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A COMPARATIVE STUDY OF MOMENTUM TRANSFER, HEAT TRANSFER, AND VAPOR TRANSFER

PART I

FORCED CONVECTION, LAMINAR CASE

by

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FOREWORD

This report is Part I of a preliminary study in connection with the Wind-tunnel Project under contract with the ONR to be carried out by the Hydraulics Laboratory of the Colorado Agricultural and Mechanical College, Fort Collins, Colorado. Although it is chiefly a review of existing literature, it also contains some original research.

Review of much of the existing literature was done by K. C. Kuo at Fort Collins in the summer of 1949. This work has greatly facilitated the preliminary preparation of the present report.

The writer wants to express his thanks to Professor T. H. Evans, Dean of Engineering of the College and Chairman of the Engineering Division of the Experiment Station and Dr. D. F. Peterson, Head of the Civil Engineering Department of the College and Chief of the Civil Engineering Section of the Experiment Station, for critical reading of the manuscript and many valuable suggestions.

The present work has been done under the supervision of Dr. M. L. Albertson, Director of the Hydraulics Laboratory of the College, to whom the writer owes many valuable discussions and suggestions, and much assistance in preparing this report.

To Mr. Fred Repper, Graduate Assistant, who has rendered indispensable assistance with his fine draftsmanship, the writer also wants to show his appreciation.

Part II covering the turbulent case of forced convection and Part III covering free convection under the same general title will be completed in the near future.

Contents

INTRODUCTION

In order to provide a theoretical basis for the experiments to be performed in connection with the ONR contract N9onr-82401, and to profit by the results of previous research workers in the field of evaporation and in related fields, a comparative study of momentum transfer, heat transfer and vapor transfer has been carried out. Such a study is not only desirable from the fundamental point of view, but will actually be of value in determining the mutual applicability of results obtained in each of the three fields, thereby reducing the experiments to be performed to a minimum by the proper utilization of existing data.

The process of evaporation is a process of mass transfer, and, like that of heat transfer, can be divided into two categories, namely conduction and convection. If the process occurs in still air and does not result in an unstable distribution of specific weight, it is called conduction, since no convection current is established and the transfer depends solely on molecular action. On the other hand, if a convectional current is induced by an unstable distribution of specific weight as a result of evaporation, the process is called free convection; and if the evaporation is mainly effected by a superposed, predominating flow, the process is called forced convection. In each of the two types of convection, the resultant flow may of course be either laminar or turbulent, or a combination of the two.

Save for a difference in the thermal and vapor diffusivities, evaporation by conduction is analogous to heat conduction, since in each of them the Laplace equation must be satisfied in the steady case and the wave equation must be satisfied in the unsteady case. As heat conduction has been thoroughly and successfully treated problems of evaporation by conduction are already solved wherever the solutions of their thermal counterparts exist.

The difficulties in convection problems lie chiefly in the non-linear character of the equations of motion and of diffusion, and in the case of turbulent flow, in the lack of a conclusive turbulent theory. As a result the existing solutions necessarily retain an individual and often a semi-empirical character. It is because of this situation and the fact that the convection types of evaporation are directly pertinent to the experimental project that this study is undertaken, which will treat these types in detail with a view to correlating the existing results and guiding the experimentation at hand, and to pointing out the directions for future research.

In the following parts dealing with the two types of convection, an effort is made to bring out, together with its limitations, the analogy between momentum transfer, heat transfer, and vapor transfer. In Part I the boundary-layer equations will be developed and forced convection in the laminar case will be treated in detail. Forced convection in the turbulent case and free convection in both the laminar and turbulent cases will be treated in Part II and Part III, respectively.

Because in free convection the distributions of velocity and of vapor concentration are interdependent, the pertaining problems are essentially more difficult than the corresponding ones in forced convection. It is for this reason that forced convection will be treated first.

PART I

FORCED CONVECTION, LAMINAR CASE

1. Boundary-Layer Theory

At the beginning of the present century, L. Prandtl (46) discovered that for the flow of a fluid with a small viscosity or, more generally, with a relatively high Reynolds number, the effect of viscosity is concentrated in a thin layer in the immediate neighborhood of any solid boundary present, and that outside that layer potential flow prevails. This discovery, suggested perhaps by experimental evidence, led to the boundary layer theory which Prandtl presented to the Third International Mathematical Congress in 1904 (46), furnishing equations which, in combination with the results of potential flow, determine the velocity distribution in the boundary layer before separation occurs. The importance of the boundary layer equations, however, is not restricted to the above application, as they are found useful also for computing the distribution of velocity in jets and wakes. After the velocity distribution has been computed by these equations, the temperature or vapor distribution can be determined by equations similar to those used for the velocity computation. In view of this, the boundary layer equations and their important solutions will be discussed first in some detail in connection with forced convection.

Denoting by u and v the velocity components in the x- and y-directions, by t, p, ρ and ν respectively the time, hydrodynamic pressure, mass density and kinematic viscosity, the

Navier-Stokes equations for two-dimensional motion

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{1}$$

and the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

can be converted by the substitutions

$$t' = \frac{tU_0}{l}, \quad x' = \frac{x}{l}, \quad y' = \frac{y}{l}, \quad p' = \frac{p}{\rho U_0^2}, \quad u' = \frac{u}{U_0}, \quad v' = \frac{v}{U_0}$$

into the dimensionless form after dropping the primes:

$$\left.\begin{array}{cccccccccccc}
\frac{\partial u}{\partial t} & + & u & \frac{\partial u}{\partial x} & + & v & \frac{\partial u}{\partial y} & = & -\frac{\partial p}{\partial x} & + & \frac{1}{R} & \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
1 & & 1 & 1 & & \delta & \frac{1}{\delta} & & 1 & & \delta^2 & \left(1 \qquad \frac{1}{\delta^2}\right) \\
\frac{\partial v}{\partial t} & + & u & \frac{\partial v}{\partial x} & + & v & \frac{\partial v}{\partial y} & = & -\frac{\partial p}{\partial y} & + & \frac{1}{R} & \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
\delta & & 1 & \delta & & \delta & 1 & & \delta & & \delta^2 & \left(\delta \qquad \frac{1}{\delta}\right)
\end{array}\right\} \quad \text{---} (3)$$

$$\begin{array}{ccc}
\frac{\partial u}{\partial x} & + & \frac{\partial v}{\partial y} = 0 \\
1 & & 1
\end{array} \tag{4}$$

where $R = \frac{U_0 l}{\nu}$ is the Reynolds number, U_0 being the free-stream velocity and l a characteristic body length. The dimensionless boundary layer thickness $\frac{\delta}{l}$ is very small compared with 1. On the basis that u is of order of magnitude 1 and that $R \sim \frac{1}{\delta^2}$ (since it has been shown from exact solutions of the Navier-Stokes equations that $\delta \sim \sqrt{\nu}$), the orders of magnitude of various terms in the above equations can be found to be those marked below them. That $\frac{\partial v}{\partial y} \sim 1$ follows from the equation of continuity. This gives $v \sim \delta$. The orders of the other terms are then obvious. That $\frac{\partial p}{\partial x} \sim 1$ and $\frac{\partial p}{\partial y} \sim \delta$ follows from Equation (3). The fact that $\frac{\partial p}{\partial y} \sim \delta$ means physically that the pressure within the boundary layer is practically constant, and hence essentially the same as that obtained for potential flow.

From the orders of magnitude of the various terms, it can be readily seen that $\frac{\partial^2 u}{\partial x^2}$ can be neglected as compared with $\frac{\partial^2 u}{\partial y^2}$. The observation that p is given by the potential flow enables $\frac{\partial p}{\partial x}$ to be known beforehand. Thus Equation (1) may be reduced to the single equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (5)$$

which together with Equation (2) and the boundary conditions

$y = 0$: $\quad u = v = 0$

$y = $: $\quad u = U$

determines the velocity in the boundary layer, where U is known from solution of the potential flow. Equation (5) is the boundary-layer equation for two-dimensional flow. For rotationally symmetric jet flow, the boundary-layer equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad (6)$$

where x and r are now the longitudinal and the radial distance respectively, and u and v the corresponding velocity components.

It may be noted that for the flow of jets $\frac{\partial p}{\partial x}$ is found to be zero. For flow around solid bodies curvilinear coordinates must be used with x measured along the body and y along a direction normal to it. The resulting equations are the same if the curvature of the body is small, but are otherwise rather complicated and difficult to use. (See 19).

2. Analogy between the Transfers of Momentum, Heat, and Vapor

For convenience the two-dimensional case will be discussed. Equation (5) can be considered as the equation of momentum transfer, with ν as the coefficient of diffusion or, for comparison, as the "momentum diffusivity." In the forced convection of heat, if the heat generated by friction and change of pressure can be neglected and if the field is free of heat sources the equation for the distribution of temperature is

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{c_p \rho}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{7}$$

where T is the temperature, k the thermal conductivity, c_p the specific heat at constant pressure, and the ratio $\frac{k}{c_p \rho}$ is called the thermal diffusivity and denoted by α which has the same dimension as ν . Equation (7) is analogous to the Navier-Stokes equation only if the pressure p is constant. Since a non-constant p can be considered as continuous sources of momentum, for the case of non-constant p the analogy can be restored only when there are continuous sources of heat in the fluid with the same distribution as p, bearing the same relation to the heat flux $c_p \rho U_0 (T_0 - T_1)$ as p to the momentum flux ρU_0^2, U_0 being the free-stream velocity, T_0 the free-stream temperature, and T_1 the constant temperature of the solid boundary. Denoting by the variable h the strength of these continuous sources, the addition of such sources will introduce a term $-\frac{1}{c_p \rho}\frac{\partial h}{\partial x}$ on the right side of Equation (7). This will restore the analogy when p is not constant. However, it must not be construed that the velocity and $(T-T_1)$ distributions are then identical. Identical distributions are obtained only when $\nu = \alpha$. The ratio $\frac{\nu}{\alpha}$ is called the Prandtl number. The following table is a summary of the foregoing discussion:

	p = constant	p ≠ constant without heat sources	with proper heat sources
$\sigma = \frac{\nu}{\alpha} \neq 1$	Analogy	Lack of nalogy	Analogy
$\sigma = \frac{\nu}{\alpha} = 1$	Complete nalogy and identical distributions of u and $T-T_1$	Lack of nalogy	Complete nalogy and identical distributions of u and $T-T_1$

Now let p be constant and the analogy exist. The product of the Reynolds number and the Prandtl number, namely $\frac{U_0 l}{\alpha}$, is called the Péclet number. It has been seen that if R is very large, Equation (5) applies. If now Pé is also very large (that is if ν and α have the same order of magnitude) by analogy the following boundary layer equation is valid:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{8}$$

(When Pé = R, the velocity and $(T-T_1)$ distributions are identical, and it is obviously true that Equation (8) is valid.)

For vapor transfer at large Péclet number, the boundary-layer equation* applies:

$$\frac{\partial P_v}{\partial t} + u\frac{\partial P_v}{\partial x} + v\frac{\partial P_v}{\partial y} = K\frac{\partial^2 P_v}{\partial y^2} = \frac{\nu}{\sigma}\frac{\partial^2 P_v}{\partial y^2} \tag{9}$$

* Here $\sigma = \frac{\nu}{K}$ where K is the vapor conductivity, and Pé = $\frac{Ubl}{K}$

It is of course assumed that P_V is small compared with the prevalent pressure in the flow, and that the flow pattern is essentially undisturbed by the motion of the vapor. Equations (8) and (9) and Equation (5) for constant pressure can be combined into the single equation

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} + v\frac{\partial \varphi}{\partial y} = \frac{\nu}{\sigma}\frac{\partial^2 \varphi}{\partial y^2} \tag{10}$$

For similar boundary conditions the distribution of φ is therefore a function of the parameter σ.

3. Exact Solutions of the Boundary-Layer Equations

(a) The Pipe

Since for the velocity distribution in established pipe flow the boundary-layer equations are exactly true, and since it is well-known to be parabolic, convection in pipes will be heated first.

Using the cylindrical coordinates r, θ and z, the velocity-components are:

$$\begin{aligned} u_r &= u_\theta = 0 \\ u_z &= 2u_m\left(1 - \frac{r^2}{a^2}\right) \end{aligned} \tag{11}$$

where u_m is the mean velocity and a the radius.

