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Singularities, Stable Surfaces, and the Repeatable Behavior of Kinematically Redundant Manipulators

Abstract

There has been significant interest in the periodic behavior, generally referred to as repeatability, exhibited by a kinematically redundant manipulator while performing a cyclic end-effector motion. Much of the early work in this area has been restricted to planar manipulators whose configuration is described in terms of absolute joint angles to simplify the problem. Unfortunately, this has resulted in the observation of certain phenomena that are unique to this special case and that do not describe the behavior of more complicated manipulators. The goal of this work is to clarify some possible misconceptions concerning the limiting behavior of a redundant manipulator under nonconservative control strategies, with particular emphasis on pseudoinverse control. In particular, stable surfaces are shown to be extremely rare, and a weaker property, referred to as repeatable trajectories, is responsible for the repeatable behavior observed in previous work. It is also shown that the Lie bracket condition need not be satisfied for this type of repeatable behavior to occur and that such trajectories need not have zero torsion, as has been previously suggested.

1. Introduction

Kinematically redundant manipulators are robotic systems that have more degrees of freedom than are required to perform a specified task. Because of this additional freedom, such manipulators have an infinite number of generalized inverse control strategies for solving the Jacobian equation. These control strategies are not, in general, repeatable in the sense that when the end effector follows a closed path, the manipulator does not necessarily return to its initial joint configuration. Klein and Huang (1983) were the first to observe this for the case of pseudoinverse control of a planar three-link manipulator. Subsequently, Klein and Kee (1989) did a numerical study of the drift in joint space for the same manipulator performing the cyclic task of repeatedly drawing a square in the workspace, showing that the drift had a numerically stable limit in some situations. These findings motivated further research into the repeatability of kinematically redundant manipulators (Bay 1992; Luo and Ahmad 1992; Mussa-Ivaldi and Hogan 1991; Maciejewski and Roberts 1990; Shamir 1990; Wampler 1989; Angeles and Mathur 1989; Shamir and Yomdin 1988).

Typically, a robotic system is described by its kinematic equation

$$\mathbf{x} = \mathbf{f}(\theta), \tag{1}$$

which relates the Cartesian coordinates of the end effector, described by the m-vector $\mathbf{x}$, to the n-dimensional vector θ of joint values. For kinematically redundant manipulators n is, of course, larger than m. Because (1) is, in general, very nonlinear, one typically works with the Jacobian equation, which, for the positional component, can be found by differentiating (1) to obtain

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\theta}. \tag{2}$$

The manipulator's task is usually specified as an end-effector path so that $\dot{\mathbf{x}}$ is given and the corresponding joint velocity $\dot{\theta}$ is to be calculated. Because the manipulator is redundant, (2) is an underdetermined system, and when the matrix $\mathbf{J}$ is of full rank, an infinite number of solutions exist. Methods from the theory of generalized inverses can be used to determine a solution of the form

$$\dot{\theta} = \mathbf{G}\dot{\mathbf{x}} + (\mathbf{I} - \mathbf{J}^{+}\mathbf{J})\mathbf{z}, \tag{3}$$

where $\mathbf{G}$ satisfies $\mathbf{JG} = \mathbf{I}$ at nonsingularities, $\mathbf{J}^{+}$ is the pseudoinverse of $\mathbf{J}$, and $\mathbf{z}$ is an arbitrary n-vector.

By imposing additional constraints on the manipulator, one can obtain repeatable control strategies in simply

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connected, singularity-free regions of the joint space. Augmenting the Jacobian with the appropriate number of kinematic constraints (Seraji 1989; Egeland 1987) is one such method of guaranteeing repeatability. The extended Jacobian (Baillieul 1985) results in a repeatable control strategy by minimizing an objective function of the joint variables. It is also possible to obtain repeatable inverses by writing G as a compliance-weighted pseudoinverse (Mussa-Ivaldi and Hogan 1991). One can even retain some of the desirable properties of a nonrepeatable inverse by selecting a repeatable inverse that is closest to the desired inverse (Roberts and Maciejewski 1992; 1993). In all cases, if a control strategy is repeatable, then no joint movement can result if the end effector is not moved. Thus the second term of (3) must be zero so that repeatable control strategies are necessarily of the form

$$\dot{\theta} = \mathbf{G}\dot{\mathbf{x}}. \quad (4)$$

An interesting problem is to characterize the control strategies that are repeatable. This differs from the existence of limit cycles first observed in Klein and Huang (1983), as no convergence is involved. This problem was solved by Shamir and Yomdin (1988) using differential geometric methods from nonlinear control theory. They have shown that a necessary and sufficient condition for a control strategy to be repeatable in an open subset of the joint space is that it satisfy the Lie bracket condition (LBC) in this region. The strategy is said to satisfy the LBC if the Lie bracket of any two columns $\mathbf{g}_i$ and $\mathbf{g}_j$ of $\mathbf{G}$ is in the column space of $\mathbf{G}$, where the Lie bracket of the two vector functions $\mathbf{g}_i$ and $\mathbf{g}_j$ is given by

$$[\mathbf{g}_i, \mathbf{g}_j] = \left(\frac{\partial \mathbf{g}_j}{\partial \theta}\right)\mathbf{g}_i - \left(\frac{\partial \mathbf{g}_i}{\partial \theta}\right)\mathbf{g}_j. \quad (5)$$

For the case of pseudoinverse control, the LBC can be applied to the transpose of the Jacobian instead of the pseudoinverse itself, which greatly simplifies the computations involved (Roberts and Maciejewski 1992).

The goal of this work is to clarify some possible misconceptions concerning the limiting behavior of a redundant manipulator under nonconservative control strategies, with pseudoinverse control used throughout as an example. The remainder of this article is organized in the following manner: Section 2 discusses stable surfaces and their relationship to kinematic singularities. In particular, it is shown that there are fundamental differences in the pseudoinverse control of planar manipulators when the joint variables are expressed in terms of absolute angles as opposed to the more realistic case of relative angles. A discussion of the applicability of stable surfaces to fully general manipulators is presented in Section 3. Section 4 discusses sufficient conditions for the existence of stable surfaces and illustrates that they do not exist for planar 3R manipulators under pseudoinverse control in terms of relative angles. Motivated by these results, Section 5 discusses a weaker property, the existence of repeatable trajectories, and illustrates that the LBC is not necessarily satisfied for this type of repeatable behavior. Section 6 discusses the property of minimum arc length, which has been attributed to repeatable trajectories, and shows it to have limited applicability for even very simple manipulators. Finally, the conclusions of this work are presented in Section 7.

2. Stable Surfaces and Singularities

Planar manipulators serve as an excellent tool for analyzing different control strategies for redundant manipulators and are a realistic model for robotic fingers. Usually such systems are described in relative angles, but sometimes the simpler case of absolute angles is used. One particular phenomenon that has been observed for the latter case is what has been called a stable surface, which is an m-dimensional hypersurface on which the manipulator is repeatable. Shamir and Yomdin (1988) have shown that a necessary condition for a stable surface is that the inverse satisfy the LBC on the surface. Because this is only a necessary condition, surfaces that satisfy the LBC will be called *candidate surfaces*. The notion of stable surfaces has been previously used to explain the cyclic behavior observed when a redundant manipulator performs the repetitive task of drawing a closed path in rectilinear space. This explanation was perhaps motivated by the fact that repeatable inverses have foliations of stable surfaces. However, much of this work will be dedicated to illustrating that the significance of the stable surface property, defined as an isolated stable surface, is extremely limited.

