable set having no finite limit points in $\mathbb{C}$, and in Chapters 7 and 8 we have given a detailed description of $\sigma(L)$. Let

$$\sigma(L) = \{\lambda_i\}_{i=1}^{\infty}$$

be any enumeration of $\sigma(L)$, let m_i $(0 < m_i < \infty)$ denote the ascent of the operator $\lambda_i I - L$ for $i = 1, 2, \ldots$, and let P_i, $i = 1, 2, \ldots$, denote the projection of $L^2[0,1]$ onto the generalized eigenspace $\mathcal{N}((\lambda_i I - L)^{m_i})$ along the range $\mathcal{R}((\lambda_i I - L)^{m_i})$.

Let $\operatorname{sp}(L)$ denote the subspace of $L^2[0,1]$ spanned by the generalized eigenfunctions of L, and let us introduce the subspaces

$$S_\infty(L) := \left\{ u \in L^2[0,1] \;\middle|\; u = \sum_{i=1}^{\infty} P_i u \right\}$$

and

$$M_\infty(L) := \bigl\{ u \in L^2[0,1] \mid P_i u = 0 \text{ for } i = 1, 2, \ldots \bigr\}.$$

Clearly $M_\infty(L)$ is closed, $\operatorname{sp}(L)$ is a subset of $S_\infty(L)$, and it is easy to check that $\overline{\operatorname{sp}(L)} = \overline{S_\infty(L)}$. Our goal in this chapter is to prove that

$$\overline{\operatorname{sp}(L)} = \overline{S_\infty(L)} = L^2[0,1] \quad \text{and} \quad M_\infty(L) = \{0\}.$$

9.1 Completeness for n Even

Assume that n is even, $n = 2\nu \geq 2$. In the representations of the characteristic determinants $\mathit{\Delta}_0$ and $\mathit{\Delta}_1$, the functions π_i, $i = 0, 1, 2$, are given by the formulas

$$\pi_2(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \quad \pi_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa, \quad \pi_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} c_\kappa \rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. See equation (7.1). In these expressions the integers p and q have been determined with $-\infty < p \leq q \leq p_0$, and the leading coefficients satisfy $a_p \neq 0$ and $c_p \neq 0$. The integer m has been fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p \leq p_0$. In Theorems 7.2, 7.3, and 7.5 the eigenvalues of L are characterized asymptotically in terms of the constants a_p, b_q, and c_p.

For the Green's function $G(t, s; \lambda)$ of the differential operator $\lambda I - L$, we have established growth rates for it in equations (6.37) and (6.77). First, in terms of the sector S_0 we have

$$|G(t, s; \lambda)| \leq \frac{2}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_0(\rho)|} \tag{9.1}$$

for $t \neq s$ in $[0, 1]$, where (9.1) is valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in S_0$ and $|\rho| > R_1$ and with $\Delta_0(\rho) \neq 0$. Second, in terms of the sector S_1 we have the analogous result

$$|G(t, s; \lambda)| \leq \frac{2}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\Delta_1(\rho)|} \tag{9.2}$$

for $t \neq s$ in $[0, 1]$, where (9.2) is now valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in S_1$ and $|\rho| > R_1$ and with $\Delta_1(\rho) \neq 0$. These growth rates depend on the sectors S_0 and S_1, but they are independent of whether L belongs to Case 1, Case 2, or Case 3. On the other hand, growth rates for the characteristic determinants Δ_0 and Δ_1 are case dependent. Specifically, assume that L belongs to Case 1 or Case 2 where $p = q \leq p_0$. From equation (7.9) we have

$$|\Delta_0(\rho)| \geq \frac{1}{2}\,|a_p||\rho|^p > 0 \tag{9.3}$$

for $\rho = a + ib \in G_0$ with $|\rho| \geq r_1$ and $b \geq d$, and from (7.10) we have

$$|\Delta_1(\rho)| \geq \frac{1}{2}\,|a_p||\rho|^p > 0 \tag{9.4}$$

for $\rho = a + ib \in G_1$ with $|\rho| \geq r_1$ and $b \leq -d$. Recall that the constant r_1 was chosen in Chapter 7 such that $r_1 > R_1 \geq R_0$. Combining (9.1) with (9.3), we obtain the estimate

$$|G(t, s; \lambda)| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.5}$$

for $t \neq s$ in $[0, 1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_0$ and with $|\rho| \geq r_1$ and $b \geq d$. Similarly, combining (9.2) with (9.4), we get

$$|G(t, s; \lambda)| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.6}$$

for $t \neq s$ in $[0, 1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_1$ and with $|\rho| \geq r_1$ and $b \leq -d$. Note that it is implicit in (9.5) and (9.6) that the point $\lambda = \rho^n$

belongs to the resolvent set $\rho(L)$, which is a consequence of (9.3) and (9.4). Thus, in Case 1 and Case 2 the resolvent satisfies the growth rate

$$\|R_\lambda(L)\| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.7}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a+ib \in S_0$, $|\rho| \geq r_1$, and $b \geq d$, or with $\rho = a+ib \in S_1$, $|\rho| \geq r_1$, and $b \leq -d$.

Next, assume that L belongs to Case 3 where $p < q \leq p_0$, the logarithmic case. Then from equation (7.31)

$$|\varDelta_0(\rho)| \geq \frac{1}{4}\,|a_p||\rho|^p > 0 \tag{9.8}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{0\infty}$ with $|\rho| \geq r_2$, and from equation (7.36)

$$|\varDelta_1(\rho)| \geq \frac{1}{4}\,|a_p||\rho|^p > 0 \tag{9.9}$$

for all $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ with $|\rho| \geq r_2$. The constant r_2 was selected in Chapter 7 such that $r_2 \geq r_1 > R_1 \geq R_0$. Note that if $\rho = a + \mathrm{i}b \in \varOmega_{0\infty}$ with $|\rho| \geq r_2$, then $\rho \in S_0$ and $a \leq \alpha \mathrm{e}^{b/n_0}$, (9.8) applies, and (9.1) is valid at the point $\lambda = \rho^n$. Similarly, if $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ with $|\rho| \geq r_2$, then $\rho \in S_1$ and $a \leq \alpha \mathrm{e}^{-b/n_0}$, (9.9) applies, and (9.2) is valid at the point $\lambda = \rho^n$. Combining (9.1) and (9.2) with (9.8) and (9.9), we obtain the estimates

$$|G(t, s; \lambda)| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.10}$$

for $t \neq s$ in $[0, 1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \varOmega_{0\infty}$ and $|\rho| \geq r_2$, and

$$|G(t, s; \lambda)| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.11}$$

for $t \neq s$ in $[0, 1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \varOmega_{1\infty}$ and $|\rho| \geq r_2$. Therefore, the resolvent in Case 3 satisfies the growth rate

$$\|R_\lambda(L)\| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.12}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho \in \varOmega_{0\infty} \cup \varOmega_{1\infty}$ and $|\rho| \geq r_2$.

With our estimates (9.7) and (9.12) for the resolvent in place, we are now ready to prove the completeness of the generalized eigenfunctions of the differential operator L. Let N be a positive integer that satisfies the condition

$$N \geq (p_0 - p - n + 1)/n.$$

Fix any real number θ_0 with either $\sigma_0 \leq \theta_0 \leq \pi$ or $-\pi \leq \theta_0 \leq -\sigma_0$, and let us consider the ray

$$\mathcal{R}_{\theta_0}\colon \quad \lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta_0}, \quad 0 < |\lambda| < \infty.$$

Assume initially that $\sigma_0 \leq \theta_0 \leq \pi$. For each $\lambda \in \mathcal{R}_{\theta_0}$ we can form the nth root

$$\rho = \sqrt[n]{\lambda} = |\lambda|^{1/n} \mathrm{e}^{\mathrm{i}\theta_0/n},$$

with $\sigma_0/n \leq \theta_0/n \leq \pi/n$ and with $\rho \in S_0$. Note that $0 < \theta_0/n \leq \pi/2$ and $\sin(\theta_0/n) > 0$, and for $\lambda \in \mathcal{R}_{\theta_0}$ and $\rho = \sqrt[n]{\lambda}$:

$$\begin{cases} \rho = a + \mathrm{i}b = |\rho|\cos(\theta_0/n) + \mathrm{i}|\rho|\sin(\theta_0/n), \\ a = |\rho|\cos(\theta_0/n) \geq 0, \qquad b = |\rho|\sin(\theta_0/n) > 0. \end{cases}$$

The geometry is slightly different for Case 1 and Case 2 where $p = q \leq p_0$ and for Case 3 where $p < q \leq p_0$. First, assume that $p = q$ (Case 1 and Case 2). Set

$$r(\theta_0) := \max\{r_1, d/\sin(\theta_0/n)\}.$$

If $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$, then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_0 with $|\rho| \geq r(\theta_0) \geq r_1$, $|\rho| \geq r(\theta_0) \geq d/\sin(\theta_0/n)$, and $b = |\rho|\sin(\theta_0/n) \geq d$. Thus, by (9.7)

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.13}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Second, assume that $p < q$ (Case 3, the logarithmic case). Set

$$r(\theta_0) := \max\{r_2, (2n_0^2/\alpha)\cot(\theta_0/n)/\sin(\theta_0/n)\}.$$

Take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_0, and $|\rho| \geq r(\theta_0) \geq r_2$ and $b = |\rho|\sin(\theta_0/n) > 0$. Also,

$$b = |\rho|\sin(\theta_0/n) \geq (2n_0^2/\alpha)\cot(\theta_0/n),$$

$$\mathrm{e}^{b/n_0} \geq b \cdot \frac{b}{2n_0^2} \geq \frac{b}{\alpha}\cot(\theta_0/n),$$

and

$$a = |\rho|\cos(\theta_0/n) = b\cot(\theta_0/n) \leq \alpha\mathrm{e}^{b/n_0},$$

and hence, $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. It follows from (9.12) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.14}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

To complete the discussion, assume that $-\pi \leq \theta_0 \leq -\sigma_0$ for the ray $\mathcal{R}_{\theta_0}$. We repeat the above argument with simple modifications. For each $\lambda \in \mathcal{R}_{\theta_0}$ we now form the nth root

$$\rho = \sqrt[n]{\lambda} = |\lambda|^{1/n} \mathrm{e}^{\mathrm{i}\theta_0/n},$$

with $-\pi/n \leq \theta_0/n \leq -\sigma_0/n$ and with $\rho \in S_1$. Note that we have $-\pi/2 \leq \theta_0/n < 0$ and $\sin(\theta_0/n) < 0$, $\cos(\theta_0/n) \geq 0$, and for $\lambda \in \mathcal{R}_{\theta_0}$ and $\rho = \sqrt[n]{\lambda}$:

$$\begin{cases} \rho = a + \mathrm{i}b = |\rho|\cos(\theta_0/n) + \mathrm{i}|\rho|\sin(\theta_0/n), \\ a = |\rho|\cos(\theta_0/n) \geq 0, \qquad b = |\rho|\sin(\theta_0/n) < 0. \end{cases}$$

First, assume that $p = q$ (Case 1 and Case 2). Set

$$r(\theta_0) := \max\{r_1, d/|\sin(\theta_0/n)|\}.$$

If $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$, then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_1 with $|\rho| \geq r(\theta_0) \geq r_1$, $|\rho| \geq r(\theta_0) \geq d/|\sin(\theta_0/n)|$, and $b = |\rho|\sin(\theta_0/n) \leq -d$. Hence, by (9.7)

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.15}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Second, assume that $p < q$ (Case 3, the logarithmic case). Set

$$r(\theta_0) := \max\{r_2, (2n_0^2/\alpha)|\cot(\theta_0/n)|/|\sin(\theta_0/n)|\}.$$

Take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_1, and $|\rho| \geq r(\theta_0) \geq r_2$ and $b = |\rho|\sin(\theta_0/n) < 0$. Also,

$$-b = |b| = |\rho||\sin(\theta_0/n)| \geq (2n_0^2/\alpha)|\cot(\theta_0/n)|,$$

$$\mathrm{e}^{-b/n_0} \geq (-b) \cdot \frac{-b}{2n_0^2} \geq \frac{-b}{\alpha}|\cot(\theta_0/n)|,$$

and

$$a = |\rho|\cos(\theta_0/n) = b\cot(\theta_0/n) = -b|\cot(\theta_0/n)| \leq \alpha\mathrm{e}^{-b/n_0},$$

and hence, $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2$. It follows from (9.12) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.16}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Recall that the constant σ_0 was selected in Chapter 5 with $0 < \sigma_0 < \pi/10$. The rays $\mathcal{R}_{\theta_0}$, with either $\sigma_0 \leq \theta_0 \leq \pi$ or $-\pi \leq \theta_0 \leq -\sigma_0$, clearly cover the sector

$$\Sigma_0\colon \text{ all } \lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \sigma_0 \leq \theta \leq 2\pi - \sigma_0$$

in the λ plane. Note that the five equally spaced rays

$$\begin{aligned}
\mathcal{R}_1\colon\ & \arg\lambda = \theta_1 := \frac{\pi}{5} = 36^\circ, \\
\mathcal{R}_2\colon\ & \arg\lambda = \theta_2 := \frac{\pi}{5} + \frac{2\pi}{5} = \frac{3\pi}{5} = 108^\circ, \\
\mathcal{R}_3\colon\ & \arg\lambda = \theta_3 := \frac{3\pi}{5} + \frac{2\pi}{5} = \pi = 180^\circ, \\
\mathcal{R}_4\colon\ & \arg\lambda = \theta_4 := \pi + \frac{2\pi}{5} = \frac{7\pi}{5} = 252^\circ, \\
\mathcal{R}_5\colon\ & \arg\lambda = \theta_5 := \frac{7\pi}{5} + \frac{2\pi}{5} = \frac{9\pi}{5} = 324^\circ
\end{aligned}$$

all lie in Σ_0, and the angle between adjacent rays is $2\pi/5 = 72^\circ < \pi/2$. From (9.13)–(9.16) we have the growth rate

$$\|R_\lambda(L)\| = O\big(|\lambda|^N\big) \quad \text{as } \lambda \to \infty \text{ along each ray } \mathcal{R}_j,$$

$j = 1, \ldots, 5$, valid for all three cases, Case 1, Case 2, Case 3. Applying the completeness theorem, Theorem 6.2 in Chapter 2 of [34], we conclude that

$$\overline{\mathrm{sp}(L)} = \overline{S_\infty(L)} = L^2[0,1] \tag{9.17}$$

and

$$M_\infty(L) = \{0\}. \tag{9.18}$$

We summarize the above results for the even order case in the following theorem.

Theorem 9.1. *Let the differential operator L be of even order $n = 2\nu$, and let L be either regular or simply irregular according to Definition 3.2. Then the spectrum $\sigma(L)$ is an infinite countable subset of $\mathbb{C}$ having no limit points in $\mathbb{C}$, and if $\sigma(L) = \{\lambda_i\}_{i=1}^\infty$ is any enumeration of $\sigma(L)$ and if $S_\infty(L)$ and $M_\infty(L)$ are the corresponding subspaces, then*

$$\overline{\mathrm{sp}(L)} = \overline{S_\infty(L)} = L^2[0,1] \quad \textit{and} \quad M_\infty(L) = \{0\}.$$

9.2 Completeness for n Odd

Assume that n is odd, $n = 2\nu - 1 \geq 3$. The functions π_1, π_0 and π_1', π_0' that appear as terms in the characteristic determinants Δ_0 and Δ_1 are given by

$$\pi_1(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} a_\kappa \rho^\kappa, \qquad \pi_0(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} b_\kappa \rho^\kappa$$

and

$$\pi_1'(\rho) = \sum_{\kappa=-(m-p_0-1)}^{q} a_\kappa' \rho^\kappa, \qquad \pi_0'(\rho) = \sum_{\kappa=-(m-p_0-1)}^{p} b_\kappa' \rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$. See equations (8.1) and (8.2). In these expressions the integers p and q have been determined with $-\infty < p\,, q \leq p_0$, and the leading coefficients satisfy $a_p \neq 0$, $b_q \neq 0$ and $a_q' \neq 0$, $b_p' \neq 0$. The integer m has been fixed with $m > n$, $m > p_0$, and $-(m - p_0 - 1) \leq p\,, q \leq p_0$. In Theorems 8.2, 8.4, and 8.6 the eigenvalues of L are characterized asymptotically in terms of the four constants a_p, b_q, a_q', b_p'.

For the Green's function $G(t, s; \lambda)$ of the differential operator $\lambda I - L$, we have established growth rates for it in equations (6.118) and (6.148) relative to the sector S_0 and in equations (6.189) and (6.219) relative to the sector S_1. Specifically, in terms of the sector S_0 we have

$$|G(t,s;\lambda)| \le \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\varDelta_0(\rho)|} \tag{9.19}$$

for $t \neq s$ in $[0,1]$, valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$, with $|\rho| > R_1$ and $b \ge 0$, and with $\varDelta_0(\rho) \neq 0$; and

$$|G(t,s;\lambda)| \le \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}\mathrm{e}^{b}|\varDelta_0(\rho)|} \tag{9.20}$$

for $t \neq s$ in $[0,1]$, valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$, with $|\rho| > R_1$ and $b \le 0$, and with $\varDelta_0(\rho) \neq 0$. In terms of the sector S_1 we have

$$|G(t,s;\lambda)| \le \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}|\varDelta_1(\rho)|} \tag{9.21}$$

for $t \neq s$ in $[0,1]$, valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$, with $|\rho| > R_1$ and $b \le 0$, and with $\varDelta_1(\rho) \neq 0$; and

$$|G(t,s;\lambda)| \le \frac{4}{|\rho|^{n-1}} + \frac{\gamma|\rho|^{p_0}}{n|\rho|^{n-1}\mathrm{e}^{-b}|\varDelta_1(\rho)|} \tag{9.22}$$

for $t \neq s$ in $[0,1]$, valid for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$, with $|\rho| > R_1$ and $b \ge 0$, and with $\varDelta_1(\rho) \neq 0$. These growth rates depend on the sectors S_0 and S_1, but they are independent of whether L belongs to Case 1, Case 2, or Case 3.

On the other hand, growth rates for the characteristic determinants $\varDelta_0$ and $\varDelta_1$ are case dependent. Specifically, assume that L belongs to Case 1 where $p = q$. From equation (8.6) we have

$$|\varDelta_0(\rho)| \ge \frac{1}{2}\,|b_p||\rho|^p > 0 \tag{9.23}$$

for $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| \ge r_1$ and $b \ge d$, and from equation (8.7) we have

$$|\varDelta_0(\rho)| \ge \frac{1}{2}\,\mathrm{e}^{-b}|a_p||\rho|^p > 0 \tag{9.24}$$

for $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| \ge r_1$ and $b \le -d$. Similarly, from equation (8.9)

$$|\varDelta_1(\rho)| \ge \frac{1}{2}\,|b_p'||\rho|^p > 0 \tag{9.25}$$

for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| \ge r_1$ and $b \le -d$, and from equation (8.10)

$$|\varDelta_1(\rho)| \ge \frac{1}{2}\,\mathrm{e}^{b}|a_p'||\rho|^p > 0 \tag{9.26}$$

for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| \ge r_1$ and $b \ge d$. Recall that the constant r_1 was chosen in Chapter 8 such that $r_1 > R_1 \ge R_0$.

Combining (9.19) and (9.23), we obtain the estimate

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.27}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_0$ and with $|\rho| \geq r_1$ and $b \geq d$; and combining (9.20) and (9.24), we obtain the estimate

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.28}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_0$ and with $|\rho| \geq r_1$ and $b \leq -d$. Similarly, combining (9.21) with (9.25), we get

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.29}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_1$ and with $|\rho| \geq r_1$ and $b \leq -d$; and combining (9.22) with (9.26), we get

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.30}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a+ib \in S_1$ and with $|\rho| \geq r_1$ and $b \geq d$. It is implicit in equations (9.27)–(9.30) that the point $\lambda = \rho^n$ belongs to the resolvent set $\rho(L)$, which is a direct consequence of (9.23)–(9.26). Thus, in Case 1 the resolvent satisfies the growth rate

$$\|R_\lambda(L)\| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.31}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + ib \in S_0$ and $|\rho| \geq r_1$ and $|b| \geq d$, or with $\rho = a + ib \in S_1$ and $|\rho| \geq r_1$ and $|b| \geq d$.

Next, assume that L belongs to Case 2 where $p < q$, a logarithmic case. Then from equation (8.31)

$$|\varDelta_0(\rho)| \geq \frac{1}{4}\,|b_q||\rho|^q > 0 \tag{9.32}$$

for $\rho = a + ib \in S_0$ with $|\rho| \geq r_1$ and $b \geq 0$, and from equation (8.34)

$$|\varDelta_0(\rho)| \geq \frac{1}{4}\,|a_p||\rho|^p \mathrm{e}^{-b} > 0 \tag{9.33}$$

for $\rho = a + ib \in \varOmega_{0\infty}$ with $|\rho| \geq r_2$. Recall that if $\rho = a + ib \in \varOmega_{0\infty}$, then $\rho \in S_0$ and $b \leq 0$. Also, from equation (8.39)

$$|\varDelta_1(\rho)| \geq \frac{1}{4}\,|a'_q||\rho|^q \mathrm{e}^{b} > 0 \tag{9.34}$$

for $\rho = a + ib \in S_1$ with $|\rho| \geq r'_1$ and $b \geq 0$, and from equation (8.42)

$$|\varDelta_1(\rho)| \geq \frac{1}{4}\,|b'_p||\rho|^p > 0 \tag{9.35}$$

for $\rho = a + ib \in \varOmega_{1\infty}$ with $|\rho| \geq r'_2$. For $\rho = a + ib \in \varOmega_{1\infty}$ we have $\rho \in S_1$ and $b \leq 0$. The constants r_1, r_2, r'_1, r'_2 were selected in Chapter 8 such that $r_2 \geq r_1 > R_1 \geq R_0$, $r'_2 \geq r'_1 > R_1 \geq R_0$, and $r_2 = r'_2$.

Combining (9.19) and (9.32), (9.20) and (9.33), (9.21) and (9.35), and (9.22) and (9.34), we obtain the following estimates:

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-q-n+1} \tag{9.36}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and with $|\rho| \geq r_1$ and $b \geq 0$;

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.37}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ and $|\rho| \geq r_2$;

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.38}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ and $|\rho| \geq r_2'$; and

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-q-n+1} \tag{9.39}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$ and with $|\rho| \geq r_1'$ and $b \geq 0$. Therefore, the resolvent in Case 2 satisfies the growth rate

$$\|R_\lambda(L)\| \leq \gamma|\rho|^{p_0-p-n+1} = \gamma|\lambda|^{(p_0-p-n+1)/n} \tag{9.40}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and $|\rho| \geq r_1$ and $b \geq 0$, or with $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ and $|\rho| \geq r_2$, or with $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ and $|\rho| \geq r_2'$, or with $\rho = a + \mathrm{i}b \in S_1$ and $|\rho| \geq r_1'$ and $b \geq 0$.