If the boundary conditions are

$$z < 0 \text{ and } r \leq a : \; T = T_0$$
$$z > 0 \text{ and } r = a : \; T = T_s$$

then defining

$$\Theta = \frac{T - T_s}{T_0 - T_s} \tag{12}$$

one has to solve the boundary-layer equation

$$2u_m\left(1 - \frac{r^2}{a^2}\right)\frac{\partial \Theta}{\partial z} = \alpha\left(\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial \Theta}{\partial r}\right) \tag{13}$$

with the boundary conditions

$$z < 0 \quad \text{and} \quad r \leq a: \quad \theta = 1 \tag{14}$$

$$z > 0 \quad \text{and} \quad r = a: \quad \theta = 0 \tag{15}$$

Putting

$$\theta = A \exp.\left(-\beta^2 \frac{\alpha z}{2u_m a^2}\right) \psi(r) \tag{16}$$

where β is an undetermined constant, Equation (13) becomes

$$\frac{d^2 x}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{\beta^2}{a^2}\left(1 - \frac{r^2}{a^2}\right)\psi = 0 \tag{17}$$

Writing $\beta \frac{r}{a} = r'$ one has

$$\frac{d^2\psi}{dr'^2} + \frac{1}{r'}\frac{d\psi}{dr'} + \left[1 - \left(\frac{r'}{\beta}\right)^2\right]\psi = 0 \tag{18}$$

The series solution without irregularities at the origin is

$$\psi(r', \beta) = 1 - \frac{r'^2}{4} + \frac{3r'^4}{2 \cdot 4^2}\left(\frac{1}{4} + \frac{1}{\beta^2}\right) + \cdots$$

where $\psi(r', \beta)$ must satisfy

$$\psi(\beta, \beta) = 0 \tag{19}$$

so that Equation (15) is satisfied. Equation (19) has infinitely many roots, of which the first three are

$$\beta_0 = 2.705, \quad \beta_1 = 6.66, \quad \beta_2 = 10.6$$

Taking the solution of (18) to be

$$\theta = \sum_{n=0}^{\infty} A_n \exp\left(-\beta_n^2 \frac{\alpha z}{2u_m a^2}\right) \psi\left(\frac{\beta_n r}{a}, \beta_n\right) \tag{20}$$

one can determine the A's from Equation (14). Nusselt (41) gave the first three to be

$$A_0 = 1.477, \quad A_1 = -0.810, \quad A_2 = 0.385$$

The solution is due to Graetz (17) and Nusselt (41), and is in good agreement with the experimental results of McAdams (38), the deviation being probably due to the distortion of the parabolic velocity distribution caused by the variation of dynamic viscosity.

(b) The Flat Plate

One of the simplest examples of the application of the boundary layer equation is the essentially parallel flow along a flat plate, treated by H, Blasius (5) in his Göttingen thesis. Let the plate begin at $x = 0$, extend parallel to the x-axis, and be infinitely long. And let the free-stream velocity be U . Here $\frac{\partial P}{\partial x} = 0$. For steady flow, Equations (5) and (2) become

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (21)$$

the boundary conditions being

$$y = 0: \; u = v = 0: \; y = \infty: \; u = U.$$

Using the stream function ψ such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

the second of Equations (21) is always satisfied. Making the substitutions

$$\eta = y\sqrt{\frac{U}{\nu x}}, \qquad \psi = \sqrt{\nu x U}\, f(\eta)$$

the velocity components are

$$u = U f'(\eta)$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,(\eta f' - f)$$

where the primes denoted differentiation with respect to η .

Furthermore

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\frac{U}{x}\eta f'', \quad \frac{\partial u}{\partial y} = U\sqrt{\frac{U}{\nu x}} f'', \quad \frac{\partial^2 u}{\partial y^2} = \frac{U^2}{\nu x} f'''$$

Substitution into the first of Equations (21) yields, after simplification:

$$ff'' + 2f''' = 0 \tag{22}$$

with the boundary conditions

$$\eta = 0: \quad f = f' = 0$$
$$\eta = \infty: \quad f' = 1$$

Blasius' solution gives

$$f = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \frac{C_n \alpha^{n+1}}{(3n+2)!} \eta^{3n+2} \tag{23}$$

where $C_0 = 1, \quad C_1 = 1, \quad C_2 = 11$

$C_3 = 375, \quad C_4 = 27{,}897, \quad C_5 = 3{,}817{,}137$

The asymptotic development near $\eta = \infty$ is formulated in the form

$$f = f_1 + f_2 + f_3 + \cdots$$

where the higher approximations are to be small in comparison with the lower approximations, for instance $f_2 \ll f_1$. The first asymptotic approximation should, of course, correspond to the potential flow, and is therefore of the form $f_1 = \eta - \beta$

As $f_1'' = 0$ and $f_1 \gg f_2$, Equation (22) can be written as

$$(\eta - \beta) f_2'' + 2 f_2''' = 0$$

the first integration of which is easily seen to be

$$\ln f_2'' = \frac{1}{2}\beta\eta - \frac{1}{4}\eta^2 + C$$

If one writes $C = -\beta^2/4 + \ln \gamma$ one obtains

$$f_2'' = \gamma e^{-\frac{1}{4}(\eta-\beta)^2}$$

which after integration, becomes

$$f_2' = \gamma \int_{\infty}^{\eta} e^{-\frac{1}{4}(\eta-\beta)^2} d\eta$$

the lower limit being chosen so that $f_2'(\infty) = 0$. Since $f_1'(\infty) = 1$ the solution $f = f_1 + f_2$ satisfies the third boundary condition $f(\infty) = 1$. Another integration gives

$$f = \eta - \beta + \gamma \int_{\infty}^{\eta} d\eta \int_{\infty}^{\eta} e^{-\frac{1}{4}(\eta-\beta)^2} d\eta \tag{24}$$

This solution still contains two integration constants β and γ, corresponding to the fact that only one of the three boundary conditions was satisfied. Further approximation with an additional f_3 was obtained by Blasius.

When the solutions given by Equations (21) and (24) are forced to make f, f' and f" from both solutions agree, at some intermediate point where both are serviceable, it is found that

$$\alpha = 0.332 \, , \; \beta = 1.730 \, , \; \gamma = 0.231$$

Because of the condition imposed by Equation (22), the higher derivatives will automatically agree. The values of f, f', f" are given by Table 1. The velocity distribution $\frac{u}{U} = f'(\eta)$ is shown in Fig. 1. It may be remarked that $f''' = 0$ when $\eta = 0$ since $f = 0$ when $\eta = 0$. Thus, the velocity profile has zero curvature at $y = 0$, since there $\frac{\partial^2 u}{\partial y^2} = U f''' = 0$. The distribution of v is plotted in Fig. 2. It is noteworthy that at $\eta = \infty$

$$V = 0.865 \, U \sqrt{\frac{\nu}{Ux}} \neq 0$$

This fact is caused by the deflection of the potential flow from the body due to the boundary layer thickness increasing downstream. This has to be tolerated as a very slight deficiency of the boundary layer solution.

As the surface drag plays an important role in the theory of transfer, it will be given here. With b denoting the width, L the length of the plate, the total friction force from one side of the plate is

$$W = b \int_0^L \tau_0 \, dx = b\mu \int_0^L \left(\frac{\partial u}{\partial y} \right)_{y=0} dx$$

But

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = U\sqrt{\frac{U}{\nu x}}\, f''(0) = \alpha\, U\sqrt{\frac{U}{\nu x}}$$

So

$$W = 2\alpha b U\sqrt{\mu\rho L U} = 0.662\, b U\sqrt{\mu\rho L U} \tag{25}$$

and

$$C_W = \frac{W}{bL\frac{\rho U^2}{2}} = \frac{1.328}{\sqrt{R}} \tag{26}$$

where $R = \frac{UL}{\nu}$

Suppose now the oncoming fluid to be of temperature T_0 and the plate of temperature T_s. For convenience let the new variable

$$\zeta = \frac{1}{2}\sqrt{\frac{U}{\nu x}}\, y = \frac{1}{2}\eta$$

be introduced. The function $f(\eta)$ is known from Blasius' solution. Setting $\varphi(\zeta) = f(\eta)$, and using the primes to indicate differentiation one has Table 2 for the values of $\varphi, \varphi', \varphi''$ as reconstructed from Table 1.

Also

$$u = \frac{U}{2}\varphi'$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U}{x}}\,(\zeta\varphi' - \varphi)$$

Using the new variable ζ and with the substitution

$$\Theta = \frac{T - T_s}{T_0 - T_s}$$

Pohlhausen (44) reduced the boundary layer equation for temperature

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{27}$$

to the following ordinary differential equation

$$\Theta'' + \sigma\varphi\Theta' = 0 \tag{28}$$

With f known, Equation (28) can be solved for different values of σ with the boundary conditions

$$\xi = 0: \quad f = 0: \quad f' = 0: \quad \theta = 0:$$
$$\xi = \infty: \quad f' = 2: \quad \theta = 1:$$

The solution of Equation (28) is obviously

$$\theta = C_1 + C_2 \int_0^\xi e^{-\sigma \int_0^\xi f\, d\xi}\, d\xi$$

where $C_1 = 0$, and

$$C_2^{-1} = \int_0^\infty e^{-\sigma \int_0^\xi f\, d\xi}\, d\xi$$

Pohlhausen determines the approximate value of C_2 to be $0.664\,\sigma^{\frac{1}{3}}$ Hence

$$\theta'_{\xi=0} = 0.644\,\sigma^{\frac{1}{3}}$$

and the heat transferred per unit width of plate per unit time is, for a length L,

$$q = -k\int_0^L \left(\frac{\partial T}{\partial y}\right)_{y=0} dx = 0.644\,K\sqrt{\frac{UL}{\nu}}\,\sigma^{\frac{1}{3}}\,(t_s - t_o)$$

This equation is in good agreement with the result of direct measurement on air by Jakob and Dow (33). The mean coefficient of heat transfer is therefore

$$\alpha = 0.644\,K\sqrt{\frac{U}{\nu L}}\,\sigma^{\frac{1}{3}}$$

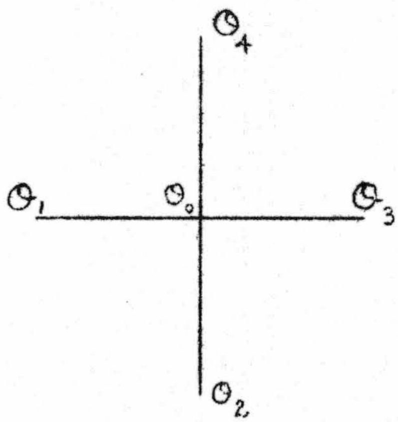
The solution of Equation (28) as outlined in the foregoing applies directly to evaporation from a plane surface if the flow is laminar. For evaporation of water in air, $\sigma = \frac{\nu}{K} = 0.6$ where K is the coefficient of vapor diffusion (or vapor diffusivity) at the prevalent temperature.

In Pohlhausen's problem the leading edge of the plate coincides with that of the diffusion surface. If this is not true, Pohlhausen's equation can no longer be solved to satisfy the boundary conditions. Equation (27) will then have to retain its partial character. Besides the independent variable ξ as defined by Pohlhausen, a second variable $\chi = \frac{x}{x_o}$ has to be introduced, where x_o is the length of approach. With Θ defined now as $\frac{c - c_s}{c_o - c_s}$ where c is the vapor concentration (dimensionless) and the subscripts retain their meanings as in the case of temperature, Equation (27) can be transformed to the dimensionless form

$$\Theta_{\xi\xi} + \sigma \xi \Theta_{\xi} = 2 \sigma \chi \xi' \Theta_{\chi} \tag{29}$$

to be solved with the boundary conditions

$$\chi = 1 \; : \; \Theta = 1$$
$$\xi = 0 \; : \; \Theta = 0$$
$$\xi = \infty \; : \; \Theta = 1$$



This at once suggests the use of the relaxation method.

Using finite differences $\Delta \xi = \frac{1}{n}$ and $\Delta \chi = \frac{1}{n}$, and designating the values of Θ at the point in question by Θ_o, and those around the point in the manner shown by the above sketch, we have

$$\Theta_{\xi\xi} = n^2 (\Theta_4 + \Theta_2 - 2\Theta_o)$$
$$\Theta_{\xi} = \frac{n}{2} (\Theta_4 - \Theta_2)$$
$$\Theta_{\chi} = \frac{n}{2} (\Theta_3 - \Theta_1)$$

Substitution into Equation (29) gives

$$\Theta_o = \frac{1}{2}\left(1 + \frac{\sigma \xi}{2n}\right)\Theta_4 + \frac{1}{2}\left(1 - \frac{\sigma \xi}{2n}\right)\Theta_2 + \frac{\sigma \eta \xi'}{2n}\Theta_1 - \frac{\sigma \eta \xi'}{2n}\Theta_3 \tag{30}$$

Equation (30) is the basis for the calculation by relaxation, which consists of its repeated use. With the boundary values fixed from the boundary conditions (the value 1 of Θ at infinity can be assumed to be attained at sufficiently large values of ξ), a plausible assumption is made for the values of Θ at staggered joints. The values of Θ at the other joints can be computed from Equation (30). The assumed values are then corrected by using Equation (30) again, the amount of correction influencing further the neighboring (first computed) values of Θ by an amount consistent with Equation (30). For instance, if Θ is assumed at the odd joints and computed at the even joints, calculation at joint 13 by Equation (30) with the Θ-values at the joints 12, 18, 14, 8 will show a discrepancy $\Delta\Theta$. This amount $\Delta\Theta$ is imposed on joint 13 to make the Θ-value identical with the newly computed value. In accordance with Equation (30), this imposition will, when the joints 12, 18, 14, 8 are considered in turn as the central joint, contribute to these joints respectively the amounts

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

$$-\frac{\sigma\eta\xi'}{2n}\Delta\Theta, \quad \frac{1}{2}\left(1+\frac{\sigma\xi}{2n}\right)\Delta\Theta, \quad \frac{\sigma\eta\xi'}{2n}\Delta\Theta, \quad \frac{1}{2}\left(1-\frac{\sigma\xi}{2n}\right)\Delta\Theta$$

This process is applied at all the odd joints. The accumulation of the contributions at the even joints will contribute back to the odd joints in the same manner, and the process continues until the contributions become insignificantly small. Occasional checks by computing anew from the neighboring joints are often helpful and a final check is always desirable. The process of relaxation does not require the joints to be divided in two groups as explained in the foregoing, but can be performed in any manner whatsoever so long as every joint is accounted for.