The presence of stable surfaces has a profound effect on the control of a manipulator, particularly with respect to reachability in the joint space. The stable surfaces in the example in Shamir and Yomdin (1988) effectively partition the joint space, as, once on the surface, the manipulator will continue to remain there. Thus two configurations in the joint space that occur on different sides of a stable surface are separated; the manipulator cannot go from one configuration to the other. Now as long as a singularity is not encountered, the manipulator can approach but cannot reach a stable surface; otherwise, by a time-reversal argument, the manipulator could leave the surface along the same trajectory traversed in the opposite direction. It is, however, possible to reach such a surface through an internal singularity as can be illustrated for the planar 3R manipulator with unit length links in absolute angles shown in Figure 1. Consider an initial joint configuration of the form $\psi = [0 \quad \pi \quad \psi_3]^T$.

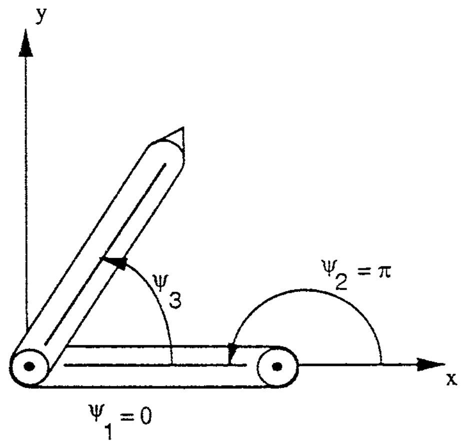

Fig. 1. Geometry of a 3R manipulator in absolute joint angles ψ. Each link has a length of 1 m. The joint angles are set at $\psi = [0 \quad \pi \quad \psi_3]^T$.

The Jacobian at this configuration is

$$\mathbf{J}(\psi) = \begin{bmatrix} 0 & 0 & -\sin\psi_3 \\ 1 & -1 & \cos\psi_3 \end{bmatrix}. \tag{6}$$

When $\sin\psi_3 \neq 0$ the Jacobian is nonsingular, and the pseudoinverse is given by

$$\mathbf{J}^+(\psi) = \frac{1}{2\sin\psi_3}\begin{bmatrix} \cos\psi_3 & \sin\psi_3 \\ -\cos\psi_3 & -\sin\psi_3 \\ -2 & 0 \end{bmatrix}. \tag{7}$$

To accomplish counterclockwise motion along the unit circle, the required end-effector velocity is perpendicular to the end-effector position and is given by

$$\dot{\mathbf{x}} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} -\sin\psi_3 \\ \cos\psi_3 \end{bmatrix}, \tag{8}$$

which, under pseudoinverse control, yields a joint velocity

$$\dot{\psi} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{9}$$

Thus, if the manipulator starts in the configuration $\psi = [0 \quad \pi \quad \pi/2]^T$, for example, and traverses a quarter of the unit circle in a counterclockwise fashion, only the third joint variable ψ_3 changes until it reaches a value of π. When $\psi_3 = \pi$, $\mathbf{J}(\psi)$ is singular, and the resulting joint velocity is

$$\dot{\psi} = \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}. \tag{10}$$

Note that at this point the manipulator is on the stable surface described by $\psi_2 = \psi_3$ and that the manipulator can leave this singularity, since the required joint motion to maintain the singularity $\dot{\psi} \propto [1 \quad 1 \quad 1]^T$ is not achieved in (10). Thus, for this example it is possible to reach a stable surface, but not to leave one, due to the discontinuity of the pseudoinverse at the singularity. At the point where the manipulator reaches the surface, the inverse switches to a form that holds on the stable surface, and the manipulator cannot return to its previous configuration.

Important information about a control strategy can be gained by considering its behavior at singular configurations. At a singularity the manipulator may get trapped, forced out of the singularity, or its behavior may depend on the particular end-effector trajectory. In particular, some additional insight into the case of multiple stable surfaces can be gained by considering the outer reach singularities of the Jacobian. An outer reach singularity occurs when the manipulator is completely extended, in which case the absolute angles have the form $\psi_{rs} = [\psi_1 \quad \psi_1 \quad \psi_1]^T$. At such a singularity the Jacobian has the form

$$\mathbf{J}(\psi_{rs}) = \begin{bmatrix} -l_1\sin\psi_1 & -l_2\sin\psi_1 & -l_3\sin\psi_1 \\ l_1\cos\psi_1 & l_2\cos\psi_1 & l_3\cos\psi_1 \end{bmatrix}. \tag{11}$$

The range of $\mathbf{J}^+(\psi)$ is given by the column space of $\mathbf{J}^T(\psi)$ so that

$$\operatorname{range}\{\mathbf{J}^+(\psi_{rs})\} = \operatorname{span}\left\{ \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \right\}. \tag{12}$$

For the special case $l_i = l$, $i = 1, 2, 3$, it follows that under pseudoinverse control the manipulator cannot escape a reach singularity. At such a singularity, the manipulator is caught between multiple stable surfaces and becomes trapped (see Figure 1 of Shamir and Yomdin [1988]). However, if the link lengths are not equal, then the manipulator can escape the singularity and, in fact, is locally forced out of the singularity.

The behavior of the planar 3R manipulator described in relative angles under pseudoinverse control is quite different from the absolute angle case just discussed. The Jacobian for this manipulator is given by

$$\mathbf{J}(\theta) = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin\theta_{12} - l_3\sin\theta_{123} & -l_2\sin\theta_{12} - l_3\sin\theta_{123} & -l_3\sin\theta_{123} \\ l_1\cos\theta_1 + l_2\cos\theta_{12} + l_3\cos\theta_{123} & l_2\cos\theta_{12} + l_3\cos\theta_{123} & l_3\cos\theta_{123} \end{bmatrix}, \tag{13}$$

where $\theta_{ij} = \theta_i + \theta_j$. For the case of relative angles, the reach singularities are given by $\theta_{rs} = [\theta_1 \quad 0 \quad 0]^T$ so that at such a singularity, the Jacobian is given by

$$\mathbf{J}(\theta_{rs}) = \begin{bmatrix} -(l_1+l_2+l_3)\sin\theta_1 & -(l_2+l_3)\sin\theta_1 & -l_3\sin\theta_1 \\ (l_1+l_2+l_3)\cos\theta_1 & (l_2+l_3)\cos\theta_1 & l_3\cos\theta_1 \end{bmatrix}. \tag{14}$$

It then follows that

$$\text{range}\{\mathbf{J}^+(\theta_{rs})\} = \text{span}\left\{ \begin{bmatrix} l_1 + l_2 + l_3 \\ l_2 + l_3 \\ l_3 \end{bmatrix} \right\}. \tag{15}$$

Thus, if there is any joint movement, all joint angles will move. This implies that the manipulator can leave the singularity, because to maintain this type of singularity, movement is only allowed in the first joint. An analogous situation exists for the other singularities, thus proving that if multiple stable surfaces exist, they do not intersect. This is a fundamental difference between the pseudoinverse behavior of the two Jacobians.

3. Likelihood of Candidate Stable Surfaces

For isolated stable surfaces to be of significant importance in practical robotic applications, one would like to assume that they are reasonably likely to exist for an arbitrary manipulator design. Previous work has implicitly made this assumption, using a planar manipulator described in absolute joint angles to justify this position. Unfortunately, it will be shown that it is highly unlikely that any fully general spatial manipulator will possess a stable surface. This will be done by first presenting an example of applying the necessary condition imposed by the LBC to a specific anthropomorphic manipulator in order to identify potential candidate surfaces, and then illustrating why these candidate surfaces cannot be stable surfaces. Insight obtained from this example will then be used to explain why it is highly unlikely that isolated stable surfaces would exist for any general manipulator design.