Finally, assume that L belongs to Case 3 where $p > q$, another logarithmic case. Then from equation (8.68)

$$|\Delta_0(\rho)| \geq \frac{1}{4}\,|a_p||\rho|^p \mathrm{e}^{-b} > 0 \tag{9.41}$$

for $\rho = a + \mathrm{i}b \in S_0$ with $|\rho| \geq r_1$ and $b \leq 0$, and from equation (8.71)

$$|\Delta_0(\rho)| \geq \frac{1}{4}\,|b_q||\rho|^q > 0 \tag{9.42}$$

for $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. Recall that if $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$, then $\rho \in S_0$ and $b \geq 0$. Also, from equation (8.76)

$$|\Delta_1(\rho)| \geq \frac{1}{4}\,|b_p'||\rho|^p > 0 \tag{9.43}$$

for $\rho = a + \mathrm{i}b \in S_1$ with $|\rho| \geq r_1'$ and $b \leq 0$, and from equation (8.79)

$$|\Delta_1(\rho)| \geq \frac{1}{4}\,|a_q'||\rho|^q \mathrm{e}^{b} > 0 \tag{9.44}$$

for $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2'$. For $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ we have $\rho \in S_1$ and $b \geq 0$. The constants r_1, r_2, r_1', r_2' were selected in Chapter 8 such that $r_2 \geq r_1 > R_1 \geq R_0$, $r_2' \geq r_1' > R_1 \geq R_0$, and $r_2 = r_2'$.

Combining (9.19) and (9.42), (9.20) and (9.41), (9.21) and (9.43), and (9.22) and (9.44), we obtain the following estimates:

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-q-n+1} \tag{9.45}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ and $|\rho| \geq r_2$;

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.46}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_0$ and with $|\rho| \geq r_1$ and $b \leq 0$;

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-p-n+1} \tag{9.47}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in S_1$ and with $|\rho| \geq r_1'$ and $b \leq 0$; and

$$|G(t,s;\lambda)| \leq \gamma|\rho|^{p_0-q-n+1} \tag{9.48}$$

for $t \neq s$ in $[0,1]$ and for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ and $|\rho| \geq r_2'$. Therefore, the resolvent in Case 3 satisfies the growth rate

$$\|R_\lambda(L)\| \leq \gamma|\rho|^{p_0-q-n+1} = \gamma|\lambda|^{(p_0-q-n+1)/n} \tag{9.49}$$

for $\lambda = \rho^n$ in $\mathbb{C}$ with $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ and $|\rho| \geq r_2$, or with $\rho = a + \mathrm{i}b \in S_0$ and $|\rho| \geq r_1$ and $b \leq 0$, or with $\rho = a + \mathrm{i}b \in S_1$ and $|\rho| \geq r_1'$ and $b \leq 0$, or with $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ and $|\rho| \geq r_2'$.

With our estimates (9.31), (9.40), and (9.49) for the resolvent in place, we are now ready to prove the completeness of the generalized eigenfunctions of the differential operator L. Let N be a positive integer that satisfies the conditions

$$N \geq (p_0 - p - n + 1)/n \quad \text{and} \quad N \geq (p_0 - q - n + 1)/n.$$

Fix any real number θ_0 satisfying one of the following four conditions: $\sigma_0 \leq \theta_0 \leq \pi/2$, $-\pi/2 \leq \theta_0 \leq -\sigma_0$, $n\pi + \sigma_0 \leq \theta_0 \leq n\pi + \pi/2$, or $n\pi - \pi/2 \leq \theta_0 \leq n\pi - \sigma_0$. Consider the ray

$$\mathcal{R}_{\theta_0}\colon \ \lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta_0}, \quad 0 < |\lambda| < \infty,$$

in the λ plane. We assert that for λ on $\mathcal{R}_{\theta_0}$ with $|\lambda|$ sufficiently large, the resolvent $R_\lambda(L)$ exists and satisfies the growth rate $\|R_\lambda(L)\| = O\big(|\lambda|^N\big)$.

Assume that $\sigma_0 \leq \theta_0 \leq \pi/2$. For each $\lambda \in \mathcal{R}_{\theta_0}$ we can form the nth root

$$\rho = \sqrt[n]{\lambda} = |\lambda|^{1/n}\mathrm{e}^{\mathrm{i}\theta_0/n},$$

with $\sigma_0/n \leq \theta_0/n \leq \pi/2n$ and with $\rho \in S_0$. Note that $0 < \theta_0/n < \pi/2$ and $\sin(\theta_0/n) > 0$, and for $\lambda \in \mathcal{R}_{\theta_0}$ and $\rho = \sqrt[n]{\lambda}$:

$$\begin{cases} \rho = a + \mathrm{i}b = |\rho|\cos(\theta_0/n) + \mathrm{i}|\rho|\sin(\theta_0/n), \\ a = |\rho|\cos(\theta_0/n) > 0, \qquad b = |\rho|\sin(\theta_0/n) > 0. \end{cases}$$

The geometry is slightly different for Case 1, Case 2, and Case 3. First, assume that L belongs to Case 1 where $p = q$. Set

$$r(\theta_0) := \max\{r_1, d/\sin(\theta_0/n)\}.$$

If $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$, then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_0 with $|\rho| \geq r(\theta_0) \geq r_1$, $|\rho| \geq r(\theta_0) \geq d/\sin(\theta_0/n)$, and $b = |\rho|\sin(\theta_0/n) \geq d$. Thus, by (9.31)

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.50}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Second, assume that L belongs to Case 2 where $p < q$. Set $r(\theta_0) := r_1$, and take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_0, $|\rho| \geq r_1$, and $b > 0$. It follows from (9.40) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.51}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Third, assume that L belongs to Case 3 where $p > q$. Set

$$r(\theta_0) := \max\{r_2, (2n_0^2/\alpha)\cot(\theta_0/n)/\sin(\theta_0/n)\}.$$

Take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_0, and $|\rho| \geq r(\theta_0) \geq r_2$ and $b = |\rho|\sin(\theta_0/n) > 0$. Also,

$$b = |\rho|\sin(\theta_0/n) \geq (2n_0^2/\alpha)\cot(\theta_0/n),$$

$$\mathrm{e}^{b/n_0} \geq b \cdot \frac{b}{2n_0^2} \geq \frac{b}{\alpha}\cot(\theta_0/n),$$

and

$$a = |\rho|\cos(\theta_0/n) = b\cot(\theta_0/n) \leq \alpha\mathrm{e}^{b/n_0},$$

and hence, $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. It follows from (9.49) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-q-n+1)/n} \leq \gamma|\lambda|^N \tag{9.52}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Next, assume that $-\pi/2 \leq \theta_0 \leq -\sigma_0$. Again for each $\lambda \in \mathcal{R}_{\theta_0}$ we can form the nth root

$$\rho = \sqrt[n]{\lambda} = |\lambda|^{1/n}\mathrm{e}^{\mathrm{i}\theta_0/n},$$

with $-\pi/2n \leq \theta_0/n \leq -\sigma_0/n$ and $\rho \in S_0$. Note that $-\pi/2 < \theta_0/n < 0$ and $\sin(\theta_0/n) < 0$, and for $\lambda \in \mathcal{R}_{\theta_0}$ and $\rho = \sqrt[n]{\lambda}$:

$$\begin{cases} \rho = a + \mathrm{i}b = |\rho|\cos(\theta_0/n) + \mathrm{i}|\rho|\sin(\theta_0/n), \\ a = |\rho|\cos(\theta_0/n) > 0, \qquad b = |\rho|\sin(\theta_0/n) < 0. \end{cases}$$

Now consider the three possible cases for L. First, assume that L belongs to Case 1 where $p = q$. Set

$$r(\theta_0) := \max\{r_1, d/|\sin(\theta_0/n)|\}.$$

If $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$, then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_0 with $|\rho| \geq r(\theta_0) \geq r_1$, $|\rho| \geq r(\theta_0) \geq d/|\sin(\theta_0/n)|$, and $b = |\rho|\sin(\theta_0/n) \leq -d$. Thus, by (9.31)

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.53}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Second, assume that L belongs to Case 2 where $p < q$. Set

$$r(\theta_0) := \max\{r_2, (2n_0^2/\alpha)|\cot(\theta_0/n)|/|\sin(\theta_0/n)|\}.$$

Take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_0, and $|\rho| \geq r(\theta_0) \geq r_2$ and $b = |\rho|\sin(\theta_0/n) < 0$. Also,

$$|b| = |\rho||\sin(\theta_0/n)| \geq (2n_0^2/\alpha)|\cot(\theta_0/n)|,$$

$$\mathrm{e}^{|b|/n_0} \geq |b| \cdot \frac{|b|}{2n_0^2} \geq \frac{|b|}{\alpha}|\cot(\theta_0/n)|,$$

and

$$a = |\rho|\cos(\theta_0/n) = |b||\cot(\theta_0/n)| \leq \alpha\mathrm{e}^{|b|/n_0} = \alpha\mathrm{e}^{-b/n_0},$$

and hence, $\rho = a + \mathrm{i}b \in \Omega_{0\infty}$ with $|\rho| \geq r_2$. It follows from (9.40) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.54}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Third, assume that L belongs to Case 3 where $p > q$. Set $r(\theta_0) := r_1$, and take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_0, $|\rho| \geq r_1$, and $b < 0$. It follows from (9.49) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-q-n+1)/n} \leq \gamma|\lambda|^N \tag{9.55}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Continuing our discussion, assume that $n\pi + \sigma_0 \leq \theta_0 \leq n\pi + \pi/2$. For each $\lambda \in \mathcal{R}_{\theta_0}$ we can form the nth root

$$\rho = \sqrt[n]{\lambda} = |\lambda|^{1/n}\mathrm{e}^{\mathrm{i}\theta_0/n},$$

with $\pi + \sigma_0/n \leq \theta_0/n \leq \pi + \pi/2n$ and with $\rho \in S_1$. Note that $\pi < \theta_0/n < 3\pi/2$ and $\sin(\theta_0/n) < 0$, and for $\lambda \in \mathcal{R}_{\theta_0}$ and $\rho = \sqrt[n]{\lambda}$:

$$\begin{cases} \rho = a + \mathrm{i}b = |\rho|\cos(\theta_0/n) + \mathrm{i}|\rho|\sin(\theta_0/n), \\ a = |\rho|\cos(\theta_0/n) < 0, \qquad b = |\rho|\sin(\theta_0/n) < 0. \end{cases}$$

First, assume that L belongs to Case 1 where $p = q$. Set

$$r(\theta_0) := \max\{r_1, d/|\sin(\theta_0/n)|\}.$$

If $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$, then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_1 with $|\rho| \geq r(\theta_0) \geq r_1$, $|\rho| \geq r(\theta_0) \geq d/|\sin(\theta_0/n)|$, and $b = |\rho|\sin(\theta_0/n) \leq -d$. Thus, by (9.31)

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.56}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Second, assume that L belongs to Case 2 where $p < q$. Set

$$r(\theta_0) := \max\{r_2', (2n_0^2/\alpha_1)\cot(\theta_0/n)/|\sin(\theta_0/n)|\}.$$

Take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_1, and $|\rho| \geq r(\theta_0) \geq r_2'$ and $b = |\rho|\sin(\theta_0/n) < 0$. Also,

$$|b| = |\rho||\sin(\theta_0/n)| \geq (2n_0^2/\alpha_1)\cot(\theta_0/n),$$

$$\mathrm{e}^{|b|/n_0} \geq |b| \cdot \frac{|b|}{2n_0^2} \geq \frac{|b|}{\alpha_1}\cot(\theta_0/n),$$

and

$$a = |\rho|\cos(\theta_0/n) = -|b|\cot(\theta_0/n) \geq -\alpha_1\mathrm{e}^{|b|/n_0} = -\alpha_1\mathrm{e}^{-b/n_0},$$

and hence, $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2'$. It follows from (9.40) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.57}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Third, assume that L belongs to Case 3 where $p > q$. Set $r(\theta_0) := r_1'$, and take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_1, $|\rho| \geq r_1'$, and $b < 0$. From (9.49) it follows that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-q-n+1)/n} \leq \gamma|\lambda|^N \tag{9.58}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Finally, assume that $n\pi - \pi/2 \leq \theta_0 \leq n\pi - \sigma_0$. For each $\lambda \in \mathcal{R}_{\theta_0}$ we can form the nth root

$$\rho = \sqrt[n]{\lambda} = |\lambda|^{1/n}\mathrm{e}^{\mathrm{i}\theta_0/n},$$

with $\pi - \pi/2n \leq \theta_0/n \leq \pi - \sigma_0/n$ and with $\rho \in S_1$. Note that $\pi/2 < \theta_0/n < \pi$ and $\sin(\theta_0/n) > 0$, and for $\lambda \in \mathcal{R}_{\theta_0}$ and $\rho = \sqrt[n]{\lambda}$:

$$\begin{cases} \rho = a + \mathrm{i}b = |\rho|\cos(\theta_0/n) + \mathrm{i}|\rho|\sin(\theta_0/n), \\ a = |\rho|\cos(\theta_0/n) < 0, \qquad b = |\rho|\sin(\theta_0/n) > 0. \end{cases}$$

First, assume that L belongs to Case 1 where $p = q$. Set

$$r(\theta_0) := \max\{r_1, d/\sin(\theta_0/n)\}.$$

If $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$, then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to the sector S_1 with $|\rho| \geq r(\theta_0) \geq r_1$, $|\rho| \geq r(\theta_0) \geq d/\sin(\theta_0/n)$, and $b = |\rho|\sin(\theta_0/n) \geq d$. Thus, by (9.31)

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.59}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Second, assume that L belongs to Case 2 where $p < q$. Set $r(\theta_0) := r_1'$, and take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_1, $|\rho| \geq r_1'$, and $b > 0$. From (9.40) it follows that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-p-n+1)/n} \leq \gamma|\lambda|^N \tag{9.60}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Third, assume that L belongs to Case 3 where $p > q$. Set

$$r(\theta_0) := \max\{r_2', (2n_0^2/\alpha_1)|\cot(\theta_0/n)|/\sin(\theta_0/n)\}.$$

Take any point $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$. Then the point $\rho = \sqrt[n]{\lambda} = a + \mathrm{i}b$ belongs to S_1, and $|\rho| \geq r(\theta_0) \geq r_2'$ and $b = |\rho|\sin(\theta_0/n) > 0$. Also,

$$b = |\rho|\sin(\theta_0/n) \geq (2n_0^2/\alpha_1)|\cot(\theta_0/n)|,$$

$$\mathrm{e}^{b/n_0} \geq b \cdot \frac{b}{2n_0^2} \geq \frac{b}{\alpha_1}\,|\cot(\theta_0/n)|,$$

and

$$a = |\rho|\cos(\theta_0/n) = b\cot(\theta_0/n) = -b|\cot(\theta_0/n)| \geq -\alpha_1\mathrm{e}^{b/n_0},$$

and hence, $\rho = a + \mathrm{i}b \in \Omega_{1\infty}$ with $|\rho| \geq r_2'$. It follows from (9.49) that

$$\|R_\lambda(L)\| \leq \gamma|\lambda|^{(p_0-q-n+1)/n} \leq \gamma|\lambda|^N \tag{9.61}$$

for $\lambda \in \mathcal{R}_{\theta_0}$ with $|\lambda| \geq r(\theta_0)^n$.

Recall that the constant σ_0 was selected in Chapter 5 with $0 < \sigma_0 < \pi/10$. The family of rays $\mathcal{R}_{\theta_0}$, with either $\sigma_0 \leq \theta_0 \leq \pi/2$, or $-\pi/2 \leq \theta_0 \leq -\sigma_0$, or $n\pi + \sigma_0 \leq \theta_0 \leq n\pi + \pi/2$, or $n\pi - \pi/2 \leq \theta_0 \leq n\pi - \sigma_0$, clearly covers the union of the two sectors

$$\begin{aligned} &\Sigma_0\colon\ \text{all } \lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \sigma_0 \leq \theta \leq \pi - \sigma_0,\\ &\Sigma_1\colon\ \text{all } \lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\pi + \sigma_0 \leq \theta \leq -\sigma_0 \end{aligned}$$

in the λ plane. Note that the five equally spaced rays

$$\begin{aligned}
\mathcal{R}_1\colon\ \arg\lambda &= \theta_1 := \frac{\pi}{10} = 18^\circ,\\
\mathcal{R}_2\colon\ \arg\lambda &= \theta_2 := \frac{\pi}{10} + \frac{2\pi}{5} = \frac{\pi}{2} = 90^\circ,\\
\mathcal{R}_3\colon\ \arg\lambda &= \theta_3 := \frac{\pi}{2} + \frac{2\pi}{5} = \frac{9\pi}{10} = 162^\circ,\\
\mathcal{R}_4\colon\ \arg\lambda &= \theta_4 := \frac{9\pi}{10} + \frac{2\pi}{5} = \frac{13\pi}{10} = 234^\circ,\\
\mathcal{R}_5\colon\ \arg\lambda &= \theta_5 := \frac{13\pi}{10} + \frac{2\pi}{5} = \frac{17\pi}{10} = 306^\circ
\end{aligned}$$

all lie in $\Sigma_0 \cup \Sigma_1$, and the angle between adjacent rays is $2\pi/5 = 72^\circ < \pi/2$. From the estimates (9.50)–(9.61) we have the growth rate

$$\|R_\lambda(L)\| = O\big(|\lambda|^N\big) \quad \text{as } \lambda \to \infty \text{ along each ray } \mathcal{R}_j,$$

$j = 1, \ldots, 5$, valid for all three cases, Case 1, Case 2, Case 3. Applying the completeness theorem, Theorem 6.2 in Chapter 2 of [34], we conclude that

$$\overline{\mathrm{sp}(L)} = \overline{S_\infty(L)} = L^2[0,1] \tag{9.62}$$

and

$$M_\infty(L) = \{0\}. \tag{9.63}$$

We summarize the above results for the odd order case in the following theorem.

Theorem 9.2. *Let the differential operator L be of odd order $n = 2\nu - 1$, and let L be either regular or simply irregular according to Definition 3.3. Then the spectrum $\sigma(L)$ is an infinite countable subset of $\mathbb{C}$ having no limit points in $\mathbb{C}$, and if $\sigma(L) = \{\lambda_i\}_{i=1}^\infty$ is any enumeration of $\sigma(L)$ and if $S_\infty(L)$ and $M_\infty(L)$ are the corresponding subspaces, then*

$$\overline{\mathrm{sp}(L)} = \overline{S_\infty(L)} = L^2[0,1] \quad \textit{and} \quad M_\infty(L) = \{0\}.$$

Remark 9.3. In view of the remarks in Section 3.4, if the differential operator L is regular, then Theorems 4.1, 5.1, and 6.1 from Chapter 6 of [34] are applicable. These three theorems give stronger versions of Theorem 9.1 and Theorem 9.2 for the special case of regularity. In the stronger versions we actually have

$$\mathbb{S}_\infty(L) = L^2[0,1] \quad \text{and} \quad \mathbb{M}_\infty(L) = \{0\}, \tag{9.64}$$

which gives a complete solution to the L^2- expansion problem.

10

Special Case $L = T$ and Degenerate Irregular Differential Operators

In the previous chapters we have developed the spectral theory of the differential operator L for the cases in which L is either *regular* or *simply irregular*. In particular, we have characterized the spectrum $\sigma(L)$ in Theorems 7.2, 7.3, and 7.5 for the case n even and in Theorems 8.2, 8.4, and 8.6 for the case n odd, and have shown that the generalized eigenfunctions of L are complete in $L^2[0,1]$ in Theorem 9.1 for n even and in Theorem 9.2 for n odd. The differential operators L that are neither regular nor simply irregular have been grouped together in the *degenerate irregular* class, for lack of a better name. This degenerate irregular class contains many strange differential operators, and has never been studied. In the future when this class is better understood, we envision it being subdivided into various subclasses, and having better names assigned to these subclasses.

10.1 The Special Case $L = T$

To illustrate some of the unusual features of the degenerate irregular differential operators, we present in this chapter some examples for the special case when the differential operator L is equal to its principal part T. Assume that $\ell = \tau$ and $\sigma = 0$, so $L = T$ and

$$\mathcal{D}(L) = \mathcal{D}(T) = \{u \in H^n[0,1] \mid B_i(u) = 0,\ i = 1, \dots, n\},$$
$$Lu = Tu = \mathrm{i}^{-n} u^{(n)}.$$

This important special case has been studied previously in Chapter 4 of [34] under the assumption that L is either regular or simply irregular, but there has been no discussion of the degenerate irregular case. Let us begin by observing some of the important simplifications that occur in Chapters 2–5 when $L = T$.

Fix any integer m with $m > n$ and $m > p_0$. Consider the approximate solutions and the approximate characteristic determinant defined in Chapter 2 and Chapter 3. First, from Example 2.4 the mth order Birkhoff approximate solutions are given by

$$z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}, \qquad k = 0, 1, \ldots, n-1,$$

for $0 \le t \le 1$ and for $\rho \neq 0$ in $\mathbb{C}$. These approximate solutions are actual solutions of the differential equation (2.1),

$$\rho^n u(t) - \mathrm{i}^{-n} u^{(n)}(t) = 0,$$

and are independent of the integer m. The associated mth order residual functions are simply

$$\eta_k(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho\omega_k t}(\rho^n I - \ell) z_k(t,\rho) = 0, \qquad k = 0, 1, \ldots, n-1,$$

for $0 \le t \le 1$ and for $\rho \neq 0$ in $\mathbb{C}$. Second, the modified Birkhoff approximate solutions are given by

$$\begin{aligned} y_k(t,\rho) &= z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}, && k = 0, 1, \ldots, \nu - 1, \\ y_k(t,\rho) &= \mathrm{e}^{-\mathrm{i}\rho\omega_k} z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k (t-1)}, && k = \nu, \ldots, n-1, \end{aligned}$$

for $0 \le t \le 1$ and for $\rho \neq 0$ in $\mathbb{C}$. These functions are also actual solutions of the differential equation (2.1).