Taking $n = 2$ the writer performed a rough relaxation. Although the result needs further refinement to be of use, the value at $\xi = \frac{1}{2}$, $\chi = \frac{1}{2}$ is found to have a rather sensitive effect on the boundary condition at $\xi = \infty$. The calculated value at that point, which is around 0.46, will be utilized as a rough check in the analytic method outlined below.

Since the point $\xi = 0, \chi = 1$ is a singularity with an infinite Θ_ξ, the change of Θ near that point is too rapid for a coarse relaxation to yield sufficiently accurate results near the singularity. A new method using the eigen-functions is applied. Let

$$\Theta(\xi,\chi) = Y(\xi)\,X(\chi)$$

Substitution into Equation (29) gives, after division by $f'Y(\xi)X(\chi)$

$$\frac{Y'' + \sigma f Y'}{f' Y} = \frac{2\sigma \chi X'}{X} = \sigma k$$

where k is an arbitrary constant. Hence

$$Y'' + \sigma f Y' - \sigma k f' Y = 0 \qquad (31)$$

$$\frac{X'}{X} = \frac{k}{2\chi} \qquad (32)$$

The solution of Equation (32) is

$$X = C\chi^{k/2}$$

The solution of Equation (31) has to satisfy the boundary conditions at $\xi = 0$ and $\xi = \infty$. When $k = 0$, the Equation (31) is obviously Equation (30), the solution of which by Pohlhausen giving

$Y_0(0) = 0$ and $Y_0(\infty) = 1$. It is desired therefore that for the other values of k, the solutions $Y_k(\xi)$ will satisfy the conditions $Y_k(0) = 0$, $Y_k(\infty) = 0$. To determine the values of k for which $Y_k(\xi)$ will satisfy the condition $Y_k(\infty) = 0$, the asymptotic behavior of $Y_k(\xi)$ must be studied. If Y" at a certain value of ξ is very small compared with the other two terms of Equation (31) (this condition is satisfied near a point of inflection) then neglecting Y" the solution of Equation (31) is

$$Y = \text{CONSTANT } \rho^k$$

From Blasius solution, $\rho \to \infty$ as the first power of ξ as $\xi \to \infty$. Therefore for $k < 0$, the situation that Y" is small compared with the other two terms of Equation (31) will maintain, Y will vary asymptotically as ρ^k, and the condition $Y_k(\infty) = 0$ will be satisfied. The solution can thus be put in the form

$$\Theta(\xi, x) = \int_0^{-\infty} f(k)\, x^{\frac{k}{2}}\, Y_k(\xi)\, dk$$

and if the function $f(k)$ is determined so that the boundary condition $\Theta(\xi, 1) = 1$ is satisfied, the problem is solved. From the considerations of the next paragraph it will be shown that $f(k)$ is everywhere zero except at a set of points of Lebesque measure zero, in fact at the points where k is equal to zero or a negative odd number. The solution may therefore be put in the form

$$\Theta(\xi, x) = Y_0(\xi) + x^{-\frac{1}{2}}\, Y_{-1}(\xi) + x^{-3/2}\, Y_{-3}(\xi) + \cdots \quad (33)$$

where $Y_0(\xi)$ is the Pohlhausen solution and $Y_{-1}(\xi)$, $Y_{-3}(\xi)$ etc. are as yet indeterminate to a constant factor (any constant multiple of these eigen-functions will satisfy Equation (31) which is linear). It must be noted that

$$\Theta(\xi, \infty) = Y_0(\xi)$$

so that the condition at $\chi=\infty$ is satisfied. The condition

$$\Theta(\xi,1)=1$$

can be replaced by

$$Y_{-1}(\xi)+Y_{-3}(\xi)+Y_{-5}(\xi)+\cdots=1-Y_0(\xi) \quad (34)$$

where $Y_0(\xi)$ is the known solution of Pohlhausen.

The following consideration justifies the selection of the eigen-values. As the value of $\frac{\partial\Theta}{\partial\xi}$ at $\xi=0$ should be asymptotically equal to 0.552 obtained by Pohlhausen and is evidently infinite at $\chi=1$, a plausible assumption for $\left.\frac{\partial\Theta}{\partial\xi}\right)_{\xi=0}$ is

$$\left.\frac{\partial\Theta}{\partial\xi}\right)_{\xi=0}=0.552+C(\chi-1)^{-\frac{1}{2}}=0.552+C\chi^{-\frac{1}{2}}(1-\chi^{-1})^{-\frac{1}{2}} \quad (35)$$

such that

$$\left.\frac{\partial\Theta}{\partial y}\right)_{y=0}=\frac{1}{2}\sqrt{\frac{U}{\nu x}}\left[0.552+C(\chi-1)^{-\frac{1}{2}}\right] \quad (36)$$

The choice of the power $-\frac{1}{2}$ is based on the assumption that the order of magnitude of $\left.\frac{\partial\Theta}{\partial y}\right)_{y=0}$ at the point $\chi=1$ is the same for the case when the approach length X_0 is occupied by the plate and the case when it is not. In the latter case, with x measured from a point with a distance X_0 ahead from the leading edge of the plate, Pohlhausen's solution gives

$$\left.\frac{\partial\Theta}{\partial y}\right)_{y=0}=\frac{0.552}{2}\sqrt{\frac{U}{\nu(x-x_0)}}=0.276\sqrt{\frac{U}{\nu x_0}}(\chi-1)^{-\frac{1}{2}} \quad (37)$$

The power $-\frac{1}{2}$ is therefore chosen. From Equation (35) it is seen that 0, -1, -3, -5 etc. are the eigen values.

Since $\Theta=0$ at $\xi=0$, the functions $Y_k(\xi)$ do not contain a constant term, but start with the first power. Denote the coefficient of the first power of ξ in $Y_k(\xi)$ by a_k.

Then by Equation (35)

$$a_0 = 0.552\,,\ a_{-1} = C\,,\ a_{-3} = \frac{C}{2}\,,\ a_{-5} = \frac{3C}{8}$$

and in general

$$a_{-n} = \frac{n}{n+1}\, a_{-(n-2)}$$

The linearity of Equation (31) permits the use of

$$1,\ \frac{1}{2},\ \frac{3}{8},\ \frac{5}{16},\ \text{etc.}$$

for the values of

$$a_{-1},\ a_{-3},\ a_{-5},\ a_{-7},\ \text{etc.}$$

Numerical integration of Equation (31) can then be performed for each k. The results are multiplied by the same constant C determined to satisfy Equation (34). The $Y_{-1}(\xi)$, $Y_{-3}(\xi)$ etc. thus obtained are then used in Equation (33) to constitute the solution.

Table 3 is the result of calculation by finite differences with $\Delta\xi = \frac{1}{4}$ up to $k = -15$. C is determined to be 0.185. The last two columns are plotted in Fig. 3 to show the extent to which the boundary condition as expressed by Equation (34) is satisfied.

It will be noted here the value 0.185 of C corresponds approximately to the value 0.40 for Θ at $\xi = 0.5$ and $\eta = 0.5$. This is roughly checked by the value 0.46 obtained with the coarse relaxation.

Integrating Equations (36) and (37) from x_0 to $x_0 + L$, the corresponding values of the total rate of evaporation are respectively

$$M_1 = K\sqrt{\frac{Ux_0}{\nu}}\left[0.552(x_L^{\frac{1}{2}} - 1) + 0.093\,\ln(2x_L - 1 + \sqrt{x_L^2 - x_L}\right](c_s - c_0)\rho \qquad (38)$$

and

$$M_2 = K\sqrt{\frac{Ux_0}{\nu}}\left[0.552(x_L - 1)^{\frac{1}{2}}\right](c_s - c_0)\,\rho \qquad (39)$$

where $\chi_L = \frac{L}{X_0}$ and ρ is the density of air, since it is assumed that for small vapor concentration ρ is essentially unaffected by the presence of vapor. The following table shows the influence of the length of approach

χ_L	1	2	3	4	5	10	20
$\frac{M_1}{M_2}$	.34	.71	.79	.83	.86	.93	.96

It may be remarked that as $\chi_L \to \infty$, $M_1/M_2 \to 1$, as can be expected.

Experimental works done in heat transfer and evaporation usually ignore the effect of the approach length. Eliás (14) used an approach length of 10 cms. and measured over a heated surface of length 50 cms. This corresponds to $\chi_L = 6$, and according to the above table introduces a deviation of around 12% from Pohlhausen's solution. However, free convection, which was also ignored, offers a compensating effect, and the resultant deviation is less than 12%. The very recent work of Yamamoto also ignores the effect of the approach length, which was not mentioned in his work. From the sketch of his paper, χ_L is about 5. His result confirms the equation (in c.g.s. system)

$$E = \left[0.298 + 0.925\, U \left(\frac{Ux}{\nu}\right)^{-\frac{1}{2}}\right](C_s - C_0)$$

where E is the average rate of evaporation, 0.298 accounts for the effect of free convection, and the other term is Pohlhausen's solution for $\nu = 0.15$, K = 0.25, and $\sigma = 0.60$. Here, deviation from Pohlhausen's solution due to the approach length is overshadowed by the effect of free convection, the net effect being expressed by the term 0.298.

So far the only consideration given to the approach length is to be found in the Iowa Dissertation of Albertson (3). He performed a series of experiments for different χ_L and different U. Defining

$$C_e = \frac{M}{Ux'(C_s - C_o)\rho}$$

Where $x' = x - x_o$, his data show the dependence of C_e on x'/x and $S = \frac{Ux'}{K}$:

$$C_e = 1.03\left(\frac{1}{S}\right)^{\frac{1}{2}\left(\frac{x'}{x}\right)^{\frac{1}{8}} - 1}$$

From equation (28), it follows that

$$C_e = \frac{K\sqrt{\frac{Ux_o}{\nu}}\left[0.552(\chi_L^{\frac{1}{2}}-1)+0.093\ln(2\chi_L-1+\sqrt{\chi_L^2-\chi_L})\right](C_s-C_o)\rho}{U_oX'(C_s-C_o)\rho}$$

$$= \left[0.6(\chi_L-1)\right]^{-\frac{1}{2}}\left[0.552(\chi_L^{\frac{1}{2}}-1)+0.093(2\chi_L-1+\sqrt{\chi_L^2-\chi_L})\right]S^{-\frac{1}{2}}$$

$$= F(\chi)S^{-\frac{1}{2}} \tag{40}$$

where $0.6 = \frac{\nu}{K}$ and $F(\chi)$ is defined by the equation. A comparison of Equation (40) with Albertson's data is shown in Fig. 4. The systematic deviation seems to decrease as S increases, and is probably due to free convection. Similarly, Equation (39) gives

$$C_e = (0.6)^{-\frac{1}{2}}(0.552)S^{\frac{1}{2}} = 0.723\,S^{-\frac{1}{2}} \tag{41}$$

which is the limiting form of Equation (40) when $x_o = o$.

(c) Wedge-shaped Bodies

Let x be measured along the wall and y in a direction normal to it, and let $\beta\pi$ be the interior angle of the wedge. If the flow is parallel to the plane of symmetry of the wedge, it can be easily shown that

$$u_1 = cx^m$$

where u_1 is the velocity of potential flow at the wall, c is a constant, and m is related to β by the following equation

$$\beta = \frac{2m}{m+1} \qquad \left(\text{or} \quad m = \frac{\beta}{2-\beta}\right)$$

Following Hartree (22), one makes the following substitutions:

$$\eta = \frac{y}{\sqrt{\frac{2}{m+1}}}\sqrt{\frac{c\,x^{m-1}}{\nu}}$$

$$\psi = \sqrt{\frac{2}{m+1}}\sqrt{c\nu x^{m+1}}\, f(\eta)$$

where ψ is the stream-function. The velocity components along the x and y directions are respectively

$$u = \frac{\partial \psi}{\partial y} = c x^m f'$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{\frac{2}{m+1}}\sqrt{c\nu x^{m-1}}\left(\frac{m+1}{2} f + \frac{m-1}{2}\eta f'\right)$$

and

$$\frac{\partial u}{\partial y} = \sqrt{\frac{m+1}{2}}\sqrt{\frac{c}{\nu}}\, c\, x^{\frac{3m-1}{2}} f''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{m+1}{2}\,\frac{c^2}{\nu}\, x^{2m-1} f'''$$

$$\frac{\partial u}{\partial x} = c\left(m x^{m-1} f' + x^{m-1}\frac{m-1}{2}\eta f''\right)$$

where the primes denote differentiation with respect to η.

