WATER RESOURCE SYSTEMS PROGRAM
Department of Civil Engineering
Colorado State University
Fort Collins, Colorado

PTIMAL CONTROL OF FLOW N COMBINED SEWER SYSTEMS

y Peter Warren Wentworth Bell

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OPTIMAL CONTROL OF FLOW
IN COMBINED SEWER SYSTEMS

prepared by

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Metropolitan Water Intelligence Systems

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FOREWORD

This is a technical report prepared under a grant by the Office of Water Resources Research for a project at Colorado State University entitled, "Metropolitan Water Intelligence Systems." The basic objective of the project was to develop criteria and information for the development of metropolitan water intelligence systems (MWIS). The MWIS is a specialized urban water system form of the management information and control system concept which is emerging as a technological innovation in industrial applications.

The project consisted of three phases, each lasting about one year. This report was prepared during Phase III. Basic objectives for Phase I were to:

1. Investigate and describe modern automation and control systems for the operation of urban water facilities with emphasis on combined sewer systems.
2. Develop criteria for managers, planners, and designers to use in the consideration and development of centralized automation and control systems for the operation of combined sewer systems.
3. Study the feasibility, both technical and social, of automation and control systems for urban water facilities with emphasis on combined sewer systems.

Basic objectives for Phase II were to:

1. Formulate a design strategy for the automation and control of combined sewer systems.
2. Develop a model of a real-time automation and control system (RTACS model).
3. Describe the requirements for computer and control equipment for automation and control systems.
4. Describe nontechnical problems associated with the implementation of automatic and control systems.

In Phase III the project objectives were focused into three basic categories:

1. Development of control strategy for automated combined sewer systems.
2. To interrelate computer and control equipment system design with the control strategy adopted.
3. To identify and describe the socio-political and economic factors to be considered in implementation.

This report describes factors associated with the first objective of Phase III.

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Maurice L. Albertson, Neil S. Grigg and George L. Smith are co-principal investigators.

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The following reports have been prepared during the CSU-OWRR project, Metropolitan Water Intelligence Systems. Copies may be obtained from the National Technical Information Service, U. S. Department of Commerce, Springfield, VA 22151. (When ordering, use the report title and the identifying number noted for each report).

Technical Report No. 1 - "Existing Automation, Control and Intelligence Systems of Metropolitan Water Facilities," by H. G. Poertner. (PB 214266)

Technical Report No. 2 - "Computer and Control Equipment," by Ken Medearis. (PB 212569)

Technical Report No. 3 - "Control of Combined Sewer Overflows in Minneapolis-St. Paul," by L. S. Tucker. (PB 212903)

Technical Report No. 4 - "Task 3 - Investigation of the Evaluation of Automation and Control Schemes for Combined Sewer Systems," by J. J. Anderson, R. L. Callery, and D. J. Anderson. (PB 212573)

Technical Report No. 5 - "Social and Political Feasibility of Automated Urban Sewer Systems," by D. W. Hill and L. S. Tucker. (PB 212574)

Technical Report No. 6 - "Urban Size and Its Relation to Need for Automation and Control," by Bruce Bradford and D. C. Taylor. (PB 212523)

Technical Report No. 7 - "Model of Real-Time Automation and Control Systems for Combined Sewers," by Warren Bell, C. B. Winn and George L. Smith. (PB 212575)

Technical Report No. 8 - "Guidelines for the Consideration of Automation and Control Systems," by L. S. Tucker and D. W. Hill. (PB 212576)

Technical Report No. 9 - "Research and Development Needs in Automation and Control of Urban Water Systems," by H. G. Poertner. (PB 212577)

Technical Report No. 10 - "Planning and Wastewater Management of a Combined Sewer System in San Francisco," by Neil S. Grigg, William R. Giessner, Robert T. Cockburn, Harold C. Coffee, Jr., Frank H. Moss, Jr. and Mark E. Noonan. (PB # to be assigned)

Technical Report No. 11 - "Optimization Techniques for Minimization of Combined Sewer Overflow," by John W. Labadie. (PB # to be assigned)

Technical Report No. 12 - "Optimal Control of Flow in Combined Sewer Systems," by P. Warren Bell. (PB # to be assigned)

COMPLETION REPORTS

"Metropolitan Water Intelligence Systems Completion Report - Phase I," by George L. Smith, Neil S. Grigg, L. Scott Tucker and Duane W. Hill. (PB 212529)

"Metropolitan Water Intelligence Systems Completion Report - Phase II," by Neil S. Grigg, John W. Labadie, George L. Smith, Duane W. Hill and Bruce H. Bradford. (PB 221992/1)

"Metropolitan Water Intelligence Systems Completion Report - Phase III," by Neil S. Grigg, John W. Labadie and Harry G. Wenzel.

ABSTRACT

OPTIMAL CONTROL OF FLOW IN COMBINED SEWER SYSTEMS

This study examines the development of a suitable control logic for the real time control of flow in combined sewer systems. The approach followed is based on continuous time optimal control theory.

The combined sewer system is modelled as a series of interconnected reservoirs having both weir and orifice controls. Using this model as a basis the state equations and inequality constraints of the system are then presented. The objective function chosen is that of minimizing weighted flow diversions from the system.

Application of the calculus of variations to the minimization problem yields necessary conditions for an optimal control. These necessary conditions are examined and solution forms for the optimal control strategies for several configurations and system inflows are derived.

The problem of numerical solution of the necessary conditions is examined and it is concluded that in general their solution is too cumbersome for practical use. An alternative control solution is proposed, based on operating rules derived from the common factors shown to exist in the previously examined solution forms. When combined with a first order gradient search technique these operating rules yield an optimal control strategy.

Results of application of this technique to systems of four reservoirs and ten reservoirs are presented. They show that a satisfactory control strategy for up to twenty control points can be obtained within the time limits imposed by real time operation. A further example is

presented showing the effects of information errors on the true optimality of a computed control strategy.

Finally the necessary modification to the necessary conditions for an optimal control in which there are time delays in the flow routing are presented. It is shown that the change in operating rules amounts to a shift in time scales between reservoirs.

It appears that the approach outlined herein is a feasible solution to the problem of real time control of flow in combined sewers.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

Symbol	Definition
$A(d)$	Area versus depth relationship for a reservoir
A_g	Area of orifice
B	Time delay in flow routing equation
C	Time delay in flow routing equation
C_D	Orifice coefficient
C_i	Shorthand representation of left side of control variable
C_W	Weir coefficient
D, D_{max}	Maximum allowable depth of storage in a reservoir
E	Time delay in flow routing equation
F	Index of performance (also used for augmented index of performance)
I	Augmented index of performance
K	Constant
L	Length of weir crest
P_i	Shorthand representation of control equation
Q	Flow rate
Q_{max}	Maximum allowable flow in conduit
R_{max}	Maximum allowable orifice opening
R_{min}	Minimum allowable orifice opening
S_i	Shorthand representation of left hand side of control variable inequality constraint
T	Time

U	Vector of control variables
V_i	Travel velocity of flow perturbation
X	Vector of state variables
X_i	Switching time for orifice control i
Y	Vector of control and state variables
Z_i	Overflow or throughput weighting factor
b	Distance between centerline of orifice and bottom of reservoir
d	Depth of water in a reservoir
f_i	Shorthand representation of right hand side of state equation
$g(x,t)$	That part of the objective function that is dependent on initial and final conditions only
g	Gravity
h	Depth of flow over weir crest
h_f	Height of weir crest above bottom of reservoir
i,j,k,l,m,n,p	Subscripts
$q(t)$	Inflow hydrograph
r	Orifice radius
s	Dummy variable
t	Time
u	Control variable
v	Velocity or flow volume
x	State variable
y	Control or state variable

α	Empirical coefficient in the equation for flow over a weir
β	Empirical coefficient in the orifice flow equation
Γ	Desired change in the objective function
Δ	Increment of time
λ	Lagrange multiplier associated with differential constraint
π	Lagrange multiplier associated with control variable inequality constraint
γ	Lagrange multiplier associated with the first derivative of a state variable inequality constraint
$\bar{\gamma}$	Lagrange multiplier associated with state variable inequality constraint
ϕ	The objective function
δ	Small time increment
ε	Small time increment
ν	Routed flow rate

SUMMARY OF THE STUDY

The study outlined below presents control logic suitable for the real time control of flow in combined sewer systems containing up to twenty regulator structures each with a controllable weir and orifice. It was assumed that presently unused storage capacity available in the conduits comprising the sewer system (particularly the trunk sewers) could be utilized by the controllable weir which would create storage and control diversions from the system to the receiving waters. The controllable orifice would regulate the flow from the trunk sewers to the interceptor sewers. It was assumed that the input run-off hydrographs to the sewer system and the physical dimensions of the sewer system were known data.

The control logic presented has the advantages that it is optimal for most practical purposes; can be used for nonlinear systems; does not require large amounts of computer memory storage; and has a relatively short computation time (approximately two minutes of IBM 60-67 CPU time for a system consisting of twenty control points). Its principal disadvantage is that at present a new control logic program must be written for each system configuration. Examination of the control logic shows that in the event of an orifice control failure an optimal control strategy can still be computed. In the event of a weir failure a sub-optimal strategy would result.

The approach taken in this study to determine a suitable control logic for control of flow in combined sewers is based on continuous time optimal control theory (calculus of variations). The necessary model for the optimization assumed that the combined sewer system could be modelled as a system of interconnected reservoirs with each reservoir representing the backwater storage at a regulator structure. (This model is discussed in detail in Chapter II). For simplicity of explanation the theoretical and numerical results derived in this study were based on standard weir and orifice equations; however, it was shown that much more general control device equations could be used without changing the basic results of the study. Likewise throughout most of the study for simplicity of explanation it was assumed that there was no time delay or modification of the flow between reservoirs. Results were derived in Chapter VI, however, to show that realistic flow routing models could be included without modifying the basic principles of the control logic.

The objective function chosen for the optimization procedure was basically that of minimizing weighted diversions from the system to the receiving waters. Weighted diversions were chosen so that consideration could be given to the effect of any required diversions from the system on the receiving waters. (This aspect is discussed in Chapter II). Initially attempts were made to determine the optimal control strategy for a given storm water input and system configuration directly from the necessary conditions for an optimal control given by the calculus of variations. Experience showed, however, that numerical solution of the necessary conditions was for most practical purposes not feasible. (Some of the computational problems are discussed in Chapter IV).

Although numerical answers were difficult to determine it was possible to determine the form of the optimal control strategies quite easily from an examination of the necessary conditions for an optimal control. In Chapter III of the study, the forms of the optimal control strategies for four basic configurations of reservoirs and several combinations of overflow weighting factors were examined. Not only did all the control strategies fall on constraint boundaries, they showed the following patterns.

a) The orifice control for any reservoir could switch from a maximum opening (as defined by the physical dimensions of the orifice or a flow constraint) to a minimum opening (as defined by a minimum allowable orifice opening). For two reservoirs feeding a common conduit the switch by one orifice control is accompanied by a switch in the opposite direction for the other orifice control. (The times at which these jumps in control occurred effectively determined when each reservoir filled to capacity).

b) The weir control (defined as the difference in elevation between the water surface elevation and the crest of the weir) could only be greater than zero when the depth in the associated reservoir was at its maximum limit.

c) As soon as a reservoir had ceased spilling, its orifice control operated in a manner that would give the maximum throughput advantage to that reservoir in the system with the highest overflow weighting factor which was still overflowing (or could still overflow) provided this advantage would be beneficial (if not the reservoir with the next highest overflow weighting factor would govern).

d) The reservoir gaining the throughput advantage then operated so as to either utilize the increased available conduit capacity or to maintain the downstream reservoir at its maximum level.

Further examination of the form of the optimal control strategies showed that given the times at which the switches in control specified in "a" above occurred for an optimal control, then by points "b", "c" and "d" above the control strategy was completely determined. Thus the problem was reduced to the determination of the optimal times at which a switch in the control as given by "a" above should occur.

As a result of the difficulties encountered in the direct solution of the necessary conditions for an optimal control strategy an alternative numerical approach was formulted in Chapter IV based on the operating procedures described above combined with a first order gradient search technique to find the optimal times at which a switch in the orifice control should occur. This involved writing a sub-program (control program) to route the given system inflows through the reservoirs according to the procedures outlined in "b", "c" and "d" above and for specified times at which the switches in the reservoir controls should occur. By perturbing each of the assumed switching times in turn and noting the associated change in the objective function the gradient of the objective function with respect to the switching times was obtained and used to compute a new set of switching times in an iterative procedure until no further reduction in the objective function could be obtained i.e., the optimal control strategy had been determined.

In Chapter V the above procedure was applied to systems consisting of four and ten reservoirs. For each system several different inflow hydrographs and weighting factors for the diversions from each reservoir were tried to ensure that the proposed procedure would operate satisfactorily. Initially for each system the control programs were written for an assumed relative order of the reservoir overflow weighting factors, however, as more experience was gained it proved possible to obtain essentially complete generality with respect to the weighting factors. Although under certain conditions sub-optimal control strategies were obtained, at no time was the degree of sub-optimality significant and in all cases the full storage capacity of the reservoir system was utilized.

For the ten reservoir system the computation times to determine an optimal control were in the order of 50 seconds using an IBM 60-67 computer. It was shown in the study, however, that this time could have been reduced to 20-25 seconds by minor modifications to the computational procedure. The entire computer program required about 5500 words of computer memory.

All the numerical results in the study were determined for cases in which the flow routing between reservoirs was considered to have no time delay or flow modification. In Chapter VI application of the variational calculus to problems including time delay showed that for routing methods such as progressive average lag or Muskingum routing the method of optimal control determination would remain essentially unchanged except that there would be a shift in the time scales for the different reservoirs in the system. The analysis also showed that if the Muskingum routing method were used the control strategy would certainly be sub-optimal.

The results of this study showed that the procedure outlined above would provide a practical method for the real time determination of an optimal control strategy suitable for control of flow combined sewer systems utilizing available in-line storage capacity. Realistic flow control device equations and flow routing methods could be incorporated into the procedure without appreciable alteration to the general procedure.

It was recommended in the study that the control strategies for cases in which the maximum allowable depth in a reservoir exceeds the maximum weir height be examined so as to allow computation of an optimal control strategy in the event of a weir failure. This would increase the range of control devices that could be considered. Following such a study it was recommended that a general control program be written to handle a wide range of system configurations so as to reduce the individual effort required by cities for implementation of the control logic.

CHAPTER I

INTRODUCTION

I.1 Subject of this Study

The purpose of this study is to develop control logic suitable for real time control of flow in combined sewers. The objective of real time control is the optimum use of available storage capacity in the conduits in order to reduce diversions from combined sewer systems to the receiving waters during periods of excess storm water runoff.

This chapter presents the necessary material for an overall understanding of the problem of combined sewer overflows. As well it attempts to orient the reader to the pertinent aspects of control logic and control systems and their use in the reduction of pollution caused by combined sewers. First a brief outline of the problem is given, followed by an overview of some of the proposed methods for reducing pollution of receiving waters resulting from undesirable overflows. Attention is then focused on automatic control of combined sewer flows as a potential solution. Following this, some of the relevant aspects of automatic control are presented. The chapter concludes with a discussion of literature relevant to the topic of control of flow in combined sewer systems.

I.2 The Problem of Combined Sewer Overflow

In most large cities of North America, at least part of the sewer system consists of combined sewers. These sewers are designed to carry both sanitary sewage and storm water runoff. Their original design was such that only a small portion of the storm water runoff, in addition to the sanitary flow, could be carried in the system. Excess flow was to be diverted to the receiving waters, without treatment, at numerous outlet points. The basic design assumption was that the storm water was clean and would sufficiently dilute the sanitary sewage so that any overflows from the system would not be a health problem. Recent studies in Cincinnatti and Tulsa (Weibel and Anderson, 1964; Cleveland, Ramsay and Walters, 1970) have shown, however, that storm water alone may be heavily polluted and thus the supposed dilution of the sanitary sewage does not exist. In some areas, the annual total BOD, suspended solids and coliforms reaching the receiving waters from storm water runoff may exceed that from the sewage tratment plant effluents. Overflows from combined sewers must therefore be considered a major source of pollutants for many waterways.

As a result of more stringent water quality requirements, combined with the fact that over 36 million people in the United States are served by combined sewers, many studies are underway to determine the most suitable methods for reducing the pollution caused by overflows from combined sewer systems.

I.3 Possible Solutions of the Combined Sewer Problem

Solutions to the problem of reduction of pollution of receiving waters by combined sewer overflows fall into two categories: removal of the pollutants from the overflows; and reduction of the volume of overflows. Some examples of these methods are discussed below.

A. Removal of the Pollutants from the Overflows.

a) Separation of storm and sanitary sewers. -This solution is not too satisfactory since in most cases the stormwater would be sent untreated to the receiving waters and, as noted above, may be heavily polluted. In addition, the cost of sewer separation may be prohibitive. For example, the cost of separating sewers in Minneapolis - St. Paul is estimated to be 300 million dollars (Minneapolis - St. Paul Sanitary District, 1970).

b) Reduced treatment of combined sewer overflows. -Methods such as screening/dissolved air flotation and micro-screening have been tested and appear to reduce pollutant concentrations of combined sewer overflows significantly (Marshe, 1970; Mason, 1970).

B. Reduction of Overflows from Combined Sewer Systems.

a) Improvement of the design and maintenance of the regulator structures. -At each point at which flow may be diverted from a combined sewer system there is a regulator structure normally consisting of a small concrete weir, over which passes any flow diverted to the receiving waters, and an orifice, through which flow remaining within the system passes. Flow through the orifice is normally controlled by some form of moveable gate. Many problems resulting from overflows at a regulator structure can be traced to poor regulator design or poor maintenance. Studies have shown that, in many cases, modification of existing structures and/or improved maintenance can lead to significant reductions in system overflows (Minneapolis - St. Paul Sanitary District, 1970;

Sullivan, 1970).

b) Utilization of the storage capacity available within the existing sewer system. -Combined sewers in a large city may range from 6 to 16 feet in diameter and during most storms flow at less than 50% of capacity (Homer and Shifrin, Inc., 1968). By installing flow control devices within the system it may be possible in many instances to utilize the normally unused sewer capacity to store a large fraction of the storm water inflows until they can be routed through the normal treatment process. Additional storage capacity may be gained by the addition of off-line storage reservoirs. The twin cities of Minneapolis - St. Paul have installed flow control devices in their combined sewer system and have shown significant reductions in combined sewer overflows (Minneapolis - St. Paul Sanitary District, 1970). Use of in-system and/or off-line storage is also planned in Seattle, San Francisco, Chicago and Detroit (Grigg et al, 1973).

c) Utilization of in-system and/or off-line storage combined with a satisfactory control scheme. -If flow control devices are installed in a sewer system, but left at pre-set positions, the full storage capacity cannot be used since the device settings must include safety allowances for possible inflow variations. It appears that more effective use of in-system storage could be obtained by sampling rainfall during a storm; computing the expected runoff inflow to various points of the sewer system; and then computing control device positions that would make maximum use of the available system storage capacity and thus reduce diversions to the receiving waters from the combined sewer system.

If in-system storage, control and reduced treatment of overflows were to be combined, it might be possible to ensure that the large majority of any diversions from the system occurred at a few specific locations. This might significantly reduce the number of reduced treatment plants necessary to maintain a specified receiving water quality.

This study focuses on the utilization of in-system storage combined with a satisfactory control scheme and its particular aim is the development of a suitable control logic for real time operation. Before proceeding to the development of the control logic it is useful to examine various degrees of sophistication of control systems and the elements of a real time automated control system for combined sewers.

I.4 Levels of Control

Several levels of refinement are possible for any control system.

The least sophisticated control might be termed "pre-set". Here control device positions are set on the basis of prior analysis of many possibilities or on the basis of "experience". Changes to the control device positions are normally made only when the system malfunctions.

Remote-supervisory control might be considered an intermediate level ofsophistication. Here, sensors, placed throughout the system which is to be controlled,relay data concerning the state of the system to a central control point. Control device changes are then made by an operator whose decisions are based upon the state of the system with possibly some prediction of future inflows. This method of control is generally limited to small scale systems where there are few decisions to be made and the amount of information to be analyzed is small.

Total automatic control is the most sophisticated and may be either feed-back (i.e. based upon the state of the system up to the time of decision), feed-forward (i.e. based upon the state of the system at some initial time and upon predictions of future inputs to the system), or a combination of the two, wherein required information is periodically updated.

Total automatic control requires knowledge of the state of the system, control logic and, for feed-forward control, mathematical models to predict future system inputs. Because machines may be used to analyze large amounts of data, total automatic control is best suited to systems where many decisions must be made in a relatively short time.

Combined sewer systems must generally be considered relatively large systems (e.g. Minneapolis - St. Paul has 36 control devices acting at 18 points in the combined sewer system (Minneapolis - St. Paul Sanitary District, 1970)); thus, if they are to be controlled, they are best controlled automatically. In addition, because of the areal variability of the system inputs, maximum use of the system storage capacity requires use of feed-forward control.

I.5 Elements of a Total Automatic Control System for Combined Sewers

Figure 1.1 shows the elements of a real time automatic control system for combined sewers. The elements of this system fall into three basic categories (Bell, Winn and Smith, 1972).

a) Physical System Components - This category contains all the elements of the system to be controlled (e.g. the sewer system), the control devices (e.g. inflatable weirs,

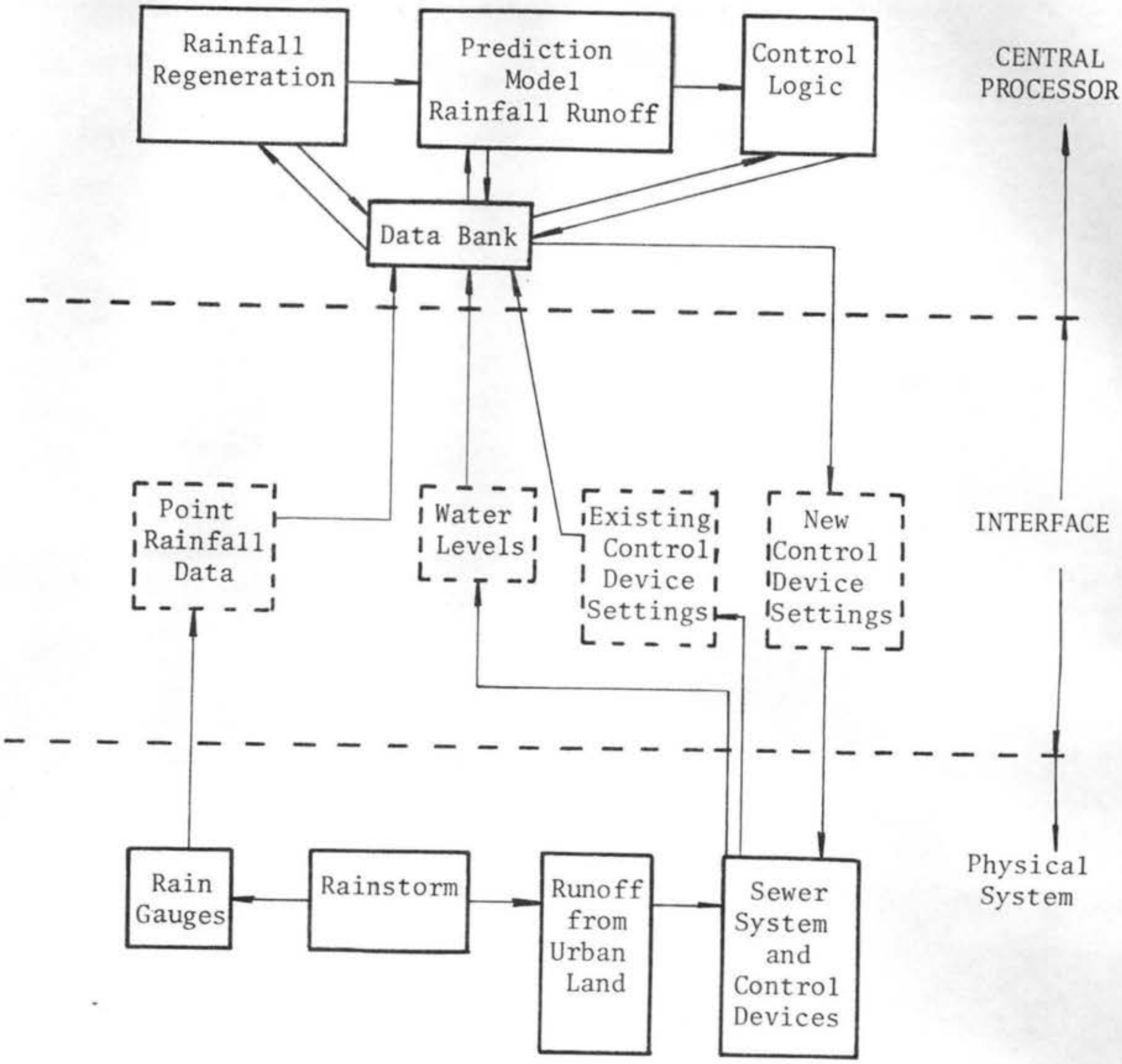


Fig. I.1 Elements of a Real Time Automated Control System for Combined Sewers

regulator gates) sensors of the state of the system (e.g. rain gauges, water-level indicators), the land area from which the runoff occurs, and the rainstorm itself. Each of these elements has an effect on the results of any control decision.

b) The Interface Elements - Any control system (other than "pre-set") requires a transfer of information both of input knowledge of the state of the system being controlled and of output of the control device positions specified by the control logic. This transfer takes place through the interface elements such as telephone lines and analogue to digital converters.

c) The Central Processor - For feed-forward control the central processor contains three basic elements. These are: a central memory for all the necessary system data; any mathematical models necessary to generate predicted system inputs and; the control logic which determines the control strategy according to the state of the system and the predicted inputs.

For combined sewer systems the mathematical models required by the central processor are (Bell, Winn and Smith, 1972):

i) a rainfall regeneration model which analyzes data from point samplers (rain-gauges) and produces a rainfall pattern varying in time and space for the entire duration of the storm,

ii) a rainfall runoff model which determines the expected runoff hydrographs for the duration of the storm using the predicted rainfall pattern,

iii) a model of the conduits, control devices and flows in the sewer system. (Usually these form a part of the control logic).

I.6 Types of Control Logic for Real Time Automatic Control of Combined Sewer Systems

The simplest type of control logic that could be used for control of flow in combined sewers is a strictly feedback control (i.e. no predictions of future inflows are used). A simple example of feedback control would be opening an orifice on the basis of the depth and rate of change of depth immediately upstream of the orifice. This logic completely ignores the effects that the outflows may have in the future at other points in the system.

An improvement on the feedback control is the use of "rule-curves" to determine control device positions. For specified future inputs specific control steps are taken. Determination of reasonable "rule-curves" is a laborious process for even a small system, as it involves consideration of a wide range of possible inputs and alternative control decisions.

One way to avoid the difficulty of arbitrarily determining reasonable "rule-curves" for each system to be controlled is to use some of the techniques of operations research. This involves first the determination of a suitable objective (e.g. minimize the volume of flow diverted from the system). The problem is then to determine the control strategy that will satisfy the objective. A discussion of the merits and shortcomings of many of the techniques of operations research available to determine this "optimal" control strategy is given in a report by Labadie (1973). Generally, the merits and disadvantages revolve around the ability of a given method to deal with non-linearities, the amount of computational time and computer storage required to solve the problem numerically, the ability of the technique to determine a global optimum (as opposed to a local optimum) and the degree of approximation required in the formulation of the problem.

The control problem can be formulated for discrete time intervals or as a continuous time problem, which is the approach taken in this study. Solution of the continuous time problem may be obtained from the calculus of variations or its subset, continuous time optimal control theory. By suitable choice of an objective function and formulation of the problem in the form required by the theory, one obtains a set of equations whose solution yields the time history of the required control device positions so as to minimize (or maximize) the given objective function.

The continuous time formulation has the advantages that it can deal with non-linear systems and will give a control device position for any point in time. Its principal disadvantages are the difficulty in solving for the control from the resulting equations and the fact that their solution yields only a set of necessary but not sufficient conditions for an optimum. There is the further disadvantage that even if the solution could be shown to be an optimum there is no assurance that the optimum would be global. This latter aspect is a problem with most techniques dealing with non-linear systems (Labadie, 1973).

I.7 The Value of Optimal Control Formulations

Although a control problem may be formulated and solved to give the "optimal" control for given input data, the control is optimal only in the sense that the system model and the information from which the control is determined are perfect. The realities of the situation are that both the system model and the input data (e.g., computed runoff hydrographs) contain many sources of error and thus the control will inevitably be sub-optimal.

If in reality any control is certain to be sub-optimal, then what is the purpose of basing the control upon optimization techniques? The principal reason is that operations research techniques bring a degree of order to the problem which may not be available in the determination of a set of "rule curves". In addition "rule-curves" are often based on local characteristics of the problem, whereas operations research techniques may be transferred from one urban area to another.

A second advantage of solving for the control strategy from optimal control formulations is its use in determining the effects of possible error sources in the overall control process, as discussed by Bell, Winn, and Smith (1972). For an exact model of a physical system, optimal control theory gives the necessary conditions for the control which will minimize a given objective function. Thus, if perfect information is available the resultant control will give the absolute minimum of the objective function that is possible for a given storm. When used in a model of a real time automated control system, this results in a standard of comparison for the effects of errors that will be introduced by sensors, information transmission systems, or computations prior to control determination (such as runoff). The adequacy of the overall system to accomplish the desired objective may also be tested and evaluated.

I.8 Literature Relevant to the Combined Sewer Flow Control Problem

As discussed earlier, background material on pollution caused by overflows from combined sewer systems is available in studies done in

Cincinnatti (Weibel and Anderson, 1964) and Tulsa (Cleveland, Ramsay and Walters, 1970) which examine the quality of urban storm water runoff; and in studies of the Detroit (Detroit Metro Water Department, 1970), Minneapolis - St. Paul (Minneapolis - St. Paul Sanitary District, 1970) and San Francisco (San Francisco Department of Public Works, 1971) combined sewer systems, which examine for each city the distribution and volume of combined sewer overflows and their water quality. The latter three reports also outline the extent of installed or planned systems for control of flow in combined sewers. Each of these systems plans real time automatic control on a feed-forward basis, but to date they have developed no satisfactory control logic.

Very little literature is available on the subject of control logic for real time operation of combined sewer systems. A paper by Bell and Winn (1972), which presents some of the early stages of this study develops solution forms for the continuous time optimal control formulation of simple sewer configurations. Bell, Winn, and Johnson (1973) in a related paper included a numerical solution of the continuous time control problem for a sewer system consisting of three regulator structures. They assumed no time delay in the flow routing between regulator structures. The numerical procedure used worked only for particular inflow conditions and thus was not satisfactory for real-life application. This paper also indicated some of the effects of errors in rainfall regeneration on the final control results. In addition, it gave an indication of the frequency of sampling and updating of control strategy necessary to give reasonable control strategies, (ten minute intervals for their example). Winn and Moore (1973) presented a formulation of the problem in the format of regulator theory, a subset of calculus of variations which requires a quadratic objective function. By suitable transformations they were able to convert the problem formulation presented by Bell and Winn into the format required by regulator theory. They presented results for two systems. The first consisted of one regulator structure only and the second of three regulator structures. The latter example used the same system configuration and input data as Bell, Johnson and Winn (1973), who solved the problem using continuous time optimal control theory.Comparison of the results obtained using the two methods shows reasonably close agreement between the time histories of the orifice controls, although regulator theory yields a smooth curve for the orifice operation while continuous time optimal control theory shows a "bang-bang" type of operation. The principle difference lies in the operation of the weir controls with regulator theory producing a much greater volume of spill and consequently less use of available storage capacity. This difference is a result of the quadratic objective function which penalizes small overflow rates only slightly. The authors felt that this problem could be overcome by the use of a dead-band on the weir operation.

Labadie (1973) has discussed optimization techniques for minimization of combined sewer overflow and has presented an approximate flow technique for a non-linear, finite dimensional solution to the control problem. The principal advantage of this technique is that it is adaptable to any method of flood routing between regulator structures. In addition it appears that it can be used for problems involving off-line storage. Labadie, Grigg and Trotta (1973) obtained numerical results using this technique for a system of three regulator structures in series and were able to show, by comparison with results obtained from a linear programming formulation, that the results were a global optimum. They suggested that this technique might be best suited to off-line development of rule curves for use in real-time system control.

Grigg et al., (1973) using Vicente Basin in San Francisco as a prototype, linearized the problem formulation and were able to obtain a solution for three storage locations in series. They also showed that time delay in the flow routing could easily be included in the formulation. Labadie (1973) pointed out that aside from possible errors introduced by linearization of an inherently non-linear problem, the linear programming method requires large amounts of computer storage for even a small number of control points.

Bell, Winn and Smith (1972) have discussed the elements of a real time automated control system for combined sewers and showed how optimal control theory could be used as part of a model of a real time automated control system to determine the effect of various sources of information error on the system operation. As part of this work, they presented a formulation of the continuous time optimal control problem for three storage locations but they assumed no time delay in the flow routing. They were unable to obtain a numerical solution to the control problem.

There is considerable literature available on the general subject of continuous time optimization. Citron (1969) in his text, has presented a derivation of the necessary conditions for an optimal control, from the point of the calculus of variations. Pontryagin et al., (1965) using the maximum principal, derived the same equations. Most work in the field has been for cases involving no time delay; however, Pontryagin et al., (1965) presented the necessary conditions for an optimal control in those cases where a problem with time delay can be stated in the correct format. L.E. EL'SGOL'C (1960) presented the Euler-Lagrange equations including time delay for the more general calculus of variations formulation. Hughes (1968) developed the theory for calculus of variations including time delay but included only end conditions, not side conditions (constraints), although it appears that certain types of constraints could easily be included.

Other relevant literature concerns various aspects of modeling sewer systems, in particular the flow, the control device parameters and the systems themselves.

Very little literature is available on flow in sewer systems. Indeed, since many systems are over fifty years old, there is often poor data on the dimensions of the conduits comprising the systems. Barnes, (1968) in a study directed at flow in sewers, measured flow parameters in a special test apparatus and compared them with mathematical results obtained by the method of characteristics. Harris (1968c) showed that using the progressive average lag method of flow routing, he could obtain good agreement between his solutions and those obtained by the method of characteristics for the Minneapolis - St. Paul interceptor system. The FWQA Storm Water Management Model (Metcalfe and Eddy, Inc., 1968a) TRANSPORT section, eliminated many of the dynamic terms of the St. Venant equations in an attempted simplification. Unfortunately, additional modifications required to overcome calculation instabilities made the method cumbersome; however, in verification tests of their model they obtained reasonable results (Metcalfe and Eddy, Inc., 1968b).

Harris (1968b) appears to have done the only studies of the numerical values of parameters of continuously variable flow control devices at regulator structures and these were limited to a few specific cases. Many hydraulic texts (e.g. Streeter (1966)) quote values for weir and orifice parameters, but these are generally not in the geometry of regulator structures.

I.9 Presentation

This chapter has presented some aspects of the problem of combined sewer overflows and the elements of a control system for control of flow in combined sewers. The remainder of this study is devoted to the development of control logic suitable for real time application to systems with in-line storage controlled by variable weirs and orifices. Application of the control logic developed herein to systems using off-line storage has not been examined .

This study assumes that the required inflow hydrographs, physical system constants and information giving the initial state of the system are available for input to the control logic. (i.e. It is assumed that the control will be of a feed-forward, feed-back form).

In Chapter II the control problem is formulated as a variational problem. First the requirements of the variational formulation are outlined. Following this the system model is developed, starting with the equations of the control devices, proceeding through the development of the state equations and constraints and ending with the examination of suitable objective functions. Finally a complete formulation of a system consisting of two control points in series is presented as an example. Modifications to include more realistic flow routing models are left until Chapter VI. Formulations for other configurations are left to the Appendices.

Chapter III presents the solution forms of the optimal control strategies for selected cases. The general procedure used to determine solution forms is given first, followed by a detailed example of two reservoirs in series. Next there is a discussion of pertinent aspects of solution forms for other configurations presented in the Appendices. The chapter concludes with an analysis of the factors common to the control strategies of all the configurations studied.

Chapter IV outlines attempts at numerical solutions of the necessary conditions for an optimal control and the reasons for their failure. An alternative approach to the control problem based on rules derived from the solution forms given by optimal control theory is then put forward.

In Chapter V the numerical procedure for application of the alternative approach to the control logic suggested in Chapter IV is given. Numerical results for systems containing four and ten regulator structures respectively are then presented. Following this, possible improvements to the computational speed of the control algorithm are suggested.

Chapter VI begins with an outline of the modified Euler-Lagrange equations for problems which include time delays. An example of the problem of two reservoirs in series with time delay in the flow routing is then analyzed. Subsequently the results are generalized for more complex problems and the necessary modifications to the no time delay numerical solution are given.

Chapter VII presents the conclusions and recommendations for further work.

CHAPTER II

FORMULATION OF THE CONTROL PROBLEM AS A VARIATIONAL PROBLEM WITHOUT TIME DELAY

II.1 Introduction

In Chapter I the advantages of using optimal control formulations in developing control logic for real time control of flow in combined sewers were outlined. In this study the approach followed in developing a suitable control logic is based upon continuous time optimal control theory (calculus of variations). Before proceeding to the development of the control logic, a brief description of the requirements of continuous time optimal control theory is given below. For a more detailed discussion of the topic the reader is referred to one of the many texts on the subject e.g. Citron (1969).

II.2 Requirements of Optimal Control Theory

Before one can begin to solve a control problem using any optimal control formulation it is necessary to have a mathematical model giving the laws of motion of the system. For the approach followed herein it is assumed that the system is deterministic and can be described with lumped parameters i.e. ordinary differential equations will suffice for the system description. No restrictions are made on system linearity.

For problems of the type being considered it is useful to divide the variables into state variables (denoted by x_i, $i = 1...n$) and control variables (denoted by u_j, $j = 1,m$). The definition is not hard and fast but may be simply described by equating state variablesto the dependent variables of the system and control variables to the independent variables. For combined sewer systems with in-line storage, the control variables are the size of the orifice openings and the depth of flow over the weirs at the various regulator structures. The state variables are the depth of storage at the regulator structures.

The system mathematical model is then presented in the form of n first order differential equations (often called the differential constraints of the system, or state equations) of the form

$$\frac{dx_i}{dt} = f(x_1,x_2 \ldots x_n, u_1, u_2 \ldots u_m, t) \quad i = 1,\ldots,n \qquad \text{II.2-1}$$

Generally this form can be obtained by suitable definition of variables and addition of new state variables if necessary.

In addition to the system differential equations there may be inequality constraints on the state and control variables. These are generally designated (using X to represent the vector of state variables and U to designate the vector of control variables) as:

$$C(X,U,t) \leq 0 \qquad \text{II.2-2}$$

for those constraints which contain at least one control variable and;

$$S(X,t) \leq 0 \qquad \text{II.2-3}$$

for those constraints in which the control variables do not appear explicitly. Equations II.2-2 are called control variable inequality constraints and equations II.2-3 are called state variable inequality constraints.

Finally, to complete the system description it is necessary to have initial conditions for the state variables at some time t_i.

The above fully describes the system but before one can determine the optimal control strategy it is necessary to have an objective function (often called an index of performance). In its most general form the objective function can be written

$$\text{Min } \phi = g(t_i, t_f, X_i, X_f) + \int_{t_i}^{t_f} F(X,U,t)dt \qquad \text{II.2-4}$$

where t_i and t_f are the initial and final times. (Note: minimization and maximization problems can be interchanged by multiplication of the objective function by -1. Therefore in what follows the minimization format will be used).

Once the mathematical model has been formulated the problem is to determine $u_i(t)$, $i = 1....m$, the optimal control strategy. A control strategy is defined to be optimal if for all other

$$u_i(t) \neq u_i(t) \qquad i = 1,...,m \qquad \text{II.2-5}$$

$$\phi \geq \phi^* \qquad \text{II.2-6}$$

where ϕ^* is the value of the objective function obtained by application of control strategy $u_i(t)$, $i = 1...m$. The necessary conditions for a control strategy to be optimal can be determined by application of the calculus of variations. Although sufficient conditions have been developed (Citron, 1969) their complexity precludes their use in most practical cases. The optimality of a solution is therefore usually based on examination of the results of application of the necessary conditions.

To determine the necessary conditions using the calculus of variations one must form an augmented index of performance by adjoining the differential constraints and the inequality constraints to the objective function by means of Lagrange multipliers, $\lambda(t)$, $\pi(t)$ and $\gamma(t)$. The $\lambda(t)$ multipliers are used to adjoin the state equations; the $\pi(t)$ multipliers to adjoin the control variable inequality constraints and the $\gamma(t)$ multipliers to adjoin the first (or higher order if necessary) derivatives of the state variable inequality constraints. (For the state variable inequaltiy constraints, the first or a higher order derivative is necessary to bring at least one control variable into the constraint). In addition, where derivatives of state variable inequality constraints are involved, the lower order derivatives become point constraints and are adjoined with multipliers $\bar{\gamma}$. It should be noted that the multipliers $\pi(t)$ and $\gamma(t)$ are defined to equal zero when their respective constraints are not binding and thus all terms added to the objective function equal zero. The augmented index of performance is then:

$$\text{Min } \phi = g(t_i X_i, t_f, X_f) + \bar{\gamma}_k S_k(X,t) + \int_{t_i}^{t_f} F(X,U,t)dt$$

$$+ \int_{t_i}^{t_f} [\lambda_j \{\dot{x}_j - f_j(X,U,t)\} + \pi_\ell C_\ell(X,U,t)$$

$$+ \gamma_k \frac{d}{dt}(S_k(X,t))]dt \qquad \text{II.2-7}$$

where $j = 1 \ldots n$: n = no. of state variables

$\ell = 1 \ldots p$: p = no. of control variable inequality constraints

$k = 1 \ldots s$: s = no. of state variable inequality constraints

With the problem formulated as outlined above the necessary conditions for a control to be optimal can be determined by application of the Euler-Lagrange equation which is

$$\frac{dI}{dy} - \frac{d}{dt}\frac{dI}{d\dot{y}} = 0 \qquad \text{II.2-8}$$

where I is the augmented index of performance and y represents the state and control variables (i.e. a set of m + n equations is obtained where m+n is the number of state and control variables).