In terms of the boundary values $B_1, \ldots, B_n$, we introduce the polynomials

$$P_i(\rho) := \sum_{p=0}^{m_i} \alpha_{ip}\rho^p, \qquad Q_i(\rho) := \sum_{p=0}^{m_i} \beta_{ip}\rho^p, \qquad i = 1, \ldots, n.$$

Then for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, \nu - 1$ we have

$$\begin{aligned} B_i(y_k(\,\cdot\,,\rho)) &= \sum_{p=0}^{m_i} \alpha_{ip}(\mathrm{i}\rho\omega_k)^p + \sum_{p=0}^{m_i} \beta_{ip}(\mathrm{i}\rho\omega_k)^p \mathrm{e}^{\mathrm{i}\rho\omega_k} \\ &= P_i(\mathrm{i}\rho\omega_k) + Q_i(\mathrm{i}\rho\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k} \\ &= \widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k} \end{aligned} \tag{10.1}$$

for $\rho \neq 0$ in $\mathbb{C}$, while for $i = 1, \ldots, n$ and $k = \nu, \ldots, n-1$

$$\begin{aligned} B_i(y_k(\,\cdot\,,\rho)) &= \sum_{p=0}^{m_i} \alpha_{ip}(\mathrm{i}\rho\omega_k)^p \mathrm{e}^{-\mathrm{i}\rho\omega_k} + \sum_{p=0}^{m_i} \beta_{ip}(\mathrm{i}\rho\omega_k)^p \\ &= Q_i(\mathrm{i}\rho\omega_k) + P_i(\mathrm{i}\rho\omega_k)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ &= \widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{aligned} \tag{10.2}$$

for $\rho \neq 0$ in $\mathbb{C}$. Thus, the functions $\widehat{P}_{ik}(\rho)$, $\widehat{Q}_{ik}(\rho)$ defined in equations (3.1), (3.2) are polynomials in ρ that are independent of the integer m, and they are identical to the polynomials $P_{ik}(\rho)$, $Q_{ik}(\rho)$ introduced in [34, pp. 99–100]. Equations (3.7) and (3.9) now simplify to

$$\widehat{P}_{ik}(\rho) = P_{ik}(\rho) = \sum_{s=0}^{m_i} p_{iks}\rho^s, \qquad \widehat{Q}_{ik}(\rho) = Q_{ik}(\rho) = \sum_{s=0}^{m_i} q_{iks}\rho^s \tag{10.3}$$

for $\rho \neq 0$ in $\mathbb{C}$ and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$. In terms of these polynomials, the approximate characteristic determinant is given by

$$\begin{aligned} \widehat{\Delta}(\rho) &= \det(B_i(y_k(\,\cdot\,,\rho))) \\ &= \det\Big(\overset{0 \le k \le \nu-1}{P_i(\mathrm{i}\rho\omega_k) + Q_i(\mathrm{i}\rho\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k}} \;\; \overset{\nu \le k \le n-1}{Q_i(\mathrm{i}\rho\omega_k) + P_i(\mathrm{i}\rho\omega_k)\mathrm{e}^{-\mathrm{i}\rho\omega_k}} \Big) \qquad (10.4) \\ &= \det\Big(\overset{0 \le k \le \nu-1}{\widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_k}} \;\; \overset{\nu \le k \le n-1}{\widehat{P}_{ik}(\rho) + \widehat{Q}_{ik}(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_k}} \Big) \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. This shows that the approximate characteristic determinant $\widehat{\Delta}(\rho)$ is independent of the integer m, and it is identical to the characteristic determinant $\Delta(\rho)$ defined for the differential operator $L = T$ in [34, p. 100].

We can also construct the (approximate) characteristic determinant using the functions $z_k(t, \rho)$, $k = 0, 1, \ldots, n-1$. Indeed, these functions form a basis for the solution space of the differential equation (2.1), and for $i = 1, \ldots, n$ and $k = 0, 1, \ldots, n-1$

$$B_i(z_k(\,\cdot\,,\rho)) = P_i(\mathrm{i}\rho\omega_k) + Q_i(\mathrm{i}\rho\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k} \tag{10.5}$$

for $\rho \neq 0$ in $\mathbb{C}$. An alternate characteristic determinant for the differential operator $L = T$ is defined by

$$\Delta_*(\rho) := \det(B_i(z_k(\,\cdot\,,\rho))) = \det\big(P_i(\mathrm{i}\rho\omega_k) + Q_i(\mathrm{i}\rho\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k}\big)$$

for $\rho \neq 0$ in $\mathbb{C}$. These characteristic determinants are related by the relation

$$\Delta(\rho) = \widehat{\Delta}(\rho) = \mathrm{e}^{\mathrm{i}\rho\eta}\Delta_*(\rho) \tag{10.6}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the constant η is defined by $\eta := -\omega_\nu - \omega_{\nu+1} - \cdots - \omega_{n-1}$. Later in this chapter it will be convenient to utilize $\Delta_*(\rho)$ in place of $\Delta(\rho)$.

Assume n is even: $n = 2\nu \geq 2$. From equation (10.3) it follows that the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, introduced in Chapter 3, are polynomials of degree $\leq p_0$ that are identical to the earlier polynomials $\pi_i(\rho)$, $i = 0, 1, 2$, defined in [34, pp. 114–115]. Also, from (3.24), (3.27), and (3.28) these polynomials have the representations

$$\widehat{\pi}_2(\rho) = \sum_{\kappa=0}^{p_0} a_\kappa \rho^\kappa, \qquad \widehat{\pi}_1(\rho) = \sum_{\kappa=0}^{p_0} b_\kappa \rho^\kappa, \qquad \widehat{\pi}_0(\rho) = \sum_{\kappa=0}^{p_0} c_\kappa \rho^\kappa$$

for $\rho \neq 0$ in $\mathbb{C}$, with $a_\kappa = b_\kappa = c_\kappa = 0$ for $\kappa = -1, -2, \ldots, -(m - p_0 - 1)$. Clearly these polynomials are independent of the integer m. Since m can be chosen arbitrarily large, we conclude that

$$a_\kappa = b_\kappa = c_\kappa = 0 \quad \text{for } \kappa = -1, -2, \ldots. \tag{10.7}$$

It is immediate that the functions $\pi_i(\rho)$, $i = 0, 1, 2$, introduced in Chapter 3 are polynomials that coincide with the polynomials $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$:

$$\begin{aligned} \pi_2(\rho) &= \widehat{\pi}_2(\rho) = \sum_{\kappa=0}^{p_0} a_\kappa \rho^\kappa, \\ \pi_1(\rho) &= \widehat{\pi}_1(\rho) = \sum_{\kappa=0}^{p_0} b_\kappa \rho^\kappa, \\ \pi_0(\rho) &= \widehat{\pi}_0(\rho) = \sum_{\kappa=0}^{p_0} c_\kappa \rho^\kappa \end{aligned} \tag{10.8}$$

for $\rho \neq 0$ in $\mathbb{C}$, and hence, the polynomials $\pi_i(\rho)$, $i = 0, 1, 2$, are independent of the integer m and are also identical to the polynomials $\pi_i(\rho)$, $i = 0, 1, 2$, defined in [34, pp. 114–115].

Assume n is odd: $n = 2\nu - 1 \geq 3$. Again from (10.3) we see that the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1$, and the functions $\pi_i(\rho)$, $i = 0, 1$, introduced in Chapter 3, are polynomials of degree $\leq p_0$ which are identical and are independent of the integer m:

$$\pi_1(\rho) = \widehat{\pi}_1(\rho) = \sum_{\kappa=0}^{p_0} a_\kappa \rho^\kappa, \qquad \pi_0(\rho) = \widehat{\pi}_0(\rho) = \sum_{\kappa=0}^{p_0} b_\kappa \rho^\kappa \tag{10.9}$$

for $\rho \neq 0$ in $\mathbb{C}$. In this case the coefficients satisfy the conditions

$$a_\kappa = b_\kappa = 0 \quad \text{for } \kappa = -1, -2, \ldots, \tag{10.10}$$

and the polynomials $\pi_i(\rho)$, $i = 0, 1$, are identical to the polynomials $\pi_i(\rho)$, $i = 0, 1$, defined in [34, pp. 123].

These results for the special case $L = T$ show that the classification scheme defined in our current work (see Definition 3.2 and Definition 3.3) is consistent with the classification scheme defined in the monograph [34] (see Definition 5.1 and Definition 6.1 in Chapter 4 of [34]): regular $\equiv$ regular, simply irregular $\equiv$ irregular, and degenerate irregular $\equiv$ degenerate.

In Chapter 4 we developed asymptotic expansions for solutions of the differential equation (2.1). Let us examine the form of these expansions in the special case $L = T$. Indeed, fix any integer m with $m > n$ and $m > p_0$, and take any integer k with $0 \leq k \leq n - 1$ and any point $\rho \neq 0$ in $\mathbb{C}$. Then the integral equations (4.9) and (4.43) reduce to the trivial equation $\psi(t) = 0$, which has the unique solution $\psi_k(t, \rho) = 0$. Applying (4.15) and

(4.48), we obtain the function $v_k(t, \rho) = z_k(t, \rho)$, which is a solution of the differential equation (2.1). The construction of the solution $v_k(\,\cdot\,, \rho)$ does not require restricting ρ to any of the sectors T_0 or T_1. Theorem 4.3, Theorem 4.4 and Theorem 4.6, Theorem 4.7 simplify to the statement that the functions

$$v_k(t, \rho) = z_k(t, \rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t}, \qquad k = 0, 1, \ldots, n-1,$$

are n linearly independent solutions of the differential equation (2.1) for all $\rho \neq 0$ in $\mathbb{C}$.

Finally, in Chapter 5 the solutions $v_k(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, are used to form the characteristic determinant of $L = T$. Note that the modified solutions $u_k(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, are the same as the modified Birkhoff approximate solutions $y_k(\,\cdot\,, \rho)$, $k = 0, 1, \ldots, n-1$, and hence, the characteristic determinant $\Delta(\rho)$ of L defined in Chapter 5 is identical to the approximate characteristic determinant $\widehat{\Delta}(\rho)$ formed above in equation (10.4):

$$\Delta(\rho) = \det(B_i(u_k(\,\cdot\,, \rho))) = \widehat{\Delta}(\rho) \tag{10.11}$$

for $\rho \neq 0$ in $\mathbb{C}$. We conclude that the characteristic determinant $\Delta(\rho)$ appearing in (10.11) is identical to the characteristic determinant $\Delta(\rho)$ defined for the differential operator $L = T$ in [34, p. 100].

10.2 Two Degenerate Irregular Examples

With the above results for the special case $L = T$ as a foundation, we next look at some examples of the degenerate irregular case.

Example 10.1. Consider the differential operator $L = T$ determined by initial value conditions at the endpoint $t = 0$:

$$B_i(u) = u^{(n-i)}(0), \qquad i = 1, \ldots, n.$$

For this model the integer p_0 is given by

$$p_0 = (n-1) + (n-2) + \cdots + 1 + 0 = \frac{n}{2}(n-1).$$

In equations (10.1) and (10.2), for $i = 1, \ldots, n$ the functions are given by

$$\begin{aligned} B_i(y_k(\,\cdot\,, \rho)) &= (\mathrm{i}\rho\omega_k)^{n-i}, && k = 0, 1, \ldots, \nu-1, \\ B_i(y_k(\,\cdot\,, \rho)) &= (\mathrm{i}\rho\omega_k)^{n-i}\mathrm{e}^{-\mathrm{i}\rho\omega_k}, && k = \nu, \ldots, n-1, \end{aligned}$$

and hence, for $i = 1, \ldots, n$

$$\begin{aligned} &\widehat{P}_{ik}(\rho) = (\mathrm{i}\rho\omega_k)^{n-i}, \quad \widehat{Q}_{ik}(\rho) \equiv 0, && k = 0, 1, \ldots, \nu-1, \\ &\widehat{P}_{ik}(\rho) \equiv 0, \quad \widehat{Q}_{ik}(\rho) = (\mathrm{i}\rho\omega_k)^{n-i}, && k = \nu, \ldots, n-1. \end{aligned} \tag{10.12}$$

Setting $\eta := -\omega_\nu - \omega_{\nu+1} - \cdots - \omega_{n-1}$, it follows from equation (10.4) that

$$\widehat{\Delta}(\rho) = \Delta(\rho) = \det\begin{array}{c} \\ \left(\right. \end{array}\hspace{-0.5em}\begin{array}{cc} {\scriptstyle 0\le k\le\nu-1} & {\scriptstyle \nu\le k\le n-1} \\ (\mathrm{i}\rho\omega_k)^{n-i} & (\mathrm{i}\rho\omega_k)^{n-i}\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{array}\hspace{-0.5em}\begin{array}{c} \\ \left. \right) \end{array}$$

$$= (\mathrm{i}\rho)^{\frac{n}{2}(n-1)}\mathrm{e}^{\mathrm{i}\rho\eta}\det\begin{pmatrix} \omega_0^{n-1} & \omega_1^{n-1} & \cdots & \omega_{n-1}^{n-1} \\ \omega_0^{n-2} & \omega_1^{n-2} & \cdots & \omega_{n-1}^{n-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

or

$$\widehat{\Delta}(\rho) = \Delta(\rho) = \gamma(\mathrm{i}\rho)^{\frac{n}{2}(n-1)}\mathrm{e}^{\mathrm{i}\rho\eta} \quad \text{for } \rho \neq 0 \text{ in } \mathbb{C}, \tag{10.13}$$

where the constant γ is the displayed Vandermonde determinant. Therefore, $\Delta(\rho) \neq 0$ for all $\rho \neq 0$ in $\mathbb{C}$, which shows that the differential operator L has no nonzero eigenvalues. Using the basis $1, t, t^2, \ldots, t^{n-1}$ for the solution space of the differential equation $-(\mathrm{i})^{-n}u^{(n)}(t) = 0$, it is easy to check that $\lambda = 0$ is not an eigenvalue of L. We conclude that

$$\sigma(L) = \emptyset \quad \text{and} \quad \rho(L) = \mathbb{C}. \tag{10.14}$$

Of course, this result follows directly from the Existence-Uniqueness Theorem for initial value problems.

Assume that n is even: $n = 2\nu \geq 2$. Using the definitions of the functions $\widehat{\pi}_i(\rho)$, $i = 0, 1, 2$, given in Chapter 3 together with equation (10.12), we see that

$$\widehat{\pi}_2(\rho) = \pi_2(\rho) \equiv 0, \qquad \widehat{\pi}_0(\rho) = \pi_0(\rho) \equiv 0, \tag{10.15}$$

and

$$\widehat{\pi}_1(\rho) = \pi_1(\rho) = \det\begin{pmatrix} (\mathrm{i}\rho\omega_0)^{n-1} & \cdots & (\mathrm{i}\rho\omega_{\nu-1})^{n-1} & (\mathrm{i}\rho\omega_\nu)^{n-1} & 0 & \cdots & 0 \\ (\mathrm{i}\rho\omega_0)^{n-2} & \cdots & (\mathrm{i}\rho\omega_{\nu-1})^{n-2} & (\mathrm{i}\rho\omega_\nu)^{n-2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{cases} 2\mathrm{i}\rho & \text{for } n = 2, \\ 0 & \text{for } n = 4, 6, 8, \ldots. \end{cases} \tag{10.16}$$

Thus, for $n = 2$ we have $p_0 = 1$ and

$$\begin{aligned} a_\kappa = c_\kappa = 0 \qquad\quad & \text{for } \kappa = 1, 0, -1, \ldots, \\ b_1 = 2\mathrm{i}, \quad b_\kappa = 0 \quad & \text{for } \kappa = 0, -1, \ldots, \end{aligned} \tag{10.17}$$

and for $n = 4, 6, 8, \ldots$ we have $p_0 = \dfrac{n}{2}(n-1)$ and

$$a_\kappa = b_\kappa = c_\kappa = 0 \quad \text{for } \kappa = p_0, \dots, 1, 0, -1, \dots. \tag{10.18}$$

According to Definition 3.2, the differential operator L is degenerate irregular.

For $n = 2$ the boundary coefficient matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

so $A_{13} = 1$ and $A_{12} = A_{14} = A_{23} = A_{24} = A_{34} = 0$. Consequently, the above results for the constants a_κ, b_κ, c_κ agree with the earlier results for the constants given in Case 5 of Chapter 5.

Assume that n is odd: $n = 2\nu - 1 \geq 3$. Using the definitions of the functions $\widehat{\pi}_1(\rho)$ and $\widehat{\pi}_0(\rho)$ given in Chapter 3 together with equation (10.12), we see immediately that

$$\widehat{\pi}_1(\rho) = \pi_1(\rho) \equiv 0, \qquad \widehat{\pi}_0(\rho) = \pi_0(\rho) \equiv 0, \tag{10.19}$$

and hence,

$$a_\kappa = b_\kappa = 0 \quad \text{for } \kappa = p_0, \dots, 1, 0, -1, \dots \tag{10.20}$$

where $p_0 = (n/2)(n-1)$. By Definition 3.3 the differential operator L is degenerate irregular.

This example illustrates the fact that the principal exponentials:

$$\mathrm{e}^{2\mathrm{i}\rho},\ \mathrm{e}^{\mathrm{i}\rho},\ 1 \quad \text{for } n \text{ even}, \qquad \mathrm{e}^{\mathrm{i}\rho},\ 1 \quad \text{for } n \text{ odd},$$

may not appear in the characteristic determinant $\Delta(\rho)$. In this situation other exponentials become the principal terms. For example, in equation (10.13) the exponential

$$\mathrm{e}^{\mathrm{i}\rho\eta} = \mathrm{e}^{-\mathrm{i}\rho(\omega_\nu + \omega_{\nu+1} + \cdots + \omega_{n-1})}$$

is the principal exponential appearing in $\Delta(\rho)$, and in fact, it is the only exponential that appears.

Example 10.2. For n even, $n = 2\nu \geq 2$, let $L = T$ be the differential operator determined by the boundary values

$$B_i(u) = u^{(n-i)}(0) + (-1)^{i+1} u^{(n-i)}(1), \qquad i = 1, \dots, n.$$

Again we have $p_0 = (n/2)(n-1)$, and for $i = 1, \dots, n$ equations (10.1) and (10.2) become

$$\begin{aligned} B_i(y_k(\,\cdot\,,\rho)) &= (\mathrm{i}\rho\omega_k)^{n-i} + (-1)^{i+1}(\mathrm{i}\rho\omega_k)^{n-i}\mathrm{e}^{\mathrm{i}\rho\omega_k}, \qquad k = 0, 1, \dots, \nu - 1, \\ B_i(y_k(\,\cdot\,,\rho)) &= (\mathrm{i}\rho\omega_k)^{n-i}\mathrm{e}^{-\mathrm{i}\rho\omega_k} + (-1)^{i+1}(\mathrm{i}\rho\omega_k)^{n-i}, \quad k = \nu, \dots, n-1. \end{aligned} \tag{10.21}$$

Hence, for $i = 1, \dots, n$

$$\begin{aligned} &\widehat{P}_{ik}(\rho) = (\mathrm{i}\rho\omega_k)^{n-i}, \quad \widehat{Q}_{ik}(\rho) = (-1)^{i+1}(\mathrm{i}\rho\omega_k)^{n-i}, \quad k = 0, 1, \ldots, \nu - 1, \\ &\widehat{P}_{ik}(\rho) = (-1)^{i+1}(\mathrm{i}\rho\omega_k)^{n-i}, \quad \widehat{Q}_{ik}(\rho) = (\mathrm{i}\rho\omega_k)^{n-i}, \quad k = \nu, \ldots, n - 1. \end{aligned} \tag{10.22}$$

Observe that in (10.21), for $i = 1, \ldots, n$ and $k = 0$ we have $\omega_0 = 1$ and

$$B_i(y_0(\,\cdot\,, \rho)) = (\mathrm{i}\rho)^{n-i}[1 + (-1)^{i+1}\mathrm{e}^{\mathrm{i}\rho}],$$

while for $i = 1, \ldots, n$ and $k = \nu$ we have $\omega_\nu = -1$ and

$$\begin{aligned} B_i(y_\nu(\,\cdot\,, \rho)) &= (-\mathrm{i}\rho)^{n-i}[\mathrm{e}^{\mathrm{i}\rho} + (-1)^{i+1}] \\ &= (-1)^{n+1}(\mathrm{i}\rho)^{n-i}[(-1)^{i+1}\mathrm{e}^{\mathrm{i}\rho} + 1] = -B_i(y_0(\,\cdot\,, \rho)). \end{aligned}$$

It follows from (10.4) that

$$\widehat{\Delta}(\rho) = \Delta(\rho) = 0 \quad \text{for } \rho \neq 0 \text{ in } \mathbb{C}. \tag{10.23}$$

Thus, each $\lambda = \rho^n \neq 0$ in $\mathbb{C}$ is an eigenvalue of L, and a corresponding eigenfunction is

$$\phi(t, \rho) = \cos\rho(t - 1/2) = \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\rho/2} \cdot \mathrm{e}^{\mathrm{i}\rho t} + \frac{1}{2}\,\mathrm{e}^{-\mathrm{i}\rho/2} \cdot \mathrm{e}^{-\mathrm{i}\rho(t-1)}.$$

For $\lambda = 0$ we use the basis $1, t, t^2, \ldots, t^{n-1}$, showing that $\lambda = 0$ is also an eigenvalue of L with eigenfunctions $\phi_j(t) = (t - 1/2)^{2j}$, $j = 0, 1, \ldots, \nu - 1$. We conclude that

$$\sigma(L) = \mathbb{C} \quad \text{and} \quad \rho(L) = \emptyset. \tag{10.24}$$

In equation (10.22) we have $\widehat{Q}_{i0}(\rho) = (-1)^{i+1}(\mathrm{i}\rho)^{n-i}$ for $i = 1, \ldots, n$ and

$$\widehat{Q}_{i\nu}(\rho) = (-\mathrm{i}\rho)^{n-i} = -(-1)^{i+1}(\mathrm{i}\rho)^{n-i} = -\widehat{Q}_{i0}(\rho) \tag{10.25}$$

for $i = 1, \ldots, n$, and $\widehat{P}_{i0}(\rho) = (\mathrm{i}\rho)^{n-i}$ for $i = 1, \ldots, n$ and

$$\widehat{P}_{i\nu}(\rho) = (-1)^{i+1}(-\mathrm{i}\rho)^{n-i} = -(\mathrm{i}\rho)^{n-i} = -\widehat{P}_{i0}(\rho) \tag{10.26}$$

for $i = 1, \ldots, n$. Therefore, from the definitions of the functions $\widehat{\pi}_2(\rho)$ and $\widehat{\pi}_0(\rho)$ given in Chapter 3, it follows that

$$\widehat{\pi}_2(\rho) = \pi_2(\rho) \equiv 0, \qquad \widehat{\pi}_0(\rho) = \pi_0(\rho) \equiv 0, \tag{10.27}$$

and hence,

$$a_\kappa = c_\kappa = 0 \quad \text{for } \kappa = p_0, \ldots, 1, 0, -1, \ldots. \tag{10.28}$$

In the definition of the function $\widehat{\pi}_1(\rho)$, if we substitute (10.25) and (10.26) into the νth columns of the two determinants appearing there, then $\widehat{\pi}_1(\rho)$ becomes the sum of two determinants that are the negatives of each other, and hence,

$$\widehat{\pi}_1(\rho) = \pi_1(\rho) \equiv 0 \tag{10.29}$$

and

$$b\kappa = 0 \quad \text{for } \kappa = p_0, \ldots, 1, 0, -1, \ldots. \tag{10.30}$$

According to Definition 3.2, the differential operator L is degenerate irregular.

The differential operators L appearing in Example 10.1 and Example 10.2 are certainly degenerate, but for very different reasons. In the first example, the spectrum $\sigma(L)$ is empty, there are no eigenvalues nor any eigenfunctions, and the resolvent $R_\lambda(L)$ exists on all of $\mathbb{C}$; in the second, the spectrum $\sigma(L)$ is equal to all of $\mathbb{C}$, there are uncountably many eigenvalues and eigenfunctions, and the resolvent $R_\lambda(L)$ fails to exist at any point of $\mathbb{C}$. What kind of spectral theory can one develop in these extreme cases?