A typical spatial redundant manipulator design is the seven-DOF anthropomorphic manipulator described in detail in Podhorodeski et al. (1991). The Jacobian for this particular manipulator is given by

$$\mathbf{J} = \begin{bmatrix} S_2C_3C_4 + C_2S_4 & -S_3C_4 & S_4 & 0 & 0 & S_5 & -C_5S_6 \\ -S_2S_3 & -C_3 & 0 & -1 & 0 & -C_5 & -S_5S_6 \\ -S_2C_3S_4 + C_2C_4 & S_3S_4 & C_4 & 0 & 1 & 0 & C_6 \\ -S_2S_3C_4g - S_2S_3h & -C_3C_4g - C_3h & 0 & -h & 0 & 0 & 0 \\ -S_2C_3g - S_2C_3C_4h - C_2S_4h & S_3g + S_3C_4h & -hS_4 & 0 & 0 & 0 & 0 \\ S_2S_3S_4g & C_3S_4g & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{16}$$

where S_i and C_i denote $\sin\theta_i$ and $\cos\theta_i$, and the parameters g and h are the nonzero lengths of the upper and lower arms, respectively. The null vector for this manipulator can also be written analytically and is given by

$$\mathbf{n_J} = \begin{bmatrix} C_3S_4S_6h \\ -S_2S_3S_4S_6h \\ -(S_2g + S_2C_4h + C_2C_3S_4h)S_6 \\ 0 \\ S_2C_4S_6g + S_2S_4C_5C_6g + S_2S_6h \\ S_2S_4S_5S_6g \\ -S_2S_4C_5g \end{bmatrix}. \tag{17}$$

A necessary condition for the existence of a candidate surface under pseudoinverse control is that the system of equations

$$\mathbf{n_J^T}[\mathbf{J}_i, \mathbf{J}_j] = 0 \qquad 1 \le i < j \le 6 \tag{18}$$

determines an m-dimensional invariant hypersurface, where $\mathbf{J}_i$ is the ith column of $\mathbf{J}^T$. For this to occur, it is necessary that (18) reduces to a set of r constraints on the surface, where $r = n - m$ is the degree of redundancy in the robotic system. In this case the degree of redundancy is one, so that on a candidate surface, the 15 equations given in (18) must reduce to one independent constraint.

To show that no stable surfaces exist for this manipulator, it is necessary to calculate some of the constraints given by (18). For example, consider the following constraint functions:

$$\mathbf{n_J^T}[\mathbf{J}_1h + \mathbf{J}_5, \mathbf{J}_6] = C_2C_3S_4^2S_6g^2h \tag{19}$$

$$\mathbf{n_J^T}[\mathbf{J}_5, \mathbf{J}_6] = C_2C_3S_4^2S_6gh(g + C_4h) - S_2S_4^3S_6gh^2(1 + C_3^2) \tag{20}$$

where each quantity would be identically zero on a candidate surface. From these two equations, one can easily conclude that the following are necessary conditions:

$$C_2C_3S_4S_6 = 0, \tag{21}$$

$$S_2S_4S_6 = 0, \tag{22}$$

where (21) follows directly from (19), while (22) follows from (20) after applying (21). By adding the square of (21) and the square of the product of C_3 with (22), one obtains that $C_3^2S_4^2S_6^2 = 0$ so that

$$C_3S_4S_6 = 0. \tag{23}$$

Using (22)–(23) one can obtain the additional constraint functions

$$\mathbf{n_J^T}[\mathbf{J}_1, \mathbf{J}_2 - \mathbf{J}_4] = S_2S_4C_5C_6g, \tag{24}$$

$$\mathbf{n_J^T}[\mathbf{J}_1, \mathbf{J}_4] = -S_2S_6(h + C_4g), \tag{25}$$

which are once again required to be zero on a candidate surface. All of the above constraints can be combined into the single vector equation given by

$$\begin{bmatrix} C_3S_4S_6 \\ S_2S_4S_6 \\ S_2S_4C_5 \\ S_2S_6(h + C_4g) \end{bmatrix} = \mathbf{0}. \tag{26}$$

For a surface determined by (26) to be a candidate surface, it must be of the same dimension as the workspace, which in this case is six. By considering the first and third elements of (26), one can easily see that this is not the case when $S_4 \neq 0$. Dividing by S_4 results in the two constraints $C_3S_6 = 0$ and $S_2C_5 = 0$, which are clearly independent and therefore define a manifold of dimension five. This would be a valid method for showing that there are no candidate surfaces except for the fact that the configurations that satisfy (26) correspond to singularities. Thus it may be possible that the surfaces satisfying (26) have the appropriate dimension after the loss of a degree of freedom due to the singularity. However, this can be shown not to be the case by considering the form of these singularities. In particular, note that the singularities of $\mathbf{J}$ occur when certain joints take on a specific value (e.g., $\theta_2 = 0$ and $\theta_3 = \pi/2$). Thus, for the manipulator to remain in a singular configuration, the joints causing the singularity must maintain their value (i.e., their time derivatives must remain zero). However, one can see that any singular configuration can be escaped under pseudoinverse control by observing that the columns of $\mathbf{J}$ are always nonzero, which in turn implies that the rows of $\mathbf{J}^+$ are never zero; hence it is always possible to induce motion in any joint by specifying an appropriate end-effector command $\dot{\mathbf{x}}$. By escaping the singularity, the manipulator automatically leaves any surface satisfying (26), proving that there are no stable surfaces when $S_4 \neq 0$. If $S_4 = 0$ because $\theta_4 = \pm\pi$, then one can show that the LBC is satisfied if and only if g is equal to h. However, this candidate surface is not a stable surface, since, once again, the fourth column of $\mathbf{J}$ is never zero, so one can always induce motion in θ_4 to escape this surface. Therefore, one can conclude that the 7-DOF manipulator whose Jacobian is given by (16) does not possess the stable surface property under pseudoinverse control.

By carefully considering the implications of the LBC, it becomes clear that one should not expect that a general manipulator would possess a stable surface. As discussed earlier, a stable surface is specified by $r = n - m$ independent equations that constrain the manipulator to lie on an m-dimensional hypersurface. The number of constraint equations determined by the LBC is $r\binom{m}{2}$. For planar manipulators, $\binom{m}{2}$ is one so that any planar manipulator will automatically result in r constraints; however, for higher dimensional workspaces, this becomes increasingly more unlikely as m becomes larger. One would then suspect that a fully general manipulator with a workspace of dimension six would not even possess a candidate surface, let alone a stable surface, as was illustrated in the above example. It would thus seem that the likelihood of the existence of a stable surface significantly diminishes for workspaces of dimension greater than two, implying that stable surfaces are much more likely to occur for planar manipulators. The next section will consider how likely it is for a planar manipulator to have the stable surface property.

4. The Existence of Stable Surfaces

The LBC only provides a necessary condition for the existence of a stable surface and thus can only be used for determining possible candidate surfaces. The previous section has shown that for fully general manipulators, the existence of a candidate surface is unlikely. This section discusses a necessary and sufficient condition for a candidate surface to be a stable surface (Shamir 1990). As would be expected, this condition further restricts the class of manipulators that may possess stable surfaces.