Application of the Euler-Lagrange equation to the augmented index of performance yields n equations of the form:

$$\frac{dF(X,U,t)}{dx_i} - \lambda_j \frac{df_j(X,U,t)}{dx_i} + \pi_\ell \frac{d}{dx_i} C_\ell(X,U,t)$$

$$+ \gamma_k \frac{d}{dx_i}\left(\frac{d}{dt}\{S_k(X,t)\}\right) = \frac{d\lambda}{dt}$$

$$i = 1,\ldots,n: \quad j = 1,\ldots,n: \quad \ell = 1,\ldots,p: \quad k = 1,\ldots,s$$

II.2-9

which are often called the adjoint equations; and m equations of the form (m is the number of control variables)

$$\frac{dF(X,U,t)}{du_i} - \lambda_j \frac{df_j(X,U,t)}{du_i} + \pi_\ell \frac{d}{du_i}\left[C_\ell(X,U,t)\right] + \gamma_k \frac{d}{du_i}\left(\frac{d}{dt}\{S_k X,t\}\right) = 0$$

$$i = 1,\ldots m: \quad j = 1,\ldots n: \quad \ell = 1,\ldots p: \quad k = 1,\ldots s. \qquad \text{II.2-10}$$

which are often called the control equations.

The remaining necessary conditions to be met are as follows:

a) Initial and/or final conditions for the state variables and time (a maximum of 2n+2 conditions):- These are normally determined from data supplied by sensors in the system. The initial and final values of time are chosen on the basis of the expected length of the storm.

b) Final conditions for the $\lambda(t)$ multipliers:- These are supplied by the so-called transversality condition at the final time t_f. For the case of fixed initial and final times this condition is written as:

$$dg(X_i,t_i,X_f,t_f) + \sum_{i=1}^{n} \lambda_i \, dx_i \Big/_{t_i}^{t_f} = 0 \qquad \text{II.2-11}$$

c) The corner conditions:- These apply when entering or leaving a constraint boundary and perform a similar function to initial and final conditions.

For the case where

$$S_k(X,t) = S_k(X) \quad , \qquad \text{II.2-12}$$

which is the case for all that follows, the corner conditions are:

i) Upon entering a state variable constraint boundary (going forward in time):

$$\left\{\sum_{i=1}^{n} \lambda_i f_i(X,U,t) - F(X,U,t)\right\}\Big/_{t_{1-}} = \left\{\sum_{i=1}^{n} \lambda_i f_i(X,U,t) - F(X,U,t)\right\}\Big/_{t_{1+}}$$

II.2-13

$$\text{and} \quad \lambda_i\Big/_{t_{1-}} + \bar{\gamma}_k \frac{\partial S_k(X)}{\partial x_i}\Big/_{t_1} = \lambda_i\Big/_{t_{1+}} \qquad i = 1,\ldots n \qquad \text{II.2-14}$$

ii) Upon entering a control variable constraint boundary or on leaving a state or control variable boundary

$$\left\{\sum_{i=1}^{n} \lambda_i f_i(X,U,t) - F(X,U,t)\right\}\Big/_{t_{2-}} = \left\{\sum_{i=1}^{n} \lambda_i f_i(X,U,t) - F(X,U,t)\Big/_{t_{2+}}\right.$$

II.2-15

$$\lambda_i\Big/_{t_{2-}} = \lambda_i\Big/_{t_{2+}} \qquad i = 1,\ldots n \qquad \text{II.2-16}$$

It is important to notice that equation II.2-14 states that on entering a state variable constraint there may be a jump in the $\lambda(t)$ multipliers.

Solution of equations II.2-9 through II.2-16 along with equation II.2-1 yields the necessary conditions for an optimal control. The control $U^*(t)$ is determined from the control equations, or from binding constraints. Note that the problem is a two point boundary value problem with the initial conditions for the state variables generally defined at t_i and the final conditions for the $\lambda(t)$ multipliers defined at t_f.

The remainder of this chapter develops the system model and the objective function for the combined sewer control problem. The application of the necessary conditions and the resulting control trajectories are left to Chapter III.

II.3 Elements of the Combined Sewer Model

The first step in setting up a mathematical model of a combined sewer system is to model the individual components relevant to the problem. This section describes the components and gives their mathematical representation.

A. Elements of Combined Sewer Systems

Figure II-1 shows an outline of a part of a typical combined sewer system.

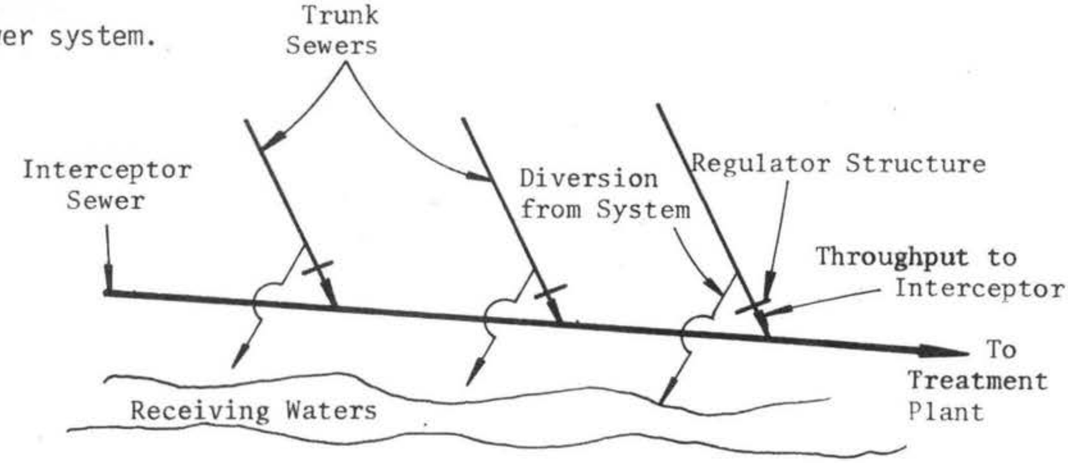


Fig. II.1 Outline of a Typical Combined Sewer System

Such a system is designed to carry the normal dry weather sanitary flow and a limited amount of storm water runoff. Any excess flow is diverted from the system to the receiving waters at the points shown in the figure. Generally, the points of diversion occur where the trunk sewers meet the main interceptor system; however, there may also be diversion points in the interceptor system itself. The diversion structures, usually called regulators, consist of some form of orifice through which passes the flow which is to remain in the system and a weir, over which passes the flow diverted from the system. In the case of a regulator within the interceptor system these controls may be reversed. In order to optimize the utilization of the storage capacity of the system it is necessary to have the capability to vary the operating positions of the control devices. In a controlled system with in-line storage most storage of the flows would take place in the trunk sewers.

B. Typical Regulator Structure

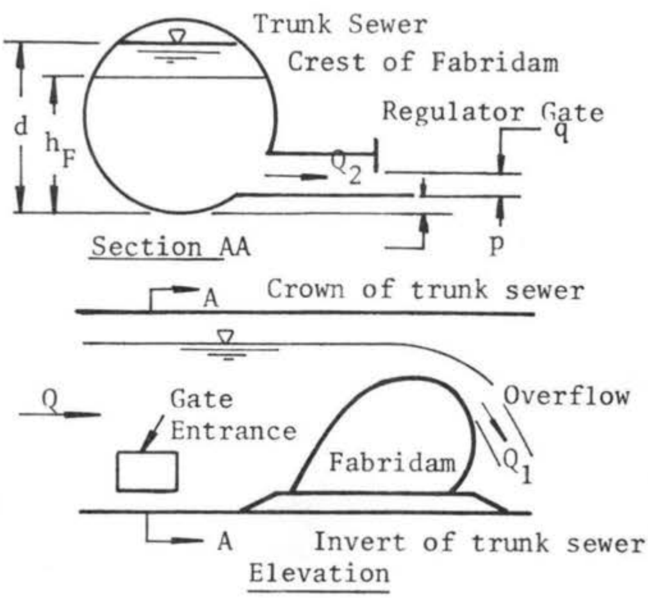


Fig. II.2 Schematic of a Typical Regulator Structure

Figure II-2 shows a schematic of a typical regulator structure installed in the Minneapolis - St. Paul combined sewer system. The weir height is varied by inflating or deflating the flexible dam and the orifice opening is controlled by hydraulically raising or lowering the regulator gate.

The mathematical equation for flow over the weir in its most general form is (Streeter 1966)

$$Q_W = C_W(h)h^{\alpha} \qquad \text{II.3-1}$$

where Q_W = flow rate vol/sec

$C_W(h)$ = a coefficient which is generally a function of the shape of the weir crest and the crest length

h = $d - h_f$ where d and h_f are as defined in figure II-2

α = an empirical coefficient.

By allowing C_W to be a function of h,the effects of variable weir width due to the curvature of the walls of the conduit and, the effects of drowning out the weir if the overflows become very large are included in the equation.

For a broad crested weir of constant width L the theoretical equation for flow over a weir is (Streeter 1966)

$$Q_W = 3.09Lh^{3/2} = C_W h^{3/2} \qquad \text{II.3-2}$$

Because of its simple form equation II.3-2 is used throughout for ease of explanation. It is shown, however, in Chapter III that the more general form, equation II.3-1, can be used without altering the general form of the optimal control solutions.

Flow through the orifice is governed by the equation

$$Q_o = C_D\, r^2 (d + b)^{\beta} \qquad \text{II.3-3}$$

where C_D is a coefficient which is generally a function of orifice shape and entrance conditions

r is the orifice diameter

d is the depth of water above the floor of the conduit

b is the vertical distance from the conduit floor to the orifice centerline (positive downward)

β is an empirical exponent

As was the case for the weir, the coefficient C_D need not be considered a constant, but can be a function of $(d + b)$ and r; thus a more general form is:

$$Q_o = \{C_D(d + b, r)\} r^2 (d + b)^{\beta} \qquad \text{II.3-4}$$

This formulation is not a true representation of the orifice at a regulator structure which is normally a rectangular shaped opening such as shown in Figure II-2. For such an orifice the governing equation for the flow is:

$$Q_o = C_D A_g (d + p - q/2)^{\beta} \qquad \text{II.3-5}$$

where A_g is the area of the opening

p is the distance from the floor of the conduit to the bottom of the orifice (positive downward)

q is the height of the opening

By ensuring that the maximum flow at all-depths using equation II.3-4 is equal to the maximum flow at the same depths based on equation II.3-5 (or a more general form) it is possible to convert any given orifice radius to an equivalent rectangular opening for less than maximum openings. If, in equation II.3-4, it is assumed that C_D is constant, b is equal to zero and, the theoretical value β equals 1/2 applies then equation II.3-4 reduces to

$$Q_o = C_D \sqrt{d}\, r^2 \qquad \text{II.3-6}$$

Again because of its simple form equation II.3-6 is used throughout for ease of explanation. It is shown in Chapter III, however, that the more general equation II.3-4 can be used without altering the general form of the optimal control solutions.

That the more general orifice and flow equations should be used in any model of a real-life system is adequately demonstrated in the analysis by Harris (1968b) of the major diversion structures in the Minneapolis - St. Paul interceptor system. It appears that equations II.3-1 and II.3-4 would be adequate to represent most of the structures he discusses, particularly since in any numerical analysis the functions need be only continuous and not necessarily mathematically smooth.

C. Reservoir Representation of In System Storage

Any reduction in outflow at a regulator structure will produce a back water curve such as shown in Figure II-3.

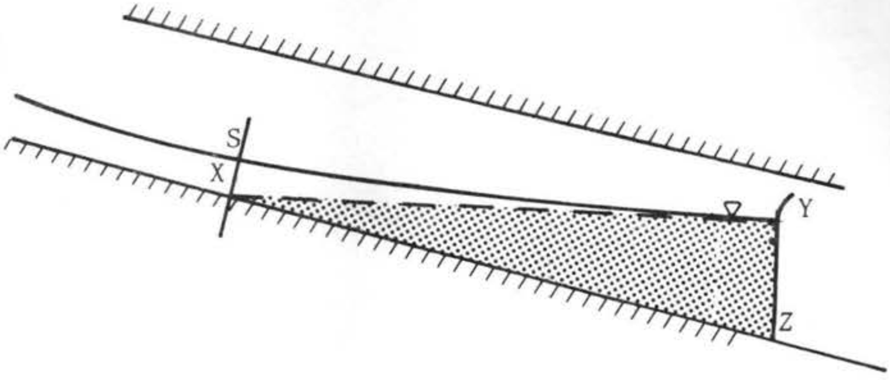


Fig. II.3 Backwater at a Regulator Structure

A simple approximation to this backwater storage is shown in Figure II-4. Here the volume of water shown shaded in Figure II-3 is concentrated in a dummy reservoir located at the regulator structure and the flow in the pipe upstream of the reservoir is assumed to act as if there were no backwater storage. If the dimensions of the reservoir are the same as those of the original conduit then the depth in the reservoir at the downstream end should be a reasonable approximation to the depth at the regulator structure in the actual backwater case. This approximation is aided by the fact that the volume of water stored in the triangle SXY of Figure II-3 is close to the volume of water remaining in the conduit between S and Y in Figure II-3.

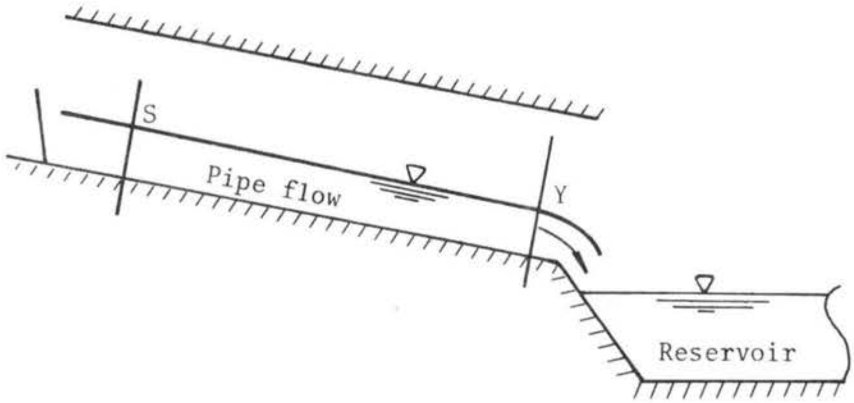


Fig. II.4 Reservoir Approximation of Backwater Storage

Preliminary studies, using a system having three control points, were reported by Bell, Johnson and Winn (1973). They compared the results from a control computed on the basis of the reservoir approximation with those for the same control used in a verified model which had a more complex backwater approximation. Table II.1 reproduced from the paper shows that reasonably good agreement was obtained between the two models.

TABLE II.1

Test Results Using the Proposed Model

Optimal Control Based on True Runoff	Control Program	Physical System Model
Maximum depth at Pt 2	13.00 ft	13.02 ft
Maximum depth at Pt 1	8.50 ft	8.60 ft
Maximum depth at Pt 3	8.50 ft	8.59 ft
Total Overflow Volume at Pt 2	26,000 ft^3	27,400 ft^3
Total Overflow Volume at Pt 1	2,740 ft^3	3,560 ft^3
Total Overflow Volume at Pt 3	18,200 ft^3	19,400 ft^3
Maximum Outflow Pt 2 & Pt 3	92 cfs	92 cfs
Maximum Outflow Pt 1	16.7 cfs	17 cfs

D. Flow Representation

In any free surface flow such as normally exists in a sewer system the velocity at which any change in flow rate is passed through the system is approximately equal to (Streeter and Wylie, 1967)

$$v_c = \sqrt{gd} \pm v$$

where g = gravity

d = depth of flow in the conduit in ft

v = velocity of flow in the conduit in f.p.s.

v_c = velocity of the perturbation in f.p.s.

The positive sign represents the effect of changes proceeding downstream and the negative sign is for changes proceeding upstream. For a depth of flow of a 4 feet and v = 4 fps

$$v_c = 11 + 4 = 15 \text{ fps.}$$

Thus, in a mile of conduit, a not uncommon distance in a sewer system, (Minneapolis - St. Paul has up to 40,000 feet between some control points) the delay quite easily amounts to 5-6 minutes or more between the time that a change is made at an upstream regulator and its effect is felt at a control point downstream.

Undoubtedly any model of the flow routing in a combined sewer system should include the effects of time delay if the computed control is to be reasonably accurate. Initially, however, it is worthwhile to assume that there is no time delay and that control changes are felt immediately throughout the system. It is shown in Chapter VI that more realistic flow routing models can be included in the formulation. This addition increases the complexity of the control and adjoint equations but does not measurably affect the numerical solution procedure.

II.4 The State Equations of the System

Although the state equations must be written for each individual system, the form is often repetitive within each system and between systems. The equations discussed below are those for a system in which the time delays in flows travelling from one point to another within the system are negligible. In addition, the reservoir analogy is assumed to apply for backwater storage at a regulator. Figure II-5 outlines the basic unit to be analyzed.

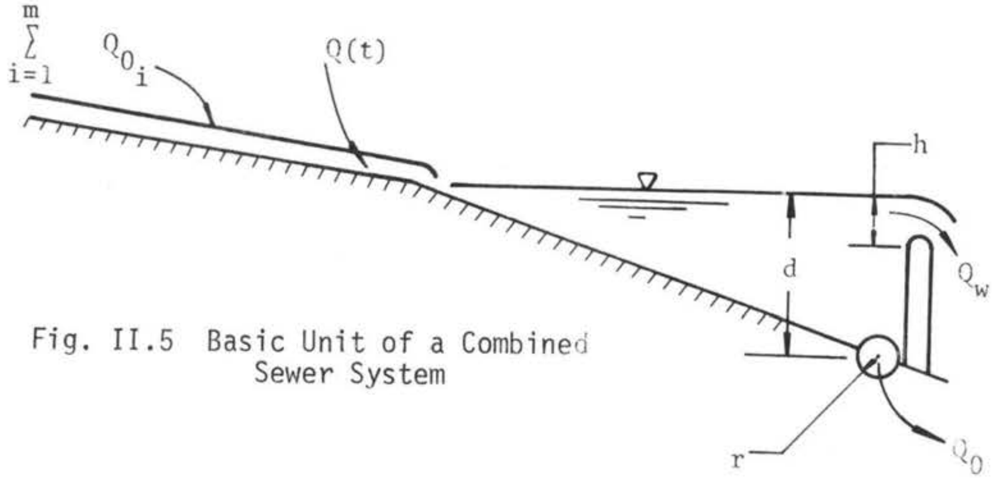


Fig. II.5 Basic Unit of a Combined Sewer System

For any reservoir in the system the inflows are: the urban land runoff plus the dry weather sanitary flow which is represented by the hydrograph q(t) and; the through-flows from parallel upstream reservoirs feeding into the reservoir under discussion which are represented by $\sum_{i=1}^{m} Q_{0_i}$, where m is the number of parallel reservoirs. The instantaneous outflow rates from the reservoir are defined by Q_w the flow over the weir and Q_0 the flow through the orifice.

The differential (state) equation of the reservoir is the continuity equation which states that the rate of change of storage in a reservoir is equal to the instantaneous inflow rate minus the instantaneous outflow rate. Thus,

$$A(d)\,\frac{d(d)}{dt} = \sum_{i=1}^{m} Q_{o_i} + q(t) - Q_o - Q_w \qquad \text{II.4-1}$$

or on rearranging and letting $(\dot{\ }) = \frac{d}{dt}$

$$\dot{d} = \frac{\sum_{i=1}^{m} Q_{o_i} + q(t) - Q_o - Q_w}{A(d)} \qquad \text{II.4-2}$$

Substituting equation II.3-2 for Q_w and equation II.3-6 for Q_o and Q_{o_i} (assuming that all system throughput is through the orifices at the regulators) yields the basic form of the state equations.

$$\dot{d} = \frac{\sum_{i=1}^{m} C_{D_i}\sqrt{d_i}\, r_i^{2} + q(t) - C_D\sqrt{d}\, r^2 - C_w h^{3/2}}{A(d)} \qquad \text{II.4-3}$$

A model of any combined sewer system meeting the requirements outlined above would have n state equations (one for each of the n regulator structures in the system) of the form of equation II.4-3.

II.5 The System Constraints

The major sources of system constraints are:

a) control device limitations - these are usually maximum or minimum settings of the control devices or may be limitations on the rate of operation;

b) flow limitations within the system - flows in the conduits cannot exceed the capacity of the individual conduits or some minimum flow must be maintained

c) storage constraints - the depth of storage at a regulator structure cannot exceed a priori limits (usually based on some safety criterion).

Each of these constraints is discussed below with the exception of the constraints on rate of operation of control devices. These constraints are discussed in an example in Chapter III.

A. Control Variable Inequality Constraints.

Constraints on the control devices and maximum allowable system flows fall in the classification of control variable inequality constraints. For the case where there is no limitation on the rate of operation of the control devices the control variables are: the depth of flow over the weir, h and; the orifice radius, r. One or both of these variables must appear in the control variable inequality constraints.

For this assumption the control variable inequality constraints are given below for the individual control variables.

a) The orifice constraint:-

$$(r - R_{min})(r - R_{max}) \leq 0 \qquad \text{II.5-1}$$

i.e., the radius of an orifice cannot be greater than some maximum value, R_{max}, dictated by the physical limitations of the orifice opening; nor can it be less than some minimum value R_{min}. The minimum limit is inserted because optimal control may require at certain times that throughput at a particular regulator be reduced to zero. Such a requirement might result in an undesirable settlement of sediment upstream of the regulator. By maintaining some minimum opening, this problem may be alleviated. In addition placing the limitation on a minimum value of r instead of a minimum flow rate through the orifice eliminates the possibility of the problem becoming infeasible as a result of a very small depth (see equation II.3-6). It also simplifies operational control procedures.

b) The weir constraint:-

$$h(h - d) \leq 0 \qquad \text{II.5-2}$$

i.e., the depth of flow over the weir cannot be less than zero (to prevent negative flows over the weir) or greater than the depth of water stored immediately upstream of the weir.

c) The flow constraints:-

In addition to the constraints imposed by physical limitations of the control devices themselves there is constraint on the instantaneous flow rate in each individual conduit of the form:

$$\sum_{i=1}^{m} Q_{o_i} + \sum_{j=1}^{p} q_j(t) - Q_{max} \leq 0 \qquad \text{II.5-3}$$

i.e., Q_{o_i} are the throughputs from m reservoirs feeding a given conduit and $q_j(t)$ are p input runoff hydrographs to the conduit. The instantaneous sum of these flows cannot exceed some maximum permissible flow, Q_{max} dictated by the size,roughness and slope of the conduit and the necessity to maintain free surface flow.

The maximum limit, Q_{max}, is premised on the assumption that steady uniform conditions exist in the conduit. In an actual system the flow will more likely be unsteady, non-uniform flow; however, as the control will tend to make maximum use of the system flow capabilities, the total flow should remain relatively constant and thus the approximation of steady uniform flow should be valid.

B. State Variable Inequality Constraints.

The remaining constraints are those on the allowable range of depths behind each regulator which have the form

$$d(d - D) \leq 0 \qquad \text{II.5-5}$$

i.e., the depth cannot be less than zero or greater than some a priori limit D.

In this constraint the control does not appear explicitly and thus it is called a state variable inequality constraint. If this constraint is differented with respect to time the control will appear and equation II.5-5 can be replaced by two constraint equations as outlined earlier. These are:

$$S^0 = d(d - D) \leq 0 \qquad \text{II.5-6}$$

which applies at the instant the constraint boundary is first reached and;

$$S' = \frac{d}{dt}[d(d - D)] = (2d - D)\dot{d} = 0 \qquad \text{II.5-7}$$

which applies along the constraint boundary.

Substituting equation II.4-3 for $\dot{d}$ into equation II.5-7 yields a constraint which includes a control variable and has the form

$$S' = (2d - D) \frac{[q(t) + \sum_{i=1}^{m} C_{D_i}\sqrt{d_i}\, r_i^{\,2} - C_w h^{3/2} - C_D\sqrt{d}\, r^2]}{A(d)} = 0$$

II.5-8

The state equations described in section II.3 and the constraints described in this section, along with the initial conditions, form the complete description of the combined sewer system model.

II.6 The Objective Function for the Problem

The overall objective of control of combined sewer systems is to reduce the amount of pollution of the receiving waters. Objectives commensurate with this overall objective are:

a) minimize the total volume of flow diverted from the combined sewer system to the receiving waters.

b) minimize the total amount of pollutants entering the receiving waters from the combined sewer system.

c) minimize the amount of those pollutants considered most harmful that reaches the receiving waters from the combined sewer system.

d) minimize the amount of pollutants reaching the receiving waters so that the reduction in the economic cost of pollution is a maximum.

Item (b) recognizes spatial and temporal differences in water quality, but does not recognize the variations in potential damage of the various pollutants.

Item (c) recognizes the variations in potential damage of the various pollutants and to some extent recognizes their spatial and temporal variation.

Item (d) is essentially the same as Item c, except that the potential damage is specified in terms of dollars instead of a value judgment. This allows an economic balance to be struck in terms of incremental benefits and the incremental costs of the system required to produce those incremental benefits. With items a, b and c, the required degree of pollution reduction and therefore the degree of system control is based on value judgments. (It should be noted that in determining economic values much value judgment may be involved).

One objective function which recognizes parts of items a, b, and d is

$$\text{Min } \phi = \int_{t_i}^{t_f} \left[\sum_{i=1}^{n} Z_i C_{w_i} h_i^{3/2} \right] dt \qquad \text{II.6-1}$$

where Z_i is a positive constant and n is the number of reservoirs.

The idea behind this objective function is that the various types of pollutants and their concentrations have a spatial variation throughout the combined sewer system. It would therefore seem desirable to reduce or prevent overflows from certain parts of the combined sewer system from reaching the receiving waters; whereas overflows from other parts of the system may be considered relatively harmless to the receiving waters. By using knowledge based on water quality tests in the combined sewer system, combined with knowledge of the effects of pollutants on the receiving waters (due both to the type of pollutant and its point of introduction) it should be possible to arrive at a set of weighting factors that will result in minimization of pollution of the receiving waters. This problem of choice of weighting factors is beyond the scope of this dissertation.

Due to present limitations of accuracy and reliability of water quality sensors (Minneapolis - St. Paul Sanitary District, 1970) along with related problems of modelling pollutant flow to the required degree of accuracy, it does not appear feasible to use an objective function based directly on pollutant overflow volumes. If reasonable water quality data is obtained from the sewer system by a point sampling program, the objective function given by equation II.6-1 should be a good compromise.

It would appear that equation II.6-1 could be improved by making the weighting factors functions of time. Although there has been shown a definite "first flush" of pollutants in the early part of a storm (e.g., studies in Tulsa (Cleveland, Ramsay and Walters, 1970)) it appears that very little would be gained by the addition of this complexity, particularly in view of the fact that the application of optimal control theory to the combined sewer problem using the objective function of equation II.6-1 shows that the control during the early stages of a storm is usually such as to maximize the throughput at each regulator.

Early studies of the optimal control problem showed that if equation II.6-1 was used as the objective function then at certain points in time the orifice control would be bounded but not unique. It was found that adding throughput terms with negative weighting factors removed most of the non-uniqueness problems. Thus an alternative objective function was devised and used for all remaining studies. It is:

$$\text{Min}\,\phi = \int_{t_i}^{t_f} \Big[\sum_{i=1}^{n} Z_i C_{w_i} h_i^{3/2} + Z_{i+n} C_{D_i} \sqrt{d_i}\, r_i^{2}\Big] dt \qquad \text{II.6-2}$$

where $Z_i > 0 \qquad i = 1,\ldots,n$

$Z_{i+n} < 0 \qquad i = 1,\ldots,n$

It was thought in addition that it might be advantageous to the sewage treatment process to receive larger flows from certain portions of a combined sewer system and that by weighting the throughputs this advantage might be realized. The type of control resulting from the application of optimal control theory, however, tends to make any such gains doubtful.

Although this dissertation is directed at controlling flow during storm periods it is worthwhile to point out that during normal dry weather operation of a sewer system there may be possible advantages to the sewage treatment process gained by using the objective function of equation II.6-2 with the weighting factors on the throughput terms being functions of time.

II.7 An Example of a Complete Problem Formulation

For completeness, a complete problem formulation for a simple system consisting of two reservoirs in series will now be given. Figure II-6 shows the system to be modelled. The variables used are all as described earlier.

In order to simplify notation in the following chapter where the forms of some optimal control strategies for this problem are derived, the following variables are introduced: C_ℓ which represents the left hand side of control variable inequality constraint ℓ, and S_k which represents the first derivative of state variable inequality constraint k.

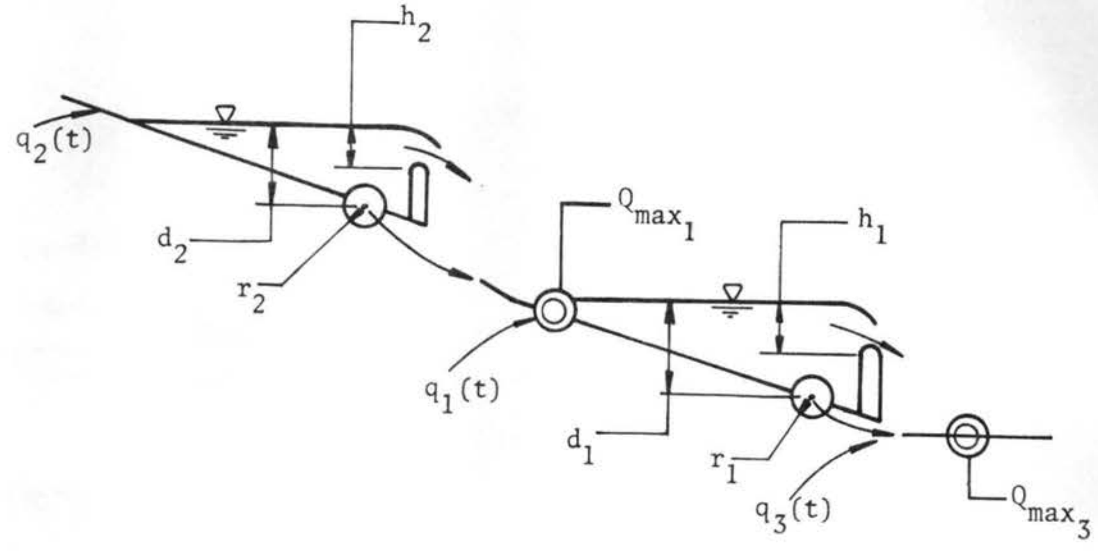


Fig. II.6 The Two Reservoirs in Series System

For this system the state equations are:

$$\dot{d}_1 = \frac{q_1(t)+C_{D_2}\sqrt{d_2}\, r_2^{\,2} - C_{D_1}\sqrt{d}\, r_1^{\,2} - C_{W_1} h_1^{\,3/2}}{A_1(d_1)} = f_1 \qquad \text{II.7-1}$$

and

$$\dot{d}_2 = \frac{q_2(t)-C_{D_2}\sqrt{d_2}\, r_2^{\,2} - C_{W_2} h_2^{\,3/2}}{A_2(d_2)} = f_2 \qquad \text{II.7-2}$$

The control variable inequality constraints of the system are; for the control variables themselves;

$$C_1 = (r_1 - R_{min_1})(r_1 - R_{max_1}) \leq 0 \quad ; \qquad \text{II.7-3}$$

$$C_2 = (r_2 - R_{min_2})(r_2 - R_{max_2}) \leq 0 \quad ; \qquad \text{II.7-4}$$

$$C_3 = h_1(h_1 - d_1) \le 0 \qquad \text{II.7-5}$$

and $$C_4 = h_2(h_2 - d_2) \le 0 \qquad \text{II.7-6}$$

and for the flow rates within the system:

$$C_5 = (q_1(t) + C_{D_2}\sqrt{d_2}\ r_2^{\,2} - Q_{max_1}) \le 0 \quad ; \qquad \text{II.7-7}$$

and $$C_6 = (q_3(t) + C_{D_1}\sqrt{d_1}\ r_1^{\,2} - Q_{max_3}) \le 0 \quad ; \qquad \text{II.7-8}$$

The final system constraints are those on the depths immediately upstream of the regulator structures. As noted earlier these are state variable inequality constraints and must be stated as a pair of constraints for each reservoir. Thus,

$$d_1(d_1 - D_1) \le 0 \quad ; \qquad \text{II.7-9}$$

and $$d_2(d_2 - D_2) \le 0 \quad ; \qquad \text{II.7-10}$$

are the two point constraints which apply at the instant the boundary is reached and:

$$S_1 = (2d_1 - D_1)\ \frac{q_1(t)+C_{D_2}\sqrt{d_2}\ r_2^{\,2} - C_{D_1}\sqrt{d_1}\ r_1^{\,2} - C_{w_1}h_1^{\,3/2}}{A_1(d_1)} = 0$$

II.7-11

and

$$S_2 = (2d_2 - D_2)\ \frac{q_2(t)-C_{D_2}\sqrt{d_2}\ r_2^{\,2} - C_{w_2}h_2^{\,3/2}}{A_2(d_2)} = 0 \qquad \text{II.7-12}$$

apply along the depth constraint.

The objective function for this problem is:

$$\text{Min } \phi = \int_{t_i}^{t_f} [Z_1 C_{w_1} h_1^{3/2} + Z_2 C_{w_2} h_2^{3/2} + Z_3 C_{D_1} \sqrt{d_1}\, r_1^2 + Z_4 C_{D_2} \sqrt{d_2}\, r_2^2] dt \qquad \text{II.7-13}$$

To complete the formulation to the point where the necessary conditions for an optimal control can be determined by the calculus of variations, it is necessary to form an augmented index of performance. Letting the right hand sides of equations II.7-1 and II.7-2 equal f_1 and f_2 respectively for ease of notation and using Lagrange multipliers as described in section II.2 of this chapter, yields as an augmented index of performance

$$\begin{aligned}
\text{Min } \phi = {} & \bar{\gamma}_1 d_1 (d_1 - D_1) + \bar{\gamma}_2 d_2 (d_2 - D_2) \\
& + \int_{t_i}^{t_f} [Z_1 C_{w_1} h_1^{3/2} + Z_2 C_{w_2} h_2^{3/2} + Z_3 C_{D_1} \sqrt{d_1}\, r_1^2 \\
& + Z_4 C_{D_2} \sqrt{d_2}\, r_2^2] dt + \int_{t_i}^{t_f} [\lambda_1 (\dot{d}_1 - f_1) + \lambda_2 (\dot{d}_2 - f_2) \\
& + \pi_1 (r_1 - R_{min_1})(r_1 - R_{max_1}) + \pi_2 (r_2 - R_{min_2})(r_2 - R_{max_2}) \\
& + \pi_3 h_1 (h_1 - d_1) + \pi_4 h_2 (h_2 - d_2) + \pi_5 (q_1(t) + C_{D_2} \sqrt{d_2}\, r_2^2 \\
& - Q_{max_1}) + \pi_6 (q_3(t) + C_{D_1} \sqrt{d_1}\, r_1^2 - Q_{max_3}) \\
& + \gamma_1 (2d_1 - D_1) f_1 + \gamma_2 (2d_2 - D_2) f_2] dt
\end{aligned} \qquad \text{II.7-14}$$

By application of the variational principles to equation II.7-14 as outlined in section II.2 of this chapter, the necessary conditions for controls $h_1(t)$, $h_2(t)$, $r_1(t)$ and $r_2(t)$ to be an optimal control strategy can be determined.

CHAPTER III

EXAMINATION OF SOLUTION FORMS FOR SELECTED CASES

III.1 Introduction

Before proceeding with attempts at numerical solution of the necessary conditions for an optimal control it is worthwhile to examine the form of the optimal control solutions. Such an examination may save much time in attempting particular types of numerical solutions which, because of their peculiarities, are doomed to failure.

This chapter outlines the procedure used to obtain solution forms of the optimal control for particular cases. The solution forms for the two reservoir problem formulated in Chapter II are examined in detail. The solution forms for other configurations are then presented and their peculiarities discussed. (The derivations of these solution forms are left to the Appendices). In all the cases examined in this chapter it is assumed that there is no time delay in the flow routing.

III.2 The Procedure Used to Determine Solution Forms

The procedure used to determine the form of a solution of the necessary conditions for a particular set of inflows consisted of five steps.

a) The problem was formulated for the system configuration to be analyzed.

b) Using the general form of necessary conditions, given in Chapter II, the control and adjoint equations were determined from the augmented objective function and the values of the λ multipliers at t_f evaluated.

c) A control strategy $U(t)$ was assumed and the state variable path that would result from the use of $U(t)$, was determined.

d) Starting at t_f the problem was worked backwards to see that all the necessary conditions could be satisfied.

e) The solution was examined to see if the control $U(t)$ could be improved and the value of the objective function reduced.

A detailed application of this procedure is given below for the problem of two reservoirs in series formulated in Chapter II.

III.3 Solution Forms for the Problem of Two Reservoirs in Series

A. The Necessary Conditions.

The complete formulation for the problem of two reservoirs in series was presented in Chapter II. The necessary conditions for an optimal control can be determined by applying the general form of the adjoint and control equations presented in the beginning of Chapter II (equations II.2-9 and II.2-10) to the augmented objective function for the two reservoir problem (equation II.7-14). This procedure results in the set of six equations (two adjoint and four control equations) given below. Note that the state variables are d_1 and d_2 (equivalent to x_i in Section II.2) and the control variables are

r_1, r_2. h_1 and h_2 (equivalent to u_i).

a) The adjoint equations for the two reservoir problem are upon simplification and rearrangement:

$$\frac{d\lambda_1}{dt} = \{Z_3 + \pi_6 + [\frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}]\} \frac{C_{D_1} r_1^{2}}{2\sqrt{d_1}}$$

$$+ \frac{\lambda_1}{A_1(d_1)} \frac{d[A_1(d_1)]}{dt} - \pi_3 h_1 \quad ; \qquad \text{III.3-1}$$

and

$$\frac{d\lambda_2}{dt} = \{Z_4 + \pi_5 + [\frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}] - [\frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}]\}$$

$$\frac{C_{D_2} r_2^{2}}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} \frac{d[A_2(d_2)]}{dt} - \pi_4 h_2 \qquad \text{III.3-2}$$

b) The control equations for the two reservoir problem are upon simplification and rearrangement:

$$P_1 = \{Z_1 + [\frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}]\} 3/2 C_{w_1} h_1^{1/2} + \pi_3(2h_1 - d_1) = 0$$

III.3-3

and

$$P_2 = \{Z_2 + [\frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}]\} 3/2 C_{w_2} h_2^{1/2} + \pi_4(2h_2 - D_2) = 0$$

III.3-4

which are the control equations for h_1 amd h_2 respectively and,

$$P_3 = \{Z_3 + [\frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}] + \pi_6\} 2C_{D_1} \sqrt{d_1}\, r_1$$

$$+ \pi_1(2r_1 - R_{max_1} - R_{min_1}) = 0 \qquad \text{III.3-5}$$

and

$$P_4 = \{Z_4 + [\frac{\lambda_2-\gamma_2(2d_2-D_2)}{A_2(d_2)}] - [\frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}] + \pi_5\}2C_{D_2}\sqrt{d_2}\, r_2$$

$$+ \pi_2(2r_2-R_{max_2}-R_{min_2}) = 0 \qquad \text{III.3-6}$$

which are the control equations for r_1 and r_2 respectively. The notation P_i on the left hand sides of equations III.3-3 to III.3-6, respectively has been introduced for later shorthand use.

Finally the values of the λ multipliers at t_f are obtained from the transversality condition (equation II.2-11). Applying this condition to the augmented index of performance for the two reservoir problem yields

$$\{\lambda_1 d(d_1) + \lambda_2 d(d_2)\}\Big/_{t_i} + \{\lambda_1 d(d_1) + \lambda_2 d(d_2)\}\Big/_{t_f} = 0$$

III.3-7

Since d_1 and d_2 are given at t equals t_i,$d(d_1)\Big/_{t_i}$ and $d(d_2)\Big/_{t_i}$ are equal to zero. At t_f,d_1 and d_2 are independent and not restrained and thus for equation III.3-7 to be satisfied:

$$\lambda_1\Big/_{t_f} = 0 \qquad \text{III.3-8}$$

$$\lambda_2\Big/_{t_f} = 0 \qquad \text{III.3-9}$$

With the control and adjoint equations determined the feasibility of various solutions can now be tested.

B. Two reservoirs in series $Z_1 > Z_2$ - CASE 1.

In the example that follows the relationships given below are assumed to exist for the weighting factors

$$Z_1 > Z_2 \qquad \text{III.3-10}$$

$$-Z_3 > -Z_4 \qquad \text{III.3-11}$$

$$Z_1 - Z_3 > Z_2 - Z_4 \qquad \text{III.3-12}$$

The last requirement is necessary to prevent the problem becoming equivalent to the case Z_2 greater than Z_1, a problem not discussed herein but whose solution should become obvious as the various configurations are studied.

Further it is assumed that for a particular set of input hydrographs the control trajectories have the form given in Figure III.1 (a and b) and the state variable trajectories that result from the application of these controls have the form given in Figure III.1 (c and d).

Finally it is assumed that the downstream flow constraint Q_{max_3} (refer to Figure II-6) is never binding, and thus r_1 is always equal to R_{max_1}.