10.3 The Special Case $n = 4$, $L = T$

Next, we give some additional examples for the special case $n = 4$ and $L = T$. The integer $n = 4$ is small enough that calculations can still be made with pencil and paper, while it is large enough to exhibit some of the more subtle features of these fourth order differential operators. In this special case the boundary values have the form

$$B_i(u) = \sum_{p=0}^{3} \alpha_{ip} u^{(p)}(0) + \sum_{p=0}^{3} \beta_{ip} u^{(p)}(1), \qquad i = 1, \ldots, 4,$$

and the associated boundary coefficient matrix becomes

$$A = \begin{pmatrix} \alpha_{13} & \beta_{13} & \alpha_{12} & \beta_{12} & \alpha_{11} & \beta_{11} & \alpha_{10} & \beta_{10} \\ \alpha_{23} & \beta_{23} & \alpha_{22} & \beta_{22} & \alpha_{21} & \beta_{21} & \alpha_{20} & \beta_{20} \\ \alpha_{33} & \beta_{33} & \alpha_{32} & \beta_{32} & \alpha_{31} & \beta_{31} & \alpha_{30} & \beta_{30} \\ \alpha_{43} & \beta_{43} & \alpha_{42} & \beta_{42} & \alpha_{41} & \beta_{41} & \alpha_{40} & \beta_{40} \end{pmatrix}.$$

Recall that we are assuming that A is in reduced row echelon form with rank 4. Note that the integer $p_0 = \sum_{i=1}^{4} m_i$ satisfies the inequalities $2 \le p_0 \le 10$. Here the fourth roots of unity are simply $\omega_0 = 1$, $\omega_1 = \mathrm{i}$, $\omega_2 = -1$, $\omega_3 = -\mathrm{i}$; the Birkhoff (approximate) solutions are given by

$$z_0(t,\rho) = \mathrm{e}^{\mathrm{i}\rho t}, \quad z_1(t,\rho) = \mathrm{e}^{-\rho t}, \quad z_2(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho t}, \quad z_3(t,\rho) = \mathrm{e}^{\rho t};$$

and the modified Birkhoff (approximate) solutions are

$$y_0(t,\rho) = \mathrm{e}^{\mathrm{i}\rho t}, \quad y_1(t,\rho) = \mathrm{e}^{-\rho t}, \quad y_2(t,\rho) = \mathrm{e}^{-\mathrm{i}\rho (t-1)}, \quad y_3(t,\rho) = \mathrm{e}^{\rho (t-1)}.$$

The characteristic determinant becomes

$$\begin{aligned} \Delta(\rho) = \widehat{\Delta}(\rho) &= \det(B_i(y_k(\,\cdot\,,\rho))) \\ &= \det\begin{pmatrix} P_1(\mathrm{i}\rho){+}Q_1(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_1(-\rho){+}Q_1(-\rho)\mathrm{e}^{-\rho} & P_1(-\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho}{+}Q_1(-\mathrm{i}\rho) & P_1(\rho)\mathrm{e}^{-\rho}{+}Q_1(\rho) \\ P_2(\mathrm{i}\rho){+}Q_2(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_2(-\rho){+}Q_2(-\rho)\mathrm{e}^{-\rho} & P_2(-\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho}{+}Q_2(-\mathrm{i}\rho) & P_2(\rho)\mathrm{e}^{-\rho}{+}Q_2(\rho) \\ P_3(\mathrm{i}\rho){+}Q_3(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_3(-\rho){+}Q_3(-\rho)\mathrm{e}^{-\rho} & P_3(-\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho}{+}Q_3(-\mathrm{i}\rho) & P_3(\rho)\mathrm{e}^{-\rho}{+}Q_3(\rho) \\ P_4(\mathrm{i}\rho){+}Q_4(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_4(-\rho){+}Q_4(-\rho)\mathrm{e}^{-\rho} & P_4(-\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho}{+}Q_4(-\mathrm{i}\rho) & P_4(\rho)\mathrm{e}^{-\rho}{+}Q_4(\rho) \end{pmatrix} \end{aligned} \tag{10.31}$$

for $\rho \neq 0$ in $\mathbb{C}$, while the alternate characteristic determinant is

$$\begin{aligned}\varDelta_*(\rho) &= \det(B_i(z_k(\,\cdot\,,\rho))) \\ &= \det\begin{pmatrix} P_1(\mathrm{i}\rho)+Q_1(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_1(-\rho)+Q_1(-\rho)\mathrm{e}^{-\rho} & P_1(-\mathrm{i}\rho)+Q_1(-\mathrm{i}\rho)\mathrm{e}^{-\mathrm{i}\rho} & P_1(\rho)+Q_1(\rho)\mathrm{e}^{\rho} \\ P_2(\mathrm{i}\rho)+Q_2(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_2(-\rho)+Q_2(-\rho)\mathrm{e}^{-\rho} & P_2(-\mathrm{i}\rho)+Q_2(-\mathrm{i}\rho)\mathrm{e}^{-\mathrm{i}\rho} & P_2(\rho)+Q_2(\rho)\mathrm{e}^{\rho} \\ P_3(\mathrm{i}\rho)+Q_3(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_3(-\rho)+Q_3(-\rho)\mathrm{e}^{-\rho} & P_3(-\mathrm{i}\rho)+Q_3(-\mathrm{i}\rho)\mathrm{e}^{-\mathrm{i}\rho} & P_3(\rho)+Q_3(\rho)\mathrm{e}^{\rho} \\ P_4(\mathrm{i}\rho)+Q_4(\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho} & P_4(-\rho)+Q_4(-\rho)\mathrm{e}^{-\rho} & P_4(-\mathrm{i}\rho)+Q_4(-\mathrm{i}\rho)\mathrm{e}^{-\mathrm{i}\rho} & P_4(\rho)+Q_4(\rho)\mathrm{e}^{\rho} \end{pmatrix}\end{aligned} \tag{10.32}$$

for $\rho \neq 0$ in $\mathbb{C}$. The two characteristic determinants are related by the relation

$$\varDelta(\rho) = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}\varDelta_*(\rho) \tag{10.33}$$

for $\rho \neq 0$ in $\mathbb{C}$. Observe that

$$\varDelta_*(\mathrm{i}\rho) = -\varDelta_*(\rho), \qquad \varDelta_*(-\rho) = \varDelta_*(\rho), \qquad \varDelta_*(-\mathrm{i}\rho) = -\varDelta_*(\rho) \tag{10.34}$$

for $\rho \neq 0$ in $\mathbb{C}$.

Suppose we expand $\varDelta_*(\rho)$ using linearity in all four columns: $\varDelta_*(\rho)$ becomes the sum of sixteen terms; each term is the product of a polynomial times an exponential, where the polynomial is the determinant of a 4×4 matrix with polynomial entries. The sixteen exponentials that appear are listed below:

$$\begin{gathered} \mathrm{e}^{0\rho} = 1, \\ \mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{\mathrm{i}\rho}, \quad \mathrm{e}^{-\rho} = \mathrm{e}^{-\rho}, \quad \mathrm{e}^{-\mathrm{i}\rho} = \mathrm{e}^{-\mathrm{i}\rho}, \quad \mathrm{e}^{\rho} = \mathrm{e}^{\rho}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\mathrm{i}\rho} = 1, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\rho} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\rho}, \\ \mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho} = \mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho}, \quad \mathrm{e}^{-\rho}\mathrm{e}^{\rho} = 1, \quad \mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho} = \mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho} = \mathrm{e}^{-\rho}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}\mathrm{e}^{\rho} = \mathrm{e}^{\mathrm{i}\rho}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho} = \mathrm{e}^{\rho}, \quad \mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho} = \mathrm{e}^{-\mathrm{i}\rho}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho} = 1. \end{gathered}$$

We note that there are only nine distinct exponentials in this list. The exponentials $\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho}$, $\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\rho}$, $\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}$, $\mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho}$ each occurs once in the list; the exponentials e^{ρ}, $\mathrm{e}^{\mathrm{i}\rho}$, $\mathrm{e}^{-\rho}$, $\mathrm{e}^{-\mathrm{i}\rho}$ each occurs twice in the list; and the exponential $\mathrm{e}^{0\rho} = 1$ occurs four times in the list. Thus, upon expansion $\varDelta_*(\rho)$ has the form

$$\begin{aligned} \varDelta_*(\rho) = \mathbb{P}_0(\rho)\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\rho} + \mathbb{P}_1(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\rho} + \mathbb{P}_2(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho} + \mathbb{P}_3(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho} \\ + \mathbb{Q}_0(\rho)\mathrm{e}^{\rho} + \mathbb{Q}_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \mathbb{Q}_2(\rho)\mathrm{e}^{-\rho} + \mathbb{Q}_3(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \mathbb{D}(\rho) \end{aligned} \tag{10.35}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the polynomials $\mathbb{P}_i(\rho)$ are each 4×4 determinants, the polynomials $\mathbb{Q}_i(\rho)$ are each the sum of two 4×4 determinants, and the polynomial $\mathbb{D}(\rho)$ is the sum of four 4×4 determinants.

Specifically, the polynomials $\mathbb{P}_0(\rho)$, $\mathbb{Q}_0(\rho)$, and $\mathbb{D}(\rho)$ are given by the equations

$$\mathbb{P}_0(\rho) = \det \begin{pmatrix} P_1(\mathrm{i}\rho) & P_1(-\rho) & Q_1(-\mathrm{i}\rho) & Q_1(\rho) \\ P_2(\mathrm{i}\rho) & P_2(-\rho) & Q_2(-\mathrm{i}\rho) & Q_2(\rho) \\ P_3(\mathrm{i}\rho) & P_3(-\rho) & Q_3(-\mathrm{i}\rho) & Q_3(\rho) \\ P_4(\mathrm{i}\rho) & P_4(-\rho) & Q_4(-\mathrm{i}\rho) & Q_4(\rho) \end{pmatrix}, \tag{10.36}$$

$$\begin{aligned} \mathbb{Q}_0(\rho) = {} & \det \begin{pmatrix} P_1(\mathrm{i}\rho) & P_1(-\rho) & P_1(-\mathrm{i}\rho) & Q_1(\rho) \\ P_2(\mathrm{i}\rho) & P_2(-\rho) & P_2(-\mathrm{i}\rho) & Q_2(\rho) \\ P_3(\mathrm{i}\rho) & P_3(-\rho) & P_3(-\mathrm{i}\rho) & Q_3(\rho) \\ P_4(\mathrm{i}\rho) & P_4(-\rho) & P_4(-\mathrm{i}\rho) & Q_4(\rho) \end{pmatrix} \\ & + \det \begin{pmatrix} Q_1(\mathrm{i}\rho) & P_1(-\rho) & Q_1(-\mathrm{i}\rho) & Q_1(\rho) \\ Q_2(\mathrm{i}\rho) & P_2(-\rho) & Q_2(-\mathrm{i}\rho) & Q_2(\rho) \\ Q_3(\mathrm{i}\rho) & P_3(-\rho) & Q_3(-\mathrm{i}\rho) & Q_3(\rho) \\ Q_4(\mathrm{i}\rho) & P_4(-\rho) & Q_4(-\mathrm{i}\rho) & Q_4(\rho) \end{pmatrix}, \end{aligned} \tag{10.37}$$

and

$$\begin{aligned} \mathbb{D}(\rho) = {} & \det \begin{pmatrix} P_1(\mathrm{i}\rho) & P_1(-\rho) & P_1(-\mathrm{i}\rho) & P_1(\rho) \\ P_2(\mathrm{i}\rho) & P_2(-\rho) & P_2(-\mathrm{i}\rho) & P_2(\rho) \\ P_3(\mathrm{i}\rho) & P_3(-\rho) & P_3(-\mathrm{i}\rho) & P_3(\rho) \\ P_4(\mathrm{i}\rho) & P_4(-\rho) & P_4(-\mathrm{i}\rho) & P_4(\rho) \end{pmatrix} \\ & + \det \begin{pmatrix} Q_1(\mathrm{i}\rho) & P_1(-\rho) & Q_1(-\mathrm{i}\rho) & P_1(\rho) \\ Q_2(\mathrm{i}\rho) & P_2(-\rho) & Q_2(-\mathrm{i}\rho) & P_2(\rho) \\ Q_3(\mathrm{i}\rho) & P_3(-\rho) & Q_3(-\mathrm{i}\rho) & P_3(\rho) \\ Q_4(\mathrm{i}\rho) & P_4(-\rho) & Q_4(-\mathrm{i}\rho) & P_4(\rho) \end{pmatrix} \\ & + \det \begin{pmatrix} P_1(\mathrm{i}\rho) & Q_1(-\rho) & P_1(-\mathrm{i}\rho) & Q_1(\rho) \\ P_2(\mathrm{i}\rho) & Q_2(-\rho) & P_2(-\mathrm{i}\rho) & Q_2(\rho) \\ P_3(\mathrm{i}\rho) & Q_3(-\rho) & P_3(-\mathrm{i}\rho) & Q_3(\rho) \\ P_4(\mathrm{i}\rho) & Q_4(-\rho) & P_4(-\mathrm{i}\rho) & Q_4(\rho) \end{pmatrix} \\ & + \det \begin{pmatrix} Q_1(\mathrm{i}\rho) & Q_1(-\rho) & Q_1(-\mathrm{i}\rho) & Q_1(\rho) \\ Q_2(\mathrm{i}\rho) & Q_2(-\rho) & Q_2(-\mathrm{i}\rho) & Q_2(\rho) \\ Q_3(\mathrm{i}\rho) & Q_3(-\rho) & Q_3(-\mathrm{i}\rho) & Q_3(\rho) \\ Q_4(\mathrm{i}\rho) & Q_4(-\rho) & Q_4(-\mathrm{i}\rho) & Q_4(\rho) \end{pmatrix}. \end{aligned} \tag{10.38}$$

From the definitions of the various polynomials we can show that

$$\mathbb{P}_1(\rho) = -\mathbb{P}_0(\mathrm{i}\rho), \qquad \mathbb{P}_2(\rho) = \mathbb{P}_0(-\rho), \qquad \mathbb{P}_3(\rho) = -\mathbb{P}_0(-\mathrm{i}\rho), \tag{10.39}$$

$$\mathbb{Q}_1(\rho) = -\mathbb{Q}_0(\mathrm{i}\rho), \qquad \mathbb{Q}_2(\rho) = \mathbb{Q}_0(-\rho), \qquad \mathbb{Q}_3(\rho) = -\mathbb{Q}_0(-\mathrm{i}\rho). \tag{10.40}$$

Indeed, to prove that $\mathbb{P}_1(\rho) = -\mathbb{P}_0(\mathrm{i}\rho)$, we see from (10.35) that $\mathbb{P}_1(\rho)$ is the coefficient of the exponential $\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\rho}$, and hence,

$$\mathbb{P}_1(\rho) = \det \begin{pmatrix} Q_1(\mathrm{i}\rho) & P_1(-\rho) & P_1(-\mathrm{i}\rho) & Q_1(\rho) \\ Q_2(\mathrm{i}\rho) & P_2(-\rho) & P_2(-\mathrm{i}\rho) & Q_2(\rho) \\ Q_3(\mathrm{i}\rho) & P_3(-\rho) & P_3(-\mathrm{i}\rho) & Q_3(\rho) \\ Q_4(\mathrm{i}\rho) & P_4(-\rho) & P_4(-\mathrm{i}\rho) & Q_4(\rho) \end{pmatrix}.$$

But from (10.36)

$$-\mathbb{P}_0(\mathrm{i}\rho) = -\det \begin{pmatrix} P_1(-\rho) & P_1(-\mathrm{i}\rho) & Q_1(\rho) & Q_1(\mathrm{i}\rho) \\ P_2(-\rho) & P_2(-\mathrm{i}\rho) & Q_2(\rho) & Q_2(\mathrm{i}\rho) \\ P_3(-\rho) & P_3(-\mathrm{i}\rho) & Q_3(\rho) & Q_3(\mathrm{i}\rho) \\ P_4(-\rho) & P_4(-\mathrm{i}\rho) & Q_4(\rho) & Q_4(\mathrm{i}\rho) \end{pmatrix},$$

and the result is now clear. In view of (10.39) and (10.40), to calculate the polynomial coefficients in equation (10.35), it is sufficient to calculate the polynomials $\mathbb{P}_0(\rho)$, $\mathbb{Q}_0(\rho)$, and $\mathbb{D}(\rho)$. This can be accomplished using (10.36), (10.37), and (10.38). Let us proceed with this calculation.

For the 4×8 boundary coefficient matrix A, denote the eight columns of A by α_3, β_3, α_2, β_2, α_1, β_1, α_0, β_0, and let $\langle a, b, c, d\rangle$ denote the determinant of the 4×4 submatrix of A formed by using columns a, b, c, d, e.g.,

$$\langle \alpha_2, \beta_3, \alpha_0, \beta_2\rangle = \det \begin{pmatrix} \alpha_{12} & \beta_{13} & \alpha_{10} & \beta_{12} \\ \alpha_{22} & \beta_{23} & \alpha_{20} & \beta_{22} \\ \alpha_{32} & \beta_{33} & \alpha_{30} & \beta_{32} \\ \alpha_{42} & \beta_{43} & \alpha_{40} & \beta_{42} \end{pmatrix}.$$

Then using a straight forward but lengthy calculation, we get

$$\begin{aligned}
\mathbb{P}_0(\rho) = {} & 2\mathrm{i}\langle \alpha_3, \beta_3, \alpha_2, \beta_2 \rangle \rho^{10} \\
& - (2 - 2\mathrm{i})[\langle \alpha_3, \beta_3, \alpha_2, \beta_1 \rangle + \langle \alpha_3, \beta_3, \beta_2, \alpha_1 \rangle]\rho^9 \\
& + 2[-\langle \alpha_3, \beta_3, \alpha_2, \beta_0 \rangle + \langle \alpha_3, \beta_3, \beta_2, \alpha_0 \rangle + 2\langle \alpha_3, \beta_3, \alpha_1, \beta_1 \rangle \\
& + \langle \alpha_3, \alpha_2, \beta_2, \beta_1 \rangle - \langle \beta_3, \alpha_2, \beta_2, \alpha_1 \rangle]\rho^8 \\
& + (2 + 2\mathrm{i})[\langle \alpha_3, \beta_3, \alpha_1, \beta_0 \rangle + \langle \alpha_3, \beta_3, \beta_1, \alpha_0 \rangle + \langle \alpha_3, \alpha_2, \beta_2, \beta_0 \rangle \\
& + \langle \alpha_3, \beta_2, \alpha_1, \beta_1 \rangle + \langle \beta_3, \alpha_2, \beta_2, \alpha_0 \rangle + \langle \beta_3, \alpha_2, \alpha_1, \beta_1 \rangle]\rho^7 \\
& + 2\mathrm{i}[-\langle \alpha_3, \beta_3, \alpha_0, \beta_0 \rangle + \langle \alpha_3, \alpha_2, \beta_1, \beta_0 \rangle + 2\langle \alpha_3, \beta_2, \alpha_1, \beta_0 \rangle \\
& + \langle \alpha_3, \beta_2, \beta_1, \alpha_0 \rangle + \langle \beta_3, \alpha_2, \alpha_1, \beta_0 \rangle + 2\langle \beta_3, \alpha_2, \beta_1, \alpha_0 \rangle \\
& + \langle \beta_3, \beta_2, \alpha_1, \alpha_0 \rangle - \langle \alpha_2, \beta_2, \alpha_1, \beta_1 \rangle]\rho^6 \\
& + (2 - 2\mathrm{i})[\langle \alpha_3, \beta_2, \alpha_0, \beta_0 \rangle + \langle \alpha_3, \alpha_1, \beta_1, \beta_0 \rangle + \langle \beta_3, \alpha_2, \alpha_0, \beta_0 \rangle \\
& + \langle \beta_3, \alpha_1, \beta_1, \alpha_0 \rangle + \langle \alpha_2, \beta_2, \alpha_1, \beta_0 \rangle + \langle \alpha_2, \beta_2, \beta_1, \alpha_0 \rangle]\rho^5 \\
& + 2[\langle \alpha_3, \beta_1, \alpha_0, \beta_0 \rangle - \langle \beta_3, \alpha_1, \alpha_0, \beta_0 \rangle - 2\langle \alpha_2, \beta_2, \alpha_0, \beta_0 \rangle \\
& - \langle \alpha_2, \alpha_1, \beta_1, \beta_0 \rangle + \langle \beta_2, \alpha_1, \beta_1, \alpha_0 \rangle]\rho^4 \\
& - (2 + 2\mathrm{i})[\langle \alpha_2, \beta_1, \alpha_0, \beta_0 \rangle + \langle \beta_2, \alpha_1, \alpha_0, \beta_0 \rangle]\rho^3 \\
& + 2\mathrm{i}\langle \alpha_1, \beta_1, \alpha_0, \beta_0 \rangle \rho^2,
\end{aligned} \tag{10.41}$$

$$\begin{aligned}
\mathbb{Q}_0(\rho) = {} & 4\mathrm{i}[\langle \alpha_3, \beta_3, \alpha_2, \alpha_1 \rangle + \langle \alpha_3, \beta_3, \beta_2, \beta_1 \rangle]\rho^9 \\
& - 4\mathrm{i}[\langle \alpha_3, \beta_3, \alpha_2, \alpha_0 \rangle - \langle \alpha_3, \beta_3, \beta_2, \beta_0 \rangle \\
& + \langle \alpha_3, \alpha_2, \beta_2, \alpha_1 \rangle - \langle \beta_3, \alpha_2, \beta_2, \beta_1 \rangle]\rho^8 \\
& + 4\mathrm{i}[\langle \alpha_3, \beta_3, \alpha_1, \alpha_0 \rangle + \langle \alpha_3, \beta_3, \beta_1, \beta_0 \rangle + \langle \alpha_3, \alpha_2, \beta_2, \alpha_0 \rangle \\
& + \langle \alpha_3, \alpha_2, \alpha_1, \beta_1 \rangle + \langle \beta_3, \alpha_2, \beta_2, \beta_0 \rangle + \langle \beta_3, \beta_2, \alpha_1, \beta_1 \rangle]\rho^7 \\
& + 4\mathrm{i}[\langle \alpha_3, \alpha_2, \alpha_1, \beta_0 \rangle + \langle \alpha_3, \alpha_2, \beta_1, \alpha_0 \rangle + \langle \alpha_3, \beta_2, \alpha_1, \alpha_0 \rangle \\
& + \langle \alpha_3, \beta_2, \beta_1, \beta_0 \rangle + \langle \beta_3, \alpha_2, \alpha_1, \alpha_0 \rangle + \langle \beta_3, \alpha_2, \beta_1, \beta_0 \rangle \\
& + \langle \beta_3, \beta_2, \alpha_1, \beta_0 \rangle + \langle \beta_3, \beta_2, \beta_1, \alpha_0 \rangle]\rho^6 \\
& - 4\mathrm{i}[\langle \alpha_3, \alpha_2, \alpha_0, \beta_0 \rangle + \langle \beta_3, \beta_2, \alpha_0, \beta_0 \rangle + \langle \alpha_3, \alpha_1, \beta_1, \alpha_0 \rangle \\
& + \langle \beta_3, \alpha_1, \beta_1, \beta_0 \rangle + \langle \alpha_2, \beta_2, \alpha_1, \alpha_0 \rangle + \langle \alpha_2, \beta_2, \beta_1, \beta_0 \rangle]\rho^5 \\
& + 4\mathrm{i}[\langle \alpha_3, \alpha_1, \alpha_0, \beta_0 \rangle - \langle \beta_3, \beta_1, \alpha_0, \beta_0 \rangle \\
& + \langle \alpha_2, \alpha_1, \beta_1, \alpha_0 \rangle - \langle \beta_2, \alpha_1, \beta_1, \beta_0 \rangle]\rho^4 \\
& - 4\mathrm{i}[\langle \alpha_2, \alpha_1, \alpha_0, \beta_0 \rangle + \langle \beta_2, \beta_1, \alpha_0, \beta_0 \rangle]\rho^3,
\end{aligned} \tag{10.42}$$

and

$$\begin{aligned}\mathbb{D}(\rho) = &-8\mathrm{i}\langle\alpha_3,\beta_3,\alpha_2,\beta_2\rangle\rho^{10}\\ &+8\mathrm{i}[\langle\alpha_3,\beta_3,\alpha_0,\beta_0\rangle+2\langle\alpha_3,\alpha_2,\alpha_1,\alpha_0\rangle+\langle\alpha_3,\alpha_2,\beta_1,\beta_0\rangle\\ &+\langle\alpha_3,\beta_2,\beta_1,\alpha_0\rangle+\langle\beta_3,\alpha_2,\alpha_1,\beta_0\rangle+\langle\beta_3,\beta_2,\alpha_1,\alpha_0\rangle\\ &+2\langle\beta_3,\beta_2,\beta_1,\beta_0\rangle+\langle\alpha_2,\beta_2,\alpha_1,\beta_1\rangle]\rho^6\\ &-8\mathrm{i}\langle\alpha_1,\beta_1,\alpha_0,\beta_0\rangle\rho^2.\end{aligned} \tag{10.43}$$