Then the boundary layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{\partial u_1}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{42}$$

can be transformed to

$$f''' + f f'' - \beta(f'^2 - 1) = 0 \tag{43}$$

with the boundary conditions

$$\eta = 0:\ f = f' = 0; \qquad \eta = \infty:\ f' = 1$$

Equation (43) was solved for different values of β by Hartree (22). The quantity $f' = \frac{u}{u_1}$ is given in Fig. 5.

When $\beta = 0$, Equation (43) is precisely Blasius' equation for the variable η. When $\beta = 1$ (and m = 1), the flow is the so-called stagnation flow, and Equation (43) becomes

$$f''' + ff'' - f'^2 + 1 = 0 \tag{44}$$

This equation was solved in 1911 by K. Hiemenz (25). Hartree's contribution is therefore the solution of Equation (43) for intermediate values of β.

Let again $\Theta = \dfrac{c - c_s}{c_o - c_s}$

In terms of the new variable η, the equation

$$u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} = K\frac{\partial^2 c}{\partial y^2} \tag{45}$$

becomes

$$\Theta'' + \sigma f \Theta' = 0 \tag{46}$$

with the boundary conditions

$$\eta = 0: \quad \Theta = 0; \quad \eta = \infty: \quad \Theta = 1$$

The solution is easily seen to be

$$\Theta = C_1 + C_2 \int_0^\eta e^{-\sigma\int_0^\eta f d\eta} d\eta \tag{47}$$

where $C_1 = 0$ and

$$C_2^{-1} = \int_0^\infty e^{-\sigma\int_0^\eta f d\eta} d\eta$$

The rate of evaporation is

$$q = -K\frac{\partial c}{\partial y}\Big)_0 = K(c_s - c_o)\frac{\partial \Theta}{\partial y}\Big)_0 = K(c_s - c_o)\sqrt{\frac{(m+1)cx^{m-1}}{2\nu}}\,\Theta'(0)$$

$$= C_2 K(c_s - c_o)\sqrt{\frac{u_1'}{\beta\nu}} \tag{48}$$

where $u_1' = \dfrac{du_1}{dx}$

(d) Arbitrarily-shaped Bodies

Let x and y again be measured along and normal to the wall respectively. For the symmetric case, the velocity $u_1(x)$ of potential flow at the wall can be expanded in the following form

$$u_1(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots \tag{49}$$

where the constants depend solely on the shape of the body, and where only the odd powers occur due to symmetry.

From Equation (49),

$$u_1 \frac{du_1}{dx} = a\left[a_1 x + 4a_3 x^3 + \left(6a_5 + \frac{3a_3^2}{a_1}\right)x^5 + \cdots\right] \tag{50}$$

Using the new variable

$$\eta = y\sqrt{\frac{a_1}{\nu}}$$

and the stream-function

$$\psi = \sqrt{\frac{\nu}{a_1}}\left\{a_1 x f_1(\eta) + 4a_3 x^3 f_3(\eta) + 6x^5\left[a_5 g_5(\eta) + \frac{a_3^2}{a_1} h_5(\eta)\right] + \cdots\right\} \tag{51}$$

one has

$$u = a_1 x f_1' + 4a_3 x^3 f_3' + 6x_5\left[a_5 g_5' + \frac{a_3^2}{a_1} h_5'\right] + \cdots$$

$$\frac{\partial u}{\partial x} = a_1 f_1' + 12 a_3 x^2 f_3' + 30 x^4\left[a_5 g_5' + \frac{a_3^2}{a_1} h_5'\right] + \cdots$$

$$\frac{\partial u}{\partial y} = \sqrt{\frac{a_1}{\nu}}\left\{a_1 x f_1'' + 4a_3 x^3 f_3'' + 6x^5\left[a_5 g_5'' + \frac{a_3^2}{a_1} h_5''\right] + \cdots\right\}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{a_1}{\nu}\left\{a_1 x f_1''' + 4a_3 x^3 f_3''' + 6x^5\left[a_5 g_5''' + \frac{a_3^2}{a_1} h_5'''\right] + \cdots\right\}$$

$$v = -\sqrt{\frac{\nu}{a_1}}\left\{a_1 f_1 + 12 a_3 x^2 f_3 + 30 x^4\left[a_5 g_5 + \frac{a_3^2}{a_1} h_5\right] + \cdots\right\}$$

Substitution of Equation (50) and the above equations in Equation (42) yields, after collecting terms containing

$a_1 x$, $4a_1 a_3 x^3$, $6a_1 a_5 x^5$, $6a_3^2 x^5$ etc.,

$$f_1'^2 - f_1 f_1'' = 1 + f_1''' \tag{52}$$

$$4f_1' f_3' - 3f_1'' f_3 - f_1 f_3'' = 1 + f_3''' \tag{53}$$

$$6f_1'g_5' - 5f_1''g_5 - f_1g_5'' = 1 + g_5''' \tag{54}$$

$$6f_1'h_5' - 5f_1''h_5 - f_1h_5'' = \frac{1}{2} + h_5''' - 8(f_3'^2 - f_3f_3'') \tag{55}$$

and so forth. The shape parameters are thus eliminated from the differential equations to be solved. The boundary conditions

$$y=0: \quad u=v=0$$
$$y=\infty: \quad u=u_1$$

can be written as

$$\eta=0: \ f_1=f_1'=0; \ f_3=f_3'=0; \ g_5=g_5'=0; \ h_5=h_5'=0$$

$$\eta=\infty: \ f_1'=1; \ f_3'=\frac{1}{4}; \ g_5'=\frac{1}{6}; \ h_5'=0 \ \ldots$$

It must be noted that Equations (52) and (44) are identical, and has been solved by Hiemenz (25), and that Equations (53) to (55) and further equations of the same series are all linear and of the third order. Hiemenz also calculated f_3 which was later improved by Howarth (30). The functions g_5 and h_5 have been calculated by Nils Frössling (16).

For slender bodies, the series for $u_1(x)$ and $u(x,y)$ converge poorly. The reason is that for such bodies $u_1(x)$ has a very steep ascent in the neighborhood of the stagnation point, while showing a rather flat curve further on. Such a function cannot be readily developed into a Taylor series. For blunt bodies, application of the method yields considerably better results.

As an example, consider the boundary layer of a parallel flow past a cylinder whose axis is perpendicular to the uniform velocity. With R denoting the radius and U the free-stream velocity, one has

$$u_1(x) = 2U\sin\phi = 2U\sin\frac{x}{R} = 2U\left\{\frac{x}{R} - \frac{1}{3!}\left(\frac{x}{R}\right)^3 + \frac{1}{5!}\left(\frac{x}{R}\right)^5 + \cdots\right\}$$

so that

$$a_1 = 2\frac{U}{R}\ ; \quad a_3 = -\frac{2}{3!}\frac{U}{R^3}\ ; \quad a_5 = \frac{2}{5!}\frac{U}{R^5}\ ; \ \cdots$$

and

$$\eta = \frac{y}{R}\sqrt{\frac{VD}{\nu}}$$

The velocity distribution is then given by

$$\frac{1}{2}\frac{u(x,y)}{U} = \frac{x}{R}f_1' - \frac{4}{3!}\left(\frac{x}{R}\right)^3 f_3' + \frac{1}{5!}\left(\frac{x}{R}\right)^5 (6g_5' + 20h_5') + \cdots$$

The point of separation is given by

$$\left.\frac{\partial u}{\partial y}\right)_{y=0} = 0$$

or

$$f_1''(0)\frac{x_s}{R} - \frac{4}{3!}f_3''(0)\left(\frac{x_s}{R}\right)^3 + \cdots = 0$$

With $f_1''(0) = 1.23264$, $f_3''(0) = 0.7246$, $g_5''(0) = 0.6348$ and $h_5''(0) = 0.1192$

$$\frac{x_s}{R} = 1.60\ ; \quad \phi_s = 92^\circ$$

Hiemenz (25) based his calculation on his experimental pressure distribution, and calculated the separation point to be at 82° whereas his measurements gave 81°. The difference between this result and that obtained with the potential-theoretical pressure distribution is due to the fact that near the separation point the actual pressure distribution deviates considerably from that obtained for potential flow.

The unsymmetrical case has been treated by Howarth (30).

In order to compute the rate of evaporation from an arbitrarily-shaped body in symmetric flow, Θ , as defined before, can be expanded in the following form:

$$\Theta = \frac{1}{a_1}\left[a_1\Theta_0 + a_3 x^2 \Theta_2 + \left(a_5\Theta_4 + \frac{a_3^2}{a_1}\Theta_5\right)x^4 + \cdots\right]$$

Then

$$a_1\frac{\partial \Theta}{\partial x} = 2a_3\Theta_2 x + 4\left[a_5\Theta_4 + \frac{a_3^2}{a_1}\Theta_5\right]x^3 + \cdots$$

$$a_1\frac{\partial \Theta}{\partial y} = \sqrt{\frac{a_1}{\nu}}\left[a_1\Theta_0' + a_3\Theta_2' x^2 + (a_5\Theta_4' + \frac{a_3^2}{a_1}\Theta_5')x^4 + \cdots\right]$$

$$a_1\frac{\partial^2 \Theta}{\partial y^2} = \frac{a_1}{\nu}\left[a_1\Theta_0'' + a_3\Theta_2'' x^2 + (a_5\Theta_4'' + \frac{a_3^2}{a_1}\Theta_5'')x^4 + \cdots\right]$$

Substitution in Equation (35) gives

$$\sigma\left[2a_1a_3f_1'\Theta_2x^2 + 4a_1a_5f_1'\Theta_4x^4 + a_3^2(4f_1'\Theta_5 + 8f_3'\Theta_2)x^4 + \cdots\right.$$
$$- a_1^2f_1\Theta_0' - a_1a_3(f_1\Theta_2' + 12f_3\Theta_0')x^2 - a_1a_5(f_1\Theta_4' + 30g_5\Theta_0')x^4$$
$$\left.- a_3^2(f_1\Theta_5' + 12f_3\Theta_2' + 30h_5\Theta_0')x^4 + \cdots\right]$$
$$= a_1^2\Theta_0'' + a_1a_3\Theta_2''x^2 + a_1a_5\Theta_4''x^4 + a_3^2\Theta_5''x^4 + \cdots$$

Equating terms with the same powers in the a's and in x, we have:

$$\Theta_0'' = -\sigma f_1\Theta_0' \tag{56}$$

$$\Theta_2'' = \sigma(2f_1'\Theta_2 - f_1\Theta_2' + 12f_3\Theta_0') \tag{57}$$

$$\Theta_4'' = \sigma(4f_1'\Theta_4 - f_1\Theta_4' - 30g_5\Theta_0') \tag{58}$$

$$\Theta_5'' = \sigma(4f_1'\Theta_5 + 8f_3'\Theta_2 - f_1\Theta_5' + 12f_3\Theta_2' + 30h_5\Theta_0') \tag{59}$$

From Equation (56), we have, corresponding to the boundary conditions $\Theta_0(0) = 0$ and $\Theta_0(\infty) = 1$

$$\Theta_0' = c\,e^{-\sigma\int_0^\eta f_1\,d\eta}$$

and

$$\Theta_0 = c\int_0^\eta e^{-\sigma\int_0^\eta f_1\,d\eta}\,d\eta \tag{60}$$

where

$$c^{-1} = \int_0^\infty e^{-\sigma\int_0^\eta f_1\,d\eta}\,d\eta$$

Using the f_1 computed by Hiemenz, Equation (60) was evaluated by E. Eckert (8) for $\sigma = 0.73$. For $\sigma = 0.60$, the present writer evaluated C to be 0.466. The values of Θ_0 and Θ_0' for different values of η are shown in Table 4, where Θ_0' is shown in more detail because of its subsequent occurrence in Equations (57)-(59) etc.

It should be noted that Equations (57)-(59) etc. are all linear and of the second order. With f_1, f_3, g_5, h_5 and Θ_0' known, they can be solved with the boundary conditions

$$\Theta_2(0) = \Theta_4(0) = \Theta_5(0) = \cdots = 0$$

and

$$\Theta_2(\infty) = \Theta_4(\infty) = \Theta_5(\infty) = \cdots = 0$$

Numerical solution of Θ_2, Θ_4 and Θ_5 by relaxation is now being undertaken by the present writer both for $\sigma = 0.60$ and $\sigma = 0.73$. Although the solutions may not be expected to give satisfactory results for slender bodies, they are important for fundamental considerations and serve as useful initial steps for further development in the already clearly indicated direction. Besides, they should give satisfactory results for blunt bodies.

Three dimensional cases have been considered by Frössling (16). An exact account of his work is not available.