The control and state variable trajectories shown in Figure III.1 may be described as follows:

$t_i - t_1$ The orifice control for the upstream reservoir is fully open and remains so until t_1 when the sum of the outflow from the upstream reservoir plus the inflow hydrograph $q_1(t)$ is equal to Q_{max_1}.

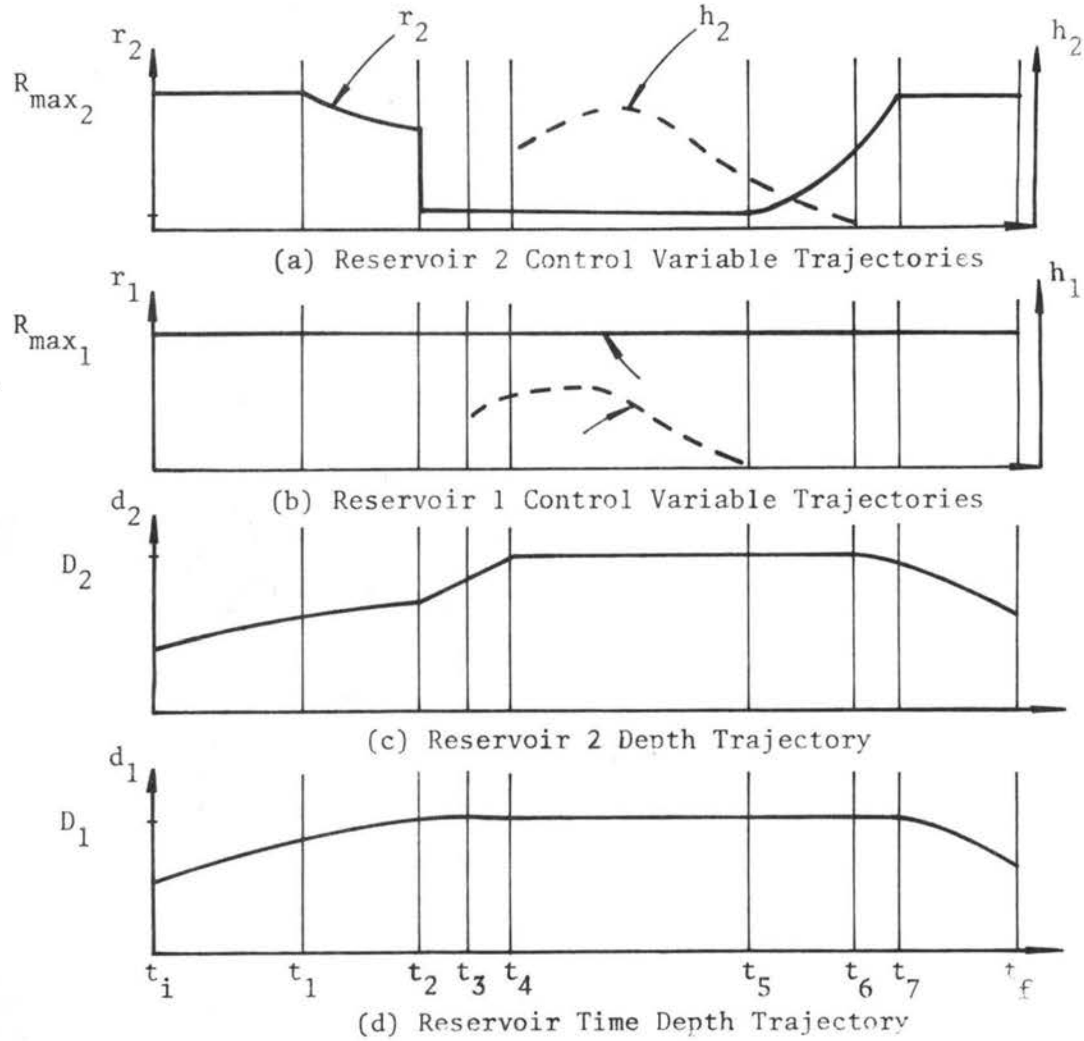


Fig. III.1 The Assumed Control and State Variable Trajectories for CASE 1

t_1-t_2 The upstream orifice control is varied to maintain the total flow at its maximum allowable limit, Q_{max_1}.

t_2-t_3 At t_2, orifice control r_2 is reduced to its minimum allowable value R_{min_2} and remains at this value until t_5.

t_3-t_4 At t_3, d_1 becomes equal to D_1 and the downstream weir control h_1 must be operated to maintain the depth D_1. This operation continues until t_5.

t_4-t_5 At t_4, d_2 becomes equal to D_2, and the weir control h_2 must be operated to maintain the depth D_2. This operation continues until t_6.

t_5-t_6 At t_5 the inflow to the downstream reservoir has decreased to the point where h_1 has been reduced to zero. The reservoir level still equals D_1. It is now possible to increase the outflow from the upstream reservoir without overflowing downstream. Thus r_2 increases until it is fully open at t_7 and is operated in a manner that keeps h_1 equal to zero and d_1 equal to D_1.

t_6-t_7 At t_6, as a result of the increased outflow from the upstream reservoir h_2 is reduced to zero. The upstream reservoir level d_2 then drops below the depth constraint D_2.

t_7-t_f When r_2 reaches its maximum value R_{max_2} at t_7, the total inflow to the downstream reservoir can no longer be maintained so as to keep d_1 equal to D_1 thus d_1 drops off the depth constraint.

It remains now to show that the control strategy outlined satisfies the necessary conditions.

At each point in time it is necessary to be able to determine values for: the four controls, r_1, r_2, h_1 and h_2; the two $\lambda(t)$ multipliers; the six $\pi(t)$ multipliers; the two $\gamma(t)$ multipliers and; the state variables d_1 and d_2. The assumed results of integration of the state equations for the problem (equations II.7-1 and II.7-2) have already been given in Figure III-1. The remaining variables can be determined by working backwards from t_f.

For example, in the interval t_7 to t_f the following relationships exist.

$$r_1 = R_{max_1} \rightarrow (r_1 - R_{min_1})(r_1 - R_{max_1}) = 0 \rightarrow \pi_1 \neq 0 \qquad \text{III.3-13}$$

$$r_2 = R_{max_2} \rightarrow (r_2 - R_{min_2})(r_2 - R_{max_2}) = 0 \rightarrow \pi_2 \neq 0 \qquad \text{III.3-14}$$

$$h_1 = 0 \rightarrow h_1(h_1 - d_1) = 0 \rightarrow \pi_3 \neq 0 \qquad \text{III.3-15}$$

$$h_2 = 0 \rightarrow h_2(h_2 - d_2) = 0 \rightarrow \pi_4 \neq 0 \qquad \text{III.3-16}$$

$$d_1(d_1 - D_1) < 0 \rightarrow \gamma_1 = 0 \qquad \text{III.3-17}$$

$$d_2(d_2 - D_2) < 0 \rightarrow \gamma_2 = 0 \qquad \text{III.3-18}$$

$$q_1(t) + C_{D_2}\sqrt{d_2}\, r_2^{\,2} - Q_{max_1} < 0 \rightarrow \pi_5 = 0 \qquad \text{III.3-19}$$

$$q_3(t) + C_{D_1}\sqrt{d_1}\, r_1^{\,2} - Q_{max_3} < 0 \rightarrow \pi_6 = 0 \qquad \text{III.3-20}$$

In the above equations there are four non-zero Lagrange multipliers and they can be used to satisfy the four control equations. Thus the value of multiplier π_1 can be determined from the control equation for r_1 (equation III.3-5) to yield

$$\pi_1 = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\} \frac{2C_{D_1}\sqrt{d_1}\ R_{max_1}}{R_{min_1} - R_{max_1}} \qquad \text{III.3-21}$$

Similarly the other non-zero multipliers can be determined from the remaining control equations for the time interval t_7 to t_f.

With the control equations satisfied in the time interval being considered all that remains is to integrate the adjoint equations III.3-1 and III.3-2.

Upon substitution of the appropriate values of the π and γ multiplers these equations become:

$$\frac{d\lambda_1}{dt} = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} \frac{d[A_1(d_1)]}{dt} \qquad \text{III.3-22}$$

and

$$\frac{d\lambda_2}{dt} = \{Z_4 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)}\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}}$$

$$+ \frac{\lambda_2}{A_2(d_2)} \frac{d[A_2(d_2)]}{dt} \qquad \text{III.3-23}$$

Since Z_3 and Z_4 are both less than zero the trajectories of the λ multipliers in the interval t_7 to t_f are as shown on Figure III.2.

At t_7 there is a corner. At t_{7-}

$$d_1(d_1 - D_1) = 0 \qquad \text{III.3-24}$$

and

$$(r_2 - R_{min_2})(r_2 - R_{max_2}) < 0 \qquad \text{III.3-25}$$

All other constraints remain as before.

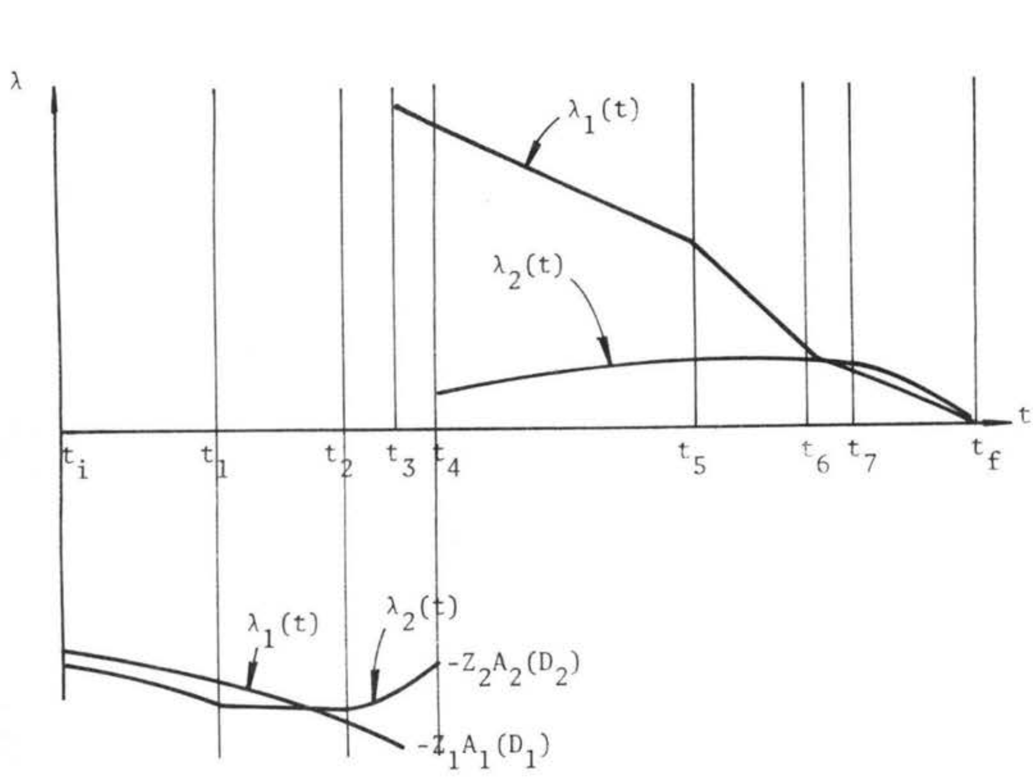


Fig. III.2 The Trajectories of the λ Multipliers for CASE 1

Application of the corner conditions (equations II.2-15 and II.2-16) yields on substitution for f_1 and f_2 and upon noting that h_1 and h_2 are zero:

$$[\lambda_1\{\frac{q_1(t)+C_{D_2}\sqrt{d_2}r_2{}^2 - C_{D_1}\sqrt{d_1}R^2_{max_1}}{A_1(d_1)}\} + \lambda_2\{\frac{q_2(t)-C_{D_2}\sqrt{d_2}r_2{}^2}{A_2(d_2)}\}-Z_3C_{D_1}\sqrt{d_1}R^2_{max_1}$$

$$-Z_4C_{D_2}\sqrt{d_2}r_2{}^2]_{t_{7-}} = [\lambda_1\{\frac{q_1(t)+C_{D_2}\sqrt{d_2}R^2_{max_2} - C_{D_1}\sqrt{d_1}R^2_{max_1}}{A_1(d_1)}\} +$$

$$\lambda_2\{\frac{q_2(t)-C_{D_2}\sqrt{d_2}R^2_{max_2}}{A_2(d_2)}\} - Z_3C_{D_1}\sqrt{d_1}R^2_{max_1} - Z_4C_{D_2}\sqrt{d_2}R^2_{max_2}]_{t_{7+}}$$

III.3-26

$$\lambda_1 \Big/_{t_{7-}} = \lambda_1 \Big/_{t_{7+}} \qquad \text{III.3-27}$$

$$\lambda_2 \Big/_{t_{7-}} = \lambda_2 \Big/_{t_{7+}} \qquad \text{III.3-28}$$

Since all the terms in equation III.3-26 are continuous at t_7 and thus approach the same limit from either the left or right it is obvious that equation III.3-26 is in fact an equality.

The remaining details of the solution are presented in Table III.1 which shows for each time interval on Figure III.1: the non-binding constraints and the resultant zero Lagrange multipliers; the binding constraints and their associated non zero Lagrange multipliers and the control equations which these multipliers are used to satisfy; and the equations used to determine each control variable. For each time interval and corner Table III.2 shows the equations of the λ multipliers and the conditions at each corner. The notation used in these tables is the same as that given in Chapter II with the formulation of the problem (i.e. C_i represents the left hand side of control variable inequality constraint i and is associated with multiplier π_i. S_i represents the left hand side of the first derivative of state variable inequality constraint i and is associated with multiplier γ_i). The notation P_i was given at the beginning of this chapter and represents the control equations. Note that it is not really necessary to know the values of the Lagrange multipliers associated with the inequality

TABLE III.1

Solution of the Constraint Multipliers, Control Equations and Controls for the Problem of Two Reservoirs in Series CASE 1

1 Time Interval	Non Binding Constraints	2 Associated Zero Multipliers	Binding Constraints	3 Associated Non Zero Multipliers	4 Control Equations Satisfied by Multipliers	5 Constraint that Controls are Determined from
$t_f > t > t_7$	C_5	π_5	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_6	π_6	C_2	π_2	P_4	$r_2 \rightarrow C_2 = 0$
	S_1	γ_1	C_3	π_3	P_1	$h_1 \rightarrow C_3 = 0$
	S_2	γ_2	C_4	π_4	P_2	$h_2 \rightarrow C_4 = 0$
$t_7 > t > t_6$	C_2	π_2	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_5	π_5	C_3	π_3	P_1	$h_1 \rightarrow C_3 = 0$
	C_6	π_6	C_4	π_4	P_2	$h_2 \rightarrow C_4 = 0$
	S_2	γ_2	S_1	γ_1	P_4	$r_2 \rightarrow \dot{S}_1 = 0$
$t_6 > t > t_5$	C_2	π_2	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_4	π_4	C_3	π_3	P_1	$h_1 \rightarrow C_3 = 0$
	C_5	π_5	S_1	γ_1	P_2	$r_2 \rightarrow \dot{S}_1 = 0$
	C_6	π_6	S_2	γ_2	P_4	$h_2 \rightarrow \dot{S}_2 = 0$
$t_5 > t > t_4$	C_3	π_3	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_4	π_4	C_2	π_2	P_4	$r_2 \rightarrow C_2 = 0$
	C_5	π_5	S_1	γ_1	P_1	$h_1 \rightarrow \dot{S}_1 = 0$
	C_6	π_6	S_2	γ_2	P_2	$h_2 \rightarrow \dot{S}_2$
$t_4 > t > t_3$	C_3	π_3	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_5	π_5	C_2	π_2	P_4	$r_2 \rightarrow C_2 = 0$
	C_6	π_6	C_4	π_4	P_2	$h_1 \rightarrow \dot{S}_1 = 0$
	S_2	γ_2	S_1	γ_1	P_1	$h_2 \rightarrow C_4 = 0$

TABLE III.1 cont'd

$t_3>t>t_2$	C_5	π_5	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_6	π_6	C_2	π_2	P_4	$r_2 \rightarrow C_2 = 0$
	S_1	γ_1	C_3	π_3	P_1	$h_1 \rightarrow C_3 = 0$
	S_2	γ_2	C_4	π_4	P_2	$h_2 \rightarrow C_4 = 0$
$t_2>t>t_1$	C_2	π_2	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_6	π_6	C_3	π_3	P_1	$r_2 \rightarrow C_5 = 0$
	S_1	γ_1	C_4	π_4	P_2	$h_1 \rightarrow C_3 = 0$
	S_2	γ_2	C_5	π_5	P_4	$h_2 \rightarrow C_4 = 0$
$t_1>t\geq t_i$	C_5	π_5	C_1	π_1	P_3	$r_1 \rightarrow C_1 = 0$
	C_6	π_6	C_2	π_2	P_4	$r_2 \rightarrow C_2 = 0$
	S_1	γ_1	C_3	π_3	P_1	$h_1 \rightarrow C_3 = 0$
	S_2	γ_2	C_4	π_4	P_2	$h_2 \rightarrow C_4 = 0$

Notes:

1. Time intervals are those shown on Figure III.1.
2. Lagrange multipliers in this column are those associated with the non binding constraints on the same line in the previous column.
3. Lagrange multipliers in this column are those associated with the binding constraints on the same line in the previous column.
4. Control equations listed in this column are satisfied by the Lagrange mutlipliers on the same line in the previous column.
5. $r_1 \rightarrow C_1 = 0$ should be read as "r_1 is determined from $C_1 = 0$. The controls in this column are listed in the order required for solution.'

TABLE III.2

THE EQUATIONS OF THE λ MULTIPLIERS AND THE CORNER CONDITIONS FOR THE PROBLEM OF TWO RESERVOIRS IN SERIES CASE 1

Time	Values of $\dot{\lambda}_i$ or λ_i
t_f	$\lambda_1 = 0$
	$\lambda_2 = 0$
$t_f > t > t_7$	$\dot{\lambda}_1 = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} \frac{d[A_1(d_1)]}{dt}$
	$\dot{\lambda}_2 = \{Z_4 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)}\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} \frac{d[A_2(d_2)]}{dt}$
Corner at t_7	All variables are continuous. Therefore the corner conditions yield
	$\lambda_1/_{t_{7-}} = \lambda_1/_{t_{7+}}$
	$\lambda_2/_{t_{7-}} = \lambda_2/_{t_{7+}}$
$t_7 > t > t_6$	$\dot{\lambda}_1 = \{Z_3 + [\frac{\lambda_1 - \gamma_1 D_1}{A_1(d_1)}]\} \frac{C_{D_1} R^2_{max_1}}{2\sqrt{D_1}} + \frac{\lambda_1}{A_1(d_1)} \frac{d[A_1(d_1)]}{dt}$
	$\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} \frac{d[A_2(d_2)]}{dt} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$
Corner at t_6	All variables are continuous. Therefore the corner conditions yield
	$\lambda_1/_{t_{6-}} = \lambda_1/_{t_{6+}}$
	$\lambda_2/_{t_{6-}} = \lambda_2/_{t_{6+}}$
$t_6 > t > t_5$	$\dot{\lambda}_1 = \{Z_3 + Z_4 - Z_2\} \frac{C_{D_1} R^2_{max_1}}{2\sqrt{D_1}} = \text{const.}$
	$\dot{\lambda}_2 = 0 \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$

TABLE III.2 con'd

Corner at t_5

All variables are continuous. Therefore the corner conditions yield

$$\lambda_1/t_{5-} = \lambda_1/t_{5+}$$

$$\lambda_2/t_{5-} = \lambda_2/t_{5+}$$

$t_5 > t > t_4$

$$\dot{\lambda}_1 = \{Z_3 - Z_1\}\frac{C_{D_1} R^2_{max_1}}{2\sqrt{D_1}} = \text{const.}$$

$$\dot{\lambda}_2 = \{Z_4 - Z_2 + Z_1\}\frac{C_{D_2} R^2_{min_2}}{2\sqrt{D_2}} = \text{const.}$$

Corner at t_4

At t_4, h_2 is discontinuous and therefore $\dot{d}_2$ is discontinuous. As this corner is the entrance to a state variable inequality constraint boundary a jump in the value of λ_2 can occur. The corner conditions yield

$$\lambda_1 \angle t_{4-} = \lambda_1/t_{4+}$$

$$\lambda_2/t_{4-} = -Z_2 A_2(D_2)$$

$t_4 > t > t_3$

$$\dot{\lambda}_1 = \{Z_3 - Z_1\}\frac{C_{D_1} R^2_{max_1}}{2\sqrt{d_1}} = \text{const.}$$

$$\dot{\lambda}_2 = \{Z_4 + Z_1 + \frac{\lambda_2}{A_2(d_2)}\}\frac{C_{D_2} R^2_{min_2}}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)}\,\frac{d[A_2(d_2)]}{dt}$$

Corner at t_3

At t_3, h_1 is discontinuous and therefore $\dot{d}_1$ is discontinuous. As this corner is the entrance to a state variable inequality constraint boundary a jump in the value of λ_1 can occur. The corner conditions yield

$$\lambda_1/t_{3-} = -Z_1 A_1(D_1)$$

$$\lambda_2/t_{3-} = \lambda_2/t_{3+}$$

TABLE III.2 cont'd

$t_3 > t > t_2$

$$\dot{\lambda}_1 = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} R^2_{max_1}}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)}\frac{d[A_1(d_1)]}{dt}$$

$$\dot{\lambda}_2 = \{Z_4 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_2} R^2_{min_2}}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)}\frac{d[A_2(d_2)]}{dt}$$

Corner at t_2

At t_2 the variable r_2 is discontinuous however the corner conditions require:

$$\lambda_1/t_{2-} = \lambda_1/t_{2+}$$

$$\lambda_2/t_{2-} = \lambda_2/t_{2+}$$

For the jump in r_2 to be permissible requires that

$$[\frac{\lambda_1}{A_1(d_1)}]_{t_2} = [\frac{\lambda_2}{A_2(d_2)} + Z_4]_{t_2}$$

$t_2 > t > t_1$

$$\dot{\lambda}_1 = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} R^2_{max_1}}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)}\frac{d[A_1(d_1)]}{dt}$$

$$\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)}\frac{d[A_2(d_2)]}{dt} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$$

Corner at t_1

All variables are continuous. Therefore the corner conditions yield

$$\lambda_1/t_{1-} = \lambda_1/t_{1+}$$

$$\lambda_2/t_{1-} \quad \lambda_1/t_{1+}$$

$t_1 > t \geq t_0$

$$\dot{\lambda}_1 = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)}\frac{d[A_1(d_1)]}{dt}$$

$$\dot{\lambda}_2 = \{Z_4 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)}\frac{d[A_2(d_2)]}{dt}$$

constraints only that they can be used to satisfy a particular control equation.

Finally note from Figure III.2 that the condition listed in Table III.2 as necessary for a jump in the control r_2 to occur at time t_2 can in fact occur. (This is most easily seen by assuming $A_i(d_i)$ is equal to 1). Thus it has been shown that the solution assumed in Figure III.1 is in fact feasible, given the assumptions and limitations outlined at the beginning of this section.

C. Discussion of the Results of CASE 1.

The first question to be answered is: is the proposed solution optimal? After t_4 on Figure III.1, when both reservoirs are storing the maximum amount of water and begin to overflow it is obvious that nothing can be done to the controls to reduce the overflow volume from the two reservoirs. The key question concerns the validity of the jump in control r_2 at t_2.

That this is reasonable may be shown by referring to Figure III.3. If the switch in r_2 takes place at $t_2 + \varepsilon$ then there *will be* an increase in throughput volume from reservoir 2 of ΔS_4 and a decrease in overflow volume from reservoir 2 of ΔS_2. By continuity

$$\Delta S_4 + \Delta S_2 = 0 \qquad \text{III.3-29}$$

As a result of the increased input to reservoir 1 there will be an increased overflow volume from that reservoir of ΔS_1 and since the depth in reservoir 1 at $t_2 + \varepsilon$ is now greater than it would have been without the addition of ΔS_4 there will be an increase in the outflow volume from reservoir 1 of ΔS_3. For this reservoir the continuity equation yields

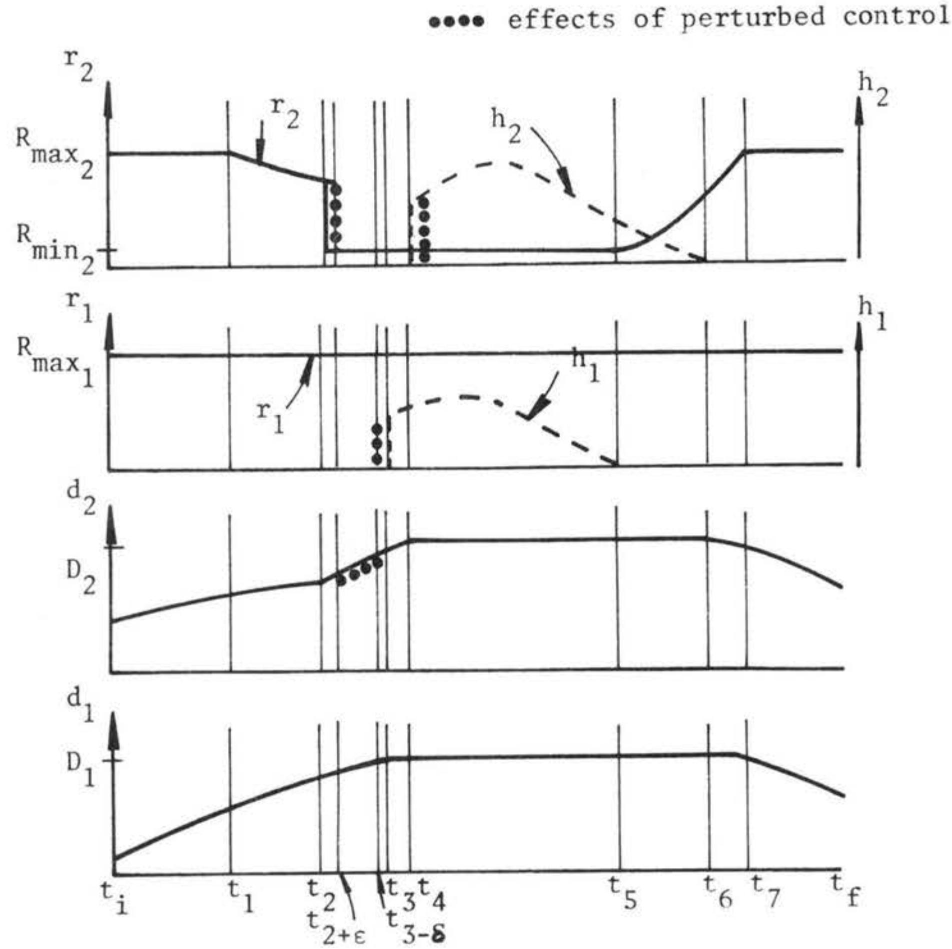


Fig. III.3 The Effects of a Perturbation of Control r_2 at time t_2

$$\Delta S_4 - \Delta S_1 - \Delta S_3 = 0 \qquad \text{III.3-30}$$

The change in the value of the objective function resulting from the change in the time that the jump in control r_2 takes place is:

$$\Delta\phi = Z_1\Delta S_1 + Z_2\Delta S_2 + Z_3\Delta S_3 + Z_4\Delta S_4 \qquad \text{III.3-31}$$

By substituting the results of equation III.3-29 and III.3-30 into equation III.3-31 this change in the objective function can be expressed as

$$\Delta\phi = Z_1(\Delta S_4-\Delta S_3)-(Z_2-Z_4)\Delta S_4-Z_3\Delta S_3 \qquad \text{III.3-32}$$

Now ΔS_3 is a function t_3-t_2 as well as the change in depth of reservoir 1. If for several values of t_2, the perturbation in r_2 is such as to keep ΔS_4 constant then ΔS_3 will vary from a value of zero when t_2 equals t_3 to a value of ΔS_4 when t_3-t_2 approaches infinity. Thus when t_2 equals t_3 equation III.3-32 reduces to

$$\Delta\phi = (Z_1-Z_2+Z_4)\Delta S_4 \qquad \text{III.3-33}$$

which by the assumptions given for the relative values of the weighting factors is greater than zero. Similarly for t_3-t_2 very large the change in the objective function is

$$\Delta\phi \approx -(Z_2-Z_4+Z_3)\Delta S_4 \qquad \text{III.3-34}$$

which by the initial assumptions is less than zero. Thus there is some value of t_2 at which $\Delta\phi$ is zero. Prior to this value of t_2 any increase in throughput from reservoir 2 is advantageous and after t_2 disadvantageous. Therefore there should be a switch in control r_2 at time t_2.

Simply stated, the switch in control occurs at that point in time, where the incremental benefits gained by increasing the throughput from the upstream reservoir are just equal to the increased costs at the downstream reservoir of overflowing a portion of that volume minus the benefits derived from the increased downstream throughput.

If there is a binding constraint on the flow rate from the downstream reservoir throughout the period t_i to t_3 then

$$\Delta S_3 = 0 \qquad \text{III.3-35}$$

and thus there would be no time at which it would be beneficial to decrease upstream overflow at the expense of increased downstream overflow.

Similarly if

$$Z_1 >> Z_2 \qquad \text{III.3-36}$$

then the time required to gain any incremental benefits from increased outflow from the upstream reservoir may be greater than that available.

The solution forms for these latter two possibilities will be examined later in this section.

Returning to the solution for CASE 1 it is interesting to note that t_3-t_2 as defined on Figure III.1 will be smaller for R_{min_2} greater than zero than for R_{min_2} equal to zero. (Since in the interval t_2 to t_3 if R_{min_2} is zero $\frac{\lambda_2^{\ 2}}{A_2(d_2)}$ is constant - see Table III.2 and Figure III.2). This is explained by the fact that any incremental increase in throughput ΔS_2 from the upstream reservoir falls on top of the throughput due to R_{min_2}. This combined effect on the downstream reservoir level means that the throughput from the downstream reservoir ΔS_3, necessary to decrease the objective function can be obtained in less time. Note that having R_{min_2} greater than zero does not mean that a better value will be obtained for the objective function than for the case R_{min_2} equals zero.

At this point it is advantageous to point out some more general aspects of the solution. In the formulation of the control problem in Chapter II it was suggested that the addition of Z_3 and Z_4, the weighting factors on the throughputs, would remove some non-uniqueness from the controls. This can be observed by examining equation III.3-5 and III.3-6, the control equations for r_1 and r_2 along with equations III.3-22 and III.3-23, the adjoint equations for the interval of time t_7 to t_f.

Since at t_f the λ multipliers are zero by equations III.2-8 and III.2-9, then if

$$Z_3 = Z_4 = 0 \qquad \text{III.3-37}$$

the values of the adjoint equations over the interval are

$$\lambda_1 = \lambda_2 = 0 \qquad \text{III.3-38}$$

Inserting the values from these two equations into the control equations for r_1 and r_2 (recalling that γ_1, γ_2, π_5 and π_6 were zero) shows that they will automatically be satisfied regardless of the values of r_1 and r_2 since

$$Z_3 + \frac{\lambda_1}{A_1(d_1)} = 0 \qquad \text{III.3-39}$$

and

$$Z_4 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)} = 0 \qquad \text{III.3-40}$$

(Note if r_2 is equal to R_{max_2} or R_{min_2}, π_2 will be zero in equation III.3-6 and likewise π_1 will be zero regardless of the value of r_1 in this interval).

Thus, there is no way to determine the control in this interval. In fact, if equation III.3-37 applies, any values for r_1 and r_2 are optimal provided there is no overflow from either reservoir in this interval.

The formulation of the general problem in Chapter II did not include limits on the rates of operation of the control devices. The jumps in controls r_2, h_1 and h_2 would all require their respective control devices to operate at infinite rates, which is impossible. A reasonable approximation to this infinite operation rate would be to operate the devices at their maximum rates and initiate their operation so that they were one half way through their required movement at the time the jump in control was to have occurred. If at any other time during the control process the optimal control required operation at a rate faster than the capabilities of the control device, then the device could be operated at its maximum speed until its position again fell on the optimal trajectory. From the form of the solution obtained for CASE 1, the above form of operation would appear to be close to that which one would expect to be required by a problem formulation including rate limitations on the control devices and most certainly would be within the accuracy of information on which the control is based. For this reason, the use of a more accurate model, including control device rate limitations cannot be justified.

The equations for the flow control devices in Chapter II were all reduced to simple forms for ease of understanding. Had the more general forms, equations II.3-1 and II.3-4 been used in CASE 1 it is clear that the form of the control equations (equations III.3-3 to III.3-6) and the adjoint equations (equations III.3-1 and III.3-2) would have remained unchanged also. This results from the fact that for Z_3 and Z_4 non zero the solution path for the control variables follows constaint boundaries throughout. For example, in the solution presented:

$$r_1 = R_{max_1} \qquad \text{III.3-40}$$

throughout and:

$$r_2 = R_{max_2} \qquad \text{III.3-41}$$

or

$$r_2 = R_{min_2} \qquad \text{III.3-42}$$

or

$$r_2 = \{[C_{d_1}\sqrt{D_1}\,R^2_{max_1} - q_1(t)]/C_{D_2}\sqrt{d_2}\}^{1/2} \qquad \text{III.3-43}$$

or

$$r_2 = \{[Q_{max_1} - q_1(t)]/C_{D_2}\sqrt{d_2}\}^{1/2} \qquad \text{III.3-44}$$

(Equation III.3-43 is determined from $d_1(d_1-D_1) = 0$), and equation III.3-44 from the flow constraint downstream of reservoir 2. Similarly h_1 and h_2 are determined from the first derivatives of the depth constraints. Thus, effectively the control drops out of the control equations, from which it is normally determined. This result is essentially true for all the solution forms to be discusssed.

The boundary aspect of the solution, which is not unexpected, as it would seem reasonable to use the maximum capacity of all parts of the system, has its drawbacks. It can be shown that the necessary conditions will be satisfied by almost any other solution for which the control is on the boundary (certain jumps in control being the exception). This creates a large number of possible optimal solutions, from which an optimum must be selected by physical understanding of the problem. It does, however, have the advantage that the more general control device equations may be used.

Finally, a peculiarity of this problem resulting from the corner conditions at t_3 and t_4 should be noted. At both these times there is a jump in the λ multipliers. Working the problem backwards as was done for CASE 1, it was easy to determine the value of the λ multipliers to the left of the jump. However, if the correct values of the λ multipliers had been known at t_i, and the problem solved going forward, it would have been impossible to determine the values of the λ multipliers to the right of the jump. In both cases they are multiplied by the first derivatives of the state variables which are zero on the depth constraint. Thus each time they drop out of the corner conditions. To solve the problem going forward it would be necessary to guess new values of λ_1 and λ_2 at t_{4+} and t_{3+} respectively and continue the problem to t_f and ensure that λ_{1/t_f} and λ_{2/t_f} reached the values required by the transversality condition. Essentially there is a separation of the problem into two separate problems, before and after the corner. What happens after the corner is independent of happenings prior to the corner. Later in this study this separation will be used to advantage.

To complete the study of the problem of two reservoirs in series, two other cases are discussed below.

D. Two Reservoirs in Series with $Z_1 >> Z_2$ - CASE 2

The assumptions concerning the relative values of the weighting factors and the non applicability of the flow constraint downstream of reservoir 1 as in CASE 1 apply. The control and adjoint equations are as derived in section III.3-A. The objective function state equations and constraints for this problem are those derived in section II.7. Figure III.4 shows the assumed control and state variable trajectories. The principle difference between these trajectories, and those of Figure III.1, result from the jump in control r_2 taking place sooner. Because of this there is no overflow from the downstream reservoir and the upstream reservoir fills earlier than in CASE 1. Since there is no overflow from the downstream reservoir, $\dot{d}_1$ is continuous at the entrance to the state variable constraint

$$d_1(d_1-D_1) \leq 0 \qquad \text{III.3-45}$$

Since all the controls are also continuous at t_4, the corner condition (equation II.2-13) is automatically satisfied regardless of the value of $\lambda_1/_{t_{4-}}$. However, by equation II.2-14, $\lambda_1/_{t_{4-}}$ may be discontinuous. This result is not as unreasonable as it may appear if it is noted that there must be some value of Z_1 say Z_1^L in CASE 1 at which the overflow from reservoir 1 is extremely small. For any values of

$$Z_1 > Z_1^L \qquad \text{III.3-46}$$

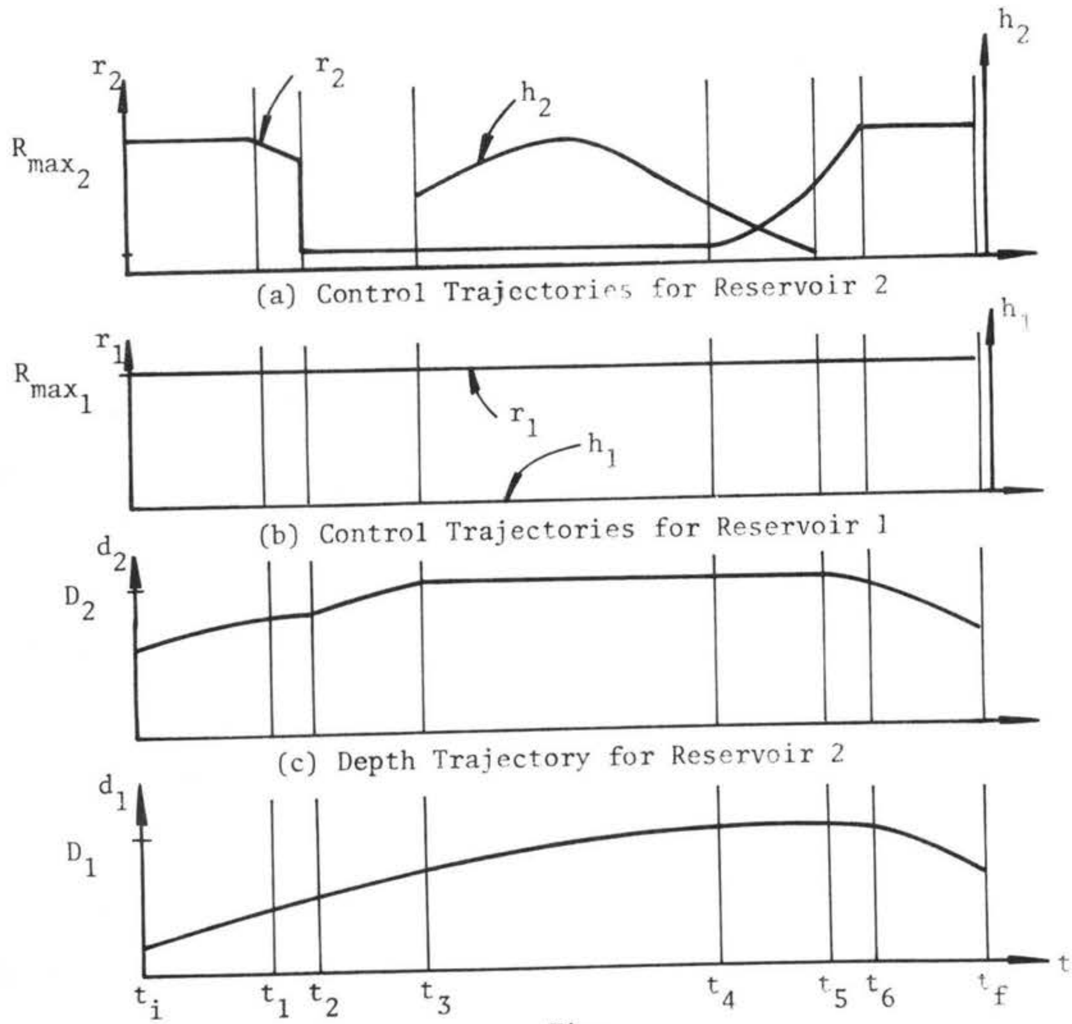


Fig. III.4 The Assumed Control and State Variable Trajectories for CASE 2

the point at which the switch in r_2 occurs must be independent of Z_1, (which was not true in CASE 1). The value of $\lambda_1/_{t_{4-}}$ must therefore be obtained by integrating forward from the value of λ_1 required by the corner conditions for the switch in control r_2, which is the same as that required for CASE 1 i.e.

$$\frac{\lambda_1}{A_1(d_1)} = \frac{\lambda_2}{A_2(d_2)} + Z_4 \qquad \text{III.3-47}$$

In addition the switch in control r_2 must be timed so that reservoir 1 fills at the point

$$q_1(t) = C_{D_1}\sqrt{D_1}\, R^2_{max_1} - C_{D_2}\sqrt{d_2}\, R^2_{min_2} \qquad \text{III.3-48}$$

which is the earliest point that reservoir 1 can fill without overflowing.

The remainder of the solution for CASE 2 is similar to CASE 1 and need not be discussed further.

E. Two Reservoirs in Series with a Downstream Flow Constraint - CASE 3.

The only difference between this case and CASE 1 is that the flow constraint (equation II.7-8) limiting the outflow from reservoir 1 is assumed to be binding for all time. The control, adjoint and state equations are as derived for CASE 1. Figure III.5 shows the assumed control and state variable trajectories.

The principle differences between Figure III.5 and Figure III.4 are:

$$R_{min_1} < r_1 < R_{max_1} \qquad \text{III.3-49}$$

for all time and; r_2 is non unique in the interval t_i to t_4 and; h_2 is non unique in the interval t_3 to t_4.