From equations (10.33) and (10.35) we see that

$$\begin{aligned}\Delta(\rho) &= \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho}\Delta_*(\rho)\\ &= \mathbb{P}_0(\rho)+\mathbb{P}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho}+\mathbb{P}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{-2\rho}+\mathbb{P}_3(\rho)\mathrm{e}^{-2\rho}\\ &\quad+\mathbb{Q}_0(\rho)\mathrm{e}^{\mathrm{i}\rho}+\mathbb{Q}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{-\rho}+\mathbb{Q}_2(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-2\rho}+\mathbb{Q}_3(\rho)\mathrm{e}^{-\rho}+\mathbb{D}(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{-\rho},\end{aligned}$$

or

$$\begin{aligned}\Delta(\rho) = &\,\mathbb{P}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho}+\mathbb{Q}_0(\rho)\mathrm{e}^{\mathrm{i}\rho}+\mathbb{P}_0(\rho)\\ &+[\mathbb{P}_2(\rho)\mathrm{e}^{-2\rho}+\mathbb{Q}_1(\rho)\mathrm{e}^{-\rho}]\mathrm{e}^{2\mathrm{i}\rho}\\ &+[\mathbb{Q}_2(\rho)\mathrm{e}^{-2\rho}+\mathbb{D}(\rho)\mathrm{e}^{-\rho}]\mathrm{e}^{\mathrm{i}\rho}\\ &+[\mathbb{P}_3(\rho)\mathrm{e}^{-2\rho}+\mathbb{Q}_3(\rho)\mathrm{e}^{-\rho}]\end{aligned} \tag{10.44}$$

for $\rho \neq 0$ in $\mathbb{C}$. This is equation (3.32) or equation (5.31) for the special case $n = 4$ and $L = T$ with

$$\pi_2(\rho) = \mathbb{P}_1(\rho) = -\mathbb{P}_0(\mathrm{i}\rho), \qquad \pi_1(\rho) = \mathbb{Q}_0(\rho), \qquad \pi_0(\rho) = \mathbb{P}_0(\rho) \tag{10.45}$$

for $\rho \neq 0$ in $\mathbb{C}$. Cf. equation (3.18).

We know that a nonzero complex number $\lambda = \rho^4$ is an eigenvalue of L if and only if $\Delta(\rho) = 0$, or equivalently, if and only if $\Delta_*(\rho) = 0$. To determine if $\lambda = 0$ is an eigenvalue for L, we use the basis $\phi_0(t) = 1$, $\phi_1(t) = t$, $\phi_2(t) = t^2$, $\phi_3(t) = t^3$ for the solution space of the differential equation $-u^{(4)}(t) = 0$. Clearly

$$\begin{aligned}B_i(\phi_0(\cdot)) &= \alpha_{i0}+\beta_{i0},\\ B_i(\phi_1(\cdot)) &= \alpha_{i1}+\beta_{i0}+\beta_{i1},\\ B_i(\phi_2(\cdot)) &= 2\alpha_{i2}+\beta_{i0}+2\beta_{i1}+2\beta_{i2},\\ B_i(\phi_3(\cdot)) &= 6\alpha_{i3}+\beta_{i0}+3\beta_{i1}+6\beta_{i2}+6\beta_{i3}\end{aligned}$$

for $i = 1, \ldots, 4$. In terms of these quantities we form the determinant

$$\Delta_0 := \det(B_i(\phi_k(\cdot))).$$

Then $\lambda = 0$ is an eigenvalue for L if and only if $\Delta_0 = 0$. Upon expansion, we obtain the following expression for the constant Δ_0:

$$\begin{aligned}
\Delta_0 = 12[&\langle \alpha_3, \alpha_2, \alpha_1, \alpha_0\rangle + \langle \alpha_3, \alpha_2, \alpha_1, \beta_0\rangle + \langle \alpha_3, \alpha_2, \beta_1, \alpha_0\rangle \\
&+ \langle \alpha_3, \beta_2, \alpha_1, \alpha_0\rangle + \langle \beta_3, \alpha_2, \alpha_1, \alpha_0\rangle + \langle \alpha_3, \alpha_2, \beta_1, \beta_0\rangle \\
&+ \langle \alpha_3, \beta_2, \alpha_1, \beta_0\rangle + \langle \alpha_3, \beta_2, \beta_1, \alpha_0\rangle + \langle \beta_3, \alpha_2, \alpha_1, \beta_0\rangle \\
&+ \langle \beta_3, \alpha_2, \beta_1, \alpha_0\rangle + \langle \beta_3, \beta_2, \alpha_1, \alpha_0\rangle + \langle \alpha_3, \beta_2, \beta_1, \beta_0\rangle \\
&+ \langle \beta_3, \alpha_2, \beta_1, \beta_0\rangle + \langle \beta_3, \beta_2, \alpha_1, \beta_0\rangle + \langle \beta_3, \beta_2, \beta_1, \alpha_0\rangle \\
&+ \langle \beta_3, \beta_2, \beta_1, \beta_0\rangle] \\
&- 12[\langle \alpha_3, \alpha_2, \alpha_0, \beta_0\rangle + \langle \alpha_3, \alpha_1, \beta_1, \alpha_0\rangle + \langle \alpha_2, \beta_2, \alpha_1, \alpha_0\rangle \\
&+ \langle \alpha_3, \alpha_1, \beta_1, \beta_0\rangle + \langle \alpha_3, \beta_2, \alpha_0, \beta_0\rangle + \langle \alpha_2, \beta_2, \alpha_1, \beta_0\rangle \\
&+ \langle \beta_3, \alpha_2, \alpha_0, \beta_0\rangle + \langle \alpha_2, \beta_2, \beta_1, \alpha_0\rangle + \langle \beta_3, \alpha_1, \beta_1, \alpha_0\rangle \\
&+ \langle \alpha_2, \beta_2, \beta_1, \beta_0\rangle + \langle \beta_3, \alpha_1, \beta_1, \beta_0\rangle + \langle \beta_3, \beta_2, \alpha_0, \beta_0\rangle] \\
&+ 6[\langle \alpha_3, \alpha_1, \alpha_0, \beta_0\rangle + \langle \alpha_2, \alpha_1, \beta_1, \alpha_0\rangle - \langle \alpha_3, \beta_1, \alpha_0, \beta_0\rangle \\
&+ \langle \alpha_2, \alpha_1, \beta_1, \beta_0\rangle + 2\langle \alpha_2, \beta_2, \alpha_0, \beta_0\rangle + \langle \beta_3, \alpha_1, \alpha_0, \beta_0\rangle \\
&- \langle \beta_2, \alpha_1, \beta_1, \alpha_0\rangle - \langle \beta_2, \alpha_1, \beta_1, \beta_0\rangle - \langle \beta_3, \beta_1, \alpha_0, \beta_0\rangle] \\
&- 2[\langle \alpha_2, \alpha_1, \alpha_0, \beta_0\rangle - 2\langle \alpha_2, \beta_1, \alpha_0, \beta_0\rangle \\
&- 2\langle \beta_2, \alpha_1, \alpha_0, \beta_0\rangle + \langle \beta_2, \beta_1, \alpha_0, \beta_0\rangle] \\
&- \langle \alpha_1, \beta_1, \alpha_0, \beta_0\rangle.
\end{aligned} \tag{10.46}$$

Example 10.3. Consider the 4th order differential operator $L = T$ determined by the boundary values

$$B_1(u) = u'''(0) + 6u(0), \quad B_2(u) = u''(0), \quad B_3(u) = u'(0), \quad B_4(u) = u(1),$$

so the boundary coefficient matrix is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly $p_0 = 6$, and $\langle \alpha_3, \alpha_2, \alpha_1, \beta_0\rangle = 1$ and $\langle \alpha_2, \alpha_1, \alpha_0, \beta_0\rangle = 6$, with all the other determinants $\langle a, b, c, d\rangle$ equal to 0. Thus, from equations (10.41)–(10.43) we have

$$\mathbb{P}_0(\rho) \equiv 0, \qquad \mathbb{Q}_0(\rho) = 4\mathrm{i}\rho^6 - 24\mathrm{i}\rho^3, \qquad \mathbb{D}(\rho) \equiv 0,$$

and then equations (10.44) and (10.39), (10.40) produce the characteristic determinant

$$\begin{aligned}
\Delta(\rho) = (4\mathrm{i}\rho^6 - 24\mathrm{i}\rho^3)\mathrm{e}^{\mathrm{i}\rho} &+ (4\mathrm{i}\rho^6 + 24\rho^3)\mathrm{e}^{-\rho}\mathrm{e}^{2\mathrm{i}\rho} \\
&+ (4\mathrm{i}\rho^6 + 24\mathrm{i}\rho^3)\mathrm{e}^{-2\rho}\mathrm{e}^{\mathrm{i}\rho} + (4\mathrm{i}\rho^6 - 24\rho^3)\mathrm{e}^{-\rho}
\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. Since $\pi_2(\rho) = -\mathbb{P}_0(\mathrm{i}\rho) \equiv 0$ and $\pi_0(\rho) = \mathbb{P}_0(\rho) \equiv 0$, the differential operator L belongs to the degenerate irregular class. Later in this chapter we will study this differential operator as a model for a more general class (Case II below), showing that it has a countably infinite spectrum that lies near the negative real axis. This is the first time that we have encountered a differential operator L of even order $n = 2\nu$ with its spectrum lying near the *negative* real axis.

Next, we reexamine our classification scheme of Chapter 3 by exploiting the explicit forms of the polynomials $\mathbb{P}_0(\rho)$, $\mathbb{Q}_0(\rho)$, and $\mathbb{D}(\rho)$. From equations (10.41)–(10.43) we see immediately that

$$\begin{aligned} &2 \leq \text{degree } \mathbb{P}_0(\rho) \leq 10 \quad \text{or} \quad \mathbb{P}_0(\rho) \equiv 0, \\ &3 \leq \text{degree } \mathbb{Q}_0(\rho) \leq 9 \quad \text{or} \quad \mathbb{Q}_0(\rho) \equiv 0, \\ &2 \leq \text{degree } \mathbb{D}_0(\rho) \leq 10 \quad \text{or} \quad \mathbb{D}_0(\rho) \equiv 0. \end{aligned}$$

Consider the following cases.

Case I. $n = 4$, $\mathbb{P}_0(\rho) \not\equiv 0$. From the relations (10.45) we see that the polynomials $\pi_0(\rho) = \mathbb{P}_0(\rho)$ and $\pi_2(\rho) = -\mathbb{P}_0(\mathrm{i}\rho)$ are both of degree p with $2 \leq p \leq 10$. It follows from equation (10.8) and Definition 3.2 that the fourth order differential operator $L = T$ is either regular or simply irregular. This case has been studied extensively in the previous chapters.

Case II. $n = 4$, $\mathbb{P}_0(\rho) \equiv 0$, $\mathbb{Q}_0(\rho) \not\equiv 0$. For this case the polynomials $\pi_0(\rho) = \mathbb{P}_0(\rho)$ and $\pi_2(\rho) = -\mathbb{P}_0(\mathrm{i}\rho)$ are identically zero, and hence, by Definition 3.2 the differential operator $L = T$ is degenerate irregular. Example 10.3 provides a model for this case. We are going to show that the differential operators belonging to Case II have some very unusual properties.

From equation (10.44) the characteristic determinant now takes the form

$$\begin{aligned} \Delta(\rho) = \mathbb{Q}_0(\rho)\mathrm{e}^{\mathrm{i}\rho} + \mathbb{Q}_1(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{2\mathrm{i}\rho} \\ + [\mathbb{Q}_2(\rho)\mathrm{e}^{-2\rho} + \mathbb{D}(\rho)\mathrm{e}^{-\rho}]\mathrm{e}^{\mathrm{i}\rho} + \mathbb{Q}_3(\rho)\mathrm{e}^{-\rho} \end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. Let q denote the degree of the polynomial $\mathbb{Q}_0(\rho)$, so $3 \leq q \leq 9$ and

$$\mathbb{Q}_0(\rho) = \pi_1(\rho) = \sum_{\kappa=3}^{q} b_\kappa \rho^\kappa, \qquad b_q \neq 0.$$

We know that $\mathbb{Q}_1(\rho) = -\mathbb{Q}_0(\mathrm{i}\rho)$, $\mathbb{Q}_2(\rho) = \mathbb{Q}_0(-\rho)$, and $\mathbb{Q}_3(\rho) = -\mathbb{Q}_0(-\mathrm{i}\rho)$. Also, from (10.41) we have $\langle\alpha_3, \beta_3, \alpha_2, \beta_2\rangle = 0$ and $\langle\alpha_1, \beta_1, \alpha_0, \beta_0\rangle = 0$, and hence, setting

$$\begin{aligned} \gamma_0 := \langle\alpha_3, \beta_3, \alpha_0, \beta_0\rangle + 2\langle\alpha_3, \alpha_2, \alpha_1, \alpha_0\rangle + \langle\alpha_3, \alpha_2, \beta_1, \beta_0\rangle \\ + \langle\alpha_3, \beta_2, \beta_1, \alpha_0\rangle + \langle\beta_3, \alpha_2, \alpha_1, \beta_0\rangle + \langle\beta_3, \beta_2, \alpha_1, \alpha_0\rangle \\ + 2\langle\beta_3, \beta_2, \beta_1, \beta_0\rangle + \langle\alpha_2, \beta_2, \alpha_1, \beta_1\rangle, \end{aligned} \qquad (10.47)$$

we see that $\mathbb{D}(\rho) = 8i\gamma_0\rho^6$. Fix a real number σ_0 with $0 < \sigma_0 < \pi/10$.

First, we introduce the sector

$$\varSigma_0\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{4} + \frac{\sigma_0}{4} \leq \theta \leq \frac{\pi}{4} - \frac{\sigma_0}{4}\,,$$

and then proceed to show that $\varDelta(\rho)$ has no zeros in the sector $\varSigma_0$ for $|\rho|$ sufficiently large. To study the behavior of $\varDelta(\rho)$ on the sector $\varSigma_0$, we first rewrite it in the alternate form

$$\begin{aligned}\varDelta(\rho) = \rho^q \mathrm{e}^{\mathrm{i}\rho}\bigg\{ &\frac{\mathbb{Q}_0(\rho)}{\rho^q} + \frac{\mathbb{Q}_1(\rho)}{\rho^q}\,\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho} \\ &+ \frac{\mathbb{Q}_2(\rho)}{\rho^q}\,\mathrm{e}^{-2\rho} + \frac{\mathbb{D}(\rho)}{\rho^q}\,\mathrm{e}^{-\rho} + \frac{\mathbb{Q}_3(\rho)}{\rho^q}\,\mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho}\bigg\}\end{aligned} \tag{10.48}$$

for $\rho \neq 0$ in $\mathbb{C}$. Set $\alpha := \sin(\pi/4+\sigma_0/4) > 0$ and $\beta := \sin(\pi/4-\sigma_0/4) > 0$, and take any point $\rho = a+\mathrm{i}b = |\rho|\mathrm{e}^{\mathrm{i}\theta} \neq 0$ in $\mathbb{C}$ with $-\pi/4+\sigma_0/4 \leq \theta \leq \pi/4-\sigma_0/4$. Clearly

$$\begin{aligned}&a = |a| = |\rho|\cos\theta \geq |\rho|\cos(\pi/4 - \sigma_0/4) = |\rho|\sin(\pi/4 + \sigma_0/4) = \alpha|\rho|,\\ &|b| = |\rho||\sin\theta| \leq |\rho|\sin(\pi/4 - \sigma_0/4) = \beta|\rho|.\end{aligned}$$

For the exponentials appearing in (10.48), we have the estimates

$$\begin{aligned}&|\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-a}\mathrm{e}^{-b} \leq \mathrm{e}^{-(\alpha-\beta)|\rho|}, && |\mathrm{e}^{-2\rho}| = \mathrm{e}^{-2a} \leq \mathrm{e}^{-2\alpha|\rho|} \leq \mathrm{e}^{-(\alpha-\beta)|\rho|},\\ &|\mathrm{e}^{-\rho}| = \mathrm{e}^{-a} \leq \mathrm{e}^{-\alpha|\rho|} \leq \mathrm{e}^{-(\alpha-\beta)|\rho|}, && |\mathrm{e}^{-\rho}\mathrm{e}^{-\mathrm{i}\rho}| = \mathrm{e}^{-a}\mathrm{e}^{b} \leq \mathrm{e}^{-(\alpha-\beta)|\rho|}.\end{aligned}$$

Thus,

$$\begin{aligned}|\varDelta(\rho)| &\geq |\rho|^q\mathrm{e}^{-b}\bigg\{|b_q| - \frac{\gamma}{|\rho|} - 4\gamma\mathrm{e}^{-\frac{1}{2}(\alpha-\beta)|\rho|}\bigg\}\\ &\geq \frac{|b_q|}{2}|\rho|^q\mathrm{e}^{-b} > 0\end{aligned} \tag{10.49}$$

for all ρ in the sector $\varSigma_0$ with $|\rho|$ sufficiently large.

Second, we introduce the sector

$$\varSigma_1\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \frac{\pi}{4} - \frac{\sigma_0}{4} \leq \theta \leq \frac{\pi}{4} + \frac{\sigma_0}{4}\,.$$

We claim that $\varDelta(\rho)$ has an infinite sequence of zeros in the sector $\varSigma_1$. In treating the sector $\varSigma_1$, we express $\varDelta(\rho)$ in the form

$$\begin{aligned}\varDelta(\rho) = \mathrm{e}^{-\rho}\big\{&\mathbb{Q}_0(\rho)\mathrm{e}^{\rho}\mathrm{e}^{\mathrm{i}\rho} + \mathbb{Q}_3(\rho)\\ &+ \mathbb{Q}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \mathbb{Q}_2(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho} + \mathbb{D}(\rho)\mathrm{e}^{\mathrm{i}\rho}\big\}\end{aligned} \tag{10.50}$$

for $\rho \neq 0$ in $\mathbb{C}$. Take any point $\rho = a + \mathrm{i}b = |\rho|\mathrm{e}^{\mathrm{i}\theta} \neq 0$ in the sector $\varSigma_1$, so $\pi/4 - \sigma_0/4 \leq \theta \leq \pi/4 + \sigma_0/4$. Then

$$\begin{aligned} a &= |a| = |\rho| \cos\theta \geq |\rho| \cos(\pi/4 + \sigma_0/4) = |\rho| \sin(\pi/4 - \sigma_0/4) = \beta|\rho|, \\ b &= |b| = |\rho| \sin\theta \geq |\rho| \sin(\pi/4 - \sigma_0/4) = \beta|\rho|, \end{aligned}$$

and the exponentials in (10.50) satisfy the conditions

$$|\mathrm{e}^{\rho}\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{a}\mathrm{e}^{-b}, \quad |\mathrm{e}^{0\rho}| = 1,$$
$$|\mathrm{e}^{2\mathrm{i}\rho}| = \mathrm{e}^{-2b} \leq \mathrm{e}^{-2\beta|\rho|} \leq \mathrm{e}^{-\beta|\rho|}, \quad |\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-a}\mathrm{e}^{-b} \leq \mathrm{e}^{-2\beta|\rho|} \leq \mathrm{e}^{-\beta|\rho|},$$
$$|\mathrm{e}^{\mathrm{i}\rho}| = \mathrm{e}^{-b} \leq \mathrm{e}^{-\beta|\rho|}.$$

On the ray $\arg\rho = \pi/4$ the exponentials $\mathrm{e}^{\rho}\mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{0\rho} = 1$ have modulus 1.

Next, we introduce the sector

$$\Sigma_{\oplus}\colon \text{ all } z = |z|\mathrm{e}^{\mathrm{i}\phi} \in \mathbb{C} \text{ with } -\frac{\sigma_0}{4} \leq \phi \leq \frac{\sigma_0}{4}\,.$$

Set $\omega := (1 + \mathrm{i})/2$, and let us make the change of variable

$$\rho = \omega z, \qquad |\rho| = \frac{|z|}{\sqrt{2}}\,.$$

Clearly $\rho \in \Sigma_1$ if and only if $z \in \Sigma_{\oplus}$. Now

$$(1+\mathrm{i})\rho = (1+\mathrm{i})\left(\frac{1+\mathrm{i}}{2}\right)z = \mathrm{i}z, \quad 0\rho = 0z = 0,$$
$$2\mathrm{i}\rho = 2\mathrm{i}\left(\frac{1+\mathrm{i}}{2}\right)z = (-1+\mathrm{i})z, \quad (-1+\mathrm{i})\rho = (-1+\mathrm{i})\left(\frac{1+\mathrm{i}}{2}\right)z = -z,$$
$$\mathrm{i}\rho = \mathrm{i}\left(\frac{1+\mathrm{i}}{2}\right)z = \frac{1}{2}(-1+\mathrm{i})z,$$

so the above exponentials transform to

$$\mathrm{e}^{\mathrm{i}z}, \;\; |\mathrm{e}^{\mathrm{i}z}| = \mathrm{e}^{-\operatorname{Im} z}, \qquad \mathrm{e}^{0z} = 1, \;\; |\mathrm{e}^{0z}| = 1,$$
$$\mathrm{e}^{-z}\mathrm{e}^{\mathrm{i}z}, \;\; |\mathrm{e}^{-z}\mathrm{e}^{\mathrm{i}z}| \leq \mathrm{e}^{-\frac{\beta}{\sqrt{2}}|z|}, \qquad \mathrm{e}^{-z}, \;\; |\mathrm{e}^{-z}| \leq \mathrm{e}^{-\frac{\beta}{\sqrt{2}}|z|},$$
$$\mathrm{e}^{-\frac{1}{2}z}\mathrm{e}^{\frac{i}{2}z}, \;\; |\mathrm{e}^{-\frac{1}{2}z}\mathrm{e}^{\frac{i}{2}z}| \leq \mathrm{e}^{-\frac{\beta}{\sqrt{2}}|z|},$$

the estimates being valid for all z in the sector $\Sigma_{\oplus}$.