(e) Wakes

The wake behind a plate of length l will be considered. Let x be measured from the leading edge. Denoting by U the free-stream velocity, u the velocity in the wake, and Δu the difference $U - u$, under the assumption that Δu is small compared with U, the equation of motion in the wake can be written as

$$U\frac{\partial \Delta u}{\partial x} = \nu \frac{\partial^2 \Delta u}{\partial y^2} \tag{61}$$

with the boundary conditions

$$y = 0 \qquad \frac{\partial \Delta u}{\partial y} = 0$$

$$y = \infty \qquad \Delta u = 0 \quad (u = U)$$

Introducing the variable

$$\eta = y\sqrt{\frac{U}{\nu x}}$$

and putting

$$\Delta u = U\left(\frac{x}{l}\right)^{-\frac{1}{2}} g(\eta)$$

one has, by substitution into Equation (61),

$$g'' + \frac{1}{2}\eta g' + \frac{1}{2}g = 0 \tag{62}$$

with the boundary conditions

$$\eta = 0: \; g' = 0; \quad \eta = \infty: \; g = 0$$

Two successive integrations give

$$g = Ce^{-\frac{\eta^2}{4}}$$

To determine the constant C , the momentum equation must be utilized. The loss of momentum per unit width is

$$\rho\int_{-\infty}^{+\infty} u(U-u)dy = \rho\int_{-\infty}^{+\infty} u\Delta u\,dy \cong \rho U\int_{-\infty}^{+\infty} \Delta u\,dy$$

since $U \cong u$. On the other hand, the total drag on the plate is, by Blasius solution,

$$2W = 1.328\,\rho U^2\sqrt{\frac{\nu l}{U}}$$

The momentum equation is

$$\rho U\int_{-\infty}^{+\infty} \Delta u\,dy = 1.328\,\rho U^2\sqrt{\frac{\nu l}{U}}$$

or

$$C\sqrt{\frac{\nu x}{U}}\,\rho U^2\left(\frac{x}{l}\right)^{-\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\eta^2/4}d\eta = 2\sqrt{\pi}\,C\rho U^2\sqrt{\frac{\nu l}{U}} = 1.328\,\rho U^2\sqrt{\frac{\nu l}{U}}$$

which yields

$$C = \frac{0.664}{\sqrt{\pi}}$$

and

$$\frac{\Delta u}{U} = \frac{0.644}{\sqrt{\pi}} \left(\frac{x}{l}\right)^{-\frac{1}{2}} e^{-\frac{U y^2}{4 \nu x}} \qquad (63)$$

Treatment of three-dimensional wakes can be found in (19).

(f) Jets

The velocity distribution in a steady, laminar air jet issuing from either a slit or a small hole in a plane wall was obtained in 1933 by Schlichting (54). Later, in 1937, Bickley (4) gave a closed solution for the two-dimensional case.

For the two-dimensional case, let x be measured along the center line of the jet and y be measured normal to it, in the plane of flow. The velocity components in the ~~x~~- and y-directions are denoted respectively by u and v.

With the substitutions

$$\xi = \left(\frac{M}{48 \rho \nu^2}\right)^{1/3} \frac{y}{x^{2/3}} \qquad (64)$$

and

$$\psi = \left(\frac{9 \nu M x}{2 \rho}\right)^{1/3} f(\xi) \qquad (65)$$

where M is the momentum flux per unit width and ψ the stream-function, the Schlichting-Bickley solution of Equation (21) gives

$$u = \left(\frac{3 M^2}{32 \rho^2 \nu x}\right)^{1/3} \operatorname{sech}^2 \xi \qquad (66)$$

$$v = \left(\frac{M \nu}{6 \rho x^2}\right)^{1/3} \left(2 \xi \operatorname{sech}^2 \xi - \tanh \xi\right) \qquad (67)$$

If the air jet is preheated, the temperature distribution can be found (67) by using the substitution

$$\Theta = \frac{T - T_0}{T_0} = \left(\frac{H}{T_0 \nu}\right)\left(\frac{\rho \nu^2}{M x}\right)^{1/3} t(\xi) \qquad (68)$$

where T_0 is the temperature of the surrounding air and H is defined to be

$$H = \int_{-\infty}^{\infty} u (T - T_0)\, dy = T_c \int_{-\infty}^{\infty} u \Theta \, dy \tag{69}$$

Substitution of Equations (64) to (68) into Equation (27) gives the differential equation

$$-2\sigma \left[t(\xi) \operatorname{sech}^2 \xi + t'(\xi) \tanh \xi \right] = t''(\xi) \tag{70}$$

with the boundary condition

$$t'(0) = 0$$

and the condition imposed by Equation (69). The solution of Equation (70) is

$$t(\xi) = c \operatorname{sech}^{2\sigma} \xi$$

where

$$c = \frac{1}{(36)^{1/3} \int_0^{\infty} \operatorname{sech}^{2+2\sigma} \xi \, d\xi}$$

So

$$T - T_0 = cH \left(\frac{\rho}{M \nu x} \right)^{1/3} \operatorname{sech}^{2\sigma} \xi$$

For the axial-symmetry case let x be measured along the center line of the jet and r be measured in the radial direction, and let u denote the longitudinal velocity component and v the radial velocity component. If the Stokes stream-function is denoted by ψ, it follows that

$$u = \frac{1}{r} \frac{\partial \psi}{\partial r}, \qquad v = -\frac{1}{r} \frac{\partial \psi}{\partial x}$$

With the substitutions

$$\eta = \frac{1}{4} \left(\frac{3M}{\pi \rho \nu^2} \right)^{1/2} \frac{r}{x}$$

and $\psi = \nu x f(\eta)$

where M now is the total momentum flux.

Schlichting obtained the solution of the boundary-layer equation

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \nu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) \tag{72}$$

as

$$u = \frac{3}{8\pi} \frac{M}{\rho \nu x} \frac{1}{(1+\frac{1}{4}\eta^2)^2} \tag{73}$$

$$V = \frac{1}{4}\left(\frac{3M}{\pi\rho}\right)^{1/2} \frac{1}{x} \frac{\eta(1-\frac{1}{4}\eta^2)}{(1+\frac{1}{4}\eta^2)^2} \tag{74}$$

If the air jet is preheated, the temperature distribution can be found (67) by using the substitution

$$\Theta = \frac{T-T_0}{T_0} = \frac{H}{T_0 \nu x} t(\eta) \tag{75}$$

where H is now defined as

$$H = \int_0^\infty 2\pi r u (T-T_0)\,dr = 2\pi T_0 \int_0^\infty r u \Theta\,dr \tag{76}$$

Substitution of Equations (71) and (73) to (75) into the boundary-layer equation

$$u \frac{\partial \Theta}{\partial x} + v \frac{\partial \Theta}{\partial r} = \frac{\nu}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \Theta}{\partial r}\right) \tag{77}$$

yields an ordinary differential equation in $t(\eta)$ the solution of which with consideration of the satisfaction of Equation (76) is

$$t(\eta) = \frac{1+2\sigma}{8\pi}\left(1+\frac{\eta^2}{4}\right)^{-2\sigma}$$

so that

$$T-T_0 = \frac{1+2\sigma}{8\pi} \frac{H}{\nu x}\left(1+\frac{\eta^2}{4}\right)^{-2\sigma}$$

What has been obtained for the temperature distribution in preheated jets applies directly to the moisture distribution in pre-moistened jets if the proper value of σ (0.60) is used.

4. Approximate Solutions for the Boundary-Layer Equations

As the exact solutions of the boundary layer equations are with a few exceptions very laborious, and since there are as yet no exact solutions that can be applied satisfactorily to arbitrarily-shaped slender bodies, resorts are often made to approximate methods in which the boundary layer equations are renounced and only certain integral relations are required to hold in consistency with the momentum equation or with the continuity principle of heat- or vapor-transfer. In 1925 papers were published by Von Kármán (36) and Pohlhausen (45) describing an approximate method to determine the distribution of velocity in the boundary layer. Similar methods applying to heat or vapor boundary-layers were certainly only extensions of the Kármán-Pohlhausen method. In the following, x and y will be the curvilinear coordinates defined before.

(a) Definitions of Various Boundary-Layer Thicknesses

It is adequate to define the boundary-layer thicknesses first, which will be used frequently in the next two sections.

The displacement thickness δ^* of the boundary layer is defined by

$$\int_0^\infty u\,dy = u_1(y-\delta^*)$$

where u_1 is the velocity of potential flow at the point in question. The above equation can be written as

$$\delta^* = \int_0^\infty \left(1-\frac{u}{u_1}\right)dy \qquad (78)$$

The momentum thickness ϑ is defined by

$$u_1 \vartheta = \int_0^{\infty} u (u_1 - u)\, dy$$

or

$$\vartheta = \int_0^{\infty} \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) dy \qquad (79)$$

We shall use the symbol δ to mean the thickness at which the velocity in the boundary-layer is essentially the same as (say 99% of) $u_{1\infty}$. Thus the value of δ is more or less arbitrary.

(b) The Kármán-Pohlhausen Method

Integration of Equation (42) with respect to y yields:

$$\frac{1}{2}\frac{d}{dx}\int_0^h u^2 dy + \int_0^h v \frac{\partial u}{\partial y} dy = h u_1 \frac{du_1}{dx} + \nu \left(\frac{du}{dy}\right)_0^h \qquad (80)$$

By partial integration and the equation of continuity

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

we have

$$\int_0^h v \frac{\partial u}{\partial y} dy = u_1 v_h - \int_0^h \frac{\partial v}{\partial y} u\, dy = -u_1 \int_0^h \frac{\partial u}{\partial x} dy + \int_0^h u \frac{\partial u}{\partial x} dy$$

Insertion in Equation (80) gives

$$\frac{d}{dx}\int_0^h u^2 dy - u_1 \int_0^h \frac{\partial u}{\partial x} dy = -\frac{h}{\rho}\frac{dp}{dx} - \frac{\tau_0}{\rho} \qquad (81)$$

where τ_0 is the shearing stress at the wall and

$$-\frac{\tau_0}{\rho} = \nu \left(\frac{\partial u}{\partial y}\right)_0^h$$

Equation (81) is the Kármán integral-condition (36).

Remembering the definitions of δ^* and ϑ, and that

$$\frac{1}{\rho}\frac{dp}{dx} = -u_1 \frac{du_1}{dx}$$

Equation (81) can be written in the form

$$\frac{\tau_0}{\rho} = u_1^2 \frac{d\vartheta}{dx} + (2\vartheta + \delta^*) u_1 \frac{du_1}{dx} \tag{82}$$

Pohlhausen (45) then assumed that

$$\frac{u}{u_1} = a\eta + b\eta^2 + c\eta^3 + d\eta^4 \tag{83}$$

where

$$\eta = \frac{y}{\delta_p(x)} \tag{84}$$

the subscript p (=Pohlhausen) being used to avoid confusion with the δ mentioned in 4(a). From the boundary conditions

$$y=0: \qquad u=0, \quad \nu\frac{\partial^2 u}{\partial y^2} = -u_1\frac{\partial u_1}{\partial x} \tag{85}$$

$$y=\delta_p: \qquad u=u_1, \qquad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0$$

the coefficients a, b, c, d are determined to be

$$a = 2 + \frac{\lambda}{6}, \; b = -\frac{\lambda}{2}, \; c = -2 + \frac{\lambda}{2}, \; d = 1 - \frac{\lambda}{6}$$

where

$$\lambda = \frac{\delta_p^2}{\nu}\frac{du_1}{dx} \tag{86}$$

The velocity distribution is then given by

$$\frac{u}{u_1} = F(\eta) + \lambda G(\eta) \tag{87}$$

where

$$F(\eta) = 2\eta - 2\eta^3 + \eta^4$$

$$G(\eta) = \frac{1}{6}(\eta - 3\eta^2 + 3\eta^3 - \eta^4)$$

Thus the only unknown is δ_p. The solution consists in finding δ_p from Equation (82). The value of λ is limited (on the positive side by the reasonableness of $\frac{u}{u_1}$ and on the negative side by separation) to the range $12 \geq \lambda \geq -12$

Using Equations (78) (79) (84) and (87) one has

$$\frac{\delta^*}{\delta_p} = \frac{3}{10} - \frac{\lambda}{120} = f_1(\lambda) \tag{88}$$

$$\frac{\vartheta}{\delta_p} = \frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072} = f_2(\lambda) \tag{89}$$

Besides, from $\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ and Equations (84) and (87)

$$\frac{\tau_0}{\mu}\frac{\delta_p}{u_1} = \frac{12+\lambda}{6} = f_3(\lambda) \tag{90}$$

Equation (82) can be written in the form

$$\frac{u_1 \vartheta \vartheta'}{\nu} + \left(2 + \frac{\delta^*}{\vartheta}\right)\frac{u_1' \vartheta^2}{\nu} = \frac{\tau_0 \vartheta}{\mu u_1} \tag{91}$$

which is a differential equation in ϑ with the unknown quantities δ^* and τ_0 expressible in terms of ϑ.