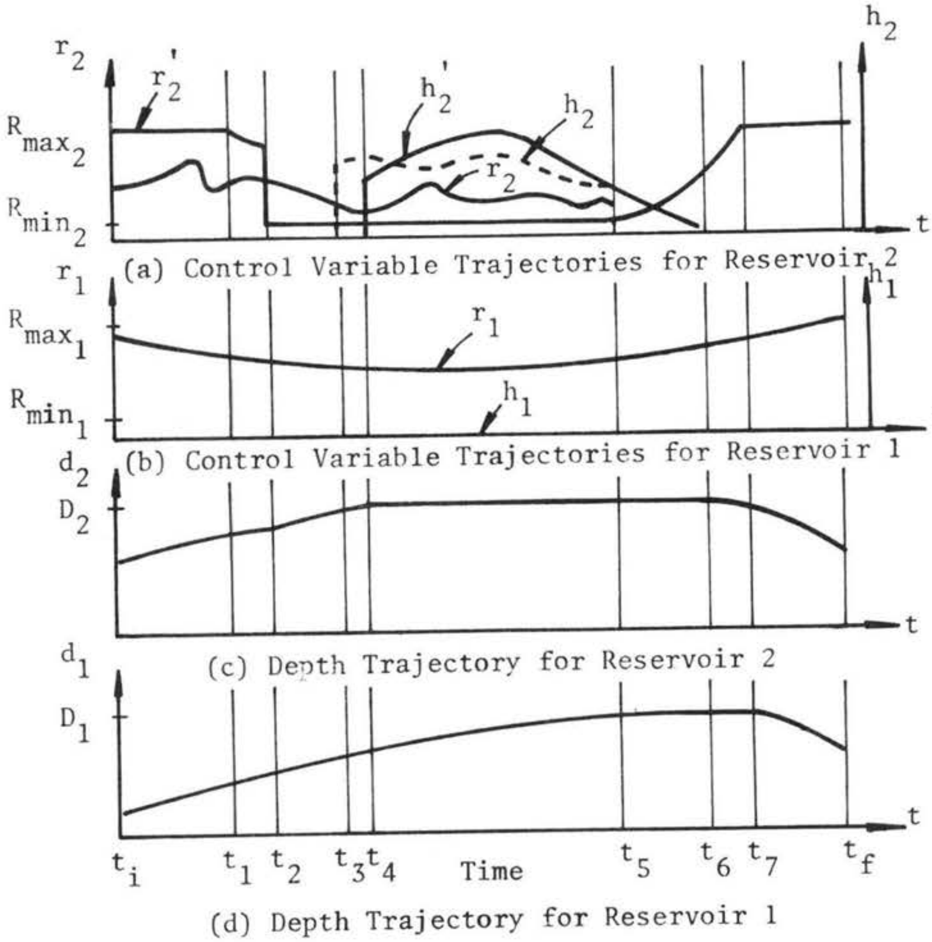


Fig. III.5 The Assumed Control and State Variable Trajectories for CASE 3

As a result of the non uniqueness of r_2, $\dot{d}_1$ is not necessarily zero at t_{4-} and thus the corner conditions yield upon simplification:

$$\lambda_1 \Big|_{t_{4-}} = (Z_4 - Z_2)A_1(D_1) \qquad \text{III.3-50}$$

Furthermore the adjoint equation for λ_1 in the interval t_3 to t_4 reduces to:

$$\frac{\lambda_1}{A_1(d_1)} = \text{constant} = Z_4 - Z_2 \qquad \text{III.3-51}$$

When this result is substituted into the control equation for r_2 (equation III.3-6) it becomes for the interval t_3 to t_4:

$$(Z_4-Z_2-Z_4+Z_2)2C_{D_2}\sqrt{D_2}\ r_2 + \pi_2(2r_2-R_{min_2}-R_{max_2}) = 0 \qquad \text{III.3-52}$$

Note that if:

$$R_{min_2} \leq r_2 \leq R_{max_2} \qquad \text{III.3-53}$$

equation III.3-52 is automatically satisfied regardless of the value of r_2 in the interval. Similarly it can be shown that equation III.3-52 applies for the interval t_i to t_3 also.

The determining factor for the control r_2 is the requirement that the volume of flow released from reservoir be such that the conditions at the corner t_4 occur. This volume of flow, ΔS_4, is by continuity;

$$\Delta S_4 = Q_{max_3} - \int_{t_i}^{t_4} q_1(t)dt - \int_{d_{1_i}}^{D_1} A_1(d_1)d(d) \qquad \text{III.3-54}$$

Whereas, in the previous two cases, without the binding flow constraint, the outflow volume from reservoir 1 could be maximized by maintaining maximum outflow from reservoir 2 as long as possible before reducing r_2 to R_{min_2}; with the binding flow constraint there is no such advantage to be gained. In addition, note that the control alternative for r_2, shown on Figure III.5, having a jump from R_{max_2} to R_{min_2} at t_2 is an allowable control and that as a result, reservoir 2 spills at t_3'.

Although there are many more possible cases for the two reservoirs in series configuration, the three examples presented herein point out some of the problems that may occur, particularly as a result of the corner conditions at a state variable constraint. Further, they show that there are many common aspects to the optimal control for the system, the principle one being the jump in the upstream reservoir control.

III.4 Two Reservoirs in Parallel - CASE 4

The schematic diagram of this configuration is shown in Figure III.6. In this case it is assumed that the flow constraint Q_{max_1} is binding for at least some portion of the time t_i to t_f, otherwise the problem reduces to one of two separate, single reservoirs, whose optimal control is obvious. Further, it is assumed that the following relationships exist for the weighting factors in the objective function:

$$Z_1 > Z_2 \qquad \text{III.4-1}$$

$$Z_1 - Z_3 > Z_2 - Z_4 \qquad \text{III.4-2}$$

and

$$-Z_3 > -Z_4 \qquad \text{III.4-3}$$

where Z_1 and Z_2 are the weighting factors on the overflows and Z_3 and Z_4 are the weighting factors (less than zero) on the throughputs from reservoirs 1 and 2 respectively. (Changing the direction of the inequalities given above effectively reverses the problem).

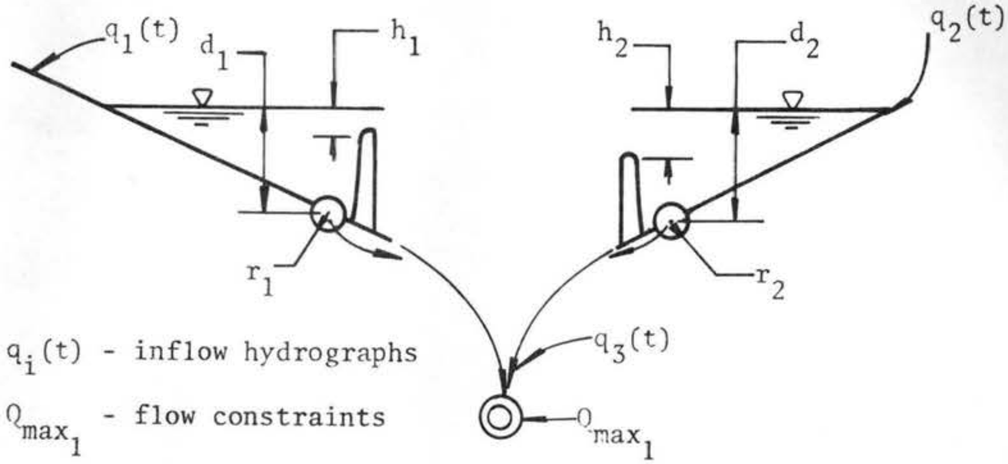


Fig. III.6 The Two Reservoirs in Parallel System

The description of the operation shown in Figure III.7 is as follows: (the verification of the solution is given in Appendix 1)

t_i-t_1 The flow constraint Q_{max_1} is not binding and both orifices can be maintained at their maximum openings.

t_1-t_2 At t_1, the flow constraint Q_{max_1} is reached and control r_2 is reduced to a level that maintains equality of the flow constraint. The orifice controls are operated in this manner until t_4.

t_2-t_3 At t_2 the depth of storage in reservoir 1 reaches its depth constraint D_1 and h_1 must be operated in a manner that keeps:

$$d_1 = D_1 \rightarrow \dot{d}_1 = 0 \qquad \text{III.4-4}$$

This operation continues up to t_4.

t_3-t_4 At t_3 the depth of storage in reservoir 2 reaches the depth constraint defined by D_2 and h_2 must be operated in a manner that maintains

$$d_2 = D_2 \quad \rightarrow \quad \dot{d}_2 = 0 \qquad \text{III.4-5}$$

This operation continues until t_6.

t_4-t_5 At t_4, the inflow $q_1(t)$ is equal to the throughput from reservoir 1. Beyond this point in time, h_1 is zero and r_1 is reduced from R_{max_1} so as to maintain the equality given by equation III.4-4. Since r_1 is decreasing, r_2 can increase to maintain the total downstream flow equal to Q_{max_1}.

t_5-t_6 At t_5, r_2 has increased to R_{max_2} and thus r_1 can begin to increase so as to maintain total flow equal to Q_{max_1}. This opening of r_1 results in outflow exceeding inflow for reservoir 1 and as a result, after t_5

$$d_1 < D_1 \qquad \text{III.4-6}$$

t_6-t_7 At t_6, the inflow to reservoir 2 equals the orifice throughput. After t_6, r_2 can be decreased to maintain d_2 equal to D_2. This allows the rate of opening of r_1 to be increased to maintain maximum system outflow.

t_7-t_f At t_7, r_1 equals R_{max_1} and r_2 can again begin to increase until at t_f, r_2 equals R_{max_2}.

The major points of interest in this sequence of operation are: the introduction of the control of r_1 in the interval t_4 to t_5 and of r_2 in the interval t_6 to t_7 dictated by the necessity to maintain

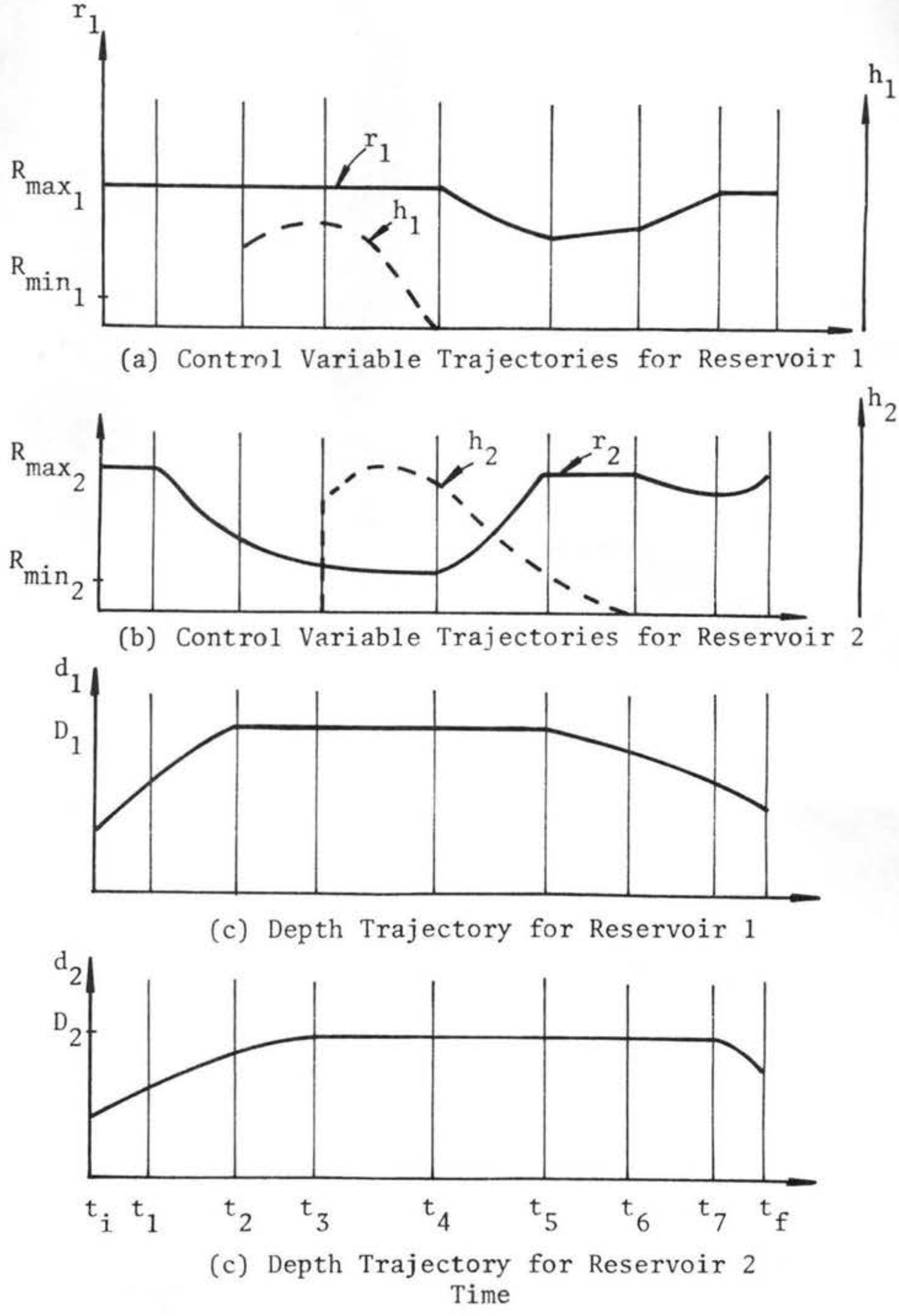


Fig. III.7 The Assumed Control and State Variable Trajectories for CASE 4

$$d = D \rightarrow \dot{d} = 0 \qquad \text{III.4-7}$$

and; the combined operation of the two reservoir orifices to keep the downstream flow constraint binding.

Although they are not discussed herein, jumps in the controls r_1 and r_2 are possible when the flow constraint is binding. A jump in the controls r_1 and r_2 must occur simultaneously in this circumstance, one control being determined from an orifice constraint and the other from the flow constraint.

Finally, following reasoning similar to that used for CASE 3 it can be argued that conditions can exist for which the controls r_1 and r_2 are coupled by the flow constraint but non unique over some interval. The constraint on their operation is the necessity to pass a given volume of flow through one of the reservoirs in a specified time.

III.5 Examples of Three Reservoir Configurations

Optimal solutions for two possible three reservoir configurations are outlined below. Detailed solutions for the state and control variable trajectories presented are given in Appendices II and III. These two configurations are called the "V" configuration, shown in Figure III-8 and the "Y" configuration, shown in Figure III-10.

A. The Three Reservoir V Configuration CASE 5.

It is assumed in this case that the flow constraint represented by Q_{max_2} in Figure III-8 is binding for most of the time period being considered and that the weighting factors in the objective function have the following relationships

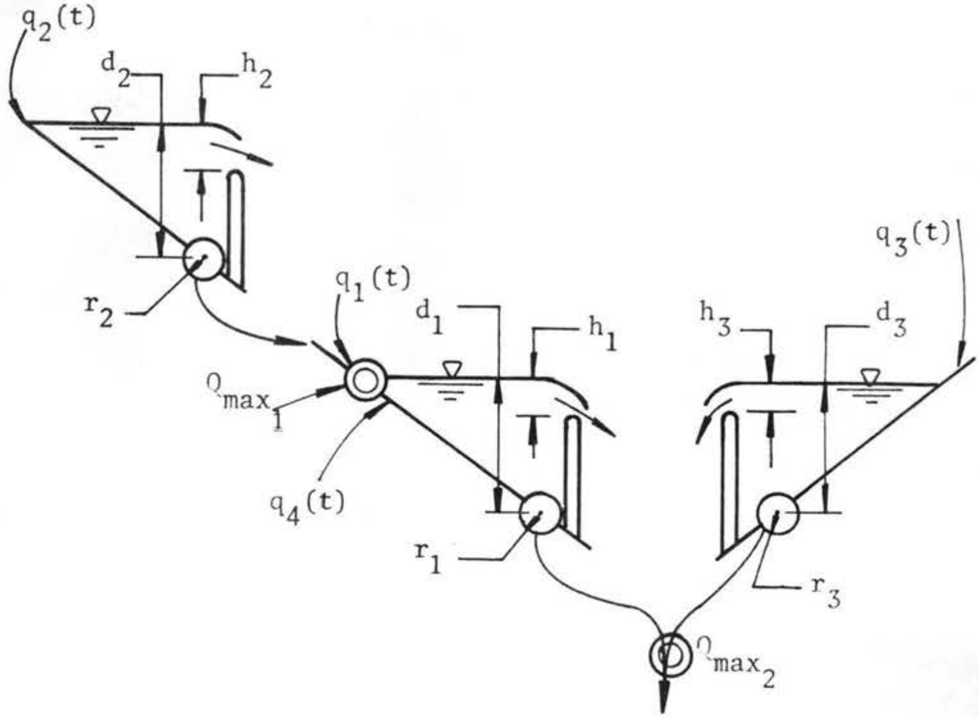


Fig. III.8 The Three Reservoir "V" Configuration

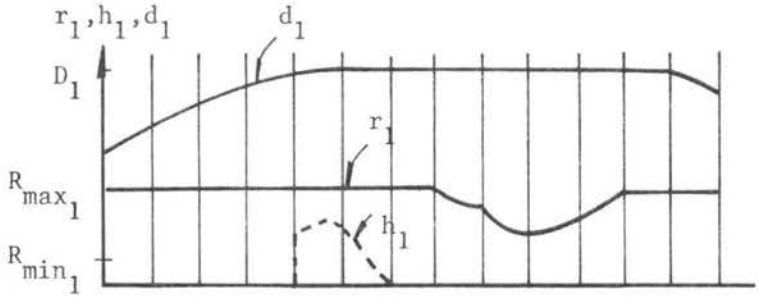
$$Z_1 > Z_2 > Z_3 \qquad \text{III.5-1}$$

$$Z_1 - Z_4 > Z_2 - Z_5 > Z - Z_6 \qquad \text{III.5-2}$$

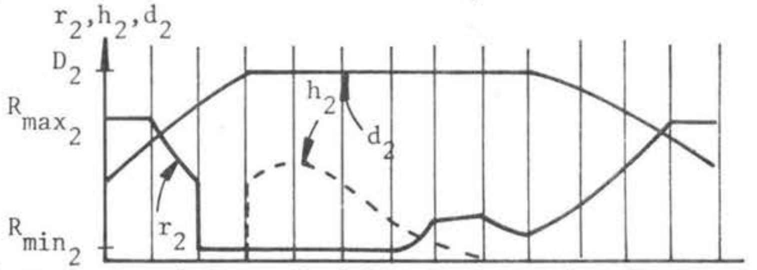
$$-Z_4 > -Z_5 > -Z_6 > 0 \qquad \text{III.5-3}$$

where Z_1, Z_2 and Z_3 are the weighting factors on the overflows, and Z_4, Z_5 and Z_6 are the weighting factors (less than zero) on the throughputs from reservoirs 1, 2 and 3 respectively.

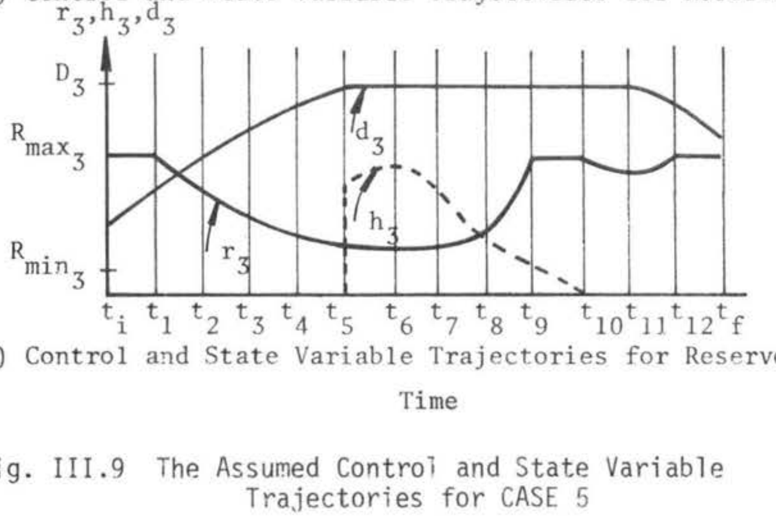
The assumed trajectories for the state and control variable are shown in Figure III-9.



(a) Control and State Variable Trajectories for Reservoir 1



(b) Control and State Variable Trajectories for Reservoir 2



(c) Control and State Variable Trajectories for Reservoir 3

Time

Fig. III.9 The Assumed Control and State Variable Trajectories for CASE 5

A description of the trajectories shown in Figure III-9 is as follows.

t_i-t_1 Up to t_1 all orifices are operated at their maximum openings.

t_1-t_2 At t_1, the two flow constraints defined by Q_{max_1} and Q_{max_2} become binding. After t_1, r_2 is operated to keep the total flow downstream of reservoir 2 equal to Q_{max_1}. Similarly r_3 is operated to the limits dictated by Q_{max_2}.

t_2-t_3 At t_2, the control r_2 jumps to R_{min_2}. Here, even though the flow constraint defined by Q_{max_2} is binding, outflow from reservoirs 1 and 2 can be increased provided that:

$$r_3 > R_{min_3} \qquad \text{III.5-4}$$

If this is not the case then conditions become similar to CASE 3 and a non unique control will result.

t_3-t_4 At t_3, the depth constraint defined by D_2 becomes binding and reservoir 2 begins to spill. After t_3, h_2 is determined by the requirement that $\dot{d}_2$ equals zero.

t_4-t_6 At t_4 and t_5, reservoirs 1 and 3 begin to spill and their weir controls are determined in a manner analogous to h_2.

t_6-t_7 At t_6, the outflow capacity of reservoir 1 exceeds the inflow and h_1 is reduced to zero. After t_6 there are two possibilities: r_1 can be decreased in a manner that maintains

$$d_1 = D_1 \quad \rightarrow \quad \dot{d}_1 = 0 \qquad \text{III.5-5}$$

thus allowing r_3 to increase or; r_1 can be kept at its maximum value and r_2 can be increased to satisfy the above equation. Since by the assumptions listed earlier:

$$Z_2-Z_5 > Z_3-Z_6 \qquad \text{III.5-6}$$

the latter control is optimal.

t_7-t_8 After t_7, control r_2 is again governed by Q_{max_1} and cannot be opened sufficiently to keep d_1 at its maximum limit. After t_7, with r_2 determined by the flow constraint, it is advantageous to decrease r_1 to satisfy equation III.5-5 and increase the outflow capacity available to r_3.

t_8-t_9 At t_8, reservoir 2 stops overflowing. The control h_2 is reduced to zero and it now becomes advantageous to operate r_2 so that

$$d_2 = D_2 \quad \rightarrow \quad \dot{d}_2 = 0 \qquad \text{III.5-7}$$

controls r_1 and r_3 are operated as in the previous interval.

t_9-t_{10} At t_9, r_3 has reached its upper limit. After this, control r_1 can be operated to satisfy the flow constraint Q_{max_2}. The control r_2 can be operated to maintain maximum storage depth in reservoir 1. As a result of this operation d_2 falls below the depth constraint D_2.

$t_{10}-t_{11}$ At t_{10}, reservoir 3 stops spilling and again there are two possible control operations. Either the operation in the previous interval can be maintained or; r_3 can be decreased so as to maintain

$$d_3 = D_3 \quad \rightarrow \quad \dot{d}_3 = 0 \qquad \text{III.5-8}$$

thus allowing r_1 to increase more rapidly to maintain maximum allowable system outflow while r_2 is operated as before. Since throughput from reservoir 3 has the least negative value, the latter control is optimal.

$t_{11}-t_{12}$ At t_{11}, r_1 reaches its upper limit. Thus after this time r_3 increases to maintain maximum system outflow, causing d_3 to fall below the limit D_3. Control r_2 is operated as before.

$t_{12}-t_f$ Finally at t_{12}, r_2 reaches its upper limit causing d_1 to fall below D_1. Control r_3 also reaches its upper limit once again at t_{12}.

The most important point to note from the above description is the operation of the reservoir controls after each reservoir stops overflowing. When the flow constraints are not binding, those reservoirs that have ceased to overflow operate in a manner that allows the maximum throughput from the reservoir in the system which is still overflowing and has the highest overflow weighting factor. If a flow constraint becomes binding then the operation of those reservoirs which have ceased overflowing is such as to maximize the throughput from that reservoir in the system which has the next highest overflow factor.

Again it should be noted that the controls are always determined from constraint boundaries.

B. The Three Reservoir Y Configuration CASE 6

It is assumed in this case that none of the flow constraints are binding; that the reservoir areas are constant; that R_{min_i} are zero and; that the weighting factors on the overflow and throughput are as given by equations III.5-1, 2 and 3. The assumed state and control variable trajectories are shown in Figure III-11. In this case there are steps in control r_2 and r_3. If the flow constraint Q_{max_4} were binding, one of the two upstream reservoirs would have a non unique orifice control in the interval t_i to t_4. (The other orifice control would either be at its maximum or minimum limit, depending upon the inflow to the system).

Immediately after reservoir 1 has stopped overflowing at t_6; r_2 is determined from the requirement that:

$$d_1 = D_1 \quad \rightarrow \quad \dot{d}_1 = 0 \qquad \text{III.5-9}$$

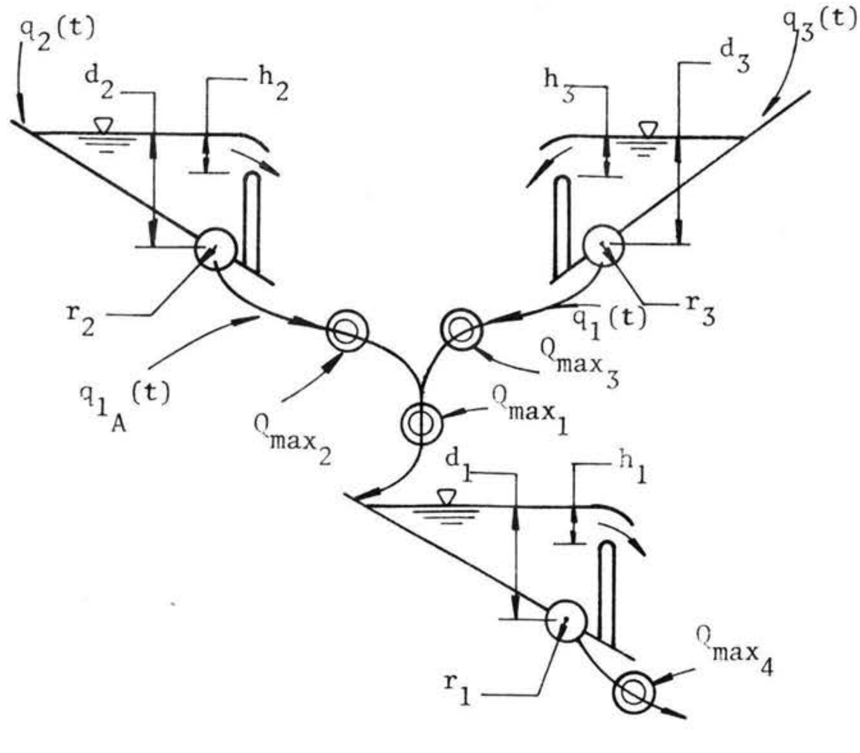
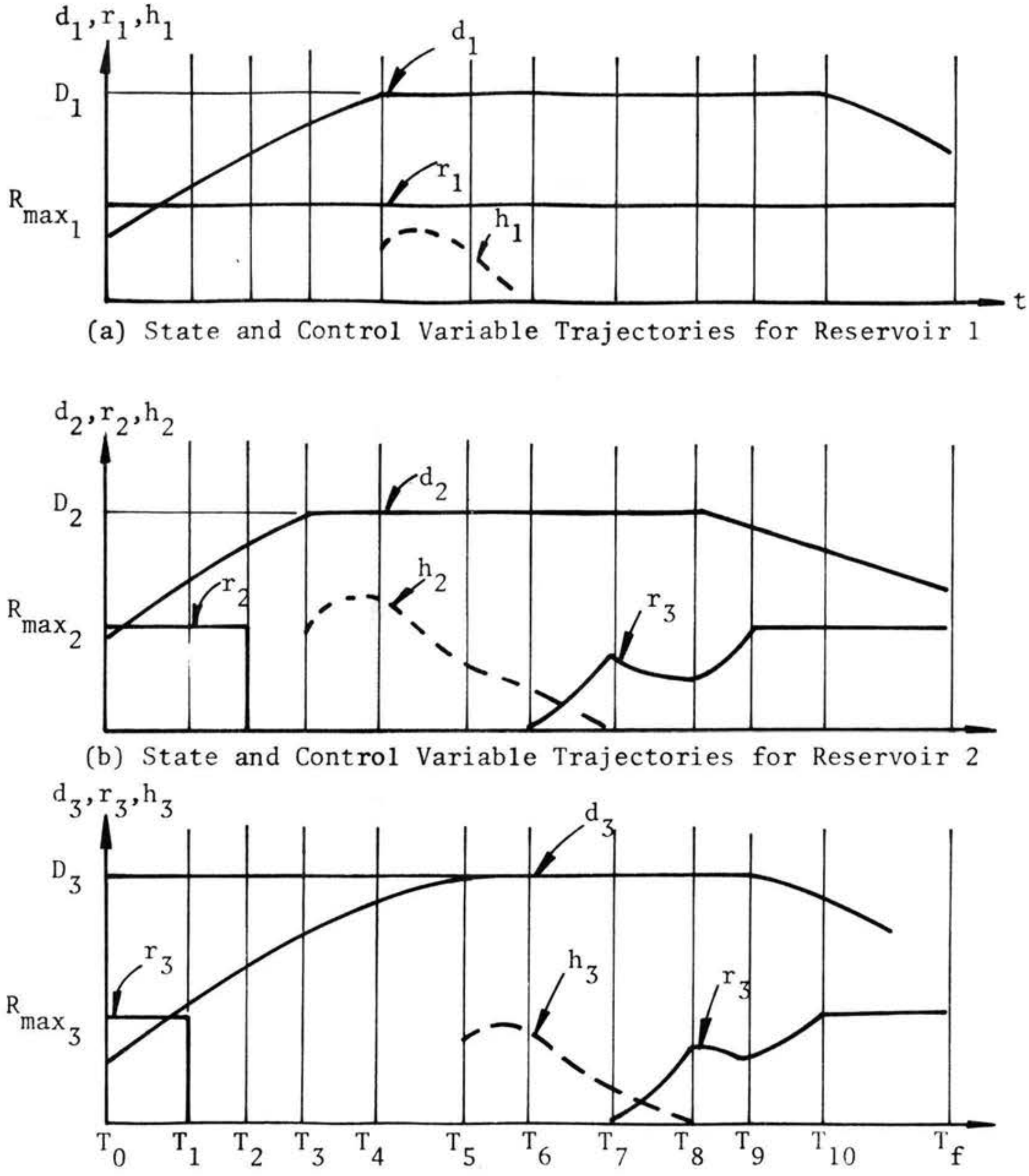


Fig. III.10 The Three Reservoir "Y" Configuration



(a) State and Control Variable Trajectories for Reservoir 1

(b) State and Control Variable Trajectories for Reservoir 2

(c) State and Control Variable Trajectories for Reservoir 3

Fig. III.11 The Assumed Control and State Variable Trajectories for CASE 6

After t_7 when reservoir 2 stops overflowing r_2 is determined by the requirement that:

$$d_2 = D_2 \quad \rightarrow \quad \dot{d}_2 = 0 \qquad \text{III.5-10}$$

and r_3 is determined from equation III.5-9. After t_7 when reservoir 3 stops overflowing r_3 is determined by the requirement that

$$d_3 = D_3 \quad \rightarrow \quad \dot{d}_3 = 0 \qquad \text{III.5-11}$$

while r_2 is again determined by equation III.5-9.

Finally, after t_9, when r_2 has reached its upper limit, r_3 is determined by equation III.5-9.

Note again in this example that the controls are only determined from system constraints.

III.6 Discussions of the Factors Common to All the Cases Examined

A. Limitations.

In all of the cases examined it was assumed that

$$h_i < d_i \qquad i = 1,\ldots,n \qquad \text{III.6-1}$$

i.e., that the system overflow capacity would never be exceeded. This is considered to be in the nature of a "disaster" situation and extremely unlikely to occur. In the event that the instantaneous inflow rate did exceed the overflow capacity of a reservoir when it was at maximum depth, it appears that the optimal strategy would be to begin operating the weir before the peak inflow occurred and before the depth constraint was reached so that enough reservoir storage would be available to absorb part of the peak inflow. This operation, however, has not been considered.

In all of the cases studied there was an implicit assumption that once a reservoir stopped overflowing, the inflow hydrographs would not be such as to cause the reservoir to overflow a second time. To be assured that this is the case the inflow hydrograph to a given reservoir must be monotonically decreasing once the reservoir stops overflowing. Inflow hydrographs not meeting this assumption may still have the control trajectories discussed but this cannot be assured. There is no restriction on the shape or the number of peaks of the inflow hydrograph prior to the reservoir ceasing to overflow other than that relating to the peak overflow capacity.

B. Common Factors in the Control Strategies.

Several factors are common to all the solution forms discussed. In each of the examples the control could always be determined from a constraint boundary. As the main result of minimizing the objective function is the minimization of weighted diversions from the system; it is reasonable to expect that this would be accomplished by maximum utilization of system throughput and system storage capacity. It follows that if the operation of the controls is always determined from constraint boundaries then sooner or later there will be jumps in the controls from one constraint boundary to another (such as were observed for the orifice controls). Further, if the operation of the controls can always be determined from constraint boundaries then there is a limited number of possible forms of operation for each control. For example, the control h, for any reservoir, can be determined from only two equations:

$$h(h-d) \leq 0 \qquad \text{III.6-2}$$

and

$$\dot{d} = \frac{\sum_{i=1}^{m} C_{D_i}\sqrt{d_i}\,r_i^{\,2} + q(t) - C_D\sqrt{d}\,r^2 - C_W h^{3/2}}{A(d)} = 0 \qquad \text{III.6-3}$$

If h is always less than d, then the only available choices for h are zero or a value that sets equation III.6-2 equal to zero.

Likewise, the control r appears in equation III.6-3; in the orifice constraint;

$$(r-R_{min})(r-R_{max}) \leq 0 \qquad \text{III.6-4}$$

in equations of the form:

$$q(t) + \sum_{i=1}^{m} C_{D_1}\sqrt{d_i}\,r_i^{\,2} \leq Q_{max} \; ; \qquad \text{III.6-5}$$

or in equations of the form;

$$\dot{d}_j = \frac{C_D\sqrt{d}\,r^2 + q(t) - C_{D_j}\sqrt{d_j}\,r_j^{\,2} - C_{W_j} h_j^{\,3/2}}{A(d_j)} = 0 \qquad \text{III.6-6}$$

(in which case r is one of the r_i of equation III.6-3).

Thus, there is a limited number of possible values of r at any one time. Which equations are used to determine h and r is a function of the available constraint multipliers. There is not necessarily a unique solution to the problem and considerable logic may be required to determine which of the possible combinations of controls is in fact optimal.

CHAPTER IV

NUMERICAL SOLUTION OF THE OPTIMAL CONTROL PROBLEM

IV.1 Introduction

This chapter outlines the problems encountered while attempting to determine an optimal control by numerical solution of the necessary conditions.

First some of the requirements that must be met by any numerical solution are presented. This is followed by a discussion of the particular numerical difficulties peculiar to the variational formulation of the problem. The actual numerical techniques attempted and the reasons for their failure are then given. Finally an alternative approach to the problem, based on examination of the control trajectories given in Chapter III, is outlined. Verification of the alternative approach is left until Chapter V.

IV.2 Requirements of the Numerical Solution

For a numerical solution of the control problem to be practical it should be capable of determining the necessary control strategies in under two minutes (this assumes that the controls will be updated approximately every twenty minutes and; that obtaining information from sensors and converting that information into a computed runoff input to the combined sewer system model will require about three minutes). In addition to determining the required control within a given time, the numerical technique used must be capable of obtaining a solution for all possible inputs within the design limits. A solution technique which converges to a solution only 95% of the time is not satisfactory. Further, it is desirable that any numerical technique used for real time control be capable of being adapted to those instances in which control device failures occur. Thus, the numerical technique should be able to determine an optimal control when at least one of the control device positions is considered fixed, (as opposed to determining a control in which the device failure is neglected). Finally it is economically desirable, but not necessary, that the control program require as little computer storage as possible.

IV.3 Problems to be considered in the Numerical Solution of the Combined Sewer Problem

The necessary conditions for a solution to the combined sewer control problem form a two point boundary value problem in which the initial conditions are known for the state variables and the final conditions are known for the λ multipliers. As a result, some iterative technique must generally be used to arrive at the optimal solution. Some of the difficulties that must be considered in selecting a numerical method suitable for the combined sewer problem are discussed below.

It was shown in Chapter III that jumps occur in the λ multipliers at the entrance to a state variable constraint boundary. It was also shown that on the t_- side of the jump there was a choice of possible values for λ multipliers. Neglecting for the moment the case where the value of λ at t_- could not be determined directly from the corner conditions, it is obvious that the problem is essentially a multiple boundary problem. Thus, the problem can be divided at t_b, the unknown time at which the state variable constraint becomes binding. Between t_{b+} and t_f there is a two point boundary value problem. Between t_i and t_{b-} the problem is still a two point boundary value problem but now the state variables are known at t_i and t_{b-} and the value of λ at t_{b-} must be one of a discrete number of possibilities (e.g. CASE 1 and CASE 3 discussed in Chapter III). Thus, the value of λ at t_{b-} is effectively an unknown except that it must satisfy the corner conditions. In reality this is determining the time t_b at which the jump in λ must occur. For the cases where the value of λ at t_{b-}, and therefore effectively t_b, cannot be determined from the corner conditions there must be some other conditions, such as the flow conditions of CASE 2, which define the time t_b. The problem then still can be subdivided into more than one two point boundary value problem.

An additional complication at the entrance to a state variable constraint boundary is the fact that even if λ is known at t_{b-}, the corner conditions will not yield the value of λ at t_{b+} (since at t_{b+}, λ is multiplied by $\dot{d}$ which is zero). Therefore, any numerical technique that requires forward integration of the λ multipliers (i.e. an initial value of λ is assumed) is faced first with the problem of the correct end point at t_{b-} and second, with making a new guess for λ at t_{b+}. A numerical technique which integrates the λ multipliers backward reduces the problems at the boundaries to the correct determination of λ at t_{b-}.

(Although it would seem that the correct value of λ at t_{b-} would be obtained for all those cases in which $\dot{d}$ at t_{b-} is non zero, the sheer fact that finite steps are used in the integration of both the state and adjoint equations may leave some doubt if the conditions at t_{b+} are

$$\dot{d} = 0 \qquad \text{IV.3-1}$$

and

$$\varepsilon > h > 0; \quad \varepsilon\text{-small} \quad . \qquad \text{IV.3-2}$$

With exact integration should h be zero or not?)

The second problem area to be considered in the determination of a suitable numerical technique is the fact, demonstrated in Chapter III, that the optimal control trajectory lies along constraint boundaries. As a result, the control cannot be determined from the control equations, which is the normal procedure. Thus, unless the problem can be formulated in a manner which ensures that the control can be determined from the control equations, the numerical solution must include some logic to determine first; which control is to be determined from which constraint boundary; and second, which Lagrange multiplier (π,γ) is to be used to satisfy which control equation (recall that one multiplier may appear in several control equations). The first part is relatively simple, even for fairly large systems. The second when combined with the first requires complex logic for even a small (e.g. three reservoir) system. In addition,should a particular series of inflows to the reservoir system occur which has not been considered in development of the logic, then the numerical method may either breakdown completely, in which case no control is obtained, or yield a sub-optimal control. For these reasons, a numerical technique which allows the determination of the control directly from the control equations would be preferred, provided the necessary reformulation did not alter the basic structure of the problem.

The final problem to be considered is that caused by non unique controls. If the problem formulation given in Chapters II and III is used, difficulties may arise in determining when the control is non unique. Given that the control is non unique there is the further difficulty of determining a control which satisfies the necessary conditions. Again it appears that it might be advantageous to reformulate the problem so that the possibility of non unique controls is avoided.

IV.4 Numerical Methods Considered in Attempts to Solve the Necessary Conditions

A. "Shooting Technique.

One common method used to solve two point boundary value problems is to guess initial values of the λ multipliers and then integrate the state and adjoint equations forward, one step at a time to t_f. At each point in time the known values of the λ multipliers and the state variables are used to determine new controls for the next integration step. The integrated values of the λ's at t_f are compared with the known values obtained from the transversality condition and if they do not agree within some tolerance limits, a new guess is made, (usually on the basis of the gradient $d\lambda_{t_f}/d\lambda_{t_i}$).

For the combined sewer problem this technique suffers from all the drawbacks discussed in the previous section and thus was considered unsuitable.

B. "Steepest Ascent Technique.

There are many numerical techniques which fall into this category. The one tried in this study consisted of: initially guessing a time history for the orifice controls; integrating the state equations forward to t_f, one step at a time using the assumed orifice control (the weir controls were determined from the first derivative of the state variable constraint when $d = D_{max}$); and then integrating the λ multipliers backward from t_f, using the previously determined state variable path and the new λ multipliers to determine a new set of controls at each time step. The procedure was then repeated from t_i using the new control until, for two successive iterations, the same control was obtained (within tolerance limits). This method was tried for the problems of two reservoirs in series and the three reservoir "V" configuration. Limited success was obtained for the two reservoir case. The first difficulty that had to be overcome was related to the controls after a reservoir started overflowing. If the state variables were not on the required boundaries for an optimal solution during this period, the γ multipliers associated with the state variable constraints were zero and the remaining constraint boundaries available for control determination at each time step were sub-optimal. This problem was surmounted by programming additional logic into the forward integration of the state variables. After each reservoir started overflowing, this logic determined the optimal orifice operation until t_f. Additional logic was also added to the reverse integration to determine the correct multipliers to be associated with each control equation. The reverse integration then verified the control backward to the time each reservoir filled. Beyond this point it was possible to determine when the switch in orifice control from minimum to maximum occurred. However, even for the two reservoir problem, the solution tended to oscillate. If during one iteration the switch in the upstream orifice control occurred too early, then the downstream reservoir filled after the optimum time. On the next iteration the reverse would be true. This oscillation was easily damped by limiting the change in the switching point. After that it was possible to obtain some numerical results. The results were limited because up to this time it was assumed that CASES 2 and 3 of Chapter III were extensions of CASE 1 and thus only one possibility for the value of the λ multipliers was assumed at t_{b-}.