At this point we introduce a modified form of the characteristic determinant based on the representation (10.50):

$$\begin{aligned} \Delta_{\oplus}(z) &:= \Delta(\omega z) \\ &= \mathrm{e}^{-\omega z}\big\{\mathbb{Q}_0(\omega z)\mathrm{e}^{\mathrm{i}z} + \mathbb{Q}_3(\omega z) \\ &\quad + \mathbb{Q}_1(\omega z)\mathrm{e}^{-z}\mathrm{e}^{\mathrm{i}z} + \mathbb{Q}_2(\omega z)\mathrm{e}^{-z} + \mathbb{D}(\omega z)\mathrm{e}^{-\frac{1}{2}z}\mathrm{e}^{\frac{\mathrm{i}}{2}z}\big\} \end{aligned} \tag{10.51}$$

for $z \neq 0$ in $\mathbb{C}$. In this last equation we have

$$\mathbb{Q}_0(\omega z) = \sum_{\kappa=3}^{q} b_\kappa \omega^\kappa z^\kappa, \quad \mathbb{Q}_3(\omega z) = -\mathbb{Q}_0(-\mathrm{i}\omega z) = -\sum_{\kappa=3}^{q} b_\kappa (-\mathrm{i})^\kappa \omega^\kappa z^\kappa.$$

Set $\xi_0 := (-\mathrm{i})^q$,

$$f(z) := b_q \omega^q \mathrm{e}^{\mathrm{i}z} - b_q(-\mathrm{i})^q \omega^q = b_q \omega^q \bigl[\mathrm{e}^{\mathrm{i}z} - \xi_0\bigr]$$

for $z \in \mathbb{C}$, and

$$\begin{aligned} g(z) :=& \sum_{\kappa=3}^{q-1} \frac{b_\kappa \omega^\kappa}{z^{q-\kappa}} \mathrm{e}^{\mathrm{i}z} - \sum_{\kappa=3}^{q-1} \frac{b_\kappa (-\mathrm{i})^\kappa \omega^\kappa}{z^{q-k}} \\ &+ \frac{1}{z^q} \bigl[\mathbb{Q}_1(\omega z)\mathrm{e}^{-z}\mathrm{e}^{\mathrm{i}z} + \mathbb{Q}_2(\omega z)\mathrm{e}^{-z} + \mathbb{D}(\omega z)\mathrm{e}^{-\frac{1}{2}z}\mathrm{e}^{\frac{\mathrm{i}}{2}z}\bigr] \end{aligned}$$

for $z \neq 0$ in $\mathbb{C}$. Then equation (10.51) can be rewritten in the simpler form

$$\Delta_\oplus(z) = z^q \mathrm{e}^{-\omega z}\bigl[f(z) + g(z)\bigr] \tag{10.52}$$

for $z \neq 0$ in $\mathbb{C}$, where

$$|\mathrm{e}^{-z}\mathrm{e}^{\mathrm{i}z}| \le \mathrm{e}^{-\frac{\beta}{\sqrt{2}}|z|}, \qquad |\mathrm{e}^{-z}| \le \mathrm{e}^{-\frac{\beta}{\sqrt{2}}|z|}, \qquad |\mathrm{e}^{-\frac{1}{2}z}\mathrm{e}^{\frac{\mathrm{i}}{2}z}| \le \mathrm{e}^{-\frac{\beta}{\sqrt{2}}|z|} \tag{10.53}$$

for z in the sector $\Sigma_\oplus$. Clearly the zeros of $f(z)$ are given by the sequence

$$\mu_k = 2\pi k + \operatorname{Arg}\xi_0, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{10.54}$$

Each μ_k is a zero of order 1 of f, and all of these zeros are real.

Proceeding as in Chapter 8, Case 1, or as in [34, pp. 146–152], it follows that $\Delta_\oplus(z)$ has a sequence of zeros z_k, $k = k_0, k_0{+}1, \ldots$, in the sector $\Sigma_\oplus$ with each z_k being a zero of order 1 of $\Delta_\oplus(z)$. These zeros satisfy the asymptotic formulas

$$|z_k - \mu_k| \le \frac{\gamma}{k}, \qquad k = k_0, k_0 + 1, \ldots, \tag{10.55}$$

and are approaching the positive real axis as $k \to \infty$. It is immediate that the characteristic determinant $\Delta(\rho)$ has a sequence of zeros

$$\rho_k = \omega z_k, \qquad k = k_0, k_0 + 1, \ldots,$$

in the sector Σ_1, each ρ_k being a zero of order 1 of $\Delta(\rho)$, with the ρ_k approaching the ray $\arg\rho = \pi/4$ as $k \to \infty$. The sequence

$$\lambda_k = (\rho_k)^4, \qquad k = k_0, k_0 + 1, \ldots,$$

is a sequence of eigenvalues for the differential operator $L = T$, with the λ_k approaching the *negative* real axis as $k \to \infty$. These eigenvalues account for all but a finite number of the eigenvalues of L. The corresponding algebraic multiplicities and ascents are

$$\nu(\lambda_k) = m(\lambda_k) = 1, \qquad k = k_0, k_0 + 1, \ldots.$$

Here in Case II we have found only one sequence of eigenvalues, with these eigenvalues approaching in the limit the negative real axis. This behavior is very different from our previous work where L is either regular or simply irregular. Cf. Theorems 7.2, 7.3, and 7.5.

Case III. $n = 4$, $\mathbb{P}_0(\rho) \equiv 0$, $\mathbb{Q}_0(\rho) \equiv 0$. For this case equation (10.44) for the characteristic determinant simplifies dramatically to

$$\Delta(\rho) = \mathbb{D}(\rho)\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho} = 8\mathrm{i}\gamma_0\rho^6\mathrm{e}^{-\rho}\mathrm{e}^{\mathrm{i}\rho} \tag{10.56}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the constant γ_0 is defined as in Case II by equation (10.47). Thus, the nonzero part of the spectrum $\sigma(L)$ is determined by the constant γ_0. Again by Definition 3.2 the differential operator $L = T$ is degenerate irregular.

Now in our previous work we have shown that $\lambda = 0$ is an eigenvalue for L if and only if $\Delta_0 = 0$, where the constant Δ_0 is given by equation (10.46). We assert that $\Delta_0 = 6\gamma_0$, and hence, $\lambda = 0$ is an eigenvalue for L if and only if $\gamma_0 = 0$. First, we already know that

$$\langle \alpha_1, \beta_1, \alpha_0, \beta_0 \rangle = 0. \tag{10.57}$$

Second, in (10.41) and (10.42) the ρ^3 coefficient must vanish:

$$\langle \alpha_2, \beta_1, \alpha_0, \beta_0 \rangle + \langle \beta_2, \alpha_1, \alpha_0, \beta_0 \rangle = 0, \quad \langle \alpha_2, \alpha_1, \alpha_0, \beta_0 \rangle + \langle \beta_2, \beta_1, \alpha_0, \beta_0 \rangle = 0,$$

and it follows that

$$\langle \alpha_2, \alpha_1, \alpha_0, \beta_0 \rangle - 2\langle \alpha_2, \beta_1, \alpha_0, \beta_0 \rangle - 2\langle \beta_2, \alpha_1, \alpha_0, \beta_0 \rangle + \langle \beta_2, \beta_1, \alpha_0, \beta_0 \rangle = 0. \tag{10.58}$$

Third, the vanishing of the ρ^4 terms in (10.41) and (10.42) yields the equations

$$\begin{gathered}\langle \alpha_3, \beta_1, \alpha_0, \beta_0 \rangle - \langle \beta_3, \alpha_1, \alpha_0, \beta_0 \rangle - 2\langle \alpha_2, \beta_2, \alpha_0, \beta_0 \rangle \\ -\langle \alpha_2, \alpha_1, \beta_1, \beta_0 \rangle + \langle \beta_2, \alpha_1, \beta_1, \alpha_0 \rangle = 0, \\ \langle \alpha_3, \alpha_1, \alpha_0, \beta_0 \rangle - \langle \beta_3, \beta_1, \alpha_0, \beta_0 \rangle + \langle \alpha_2, \alpha_1, \beta_1, \alpha_0 \rangle - \langle \beta_2, \alpha_1, \beta_1, \beta_0 \rangle = 0,\end{gathered}$$

and hence,

$$\begin{gathered}\langle \alpha_3, \alpha_1, \alpha_0, \beta_0 \rangle + \langle \alpha_2, \alpha_1, \beta_1, \alpha_0 \rangle - \langle \alpha_3, \beta_1, \alpha_0, \beta_0 \rangle \\ +\langle \alpha_2, \alpha_1, \beta_1, \beta_0 \rangle + 2\langle \alpha_2, \beta_2, \alpha_0, \beta_0 \rangle + \langle \beta_3, \alpha_1, \alpha_0, \beta_0 \rangle \\ -\langle \beta_2, \alpha_1, \beta_1, \alpha_0 \rangle - \langle \beta_2, \alpha_1, \beta_1, \beta_0 \rangle - \langle \beta_3, \beta_1, \alpha_0, \beta_0 \rangle = 0.\end{gathered} \tag{10.59}$$

Fourth, the vanishing of the ρ^5 terms in (10.41) and (10.42) implies that

$$\begin{gathered}\langle \alpha_3, \beta_2, \alpha_0, \beta_0 \rangle + \langle \alpha_3, \alpha_1, \beta_1, \beta_0 \rangle + \langle \beta_3, \alpha_2, \alpha_0, \beta_0 \rangle \\ +\langle \beta_3, \alpha_1, \beta_1, \alpha_0 \rangle + \langle \alpha_2, \beta_2, \alpha_1, \beta_0 \rangle + \langle \alpha_2, \beta_2, \beta_1, \alpha_0 \rangle = 0, \\ \langle \alpha_3, \alpha_2, \alpha_0, \beta_0 \rangle + \langle \beta_3, \beta_2, \alpha_0, \beta_0 \rangle + \langle \alpha_3, \alpha_1, \beta_1, \alpha_0 \rangle \\ +\langle \beta_3, \alpha_1, \beta_1, \beta_0 \rangle + \langle \alpha_2, \beta_2, \alpha_1, \alpha_0 \rangle + \langle \alpha_2, \beta_2, \beta_1, \beta_0 \rangle = 0,\end{gathered}$$

and it follows that

$$\begin{aligned}&\langle\alpha_3,\alpha_2,\alpha_0,\beta_0\rangle+\langle\alpha_3,\alpha_1,\beta_1,\alpha_0\rangle+\langle\alpha_2,\beta_2,\alpha_1,\alpha_0\rangle\\+&\langle\alpha_3,\alpha_1,\beta_1,\beta_0\rangle+\langle\alpha_3,\beta_2,\alpha_0,\beta_0\rangle+\langle\alpha_2,\beta_2,\alpha_1,\beta_0\rangle\\+&\langle\beta_3,\alpha_2,\alpha_0,\beta_0\rangle+\langle\alpha_2,\beta_2,\beta_1,\alpha_0\rangle+\langle\beta_3,\alpha_1,\beta_1,\alpha_0\rangle\\+&\langle\alpha_2,\beta_2,\beta_1,\beta_0\rangle+\langle\beta_3,\alpha_1,\beta_1,\beta_0\rangle+\langle\beta_3,\beta_2,\alpha_0,\beta_0\rangle=0.\end{aligned}\tag{10.60}$$

Fifth, the vanishing of the ρ^6 terms in (10.42) and (10.41) gives

$$\begin{aligned}&\langle\alpha_3,\alpha_2,\alpha_1,\beta_0\rangle+\langle\alpha_3,\alpha_2,\beta_1,\alpha_0\rangle+\langle\alpha_3,\beta_2,\alpha_1,\alpha_0\rangle\\+&\langle\alpha_3,\beta_2,\beta_1,\beta_0\rangle+\langle\beta_3,\alpha_2,\alpha_1,\alpha_0\rangle+\langle\beta_3,\alpha_2,\beta_1,\beta_0\rangle\\+&\langle\beta_3,\beta_2,\alpha_1,\beta_0\rangle+\langle\beta_3,\beta_2,\beta_1,\alpha_0\rangle=0\end{aligned}\tag{10.61}$$

and

$$\begin{aligned}2\langle\alpha_3,\beta_2,&\alpha_1,\beta_0\rangle+2\langle\beta_3,\alpha_2,\beta_1,\alpha_0\rangle\\&=\langle\alpha_3,\beta_3,\alpha_0,\beta_0\rangle-\langle\alpha_3,\alpha_2,\beta_1,\beta_0\rangle-\langle\alpha_3,\beta_2,\beta_1,\alpha_0\rangle\\&\quad-\langle\beta_3,\alpha_2,\alpha_1,\beta_0\rangle-\langle\beta_3,\beta_2,\alpha_1,\alpha_0\rangle+\langle\alpha_2,\beta_2,\alpha_1,\beta_1\rangle.\end{aligned}\tag{10.62}$$

If we now substitute (10.57)–(10.61) into the expression (10.46) for the constant Δ_0, then it simplifies to give

$$\begin{aligned}\Delta_0=6[&2\langle\alpha_3,\alpha_2,\alpha_1,\alpha_0\rangle+2\langle\alpha_3,\alpha_2,\beta_1,\beta_0\rangle+2\langle\alpha_3,\beta_2,\alpha_1,\beta_0\rangle\\&+2\langle\alpha_3,\beta_2,\beta_1,\alpha_0\rangle+2\langle\beta_3,\alpha_2,\alpha_1,\beta_0\rangle+2\langle\beta_3,\alpha_2,\beta_1,\alpha_0\rangle\\&+2\langle\beta_3,\beta_2,\alpha_1,\alpha_0\rangle+2\langle\beta_3,\beta_2,\beta_1,\beta_0\rangle],\end{aligned}\tag{10.63}$$

and if we finally substitute (10.62) into (10.63), then we arrive at the result

$$\begin{aligned}\Delta_0=6[&2\langle\alpha_3,\alpha_2,\alpha_1,\alpha_0\rangle+\langle\alpha_3,\alpha_2,\beta_1,\beta_0\rangle+\langle\alpha_3,\beta_3,\alpha_0,\beta_0\rangle\\&+\langle\alpha_2,\beta_2,\alpha_1,\beta_1\rangle+\langle\alpha_3,\beta_2,\beta_1,\alpha_0\rangle+\langle\beta_3,\alpha_2,\alpha_1,\beta_0\rangle\\&+\langle\beta_3,\beta_2,\alpha_1,\alpha_0\rangle+2\langle\beta_3,\beta_2,\beta_1,\beta_0\rangle]=6\gamma_0.\end{aligned}\tag{10.64}$$

This establishes the assertion.

In view of the above, we conclude that if $\gamma_0 \neq 0$, then

$$\sigma(L)=\emptyset\quad\text{and}\quad\rho(L)=\mathbb{C},\tag{10.65}$$

while if $\gamma_0 = 0$, then

$$\sigma(L)=\mathbb{C}\quad\text{and}\quad\rho(L)=\emptyset.\tag{10.66}$$

Example 10.1 and Example 10.2 with $n = 4$ are models for these two extremely degenerate irregular cases.

10.4 Some Results for the Special Case $n = 2\nu$, $L = T$

Some of the above results for the case $n = 4$ can be generalized to higher order n. We will only sketch the generalizations here. Assume that $n = 2\nu \geq 6$ and $L = T$. From equation (10.4) the characteristic determinant is given by

$$\begin{aligned}\Delta(\rho) &= \det(B_i(y_k(\,\cdot\,,\rho))) \\ &\qquad\qquad \begin{array}{cc} 0 \leq k \leq \nu-1 & \qquad\qquad \nu \leq k \leq n-1 \end{array} \\ &= \det\begin{pmatrix} P_1(\mathrm{i}\rho\omega_k) + Q_1(\mathrm{i}\rho\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k} & Q_1(\mathrm{i}\rho\omega_k) + P_1(\mathrm{i}\rho\omega_k)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \\ \vdots & \vdots \\ P_n(\mathrm{i}\rho\omega_k) + Q_n(\mathrm{i}\rho\omega_k)\mathrm{e}^{\mathrm{i}\rho\omega_k} & Q_n(\mathrm{i}\rho\omega_k) + P_n(\mathrm{i}\rho\omega_k)\mathrm{e}^{-\mathrm{i}\rho\omega_k} \end{pmatrix}\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where $\mathrm{e}^{\mathrm{i}\rho\omega_0} = \mathrm{e}^{-\mathrm{i}\rho\omega_\nu} = \mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{-\mathrm{i}\rho\omega_{n-1}}$. Suppose we expand the determinant for $\Delta(\rho)$ using linearity in the 0th column, $(\nu-1)$st column, νth column, and $(n-1)$st column. This expansion expresses $\Delta(\rho)$ as a sum of sixteen terms: each term consists of an exponential formed from $\mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$ multiplied by a determinant that is independent of $\mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$. The sixteen exponentials that appear are as follows:

$$\begin{gathered} \mathrm{e}^{0\rho} = 1, \\ \mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{\mathrm{i}\rho}, \quad \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \quad \mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{\mathrm{i}\rho}, \quad \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{2\mathrm{i}\rho}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \\ \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \quad \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} = \mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \quad \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}, \quad \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}, \\ \mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = \mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}. \end{gathered}$$

There are nine distinct exponentials appearing in this list: the exponentials $\mathrm{e}^{0\rho} = 1$, $\mathrm{e}^{2\mathrm{i}\rho}$, $\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}$, $\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}$ each occurs once in the list; the exponentials $\mathrm{e}^{\mathrm{i}\rho}$, $\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$, $\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}$, $\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$ each occurs twice in the list; and the exponential $\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$ occurs four times in the list. Therefore, upon expansion $\Delta(\rho)$ takes the form

$$\begin{aligned}\Delta(\rho) &= D_{10}(\rho) + D_{11}(\rho)\mathrm{e}^{2\mathrm{i}\rho} + D_{12}(\rho)\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + D_{13}(\rho)\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} \\ &\quad + D_{20}(\rho)\mathrm{e}^{\mathrm{i}\rho} + D_{21}(\rho)\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} + D_{22}(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + D_{23}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ &\quad + D_{40}(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\end{aligned} \tag{10.67}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the function $D_{ij}(\rho)$ is an entire function that is the sum of i determinants.

Next, in the representation of $D_{ij}(\rho)$ as a sum of determinants, we proceed to expand each determinant using linearity in columns 1 through $\nu - 2$ and

$\nu + 1$ through $n - 2$. This is the same expansion used earlier in Chapter 3 and Chapter 5; it expresses each $D_{ij}(\rho)$ in the form

$$D_{ij}(\rho) = \mathcal{P}_{ij}(\rho) + \mathcal{Q}_{ij}(\rho),$$

where $\mathcal{P}_{ij}(\rho)$ is a polynomial of degree $\le p_0$ that has a well-defined form and structure, and where the entire function $\mathcal{Q}_{ij}(\rho)$ is a sum of terms each being the product of a polynomial of degree $\le p_0$ times a product of some of the exponentials $\mathrm{e}^{\mathrm{i}\rho\omega_k}$, $k = 1, \ldots, \nu - 2$, or $\mathrm{e}^{-\mathrm{i}\rho\omega_k}$, $k = \nu + 1, \ldots, n - 2$ (at least one of these exponentials appears in each such product). Upon substituting these forms into equation (10.67), we arrive at our principal representation of the characteristic determinant:

$$\begin{aligned} \Delta(\rho) = {} & [\mathbb{P}_0(\rho) + \Theta_0(\rho)] + [\mathbb{P}_1(\rho) + \Theta_1(\rho)]\mathrm{e}^{2\mathrm{i}\rho} + [\mathbb{P}_2(\rho) + \Theta_2(\rho)]\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} \\ & + [\mathbb{P}_3(\rho) + \Theta_3(\rho)]\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{Q}_0(\rho) + \Psi_0(\rho)]\mathrm{e}^{\mathrm{i}\rho} \\ & + [\mathbb{Q}_1(\rho) + \Psi_1(\rho)]\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{Q}_2(\rho) + \Psi_2(\rho)]\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} \\ & + [\mathbb{Q}_3(\rho) + \Psi_3(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{D}(\rho) + \Upsilon(\rho)]\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \end{aligned} \tag{10.68}$$

for $\rho \ne 0$ in $\mathbb{C}$, where the functions $\mathbb{P}_i(\rho)$, $\mathbb{Q}_i(\rho)$, $\mathbb{D}(\rho)$ are polynomials of degree $\le p_0$ that have a well-defined form and structure, and where the functions $\Theta_i(\rho)$, $\Psi_i(\rho)$, $\Upsilon(\rho)$ are entire functions that go to 0 very rapidly as $|\rho| \to \infty$ on various sectors (to be specified below). In (10.68) the polynomials satisfy the relations

$$\mathbb{P}_1(\rho) = -\mathbb{P}_0(\mathrm{i}\rho), \qquad \mathbb{P}_3(\rho) = -\mathbb{P}_0(-\mathrm{i}\rho), \qquad \mathbb{Q}_3(\rho) = -\mathbb{Q}_0(-\mathrm{i}\rho). \tag{10.69}$$

$\mathbb{P}_2(\rho)$ is not related to $\mathbb{P}_0(\rho)$, nor are $\mathbb{Q}_1(\rho)$ and $\mathbb{Q}_2(\rho)$ related to $\mathbb{Q}_0(\rho)$, as in the case $n = 4$. Again we have

$$\pi_2(\rho) = \mathbb{P}_1(\rho) = -\mathbb{P}_0(\mathrm{i}\rho), \qquad \pi_1(\rho) = \mathbb{Q}_0(\rho), \qquad \pi_0(\rho) = \mathbb{P}_0(\rho) \tag{10.70}$$

for $\rho \ne 0$ in $\mathbb{C}$.

The representation (10.68) of the characteristic determinant $\Delta(\rho)$ generalizes the previous representations (3.32) and (5.31), and the representation (10.44) for the case $n = 4$. It displays not only the primary terms

$$\pi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho} = \mathbb{P}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho} = -\mathbb{P}_0(\mathrm{i}\rho)\mathrm{e}^{2\mathrm{i}\rho}, \quad \pi_1(\rho)\mathrm{e}^{\mathrm{i}\rho} = \mathbb{Q}_0(\rho)\mathrm{e}^{\mathrm{i}\rho}, \quad \pi_0(\rho) = \mathbb{P}_0(\rho),$$

but also some of the secondary terms which come into play in the degenerate irregular case. In particular, we will show below the important role played by the term

$$\mathbb{Q}_3(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} = -\mathbb{Q}_0(-\mathrm{i}\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}.$$

Let us now consider some cases which are higher order analogues of the previous cases for $n = 4$.

Case I. $n = 2\nu \geq 6$, $\mathbb{P}_0(\rho) \not\equiv 0$. For this case the two polynomials $\pi_0(\rho) = \mathbb{P}_0(\rho)$ and $\pi_2(\rho) = -\mathbb{P}_0(\mathrm{i}\rho)$ are both of degree p with $0 \leq p \leq p_0$, and hence, the differential operator $L = T$ is either regular or simply irregular. The spectrum of L is characterized in Theorems 7.2, 7.3, and 7.5, and the generalized eigenfunctions of L are shown to be complete in $L^2[0,1]$ in Theorem 9.1.