Now set $K = Z u_1' = \frac{\vartheta^2}{\nu} u_1'$

with

$$Z = \frac{\vartheta^2}{\nu} \tag{92}$$

Then from Equations (86) and (89):

$$K = \lambda\left[f_2(\lambda)\right]^2$$

Also, from Equations (88) and (89):

$$\frac{\delta^*}{\vartheta} = \frac{f_1(\lambda)}{f_2(\lambda)} = F_1(K)$$

and, from Equations (89) and (90)

$$\frac{\tau_0}{\mu}\frac{\vartheta}{u_1} = \frac{\tau_0}{\mu}\frac{\delta_p}{u_1}\frac{\vartheta}{\delta_p} = f_3(\lambda) f_2(\lambda) = F_2(K)$$

From Equations (91) and (92)

$$\frac{1}{2} u_1 \frac{dZ}{dx} + \left[2 + F_1(K)\right] K - F_2(K) = 0$$

If one sets

$$2F_2(K) - 4K - 2F_1(K)K = F(K)$$

one has

$$\frac{dZ}{dx} = \frac{F(K)}{u_1}, \qquad K = Z u_1' \tag{93}$$

where

$$F(K) = 2f_2(\lambda)\left\{2 - \frac{116}{315}\lambda + \left(\frac{2}{945} + \frac{1}{120}\right)\lambda^2 + \frac{2}{9072}\lambda^3\right\}$$

Equation (93) should be solved with $K_0 = 0.0770$ corresponding to $\lambda_0 = 7.052$ where K_0 and λ_0 are the values of K and λ at the stagnation point, 0.0770 being a zero of $F(K)$.

For the flat plate, solution by this method yields

$$\vartheta\sqrt{\frac{u_1}{\nu x}} = 0.685 \quad \text{and} \quad \frac{\tau_0}{\mu U}\sqrt{\frac{\nu x}{u_1}} = 0.343,$$

whereas the exact values given by Blasius' calculation are respectively 0.664 and 0.332. The agreement is good.

For the stagnation point profile the exact solution was discussed in 3(c). The approximate method gives

$$\vartheta\sqrt{\frac{u_1}{\nu}} = \sqrt{K_0} = 0.278$$

$$\vartheta\sqrt{\frac{u_1}{\nu}} = F_1(K)\sqrt{K_0} = 0.641$$

$$\frac{\tau_0}{\mu u_1}\sqrt{\frac{\nu}{u_1}} = \frac{F_2(K)}{\sqrt{K_0}} = 1.19$$

whereas the values by the exact solution are respectively 0.292, 0.648 and 1.234. The agreement is again very good.

(c) An Approximate Method for Calculations in Vapor Transfer or Heat Transfer from Arbitrarily-shaped Bodies.

In 3(c), exact methods for determining the velocity and moisture fields in the laminar boundary-layer attached to a wedge-shaped body were discussed. For arbitrarily-shaped bodies, H. Schuh (61) mentioned a method for determining the temperature in the field of flow. The method is similar to the Kármán-Pohlhausen method. Maintaining the potential velocity and its derivative with respect to x, it requires only the balance of heat. In the following, Schuh's method will be presented, with the necessary changes to render the method applicable to evaporation.

Remembering $v = -\int_0^y \frac{\partial u}{\partial x} dy$, integration of Equation (45) with respect to y with the condition $\frac{\partial c}{\partial y} = 0$ at the outside of the boundary-layer (or at $y = \infty$) gives

$$\frac{\partial}{\partial x} \int_0^\infty u(c - c_o) dy = -k\left(\frac{\partial c}{\partial y}\right)_o \tag{94}$$

Now imagine the body to be replaced in successive ranges by wedge-shaped bodies with the same velocity and the same x-derivative thereof just outside of the boundary layer. The stagnation point of the given body and those of the elemental wedges naturally do not coincide. Denoting by x_o the difference between the stagnation points of the body and the wedge replacing it at the point in question, from the following relations

$$\eta = \frac{y}{\sqrt{2-\beta}} \sqrt{\frac{u_1}{\nu(x+x_o)}}$$

$$u_1 = C(x+x_o)^{\frac{\beta}{2-\beta}}$$

$$u_1' = \frac{du_1}{dx} = \frac{\beta}{2-\beta} \frac{u_1}{x+x_o}$$

x and x_o can be eliminated to give

$$\eta = \frac{y\sqrt{u_1'}}{\sqrt{\beta\nu}}$$

where it must be noted that the definition of η is exactly the same as that given in 3(c).

With Θ defined as in Equation (46),

$$\int_0^\infty u(c - c_o)\, dy = u_1(c_s - c_o) \int_0^\infty \frac{u}{u_1}(1-\Theta)\, dy$$
$$= u_1(c_s - c_o)\frac{\sqrt{\beta\nu}}{\sqrt{u_1'}} \int_0^\infty \frac{u}{u_1}(1-\Theta)\, d\eta$$

As $\frac{u}{u_1}$ is the f' in Equation (43) the solution of which is known for a certain β and as $1-\theta$ is known from the solution of Equation (46) for the same β, the integral $\int_0^\infty \frac{u}{u_1}(1-\theta)dy$ is a function of β only. Writing

$$\int_0^\infty \frac{u}{u_1}(1-\theta)dy = J(\beta)$$

one has

$$\int_0^\infty u(c-c_0)dy = u_1(c_s-c_0)\frac{\sqrt{\beta\nu}}{\sqrt{u_1'}}J(\beta)$$

Equation (94) can then be written as

$$\frac{d}{dx}\left(\frac{u_1\sqrt{\beta}}{\sqrt{u_1'}}J(\beta)\right) = \frac{1}{\sigma}\frac{\sqrt{u_1'}}{\sqrt{\beta}}(\theta')_0 \qquad (95)$$

where $\sigma = \frac{\nu}{\kappa}$. Let $x' = \frac{x}{l}$ where l is a characteristic length of the body, and let the free-stream velocity be U, then multiplying Equation (95) by $\sqrt{\frac{l}{U}}$ gives

$$\frac{d}{dx'}W = \frac{1}{\sigma}\frac{\sqrt{\left(\frac{u_1}{U}\right)'}}{\sqrt{\beta}}(\theta')_0$$

where

$$W = \frac{\frac{u_1}{U}\sqrt{\beta}}{\sqrt{\left(\frac{u_1}{U}\right)'}}J(\beta)$$

and $\left(\frac{u_1}{U}\right)'$ denotes the derivative of $\frac{u_1}{U}$ with respect to x'.

At the stagnation point, $u_1 = 0$, and hence $W = 0$. Thus

$$W = \frac{1}{\sigma}\int_0^{x'}\frac{\sqrt{\left(\frac{u_1}{U}\right)'}}{\sqrt{\beta}}(\theta')_0\,dx' \qquad (96)$$

which can be solved step by step, giving as the main result β as a function of x'.

The local rate of evaporation is

$$q = -K\left(\frac{\partial c}{\partial y}\right)_{y=0} = -K\,(c_s - c_0)\,\frac{\sqrt{u_1}}{\sqrt{\beta\nu}}\,(\Theta')_0 \tag{97}$$
$$= \frac{-K(c_s - c_0)\sqrt{u_1}}{\sqrt{\nu}}\,J_1(\beta)$$

and the total rate of evaporation can be found from Equation (97) by integration.

5. Concluding Remarks

(a) Since the equations of motion and of diffusion have been well formulated, the difficulties encountered in finding the distributions of velocity, temperature and moisture in a field of flow are chiefly mathematical. With the use of the approximate boundary-layer equations, solutions for these distributions have in many instances been found where the use of the exact equations would present prohibitive difficulties.

(b) As has been mentioned before, a complete analogy between momentum transfer, heat transfer, and vapor transfer does not exist whenever there is a pressure gradient in the field of flow. It can be further stated that this is also true whenever the internal friction in the case of heat transfer or the velocity perpendicular to a solid boundary in the case of vapor transfer cannot be neglected, or the physical constants suffer variations that are not negligible. Aside from these limitations, the distribution of temperature or moisture for forced convection in a certain field of laminar flow is a function of the velocity distribution and the Prandtl number σ alone.

(c) For problems wherein the differences in velocity, temperature, and moisture are not exceedingly large, the physical properties of the fluid can be considered as constant and the interdependence of these quantities can be neglected. From the foregoing it can be seen that these problems may be considered as essentially solved whenever the boundary-layer equations can be applied, the method of solution of the yet unsolved problems being clearly indicated by the existing literature. In view of the non-linear character of the boundary-layer equations, this fact offers a considerable satisfaction indeed.

(d) Variable density has been extensively treated in the transomic and supersonic theories in aerodymamics. Generation of heat by internal friction has been more or less rigorously dealt with in (8), (9), (10), (11), (12), (32), (39), (61), and possibly (60). Treatment of problems involving large differences in temperature and moisture can be found in (2) and (60). As problems of this kind are important and represent a field not yet sufficiently explored, it can be safely predicted that future research will be greatly concerned with them.

(e) It must be noted that since the boundary-layer equations are approximate, and since the pressure corresponding to potential flow can be disturbed by the growth of a boundary layer, experimental data are often desirable to check the theoretical deductions. Moreover, in nature pure laminar flow seldom occurs; whenever turbulence is present or there is a possibility for its occurrence, experiments are indispensable. Thus, theoretical considerations contained in this report facilitate experimentation without eliminating its necessity.

6. Bibliography

1. Ackeret, J., Pfenninger, W., Kas, M.: Verhinderung des Turbulentwerdens einer Reibungsschicht durch Absaugung. Naturw. 1941, p. 622.

2. Ackermann, G.: Wärmeübertragung in gleichen Feld bei grossen Temperatur. und Partielldruckdifferenz. Forschung a.d. Geb. d. Ing. Vol. 8, 1937, Forschungsheft No. 382. (Theoretical)

3. Albertson, M. L.: Analysis of evaporation as a boundary-layer phenomenon. Dissertation, Univ. of Iowa, 1948. (Theoretical and experimental).

4. Bickley, W.: The Plane Jet. Phil. Mag., ser. 7, vol. 23, No. 156, p. 727, Apr. 1937. (Th.)

5. Blasius, H.: Grenzschichten in Flüssigkeiten mit kleiner Reibung. Zs. Math. u. Phys. Bd. 56, p. 1, 1908. (Th.)

6. Boltze, E.: Grenzschichten an Rotationskörpern in Flüssigkeiten mit kleiner Reibung. Diss. Göttingen 1908. (Th.)

7. Bussmann, K.: Die laminare Reibungsschicht an Joukowsky-Profilen. (not yet published).

8. Eckert, E.: The computation of the heat transfer in the laminar boundary layer of bodies in a flow. VDI - Forschrengsheft No. 416, Berlin 1942. (Th.)

9. Eckert, E.: Temperature measurements in gases at high velocities. VDI-Zeitschrift Vol. 84, no. 43, p. 813, 1940. (Th.)

10. Eckert, E., and Drewitz, O.: The computation of the field of temperature in the laminar boundary layer of unheated bodies in a flow at high velocities. Luftf. Forsch., vol. 19, p. 189, 1942. (Th.)

11. Eckert, E., and Weise, W.: Measurements of the temperature distribution on the surface of unheated bodies in a flow at high velocity. Dtsch. Luftfahrtforsch. Jahrbuch 1940, II, p. 25. (Exp.)

12. Eckert, E. & Drewitz, O.: Der Wärmeübergang an eine mit grosser Geschwindigkeit längs Angeströme Platte. Forschung a.d. Geb. d. Ingenieurwes. Vol. 11, pp. 116-124, 1940. (Th.)

13. Elenbaas, W.: Über eine Kombination der hydrodynamischen Theorie des Langmuirschen Theorie. Physica, vol. 4, pp. 761-765, 1937. (Th.)

14. Éliás, F.: The Transference of heat from a hot plate to an air stream. NACA TM NO. 614, (Exp.)

15. Falkner, V. N., and Skan, S. W. : Some approximate solutions of the boundary layer equations. Phil. Mag., vol. 12, p. 865, 1931; ARC Rep. A. Mem. No. 1314, 1930. (Th.)

16. Frössling, Nils: Verdunstung, Wärmeübergang und Geschwindigkeitsverteilung bei zweidimensionaler und rotations symmetrischer laminarer Grenzschichtströmung. Lunds. Univ. Arsskr. N.F. Avd. 2, Bd. 35, Nr. 4, 1940. (Th.)

17. Graetz: Ann. d. Phys., 18, pp. 79-94, 1883; 25, pp. 337-357, 1885. (Th.)

18. Goldstein, S.: Concerning Some Solutions of the Boundary Layer Equations in Hydrodynamics. Proc. Phil. Soc. Vol. 26, I, 1936. (Th.)

19. Goldstein, S.: Modern Developments in Fluid Dynamics, Vol. I and Vol. II. Oxford, 1938. (Th. & exp.)

20. Hamel, G.: Spiralförmige Bewegungen zäher Flüssigkeiten. Jahresber. Mathem. Vereinigung 1916, p. 34. (Th.)

21. Hansen, M.: Die Geschwindigkeitsverteilung in der Grenzschicht an einer eingetauchten Platte. Zs. f. angew. Math. u. Mech. Bd. 8, p. 185, 1928. (Exp.)

22. Hartree, D. R.: On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer. Proc. Camb. Phil. Soc., vol. 33, P. II, p. 223, 1937. (Th.)