Further work using this technique was stopped when it was realized that there were other possibilities for the values of the λ multipliers at t_{b-}.

Aside from the difficulties in programming the logic, particularly that associated with the determination of which multiplier to use to satisfy which control equation, the program was computationally slow (partly as a result of the complex logic). Even if the technique could have been made to work successfully, it is doubtful that the computational time for a reasonable number of reservoirs would have been short enough for use in real-time control. (For the three reservoir problem, 1-2 minutes CDC 6400 computer time were required).

C. "Penalty Function" Technique.

As pointed out earlier it might be beneficial to reformulate the problem in such a manner that the control can be determined from the control equations. In addition it would be desirable to eliminate the jumps in the λ multipliers and, if possible eliminate the possibility of non unique controls. Such a reformulation, which maintains the essential points of the formulation given in Chapter II, is possible. It is accomplished by replacing each of the inequality constraints by new terms in the objective function. These terms are zero if the constraint that they replace is not violated; and increase in value very rapidly for any small violation of the constraint.

For this study, all the control variable inequality constraints, which had the form:

$$(X-X_{min})(X-X_{max}) \leq 0 \qquad \text{IV.4-1}$$

were replaced by terms in the objective function having the form

$$K(X)_{[X_{min},X_{max}]} \{(X-X_{min})(X-X_{max})\}^2 \qquad \text{IV.4-2}$$

where

$$K(X)_{[X_{min},X_{max}]} = \begin{cases} 0; & \text{if } X_{min} \leq X \leq X_{max} \\ K; & \text{if } X < X_{min} \text{ or } X > X_{max} \end{cases} \qquad \text{IV.4-3}$$

(The second term in equation IV.4-2 was squared to insure continuous first derivatives).

In order to ensure that the control could be determined from the control equations it was necessary to replace the state variable inequality constraints which have the form

$$d(d-D_{max}) \leq 0$$

by penalty functions of the form

$$K(d+\dot{d}\delta t)_{[D_{min},D_{max}]}\{(d+\dot{d}\delta t-D_{min})(d+\dot{d}\delta t-D_{max})\}^2 \qquad \text{IV.4-4}$$

where

$$\dot{d} = \frac{\sum_{i=1}^{m} C_{D_i}\sqrt{d}_i r_i^{\,2} + q(t) - C_D\sqrt{d}\,r^2 - C_w h^{3/2}}{A(d)} \qquad \text{IV.4-5}$$

(which is equation II.4-3) and; δt is a small fixed time increment.

Minimum limits greater than zero were required in all the penalty functions in order to avoid a saddle point solution when r, h or d equalled zero.

The modified objective function now included a penalty function of the form outlined above for each inequality constraint in the formulation given in Chapter II. The differential constraints remained the same as those in Chapter II and were adjoined to the objective function with Lagrange multipliers (λ). Because the state variable inequality constraints were eliminated,the λ multipliers in the penalty function formulation were now continuous at all times.

The penalty function formulation was combined with the steepest ascent technique in an attempt to solve the three reservoir "V" problem. With this combination, no complex logic was required in the computer program to adjust the controls during the forward integration.

Although the penalty function formulation appeared to eliminate many of the problems of the formulation presented in Chapter II, it brought forward new problems. The determination of the controls at each point in time required the simultaneous solution of six highly non-linear equations. Solution of these equations was computationally slow (using Newton's Method) and it was very difficult to ensure convergence to a solution. This, however, was not as serious as the problems encountered with the integration of the λ multipliers. The discontinuities in the λ multipliers in the inequality constraint formulation were replaced by very rapid changes in the values of the λ multipliers in the penalty function formulation. As the optimal solution was very sensitive to the values of the λ multipliers, particularly if the magnitude of the multiplier became too great, it would have been necessary to reduce the time increment used for the integration to a very small value in order to obtain the required accuracy. This would have greatly increased the computational time as well as computer memory storage requirements. Even when satisfactory valuesof the λ multipliers were obtained, the solution technique showed evidence of the same oscillation problems that appeared in the steepest ascent technique. As the numerical solution technique was already computationally too slow to be feasible for real time automatic control, it was felt that further work was not justified and this approach was abandoned.

IV.5 An Alternative Approach to the Determination of an Optimal Control

Another approach to the problem of solution of the necessary conditions is to divide the problem into two steps. The first step is to try and determine an optimal control for a given set of reservoirs and inflow data. In the second step, once this "optimal" control has been obtained and the resultant state variable trajectories determined, the necessary conditions can be applied to ensure that the solution is in fact optimal. This method should reduce the oscillation problems that occurred with the steepest ascent technique and as the "optimum" conditions should exist on both sides of a state variable boundary, there should be less problems determining the value of the λ multipliers at the t_{b-} side of a jump. Also, it might be computationally much faster than the steepest ascent technique as the reverse integration is performed only once to verify that the control satisfies the necessary conditions. The principle disadvantage of this method is that it would still require most of the logic of the steepest ascent technique. This means that the programmer must consider all possibilities of operation. To reduce the number of possibilities to a managable size this may mean that it would be necessary to write a new program for each reservoir system configuration and, for any system configuration, a new program for each different set of relative values of the weighting factors (e.g. if for the same reservoir configuration one case has Z_1 greater than Z_2 and another case to be considered has Z_2 greater than Z_1, then separate control programs might be required). For real time control however this is not a serious disadvantage as it is reasonable to assume that the reservoir configuration, and the overflow and throughput weighting factors, once determined, would remain fixed. It would be a disadvantage in the design stages of a project where both the configuration and the weighting factors might be altered.

The problem of determining an "optimal" control in the first step can be broken into two stages. First, for each reservoir orifice control $r_j(t)$ it is assumed that a jump in control from R_{max_j} (or the maximum conduit capacity) to R_{min_j} occurs at time X_j. Second, if any reservoir fills to the point d_j equals D_{max_j} say at time t_{b_j}, then after t_{b_j}, a set of reservoir operation rules must be devised which minimizes the sum of the weighted diversions from the system. The basis of these rules is given in Chapter III and is discussed in more detail in the next section. Given the set of operating rules, the problem is now reduced to the determination of the optimal switching times, X_j $(j = 1,\ldots,n)$.

That is, the problem is now

$$\text{Min } \phi = \int_{t_i}^{t_r} [\sum_{i=1}^{n} Z_i C_{w_i} h_i^{3/2} + \sum_{j=1}^{n} Z_{j+n} C_{D_j} \sqrt{d_j}\, r_j^2] dt \qquad \text{IV.5-1}$$

$$= f(Z_1, Z_2 .. Z_{2n}, X_1, X_2 \ldots X_n)$$

and for given values of the Z weighting factors this becomes

$$\text{Min } \phi = F(X_1, X_2, \ldots, X_n) \qquad \text{IV.5-2}$$

This problem can be solved by a gradient search procedure (see for example Wilde and Beightler, 1967) to obtain those values of X_j that minimize the objective function. With the known values of the switching times X_j and the given operating rules, the control is fully determined. It remains to show that it satisfies the necessary conditions for an optimal control. Examples of the use of this technique are given in Chapter V. The basis of the operating rules is discussed in section 6 of this chapter.

If the operating rules are correctly formulated then the obvious question arises: is it necessary to ensure that the control satisfies the necessary conditions? First of all, the logic that determines which of the π and γ multipliers should be used to satisfy which control equation during the reverse integration would be based on the known form of the operating rules. Therefore, any error in the forward integration would be reflected in a compensating error in the reverse integration. Second, the final check, which, by the above statements is almost assured of being satisfied, would increase the computational time and computer memory requirements and produce effectively no new usable information. Finally, even if the check did reveal that the operating rules did not produce an optimal control, there is not much that could be done during real time operation. For these reasons it was decided in this study not to proceed with the development of the check routine.

IV.6 The Operating Rules for a System of Reservoirs

A. The Weir Control.

As noted in the examples given in Chapter III the weir control h_j for reservoir j only operates when

$$d_j = D_j \qquad \text{IV.6-1}$$

Also, as noted in Chapter III, the operation of the weir under this condition is always such as to keep

$$\dot{d}_j = 0 \quad . \qquad \text{IV.6-2}$$

This was true for all the examples cited in Chapter III, provided the assumption holds that for all time:

$$h_j < d_j \qquad \text{IV.6-3}$$

(Cases where this assumption does not hold have not been discussed in this study). Under the above conditions, in all the examples cited, the orifice control was determined from some other equation.

As any other method of weir operation leaves open the possibility that the full storage capacity of a reservoir may not be used, there is no reason to doubt the generality of the above procedure.

B. The Orifice Controls.

As demonstrated in Chapter III the optimal solution is always on constraint boundaries. The orifice control r_j for any reservoir j appears only in the following constraints:

$$(r_j - R_{min_j})(r_j - R_{max_j}) \leq 0 \qquad \text{IV.6-4}$$

$$q(t) + \sum_{k=1}^{m} (C_{D_k} \sqrt{d_k}\, r_k^2) + C_{D_j} \sqrt{d_j}\, r_j^2 \leq Q_{max} \qquad \text{IV.6-5}$$

$$\dot{d}_j = \frac{q(t) + \sum_{k=1}^{m} C_{D_k} \sqrt{d_k} r_k^2 - C_{D_j} \sqrt{d_j} r_j^2 - C_{w_j} h_j^{3/2}}{A_j(d_j)} \qquad \text{IV.6-6}$$

and

$$\dot{d}_k = \frac{q(t) + C_{D_j} \sqrt{d_j} r_j^2 + \sum_{p=1}^{m} C_{D_p} \sqrt{d_p} r_p^2 - C_{D_k} \sqrt{d_k} r_k^2 - C_{w_k} h_k^{3/2}}{A_k(d_k)} \qquad \text{IV.6-7}$$

where the subscript k in the last equation represents the reservoir immediately downstream. It was pointed out in Chapter III, that multipliers are only available to allow the determination of r_j from equation IV.6-6 when

$$h_j = 0 \qquad \text{IV.6-8}$$

and

$$d_j = D_j \qquad \text{IV.6-9}$$

Thus, unless reservoir j has filled and stopped overflowing the control r_j must be determined from either equations IV.6-4, IV.6-5 or IV.6-7.

For reasons equivalent to the above, r_j can only be determined from equation IV.6-7 if

$$h_k = 0 \qquad \text{IV.6-10}$$

and

$$d_k = D_k \qquad \text{IV.6-11}$$

As the reservoirs do not fill until sometime after the start of a storm, this means that initially the control r_j must be determined from equations IV.6-4 or IV.6-5 (the latter may represent more than one equation). If the optimal control specifies the maximum output from reservoir j then the possible values for r_j are

$$r_j = R_{max_j} \qquad \text{IV.6-12}$$

or r_j equals the limiting value determined by equation IV.6-5. If the optimal control specifies the minimum outflow from reservoir j then only equation IV.6-4 applies and

$$r_j = R_{min_j} \qquad \text{IV.6-13}$$

The examples worked in Chapter III showed that a jump in the control r_j from a maximum value to a minimum value could occur at time X_j where

$$t_i \leq X_j \leq t_f \qquad \text{IV.6-14}$$

Therefore, if the control is operated at its maximum value in the interval t_i to X_j and then switched to its minimum value after X_j and, if in searching for an optimum, X_j is allowed to vary from t_i to t_f then there is no loss of generality. The control r_j can be at a maximum value for all time, or until such time as it can be determined from equations IV.6-6 or IV.6-7; or r_j can be a combination of the above as determined by the switching time X_j. It should be noted from the examples given in Chapter III that for reservoirs in series having the downstream overflow weighting factor greater than the upstream weighting factor (and assuming that the throughput weighting factors are negligible in comparison) the necessary conditions required the switch in control X_j for the upstream reservoir to occur before the downstream reservoir fills. If this were not the case, then water would be diverted from the downstream reservoir that could have been diverted from the upstream reservoir at a smaller cost.

For reservoirs operating in parallel and governed by a common flow constraint (equation III.6-5) at the outlet of the entire system, the examples of Chapter III showed that the total overflow from the system outlet is always the maximum possible. Thus, the switch in control is not from a maximum to a minimum position but from a dominant (i.e., first call on the conduit capacity) to a subservient position. Therefore, for two reservoirs in parallel it is only necessary to consider the switching time for one reservoir. The switch for the other reservoir must occur at the same time in the opposite direction. As demonstrated in the examples in Chapter III the reservoir with the highest overflow weighting factor will start at t_i in the dominant position.

In the examples examined in Chapter III, the only time when control r_j could move off the boundaries given by equations IV.6-4 and IV.6-5 was when reservoir j stopped overflowing, in which case the control could be determined from equation IV.6-6, or when the reservoir immediately downstream stopped overflowing, in which case the control could be determined from equation IV.6-7. The former, which results in a decrease in r_j, can only occur if: r_j was in a maximum position or; if r_j had previously been determined from equation IV.6-7 which causes an increase in r_j (this is a result of the assumption that once a reservoir stops overflowing the inflow hydrograph is monotonically decreasing). When reservoir j stops overflowing, the choice of decreasing r_j (if possible) or increasing $r_{upstream}$ depends upon the location in the system of the reservoir with the highest overflow weighting factor that is still overflowing (or may overflow) and whose throughput can be increased by application of either equations IV.6-6 or IV.6-7. Note that in either case this may cause a chain reaction throughout the system as for example in CASE 5 (the three reservoir "V" configuration) discussed in Chapter III. When reservoir 2 stops overflowing at t_8, r_2 is determined from

$$\dot{d}_2 = 0 \qquad \text{IV.6-15}$$

This reduces the outflow from reservoir 2 and thus r_1 is determined from

$$\dot{d}_1 = 0 \qquad \text{IV.6-16}$$

These two operations allow increased outflow from reservoir 3, the only reservoir in the system that is still overflowing.

Finally, if a reservoir stops overflowing, and cannot be operated to increase the outflow of a reservoir that is still overflowing (or may overflow), then it operates in a manner that maximizes the weighted system throughput (for example CASE 5, the interval t_{10} to t_f).

The above operating rules have been based on the limited possibilities shown to exist for solution of the necessary conditions as exemplified by the examples given in Chapter III. By suitable choice of the switching times X_j for the switch in each reservoir control, each reservoir can be made to fill if possible (i.e., if the inflow $q_j(t)$ is great enough). Thus, the maximum storage capacity of each reservoir will be utilized. The operational procedure after each reservoir stops overflowing, assures that the storage in each reservoir is not reduced to the detriment of reservoirs that are still overflowing, or may still overflow. At their worst, (i.e., violation of the requirement of monotonicity of the inflow hydrograph after a reservoir has stopped overflowing) these rules will minimize total system overflow. At best, they will insure that the weighted overflow from the system is minimized once the optimum switching times have been determined.

IV.7 Comments on the Alternative Approach to the Determination of an Optimal Control

At the start of this chapter several criteria were outlined that should be met by a numerical solution to the control problem. It is worthwhile to compare the proposed solution technique with these criteria.

In the following chapter it is shown that using the proposed system for determination of an optimal control the time available for the determination of an optimal control (two minutes) would allow real time control for a system having up to twenty reservoirs. (The Minneapolis - St. Paul system has 18 control points (Minneapolis - St. Paul Sanitary District, 1970)). Although the rules proposed do not guarantee a global optimum, they do ensure a reasonable control. The studies outlined in the next chapter revealed no problems with convergence to a solution. In addition the computer programs required very little computer memory storage.

Although the proposed procedure is easily adaptable to complex weirs and orifices at regulator structures, it does not have complete adaptability to control device failures. If an orifice became locked in one position there is no problem of immediate adaptation since the problem becomes for r_k, the locked orifice.

$$R_{min_k} = r_k = R_{max_k} \qquad \text{IV.7-1}$$

which is still within the original format of the problem. If a weir control becomes blocked in one position, the best adaptation might be to reduce D_{max} for that reservoir so as to increase the storage safety margin; however, the computed control would not be optimal. If it suddenly became desirable to change the relative values of the weighting factors, then unless the control programs were completely general (i.e., capable of determining an optimum for any set of overflow weighting factors) such a change might not always be immediately possible. (Experience indicates, however, that at least some generality is possible).

CHAPTER V

NUMERICAL RESULTS

V.1 Introduction

In this chapter,optimal control strategies obtained numerically for two different reservoir configurations are presented and discussed. The control logic used for these examples was based on the operational rules discussed in Chapter IV in combination with a first order gradient search technique. The first example presented is a control strategy for a system of four reservoirs in series. This system, although relatively simple, was chosen in order to gain experience in programming the necessary logic. The second example consists of a system of ten reservoirs and was chosen to illustrate the fact that the optimal control for a reasonably complex system could be determined rapidly enough for real time operation. A final example is presented showing the optimal control for a system of three reservoirs in the "V" configuration. This solution, obtained by use of a steepest ascent technique, is included to show the effect that information errors may have on the optimality of a control.

V.2 Optimal Control Strategies for Four Reservoirs in Series

The system analyzed is shown in Figure V.1. The program logic was based on the operating procedures discussed in Chapter IV and the optimization of the switching times (X_i) was accomplished by means of a first order gradient search technique. The experience and insight gained using the four reservoirs in series program resulted in less complex logic in the ten reservoir program. Therefore only the logic and solution procedure for the ten reservoir problem are discussed in detail.

The relative values of the weighting factors considered for the four reservoir problem were

$$Z_1 > Z_2 > Z_4 > Z_3 \qquad \text{V.2-1}$$

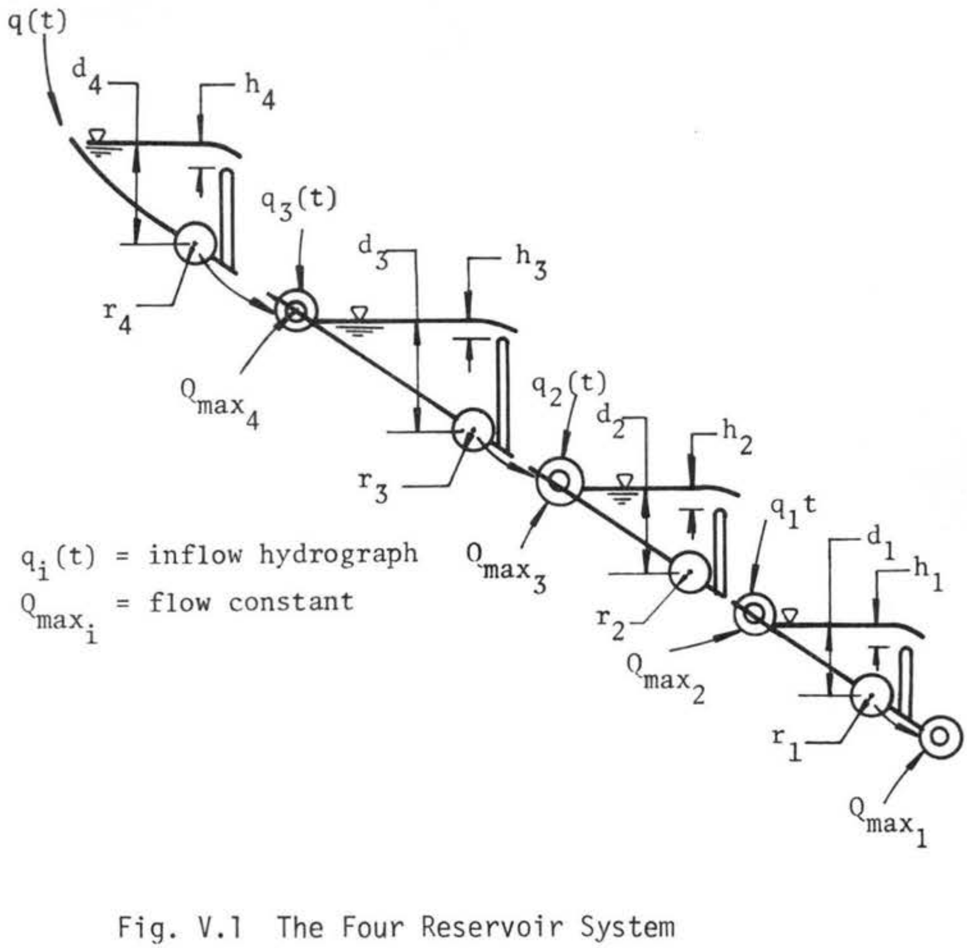


Fig. V.1 The Four Reservoir System

where $Z_1, \ldots, Z_4$ were the overflow weighting factors; and

$$Z_1 - Z_5 > Z_2 - Z_6 > Z_4 - Z_8 > Z_3 - Z_7 \qquad \text{V.2-2}$$

where $Z_5,\ldots,Z_8$ were the throughput weighting factors for reservoirs 1 to 4 respectively. In addition

$$-Z_5 > -Z_6 > -Z_8 > -Z_7 \qquad \text{V.2-3}$$

These were equivalent to the limitations given in Chapter III to prevent the possibility of inadvertently changing the overflow priorities of the system by making the magnitude of a throughput weighting factor too large. Note in this example that the most upstream overflow weighting factor was greater than that for the reservoir directly downstream. In this case the switch in control for reservoir 4 could occur after reservoir 3 started overflowing. It was assumed that

$$R_{min_i} = 0 \qquad (i = 1,\ldots,4) \qquad \text{V.2-4}$$

for all reservoirs.

Twelve trials were made with this program using different input hydrographs, overflow and throughput weighting factors, reservoir sizes, and initial switching times. Typical solution times for fifty time increments were in the order of eight seconds using an IBM model 60-67 computer under the control of an MTS operating system. No attempts were made to decrease these computational times although, as pointed out later in the discussion of the ten reservoir example, there were several obvious ways in which the computational time could have been considerably reduced. The maximum number of iterations required for convergence of any individual control determination was ten. For all cases the convergence was very stable, even when the optimal control required that a reservoirfill but not overflow. (In actual fact, there was always a slight overflow in these cases but it was always the minimum that could be obtained for the given integration step size).

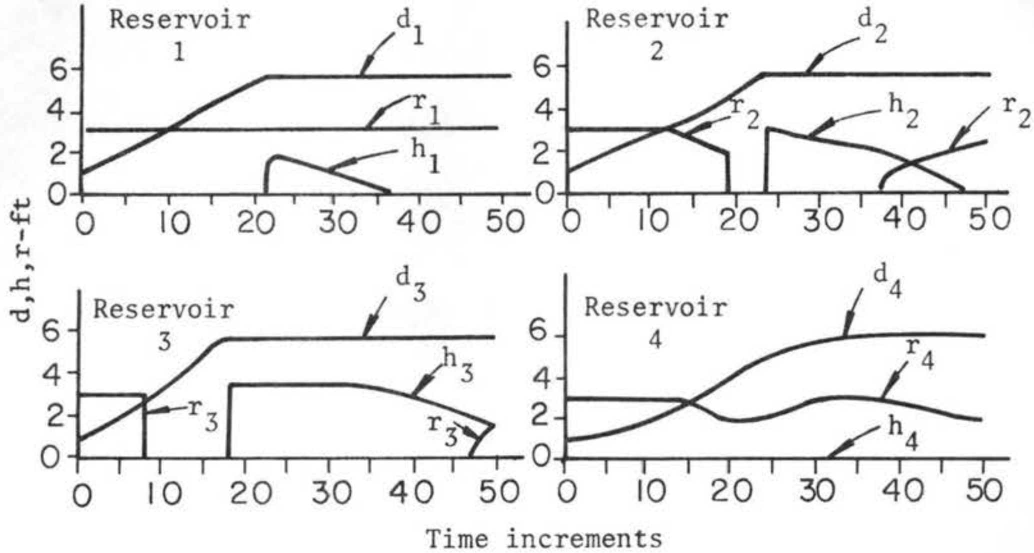


Fig. V.2 Control Strategy for Four Reservoirs in Series with a Small Difference Between Overflow Weighting Factors

Figure V.2 shows the results of a typical optimal control determination. Pertinent data for the example is listed in Table V.1. In this example the overflow weighting factors were such that it was

Table V.1

Data for Example #1

Reservoir Number	1	2	3	4
Overflow Weighting Factor	3.00	2.75	2.50	2.67
Throughput Weighting Factor	-0.30	-0.27	-0.25	-0.26
Maximum Allowable Depth	5.50	5.50	5.75	6.0
Maximum Allowable Orifice Opening	3.0	3.0	3.0	3.0

advantageous to increase the outflow from reservoir 2 so that reservoir 1 spilled, even though reservoir 1 had a higher overflow weighting factor. A similar effect is evident for the flow between reservoirs 2 and 3. It was not advantageous to allow overflow from reservoir 4 and its outflow was the maximum permissable until t equalled 35 units at which point it could begin withholding outflow to the advantage of the downstream reservoirs. Although the inflows to reservoir 3 were still too great to prevent overflow when t equalled 50 units, the remaining reservoirs in the system were being operated to the maximum advantage of reservoir 3 at this time. Eventually as the inflows decreased with time, all the orifices would have opened to their maximum limits.

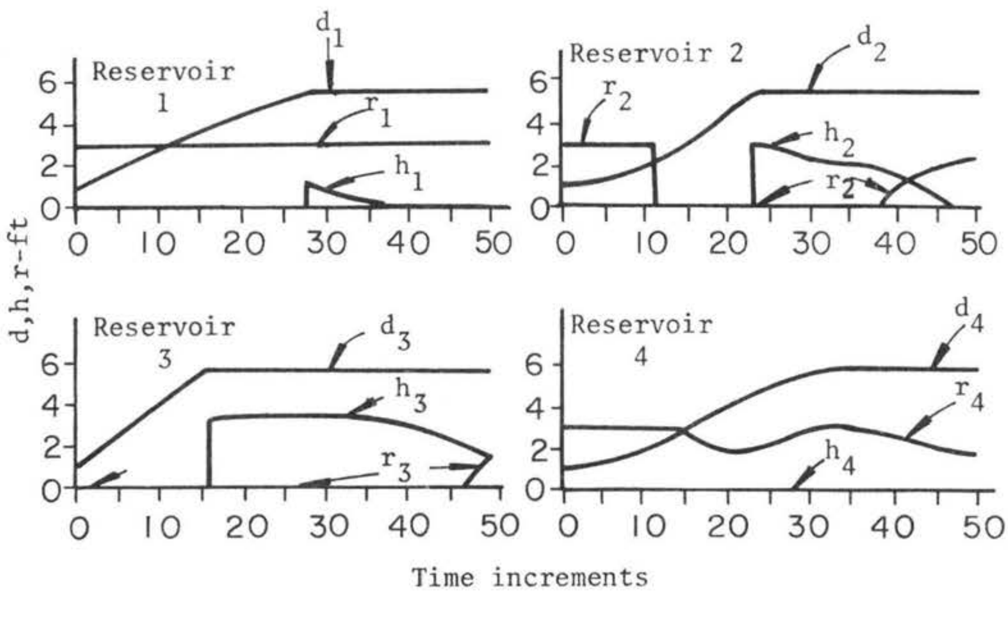


Fig. V.3 Control Strategy for Four Reservoirs in Series with a Large Difference Between Overflow Weighting Factors

Figure V.3 shows the control solution when the differences between the individual weighting factors were increased to the values shown in Table V.2. All other data was the same as in the previous example. In this case it was not advantageous to spill from reservoir 1. The spill that is shown was a result of the numerical inaccuracy. Had orifice 2 closed one time increment sooner, reservoir 1 would

Table V.2

Weighting Factors for Example #2

Reservoir Number	1	2	3	4
Overflow Weighting Factor	6.0	4.0	2.0	3.0
Throughput Weighting Factor	-0.60	-0.40	-0.20	-0.30

not have filled. Because the logic was programmed so that once orifice 2 closed it could only be opened if reservoir 1 filled, the optimal solution required small overflow from reservoir 1. If smaller time steps had been used, this overflow would have been reduced. Note that orifice 3 remained closed for almost the entire time for which the control was computed and as a result filled to capacity much sooner than in the previous example. In this example the spill from reservoirs 2 and 3 was increased over that in the previous example while that from reservoir 1 was reduced, as would be expected.

V.3 The Ten Reservoir Example

A. The System.

As the preliminary results from the four reservoir case indicated that the control logic could be programmed without too much difficulty and that the gradient search technique would find the optimal switching times in a stable manner, experiments were tried using a larger and more complex configuration. This configuration, consisting of ten storage locations, is shown in Figure V.4. Note that it contains as subsets the three reservoir "V" and "Y" configurations, and series configurations. The relative values of the overflow weighting factors for which this program was written are shown in Table V.3.

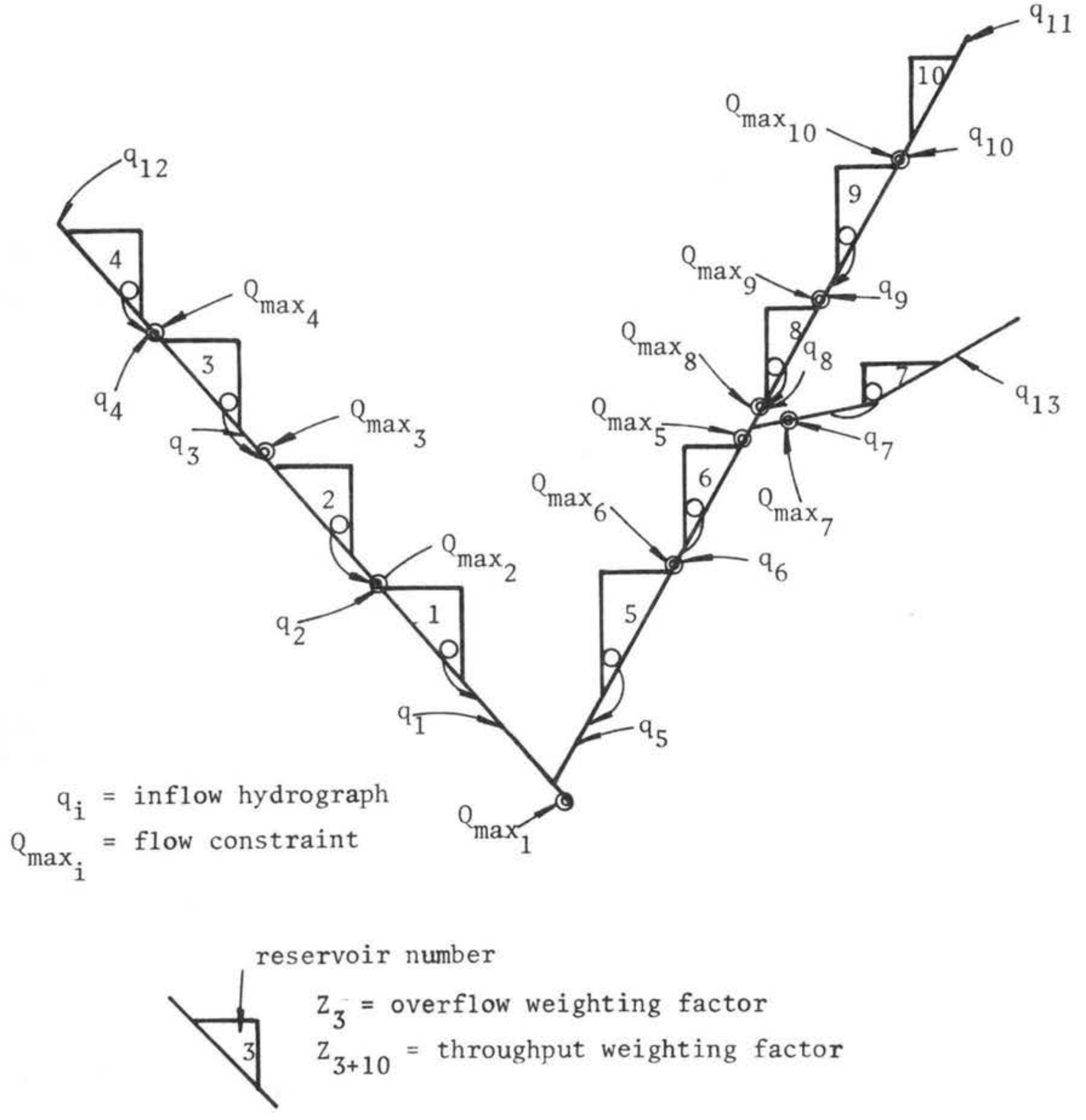


Fig. V.4 The Ten Reservoir System

Table V.3

Relative Values of the Overflow Weighting Factors
for the Ten Reservoir Problem

For left hand branch	$Z_1 > Z_2 > Z_4 > Z_3$
For the right hand branch	$Z_5 > Z_6 > Z_7 > Z_9 > Z_8 > Z_{10}$
For the entire system	$Z_1 > Z_2 > Z_5 > Z_4 > Z_6 > Z_3 > Z_7 > Z_9 > Z_8 > Z_{10}$

As was the case for the previous examples, the absolute values of the throughput weighting factors were made small in comparison to the overflow weighting factors for the individual reservoirs.

Note that the left hand branch is similar to the four reservoir example; however, some of the overflow weighting factors for the right hand branch are larger than some of the overflow weighting factors in the left hand branch.

B. The Control Logic.

Figure V.5 shows a simplified flow diagram of the logic used to determine the control, given the times X_i at which jumps occur in the controls. For reservoirs 1 and 5 which are in a "V" configuration the switching times for reservoirs 1 and 5 were both included to allow more generality. If at any time both controls were specified to be dominant or both subservient then the control for reservoir 1 assumed the dominant position. Reservoirs 7 and 8 which form part of a "Y" configuration each required the specification of a switching time as reservoir 6 downstream could require them to shut down at separate times (recall CASE 6 in Chapter III). Thus a total of ten switching times had to be specified.

The integration of the state equations was accomplished by a

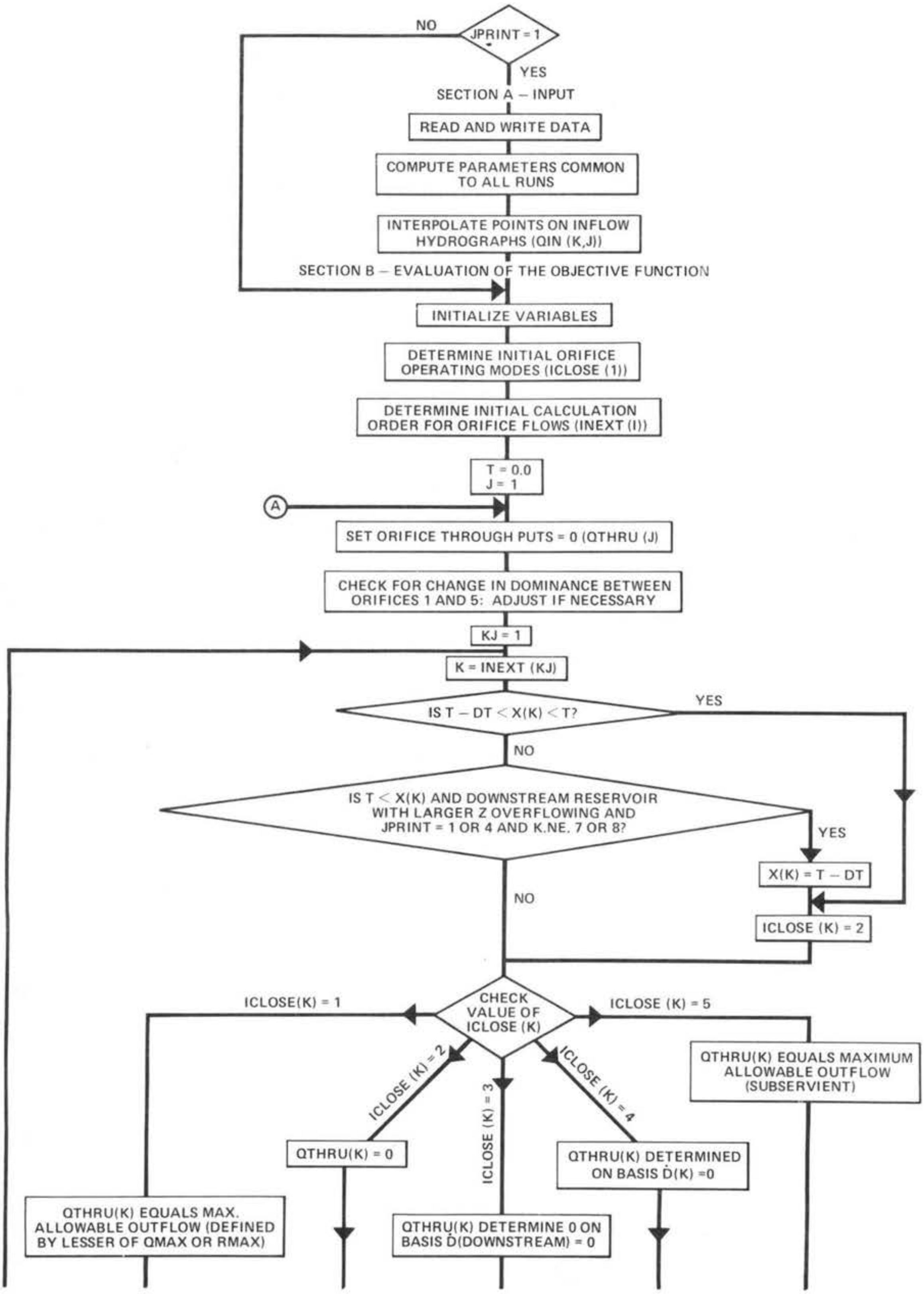


Fig. V.5 Simplified Flow Diagram of Logic Used to Determine Control Strategy for Given X_i

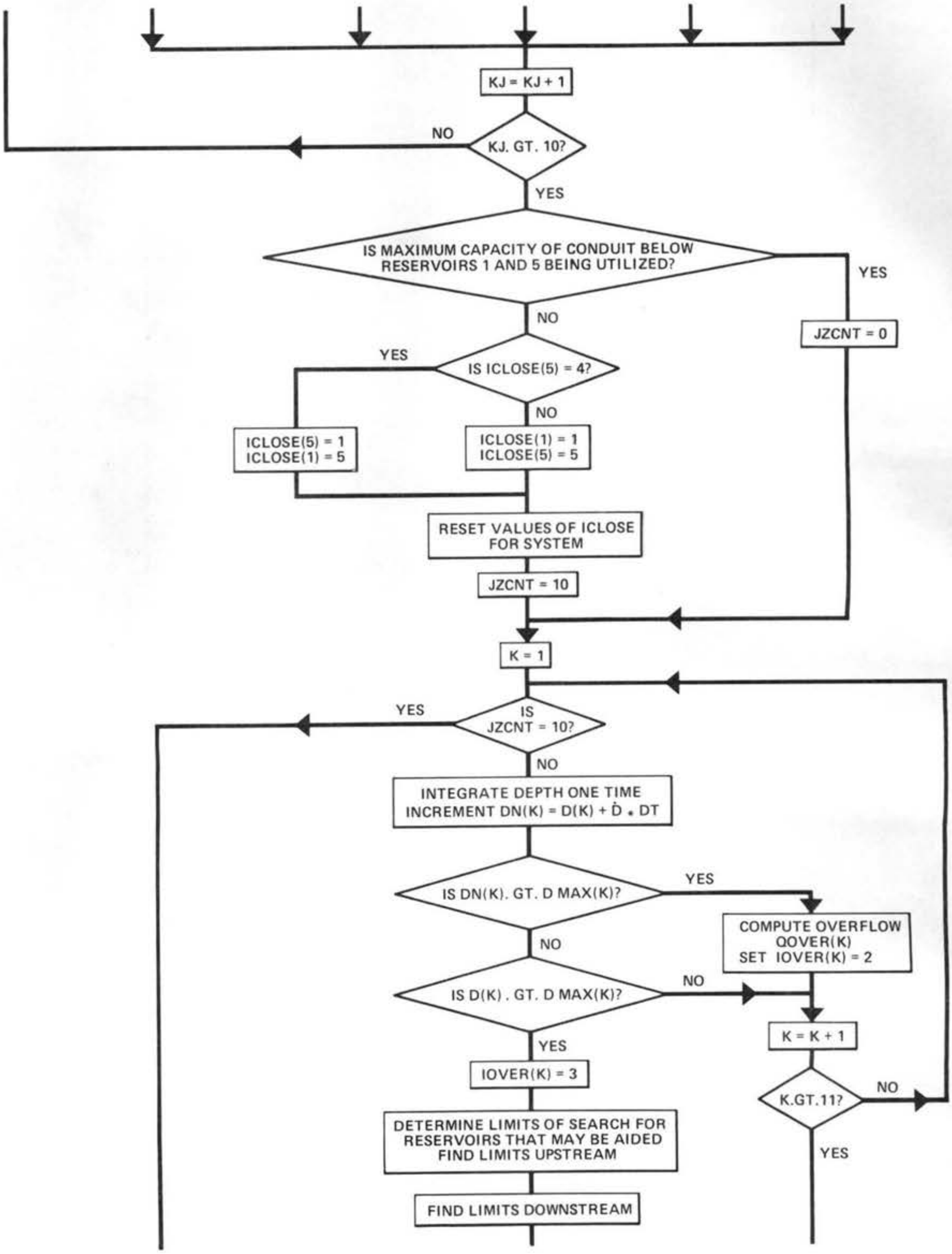


Fig. V.5 continued

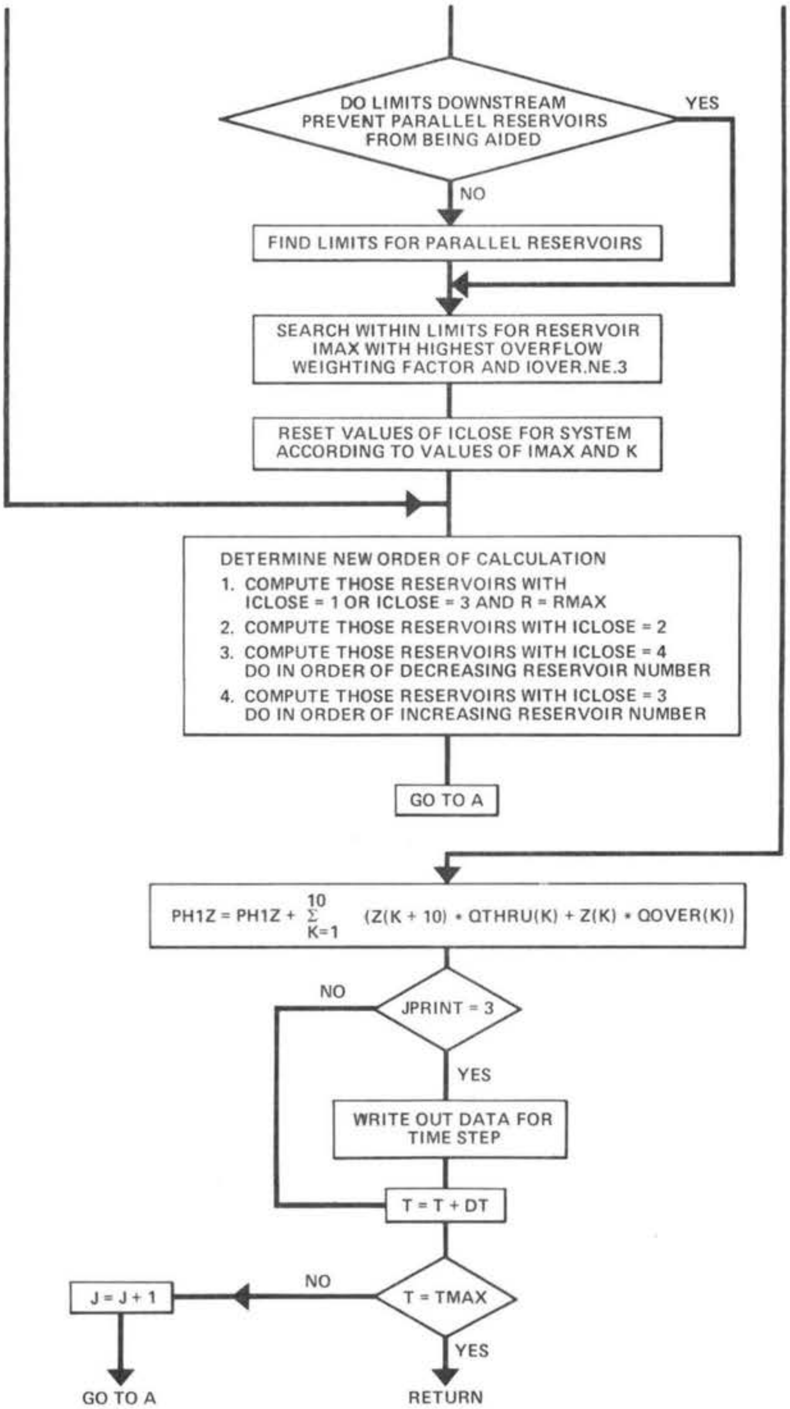


Fig. V.5 continued

first order technique (i.e. the known state and control variables at time T were used to compute the state variables at time T + DT). This meant that the controls at time T first had to be determined. To do this five forms of orifice operation were defined:

type 1 - the orifice was at its maximum allowable opening or at a limit determined by a downstream flow constraint;

type 2 - the orifice was at its minimum opening (which was assumed to be zero in this example);

type 3 - the orifice was controlled by the downstream reservoir in a manner that maintained $\dot{d}_{(\text{downstream})}=0$;

type 4 - the orifice was controlled to maintain $\dot{d} = 0$;

type 5 - the orifice was maintained at the maximum opening that would ensure the full use of the conduit capacity remaining after another parallel reservoir had taken first priority on the flow capacity. This was used for reservoirs 1 and 5.