Case II. $n = 2\nu \geq 6$, $\mathbb{P}_0(\rho) \equiv 0$, $\mathbb{Q}_0(\rho) \not\equiv 0$. Since the polymonials $\pi_0(\rho) = \mathbb{P}_0(\rho)$ and $\pi_2(\rho) = -\mathbb{P}_0(\mathrm{i}\rho)$ are both identically zero, the differential operator $L = T$ is degenerate irregular. From equation (10.68) the characteristic determinant now takes the simpler form

$$\begin{aligned}\Delta(\rho) = {} & [\mathbb{Q}_0(\rho) + \Psi_0(\rho)]\mathrm{e}^{\mathrm{i}\rho} + [\mathbb{Q}_3(\rho) + \Psi_3(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ & + [\mathbb{P}_2(\rho) + \Theta_2(\rho)]\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{Q}_1(\rho) + \Psi_1(\rho)]\mathrm{e}^{2\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ & + [\mathbb{Q}_2(\rho) + \Psi_2(\rho)]\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{D}(\rho) + \Upsilon(\rho)]\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ & + \Theta_0(\rho) + \Theta_1(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \Theta_3(\rho)\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}\end{aligned} \tag{10.71}$$

for $\rho \neq 0$ in $\mathbb{C}$. Let q denote the degree of the polynomial $\mathbb{Q}_0(\rho) = \pi_1(\rho)$, so $0 \leq q \leq p_0$ and $\mathbb{Q}_0(\rho) = \sum_{\kappa=0}^{q} b_\kappa \rho^\kappa$ with $b_q \neq 0$. Fix a real number σ_0 with $0 < \sigma_0 < \pi/10$.

First, let us study the behavior of $\Delta(\rho)$ on the sector

$$\Sigma_0\colon \ \text{all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } -\frac{\pi}{n} + \frac{\sigma_0}{n} \leq \theta \leq \frac{\pi}{n} - \frac{\sigma_0}{n}\,.$$

Let $h(\rho)$ be the analytic function defined by

$$\begin{aligned}h(\rho) := {} & \sum_{\kappa=0}^{q-1} \frac{b_\kappa}{\rho^{q-\kappa}} + \frac{1}{\rho^q}\Big\{\Psi_0(\rho) + [\mathbb{Q}_3(\rho) + \Psi_3(\rho)]\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ & + [\mathbb{P}_2(\rho) + \Theta_2(\rho)]\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{Q}_1(\rho) + \Psi_1(\rho)]\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ & + [\mathbb{Q}_2(\rho) + \Psi_2(\rho)]\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}} + [\mathbb{D}(\rho) + \Upsilon(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}} \\ & + \Theta_0(\rho)\mathrm{e}^{-\mathrm{i}\rho} + \Theta_1(\rho)\mathrm{e}^{\mathrm{i}\rho} + \Theta_3(\rho)\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}\Big\}\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. Then equation (10.71) can be rewritten in the compact form

$$\Delta(\rho) = \rho^q \mathrm{e}^{\mathrm{i}\rho}\big[b_q + h(\rho)\big] \tag{10.72}$$

for $\rho \neq 0$ in $\mathbb{C}$. This is the form best-suited for treating the characteristic determinant on the sector Σ_0.

Let $\alpha := \sin(\pi/n + \sigma_0/n) > 0$ and $\beta := \sin(\pi/n - \sigma_0/n) > 0$, and then choose a constant ξ such that $\beta < \xi < \alpha$ and set $\eta := \xi - \beta > 0$. For the exponentials appearing in (10.71) and (10.72) either explicitly or implicitly, we have the estimates

$$|\mathrm{e}^{\mathrm{i}\rho\omega_k}| \leq \mathrm{e}^{-\alpha|\rho|}, \qquad k = 1, \ldots, \nu - 1,$$

and

$$|\mathrm{e}^{-\mathrm{i}\rho\omega_k}| \le \mathrm{e}^{-\alpha|\rho|}, \qquad k = \nu+1, \dots, n-1,$$

for $\rho \in \Sigma_0$, which leads to the estimates

$$|\Theta_i(\rho)| \le \gamma_0 \mathrm{e}^{-\xi|\rho|}, \quad |\Psi_i(\rho)| \le \gamma_0 \mathrm{e}^{-\xi|\rho|}, \quad |\Upsilon(\rho)| \le \gamma_0 \mathrm{e}^{-\xi|\rho|}$$

for $\rho \in \Sigma_0$ and for $i = 0, 1, 2, 3$. The exponentials $\mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{-\mathrm{i}\rho}$ are not bounded on the sector Σ_0, but they do satisfy the growth rates

$$|\mathrm{e}^{\mathrm{i}\rho}| \le \mathrm{e}^{\beta|\rho|} \quad \text{and} \quad |\mathrm{e}^{-\mathrm{i}\rho}| \le \mathrm{e}^{\beta|\rho|}$$

for $\rho \in \Sigma_0$.

Choose a constant $\gamma_1 > 0$ such that

$$|\mathbb{P}_i(\rho)|\mathrm{e}^{-(\alpha-\beta)} \le \gamma_1 \mathrm{e}^{-\eta|\rho|}, \quad |\mathbb{Q}_i(\rho)|\mathrm{e}^{-(\alpha-\beta)} \le \gamma_1 \mathrm{e}^{-\eta|\rho|},$$
$$|\mathbb{D}(\rho)|\mathrm{e}^{-(\alpha-\beta)} \le \gamma_1 \mathrm{e}^{-\eta|\rho|}$$

for all $\rho \in \mathbb{C}$ and for $i = 0, 1, 2, 3$. In terms of these constants we have

$$|\mathbb{Q}_3(\rho)\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_1 \mathrm{e}^{-\eta|\rho|}, \quad |\mathbb{P}_2(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_1 \mathrm{e}^{-\eta|\rho|},$$
$$|\mathbb{Q}_1(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_1 \mathrm{e}^{-\eta|\rho|}, \quad |\mathbb{Q}_2(\rho)\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_1 \mathrm{e}^{-\eta|\rho|},$$
$$|\mathbb{D}(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_1 \mathrm{e}^{-\eta|\rho|}$$

for $\rho \in \Sigma_0$, and

$$|\Psi_0(\rho)| \le \gamma_0 \mathrm{e}^{-\eta|\rho|}, \quad |\Psi_3(\rho)\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|},$$
$$|\Theta_2(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|}, \quad |\Psi_1(\rho)\mathrm{e}^{\mathrm{i}\rho}\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|},$$
$$|\Psi_2(\rho)\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|}, \quad |\Upsilon(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|},$$
$$|\Theta_0(\rho)\mathrm{e}^{-\mathrm{i}\rho}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|}, \quad |\Theta_1(\rho)\mathrm{e}^{\mathrm{i}\rho}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|},$$
$$|\Theta_3(\rho)\mathrm{e}^{-\mathrm{i}\rho}\mathrm{e}^{2\mathrm{i}\rho\omega_{\nu-1}}| \le \gamma_0 \mathrm{e}^{-\eta|\rho|}$$

for $\rho \in \Sigma_0$.

From the above it is immediate that $|h(\rho)| \le \gamma_2/|\rho|$ for all $\rho \in \Sigma_0$ with $|\rho| \ge 1$, which implies that

$$|\Delta(\rho)| \ge |\rho|^q \mathrm{e}^{-b}\left\{|b_q| - \frac{\gamma_2}{|\rho|}\right\} \ge \frac{|b_q|}{2}|\rho|^q \mathrm{e}^{-b} > 0 \tag{10.73}$$

for all ρ in Σ_0 with $|\rho|$ sufficiently large. We conclude that the characteristic determinant $\Delta(\rho)$ has no zeros in the sector Σ_0 when $|\rho|$ is sufficiently large. Where does $\Delta(\rho)$ have its zeros?

Second, we introduce the sector

$$\Sigma_1\colon \text{ all } \rho = |\rho|\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{C} \text{ with } \frac{\pi}{n} - \frac{\sigma_0}{n} \le \theta \le \frac{\pi}{n} + \frac{\sigma_0}{n}\,,$$

and proceed to look for zeros in the sector Σ_1. We begin by rewriting equation (10.71) in the form

$$\begin{aligned}\Delta(\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\Big\{&[\mathbb{Q}_0(\rho) + \Psi_0(\rho)]\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} + [\mathbb{Q}_3(\rho) + \Psi_3(\rho)]\\ &+ [\mathbb{P}_2(\rho) + \Theta_2(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho} + [\mathbb{Q}_1(\rho) + \Psi_1(\rho)]\mathrm{e}^{2\mathrm{i}\rho}\\ &+ [\mathbb{Q}_2(\rho) + \Psi_2(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} + [\mathbb{D}(\rho) + \Upsilon(\rho)]\mathrm{e}^{\mathrm{i}\rho}\\ &+ \Theta_0(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}} + \Theta_1(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho} + \Theta_3(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\Big\}\end{aligned} \tag{10.74}$$

for $\rho \neq 0$ in $\mathbb{C}$. Note that $\pi/n + \sigma_0/n < 3\pi/n - \sigma_0/n < \pi/2$ because $n \geq 6$. Let $\alpha_1 := \sin(3\pi/n - \sigma_0/n) > \alpha$, and choose constants ξ_1 and η_1 such that $\alpha < \xi_1 < \alpha_1$ and $0 < \eta_1 < \beta$ and $\eta_1 \leq \xi_1 - \alpha$. Then we obtain the estimates

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}\rho\omega_k}| &\leq \mathrm{e}^{-\alpha_1|\rho|}, \qquad && k = 1, \dots, \nu - 2,\\ |\mathrm{e}^{-\mathrm{i}\rho\omega_k}| &\leq \mathrm{e}^{-\alpha_1|\rho|}, \qquad && k = \nu + 1, \dots, n - 2,\end{aligned}$$

for $\rho \in \Sigma_1$, and

$$|\Theta_i(\rho)| \leq \gamma_3\mathrm{e}^{-\xi_1|\rho|}, \quad |\Psi_i(\rho)| \leq \gamma_3\mathrm{e}^{-\xi_1|\rho|}, \quad |\Upsilon(\rho)| \leq \gamma_3\mathrm{e}^{-\xi_1|\rho|}$$

for $\rho \in \Sigma_1$ and for $i = 0, 1, 2, 3$. For the exponentials $\mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}$, we find that

$$\mathrm{e}^{-\alpha|\rho|} \leq |\mathrm{e}^{\mathrm{i}\rho}| \leq \mathrm{e}^{-\beta|\rho|} \quad \text{and} \quad \mathrm{e}^{-\alpha|\rho|} \leq |\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \leq \mathrm{e}^{-\beta|\rho|}$$

for $\rho \in \Sigma_1$. Thus, these two exponentials are going to 0 on Σ_1 as $|\rho| \to \infty$, but at a slower rate than the other exponentials. On the ray $\arg\rho = \pi/n$ we actually have

$$|\mathrm{e}^{\mathrm{i}\rho}| = |\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| = \mathrm{e}^{-|\rho|\sin\pi/n},$$

so the two exponentials are decaying at exactly the same rate on this ray. Also, on the ray $\arg\rho = \pi/n$ the exponentials $\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho}$ and $\mathrm{e}^{0\rho} = 1$ both have modulus 1.

Choose a constant $\gamma_4 > 0$ such that

$$|\mathbb{P}_i(\rho)|\mathrm{e}^{-\beta|\rho|} \leq \gamma_4\mathrm{e}^{-\eta_1|\rho|}, \quad |\mathbb{Q}_i(\rho)|\mathrm{e}^{-\beta|\rho|} \leq \gamma_4\mathrm{e}^{-\eta_1|\rho|}, \quad |\mathbb{D}(\rho)|\mathrm{e}^{-\beta|\rho|} \leq \gamma_4\mathrm{e}^{-\eta_1|\rho|}$$

for all $\rho \in \mathbb{C}$ and for $i = 0, 1, 2, 3$. Combining the above estimates, we get

$$\begin{aligned}|\mathbb{P}_2(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho}| &\leq \gamma_4\mathrm{e}^{-\eta_1|\rho|}, \qquad & |\mathbb{Q}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho}| &\leq \gamma_4\mathrm{e}^{-\eta_1|\rho|},\\ |\mathbb{Q}_2(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho}| &\leq \gamma_4\mathrm{e}^{-\eta_1|\rho|}, \qquad & |\mathbb{D}(\rho)\mathrm{e}^{\mathrm{i}\rho}| &\leq \gamma_4\mathrm{e}^{-\eta_1|\rho|}\end{aligned}$$

for $\rho \in \Sigma_1$, and

$$\begin{aligned}|\Psi_0(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|}, \qquad & |\Psi_3(\rho)| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|},\\ |\Theta_2(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|}, \qquad & |\Psi_1(\rho)\mathrm{e}^{2\mathrm{i}\rho}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|},\\ |\Psi_2(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|}, \qquad & |\Upsilon(\rho)\mathrm{e}^{\mathrm{i}\rho}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|},\\ |\Theta_0(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|}, \qquad & |\Theta_1(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho}| &\leq \gamma_3\mathrm{e}^{-\eta_1|\rho|},\end{aligned}$$

$$|\Theta_3(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}| \leq \gamma_3\mathrm{e}^{-\eta_1|\rho|}$$

for $\rho \in \Sigma_1$. Let $\Phi(\rho)$ be the entire function defined by

$$\begin{aligned}\Phi(\rho) := {} & \Psi_0(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} + \Psi_3(\rho) \\ & + [\mathbb{P}_2(\rho) + \Theta_2(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho} + [\mathbb{Q}_1(\rho) + \Psi_1(\rho)]\mathrm{e}^{2\mathrm{i}\rho} \\ & + [\mathbb{Q}_2(\rho) + \Psi_2(\rho)]\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} + [\mathbb{D}(\rho) + \Upsilon(\rho)]\mathrm{e}^{\mathrm{i}\rho} \\ & + \Theta_0(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}} + \Theta_1(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{2\mathrm{i}\rho} + \Theta_3(\rho)\mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\end{aligned}$$

for $\rho \in \mathbb{C}$. Then the representation (10.74) of the characteristic determinant can be rewritten as

$$\Delta(\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_{\nu-1}}\big\{\mathbb{Q}_0(\rho)\mathrm{e}^{-\mathrm{i}\rho\omega_{\nu-1}}\mathrm{e}^{\mathrm{i}\rho} - \mathbb{Q}_0(-\mathrm{i}\rho) + \Phi(\rho)\big\} \tag{10.75}$$

for $\rho \neq 0$ in $\mathbb{C}$, where $\mathbb{Q}_0(\rho)$ is a polynomial of degree q with $0 \leq q \leq p_0$ and where $\Phi(\rho)$ is an entire function with

$$|\Phi(\rho)| \leq \gamma_5 \mathrm{e}^{-\eta_1|\rho|} \quad \text{for } \rho \in \Sigma_1. \tag{10.76}$$

Finally, we introduce the sector

$$\Sigma_{\oplus}\colon \text{ all } z = |z|\mathrm{e}^{\mathrm{i}\phi} \in \mathbb{C} \text{ with } -\frac{\sigma_0}{n} \leq \phi \leq \frac{\sigma_0}{n}\,.$$

Set $\omega := \mathrm{e}^{\mathrm{i}\pi/n}/(2\cos\pi/n)$, and make the change of variable

$$\rho = \omega z, \qquad |\rho| = |\omega||z| = \frac{|z|}{2\cos\pi/n}\,.$$

Clearly $\rho \in \Sigma_1$ if and only if $z \in \Sigma_{\oplus}$. Note that

$$\begin{aligned}-\omega_{\nu-1} + 1 = {} & \cos\frac{2\pi}{n} - \mathrm{i}\sin\frac{2\pi}{n} + 1 \\ = {} & \Big(2\cos^2\frac{\pi}{n} - 1\Big) - \mathrm{i}\Big(2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\Big) + 1 \\ = {} & \Big(2\cos\frac{\pi}{n}\Big)\Big(\cos\frac{\pi}{n} - \mathrm{i}\sin\frac{\pi}{n}\Big) = \Big(2\cos\frac{\pi}{n}\Big)\mathrm{e}^{-\mathrm{i}\pi/n},\end{aligned}$$

and hence,

$$\mathrm{i}\rho(-\omega_{\nu-1} + 1) = \mathrm{i}z.$$

Next, we introduce a modified form of the characteristic determinant:

$$\Delta_{\oplus}(z) := \Delta(\omega z) = \mathrm{e}^{\mathrm{i}\omega z\omega_{\nu-1}}\big\{\mathbb{Q}_0(\omega z)\mathrm{e}^{\mathrm{i}z} - \mathbb{Q}_0(-\mathrm{i}\omega z) + \Phi(\omega z)\big\} \tag{10.77}$$

for $z \neq 0$ in $\mathbb{C}$. Define the constant ξ_0 by $\xi_0 := (-\mathrm{i})^q$, the entire function $f(z)$ by

$$f(z) := b_q\omega^q\mathrm{e}^{\mathrm{i}z} - b_q(-\mathrm{i})^q\omega^q = b_q\omega^q\big[\mathrm{e}^{\mathrm{i}z} - \xi_0\big]$$

for $z \in \mathbb{C}$, and the analytic function $g(z)$ by

$$g(z) := \sum_{\kappa=0}^{q-1} \frac{b_\kappa \omega^\kappa}{z^{q-\kappa}} \, \mathrm{e}^{\mathrm{i}z} - \sum_{\kappa=0}^{q-1} \frac{b_\kappa (-\mathrm{i})^\kappa \omega^\kappa}{z^{q-k}} + \frac{1}{z^q} \, \Phi(\omega z)$$

for $z \neq 0$ in $\mathbb{C}$. In terms of these quantities equation (10.77) becomes

$$\Delta_\oplus(z) = z^q \mathrm{e}^{\mathrm{i}\omega z \omega_{\nu-1}} \big[f(z) + g(z) \big] \tag{10.78}$$

for $z \neq 0$ in $\mathbb{C}$. From equation (10.76) it follows that $|g(z)| \leq \gamma_6/|z|$ for all $z \in \Sigma_\oplus$ with $|z| \geq 1$.

The rest of the discussion is identical to the case $n = 4$. First, the zeros of the function $f(z)$ are given by the sequence

$$\mu_k = 2\pi k + \operatorname{Arg} \xi_0, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Each of these zeros is a zero of order 1, and all are real-valued. Second, the function $\Delta_\oplus(z)$ has a sequence of zeros z_k, $k = k_0, k_0 + 1, \ldots$, in the sector $\Sigma_\oplus$, with each a zero of order 1. The z_k satisfy the asymptotic formulas

$$|z_k - \mu_k| \leq \frac{\gamma}{k}, \qquad k = k_0, k_0 + 1, \ldots,$$

and are approaching the positive real axis as $k \to \infty$. They account for all but a finite number of the zeros of $\Delta_\oplus(z)$ in the sector $\Sigma_\oplus$. Third, the sequence

$$\rho_k = \omega z_k, \qquad k = k_0, k_0 + 1, \ldots,$$

is a sequence of zeros for the characteristic determinant $\Delta(\rho)$ in the sector Σ_1. Each ρ_k is a zero of order 1, with the ρ_k approaching the ray $\arg \rho = \pi/n$ as $k \to \infty$. Fourth, the sequence

$$\lambda_k = (\rho_k)^n, \qquad k = k_0, k_0 + 1, \ldots,$$

is a sequence of eigenvalues for the differential operator $L = T$, with the λ_k approaching the *negative* real axis as $k \to \infty$. These eigenvalues account for all but a finite number of the eigenvalues of L. The corresponding algebraic multiplicities and ascents are

$$\nu(\lambda_k) = m(\lambda_k) = 1, \qquad k = k_0, k_0 + 1, \ldots .$$

11

Unsolved Problems

In the previous chapters we have established important results in the spectral theory of two-point differential operators L that are either regular or simply irregular. For these two cases there still remain many unsolved problems, while for the degenerate irregular differential operators the spectral theory is completely wide open. It seems only appropriate to list here some of the important unsolved problems.

Problem 1. Do the formal solutions

$$Z_k(t,\rho) = \mathrm{e}^{\mathrm{i}\rho\omega_k t} \sum_{j=0}^{\infty} Z_{kj}(t)\rho^{-j}, \qquad k = 0, 1, \ldots, n-1,$$

introduced in Chapter 2 converge to actual solutions of the differential equation (2.1)?

Problem 2. Can the characteristic determinants $\Delta_0(\rho)$ and $\Delta_1(\rho)$ of Chapter 5 be obtained as limits of the approximate characteristic determinant $\widehat{\Delta}(\rho, m)$ of Chapter 3 as $m \to \infty$?

Problem 3. For the characteristic determinants in Theorems 5.1 and 5.2, is it true that $\Delta_1(\rho) = \mathrm{e}^{-2\mathrm{i}\rho}\Delta_0(\rho)$ on the horizontal strip $T_0 \cap T_1$? Can the characteristic determinant $\Delta_0(\rho)$ be continued to the region consisting of all points $\rho \in \mathbb{C}$ with $|\rho| > R_0$?

Problem 4. If the differential operator L is simply irregular, is it true that $S_\infty(L) \neq \overline{S_\infty(L)}$? Are the associated projections P_i unbounded? This is true in certain situations for the case $n = 2$, as shown in the two series [25, 26] and [30, 31, 32, 33].

Problem 5. If the differential operator L is degenerate irregular and the spectrum $\sigma(L)$ is a countably infinite set, can it be shown that $\overline{S_\infty(L)} = L^2[0, 1]$?

Models for this type of problem are given in Case II of Chapter 10.

Problem 6. What is the subspace $S_\infty(L)$ when the differential operator L is simply irregular or degenerate irregular? What is $S_\infty(L)$ when $n = 2$, $L = T$, and L is simply irregular? In general, is the domain $\mathcal{D}(L)$ a subset of $S_\infty(L)$?

Problem 7. Is there a natural subdivision of the degenerate irregular case into disjoint subcases having different spectral properties? The cases where the spectrum $\sigma(L)$ is countably infinite, or is empty, or is equal to all of $\mathbb{C}$, should go into different subcases. Is there a spectral theory available for any of these subcases?

Problem 8. For the case $n = 2\nu$ even, it is easy to verify that Dirichlet and Neumann boundary conditions determine a differential operator L that is regular. This is also true for periodic boundary conditons and boundary conditons of Sturm Type [36, pp. 60–63]. Are all self-adjoint differential operators regular? In [25, 26] we have shown this to be true for the case $n = 2$ and $L = T$, because in the irregular cases either the associated projections are unbounded (Case IX and Case XI) or the spectrum is either empty (Case XII) or is equal to all of $\mathbb{C}$ (Case XIII). See [25, p. 556].

Problem 9. For $n = 2\nu$ even, are there examples of differential operators L belonging to Case 2 that are simply irregular, i.e., where the integers p and q satisfy the conditions $p = q < p_0$?

Problem 10. In general, can the spectrum $\sigma(L)$ be a nonempty *finite* set? This is impossible in the special case $n = 2$, $L = T$ (see [25, p. 556]) and in the special case $n = 4$, $L = T$ (see Cases I, II, III in the previous chapter).