23. v.d. Hegge-Zynen: Thesis Delft 1924, Burgers Proc. I. Intern. Congress appl. Mech. Delft 1924.

24. Herman, R.: Wärmeübergang bei freier Strömung am Waagerechten Zylinder in Zwei-atomigen Gassen. Forschung a.d. Geb. d. Inf. vol. B7, Forschungsheft No. 379, 1936. (Th.)

25. Hiemnz, K.: Die Grenzschicht an einem in den gleich-förmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder. (Diss Gottingen, 1911). Dingl. Polytechn. Journ. Bd. 326, p. 321, 1911. (Th.)

26. Hoffmann, W.: Wärmeübergang und Diffusion. Forschung a. d. Get. d. Inf., Vol. 6, pp. 293-304, 1935. (Th.)

27. Hofmann, E.: Über die Gesetzmässigkeit der Wärme und der Stoffübertragang auf Grund des Strömun svorgangs in Rohr. Forschung a.d. Geb. d. Inf., vol. 11, pp. 159-169, 1940. (Th.)

28. Holstein-Bohlen: Ein vereinfachtes Verfahren zue Berechnung laminarer Reibungsschichten, die dem Näherungsansatz von K. Pohlhausen genügen. (Not yet published.)

29. Homann, F.: Der Einfluss grosser Zähigkeit bei der Strömung um den Zylinder und um die Kugel. Zs. f. angew. Math. u. Mech. Bd. 16, p. 153, 1936.

30. Howarth, L.: On the Calculation of Steady Flow in the Boundary Layer Near the Surface of a Cylinder in a Stream. ARC Rep. 1632, 1935. (Th.)

31. Howarth, L.: On the Solution of the Laminar Boundary Layer Equations. Proc. Roy. Soc. Lond.A No. 919, vol. 164, 1938, p. 547. (Th.)

32. Jacob, M.: Heat Transfer. John Wiley & Sons, 1949. (Th. & Exp.)

33. Jacob, M., and Dow, W. M.: Trans. ASME vol. 68, p. 123, 1946, (Exp.)

34. Jakob, M. & Hawkins, G. A.: Heat Transfer. John Wiley & Sons, 1947. (Th. & Exp.)

35. Kármán, Th.: The analogy between fluid friction and heat transfer. Trans. A.S.M.E. vol. 61, pp. 705-710, 1939. (Th.)

36. v. Kármán, Th.: Laminare und turbulente Reibung. Zs. f. angew. Math. u. Mech. Bd. 1, p. 235, 1921, (Th.)

37. Lamb, H.: Hydrodynamics, 6th edition. Dover, New York, 1945.

38. McAdams, W. H. Heat Transmission. McGraw-Hill, 1942. (Th. & exp.)

39. Mangler, W.: The similar solutions of Prandtl's boundary layer equation. ZAMM, vol. 23, p. 241, 1943. (Th.)

40. Müller, W.: Einführung in die Theorie der zähen Flüssigkeiten. Liepzig, 1932. (Th.)

41. Nusselt, W.: Zs. d. VDI, 54, pp. 1154-1158, 1910. (Th.)

42. Nusselt, W.: Wärmeübergang, Diffusion, und Verdunstung, ZAMM, vol. 10, pp. 105-121, 1930. (Th.)

43. Piercy, N. A. V., and Preston, J. H.: A simple solution of the flat plate problem of skin friction and heat transfer. Phil. Mag. (7) 21, p. 995, 1936. (Th.)

44. Pohlhausen, E.: The heat-exchange between solid bodies and fluids with small friction and small heat conductivity. ZAMM Vol. 1, p. 115, 1921. (Th.)

45. Pohlhausen, K.: Zur näherungsweisen Integration der Differentialgleichungen der laminaren Reibungsschicht. Zs. f. angew. Math. y. Mech. Bd. 1, p. 252, 1921. (Th.)

46. Prandtl, L.: Ueber Flüssigkeitsbewegung bei sehr kleiner Reibung. Verhandlung III. Intern. Math. Kongress, Heidelberg, 1904, represented in "View Abhdnlg. zur Hydrodynamik", Göttingen, 1927. (Th.)

47. Prandtl, L.: The Mechanics of Viscous Fluids. In F. W. Durand Aerodynamic Theory, Bd. III, 1935. (Th. & exp.)

48. Prandtl, L.: Anschauliche und nutzliche Mathematik. Vorlesung W. S. 1931/32. (Th.)

49. Prandtl-Tietjens: Hydro- und Aeromechanik; Bd. I und II. Berlin 1929 und 1931. (Th.).

50. Pretsch, J.: Die laminare Reibungsschicht an elliptischen Zylindern und Rotationsellipsoiden bei symmetrischer Umströmung. Luftfahrtforschung Bd. 18, p. 397, 1941.

51. Schrenk, O.: Versuche mit Absaugeflügeln. Luftfahrtforschung 1935, p. 16.

52. Schiller, L.: Forschungsarbeiten auf dem Gebiet des Ingenieurwesens. Heft 248, 1922.

53. Schlichting, H.: Berechnung ebener periodischer Grenzschichtströmungen. Physik, Zeitzchrift, 1932, p. 327. (Th.)

54. Schlichting, H.: Laminare Strahlausbreitung. Zs. f. angew. Math u. Mech. Bd. 13, p. 260, 1933. (Th.)

55. Schlichting, H.: Laminare Kanaleinlaufströmung. Zs. f. angew. Math. u. Mech. Bd. 14, p. 368, 1934. (Th.)

56. Schlichting, H.: Die Grenzschicht mit Absaugung und Ausblasen Luftfahrtforschung Bd. 19, p. 179, 1942.

57. Schlichting, H.: Boundary-Layer Theory. NACA TM No. 1217, 1949. (Th.)

58. Schlichting, H., and Ulrich, A.: Zur Berechnung des Umschlages laminar - turbulent. (Not yet published.)

59. Schmidt, E., and Wenner, K.: Heat Delivery over the circumference of a heated cylinder in a flow. Forsch. a.d. Geb. d. Inf., Vol. 12, p. 65, 1941. (Th. & exp.)

60. Schuh, H.: On the solution of the laminar boundary-layer equation for the flat plate for the fields of velocity and temperature at variable material data of the plate and for the field of diffusion at higher concentrations. FB No. 1980. (Th.)

61. Schuh, H.: Laminarer Wärmeübergang in Grenzschichten bei hohen Geschwindigkeiten. Zantrale f. wissenschaftlichen Berichtswesen FB 2019, 1944. (Th.)

62. Szymanski: Qualques solutions exactes des équations de l'hydrodynamique du fluide visqueux dans le cas d'un tube cylindrique. Journ. de math. pures et appliquées. Reihe 9, Bd. 11, 1932, p. 67. See also: Intern. Congr. of Mechanics Stockholm, 1930. (Th.)

63. Taylor, G. I.: The application of Osborne Reynolds' theory of heat transfer to flow through a pipe. Proc. Roy. Soc. London, vol. 129, p. 25, 1930. (Th.)

64. Tollmien, W.: Laminare Grenzschichten. Handbuck der Experimental physik, Bd. IV, Teil I, Leipzig, 1931. (Th.)

65. Tomotika, S.: The laminar boundary layer on the surface of a sphere in a uniform stream. ARC Rep. 1678, 1936.

66. Yamamoto, Gi-ichi: Investigation of evaporation from pans. Trans. Am. Geophys, Union, vol. 31, No. 3, pp. 349-356, 1950. (Exp.)

67. Yih, C. S.: Temperature distribution in a steady, laminar, preheated air-jet. Paper presented to the 1950 National Conference of the Applied Mechanics Division of the ASME at Purdue University. (Th.)

68. Results of the Aerodynamics Test Institute at Göttingen. Aerodynamischen Versuchsanstalt zu Göttingen. 2. Lieferung, p. 29. (Exp.)

Table 1

η	f	f'	f"
0	0	0	0.33206
0.2	0.00664	0.06641	0.33199
0.4	0.02656	0.13277	0.33147
0.6	0.05974	0.19894	0.33008
0.8	0.10611	0.26471	0.32739
1.0	0.16557	0.32979	0.32301
1.2	0.23795	0.39378	0.31659
1.4	0.32298	0.45627	0.30787
1.6	0.42032	0.51676	0.29917
1.8	0.52952	0.57477	0.28293
2.0	0.65003	0.62977	0.26675
2.2	0.78120	0.68132	0.24835
2.4	0.92230	0.72899	0.22809
2.6	1.07252	0.77246	0.20646
2.8	1.23099	0.81152	0.18401
3.0	1.39682	0.84605	0.16136
3.2	1.56911	0.87609	0.13913
3.4	1.74696	0.90177	0.11788
3.6	1.92954	0.92333	0.09809
3.8	2.11605	0.94112	0.08013
4.0	2.30576	0.95552	0.06424
4.2	2.49806	0.96696	0.05052
4.4	2.69238	0.97587	0.03897
4.6	2.88826	0.98269	0.02948
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543
5.8	4.07990	0.99838	0.00365
6.0	4.27964	0.99898	0.00240
6.2	4.47948	0.99937	0.00155
6.4	4.67938	0.99961	0.00098
6.6	4.87931	0.99977	0.00061
6.8	5.07928	0.99987	0.00037
7.0	5.27926	0.99992	0.00022
7.2	5.47925	0.99996	0.00013
7.4	5.67924	0.99998	0.00007
7.6	5.87924	0.99999	0.00004
7.8	6.07923	1.00000	0.00002
8.0	6.27923	1.00000	0.00001
8.2	6.47923	1.00000	0.00001
8.4	6.67923	1.00000	0
8.6	6.87923	1.00000	0
8.8	7.07923	1.00000	0

Table 2

ξ	f	f'	f''
0	0	0	1.33
0.2	0.04	0.29	1.33
0.4	0.13	0.55	1.32
0.6	0.26	0.80	1.27
0.8	0.44	1.04	1.18
1.0	0.66	1.27	1.05
1.2	0.93	1.47	0.90
1.4	1.24	1.64	0.70
1.6	1.58	1.77	0.53
1.8	1.96	1.88	0.39
2.0	2.37	1.94	0.27
2.2	2.8	1.98	0.17
2.4		2.00	0.10
2.6		2.00	0.04
2.8		2.00	0.00
3.0		2.00	0.00
3.2		2.00	0.00

Table 3

ξ	$Y_0(\xi)$	$Y_{-1}(\xi)$	$Y_{-3}(\xi)$	$Y_{-5}(\xi)$	$Y_{-7}(\xi)$	$Y_{-9}(\xi)$	$Y_{-11}(\xi)$	$Y_{-13}(\xi)$	$Y_{-15}(\xi)$	$\sum_{k=-1}^{-15} Y_k(\xi)$	$\sum_{k=-1}^{-\infty} Y_k(\xi)$ (Approx.)	$1-Y_0(\xi)$	$C\sum_{k=-1}^{-\infty} Y_k(\xi)$ (approx.) ($C=0.185$)
0	0	0	0	0	0	0	0	0	0	0	0	1	0
0.25	.138	.25	.125	.094	.078	.068	.062	.057	.053	.787	3.945*	.862	.730
0.50	.274	.494	.245	.181	.149	.128	.116	.104	.094	1.511	4.127*	.726	.763
0.75	.406	.718	.343	.243	.191	.157	.136	.113	.098	1.999	3.376*	.594	.625
1.00	.533	.905	.398	.257	.183	.133	.101	.067	.048	2.092	2.592*	.467	.480
1.25	.648	1.033	.393	.212	.118	.057	.019	-.013	-.030	1.789	1.789	.352	.331
1.50	.747	1.089	.327	.122	.020	-.035	-.062	-.073	-.073	1.315	1.315	.253	.243
1.75	.828	1.071	.217	.014	-.068	-.092	-.089	-.066	-.044	.943	.943	.172	.174
2.00	.889	.993	.095	-.071	-.105	-.086	-.051	-.009	.018	.784	.784	.111	.145
2.25	.932	.877	-.008	-.112	-.087	-.035	.009	.037	.045	.726	.726	.068	.134
2.50	.960	.749	-.073	-.104	-.038	.017	.042	.038	.022	.653	.653	.040	.121
2.75	.977	.627	-.099	-.068	.007	.039	.034	.009	-.011	.538	.538	.023	.100
3.00	.987	.523	-.096	-.028	.029	.031	-.008	-.014	-.020	.417	.417	.013	.077

* Extrapolated

Table 4

ξ	Θ_0'	Θ_0	ξ	Θ_0'	Θ_0
0	0.4663	0	2.1	0.2268	
0.1	0.4663		2.2	0.2071	0.8417
0.2	0.4659	0.0932	2.3	0.1889	
0.3	0.4649		2.4	0.1698	0.8794
0.4	0.4629	0.1862	2.5	0.1524	
0.5	0.4599		2.6	0.1359	0.9100
0.6	0.4555	0.2781	2.7	0.1205	
0.7	0.4496		2.8	0.1062	0.9341
0.8	0.4422	0.3680	2.9	0.0931	
0.9	0.4330		3.0	0.0811	0.9527
1.0	0.4222	0.4545	3.1	0.0702	
1.1	0.4098		3.2	0.0604	0.9668
1.2	0.3958	0.5364	3.3	0.0517	
1.3	0.3803		3.4	0.0439	0.9772
1.4	0.3635	0.6124	3.5	0.0371	
1.5	0.3456		3.6	0.0312	0.9846
1.6	0.3268	0.6815	3.7	0.0261	
1.7	0.3073		3.8	0.0216	0.9899
1.8	0.2873	0.7429	3.9	0.0179	
1.9	0.2670		4.0	0.0146	0.9935
2.0	0.2468	0.7963	4.1	0.0120	