Consider a manipulator with a single degree of redundancy that has a candidate surface given by

$$S(\boldsymbol{\theta}) = 0, \tag{27}$$

where S can be found by applying the LBC. If the equation does in fact describe a stable surface, then as the manipulator moves, equation (27) must continue to be satisfied. Thus the directional derivative of S along any joint movement determined by the pseudoinverse is zero; i.e.,

$$\dot{S} = \nabla S \cdot \dot{\boldsymbol{\theta}}_p = 0, \tag{28}$$

where $\dot{\boldsymbol{\theta}}_p$ represents the pseudoinverse solution of $\dot{\boldsymbol{\theta}}$ for a particular end-effector motion. Equation (28) essentially means that on a stable surface, the LBC continues to hold (i.e., the time derivative of $S(\boldsymbol{\theta})$ is zero). In some cases, the function $S(\boldsymbol{\theta})$ can be viewed as a kinematic constraint that guarantees repeatability and on which the stable surface results in a control that corresponds to the pseudoinverse. Since $\dot{\boldsymbol{\theta}}_p$ is exactly determined by the column space of $\mathbf{J}^T$, it follows that

$$\mathbf{J}\nabla S = \mathbf{0}, \tag{29}$$

so that ∇S is in the null space of $\mathbf{J}$. For manipulators with one degree of redundancy, this null space is characterized by the null vector $\mathbf{n_J}$. If ∇S is not a multiple of $\mathbf{n_J}$, then it is possible to choose an end-effector movement that produces a joint movement with a component in the ∇S direction, thus pushing it off the surface. Therefore, for the manipulator to be on a stable surface described by (27), it is necessary for ∇S to be a multiple of $\mathbf{n_J}$. This provides an additional tool for checking candidate surfaces (Shamir 1990).

With this additional constraint, one can easily show that stable surfaces are not as common as previously thought. In particular, there do not exist any stable surfaces for the planar 3R manipulator under pseudoinverse control in relative angles, which is in fact the manipulator that first motivated the problem of repeatability. The necessary relationship of the joint variables on a stable surface for this type of manipulator is found using the LBC and is given by

$$S(\theta) = -l_3 \sin\theta_2 + l_2 \cos\theta_2 \sin\theta_3 + l_3 \sin\theta_3 \cos(\theta_2+\theta_3) = 0. \tag{30}$$

For this particular manipulator, the null space of $\mathbf{J}(\theta)$ is characterized by the vector

$$\mathbf{n_J} = \begin{bmatrix} l_2 l_3 \sin\theta_3 \\ -l_2 l_3 \sin\theta_3 - l_1 l_3 \sin(\theta_2+\theta_3) \\ l_1 l_2 \sin\theta_2 + l_1 l_3 \sin(\theta_2+\theta_3) \end{bmatrix}. \tag{31}$$

For (30) to describe a stable surface, its gradient must be a multiple of $\mathbf{n_J}$. Differentiating (30) yields

$$\nabla S = \begin{bmatrix} 0 \\ -l_3 \cos\theta_2 - l_2 \sin\theta_2 \sin\theta_3 - l_3 \sin\theta_3 \sin(\theta_2+\theta_3) \\ l_2 \cos\theta_2 \cos\theta_3 + l_3 \cos\theta_3 \cos(\theta_2+\theta_3) - l_3 \sin\theta_3 \sin(\theta_2+\theta_3) \end{bmatrix}. \tag{32}$$

For (29) to be satisfied, either $\mathbf{n_J}$ and ∇S must be proportional to each other or ∇S must be $\mathbf{0}$. By comparing the first element of (31) with (32), it is easily seen that the first case can only occur when $\sin\theta_3 = 0$. If this is true, then from (30) it can be seen that $\sin\theta_2$ is also zero, which would imply that the manipulator is in a singular configuration; however, it was pointed out in Section 2 that the manipulator can escape this singularity using pseudoinverse control for relative joint angles. Now consider the case when $\nabla S = \mathbf{0}$. One can show that the second element of ∇S cannot be zero on a surface described by S by noting that

$$S^2 + \left(\frac{\partial S}{\partial \theta_2}\right)^2 = l_3^2 + 2l_3^2 \sin^2\theta_3 + (l_2 - l_3)^2 \sin^2\theta_3 > 0, \tag{33}$$

where $\partial S/\partial\theta_2$ is the second element of ∇S. Because (33) holds for the entire joint space, there is no value of θ such that $S = 0$ and $\nabla S = \mathbf{0}$. Thus the surfaces satisfying (30) are not stable surfaces.

Even for manipulators where it is possible to have stable surfaces, such surfaces only exist under extremely restrictive conditions. Consider the n-link planar manipulator shown in Figure 2 described in terms of its absolute angles. The Jacobian corresponding to this manipulator is given by

$$\mathbf{J}(\psi) = \begin{bmatrix} -l_1 \sin\psi_1 & -l_2 \sin\psi_2 & \cdots & -l_n \sin\psi_n \\ l_1 \cos\psi_1 & l_2 \cos\psi_2 & \cdots & l_n \cos\psi_n \end{bmatrix}. \tag{34}$$

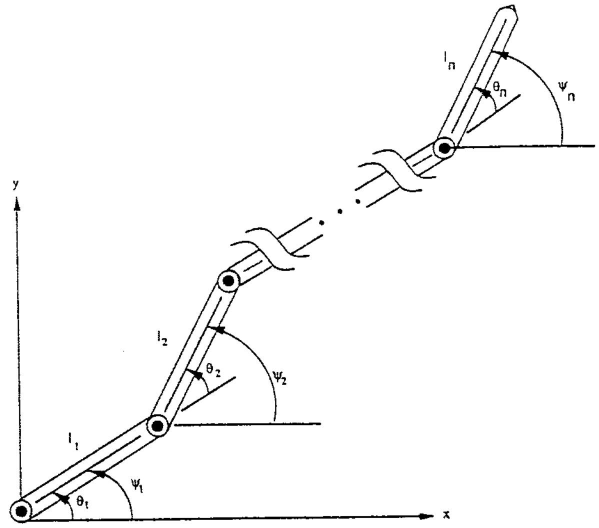

Fig. 2. Geometry of a planar revolute n-link manipulator in terms of relative joint angles θ and absolute joint angles ψ.

For the manipulator Jacobian given in (34), the Lie bracket of the columns $\mathbf{J}_1(\psi)$ and $\mathbf{J}_2(\psi)$ of $\mathbf{J}^T(\psi)$ is

$$[\mathbf{J}_1(\psi), \mathbf{J}_2(\psi)] = \begin{bmatrix} l_1^2 \\ l_2^2 \\ \vdots \\ l_n^2 \end{bmatrix}. \tag{35}$$

Thus, on a candidate surface of the manipulator described by (34), the matrix

$$\begin{bmatrix} l_1^2 & l_2^2 & \cdots & l_n^2 \\ l_1 \sin\psi_1 & l_2 \sin\psi_2 & \cdots & l_n \sin\psi_n \\ l_1 \cos\psi_1 & l_2 \cos\psi_2 & \cdots & l_n \cos\psi_n \end{bmatrix} \tag{36}$$

is not of full rank. This will precisely occur when all of the minors of (36) are zero; i.e., when

$$\begin{vmatrix} l_i^2 & l_j^2 & l_k^2 \\ l_i \sin\psi_i & l_j \sin\psi_j & l_k \sin\psi_k \\ l_i \cos\psi_i & l_j \cos\psi_j & l_k \cos\psi_k \end{vmatrix} = 0 \qquad 1 \le i < j < k \le n. \tag{37}$$

Expanding the determinant yields

$$l_i \sin(\psi_j - \psi_k) + l_j \sin(\psi_k - \psi_i) + l_k \sin(\psi_i - \psi_j) = 0, \tag{38}$$

which is a necessary relationship between the joint angles ψ_i on a stable surface. This highly restrictive condition on the existence of isolated stable surfaces for planar manipulators can be used to prove the following theorem:

THEOREM 1. The n-link planar revolute manipulator described in absolute angles (as given in (34)) has a stable surface under pseudoinverse control if and only if there are no more than two distinct link lengths.