Problem 11. In Example 10.2 we gave a model of an even order differential operator $L = T$ with $\sigma(L) = \mathbb{C}$ and $\rho(L) = \emptyset$. Can we find an analogous model of an odd order differential operator $L = T$? Are there more general models with $\sigma(L) = \mathbb{C}$ and $\rho(L) = \emptyset$ where L has variable coefficients?

Problem 12. The rows of the $n \times 2n$ boundary coefficient matrix A,

$$(\alpha_{i\,n-1}, \beta_{i\,n-1}, \alpha_{i\,n-2}, \beta_{i\,n-2}, \dots, \alpha_{i0}, \beta_{i0}), \qquad i = 1, \dots, n,$$

span an $(n-1)$-dimensional linear space $\mathbb{L}$ in the projective space $\mathbb{P}^{2n-1}$. Let the columns of A be denoted by $\gamma_1, \dots, \gamma_{2n}$, and for integers $j_1, \dots, j_n$ with $1 \le j_1 < \cdots < j_n \le 2n$, let $\langle \gamma_{j_1}, \dots, \gamma_{j_n} \rangle$ denote the determinant of the $n \times n$ submatrix of A formed by using the columns $\gamma_{j_1}, \dots, \gamma_{j_n}$. Then the $\binom{2n}{n}$ determinants $\langle \gamma_{j_1}, \dots, \gamma_{j_n} \rangle$ are the Plücker coordinates of the linear space $\mathbb{L}$. For the special case $n = 4$, $L = T$, the 70 Plücker coordinates appear very prominently in equations (10.41), (10.42), and (10.43) for the polynomial coefficients $\mathbb{P}_0(\rho)$, $\mathbb{Q}_0(\rho)$, and $\mathbb{D}(\rho)$ that determine the characteristic determinant $\Delta(\rho)$. For the general case $L = T$, can we use the Plücker coordinates associated with the boundary coefficient matrix A to effectively construct the characteristic determinant?

Problem 13. The spectral theory for the case $n = 1$ and $\ell = (1/\mathrm{i})(d/dt) + q(t)$ has never been developed. It would be instructive to have this elementary case included in the overall picture; it might shed some light on some of the other problems.

Problem 14. For the special case $L = T$, in forming the alternate characteristic determinant $\Delta_*(\rho) = \det(B_i(z_k(\,\cdot\,,\rho)))$ the exponentials

$$\mathrm{e}^{\mathrm{i}\rho(\delta_0\omega_0+\delta_1\omega_1+\cdots+\delta_{n-1}\omega_{n-1})}$$

appear, where the constants δ_i are either 0 or 1. How many *distinct* exponentials are formed from these 2^n exponentials? We know the following results:

$n = 2:$	4 exponentials,	2 distinct exponentials,
$n = 3:$	8 exponentials,	7 distinct exponentials,
$n = 4:$	16 exponentials,	9 distinct exponentials,
$n = 6:$	64 exponentials,	19 distinct exponentials,
$n = 8:$	256 exponentials,	81 distinct exponentials.

Is there a practical way to construct the distinct exponentials so that they can be ordered according to their moduli relative to the sector S_0? Can we calculate the polynomial coefficient of each distinct exponential? The case $n = 4$ has been treated in detail in Chapter 10.

Problem 15. For the special case $n = 8$, $L = T$, we have done some preliminary work with help from Maple. When the characteristic determinant $\Delta(\rho)$ is expanded using linearity in all eight columns, the 256 exponentials that are produced reduce down to 81 distinct exponentials. First, there are the three primary exponentials $\mathrm{e}^{0\rho} = 1$, $\mathrm{e}^{\mathrm{i}\rho}$, $\mathrm{e}^{2\mathrm{i}\rho}$. Second, there are the six secondary exponentials

$$\begin{array}{ccc}
\mathrm{e}^{[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}]\rho}, & \mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1-\frac{\sqrt{2}}{2})\mathrm{i}]\rho}, & \mathrm{e}^{[-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}]\rho}, \\
\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(2-\frac{\sqrt{2}}{2})\mathrm{i}]\rho}, & \mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1+\frac{\sqrt{2}}{2})\mathrm{i}]\rho}, & \mathrm{e}^{[-\frac{\sqrt{2}}{2}+(2+\frac{\sqrt{2}}{2})\mathrm{i}]\rho}.
\end{array}$$

And third, there are 72 additional exponentials of lower order. The ordering of these 81 exponentials is according to their moduli relative to the sector S_0. In terms of these exponentials the characteristic determinant takes the form

$$\begin{aligned}
\Delta(\rho) = {} & \mathbb{P}_0(\rho) + \mathbb{Q}_0(\rho)\mathrm{e}^{\mathrm{i}\rho} + \mathbb{P}_1(\rho)\mathrm{e}^{2\mathrm{i}\rho} + \mathbb{Q}_1(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}]\rho} \\
& + \mathbb{R}_0(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1-\frac{\sqrt{2}}{2})\mathrm{i}]\rho} + \mathbb{S}_0(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}]\rho} + \mathbb{T}_0(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(2-\frac{\sqrt{2}}{2})\mathrm{i}]\rho} \\
& + \mathbb{R}_1(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1+\frac{\sqrt{2}}{2})\mathrm{i}]\rho} + \mathbb{Q}_2(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(2+\frac{\sqrt{2}}{2})\mathrm{i}]\rho} + \cdots
\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$, where the functions $\mathbb{P}_0(\rho), \dots, \mathbb{Q}_2(\rho), \dots$ are polynomials of degree ≤ 44. For the special case $\mathbb{P}_0(\rho) \equiv 0$ and $\mathbb{Q}_0(\rho) \equiv 0$, this representation simplifies to

$$\begin{aligned}\Delta(\rho) &= \mathbb{R}_0(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1-\frac{\sqrt{2}}{2})\mathrm{i}]\rho} + \mathbb{S}_0(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}]\rho} \\ &\qquad + \mathbb{T}_0(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(2-\frac{\sqrt{2}}{2})\mathrm{i}]\rho} + \mathbb{R}_1(\rho)\mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1+\frac{\sqrt{2}}{2})\mathrm{i}]\rho} + \cdots \\ &= \mathrm{e}^{[-\frac{\sqrt{2}}{2}+(1-\frac{\sqrt{2}}{2})\mathrm{i}]\rho}\big\{\mathbb{R}_0(\rho) + \mathbb{S}_0(\rho)\mathrm{e}^{(-1+\sqrt{2})\mathrm{i}\rho} \\ &\qquad + \mathbb{T}_0(\rho)\mathrm{e}^{\mathrm{i}\rho} + \mathbb{R}_1(\rho)\mathrm{e}^{\sqrt{2}\mathrm{i}\rho} + \cdots\big\}\end{aligned}$$

for $\rho \neq 0$ in $\mathbb{C}$. Clearly the differential operator $L = T$ becomes degenerate irregular. How does one calculate the zeros of $\Delta(\rho)$ in this special case? See the paper by Langer [27]. Are the generalized eigenfunctions of L complete in $L^2[0, 1]$? Is it possible that there may be more than two sequences of eigenvalues? Could there be algebraic multiplicities greater than 2? Can we give a specific example that falls within this special case?

12

Appendix

The appendix contains two lemmas which are used to develop the regularity properties for our solutions of the differential equation (2.1). These lemmas generalize well-known results from classical analysis. The first is a complex version of Leibniz's rule from advanced calculus. See [7, p. 68 and p. 73].

Lemma 12.1. *Let $\varDelta_1$ and $\varDelta_2$ be the triangles in $\mathbb{R}^2$ consisting of all points (t, s) satisfying the inequalities $0 \le s \le t \le 1$ and $0 \le t \le s \le 1$, respectively, and let G be an open set in the complex ρ plane; let $\phi_1 \colon \varDelta_1 \times G \to \mathbb{C}$ and $\phi_2 \colon \varDelta_2 \times G \to \mathbb{C}$ be continuous functions, and let ϕ be the function defined by $\phi(t, s, \rho) := \phi_1(t, s, \rho)$ for $0 \le s < t \le 1$, $\rho \in G$, and $\phi(t, s, \rho) := \phi_2(t, s, \rho)$ for $0 \le t < s \le 1$, $\rho \in G$; and let $u \colon [0, 1] \times G \to \mathbb{C}$ be the function defined by*

$$u(t, \rho) := \int_0^1 \phi(t, s, \rho)\, ds \quad \text{for } 0 \le t \le 1,\ \rho \in G.$$

Then u is continuous on $[0, 1] \times G$. Moreover, if $\partial\phi_1/\partial\rho$ exists and is continuous on $\varDelta_1 \times G$ and if $\partial\phi_2/\partial\rho$ exists and is continuous on $\varDelta_2 \times G$, then $\partial u/\partial\rho$ exists and is continuous on $[0, 1] \times G$ with

$$\frac{\partial u}{\partial \rho}(t, \rho) = \int_0^1 \frac{\partial \phi}{\partial \rho}(t, s, \rho)\, ds \quad \text{for } 0 \le t \le 1,\ \rho \in G.$$

Note: we do not make any assumptions about the values of $\phi_1(t, s, \rho)$ and $\phi_2(t, s, \rho)$ when $t = s$, so we can have a jump across the diagonal $t = s$.

Proof. First, fix points $t_0 \in [0, 1]$ and $\rho_0 \in G$, and let us show that u is continuous at the point (t_0, ρ_0). Take any $\epsilon > 0$. Choose $r_0 > 0$ such that the closed disk $D_0 \colon |\rho - \rho_0| \le r_0$ lies in G. Then ϕ_1 and ϕ_2 are continuous on the compact sets $\varDelta_1 \times D_0$ and $\varDelta_2 \times D_0$, respectively. Select a constant $M > 0$ such that $|\phi_1(t, s, \rho)| \le M$ for $(t, s, \rho) \in \varDelta_1 \times D_0$ and such that $|\phi_2(t, s, \rho)| \le M$ for $(t, s, \rho) \in \varDelta_2 \times D_0$. Choose $\delta > 0$ such that $\delta \le r_0$, $\delta \le \epsilon/(6M)$, and such that

$$|\phi_1(t', s', \rho') - \phi_1(t, s, \rho)| \le \epsilon/3$$

for $(t', s', \rho'), (t, s, \rho) \in \varDelta_1 \times D_0$ with $|t' - t|^2 + |s' - s|^2 + |\rho' - \rho|^2 \le \delta^2$, and such that

$$|\phi_2(t', s', \rho') - \phi_2(t, s, \rho)| \le \epsilon/3$$

for $(t', s', \rho'), (t, s, \rho) \in \varDelta_2 \times D_0$ with $|t' - t|^2 + |s' - s|^2 + |\rho' - \rho|^2 \le \delta^2$.

Now take $t \in [0, 1]$ and $\rho \in G$ with $t_0 \le t$ and $|t - t_0|^2 + |\rho - \rho_0|^2 \le \delta^2$. Clearly $|\rho - \rho_0| \le \delta \le r_0$, so $\rho \in D_0$. Then

$$\begin{aligned} u(t, \rho) - u(t_0, \rho_0) &= \int_0^1 [\phi(t, s, \rho) - \phi(t_0, s, \rho_0)]\, ds \\ &= \int_0^{t_0} [\phi_1(t, s, \rho) - \phi_1(t_0, s, \rho_0)]\, ds \\ &\quad + \int_{t_0}^{t} [\phi_1(t, s, \rho) - \phi_2(t_0, s, \rho_0)]\, ds \\ &\quad + \int_{t}^{1} [\phi_2(t, s, \rho) - \phi_2(t_0, s, \rho_0)]\, ds, \end{aligned}$$

and hence, $|u(t, \rho) - u(t_0, \rho_0)| \le \epsilon/3 + 2M \cdot (\epsilon/6M) + \epsilon/3 = \epsilon$. The same estimate is also valid for the case $t \le t_0$. This establishes the continuity of u at (t_0, ρ_0).

Second, assume that $\partial\phi_1/\partial\rho$ exists and is continuous on $\varDelta_1 \times G$ and that $\partial\phi_2/\partial\rho$ exists and is continuous on $\varDelta_2 \times G$. Fix $t_0 \in [0, 1]$ and $\rho_0 \in G$, and let us show that $u(t_0, \rho)$ is a differentiable function of ρ at the point $\rho = \rho_0$. Take any $\epsilon > 0$. As above choose $r_0 > 0$ such that the disk $D_0\colon |\rho - \rho_0| \le r_0$ lies in G. Set

$$\begin{aligned} \psi_1(s, \rho) &:= \frac{\partial\phi_1}{\partial\rho}(t_0, s, \rho) \quad \text{for } 0 \le s \le t_0,\ \rho \in G, \\ \psi_2(s, \rho) &:= \frac{\partial\phi_2}{\partial\rho}(t_0, s, \rho) \quad \text{for } t_0 \le s \le 1,\ \rho \in G, \end{aligned}$$

and then set

$$\begin{aligned} \psi(s, \rho) &:= \psi_1(s, \rho) = \frac{\partial\phi}{\partial\rho}(t_0, s, \rho) \quad \text{for } 0 \le s < t_0,\ \rho \in G, \\ \psi(s, \rho) &:= \psi_2(s, \rho) = \frac{\partial\phi}{\partial\rho}(t_0, s, \rho) \quad \text{for } t_0 < s \le 1,\ \rho \in G. \end{aligned}$$

Choose $\delta > 0$ such that $\delta \le r_0$, such that

$$|\psi_1(s', \rho') - \psi_1(s, \rho)| \le \epsilon$$

for all $(s', \rho'), (s, \rho) \in [0, t_0] \times D_0$ with $|s' - s|^2 + |\rho' - \rho|^2 \le \delta^2$, and such that

$$|\psi_2(s', \rho') - \psi_2(s, \rho)| \le \epsilon$$

for all $(s', \rho'), (s, \rho) \in [t_0, 1] \times D_0$ with $|s' - s|^2 + |\rho' - \rho|^2 \leq \delta^2$.

Take any point $\rho \in G$ with $0 < |\rho - \rho_0| \leq \delta$. Clearly $\rho \in D_0$, and the trace of the oriented interval $\gamma = [\rho_0, \rho]$ lies in D_0. Now for any fixed s with $0 \leq s < t_0$, the function $\Phi_1(\rho) := \phi_1(t_0, s, \rho) - \rho\psi_1(s, \rho_0)$ is an anti-derivative for the function $\psi_1(s, \rho) - \psi_1(s, \rho_0)$ (as functions of the ρ variable), while for fixed s with $t_0 < s \leq 1$ the function $\Phi_2(\rho) := \phi_2(t_0, s, \rho) - \rho\psi_2(s, \rho_0)$ is an anti-derivative for the function $\psi_2(s, \rho) - \psi_2(s, \rho_0)$. Thus, for s with $0 \leq s < t_0$ we have

$$\begin{aligned}
&\phi_1(t_0, s, \rho) - \phi_1(t_0, s, \rho_0) - (\rho - \rho_0)\psi_1(s, \rho_0) \\
&\qquad = \phi_1(t_0, s, \rho) - \rho\psi_1(s, \rho_0) - \phi_1(t_0, s, \rho_0) + \rho_0\psi_1(s, \rho_0) \\
&\qquad = \Phi_1(\rho) - \Phi_1(\rho_0) = \int_\gamma [\psi_1(s, \xi) - \psi_1(s, \rho_0)]\, d\xi,
\end{aligned}$$

and hence,

$$\begin{aligned}
|\phi_1(t_0, s, \rho) - \phi_1(t_0, s, \rho_0) - (\rho - \rho_0)\psi_1(s, \rho_0)| &\leq \int_\gamma |\psi_1(s, \xi) - \psi_1(s, \rho_0)||d\xi| \\
&\leq \epsilon|\rho - \rho_0|.
\end{aligned}$$

In the case for s with $t_0 < s \leq 1$ the same argument shows that

$$|\phi_2(t_0, s, \rho) - \phi_2(t_0, s, \rho_0) - (\rho - \rho_0)\psi_2(s, \rho_0)| \leq \epsilon|\rho - \rho_0|.$$

Combining all the above pieces, we have

$$\begin{aligned}
&\left|\frac{u(t_0, \rho) - u(t_0, \rho_0)}{\rho - \rho_0} - \int_0^1 \psi(s, \rho_0)\, ds\right| \\
&\quad \leq \frac{1}{|\rho - \rho_0|} \int_0^{t_0} |\phi_1(t_0, s, \rho) - \phi_1(t_0, s, \rho_0) - (\rho - \rho_0)\psi_1(s, \rho_0)|\, ds \\
&\qquad + \frac{1}{|\rho - \rho_0|} \int_{t_0}^1 |\phi_2(t_0, s, \rho) - \phi_2(t_0, s, \rho_0) - (\rho - \rho_0)\psi_2(s, \rho_0)|\, ds \\
&\quad \leq \frac{1}{|\rho - \rho_0|} \cdot \epsilon\, |\rho - \rho_0| t_0 + \frac{1}{|\rho - \rho_0|} \cdot \epsilon\, |\rho - \rho_0|(1 - t_0) = \epsilon
\end{aligned}$$

for $\rho \in G$ with $0 < |\rho - \rho_0| \leq \delta$. We conclude that the function $u(t_0, \rho)$ is indeed differentiable at $\rho = \rho_0$. The continuity of $\partial u/\partial\rho$ on $[0, 1] \times G$ follows from the first part of the proof. □

The second lemma generalizes a well-known result on the analyticity of the limit of a sequence of analytic functions. See [37, p. 256].

Lemma 12.2. *Let G be an open set in the ρ plane, let $u_k\colon [0, 1] \times G \to \mathbb{C}$, $k = 1, 2, \ldots$, be a sequence of functions, and let $u\colon [0, 1] \times G \to \mathbb{C}$ be a function, where for each compact subset K of G, it is assumed that the u_k converge* *uniformly on* $[0,1] \times K$ *to* u*. Assume that each* u_k *is continuous on* $[0,1] \times G$ *and that* $\partial u_k/\partial\rho$ *exists and is continuous on* $[0,1] \times G$*. Then* u *is continuous on* $[0,1] \times G$*,* $\partial u/\partial\rho$ *exists and is continuous on* $[0,1] \times G$*, and for each compact subset* K *of* G*, the functions* $\partial u_k/\partial\rho$ *converge uniformly on* $[0,1] \times K$ *to* $\partial u/\partial\rho$*.*

Proof. First, fix $t_0 \in [0,1]$ and $\rho_0 \in G$. To show that u is continuous at (t_0, ρ_0), take any $\epsilon > 0$. Choose $r_0 > 0$ such that the closed disk $D_0\colon |\rho - \rho_0| \le r_0$ lies in G. Appealing to the uniform convergence on $[0,1] \times D_0$, choose a positive integer k such that

$$|u_k(t,\rho) - u(t,\rho)| \le \epsilon/3 \quad \text{for all } (t,\rho) \in [0,1] \times D_0.$$

Using the continuity of u_k at (t_0, ρ_0), select $\delta > 0$ such that $\delta \le r_0$ and such that

$$|u_k(t,\rho) - u_k(t_0,\rho_0)| \le \epsilon/3$$

for all $(t,\rho) \in [0,1] \times G$ with $|t - t_0|^2 + |\rho - \rho_0|^2 \le \delta^2$. Then for $(t,\rho) \in [0,1] \times G$ with $|t - t_0|^2 + |\rho - \rho_0|^2 \le \delta^2$, we have $|\rho - \rho_0| \le \delta \le r_0$, $\rho \in D_0$, and

$$\begin{aligned} |u(t,\rho) - u(t_0,\rho_0)| &\le |u(t,\rho) - u_k(t,\rho)| + |u_k(t,\rho) - u_k(t_0,\rho_0)| \\ &\qquad + |u_k(t_0,\rho_0) - u(t_0,\rho_0)| \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

We conclude that u is continuous at (t_0, ρ_0), and hence, u is continuous on $[0,1] \times G$.

Second, fix any point $\rho_0 \in G$. Choose $r_0 > 0$ such that the closed disk $D_0\colon |\rho - \rho_0| \le r_0$ lies in G. In terms of the open disk $\Delta_0\colon |\rho - \rho_0| < r_0$, we will show that $\partial u/\partial\rho$ exists on $[0,1] \times \Delta_0$. Because ρ_0 is an arbitrary point of G, it then follows that $\partial u/\partial\rho$ exists on $[0,1] \times G$.

Form the circle $\gamma(\tau) = \rho_0 + r_0 \mathrm{e}^{\mathrm{i}\tau}$, $0 \le \tau \le 2\pi$, which lies in the disk D_0 and in G. Clearly the u_k converge uniformly on $[0,1] \times D_0$ and on $[0,1] \times \{\gamma\}$ to u. Now for $t \in [0,1]$ and $\rho \in \Delta_0$ Cauchy's integral formula gives

$$u_k(t,\rho) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{u_k(t,\xi)}{\xi - \rho}\, d\xi \quad \text{and} \quad \frac{\partial u_k}{\partial\rho}(t,\rho) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{u_k(t,\xi)}{(\xi - \rho)^2}\, d\xi$$

for $k = 1, 2, \ldots$. Letting $k \to \infty$ and appealing to the uniform convergence on $[0,1] \times \{\gamma\}$, we obtain the result

$$u(t,\rho) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{u(t,\xi)}{\xi - \rho}\, d\xi \quad \text{for } (t,\rho) \in [0,1] \times \Delta_0.$$

From the classical theory it follows that $\partial u/\partial\rho$ exists at each point (t,ρ) in $[0,1] \times \Delta_0$ with

$$\frac{\partial u}{\partial\rho}(t,\rho) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{u(t,\xi)}{(\xi - \rho)^2}\, d\xi.$$

Next, introduce the closed disk $D_1\colon |\rho - \rho_0| \le r_0/2$, and let us show that the $\partial u_k/\partial\rho$ converge uniformly on $[0,1]\times D_1$ to $\partial u/\partial\rho$. Indeed, take any $\epsilon > 0$. Choose a positive integer N such that

$$|u_k(t,\xi) - u(t,\xi)| \le \frac{\epsilon r_0}{4}$$

for all $k \ge N$, $t \in [0,1]$, and $\xi \in \{\gamma\}$. Then for $k \ge N$ and $(t,\rho) \in [0,1] \times D_1$, from the above we have

$$\begin{aligned}\left|\frac{\partial u_k}{\partial\rho}(t,\rho) - \frac{\partial u}{\partial\rho}(t,\rho)\right| &\le \frac{1}{2\pi}\int_\gamma \frac{|u_k(t,\xi) - u(t,\xi)|}{|\xi-\rho|^2}\,|d\xi| \\ &\le \frac{1}{2\pi}\cdot\frac{\epsilon r_0}{4}\cdot\frac{1}{\left(\frac{r_0}{2}\right)^2}\cdot 2\pi r_0 = \epsilon.\end{aligned}$$

This establishes the desired uniform convergence on $[0,1] \times D_1$.

If K is a compact subset of G, then K can be covered by a finite number of closed disks such as D_1, and it follows that the functions $\partial u_k/\partial\rho$ converge uniformly on $[0,1] \times K$ to $\partial u/\partial\rho$. Applying the first part of the proof to the sequence $\partial u_k/\partial\rho$, $k = 1, 2, \ldots$, and to the function $\partial u/\partial\rho$, we see that $\partial u/\partial\rho$ is continuous on $[0,1] \times G$. $\square$

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Index