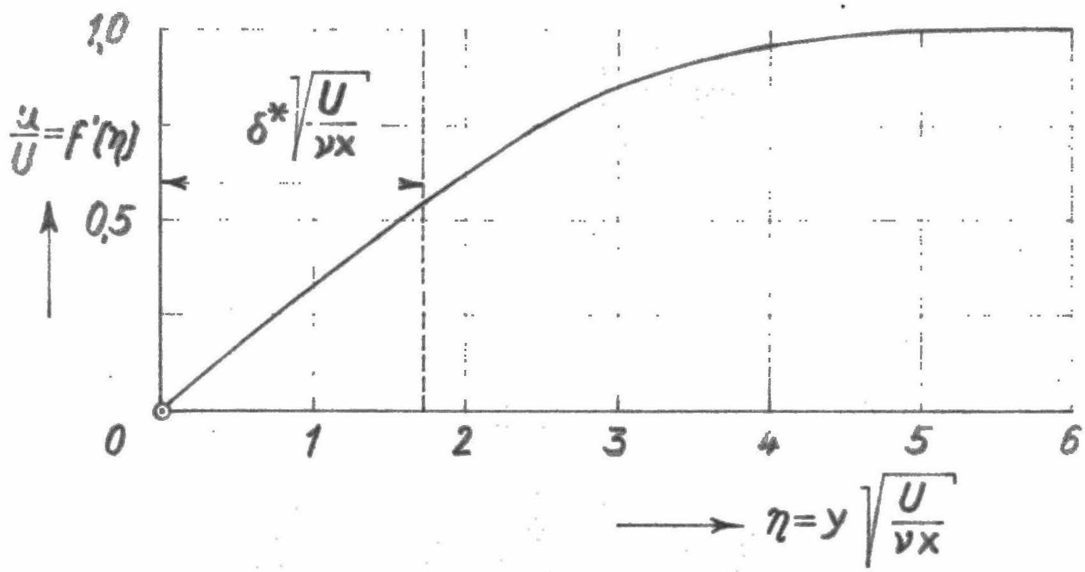


Figure 1.--Velocity distribution u(x,y) in the boundary layer on the flat plate (according to Blasius).

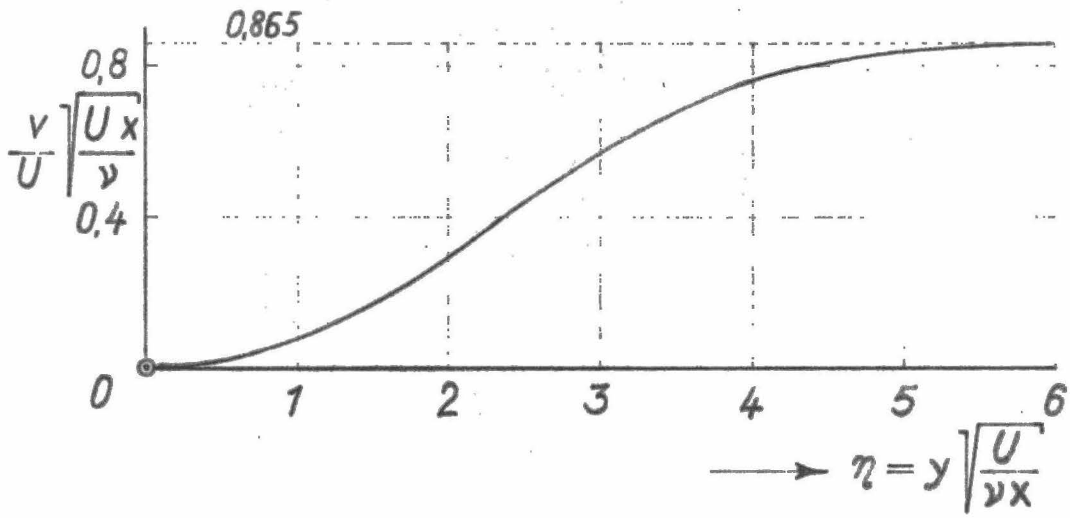


Figure 2.--The transverse velocity v(x,y) in the boundary layer on the flat plate.

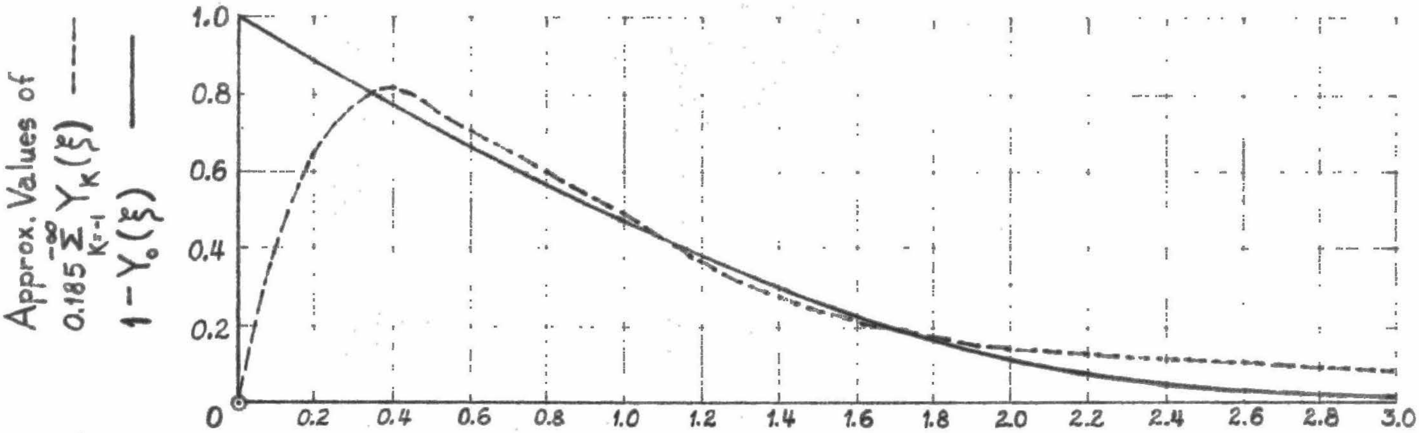


Figure 3.—Comparison of the last two columns in Table 3.

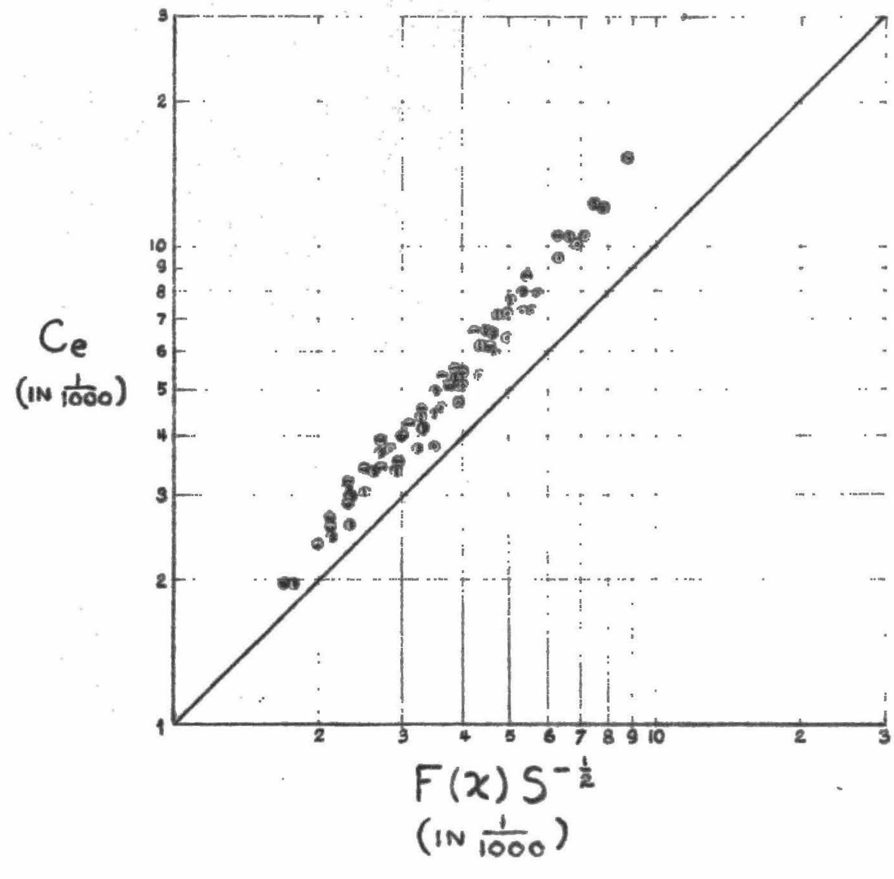


Figure 4.—Comparison of Equation (40) with Albertson's data

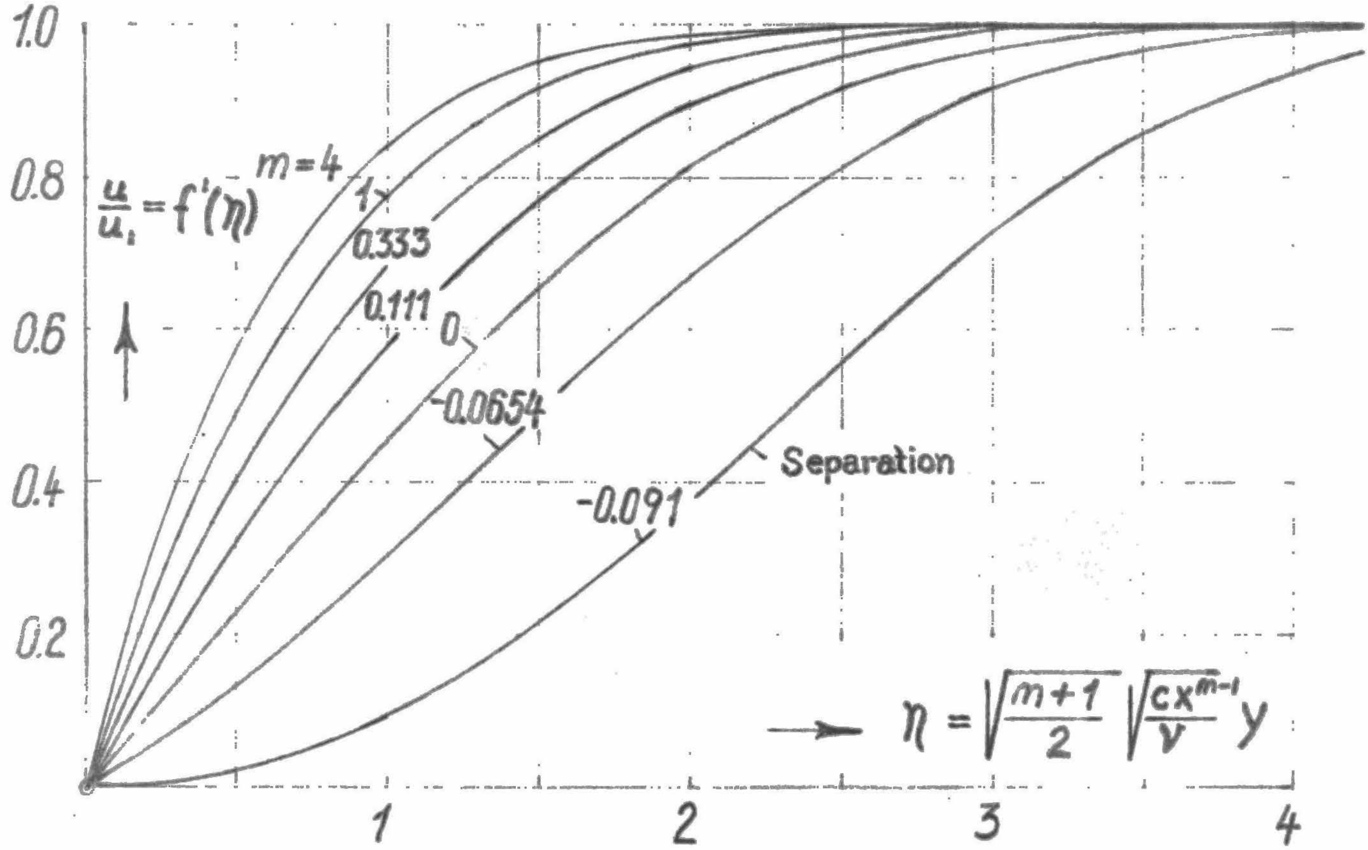


Figure 5.--Boundary-layer profiles for the potential flow $u_1 = cx^n$