Proof. See Appendix A.

Thus this type of manipulator has a stable surface when there are no more than two distinct link lengths. If the two distinct link lengths are L_1 and L_2, then the absolute angles associated with links of length L_1 are all equal and similarly for L_2. When all link lengths are equal, there are multiple (but a finite number of) stable surfaces; in fact, for equal link lengths, there are $2^n - 2$ stable surfaces. For this case the manipulator is on a stable surface if and only if the absolute angles take on exactly two distinct values. Physically this means that all but two of the joints are frozen on a stable surface so that the manipulator behaves like a planar 2R manipulator, which is not a very useful control strategy, particularly with respect to utilizing redundancy. In addition, because one can never guarantee that the manipulator links are exactly equal, it is, for all practical purposes, impossible to even build a planar n-link manipulator that has the stable surface property under pseudoinverse control.

Although describing a manipulator in terms of absolute angles simplifies the problem, from a physical point of view, one is typically more interested in minimizing joint velocities in terms of relative angles. It is important to note that the pseudoinverse for the Jacobian in absolute angles given in (34) corresponds to a weighted pseudoinverse for the Jacobian in relative angles given by (13). The weighting matrix in this case is given by $\mathbf{Q} = \mathbf{T}^T\mathbf{T}$, where $\mathbf{T}$ is the matrix transforming θ into ψ (i.e., $\psi = \mathbf{T}\theta$) and is given by

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \qquad (39)$$

This implies the important observation that by applying a weighting to the pseudoinverse, one does not simply warp an existing stable surface but may cause it to cease to exist. Thus the stable surface property is not preserved under joint transformations. The popularity of stable surfaces in the literature is most likely due to the fact that the periodic behavior of the equal link length planar manipulator described in absolute angles was due to the existence of stable surfaces, and that this was unfortunately attributed to the periodic behavior of the relative angle case, which does not have a stable surface.

5. Repeatable Trajectories and the LBC

The above section has shown that the candidate surfaces for the planar 3R manipulator under pseudoinverse control using relative joint angles are not stable surfaces. Therefore, the periodic behavior exhibited by the joint angle trajectories is due not to the presence of a stable surface but to the nature of the specified cyclic end-effector trajectory. This type of behavior will be referred to as the existence of repeatable trajectories. It is natural to ask whether the LBC holds on such repeatable trajectories, as much of the work on generating repeatable behavior relies on this condition as a foundation. It is easy to show that for some types of manipulators, there is a large class of end-effector trajectories that always result in a repeatable joint space trajectory. Consider a revolute planar three-link manipulator in $I\!R^3$ with an additional prismatic joint that moves along the z-axis. An example of such a manipulator is a SCARA manipulator (e.g., the ADEPT) if the last link length is modified to be nonzero (Figure 3). The manipulator satisfies the LBC under the same conditions as the 3R manipulator without the prismatic joint—namely, when (30) is satisfied. Take any small trajectory in the $z = 0$ plane that does not intersect itself and that does not encounter a singularity. Call the end points of this trajectory A and B, where A denotes the starting point. When the end effector reaches point B, let the path continue straight up from B one unit in the z direction to a point that will be denoted C. Because movement in the z direction is only realized by the prismatic joint, the revolute joints are unaffected by the movement from B to C. Now let the manipulator traverse the previous path in the opposite direction except at $z = 1$, stopping at D, a unit translation along the z axis from A. By a time-reversal argument, it readily follows that the revolute joints of the manipulator at point D would be the same as they were at point A. Finally, let the end effector move from D to A. The manipulator would return to its initial configuration. Any such trajectory for this manipulator possesses the property that it introduces no drift in the joint space and results

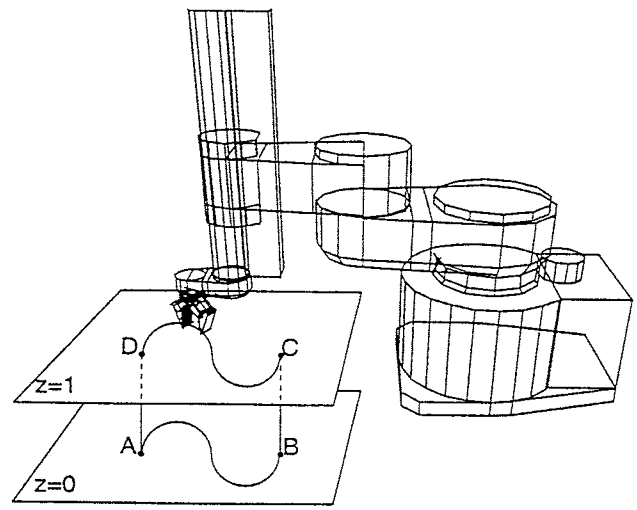

Fig. 3. An example of a redundant manipulator that possesses a large class of repeatable trajectories that do not satisfy the LBC.

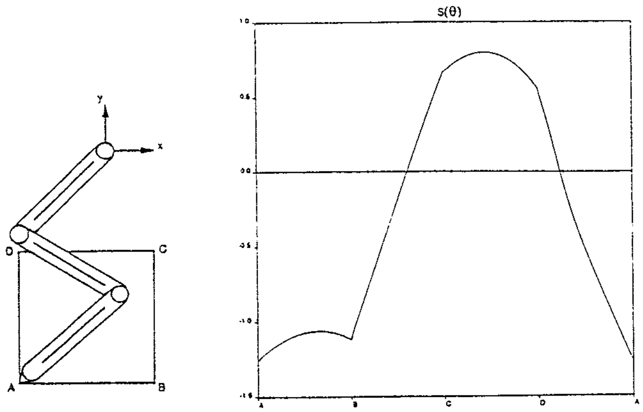

Fig. 4. A graph of $S(\boldsymbol{\theta})$ along a repeatable trajectory corresponding to the square end-effector path shown. Note that $S(\boldsymbol{\theta})$ is not identically zero along the trajectory, thus showing that the LBC does hold on this particular repeatable trajectory.

in a closed joint space trajectory regardless of whether the joint space trajectory intersects a candidate surface. Such "no drift" trajectories are always present when the joint space–to-workspace mapping is decoupled into redundant and nonredundant components. Describing the same SCARA manipulator in absolute angles provides an example of where there is a stable hypersurface but where there are repeatable trajectories which do not intersect the hypersurface.

Although this example may be considered a special case, in practice it is quite simple to verify that the LBC is not satisfied for general trajectories, including those used in Klein and Huang (1983), by simply calculating the value of $S(\boldsymbol{\theta})$ along a repeatable trajectory, as illustrated in Figure 4. This graph represents the function $S(\boldsymbol{\theta})$ given in (30) for a repeatable trajectory of the 3R manipulator in relative angles where the link lengths are 1 m. The repeatable trajectory corresponds to a limit cycle for a square end-effector motion. It can clearly be seen that $S(\boldsymbol{\theta})$ is not identically zero, so that the LBC is not satisfied for this repeatable trajectory. The significance of this observation lies in the fact that techniques that have been proposed for obtaining repeatable behavior that rely on the LBC (Luo and Ahmad 1992) will not be able to identify these repeatable trajectories. This is particularly unfortunate in light of the results of the previous section, which show that stable surfaces rarely exist for revolute manipulators. Thus the LBC, while providing an elegant test for repeatability, does not provide as powerful a tool for locating periodic solutions as previously thought. In particular, one cannot rely on using the LBC for setting appropriate initial configurations to guarantee a closed joint space trajectory for a closed end-effector motion.