If the orifice operation was type 1, which was always the case if:

$$T \leq X_i \qquad \text{V.3-1}$$

(where X_i is the time at which a jump in the orifice control for reservoir i could occur) then no change to any other form of operation could occur. This was done to ensure that during the gradient search process any perturbation of X_i would produce a meaningful derivative.

To aid the convergence procedure, and reduce the programming problems, use was made of the fact shown in the examples of Chapter III that if a reservoir was overflowing, and the reservoir upstream had a lower overflow weighting factor, then the upstream orifice would be closed. This lead to inclusion in the program of a section that adjusted the switching times X_i at the start of each iteration in the gradient search process to ensure that such would be the case. (An exception to this adjustment was made in the case of reservoirs seven and eight since the possibility existed that one orifice might not be required to close - see example in Appendix lV).

Given the type of orifice operation, the order of calculation for each reservoir outflow was then determined to ensure that all necessary information was available before a reservoir outflow was calculated. Thus reservoir outflows for orifices with operations type 1, or type 2 were computed first followed by those orifices with operation type 4. The order of calculation for orifices with type 4 operation had to be further refined to progress downstream. Reservoirs with orifice operation type 5 could then be computed followed, finally, by those with type 3 operation which were calculated in order progressing upstream.

With the orifice operation types determined, the throughputs for each reservoir were determined and checks were made to ensure that no constraints were violated. The state equations were then integrated one time step (DT) forward by assuming that all the weir controls were zero. If the new value of d_i for any reservoir i exceeded the limit D_{max_i}, then h_i was increased to ensure that:

$$d_i = D_{max_i}$$

If the old value of d_i was equal to D_{max_i} and the new value of d_i was less than D_{max_i} then reservoir i had stopped overflowing and its orifice operation could be altered, if necessary, to aid some other reservoir in the system. First it was necessary to determine which reservoirs could be aided.(E.g. an upstream reservoir with its orifice already fully open could not be aided, nor could any reservoir further upstream. Likewise reservoirs downstream of a reservoir with its orifice fully open could not be aided unless T was greater than the switching time for that reservoir). Having defined the range of reservoirs whose overflow might possibly be reduced, a search was made among those reservoirs for the one with the highest overflow weighting factor, that had not stopped overflowing. (This included those reservoirs that had not yet begun to overflow). The orifice operations were then adjusted accordingly (e.g., if the reservoir that could be aided was in a parallel branch then the orifices of all reservoirs downstream of reservoir i to the junction of the parallel group of reservoirs were adjusted to type 4 operation. Those parallel reservoirs upstream from the junction and including the reservoir to be aided had their orifice operations changed to type 3). With the change in the orifice operations it was necessary to change the order of calculation to the form outlined earlier. After this step was completed all the throughputs for the time step were recomputed and new integrated values of the state variables determined. At the completion of each time step, the weighted overflows and throughputs for the time step were added to the objective function. The entire process was then repeated successively until the final time limit. At this time the value of the objective function for given values of X_i was known.

A listing of this subroutine (SUBROUTINE PHI) is given in Appendix VI).

C. The Gradient Search Routine.

A complete description of the first order gradient search procedure is given by Wilde and Beightler (1967). A brief outline of this procedure as used to determine the optimal control strategy is given below.

To determine the optimum values of the switching times X_i, a first order gradient search routine was programmed for the computer. For a given set of switching times (X_i^0) this routine first determined the rate of change of the objective function (ϕ) with respect to each X_i. This was accomplished by perturbing each X_i by -DT and computing a new value of the objective function for each perturbation (-DT was used because of the requirement that an orifice close if the reservoir downstream had a higher overflow weighting factor and began to overflow).

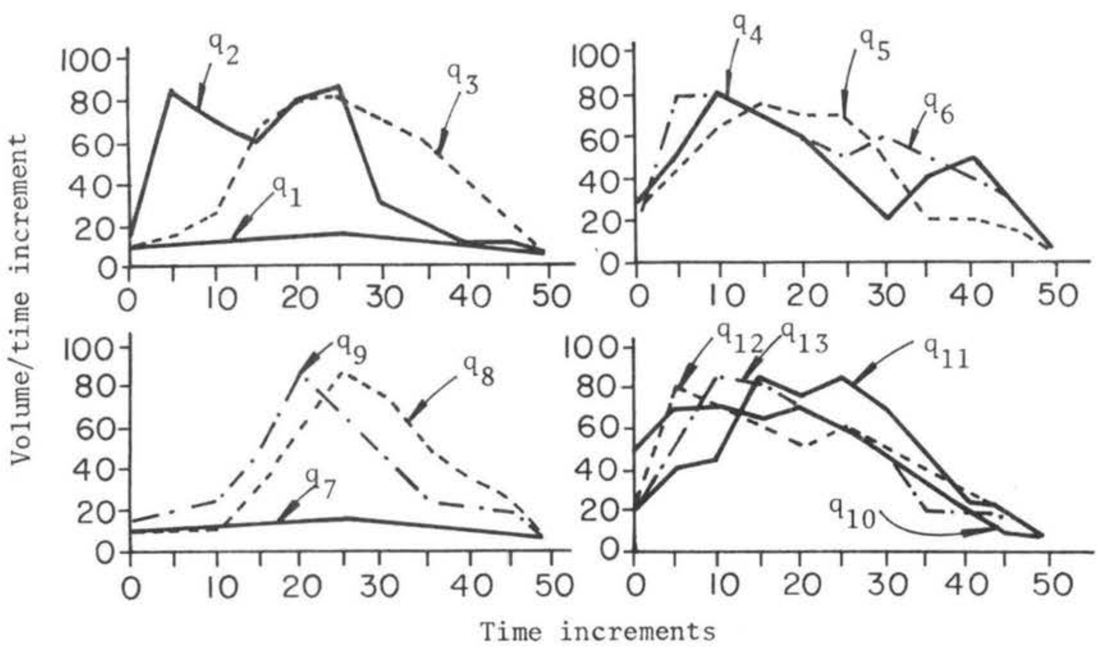


Fig. V.6 Inflow Hydrographs for the Ten Reservoir Example

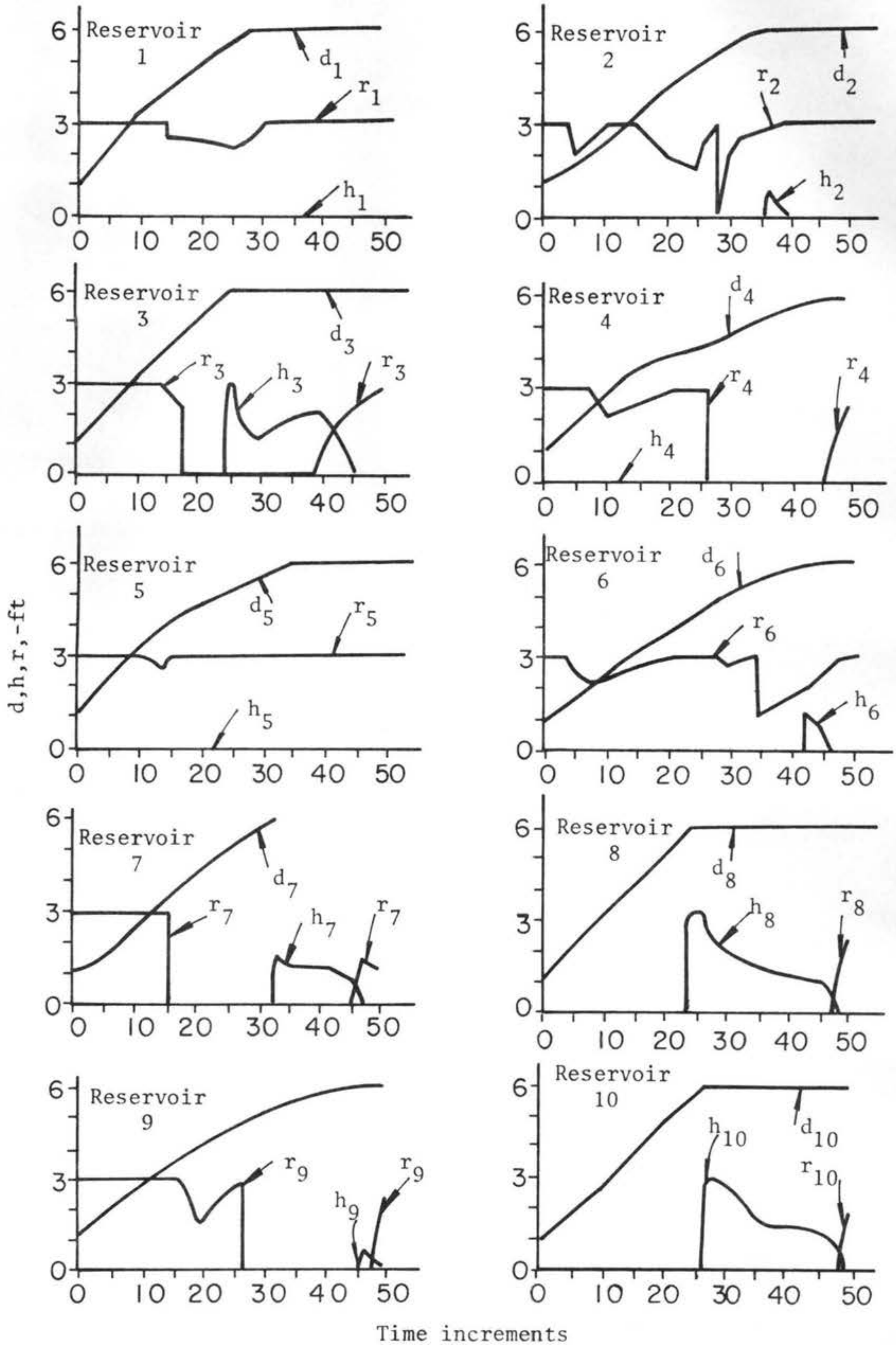


Fig. V.7 The Optimal Control Strategy for the Ten Reservoir Example

For a given reduction Γ in the objective function, this routine then determined new values of X_i (X_i') on the basis of the gradient at X_i^o and checked that the objective function had in fact been reduced. If it happened that the objective function was not reduced, then the amount of the desired reduction, Γ, was decreased until either: a reduction in the objective function was obtained or; Γ had been reduced to a size such that only one of the X_i', say X_i^p, differed from X_i^o by an amount DT. In the latter case, if the objective function had still not been reduced, the derivative of X_i^p was assumed to be zero. The desired reduction Γ was then increased and the iteration was continued with the remaining X_i until either: the objective function was decreased or; until all the derivatives were zero. (Because of the finite time steps, zero in this case meant that $\frac{d\phi}{dX_i}$ changed signs for perturbations either side of each variable X_i).

Several test runs showed that if a derivative was zero during one iteration, then in successive iterations, it was likely to remain so, even though other variables might change by relatively large amounts. As a result, once the derivative of a variable became zero, that variable was dropped from further computations until all derivatives were zero. At this point computations were again made for all the switching times and any final adjustments in the variables X_i were made. The net effect of this change was to reduce the computation time by about one half.

D. Numerical Results.

Figure V.6 shows the input hydrographs and Figure V.7 shows the resulting computed optimal control strategy for the ten reservoir system. The necessary constants for each individual reservoir are shown in Table V.4. All the flow constraints (Q_{max_i}) were set at 95 cfs except Q_{max_1}, which was set at 165 cfs and Q_{max_5}, which was set at 195 cfs.

Table V.4

Data for the Ten Reservoir Example

Reservoir No:	1	2	3	4	5	6	7	8	9	10
Overflow Weighting Factor	10.0	9.30	6.50	7.90	8.60	7.20	5.80	4.40	5.10	3.70
Throughput Weighting Factor	-.060	-.059	-.055	-.057	-.058	-.056	-.054	-.052	-.053	-.051
C_D	2.50	2.00	2.00	2.00	2.50	2.00	2.00	2.00	2.00	2.00
C_W	15.0	15.0	15.0	15.0	15.0	15.0	15.0	15.0	15.0	15.0
Area	80+50d	80+50d	50+80d	50+80d	50+80d	50+80d	50+80d	50+80d	50+80d	50+80d

The results showed that the overflows from reservoirs 1, 2, 4, 5, 6 and 9 were as close to zero as the numerical accuracy of the program would allow. The overflow from the system occurred from reservoirs 3, 7, 8 and 10. At first glance it appeared that the overflow from reservoir 3 could be reduced at the expense of reservoir 7, which had a lower overflow weighting factor, by maintaining orifice 1 in the dominant position for a longer time and thus increasing the flow through reservoirs 1, 2 and 3. In fact this would probably have been the case if shorter time steps were used; however, any attempts to improve on this control were offset by the numerical accuracy of the program. It appeared that the objective function might have been reduced by about 50 units if the control had been exact, which was a small amount when compared to the total reduction in the objective function for this run of about 16,500 units. Note that all reservoirs in the system filled and remained filled until all system overflow had ceased; thus any effects of decreasing the spill from one reservoir would have been at the expense of overflowing from another reservoir in the system. The net effect would therefore have been equal to the reduction in overflow volume from the first reservoir times the difference between the overflow weighting factors of the two reservoirs.

It is worth noting in this example that the inflow q_4 to reservoir 3 was not monotonic decreasing after reservoir 3 stopped overflowing. Because reservoir 4 was completely shut down while q_4 was increasing, this did not lead to a sub-optimal control.

The above example was typical of the type of control strategies obtained. Other examples showing the results of different input hydrographs, flow constraints and weighting factors are shown in Appendices IV and V.

E. Operating Experiences and Computational Times for the Ten Reservoir Control Program.

The only difficulties experienced with the control program were the sub-optimality that could be attributed to numerical accuracy and certain cases of sub-optimality that could be attributed to the use of a first-order gradient search technique. These latter cases, invariably occurred when the relative values of the overflow weighting factors between two reservoirs in series were such that the optimal control required that the downstream reservoir fill but not overflow. When this condition was reached during the iterative process, any perturbation of the upstream switching time alone would result in an increase in the objective function. If the switching time were decreased there would be increased overflow for the upstream reservoir and incomplete utilization of the storage downstream.. If the switching time were increased overflow would occur downstream at a higher penalty than if it had occurred upstream. Had higher order derivatives been used, the simultaneous movement of several controls might have resulted in an overall reduction of the objective function. The final control obtained in these cases, although sub-optimal, still resulted in the full use of the system storage capacity and the majority of the overflow occurred from the reservoirs with the lowest overflow weighting factors; however, the distribution of overflow between those reservoirs that overflowed could have been improved. The example shown in Figure V.7 shows this effect although in this case numerical accuracy was the over-riding factor.

The normal operating times for the ten reservoir control program for 50 time steps were about 50 seconds using an IBM Model 60-67 computer with an MTS operating system. The maximum time for any run was 58 seconds. The core storage required by the gradient search routine and the control program was a total of 22,406 bytes (Approximately 5,500 words).

As mentioned previously the computational time could be significantly reduced. The reduction obtained by modifications to the gradient search routine have previously been discussed. Further reductions in the computational time could have been obtained by:

a) revising the procedure for calculating the numerical derivatives. Each derivative calculation required operation of the control program from time zero. A reduction of 30-40% in computational time could have been obtained by storing the results up to each switching time when computing the base value of the objective function. Computation of the perturbed values of the objective function could have then been computed for only the time span after the base value switch in control occurred.

b) using longer time steps during the initial stages of the optimization process. Only during the latter stages of the optimization was it necessary to compute the control to the final accuracy desired. Tests indicated that this would have lead to a further reduction in computation time of about 30%.

c) increasing the convergence tolerance. For the results presented herein, the convergence criterion was that all derivatives with respect to the switching times be zero (i.e. a perturbation either way from the base value of the switching times resulted in a change in sign of the derivative). Typically a first order gradient technique will get close to the optimum very rapidly and then its approach is very slow. For example, in obtaining the results shown in Figure V.7, 80% of the reduction in the objective function was obtained in the first 10 iterations. Another 10 iterations were required to reach the minimum value.

Thus some further saving in computational time could have been obtained by relaxing the convergence criterion.

Although optimization techniques other than the first order gradient technique may have been more efficient, the results obtained using the first order gradient technique showed that the computational times would be within the limits required for real-time operation. Further examination of optimization techniques was beyond the scope of this study.

The computational time required for systems larger than 10 reservoirs can only be estimated. Indications are that the computational time would increase in the order of the 2.5 power of the number of orifice controls. Thus if no improvements are made in the operating efficiency of the optimization process, the computational time for 50 time steps for a system having 20 orifice controls would be in the order of 280 seconds. With the suggested operating improvements this could be reduced to 140 seconds for a system of 20 reservoirs. This is about the upper limit of the time available for real-time control determination.

F. Development of a More General Control Program.

The control program described above was initially designed to determine the control for the specific configuration and relative values of the overflow weighting factors discussed at the beginning of this section. As the program was developed, it appeared that it could be modified to be more flexible in the choice of the relative values of the weighting factors. The flow chart shown in Figure V.5 is that of the revised program. For the given reservoir configuration it is believed to be completely general with respect to the relative values of the overflow weighting factors except for the restriction that Z_7 be greater than Z_8 (assuming that the throughput factors are very small in comparison). This former restriction was not removed because the manner in which the original program was set up was not amenable to simple change. Some results using relative values of the overflow weighting factors other than those listed in Table V.3 are given in Appendix V.

Although no attempt was made to write a control program which would be capable of determining the control for any given configuration of reservoirs, there do not appear to be any serious difficulties preventing the writing of such a program. The greatest difficulty would seem to be in determining the dominance and subservience of the orifice controls for more than two reservoirs in parallel. This condition, which has not been investigated in this study should be examined before any attempt is made to write a more general control program.

V.4 An Example of the Effect of Information Errors on Control Optimality

The results discussed below are presented in more detail in a paper by Bell, Johnson and Winn (1973). In this part of the study an attempt was made to gain some insight into the effects of information errors on the accuracy of the computed control.

To determine the effects of information errors it was necessary to develop a model of a real-time automated control system (RTACS). This development is discussed by Bell, Winn and Smith (1972). The flow chart of the RTACS model is shown in Figure V.8. The information errors considered were those resulting from the regeneration of a time varying field of rainfall data from point rainfall data. The prediction model for rainfall runoff in the control algorithm and the rainfall runoff model in the physical system were identical.

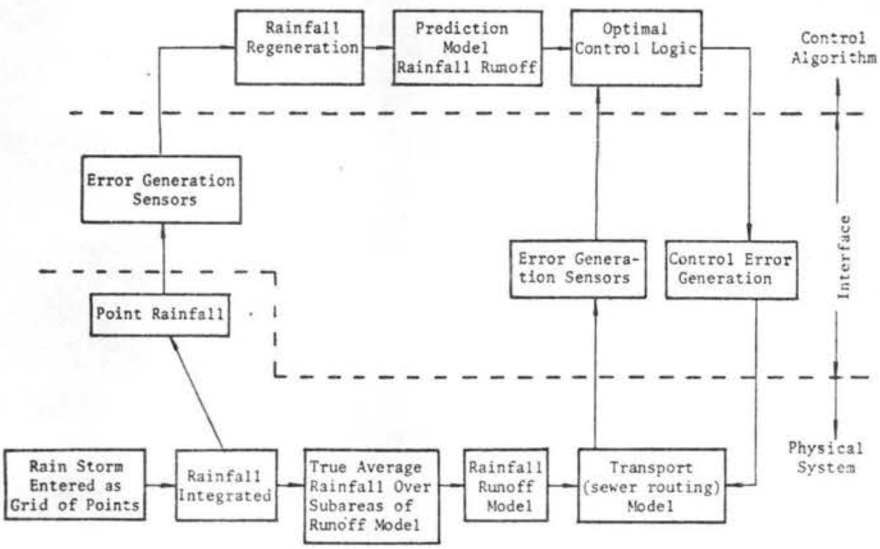


Fig. V.8 Elements of the RTACS Model

The basin analyzed is shown in Figure V.9. The sewer system draining this basin has three flow control points and is analogous to the three reservoir "V" configuration discussed in Chapter III. The conduits between reservoirs are short enough that the time delays in the flow can be neglected.

The assumed rainstorm shown in Figure V.10 travelled westward across the basin at a rate of 250 feet per minute. The rainfall intensity was assumed to be constant in the north-south direction and the western edge of the storm was assumed to be on the eastern edge of the catchment at t = 0. It took 50 minutes for the storm to pass completely over the basin.

The rainfall was recorded at raingauge B on Figure V.9. To determine the rainfall to be used for input to the prediction model for rainfall runoff in the control algorithm, two rainfall regeneration models were used. The first model assumed that the point rainfall recorded at raingauge B was constant over the entire basin. The second model assumed that the rainfall intensity recorded at raingauge B would occur over each of the subcatchments at time $t + T_i$ where T_i

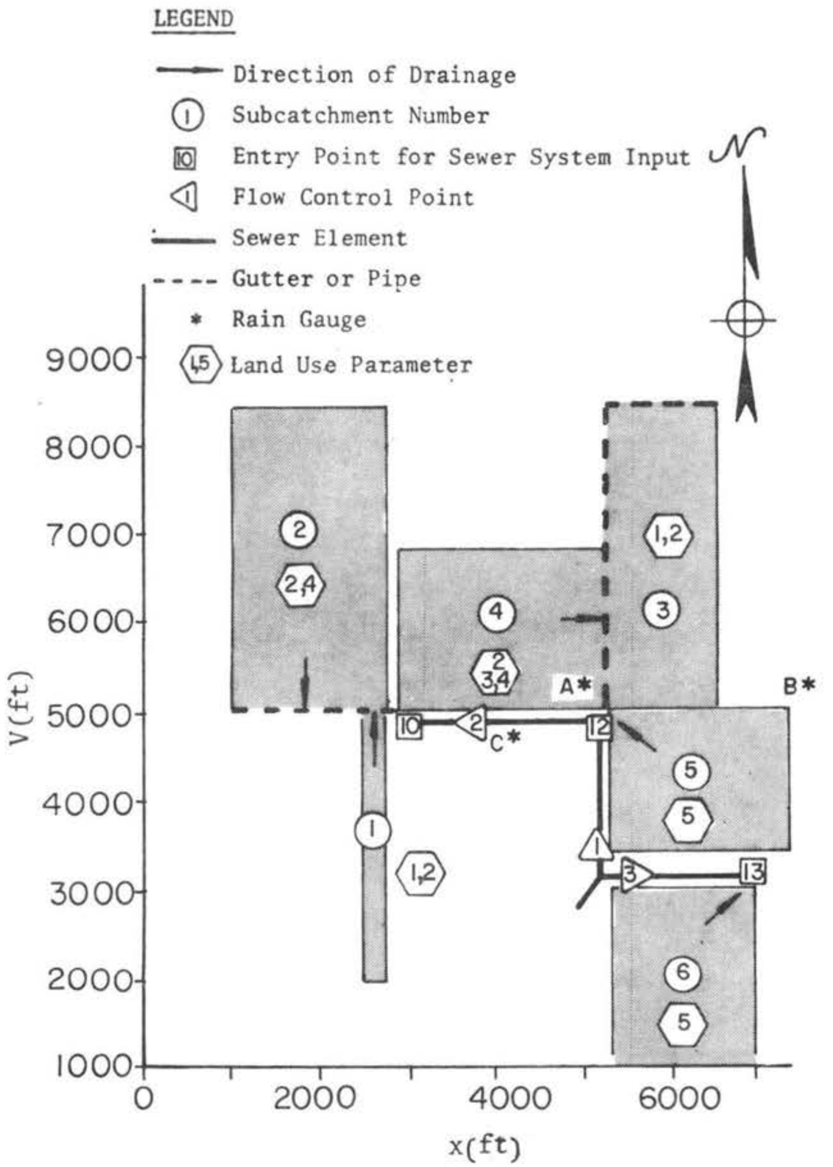


Fig. V.9 The Physical System

was defined for each subcatchment as the time for the storm to move from over the raingauge to the center of subcatchment i. These time delays are listed in Table V.5. The resultant input hydrographs to the control model resulting from these regeneration models are shown in Figure V.11. The true runoff input to the transport (sewer routing) model is shown in Figure V.12. It is the sum of the dry weather flow plus the runoff computed using the true average rainfall over the subareas of the runoff model. The true runoff was used in all tests as input to the transport model.

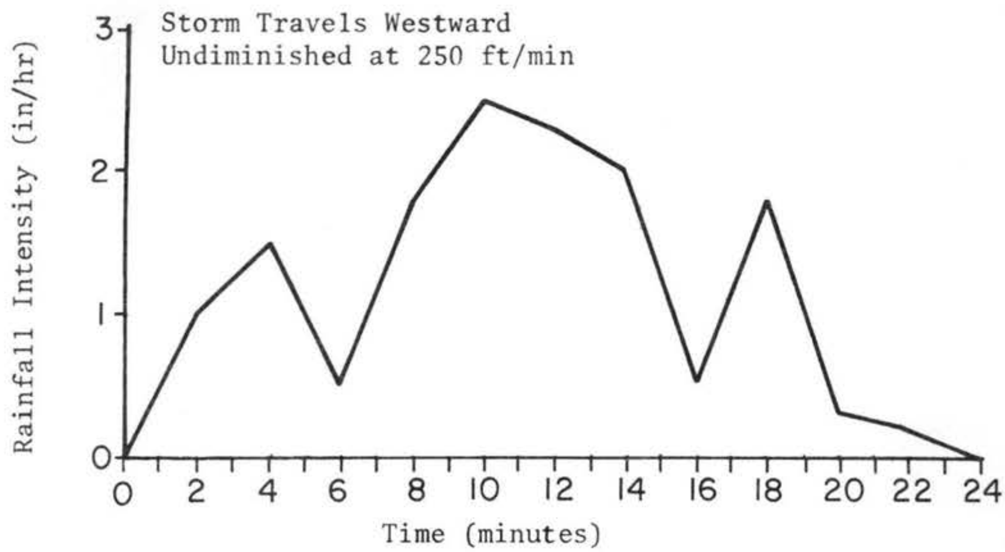


Fig. V.10 The Assumed Rain Storm

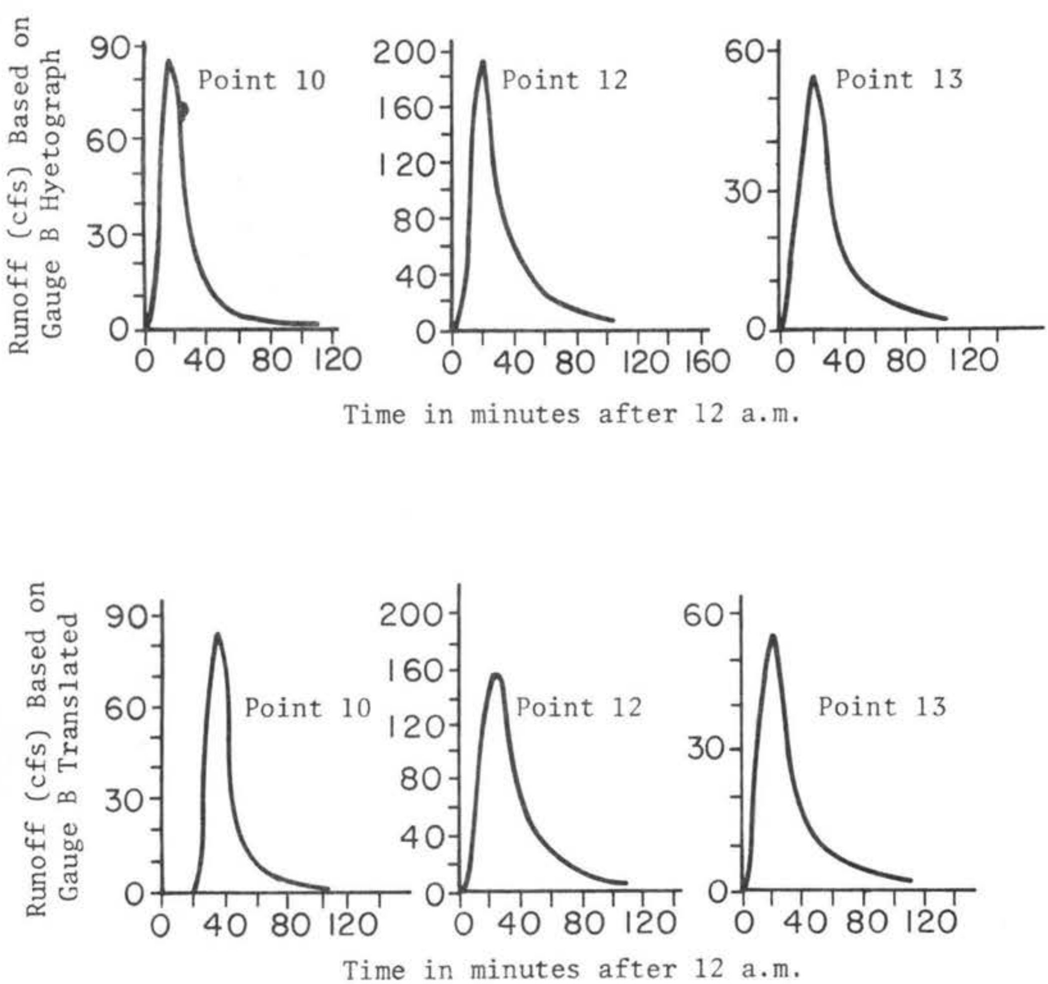


Fig. V.11 The Computed Runoff Used for Control

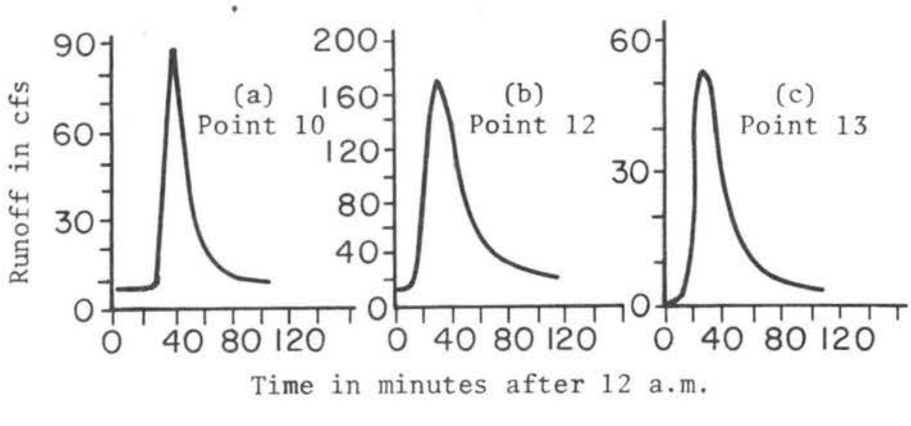


Fig. V.12 The True Runoff Used for Control

Table V.5

Time Delays for Rainfall Regeneration Based on Rainfall at Gauge B

Subcatchment Number	Distance of Subcatchment from Rain Gauge B (ft)	Time Delay T_i (min)
1	5350	22
2	6100	24
3	2100	8
4	3900	16
5	1700	6
6	1900	8

Three tests were made. In the first test the optimal control was computed using the true runoff and then this control was used to operate the controls in the transport model. The second test used the runoff computed from raingauge B data (untranslated) as input to the optimal control logic. Again the computed control was used to operate the transport model. The final test followed the same procedure using the runoff computed from translated raingauge B data. In each case the optimal control was determined using a steepest ascent technique.

Table V.6 shows a comparison of the results predicted by the control program and the actual results obtained when the control was used in the transport model for each of the cases examined.

Table V.6

Comparison of the Effects of Information Errors on Control Strategies

Optimal Control Based on True Runoff	Control Program	Physical System Model
Maximum depth at Pt 2	13.00 ft	13.02 ft
Maximum depth at Pt 1	8.50 ft	8.60 ft
Maximum depth at Pt 3	8.50 ft	8.59 ft
Total Overflow Volume at Pt 2	26,000 ft^3	27,400 ft^3
Total Overflow Volume at Pt 1	2,740 ft^3	3,560 ft^3

Table V.5 continued

Total Overflow Volume at Pt 3	18,200 ft^3	19,400 ft^3
Maximum Outflow Pt 2 & Pt 3	92 cfs	92 cfs
Maximum Outflow Pt 1	16.7 cfs	17 cfs
Optimal Control Based on Rain Gauge B Data		
Maximum Depth at Pt 2	13.00 ft	14.12 ft
Maximum Depth at Pt 1	8.50 ft	9.08 ft
Maximum Depth at Pt 3	8.50 ft	8.96 ft
Total Overflow Volume at Pt 2	7,650 ft^3	30,600 ft^3
Total Overflow Volume at Pt 1	0 ft^3	12,800 ft^3
Total Overflow Volume at Pt 3	23,700 ft^3	18,100 ft^3
Maximum Outflow Pt 1 & Pt 3	92 cfs	94.5 cfs
Maximum Outflow Pt 2	16.7 cfs	21 cfs
Optimal Control Based on Rain Gauge B Data Translated		
Maximum Depth at Pt 2	13.00 ft	13.64 ft
Maximum Depth at Pt 1	8.17 ft	8.73 ft
Maximum Depth at Pt 3	8.50 ft	8.87 ft
Total Overflow Volume at Pt 2	370 ft^3	14,700 ft^3
Total Overflow Volume at Pt 1	0 ft^3	3,580 ft^3
Total Overflow Volume at Pt 3	15,100 ft^3	16,700 ft^3
Maximum Outflow Pt 1 & Pt 3	92.0 cfs	95 cfs
Maximum Outflow Pt 2	16.7 cfs	20.4 cfs

Considering the differences between the mathematical model in the control logic and the transport model, the agreement obtained between the two models when the true runoff was used for control is surprisingly good.

Although the control using data from raingauge B translated produced the lowest overflow volumes, it did so at the expense of violating all the depth and flow constraints. This resulted from the fact that the control runoff model did not include the dry weather flow and thus produced hydrographs that were lower than the true hydrographs. Therefore all the orifices were maintained at their full opening and the weirs at control points 1 and 2 were maintained at their maximum level.

The control using the untranslated raingauge B data resulted in the greatest system overflows because the timing of the runoff peaks was generally early. This resulted in all the orifice and weir adjustments being made too early. The failure to include dry weather flow compounded the problem but timing was the main problem.

Although these tests are far from definitive they do indicate that information errors resulting in mistiming of the input hydrographs to the control logic, or in underestimation of the flows, may result in considerable deviations from true optimality for any computed control. The situation represented by this example was highly simplified and many more sources of information error exist within the system. In addition, the control in this instance was computed after the fact i.e., it assumed that complete knowledge of the storm was available for the control program. In actual operation some inference as to the course of the storm must be made each time the control is updated.

CHAPTER VI

MODIFICATIONS TO THE VARIATIONAL FORMULATION TO INCLUDE MORE REALISTIC FLOW ROUTING MODELS

VI.1 Introduction

To this point in the study, it has been assumed that the flow leaving one reservoir appears instantly at a reservoir downstream or at the junction of parallel reservoirs. As shown in CHAPTER V, this assumption is valid for relatively short conduits; however, conduit lengths up to three miles are not uncommon in combined sewer systems in major urban centers. In these cases, the downstream reservoir observes a time delay, as well as modification of the flow regime in the flow that leaves the upstream reservoir.

This chapter presents the modified Euler-Lagrange equations applicable to problem formulations including time delay. This is followed by a discussion of some of the flow routing models suitable for combined sewers. Finally, to illustrate the effects of time delay on the optimal control an example of two reservoirs in series is presented.

VI.2 The Modified Euler-Lagrange Equations

To date there is very little discussion in the literature of the modifications necessary to the Euler-Lagrange equations to include time delay. Hughes (1968) discussed the variational problem with time delay but did not consider side conditions (constraints). Pontryagin et al.

(1965) discusses briefly the solution of problems with time delay using the maximum principle. EL'SGOL'C (1960) presented the modified Euler-Lagrange equations without discussion. It does not appear that anyone has given a complete discussion of the variational problem with time delay, which includes differential constraints, state and control variable inequality constraints, corner and end conditions. Useful information can be obtained by applying the modified Euler-Lagrange equations presented by EL'SGOL'C to a variational formulation of the control problem which includes time delay.

The modified Euler-Lagrange equations as presented by EL'SGOL'C are

$$\left(F'x_i(s) - \frac{d}{ds} F'\dot{x}_i(s)\right)_{s=t} + \left(F'x_i(s-\tau) - \frac{d}{ds} \dot{x}_i(s-\tau)\right)_{s=t+\tau} = 0$$

$$(i = 1,\ldots,n) \qquad \text{IV.1-1}$$

where: $F'x(s)$ is the first derivative of the augmented index performance F with respect to $x(s)$

τ is the time delay

s is a dummy variable

The terms in the first set of brackets are the normal form of the Euler-Lagrange equation. The terms in the second set of brackets are the modifications due to time delay and would be repeated if more than one time delay was applicable.

VI.3 Flow Routing Models Suitable for Flow in Combined Sewers

This discussion will be limited to two possible flow routing models suggested in the literature as suitable for flow in combined sewers although there are undoubtedly other models that may be suitable.

A. The Muskingum Routing Method.

This method is discussed by Lawler (1964). It has been proposed for use in routing flow in combined sewers by Labadie (1973) and Grigg et al. (1973).

A simplified form of the Muskingum equation given by Lawler is

$$O_2 = C_1 I_2 + C_2 I_1 + C_3 O_1 \qquad \text{VI.3-1}$$

where: I_1 and I_2 are the inflows to the conduit at times t_1 and t_2 respectively $(t_2 > t_1)$;

O_1 and O_2 are the outflows from the conduit at times t_1 and t_2;

C_1, C_2, C_3 are empirical coefficients.

The more general form of the Muskingum equation used by Labadie and Grigg et al includes only the above variables but allows a more general relationship between them. For simplicity of discussion equation VI.3-1 will be used herein.