6. Minimum Arc Length Trajectories

It has been suggested that the joint space trajectory that minimizes its arc length subject to satisfying a given closed end-effector motion is repeatable and that this optimal trajectory is obtained under nonweighted pseudoinverse control (Bay 1992). A trajectory that minimizes its arc length subject to staying on the manifold determined by the end-effector motion would have zero torsion. If the repeatable trajectories of the nonweighted pseudoinverse have this property of zero torsion, this can be used for finding repeatable trajectories.

Minimizing the arc length of the joint space trajectory corresponds to minimizing

$$\int_{t_0}^{t_f} \sqrt{\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}}}\, dt \tag{40}$$

subject to

$$\mathbf{x}(t) - \mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}, \tag{41}$$

where $\mathbf{x}(t)$ is the given end-effector motion, and t_0 and t_f are, respectively, the initial and final times of the trajectory. Application of the Euler-Lagrange equation results in the following requirement of an optimal solution:

THEOREM 2. A necessary condition for an extremal of the problem of minimizing $\int_{t_0}^{t_f} \sqrt{\dot{\boldsymbol{\theta}}^T \dot{\boldsymbol{\theta}}}\, dt$ subject to satisfying $\mathbf{x}(t) - \mathbf{f}(\boldsymbol{\theta}) = \mathbf{0}$ is

$$\frac{d}{dt}(\mathbf{n_J} \cdot \dot{\boldsymbol{\theta}}) - \dot{\boldsymbol{\theta}}^T \left(\frac{\partial \mathbf{n_J}}{\partial \boldsymbol{\theta}}\right) \dot{\boldsymbol{\theta}} - \frac{\dot{\boldsymbol{\theta}} \cdot \ddot{\boldsymbol{\theta}}}{\|\dot{\boldsymbol{\theta}}\|^2} \mathbf{n_J} \cdot \dot{\boldsymbol{\theta}} = 0. \tag{42}$$

Proof. See Appendix B.

It can be shown that the trajectories determined by pseudoinverse control of the planar 3R manipulator with unit length links described in absolute angles do satisfy (42). Because the control is the pseudoinverse, $\mathbf{n_J} \cdot \dot{\boldsymbol{\theta}} = 0$, which simplifies (42) to

$$\dot{\boldsymbol{\theta}}^T \left(\frac{\partial \mathbf{n_J}}{\partial \boldsymbol{\theta}}\right) \dot{\boldsymbol{\theta}} = 0. \tag{43}$$

Once again, because pseudoinverse control is being used, the joint velocities have the form $\dot{\boldsymbol{\psi}} = \mathbf{J}^T(\boldsymbol{\psi})\mathbf{w}$, where $\mathbf{w}$ is a 2-vector. With this substitution, (43) becomes

$$\mathbf{w}^T \mathbf{M} \mathbf{w} = 0, \tag{44}$$

where

$$\mathbf{M} = \mathbf{J}\left(\frac{\partial \mathbf{n_J}}{\partial \boldsymbol{\psi}}\right)\mathbf{J}^T, \tag{45}$$

$$\mathbf{J} = \begin{bmatrix} -\sin\psi_1 & -\sin\psi_2 & -\sin\psi_3 \\ \cos\psi_1 & \cos\psi_2 & \cos\psi_3 \end{bmatrix}, \tag{46}$$

$$\frac{\partial \mathbf{n_J}}{\partial \psi} = \begin{bmatrix} 0 & -\cos(\psi_3 - \psi_2) & \cos(\psi_3 - \psi_2) \\ \cos(\psi_1 - \psi_3) & 0 & -\cos(\psi_1 - \psi_3) \\ -\cos(\psi_2 - \psi_1) & \cos(\psi_2 - \psi_1) & 0 \end{bmatrix}, \tag{47}$$

and where $\mathbf{n_J} = [\sin(\psi_3 - \psi_1) \quad \sin(\psi_1 - \psi_2) \quad \sin(\psi_2 - \psi_1)]^T$. It is not hard to verify that the matrix **M** is zero on the stable surfaces $\psi_i = \psi_j$, $i \neq j$. Thus, the necessary condition for minimal arc length subject to a specified end-effector motion is satisfied for those repeatable trajectories that are on a stable surface of this particular manipulator. It should be noted that because the stable surfaces for this manipulator are, in fact, planar, the torsion of any trajectory on the surface is zero.

Because the minimum arc length and zero torsion arguments are crucial to the method described in Bay (1992), an analysis of pseudoinverse control for the 3R manipulator described in the more realistic relative angles will now be done. For this case the necessary condition can be easily checked through a simulation. Let

$$\mathbf{L} = \frac{1}{2}\left[\left(\frac{\partial \mathbf{n_J}}{\partial \theta}\right) + \left(\frac{\partial \mathbf{n_J}}{\partial \theta}\right)^T\right], \tag{48}$$

and consider the quantity

$$\frac{\dot{\theta}^T \mathbf{L} \dot{\theta}}{\|\dot{\theta}\|^2}. \tag{49}$$

Now because **L** is symmetric, its eigenvalues are real, and by the Rayleigh-Ritz theorem, (49) is bounded above and below by the maximum and minimum eigenvalues of **L**, respectively. As above, if the minimum arc length trajectory is determined by pseudoinverse control, then (49) must be zero. Figure 5 shows (49) along a repeatable trajectory corresponding to a square in the workspace. It can be seen from this graph that the necessary condition for minimum arc length is not satisfied. The actual joint space trajectory for this specified end-effector trajectory is given in Figure 6. Note that from this figure, one can see that the repeatable trajectory is not a simple planar curve, since its projection onto the $\theta_1 - \theta_3$ plane appears to cross over itself. Thus, the minimum arc length property of repeatable trajectories appears to be restricted to a very limited class of manipulators and then only when using the less useful formulation of absolute joint angles.

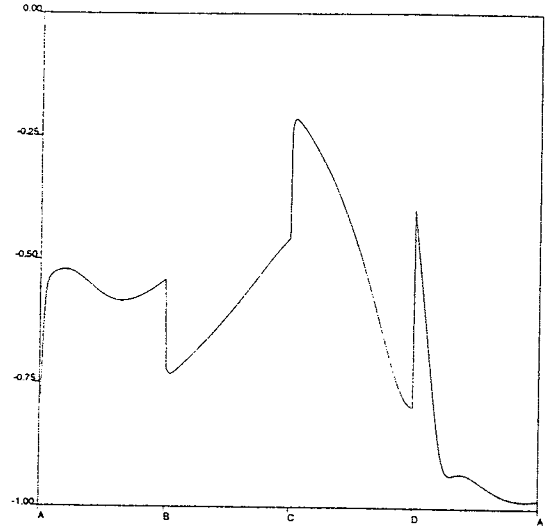

Fig. 5. A graph of the necessary condition for a minimum arc length trajectory along a repeatable trajectory corresponding to the square end-effector path in Fig. 4. Note that because this condition is not identically zero along the trajectory, this repeatable trajectory is not of minimum arc length.

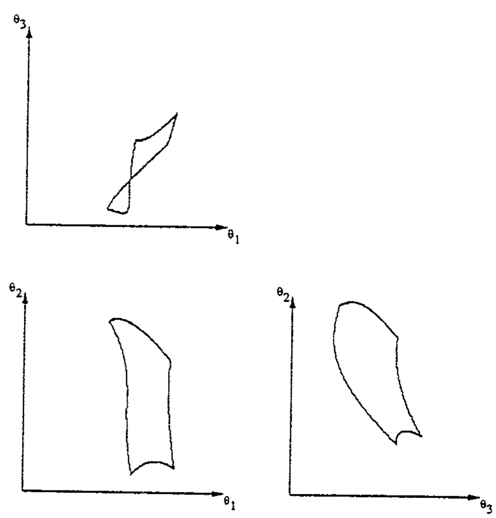

Fig. 6. Three orthogonal views of a repeatable trajectory corresponding to the end-effector trajectory in Figure 4, illustrating that it is not of zero torsion.

7. Conclusions

This article has illustrated that the issue of repeatability has many subtle aspects that must be considered to obtain practical results. Some of the properties previously associated with repeatable trajectories, while true for a small class of redundant manipulators, have been shown not to hold in general. In particular, it was shown that the existence of stable surfaces for revolute manipulators under pseudoinverse control can only be verified for planar mechanisms with severely constrained link lengths, and then only when the problem is cast in terms of absolute angles. The relationship between kinematic singularities and the possible intersection of stable surfaces was also used to illustrate the radical difference between problems specified in absolute angles and those specified in more realistic relative angles. It was also shown that repeatable trajectories under pseudoinverse control need not have zero torsion. The fundamental conclusion from these results is that one cannot rely on merely setting the initial joint configuration of a manipulator to lie on a particular surface to guarantee repeatable behavior. In addition, this work has shown that the LBC is not, in general, satisfied for repeatable trajectories, thus rendering it ineffective as a method of identifying such trajectories.

Appendix A: Proof of Theorem 1

LEMMA 1. Suppose that $\sin\alpha\sin\beta = \sin\alpha\sin\gamma = \sin\beta\sin\gamma = 0$ and that $\alpha+\beta+\gamma=0$. Then $\sin\alpha = \sin\beta = \sin\gamma = 0$.

Proof. Because $\sin\alpha\sin\beta = 0$, it follows that $\sin\alpha = 0$ or $\sin\beta = 0$. Without loss of generality, one can assume that $\sin\alpha = 0$. Now consider $\sin\beta\sin\gamma = 0$. Because $\alpha+\beta+\gamma = 0$, one has that

$$\begin{aligned}\sin\beta\sin\gamma &= \sin\beta\sin(-\alpha-\beta)\\ &= \sin\beta[-\sin\alpha\cos\beta - \cos\alpha\sin\beta] \qquad \text{(A1)}\\ &= -\cos\alpha\sin^2\beta,\end{aligned}$$

implying $\cos\alpha\sin^2\beta = 0$. Because $\cos\alpha = \pm 1$, it follows that $\sin\beta = 0$. Finally, note that

$$\sin\gamma = -\sin\alpha\cos\beta - \cos\alpha\sin\beta = 0. \qquad \text{(A2)}$$

□

THEOREM 1. The n-link planar revolute manipulator described in absolute angles (as given in (34)) has a stable surface under pseudoinverse control if and only if there are no more than two distinct link lengths.

Proof. (⇐) Because the manipulator is operating under pseudoinverse control, the joint velocities are in the column space of $\mathbf{J}^T$; i.e.,

$$\dot{\psi} \in \text{Range}\left\{\begin{bmatrix} -l_1\sin\psi_1 & l_1\cos\psi_1 \\ -l_2\sin\psi_2 & l_2\cos\psi_2 \\ \vdots & \vdots \\ -l_n\sin\psi_n & l_n\cos\psi_n \end{bmatrix}\right\}. \qquad \text{(A3)}$$

Assume that there are no more than two distinct link lengths, given here as L_1 and L_2, where $L_1 = L_2$ when all link lengths are the same. Let $\psi_{i_1}, \psi_{i_2}, \ldots, \psi_{i_p}$ be the joint variables associated with the links of length L_1 and let $\psi_{j_1}, \psi_{j_2}, \ldots, \psi_{j_{n-p}}$ be the joint variables associated with the links of length L_2 where $1 \le p < n$. We claim that the surface parameterized by

$$\psi_{i_1} = \psi_{i_2} = \cdots = \psi_{i_p} = \phi_1 \qquad \text{(A4)}$$

$$\psi_{j_1} = \psi_{j_2} = \cdots = \psi_{j_{n-p}} = \phi_2 \qquad \text{(A5)}$$

is invariant under (A3). This will be proven by showing that for any configuration on the surface, the manipulator continues to remain on the surface under pseudoinverse control. With the given assumptions on link lengths, (A3) becomes

$$\dot{\psi}_{i_k} = -\alpha L_1\sin\psi_{i_k} + \beta L_1\cos\psi_{i_k} \quad 1\le k\le p \qquad \text{(A6)}$$

$$\dot{\psi}_{j_l} = -\alpha L_2\sin\psi_{j_l} + \beta L_2\cos\psi_{j_l} \quad 1\le l\le n-p \qquad \text{(A7)}$$

where, for nonsingular configurations, $\alpha(t,\psi)$ and $\beta(t,\psi)$ are given by $(\mathbf{J}\mathbf{J}^T)^{-1}\dot{\mathbf{x}}$. Because each ψ_{i_k} is equal on the surface described by (A4) and (A5), it follows that the joint velocities $\dot{\psi}_{i_k}$ are equal. Similarly, the $\dot{\psi}_{j_l}$ are also equal. It then follows that the manipulator remains on the surface under pseudoinverse control.

(⇒) Suppose that the three joint links l_i, l_j, and l_k are distinct. From equation (38) of the text, it follows that a candidate surface must satisfy

$$S(\psi) = -l_i s(\psi_j - \psi_k) - l_j s(\psi_k - \psi_i) - l_k s(\psi_i - \psi_j) = 0. \qquad \text{(A8)}$$

The gradient of S is

$$\nabla S = \begin{bmatrix} l_j c(\psi_i - \psi_k) - l_k c(\psi_j - \psi_i) \\ -l_i c(\psi_k - \psi_j) + l_k c(\psi_j - \psi_i) \\ l_i c(\psi_k - \psi_j) - l_j c(\psi_i - \psi_k) \end{bmatrix}, \qquad \text{(A9)}$$

where we abuse notation by only showing the ith, jth, and kth components. For the candidate surface to be a stable surface, $\mathbf{J}\nabla S = \mathbf{0}$; i.e.,

$$\begin{bmatrix} -l_i s\psi_i & -l_j s\psi_j & -l_k s\psi_k \\ l_i c\psi_i & l_j c\psi_j & l_k c\psi_k \end{bmatrix} \times \begin{bmatrix} l_j c(\psi_i - \psi_k) - l_k c(\psi_j - \psi_i) \\ -l_i c(\psi_k - \psi_j) + l_k c(\psi_j - \psi_i) \\ l_i c(\psi_k - \psi_j) - l_j c(\psi_i - \psi_k) \end{bmatrix} = \mathbf{0}. \qquad \text{(A10)}$$