B. The Progressive Average Lag Technique.

Harris (1968c) has suggested the use of this technique for routing flows in a model of the combined sewer system of Minneapolis - St. Paul. In tests of his model he was able to obtain good agreement between flows routed using the progressive average lag technique and the same flows routed using the method of characteristics which is generally considered to be the most accurate method of flow routing. The progressive average lag method as presented by Lawler (1964) has the basic equation:

$$O_t = \frac{1}{n}\left(I_{t-\tau} + I_{t-\tau-dt} + I_{t-\tau-2dt} + \ldots + I_{t-\tau-(n-1)dt}\right) \qquad \text{VI.3-2}$$

where O_t is the outflow from the conduit at time t;

$I_{t-\tau} \ldots I_{t-\tau-(n-1)dt}$ are the inflows to the conduit at times $t-\tau \ldots . t-\tau-(n-1)dt$;

τ is an empirical time lag;

n is the number of inflows averaged.

Both the Muskingum and Progressive average lag techniques were originally developed for use in flood routing in river channels. Both techniques apply only to free surface flow and both could not be expected to be accurate under conditions of rapidly varied flow (e.g. surges).

VI.4 The Two Reservoir Problem with Time Delay

To demonstrate the effects of time delay on the optimal control strategy, the two reservoir problem with time delay is formulated below and some facets of the effect of time delay examined.

A. Formulation of the Two Reservoir Problem with Time Delay.

This system is the same as that shown in Figure II-6 except that it is now assumed that the flow leaving reservoir 2 is delayed and modified before it reaches reservoir 1.

The differential constraints of the system are

$$\dot{d}_1 = (q_1 + \dot{v}_3 - C_{D_1}\sqrt{d_1}\, r_1^{\,2} - C_{W_1} h_1^{\,3/2})/A_1(d_1) = f_1 \qquad \text{IV.4-1}$$

$$\dot{d}_2 = (q_2 - C_{D_2}\sqrt{d_2}r_2{}^2 - C_{W_2}h_2{}^{3/2})/A_2(d_2) = f_2 \qquad \text{VI.4-2}$$

$$\dot{v}_3 = f\{C_{D_2}\sqrt{d_2(t-B)}[r_2(t-B)]^2,\ C_{D_2}\sqrt{d_2(t-C)}[r_2(t-C)]^2,\ \dot{v}_3(t-E)\} = f_3 \qquad \text{VI.4-3}$$

Equation VI.4-1 is similar to equation II.7-1 except that the outflow from the upstream reservoir in the latter equation has been replaced by $\dot{v}_3$. Equation VI.4-2 is identical to equation II.7-2. The last equation VI.4-3 represents the inflow to reservoir 1 ($\dot{v}_3$) as a function of the inflow $\dot{v}_3$, E time units earlier and the outflows from reservoir 2, B and C time units earlier. Note that if

$$E = C \qquad \text{VI.4-3}$$

and

$$B = 0 \qquad \text{VI.4-4}$$

the above formulation is equivalent to the use of a muskingum routing. If $\dot{v}_3$ is eliminated from the right hand side of equation VI.4-3 the formulation is equivalent to a progressive average lag routing with $n = 2$.

The control variable inequality constraints for the problem are:

$$(r_i - R_{min_i})(r_i - R_{max_i}) \leq 0 \qquad i = 1, 2 \qquad \text{VI.4-5}$$

$$h_i(h_i - d_i) \leq 0 \qquad i = 1, 2 \qquad \text{VI.4-6}$$

$$C_{D_1}\sqrt{d_1}r_1{}^2 + q_3(t) - Q_{max_3} \leq 0 \qquad \text{VI.4-7}$$

and

$$\dot{v}_3 + q_1(t) - Q_{max_2} \leq 0 \qquad \text{VI.4-8}$$

Equations VI.4-5, - VI.4-7 are identical to equations II.7-3 - II.7-6 and II.7-8 respectively. Equation VI.4-8 is a modified form of equation II.7-7 in which $\dot{v}_3$ represents the delayed output from reservoir 2.

The state variable inequality constraints are:

$$(d_i - D_{max_i})d_i \leq 0 \; . \qquad i = 1,2 \qquad \text{VI.4-9}$$

Using the same objective function given by equation II.7-13 and adjoining the differential and inequality constraints in the usual manner yields as an augmented index of performance

$$\begin{aligned}
\text{Min}\phi = {} & \bar{\gamma}_1 d_1(d_1-D_1) + \bar{\gamma}_2 d_2(d_2-D_2) \\
& + \int_0^{t_f} \{Z_1 C_{W_1} h_1^{3/2} + Z_2 C_{W_2} h_2^{3/2} + Z_3 C_{D_1}\sqrt{d_1}\, r_1^2 \\
& + Z_4 C_{D_2}\sqrt{d_2}\, r_2^2\} dt + \int_0^{t_f} \{\lambda_1(\dot{d}_1-f_1) + \lambda_2(\dot{d}_2-f_2) + \lambda_3(\dot{\nu}_3-f_3) \\
& + \pi_1(r_1-R_{max_1})(r_1-R_{min_1}) + \pi_2(r_2-R_{max_2})(r_2-R_{min_2}) \\
& + \pi_3(h_1)(h_1-d_1) + \pi_4(h_2)(h_2-d_2) \\
& + \pi_5(C_{D_1}\sqrt{d_1}\, r_1^2 + q_3(t) - Q_{max_3}) + \pi_6(\dot{\nu}_3 + q(t) - Q_{max_2}) \\
& + \gamma_1(2d_1-D_1)f_1 + \gamma_2(2d_2-D_2)f_2\} dt \qquad \text{VI.4-10}
\end{aligned}$$

In the above f_1, f_2 and f_3 represent the right hand sides of equations VI.4-1, VI.4-2 and VI.4-3 respectively. There are 7 variables(d_1, d_2, r_1, r_2 h_1, h_2 and ν_3) in equation VI.4-10 and therefore the application of the modified Euler-Lagrange equation (VI.2-1) should yield 7 equations (3 adjoint equations and 4 control equations).

B. The Necessary Conditions for the Problem

The resultant control equations for this example upon taking derivatives with respect to r_1, r_2, h_1 and h_2 become:

$$\{Z_3 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)} + \pi_5\} 2C_{D_1} \sqrt{d_1}\, r_1 + \pi_1(2r_1 - R_{max_1} - R_{min_1}) = 0 \qquad \text{VI.4-11}$$

which is identical in form to equation III.3-5;

$$\{Z_4 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}\} 2C_{D_2} \sqrt{d_2}\, r_2 + \pi_2(2r_2 - R_{max_2} - R_{min_2}) - [\lambda_3(s) \frac{d\, f_3(r_2(s-B))}{dr_2(s-B)}]_{s=t+B} - [\lambda_3(s) \frac{d\, f_3(r_2(s-C))}{dr_2(s-C)}]_{s=t+C} = 0 \qquad \text{VI.4-12}$$

which bears some resemblance to equation III.3-6 but which requires further modification;

$$\{Z_1 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}\} 3/2\, C_{W_1} h_1^{1/2} + \pi_3(2h_1 - d_1) = 0 \qquad \text{VI.4-13}$$

which is identical in form to equation III.3-3; and

$$\{Z_2 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}\} 3/2C_{W_2} h_2^{1/2} + \pi_4(2h_2 - d_2) = 0 \qquad \text{VI.4-14}$$

As would be expected the only control equation to differ from those in the example in Chapter III is equation VI.4-12.

Taking the derivatives with respect to the state variables d_1, d_2 and ν_3 yields the adjoint equations for the problem which are:

$$\{Z_3 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)} + \pi_5\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1}{dd_1} - \pi_3 h_1 = \frac{d\lambda_1}{dt}$$

VI.4-15

$$\{Z_4 + \frac{\lambda_2-\gamma_2(2d_2-D_2)}{A_2(d_2)}\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2}{dd_2} - \pi_4(h_2)$$

$$-[\lambda_3(s) \frac{df_3(d_2(s-B))}{dd_2(s-B)}]_{s=t+B} - [\lambda_3(s) \frac{df_3(d_2(s-C))}{dd_2(s-C)}]_{s=t+C} = \frac{d\lambda_2}{dt}$$

VI.4-16

and

$$-\frac{d}{dt}[\pi_6 - \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}] - [\frac{d}{ds}\{\frac{df_3(\dot{v}_3(s-E))}{d\dot{v}_3(s-E)}\}]_{s=t+E} = \frac{d\lambda_3}{dt}$$

VI.4-17

By substituting the results of either equations VI.3-1 or VI.3-2 into equation VI.4-8 and applying this result to equation VI.4-17 the last term of equation VI.4-17 falls out and λ_3 becomes

$$\lambda_3 = -\pi_6 + \frac{(\lambda_1-\gamma_1(2d_1-D_1))}{A_1(d_1)} \qquad \text{VI.4-18}$$

When the results of substitution of the routing equations into equation VI.4-8 and the value of λ_3 are applied to equations VI.4-12 and VI.4-16, these latter equations become:

$$\{Z_4 + \frac{\lambda_2-\gamma_2(2d_2-D_2)}{A_2} - K_1[-\pi_6 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}]_{t+B}$$

$$- K_2[-\pi_6 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}]_{t+C}\}2C_{D_2}\sqrt{d_2}\, r_2$$

$$+ \pi_2(2r_2-R_{max_2}-R_{min_2}) = 0 \qquad \text{VI.4-19}$$

and;

$$\{Z_4 + \frac{\lambda_2-\gamma_2(2d_2-D_2)}{A_2(d_2)} - K_1[-\pi_6 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}]_{t+B}$$

$$- K_2[-\pi_6 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}]_{t+C}\} \frac{C_{D_2} r_2^{\,2}}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2}{dd_2}$$

$$- \pi_4 h_2 = \frac{d\lambda_2}{dt} \qquad \text{VI.4-20}$$

In the above two equations, K_1 and K_2 are constants determined by the type of flow routing used. Observe that if a progressive average lag technique were used with n equal one, then

$$K_1 = 1 \qquad \text{VI.4-21}$$
$$K_2 = 0$$

In addition if

$$B = 0 \qquad \text{VI.4-22}$$

equations VI.4-19 and VI.4-20 reduce to the same form as equations III.2-9 and III.2-3.

The control equations for the two reservoir problem with time delay are now equations VI.4-11, 13, 14 and 19 and the adjoint equations VI.4-15, 18 and 20. The effect of the routing with time delay amounts to a shift in time scales as would be expected.

C. Analysis of the Necessary Conditions.

Although a complete analysis of the two reservoirs in series problem with time delay will not be attempted here, it is worthwhile to consider the condition analogous to the no time delay case in which the outflow from the upstream reservoir is governed by

$$\dot{d}_1 = 0 \qquad \text{VI.4-23}$$

(i.e. the downstream reservoir is full but not overflowing).

Assume that:

$$C > B \geq 0 \qquad \text{and} \quad C \geq E \qquad \text{VI.4-24}$$

If the downstream reservoir stops overflowing at $t = t_2$ and if it is assumed that the upstream reservoir behaves similarly in the delay and no time delay cases and allows only the minimum flow determined by R_{min_2} into reservoir 1 when reservoir 1 is overflowing then over some interval t_2-Δ to t_2 for R_{min_2} equal to zero

$$\dot{\upsilon}_3 = 0 \qquad \text{VI.4-25}$$

This leads to the requirements that

$$C_{D_2}\sqrt{d_2(t-B)}(r_2(t-B))^2 = 0 \qquad \text{VI.4-26}$$

$$C_{D_2}\sqrt{d_2(t-C)}(r_2(t-C)) = 0 \qquad \text{VI.4-27}$$

and $$\dot{\upsilon}_3(t-E) = 0 \qquad \text{VI.4-28}$$

in the interval

$$t_2-\Delta \leq t \leq t_2 \qquad \text{VI.4-29}$$

(This assumes that it is not possible to combine positive values of the upstream reservoir throughput to obtain a zero value for $\dot{v}_3$).

If r_2 is to increase from its minimum position, at sometime t_α, to maintain maximum storage in reservoir 1 after it ceases to overflow, then examination of equation VI.4-19 shows that in general the only Lagrange multipliers available to satisfy this equation are $\gamma_1/t_{\alpha+B}$ and $\gamma_1/t_{\alpha+C}$. That is either:

$$t_2 = t_\alpha + B \qquad \text{VI.4-30}$$

or

$$t_2 = t_\alpha + C \qquad \text{VI.4-31}$$

If equation VI.4-31 applies then flows above the minimum will reach reservoir 1 at

$$t_3 = t_2 - C + B$$

which is prior to t_2, thus violating the assumed operation and requirement given by equation VI.4-26.

If equation VI.4-30 applies then all the requirements given by equations VI.4-26 to 29 are met and the inflow to reservoir 1 resulting from the operation of reservoir 2 will begin to increase at t_{2+} as desired. Thus the shift in time scales is equal to B.

By similar reasoning it can be shown that the multiplier available to satisfy equation VI.4-19 when the flow constraint (equation VI.4-8) is binding is $\pi_6/t_{\alpha+B}$.

The multiplier available to satisfy equation VI.4-19 when h_2 is greater than zero and d_2 equals D_2 is clearly γ_2/t_α.

Observe that if a Muskingum type flow routing is used that B equals zero and thus the physical reality of the delay in the flow routing is ignored by the mathematics.

VI.5 Adjustments to the Numerical Technique to Include Realistic Flow Routing Models

The above examination of the two reservoir problem with time delay is admittedly far from complete; however, the similarity between the control and adjoint equations for the delay and no delay cases would indicate that the general form of the solutions with no time delay for other configurations should be equally applicable to problems with time delay by shifting the time scales for each reservoir by B_j, where B_j is the smallest time lag applicable to conduit j. Provided the upstream flow conditions (i.e. d and r) are known for the time interval

$$t_o - C_j \leq t \leq t_o \qquad \text{VI.5-1}$$

where C_j is the greatest time lag associated with conduit j then all the necessary information is available to compute the inflows at time t. Thus the only alterations necessary to include flow routing with time delay in the numerical technique proposed in this study are:

a) the upstream flow conditions in the interval given by equation VI.5-1;

b) a shift in the time scales for each upstream reservoir of B_j;

c) addition of the flow routing equations.

CHAPTER VII

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

VII.1. Conclusions

In this study the calculus of variations was applied to the problem of minimizing the weighted diversions of flow to the receiving waters from a combined sewer system.

In Chapter I some possible solutions to the combined sewer problem were examined. It was shown that use of existing system storage capacity may be considered a feasible solution to the overflow problem. The problems related to the determination of control logic that would maximize the use of available system storage capacity were outlined and the advantages of optimal formulations presented. The information presented in this chapter established the fact that the study of suitable control logic for control of flows in combined sewers is a practical and relevant problem in North America today.

In Chapter II the definition of an optimal control was given along with the necessary conditions given by the calculus of variations for a control to be optimal. A mathematical representation of the backwater storage in a controlled combined sewer system as a series of interconnected reservoirs and orifices was presented. In this chapter standard weir and orifice equations were used; however, the results of Chapter III showed that much greater generality of weir and orifice representations could be substituted in the formulation without affecting the overall results. Constraints were placed on the range of operation of the flow control devices, the storage available in each reservoir and the flow capacities of the conduits. No constraints were placed on the rate of operation of the control devices but it was pointed out in Chapter III that a reasonable approximation to the rate limited case could be made very easily. Throughout most of this study it was assumed that there was no time delay in the flow routing. Considering the accuracy of the data that will be available with present technology, and the results of the real time automatic control simulation presented in Chapter V, the system model presented in Chapter II can be considered suitable for the determination of an optimal control strategy in those cases where the time delays in flow routing are small. The results of Chapter VI extended this model for arbitrary time delays in the flow routing. In Chapter IV it was shown that the orifice control had sufficient flexibility to allow an optimal control determination in the event that an orifice control became inoperable and thus fixed in one position. The flexibility of the weir control to allow an optimal control determination was shown to be somewhat limited in the case of its failure.

Two forms of the objective function for the problem were also presented in Chapter II, one including only the weighted overflows, the other including weighted overflows and throughputs. It was shown in Chapter III that the latter objective function eliminated some possibilities of non unique orifice controls from the control problem. The latter objective function was thus used for the remainder of this study. It was always assumed that the throughput weighting factors were very small in comparison to the overflow weighting factors to avoid inadvertently making overflow from one reservoir greater than desired.

In Chapter III and Appendices I-III the forms of the optimal control trajectories obtained from the calculus of variations were presented for four configurations and various assumed input hydrographs. It was assumed that the depth of flow over the weir was always less than the maximum allowable. It was further assumed that after a reservoir stopped overflowing the inflow hydrograph to the reservoir was non increasing. The results demonstrated that the optimal control trajectories were always obtained from constraint boundaries except for those cases in which the orifice control was non unique. For these latter cases an optimal solution determined entirely by constraint boundaries was always feasible. It was shown that as a result of the optimal solution lying along constraint boundaries the possible methods of operation of the control devices and their sequence of operation was highly limited. This important result was used to advantage in Chapters IV and V to determine a numerical technique to solve the control problem.

In Chapter III the possibilities for several different feasible values of the Lagrange multiplier at the t_- side of the entrance to a state variable constraint boundary were demonstrated. The division of the problem into two separate problems by this constraint boundary entrance was also explained.

In Chapter IV the attempts to determine the optimal control strategy by numerical solution of the necessary conditions were discussed. The problems caused by the combination of numerical inaccuracy and the possible values of the Lagrange multipliers to the left of the entrance on to a state variable constraint boundary were outlined (both as a result of a jump in the multipliers, as in the case of the steepest ascent technique, or as a result of the equivalent rapid change in value of the multipliers in the penalty function technique). Although the penalty function formulation reduced the logic required for control operation after the reservoirs stopped overflowing when compared to the steepest ascent technique, it appeared that even if the problems at the state variable constraint boundaries could be overcome, the computational time required to obtain the optimal control for a system of reasonable size would be too great for practical use. In general it would appear from the results presented in this chapter that the problems caused by jumps in the Lagrange multipliers preclude the determination of the optimal control strategy for the combined sewer problem by direct solution of the necessary conditions, at least in the time required for real time operation.

As a result of the above findings, it was proposed in Chapter IV that the results obtained in Chapter III be formulated into a set of operating rules and the problem reduced to that of determining the optimal switching times X_i. Even though the resulting control would not be optimal if the limitations of non increasing flows on the latter parts of the input hydrographs were violated, the control would at worst assure maximum utilization of the available system storage capacity. This methodology in which the results of the analysis of solution forms obtained by examination of the necessary conditions are used to obtain reasonable operating rules with a resultant major reduction in the dimensionality of the overall problem is another important result of this study and would appear to extend the usefulness of the application of variational calculus to practical problems.

Chapter V presented optimal control trajectories for two system configurations obtained by optimizing the switching times X_i for the given operating rules. The results showed that for a system of up to twenty reservoirs, the optimal control (optimal for the given operating rules) could be obtained in approximately two minutes provided relatively minor modifications were made to the optimization procedure. This time is believed to be reasonable for real time control determination. The control trajectories presented always made maximum utilization of the available system storage capacities and generally permitted overflow from only those reservoirs with the highest overflow weighting factors. The use of a first order gradient search technique to obtain the optimal switching times X_i gave satisfactory results but no claim is made that it is the best technique for this application.

The results noted in Chapter V and presented in Appendix V show that at least generality with respect to the relative values of the overflow weighting factors can be obtained from a program written to obtain the optimal control trajectories for a given configuration of reservoirs and arbitrary input hydrographs. There does not appear to be any problems that would prevent the writing of a program capable of determining the optimal control trajectories for an arbitrary configuration of reservoirs having arbitrary overflow weighting factors and input hydrographs.

The simulation example presented at the end of Chapter V demonstrated the use of the control logic, the reasonableness of the reservoir representation of backwater storage and the lack of optimality that will invariably exist when the computed optimal control is applied to an actual operating system. This lack of true optimality serves as further justification for the approach proposed in Chapter IV and demonstrated in Chapter V.

In Chapter VI it was demonstrated that the solution procedure presented in Chapters IV and V could be easily modified to include more realistic flow routing methods. These modifications should have minimal effect on the computational times presented in Chapter V. It was also demonstrated that the use of a Muskingum routing technique in optimal control formulations will produce unrealistic results.

Generally it can be concluded that the methodology presented herein represents a feasible and practical approach to the determination of reasonable control strategies for the minimization of weighted overflow from combined sewer systems having weir and orifice controls and utilizing in-line storage.

VII.2 Recommendations for Further Work

In this study only cases involving two reservoirs in parallel were examined. The optimal control trajectories for more than two reservoirs in parallel should be examined to determine the dominance-subservience relationships that may exist for the orifice controls in these configurations. It is expected that the control strategies will have a similar form to those presented herein.

In Chapter IV the lack of adaptability of the weir controls in those instances in which they become inoperable was noted. It would be fruitful to reformulate the orifice controls in the optimal control problem in terms of the weir height ω instead of the depth of flow over the weir h. This would not only allow better control determination in the event of a device failure but would increase the generality of the overall solution to those cases in which the maximum allowable depth at a control point is greater than the maximum height of the weir. At present the maximum weir height must be greater than or equal to the maximum allowable depth.

In Chapter V it was noted that the possibility of writing a general control program existed. Following the examination of the two areas suggested above, it would appear that such a program would be a useful contribution. Undoubtedly it would be computationally slower than a program written for a specific configuration but it would aid greatly in the determination of an optimum configuration.

The first order gradient search technique used in this study to determine the optimum switching times was not claimed to be the best for this purpose. It is possible that other optimization techniques might produce substantial reductions in computational time and at the same time eliminate some of the problems shown in Chapter V to exist with the first order gradient technique.

The effect of information errors on the overall control results was examined only briefly in this study. It would appear fruitful to examine not only the effects of these errors but the major sources. The problem of regeneration of a field of rainfall data in space and forward in time from a series of point rainfall readings would appear to be one of the most pressing in this area.

In Chapter IV it was stated that approximately two minutes would be available for control determination. This was an estimate based on known times for the determination of runoff inputs and, estimates of the times that the other data gathering and control supervision functions would require. The validity of this estimate should be checked to determine the seriousness of the time constraint. This examination would also of necessity include a study of the optimal time span between control updates.

It appears that there might be some benefit to the sewage treatment process to examine the companion dry weather problem to the problem of minimization of weighted diversions from combined sewer systems. This is the determination of the throughputs from the reservoirs to maximize overall sewage treatment. This would allow increased benefits to be obtained from the same control equipment possibly in the areas of reduced treatment plant capacity requirements and improved treatment.

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APPENDIX I

THE PROBLEM OF TWO RESERVOIRS IN PARALLEL

A.I.1 Introduction

This Appendix gives the complete solution for CASE 4 in Chapter III; the problem of the optimal control of two reservoirs in parallel each of which overflows.

The format followed is similar to that of CASE 1 in Chapter III except that the algebraic equations for the Lagrange multipliers associated with the inequality constraints are not given as it is only important to know that a particular multiplier is available to satisfy a particular control equation.

To simplify notation, the following variables have been introduced; f_1 and f_2 which represent the right hand sides of the state equations; C_1-C_5 which represent the left hand side of the control variable inequality constraints; S_1 and S_2 which represent the left hand side of the state variable inequality constraints and; P_1 - P_4 which represent the control equations.

A.I.2 The State Equations

The two reservoirs in parallel system is shown in figure III-6. For this system, following the formulation given in Chapter II the state equations are:

$$\dot{d}_1 = \frac{q_1(t) - C_{D_1}\sqrt{d_1}r_1^{\,2} - C_{W_1}h_1^{\,3/2}}{A_1(d_1)} = f_1 \qquad \text{A.I-1}$$

and;

$$\dot{d}_2 = \frac{q_2(t) - C_{D_2}\sqrt{d_2}r_2^{\,2} - C_{W_2}h_2^{\,3/2}}{A_2(d_2)} \qquad \text{A.I-2}$$

A.I.3 The System Inequality Constraints

A. The control variable inequality constraints

$$C_i = (r_i - R_{min_i})(r_i - R_{max_i}) \leq 0 \qquad i = 1,2 \qquad \text{A.I-3}$$

$$C_{i+2} = h_i(h_i - d_i) \leq 0 \qquad i = 1,2 \qquad \text{A.I-4}$$

$$C_5 = C_{D_1}\sqrt{d_1}r_1^{\,2} + C_{D_2}\sqrt{d_2}r_2^{\,2} + q_3(t) - Q_{max} \leq 0 \qquad \text{A.I-5}$$

B. The state variable inequality constraints

$$S_i = d_i(d_i - D_i) \leq 0 \qquad i = 1,2 \quad \text{A.I.2-4} \qquad \text{A.I-6}$$

A.I.4 The Augmented Objective Function

Using the abbreviations given in equations A.I-1 - A.I-5 and taking the first derivative of the state variable inequality constraint to adjoin the state variable inequality constraints in the normal manner, the augmented objective function for the problem becomes

$$\text{Min}\phi = \int_0^{t_f} \{Z_1 C_{W_1} h_1^{3/2} + Z_2 C_{W_2} h_2^{3/2} + Z_3 C_{D_1} \sqrt{d_1} r_1^2$$

$$+ Z_4 C_{D_2} \sqrt{d_2} r_2^2 + \lambda_1(\dot{d}_1 - f_1) + \lambda_2(\dot{d}_2 - f_2)$$

$$+ \pi_1 C_1 + \pi_2 C_2 + \pi_3 C_3 + \pi_4 C_4 + \pi_5 C_5 + \gamma_1(2d_1 - D_1) f_1$$

$$+ \gamma_2(2d_2 - D_2) f_2\} dt + \bar{\gamma}_1 S_1 + \bar{\gamma}_2 S_2 \qquad \text{A.I-7}$$

A.I.5 The Control and Adjoint Equations

Applying the Euler-Lagrange equation to the augmented objective function yields the following necessary conditions for an optimal control.

A. The control equations

$$P_1 = \{Z_3 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)} + \pi_5\} 2C_{D_1} \sqrt{d_1} r_1 + \pi_1(2r_1 - R_{max_1} - R_{min_1}) = 0 \qquad \text{A.I-8}$$

$$P_2 = \{Z_4 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)} + \pi_5\} 2C_{D_2} \sqrt{d_2} r_2 + \pi_2(2r_2 - R_{max_2} - R_{min_2}) = 0 \qquad \text{A.I-9}$$

$$P_3 = \{Z_1 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}\} 3/2 C_{W_1} h_1^{1/2} + \pi_3(2h_1 - d_1) = 0 \qquad \text{A.I-10}$$

$$P_4 = \{Z_2 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}\} 3/2 C_{W_2} h_2^{1/2} + \pi_4(2h_2 - d_2) = 0 \qquad \text{A.I-11}$$

B. The adjoint equations

$$\dot{\lambda}_1 = \{Z_3 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)} + \pi_5\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1} - \pi_3 h_1 \qquad \text{A.I-12}$$

$$\dot{\lambda}_2 = \{Z_4 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)} + \pi_5\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} - \pi_4 h_2 \qquad \text{A.I-13}$$

In equations A.I-12 and A.I-13 any terms in which γ_i and f_i appear as a product have been eliminated since one or the other is always zero.

C. The transversality condition

The values of λ_i (i = 1,2) at t_f as given by the transversality condition are:

$$\lambda_1/t_f = 0 \qquad \text{A.I-14}$$

and;

$$\lambda_2/t_f = 0 \qquad \text{A.I-16}$$

A.I.6 The Solution Form When Both Reservoirs Overflow

In determining the solution form for a case when both reservoirs overflow, the following assumptions were made:

(a) $Z_1 > Z_2 > 0$ A.I-16

(b) $-Z_3 > -Z_4 > 0$ A.I-17

(c) $-Z_3 + Z_1 > -Z_4 + Z_2$ A.I-18

(d) $h_i < d_i$ $i = 1,2$ A.I-19

The assumed state and control variable trajectories for the optimal solution are shown in figure III-7. The description of the control operations is the same as that given for CASE 4 in Chapter III.

Following the procedure outlined in Chapter III the verification of the assumed trajectories can proceed backward from t_f. For each time interval on figure III.7, Table A1.1 shows: the non binding constraints and the resultant zero Lagrange multipliers; the binding constraints, their associated non-zero Lagrange multipliers and the control equations which these multipliers are used to satisfy; and the equations used to determine each control variable. For each time interval and corner, Table A1.2 shows the equations of the λ multipliers. The form of the λ multipliers is plotted on figure A1.1 in terms of $\lambda_i/A_i(d_i)$.

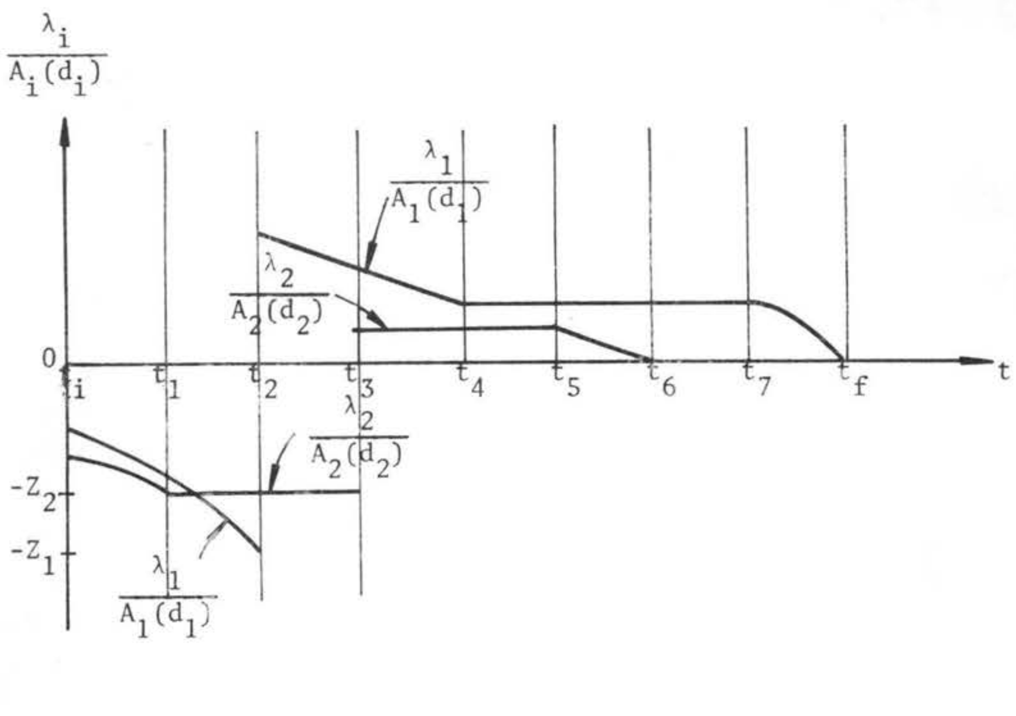

Figure A1.1 The Trajectories of the λ Multipliers

TABLE A1.1

Solution of the Constraint Multipliers, Control Equations and Controls for the Two Reservoirs in Parallel.

1		2		3	4	5
Time Interval	Non Binding Constraints	Associated zero Multipliers	Binding Constraints	Associated Non Zero Multipliers	Control Equations satisfied by Multipliers	Constraint Controls are Determined From
	C_2	π_2	C_1	π_1	P_1	$r_1 \rightarrow C_1 = 0$
$t_f > t > t_7$	S_1	γ_1	C_3	π_3	P_3	$r_2 \rightarrow C_5 = 0$
	S_2	γ_2	C_4	π_4	P_4	$h_1 \rightarrow C_3 = 0$
			C_5	π_5	P_2	$h_2 \rightarrow C_4 = 0$
	C_1	π_1	C_3	π_3	P_3	$r_2 \rightarrow \dot{S}_2 = 0$
$t_7 > t > t_6$	C_2	π_2	C_4	π_4	P_4	$r_1 \rightarrow C_5 = 0$
	S_1	γ_1	C_5	π_5	P_1	$h_1 \rightarrow C_3 = 0$
			S_2	γ_2	P_2	$h_2 \rightarrow C_4 = 0$
	C_1	π_1	C_2	π_2	P_2	$r_2 \rightarrow C_2 = 0$
$t_6 > t > t_5$	C_4	π_4	C_3	π_3	P_3	$r_1 \rightarrow C_5 = 0$
	S_1	γ_1	C_5	π_5	P_1	$h_1 \rightarrow C_3 = 0$
			S_2	γ_2	P_4	$h_2 \rightarrow \dot{S}_2 = 0$
			C_3	π_3	P_3	$r_1 \rightarrow \dot{S}_1 = 0$
$t_5 > t > t_4$	C_1	π_1				
			C_5	π_5	P_2	$r_2 \rightarrow C_5 = 0$
	C_2	π_2				
			S_1	γ_1	P_1	$h_1 \rightarrow C_3 = 0$
	C_4	π_4				
			S_2	γ_2	P_4	$h_2 \rightarrow \dot{S}_2 = 0$

TABLE A1.1 continued

$t_4>t>t_3$	C_2 C_3 C_4	π_2 π_3 π_4	C_1 C_5 S_1 S_2	π_1 π_5 γ_1 γ_2	P_1 P_2 P_3 P_4	$r_1 \rightarrow C_1 = 0$ $r_2 \rightarrow C_5 = 0$ $h_1 \rightarrow \dot{S}_1 = 0$ $h_2 \rightarrow \dot{S}_2 = 0$
$t_3>t>t_2$	C_2 C_3 S_2	π_2 π_3 γ_2	C_1 C_4 C_5 S_1	π_1 π_4 π_5 γ_1	P_1 P_4 P_2 P_3	$r_1 \rightarrow C_1 = 0$ $r_2 \rightarrow C_5 = 0$ $h_1 \rightarrow \dot{S}_1 = 0$ $h_2 \rightarrow C_4 = 0$
$t_2>t>t_1$	C_2 S_1 S_2	π_2 γ_1 γ_2	C_1 C_3 C_4 C_5	π_1 π_3 π_4 π_5	P_1 P_3 P_4 P_2	$r_1 \rightarrow C_1 = 0$ $r_2 \rightarrow C_5 = 0$ $h_1 \rightarrow C_3 = 0$ $h_2 \rightarrow C_4 = 0$
$t_1>t\geq t_0$	C_5 S_1 S_2	π_5 γ_1 γ_2	C_1 C_2 C_3 C_4	π_1 π_2 π_3 π_4	P_1 P_2 P_3 P_4	$r_1 \rightarrow C_1 = 0$ $r_2 \rightarrow C_2 = 0$ $h_1 \rightarrow C_3 = 0$ $h_4 \rightarrow C_4 = 0$

Notes

1. Time intervals are those shown on figure A1.2.
2. Lagrange Multipliers in this column are those associated with non-zero constraints on the same line in the previous column.
3. Lagrange Multipliers in this column are those associated with binding constraints on the same line in the previous column.
4. Control equations listed in this column are satisfied by the Lagrange multipliers on the same line in the previous column.
5. $r_1 \rightarrow C_1 = 0$ should be read as "r_1 is determined from $C_1 = 0$". The controls in this column are listed in the order required for solution.

TABLE A1.2

The Equations of the λ Multipliers and the Corner Conditions for the Problem of Two Reservoirs in Parallel

Time	Values of $\dot{\lambda}_i$ or λ_i
t_f	$\lambda_1 = 0$ $\lambda_2 = 0$
$t_f > t > t_7$	$\dot{\lambda}_1 = \{Z_3 - Z_4 + \frac{\lambda_1}{A_1(d_1)} - \frac{\lambda_2}{A_2(d_2)}\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$ $\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$
Corner at t_7	All variables are continuous. Therefore the corner conditions yield: $\lambda_1/t_{7-} = \lambda_1/t_{7+}$ $\lambda_2/t_{7-} = \lambda_2/t_{7+}$
$t_7 > t > t_6$	$\dot{\lambda}_1 = \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1} \rightarrow \frac{\lambda_1}{A_1(d_1)} = \text{const.}$ $\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$
Corner at t_6	All variables are continuous. Therefore the corner conditions yield: $\lambda_1/t_{6-} = \lambda_1/t_{6+}$ $\lambda_2/t_{6-} = \lambda_2/t_{6+}$

TABLE A1.2 continued

$t_6>t>t_5$

$$\dot{\lambda}_1 = \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1} \rightarrow \frac{\lambda_1}{A_1(d_1)} = \text{const.}$$

$$\dot{\lambda}_2 = \{(Z_4-Z_2) - (Z_3 + \frac{\lambda_1}{A_1(d_1)})\} \frac{C_{D_2} r_2^{\,2}}{2\sqrt{d_2}} f_2 \frac{dA_2(d_2)}{dd_2}$$

Corner at t_5

All variables are continuous. Therefore the corner conditions yield:

$$\lambda_1/t_{5-} = \lambda_1/t_{5+}$$

$$\lambda_2/t_{5-} = \lambda_2/t_{5+}$$

$t_5>t>t_4$

$$\dot{\lambda}_1 = \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1} \rightarrow \frac{\lambda_1}{A_1(d_1)} = \text{const.}$$

$$\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$$

Corner at t_4

All variables are continuous. Therefore the corner conditions yield:

$$\lambda_1/t_{4-} = \lambda_1/t_{4+}$$

$$\lambda_2/t_{4-} = \lambda_2/t_{4+}$$

$t_4>t>t_3$

$$\dot{\lambda}_1 = \{(Z_3-Z_1) - (Z_2-Z_4)\} \frac{C_{D_1} r_1^{\,2}}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$$

Corner at t_3

At t_3 the control h_2 is discontinuous and the derivative of the state variable d_2 is discontinuous. In this case the corner conditions yield:

$$\lambda_1/t_3 = \lambda_1/t_{3+}$$

$$\lambda_2/t_{3-} = -Z_2A_2(D_2)$$

TABLE A1.2 continued

$t_3>t>t_2$

$$\dot{\lambda}_1 = \{(Z_3-Z_1) - (\frac{\lambda_2}{A_2(d_2)} + Z_4)\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2} = \text{const.}$$

Corner at t_2

At t_2 the control h_1 is discontinuous and the derivative of the state variable d_1 is discontinuous. In this case the corner conditions yield:

$$\lambda_1/t_{2-} = -Z_1A_1(D_1)$$

$$\lambda_2/t_{2-} = \lambda_2/t_{2+}$$

$t_2>t>t_1$

$$\dot{\lambda}_1 = \{(Z_3 + \frac{\lambda_1}{A_1(d_1)}) - (Z_4 + \frac{\lambda_2}{A_2(d_2)})\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2} = \text{const.}$$

Corner at t_1

All variables are continuous. Therefore the corner conditions yield:

$$\lambda_1/t_{1-} = \lambda_1/t_{1+}$$

$$\lambda_2/t_{1-} = \lambda_2/t_{1+}$$

$t_1>t\geq t_0$

$$\dot{\lambda}_1 = \{Z_3 + \frac{\lambda_1}{A_1(d_1)}\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \{Z_4 + \frac{\lambda_2}{A_2(d_2)}\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2}$$

A.I.7 Possibilities for a Jump in Control

In the interval t_i-t_1 both orifice controls are at their maximum and so a jump is meaningless.

In the interval t_1-t_2 when the flow constraint is binding examination of control equtions A.I-8 and A.I-9 shows that for a switch in the controls to occur would require:

$$Z_3 + \frac{\lambda_1}{A_1(d_1)} = Z_4 + \frac{\lambda_2}{A_2(d_2)} \qquad \text{A.I-20}$$

Examination of the equations for the λ's in this interval shows that this would require:

$$\dot{\lambda}_1 = 0 \qquad \text{A.I-21}$$

which it can only approach assymptotically. Therefore no switch in control can occur in this interval.

Following similar reasoning, and noting that when a reservoir is overflowing, a jump in the control r_i will require a jump in the weir control h_i, it can be shown that there are no possibilities for jumps in the orifice controls for this problem.

For inflows which will not fill reservoir 1 if orifice 1 remains the dominant control, different corner conditions, analogous to those of CASE 2 or CASE 3 given in Chapter III for reservoirs in series may occur and thus lead to a switch in the dominance of the orifice controls.

APPENDIX II

THE PROBLEM OF THREE RESERVOIRS IN THE "V" CONFIGURATION

A.II.1 Introduction

This Appendix gives the complete solution for CASE 5 in Chapter III; the problem of three reservoirs in the V configuration, each of which overflows.

The format followed is similar to that of CASE 1 in Chapter III except that the algebraic equations of the Lagrange multipliers associated with the inequality constraints are not given as it is only important to know that a particular multiplier is available to satisfy a particular control equation.

To simplify notation the following variables have been introduced: f_i, which represents the right hand sides of the state equations; C_j, which represents the left hand side of the control variable inequality constraints; S_k which represents the left hand side of the state variable inequality constraints and; P_j which represents the control equations.