After multiplying and applying the trigonometric substitutions

$$\sin\alpha\cos(\gamma-\alpha) - \sin\beta\cos(\gamma-\beta) = \cos\beta\sin(\gamma-\beta) - \cos\alpha\sin(\gamma-\alpha) \qquad \text{(A11)}$$

and

$$\cos\alpha\cos(\gamma-\alpha) - \cos\beta\cos(\gamma-\beta) = \sin\alpha\sin(\gamma-\alpha) - \sin\beta\sin(\gamma-\beta), \qquad \text{(A12)}$$

one obtains

$$\begin{bmatrix} -l_j c\psi_j - l_k c\psi_k & -l_i c\psi_i - l_k c\psi_k & -l_i c\psi_i - l_j c\psi_j \\ -l_j s\psi_j - l_k s\psi_k & -l_i s\psi_i - l_k s\psi_k & -l_i s\psi_i - l_j s\psi_j \end{bmatrix} \times \begin{bmatrix} l_i s(\psi_k - \psi_j) \\ l_j s(\psi_i - \psi_k) \\ l_k s(\psi_j - \psi_i) \end{bmatrix} = \mathbf{0}, \qquad \text{(A13)}$$

where s and c denote sine and cosine, respectively. Adding the following vector multiple of (A8),

$$\begin{bmatrix} l_i c\psi_i + l_j c\psi_j + l_k c\psi_k \\ l_i s\psi_i + l_j s\psi_j + l_k s\psi_k \end{bmatrix} S(\psi) = \mathbf{0}, \qquad (A14)$$

yields

$$\begin{bmatrix} l_i c\psi_i & l_j c\psi_j & l_k c\psi_k \\ l_i s\psi_i & l_j s\psi_j & l_k s\psi_k \end{bmatrix} \begin{bmatrix} l_i s(\psi_k - \psi_j) \\ l_j s(\psi_i - \psi_k) \\ l_k s(\psi_j - \psi_i) \end{bmatrix} = \mathbf{0}. \qquad \text{(A15)}$$

Premultiplying by

$$\begin{bmatrix} -s\psi_i & c\psi_i \\ -s\psi_j & c\psi_j \\ -s\psi_k & c\psi_k \end{bmatrix} \qquad \text{(A16)}$$

gives

$$\mathbf{0} = \begin{bmatrix} 0 & l_j s(\psi_j - \psi_i) & -l_k s(\psi_i - \psi_k) \\ -l_i s(\psi_j - \psi_i) & 0 & l_k s(\psi_k - \psi_j) \\ l_i s(\psi_i - \psi_k) & -l_j s(\psi_k - \psi_j) & 0 \end{bmatrix} \times \begin{bmatrix} l_i s(\psi_k - \psi_j) \\ l_j s(\psi_i - \psi_k) \\ l_k s(\psi_j - \psi_i) \end{bmatrix} \qquad \text{(A17)}$$

$$= \begin{bmatrix} (l_j^2 - l_k^2) s(\psi_j - \psi_i) s(\psi_i - \psi_k) \\ (l_k^2 - l_i^2) s(\psi_j - \psi_i) s(\psi_k - \psi_j) \\ (l_j^2 - l_i^2) s(\psi_i - \psi_k) s(\psi_k - \psi_j) \end{bmatrix}.$$

Because l_i, l_j, and l_k are assumed to be distinct, it follows that

$$\mathbf{0} = \begin{bmatrix} s(\psi_j - \psi_i) s(\psi_i - \psi_k) \\ s(\psi_j - \psi_i) s(\psi_k - \psi_j) \\ s(\psi_i - \psi_k) s(\psi_k - \psi_j) \end{bmatrix}. \qquad \text{(A18)}$$

By the above lemma, $s(\psi_k - \psi_j) = s(\psi_i - \psi_k) = s(\psi_j - \psi_i) = 0$, so that $\psi_j = \psi_i + m\pi$ and $\psi_k = \psi_i + n\pi$. Note that this implies that for the planar 3R manipulator in absolute angles with three distinct link lengths, the only configurations satisfying (A18) are singular, and the proof would be finished. Now for this relation to continue to be satisfied, each joint velocity $\dot{\psi}_i$, $\dot{\psi}_j$, and $\dot{\psi}_k$ as determined by pseudoinverse control must be equal throughout the candidate surface. However, the joint velocities must satisfy

$$\begin{bmatrix} \dot{\psi}_i \\ \dot{\psi}_j \\ \dot{\psi}_k \end{bmatrix} \in \text{Range} \begin{bmatrix} -l_i s\psi_i & l_i c\psi_i \\ -l_j s\psi_j & l_j c\psi_j \\ -l_k s\psi_k & l_k c\psi_k \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} l_i \\ \pm l_j \\ \pm l_k \end{bmatrix} \right\}. \qquad \text{(A19)}$$

Because we have assumed that l_i, l_j, and l_k are distinct, the joint velocities must also be distinct, contradicting the requirement that the joint velocities must be equal. □

Appendix B: Proof of Theorem 2

THEOREM 2. A necessary condition for an extremal of the problem of minimizing $\int_{t_0}^{t_f} \sqrt{\dot{\theta}^T \dot{\theta}}\, dt$ subject to satisfying $\mathbf{x}(t) - \mathbf{f}(\theta) = \mathbf{0}$ is

$$\frac{d}{dt}(\mathbf{n_J} \cdot \dot{\theta}) - \dot{\theta}^T \left(\frac{\partial \mathbf{n_J}}{\partial \theta} \right) \dot{\theta} - \frac{\dot{\theta} \cdot \ddot{\theta}}{\|\dot{\theta}\|^2} \mathbf{n_J} \cdot \dot{\theta} = 0. \qquad \text{(B1)}$$

Proof. The trajectory θ that minimizes the arc length of the joint space trajectory subject to satisfying the desired end-effector motion must satisfy the Euler-Lagrange equation

$$\frac{\partial H}{\partial \theta} - \frac{d}{dt} \frac{\partial H}{\partial \dot{\theta}} = \mathbf{0}, \qquad \text{(B2)}$$

where

$$H(t, \theta, \dot{\theta}) = \sqrt{\dot{\theta}^T \dot{\theta}} + \lambda^T [\mathbf{x}(t) - \mathbf{f}(\theta)], \qquad \text{(B3)}$$

and $\mathbf{x}(t)$ is the given end-effector motion. Substituting the expression for H into (B2) yields

$$-\mathbf{J}^T \lambda - \frac{\ddot{\theta}}{\|\dot{\theta}\|} + \frac{\dot{\theta} \cdot \ddot{\theta}}{\|\dot{\theta}\|^3} \dot{\theta} = \mathbf{0}, \qquad \text{(B4)}$$

which, on premultiplying each side by $-\|\dot{\theta}\| \mathbf{n_J}^T$, gives the condition

$$\mathbf{n_J} \cdot \ddot{\theta} - \frac{\dot{\theta} \cdot \ddot{\theta}}{\|\dot{\theta}\|^2} \mathbf{n_J} \cdot \dot{\theta} = 0. \qquad \text{(B5)}$$

Note that

$$\frac{d}{dt}(\mathbf{n_J} \cdot \dot{\theta}) = \mathbf{n_J} \cdot \ddot{\theta} + \dot{\mathbf{n}}_\mathbf{J} \cdot \dot{\theta} \qquad \text{(B6)}$$

and

$$\dot{\mathbf{n}}_\mathbf{J} = \left(\frac{\partial \mathbf{n_J}}{\partial \theta}\right) \dot{\theta}. \qquad \text{(B7)}$$

Making the appropriate substitutions, (B5) can now be rewritten as

$$\frac{d}{dt}(\mathbf{n_J} \cdot \dot{\theta}) - \dot{\theta}^T \left(\frac{\partial \mathbf{n_J}}{\partial \theta} \right) \dot{\theta} - \frac{\dot{\theta} \cdot \ddot{\theta}}{\|\dot{\theta}\|^2} \mathbf{n_J} \cdot \dot{\theta} = 0. \qquad \text{(B8)}$$

□

Acknowledgments

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