A.II.2 The State Equations

The three reservoir "V" configuration is shown in figure III-8. For this system, following the formulation given in Chapter II the state equations are:

$$\dot{d}_1 = \frac{q_1(t) + q_4(t) + C_{D_2}\sqrt{d_2}\,r_2^{\,2} - C_{D_1}\sqrt{d_1}\,r_1^{\,2} - C_{W_1}h_1^{\,3/2}}{A_1(d_1)} = f_1 \qquad \text{A.II-1}$$

$$\dot{d}_2 = \frac{q_2(t) - C_{D_2}\sqrt{d_2}\,r_2^{\,2} - C_{W_2}h_2^{\,3/2}}{A_2(d_2)} = f_2 \qquad \text{A.II-2}$$

and

$$\dot{d}_3 = \frac{q_3(t) - C_{D_3}\sqrt{d_3}\,r_3^{\,2} - C_{W_3}h_3^{\,3/2}}{A_3(d_3)} = f_3 \qquad \text{A.II-3}$$

A.II.3 The System Inequality Constraints

A. The Control Variable Inequality Constraints

$$C_i = h_i(h_i - d_i) \leq 0 \qquad i = 1,2,3 \qquad \text{A.II-4}$$

$$C_{i+3} = (r_i - R_{max_i})(r_i - R_{min_i}) \leq 0 \quad i = 1,2,3 \qquad \text{A.II-5}$$

$$C_7 = q_1(t) + C_{D_2}\sqrt{d_2}\,r_2^{\,2} - Q_{max_1} \leq 0 \qquad \text{A.II-6}$$

$$C_8 = C_{D_1}\sqrt{d_1}\,r_1^{\,2} + C_{D_3}\sqrt{d_3}\,r_3^{\,2} - Q_{max_2} \leq 0 \qquad \text{A.II-7}$$

B. The State Variable Inequality Constraints

$$S_i = d_i(d_i - D_i) \leq 0 \qquad i = 1,2,3 \qquad \text{A.II-8}$$

A.II.4 The Augmented Objective Function

Using the abbreviations given in equations A.II-1 through A.II-8 and taking the first time derivative of the state variable inequality constraints and adjoining them in the usual manner, the augmented objective function for the problem becomes:

$$\begin{aligned}\text{Min}\phi = \int_0^{t_f} \{ & Z_1 C_{W_1} h_1^{3/2} + Z_2 C_{W_2} h_2^{3/2} + Z_3 C_{W_3} h_3^{3/2} + Z_4 C_{D_1} \sqrt{d_1} r_1^2 \\ & + Z_5 C_{D_2} \sqrt{d_2} r_2^2 + Z_6 C_{D_3} \sqrt{d_3} r_3^2 + \lambda_1(\dot{d}_1 - f_1) + \lambda_2(\dot{d}_2 - f_2) \\ & + \lambda_3(\dot{d}_3 - f_3) + \pi_1 C_1 + \pi_2 C_2 + \pi_3 C_3 + \pi_4 C_4 + \pi_5 C_5 \\ & + \pi_6 C_6 + \pi_7 C_7 + \pi_8 C_8 + \gamma_1(2d_1 - D_1) f_1 + \gamma_2(2d_2 - D_2) f_2 \\ & + \gamma_3(2d_3 - D_3) f_3 \} dt + \bar{\gamma}_1 S_1 + \bar{\gamma}_2 S_2 \end{aligned} \qquad \text{A.II-9}$$

A.II.5 <u>The Control and Adjoint Equations</u>

Applying the Euler-Lagrange equation to the augmented objective function yields the following necessary conditions for an optimal control.

A. The Control Equations

$$P_1 = \{Z_1 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}\} 3/2 C_{W_1} h_1^{3/2} + \pi_1(2h_1 - d_1) = 0 \qquad \text{A.II-10}$$

$$P_2 = \{Z_2 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}\} 3/2 C_{W_2} h_2^{3/2} + \pi_2(2h_2 - d_2) = 0 \qquad \text{A.II-11}$$

$$P_3 = \{Z_3 + \frac{\lambda_3 - \gamma_3(2d_3 - D_3)}{A_3(d_3)}\} 3/2 C_{W_3} h_3^{3/2} + \pi_3(2h_3 - d_3) = 0 \qquad \text{A.II-12}$$

$$P_4 = \{Z_4 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)} + \pi_8\} 2 C_{D_1} \sqrt{d_1} r_1 + \pi_4(2r_1 - R_{max_1} - R_{min_1}) = 0$$

A.II-13

$$P_5 = \{Z_5 + (\frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}) - (\frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}) + \pi_7\}2C_{D_2}\sqrt{d_2}r_2$$

$$+ \pi_5(2r_2 - R_{max_2} - R_{min_2}) = 0 \qquad \text{A.II-14}$$

$$P_6 = \{Z_6 + \frac{\lambda_3 - \gamma_3(2d_3 - D_3)}{A_3(d_3)} + \pi_8\}2C_{D_3}\sqrt{d_3}r_3 + \pi_6(2r_3 - R_{max_1} - R_{min_1}) = 0$$

A.II-15

B. The Adjoint Equations

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)} + \pi_8\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1} - \pi_1 h_1$$

A.II-16

$$\dot{\lambda}_2 = \{Z_5 + (\frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}) - (\frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}) + \pi_7\} \frac{C_{D_2} r_2^2}{2\sqrt{d_2}}$$

$$+ \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} - \pi_2 h_2 \qquad \text{A.II-17}$$

$$\dot{\lambda}_3 = \{Z_6 + \frac{\lambda_3 - \gamma_3(2d_3 - D_3)}{A_3(d_3)} + \pi_8\} \frac{C_{D_3} r_3^2}{2\sqrt{d_3}} + \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3} - \pi_3 h_3$$

A.II-18

In equations A.II-16 - A.II-18, any terms in which γ_i and f_i appear as a product have been eliminated since one of the other is always zero.

C. The Transversality Condition

The values of $\lambda_i/_{t_f}$ as given by the transversality condition are:

$$\lambda_1/_{t_f} = 0 \qquad \text{A.II-19}$$

$$\lambda_2/t_f = 0 \quad ; \tag{A.II-20}$$

and

$$\lambda_3/t_f = 0 \quad . \tag{A.II-21}$$

A.II.6 <u>The Solution Forms When All Reservoirs Overflow</u>

In determining a solution form for a case where all three reservoirs overflow, the following assumptions were made:

(a) $Z_1 > Z_2 > Z_3 > 0$ A.II-22

(b) $-Z_4 > -Z_5 > -Z_6 > 0$ A.II-23

(c) $Z_1 - Z_4 > Z_2 - Z_5 > Z_3 - Z_6 > 0$ A.II-24

(d) $h_i < d_i \qquad i = 1,2,3$ A.II-25

The assumed state and control variable trajectories for the optimal solution are shown in figure III-9.

The description of the control operations is the same as that given for CASE 5 in Chapter III.

Following the procedure outlined in Chapter III the verification of the assumed trajectories can proceed backward from t_f. For each time interval on figure III-9, Table A2.1 shows the non-binding constraints and the resulting zero Lagrange multipliers; the binding constraints and their associated non-zero Lagrange multipliers and the control equations which these multipliers are used to satisfy; and the equations used to determine each control variable. For each time interval and corner Table A2.2 shows the equations of the λ multipliers. The form of the λ multipliers is plotted on figure A.2.1 in terms of $\lambda_i/A_i(d_i)$.

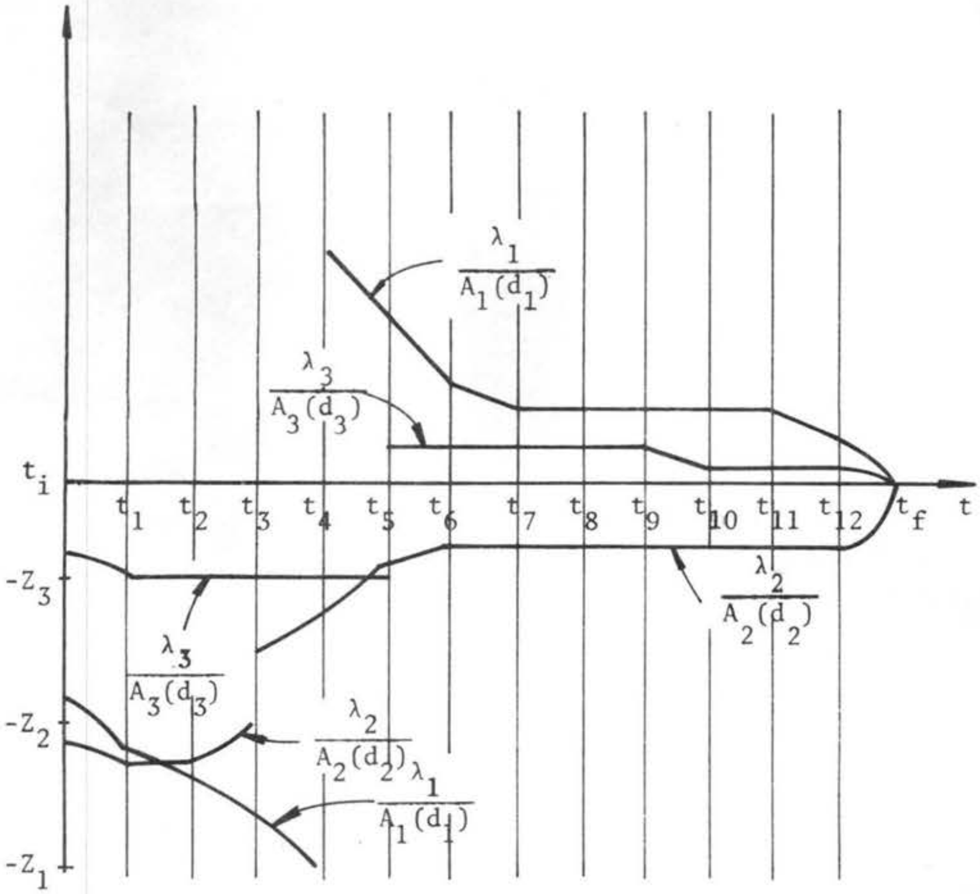

Fig. A2.1 The Trajectories of the λ Multipliers for the Three Reservoir "Y" Configuration

A.II.7 Possibilities for a Jump in Control

For the reasons given in section A.I.7 of Appendix I, a switch in control between r_1 and r_3 is not possible for the three reservoir "V" configuration when all three reservoirs overflow. If neither reservoir 1 or 2 overflows, or if the outflow from reservoir 2 is limited by the flow constraint $C_7 = 0$ and reservoir 1 does not overflow, then the possibility

TABLE A2.1

Solution of the Constraint Multipliers, Control Equations and Controls for the Case of Three Reservoirs in the "V" Configuration.

1 Time Interval	Non-Binding Constraints	2 Associated Zero Multipliers	Binding Constraints	3 Associated Non Zero Multipliers	4 Control Equations Satisfied by Multipliers	5 Constraint Controls Are Determined From
$t_f > t > t_{12}$	C_7	π_7	C_1	π_1	P_1	$r_1 \rightarrow C_4 = 0$
	C_8	π_8	C_2	π_2	P_2	$r_2 \rightarrow C_5 = 0$
	S_1	γ_1	C_3	π_3	P_3	$r_3 \rightarrow C_6 = 0$
	S_2	γ_2	C_4	π_4	P_4	$h_1 \rightarrow C_1 = 0$
	S_3	γ_3	C_5	π_5	P_5	$h_2 \rightarrow C_2 = 0$
			C_6	π_6	P_6	$h_3 \rightarrow C_3 = 0$
$t_{12} > t > t_{11}$	C_5	π_5	C_1	π_1	P_1	$h_1 \rightarrow C_1 = 0$
	C_6	π_6	C_2	π_2	P_2	$h_2 \rightarrow C_2 = 0$
	C_7	π_7	C_3	π_3	P_3	$h_3 \rightarrow C_3 = 0$
	S_2	γ_2	C_4	π_4	P_4	$r_1 \rightarrow C_4 = 0$
	S_3	γ_3	C_8	π_8	P_6	$r_2 \rightarrow \dot{S}_1 = 0$
			S_1	γ_1	P_5	$r_3 \rightarrow C_8 = 0$
$t_{11} > t > t_{10}$	C_4	π_4	C_1	π_1	P_1	$h_1 \rightarrow C_1 = 0$
	C_5	π_5	C_2	π_2	P_2	$h_2 \rightarrow C_2 = 0$
	C_6	π_6	C_3	π_3	P_3	$h_3 \rightarrow C_3 = 0$
	C_7	π_7	C_8	π_8	P_4	$r_3 \rightarrow \dot{S}_3 = 0$
	S_2	γ_2	S_1	γ_1	P_5	$r_1 \rightarrow C_8 = 0$
			S_3	γ_3	P_6	$r_3 \rightarrow \dot{S}_1 = 0$

TABLE A2.1 continued

$t_{10}>t>t_9$	c_3	π_3	c_1	π_1	P_1	$h_1 \rightarrow c_1 = 0$
	c_4	π_4	c_2	π_2	P_2	$h_2 \rightarrow c_2 = 0$
	c_5	π_5	c_6	π_6	P_6	$r_3 \rightarrow c_6 = 0$
	c_7	π_7	c_8	π_8	P_4	$r_1 \rightarrow c_8 = 0$
	s_2	γ_2	s_1	γ_1	P_5	$r_2 \rightarrow \dot{s}_1 = 0$
			s_3	γ_3	P_3	$h_3 \rightarrow c_3 = 0$
$t_9>t>t_8$	c_3	π_3	c_1	π_1	P_1	$h_1 \rightarrow c_1 = 0$
	c_4	π_4	c_2	π_2	P_2	$h_2 \rightarrow c_2 = 0$
	c_5	π_5	c_8	π_8	P_6	$r_2 \rightarrow \dot{s}_2 = 0$
	c_6	π_6	s_1	γ_1	P_4	$r_1 \rightarrow \dot{s}_1 = 0$
	c_7	π_7	s_2	γ_2	P_5	$r_3 \rightarrow c_8 = 0$
			s_3	γ_3	P_3	$h_3 \rightarrow \dot{s}_3 = 0$
$t_8>t>t_7$	c_2	π_2	c_1	π_1	P_1	$h_1 \rightarrow c_1 = 0$
	c_3	π_3	c_7	π_7	P_2	$h_2 \rightarrow \dot{s}_2 = 0$
	c_4	π_4	c_8	π_8	P_6	$r_2 \rightarrow c_7 = 0$
	c_5	π_5	s_1	γ_1	P_4	$r_1 \rightarrow \dot{s}_1 = 0$
	c_6	π_6	s_2	γ_2	P_5	$r_3 \rightarrow c_8 = 0$
			s_3	γ_3	P_3	$h_3 \rightarrow \dot{s}_3 = 0$
$t_7>t>t_6$	c_2	π_2	c_1	π_1	P_1	$h_1 \rightarrow c_1 = 0$
	c_3	π_3	c_4	π_4	P_4	$r_1 \rightarrow c_4 = 0$
	c_5	π_5	c_8	π_8	P_6	$r_2 \rightarrow \dot{s}_1 = 0$
	c_6	π_6	s_1	γ_1	P_5	$h_2 \rightarrow \dot{s}_2 = 0$
	c_7	π_7	s_2	γ_2	P_2	$r_3 \rightarrow c_8 = 0$
			s_3	γ_3	P_3	$h_3 \rightarrow \dot{s}_3 = 0$

TABLE A2.1 continued

$t_6 > t > t_5$	C_1	π_1	C_4	π_4	P_4	$r_1 \rightarrow C_4 = 0$
	C_2	π_2	C_5	π_5	P_5	$r_2 \rightarrow C_5 = 0$
	C_3	π_3	C_8	π_8	P_6	$h_1 \rightarrow \dot{S}_1 = 0$
	C_6	π_6	S_1	γ_1	P_1	$h_2 \rightarrow \dot{S}_2 = 0$
	C_7	π_7	S_2	γ_2	P_2	$r_3 \rightarrow C_8 = 0$
			S_3	γ_3	P_3	$h_3 \rightarrow \dot{S}_3 = 0$
$t_5 > t > t_4$	C_1	π_1	C_3	π_3	P_3	$r_1 \rightarrow C_4 = 0$
	C_2	π_2	C_4	π_4	P_4	$r_2 \rightarrow C_5 = 0$
	C_6	π_6	C_5	π_5	P_5	$r_3 \rightarrow C_8 = 0$
	C_7	π_7	C_8	π_8	P_6	$h_1 \rightarrow \dot{S}_1 = 0$
	S_3	γ_3	S_1	γ_1	P_1	$h_2 \rightarrow \dot{S}_2 = 0$
			S_2	γ_2	P_2	$h_3 \rightarrow C_3 = 0$
$t_4 > t > t_3$	C_2	π_2	C_1	π_1	P_1	$h_1 \rightarrow C_1 = 0$
	C_6	π_6	C_3	π_3	P_3	$r_1 \rightarrow C_4 = 0$
	C_7	π_7	C_4	π_4	P_4	$r_2 \rightarrow C_5 = 0$
	S_1	γ_1	C_5	π_5	P_5	$h_2 \rightarrow \dot{S}_2 = 0$
	S_3	γ_3	C_8	π_8	P_6	$r_3 \rightarrow C_8 = 0$
			S_2	γ_2	P_2	$h_3 \rightarrow C_3 = 0$
$t_3 > t > t_2$	C_6	π_6	C_1	π_1	P_1	$r_1 \rightarrow C_4 = 0$
	C_7	π_7	C_2	π_2	P_2	$h_1 \rightarrow C_1 = 0$
	S_1	γ_1	C_3	π_3	P_3	$r_2 \rightarrow C_5 = 0$
	S_2	γ_2	C_4	π_4	P_4	$h_2 \rightarrow C_2 = 0$
	S_3	γ_3	C_5	π_5	P_5	$r_3 \rightarrow C_8 = 0$
			C_8	π_6	P_6	$h_3 \rightarrow C_3 = 0$

TABLE A2.1 continued

$t_2>t>t_1$	C_5	π_4	C_1	π_1	P_1	$r_1 \rightarrow C_4 = 0$
	C_6	π_5	C_2	π_2	P_2	$h_1 \rightarrow C_1 = 0$
	S_1	γ_1	C_3	π_3	P_3	$r_2 \rightarrow C_7 = 0$
	S_2	γ_2	C_4	π_4	P_4	$h_2 \rightarrow C_2 = 0$
	S_3	γ_3	C_7	π_7	P_5	$r_3 \rightarrow C_8 = 0$
			C_8	π_8	P_6	$h_3 \rightarrow C_3 = 0$
$t_1>t>t_0$	C_7	π_7	C_1	π_1	P_1	$r_1 \rightarrow C_4 = 0$
	C_8	π_8	C_2	π_2	P_2	$r_2 \rightarrow C_5 = 0$
	S_1	γ_1	C_3	π_3	P_3	$r_3 \rightarrow C_6 = 0$
	S_2	γ_2	C_4	π_4	P_4	$h_1 \rightarrow C_1 = 0$
	S_3	γ_3	C_5	π_5	P_5	$h_2 \rightarrow C_2 = 0$
			C_6	π_6	P_6	$h_3 \rightarrow C_3 = 0$

Notes

1. Time intervals are those shown on figure A2.2.
2. Lagrange multipliers inthis column are those associated with non zero constraints on the same line in the previous column.
3. Lagrange multipliers in this column are those associated with the binding constraints on the same lines in the previous column.
4. Control equations listed in this column are satisfied by the Lagrange multipliers on the same line in the previous column.
5. $r_1 \rightarrow C_4 = 0$ should be read as "r_1 is determined from $C_4 = 0$". The controls in this column are listed in the order required for solution.

TABLE A2.2

The Equations of the λ Multipliers and the Corner Conditions for the Problem of Three Reservoirs in the "V" Configuration.

Time	Values of $\dot{\lambda}_i$ or λ_i
t_f	$\lambda_1 = 0$ $\lambda_2 = 0$ $\lambda_3 = 0$
$t_f > t > t_{12}$	$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$ $\dot{\lambda}_2 = \{Z_5 + \frac{\lambda_2}{A_2(d_2)} - Z_4 - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2}$ $\dot{\lambda}_3 = \{Z_6 + \frac{\lambda_3}{A_3(d_3)}\}\frac{C_{D_3} r_3^2}{2\sqrt{d_3}} + \frac{\lambda_3}{A_3(d_3)} f_1 \frac{dA_1(d_1)}{dd_1}$
Corner at t_{12}	All variables are continuous. Therefore the corner conditions yield: $\lambda_i/_{t_{12-}} = \lambda_i/_{t_{12+}} \qquad i = 1,2,3$
$t_{12} > t > t_{11}$	$\dot{\lambda}_1 = \{Z_4 + Z_5 + \frac{\lambda_2}{A_2(d_2)} - Z_6 - \frac{\lambda_3}{A_3(d_3)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$ $\dot{\lambda}_2 = \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} \rightarrow \frac{\lambda_2}{A_2(d_2)} = \text{const.}$ $\dot{\lambda}_3 = \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3} \rightarrow \frac{\lambda_3}{A_3(d_3)} = \text{const.}$
Corner at t_{11}	All variables are continuous. Therefore the corner conditions yield: $\dot{\lambda}_i/_{t_{11-}} = \dot{\lambda}_i/_{t_{11+}} \qquad i = 1,2,3$
$t_{11} > t > t_{10}$	$\dot{\lambda}_i = \frac{\lambda_i}{A_i(d_i)} f_i \frac{dA_i(d_i)}{dd_i} \rightarrow \frac{\lambda_i}{A_i(d_i)} = \text{const.} \qquad i = 1,2,3$

TABLE A2.2 cont'd

Corner at t_{10} All variables are continuous. Therefore the corner conditions yield:

$$\lambda_i/t_{10-} = \lambda_i/t_{10+} \qquad i = 1,2,3$$

$t_{10}>t>t_9$

$$\dot{\lambda}_i = \frac{\lambda_i}{A_i(d_i)} f_i \frac{dA_i(d_i)}{dd_i} \rightarrow \frac{\lambda_i}{A_i(d_i)} = \text{const.} \qquad i = 1,2$$

$$\dot{\lambda}_3 = \{Z_6 - Z_3 - Z_4 - Z_5 - \frac{\lambda_2}{A_2(d_2)}\}\frac{C_{D_3} r_3^2}{2\sqrt{d_3}} + \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3}$$

Corner at t_9 All variables are continuous. Therefore the corner conditions yield:

$$\lambda_i/t_{9-} = \lambda_i/t_{9+} \qquad i = 1,2,3$$

$t_9>t>t_8$

$$\dot{\lambda}_i = \frac{\lambda_i}{A_i(d_i)} f_i \frac{dA_i(d_i)}{dd_i} \rightarrow \frac{\lambda_i}{A_i(d_i)} = \text{const.} \qquad i = 1,2,3$$

Corner at t_8 All variables are continuous. Therefore the corner conditions yield:

$$\lambda_i/t_{8-} = \lambda_i/t_{8+} \qquad i = 1,2,3$$

$t_8>t>t_7$

$$\lambda_i = \frac{\lambda_i}{A_i(d_i)} f_i \frac{dA_i(d_i)}{dd_i} \rightarrow \frac{\lambda_i}{A_i(d_i)} = \text{const.} \qquad i = 1,2,3$$

Corner at t_7 All variables are continuous. Therefore the corner conditions yield:

$$\lambda_i/t_{7-} = \lambda_i/t_{7+}$$

$t_7>t>t_6$

$$\dot{\lambda}_1 = \{Z_4 + Z_5 - Z_2 - Z_6 + Z_3\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\lambda_i = \frac{\lambda_i}{A_i(d_i)} f_i \frac{dA_i(d_i)}{dd_i} \rightarrow \frac{\lambda_i}{A_i(d_i)} = \text{const.} \qquad i = 2,3$$

Corner at t_6 All variables are continuous. Therefore the corner conditions yield:

$$\lambda_i/t_{6-} = \lambda_i/t_{6+} \qquad i = 1,2,3$$

TABLE A2.2 cont'd

$t_6 > t > t_5$

$$\dot{\lambda}_1 = \{(Z_4 - Z_1) - (Z_6 - Z_3)\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \{(Z_5 - Z_2) + Z_1\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2}$$

$$\dot{\lambda}_3 = \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3} \qquad \rightarrow \frac{\lambda_3}{A_3(d_3)} = \text{const.}$$

Corner at t_5

At t_5, h_3 is discontinuous and therefore $\dot{d}_3$ is discontinuous. Therefore the corner conditions yield:

$$\lambda_i/_{t_{5-}} = \lambda_i/_{t_{5+}} \qquad i = 1,2$$

and

$$\lambda_3/_{t_{5-}} = -Z_3 A_3(D_3)$$

$t_5 > t > t_4$

$$\dot{\lambda}_1 = \{(Z_4 - Z_1) - (Z_6 - Z_3)\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \{Z_5 - Z_2 + Z_1\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2}$$

$$\dot{\lambda}_3 = \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3} \qquad \rightarrow \frac{\lambda_3}{A_3(d_3)} = \text{const.}$$

Corner at t_4

At t_4, h_1 is discontinuous and therefore $\dot{d}_1$ is discontinuous. Therefore the corner conditions yield:

$$\lambda_1/_{t_4} = -Z_1 A_1(D_1)$$

$$\lambda_i/_{t_{4-}} = \lambda_i/_{t_{4+}} \qquad i = 2,3$$

TABLE A2.2 cont'd

$t_4 > t > t_3$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)} - (Z_6 + \frac{\lambda_3}{A_3(d_3)})\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \{(Z_5 - Z_2) - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2}$$

$$\dot{\lambda}_3 = \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3} \quad \rightarrow \quad \frac{\lambda_3}{A_3(d_3)} = \text{const.}$$

Corner at t_3 — At t_3, h_2 is discontinuous and therefore $\dot{d}_2$ is discontinuous. Therefore the corner conditions yield:

$$\lambda_i/_{t_{3-}} = \lambda_i/_{t_{3+}} \qquad i = 1,3$$

and

$$\lambda_2/_{t_{3-}} = -Z_2 A_2(D_2)$$

$t_3 > t > t_2$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)} - (Z_6 + \frac{\lambda_3}{A_3(d_3)})\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \{Z_5 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2}$$

$$\dot{\lambda}_3 = \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{dd_3}$$

Corner at t_2 — At this corner there is no state variable constraint involved, thus even though the control r_1 is discontinuous the corner conditions require

$$\lambda_i/_{t_{2-}} = \lambda_i/_{t_{2+}} \qquad i = 1,2,3$$

$t_2 > t > t_1$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)} - (Z_6 + \frac{\lambda_3}{A_3(d_3)})\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_i = \frac{\lambda_i}{A_i(d_i)} f_i \frac{dA_i(d_i)}{dd_i} \quad \rightarrow \quad \frac{\lambda_i}{A_i(d_i)} = \text{const.} \qquad i = 2,3$$

TABLE A2.2 cont'd

Corner at t_1 — All variables are continuous. Therefore the corner conditions yield:

$$\lambda_i/_{t_{1-}} = \lambda_i/_{t_{1+}} \qquad i = 1,2,3$$

$t_1 > t \geq t_0$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^{2}}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)}\, f_1\, \frac{dA_1(d_1)}{dd_1}$$

$$\dot{\lambda}_2 = \{(Z_5 + \frac{\lambda_2}{A_2(d_2)}) - (Z_4 + \frac{\lambda_1}{A_1(d_1)})\}\frac{C_{D_2} r_2^{2}}{2\sqrt{d_2}} + \frac{\lambda_2}{A_2(d_2)}\, f_2\, \frac{dA_2(d_2)}{dd_2}$$

$$\dot{\lambda}_3 = \{Z_6 + \frac{\lambda_3}{A_3(d_3)}\}\frac{C_{D_3} r_3^{2}}{2\sqrt{d_3}} + \frac{\lambda_3}{A_3(d_3)}\, f_3\, \frac{dA_3(d_3)}{dd_3}$$

of a jump in control between r_1 and r_3 exists.

To show that the jump in control r_2 can occur at t_2, the corner conditions given by equations II.2-13 and 14 of Chapter II can be applied (observe that $\bar{\gamma}_k$ equals zero) or the control equation for r_2 (equation A.2-14) can be examimed. Following the latter approach, if a jump in control r_2 is to occur then equation A.2-14 must at some instant in time be zero for any value of r_2. This leads to the requirement that:

$$Z_5 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)} = 0 \qquad \text{A.II-26}$$

when neither reservoir 1 nor reservoir 2 is overflowing. By considering the case when the absolute value of Z_5 is very small and referring to figure A2.1, it is seen that this condition can occur at t_2 as shown.

Another possibility of interest that could occur if reservoir 2 started overflowing much sooner,is that the control r_2 could switch while reservoir 2 is overflowing. The corner conditions show that the necessary condition for a switch in control r_2 to occur in this situation is

$$-Z_2 + Z_5 = \frac{\lambda_1}{A_1(d_1)} \qquad \text{A.II-27}$$

which, provided reservoir 1 is not overflowing, can occur (refer to figure A2.1).

By following similar procedures it can be shown that for the relative values of the weighting factors given by equations A.II-22 to 25, the switch in control r_2 cannot occur while reservoir 1 is overflowing.

Other possibilities related to the discussion of the problem of two reservoirs in series, discussed in Chapter III, apply to the three reservoir problem.

APPENDIX III

THE PROBLEM OF THREE RESERVOIRS IN THE "Y" CONFIGURATION

A.III.1 Introduction

This appendix gives the complete solution for CASE 6 in Chapter III; the problem of three reservoirs in the Y configuration, each of which overflows.

The format followed is similar to that of CASE 1 in Chapter III except that the algebraic equations of the Lagrange multipliers associated with the inequality constraints are not given as it is only important to know that a particular multiplier is available to satisfy a particular control equation.

To simplify notation the following variables have been introduced: f_i, which represents the right hand side of the state equations; C_j which represents the left hand side of the control variable inequality constraints; S_k, which represents the left hand side of the state variable inequa constraints and; P_j which represents the control equations.

A.III.2 The State Equations

The three reservoir "Y" configuration is shown in figure III-10. For this system, following the formulation given in Chapter II, the state equations are:

$$\dot{d}_1 = \frac{q_{1A}(t) + q_{1B}(t) + C_{D_2}\sqrt{d_2}\,r_2^{\,2} + C_{D_3}\sqrt{d_3}\,r_3^{\,2} - C_{D_1}\sqrt{d_1}\,r_1^{\,2} - C_{W_1}h_1^{\,3/2}}{A_1(d_1)} \quad \text{A.III-1}$$

$$\dot{d}_2 = \frac{q_2(t) - C_{D_2}\sqrt{d_2}\,r_2^{\,2} - C_{W_2}h_2^{\,3/2}}{A_2(d_2)} \quad \text{A.III-2}$$

$$\dot{d}_3 = \frac{q_3(t) - C_{D_3}\sqrt{d_3}\,r_3^{\,2} - C_{W_3}h_3^{\,3/2}}{A_3(d_3)} \quad \text{A.III-3}$$

A.III.3 The System Inequality Constraints

A. The Control Variable Inequality Constraints

$$C_i = (r_i - R_{min_i})(r_i - R_{max_i}) \leq 0 \qquad i = 1,2,3 \quad \text{A.III-4}$$

$$C_{i+3} = h_i(h_i - d_i) \leq 0 \qquad i = 1,2,3 \quad \text{A.III-5}$$

$$C_7 = q_{1A}(t) + C_{D_2}\sqrt{d_2}\,r_2^{\,2} \leq Q_{max_2} \quad \text{A.III-6}$$

$$C_8 = q_{1B}(t) + C_{D_3}\sqrt{d_3}\,r_3^{\,2} \leq Q_{max_3} \quad \text{A.III-7}$$

$$C_9 = q_{1A}(t) + q_{1B}(t) + C_{D_2}\sqrt{d_2}\,r_2^{\,2} + C_{D_3}\sqrt{d_3}\,r_3^{\,2} \leq Q_{max_1} \quad \text{A.III-8}$$

$$C_{10} = C_{D_1}\sqrt{d_1}\,r_1^{\,2} \leq Q_{max_4} \quad \text{A.III-9}$$

B. The State Variable Inequality Constraints

$$S_i = d_i(d_i - D_i) \leq 0 \quad \text{A.III-10}$$

A.III.4 The Augmented Objective Function

Using the abbreviated notation given in equations A.III-1 through A.III-10 and; taking the first time derivative of the state variable inequality constraints and adjoining them in the usual manner, the augmented objective function for the problem becomes:

$$\mathrm{Min}\phi = \int_{0}^{T_f} \{\sum_{i=1}^{3} (Z_i C_{W_i} h_i^{3/2} + Z_{i+3} C_{D_i} \sqrt{d_i} r_1^{2})$$

$$+ \sum_{j=1}^{3} (\lambda_j(\dot{d}_j - f_j) + \sum_{j=1}^{10} \pi_j C_j + \sum_{R=1}^{3} \gamma_j (2d_j - D_j) f_j\} dt$$

$$+ \sum_{i=1}^{3} \bar{\gamma}_j S_j \qquad \text{A.III-11}$$

A.III.5 The Control and Adjoint Equations

Applying the Euler-Lagrange equations to the augmented index of performance yields the following necessary conditions for an optimal control.

A. The Control Equations

$$P_1 = [Z_1 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}] 3/2 C_{W_1} h_1^{1/2} + \pi_4(2h_1 - d_1) = 0 \qquad \text{A.III-12}$$

$$P_2 = [Z_2 + \frac{\lambda_2 - \gamma_2(2d_2 - D_2)}{A_2(d_2)}] 3/2 C_{W_2} h_2^{1/2} + \pi_5(2h_2 - d_2) = 0 \qquad \text{A.III-13}$$

$$P_3 = [Z_3 + \frac{\lambda_3 - \gamma_3(2d_3 - D_3)}{A_3(d_3)}] 3/2 C_{W_3} h_3^{1/2} + \pi_6(2h_3 - d_3) = 0 \qquad \text{A.III-14}$$

$$P_4 = [Z_4 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)} + \pi_{10}]2C_{D_1}\sqrt{d_1}r_1 + \pi_1(2r_1-R_{max_1}-R_{min_1}) = 0$$

A.III-15

$$P_5 = \{Z_5 + [\frac{\lambda_2-\gamma_2(2d_2-D_2)}{A_2(d_2)}] - [\frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}]$$

$$+ \pi_7 + \pi_9\}2C_{D_2}\sqrt{d_2}r_2 + \pi_2(2r_2-R_{max_2}-R_{min_2}) = 0 \qquad \text{A.III-16}$$

$$P_6 = \{Z_6 + [\frac{\lambda_3-\gamma_3(2d_3-D_3)}{A_3(d_3)}] - [\frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}]$$

$$+ \pi_8 + \pi_9\}2C_{D_3}\sqrt{d_3}r_3 + \pi_3(2r_3-R_{max_3}-R_{min_3}) = 0 \qquad \text{A.III-17}$$

B. The Adjoint Equations

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)} + \pi_{10}\} \frac{C_{D_1}r_1^2}{2\sqrt{d_1}} + \frac{\lambda_1}{A_1(d_1)} f_1 \frac{dA_1(d_1)}{dd_1}$$

$$-\pi_4h_1 \qquad \text{A.III-18}$$

$$\dot{\lambda}_2 = \{Z_5 + [\frac{\lambda_2-\gamma_2(2d_2-D_2)}{A_2(d_2)}] - [\frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}] + \pi_7 + \pi_9\}\frac{C_{D_2}r_2^2}{2\sqrt{d_2}}$$

$$+ \frac{\lambda_2}{A_2(d_2)} f_2 \frac{dA_2(d_2)}{dd_2} - \pi_5h_2 \qquad \text{A.III-19}$$

$$\dot{\lambda}_3 = \{Z_6 + [\frac{\lambda_3-\gamma_3(2d_3-D_3)}{A_3(d_3)}] - [\frac{\lambda_1-\gamma_1(2d_1-D_1)}{A_1(d_1)}] + \pi_8 + \pi_9\}\frac{C_{D_3}r_3^2}{2\sqrt{d_3}}$$

$$+ \frac{\lambda_3}{A_3(d_3)} f_3 \frac{dA_3(d_3)}{d(d_3)} - \pi_6h_3 \qquad \text{A.III-20}$$

In equations A.III-18 to A.III-20, any terms in which γ_i and f_i appear as a product have been eliminated as one or the other is always zero.

C. The Transversality Condition

The values of λ_i/t_f as given by the transversality condition are:

$$\lambda_1/t_f = 0 \qquad \text{A.III-21}$$

$$\lambda_2/t_f = 0 \qquad \text{A.III-22}$$

$$\lambda_3/t_f = 0 \qquad \text{A.III-23}$$

A.III.6 The Solution Forms When All Reservoirs Overflow

In determining a solution form for the case where all three reservoirs overflow, the following assumptions were made

(a) $Z_1 > Z_2 > Z_3 > 0$ A.III-24

(b) $-Z_4 > -Z_5 > -Z_6 > 0$ A.III-25

(c) $Z_1-Z_4 > Z_2-Z_5 > Z_3-Z_6 > 0$ A.III-26

(d) $h_i < d_i \qquad i = 1,2,3$ A.III-27

(e) $R_{min_i} = 0 \qquad i = 1,2,3$ A.III-28

(f) $\pi_i = 0 \qquad i = 7,8,9,10$ A.III-29

(g) $A_i(d_i) = K_i \qquad i = 1,2,3$ A.III-30

Assumption f means that it was assumed that none of the flow constraints were violated.

The assumed state and control variable trajectories for the optimal solution are shown in figure III-11. A brief description of the control operations is given with the discussion of CASE 6 in Chapter III.

Following the procedure outlined in Chapter III the verification of the assumed trajectories can proceed backward from t_f. For each time interval on figure III-11, Table A3.1 shows the non binding constraints and the resultant zero Lagrange multipliers; the binding constraints and their associated non zero Lagrange multipliers and the control equations which these multipliers are used to satisfy; and the equations used to determine each control variable. For each time interval and corner, Table A3.2 shows the equations of the λ multipliers. The form of the λ multipliers is plotted in figure A3.1 in terms of $\frac{\lambda_i}{A_i(d_i)}$.

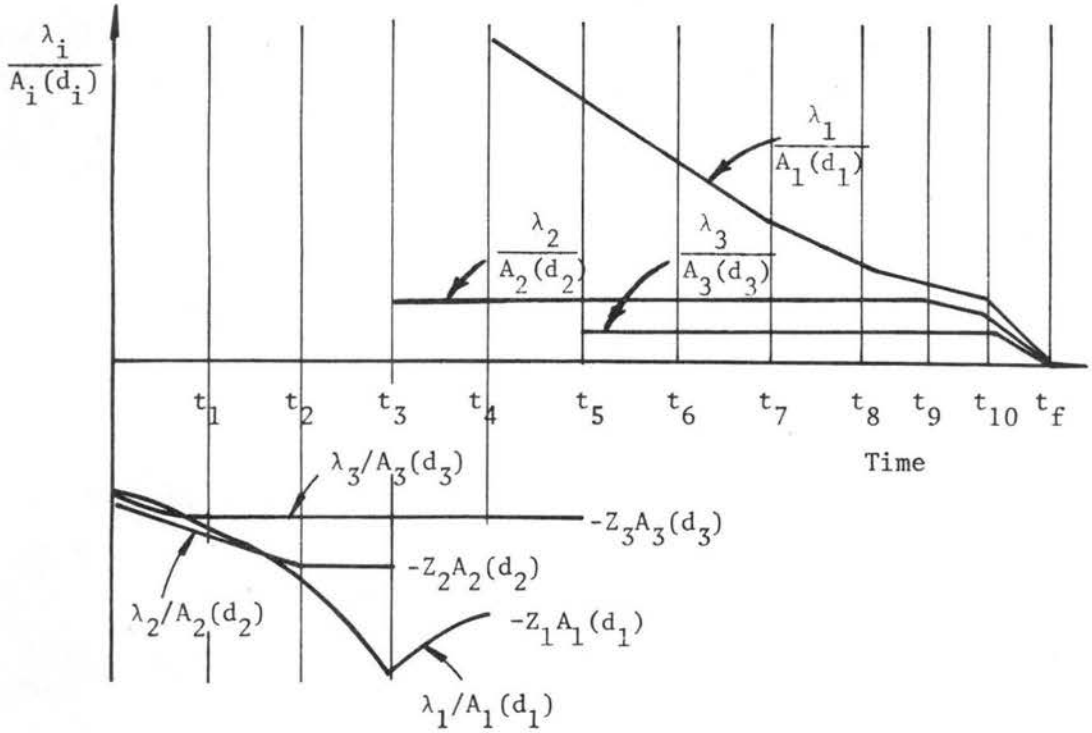

Fig. A3.1 The Trajectories of the λ Multipliers for the Three Reservoir "Y" Configuration

TABLE A3.1

Solution of the Constraint Multipliers, Control Equations and Controls for the Case of Three Reservoirs in the "Y" Configuration

Time Interval[1]	Non Binding Constraints	Associated Zero Multipliers[2]	Binding Constraints	Associated Non Zero Multipliers[3]	Control Equations Satisfied by Multipliers[4]	Constraint Controls Are Determined From[5]
$t_f > t > t_{10}$	C_7	π_7	C_1	π_1	P_4	$r_1 \rightarrow C_1 = 0$
	C_8	π_8	C_2	π_2	P_5	$r_2 \rightarrow C_2 = 0$
	C_9	π_9	C_3	π_3	P_6	$r_3 \rightarrow C_3 = 0$
	C_{10}	π_{10}	C_4	π_4	P_1	$h_1 \rightarrow C_4 = 0$
	S_1	γ_1	C_5	π_5	P_2	$h_2 \rightarrow C_5 = 0$
	S_2	γ_2	C_6	π_6	P_3	$h_6 \rightarrow C_6 = 0$
	S_3	γ_3				
$t_{10} > t > t_9$	C_3	π_3	C_1	π_1	P_4	$h_1 \rightarrow C_4 = 0$
	C_7	π_7	C_2	π_2	P_5	$h_2 \rightarrow C_5 = 0$
	C_8	π_8	C_4	π_4	P_1	$h_3 \rightarrow C_6 = 0$
	C_9	π_9	C_5	π_5	P_2	$r_1 \rightarrow C_1 = 0$
	C_{10}	π_{10}	C_6	π_6	P_3	$r_2 \rightarrow C_2 = 0$
	S_2	γ_2	S_1	γ_1	P_6	$r_3 \rightarrow \dot{S}_1 = 0$
	S_3	γ_3				
$t_9 > t > t_8$	C_2	π_2	C_1	π_1	P_4	$h_1 \rightarrow C_4 = 0$
	C_3	π_3	C_4	π_4	P_1	$h_2 \rightarrow C_5 = 0$
	C_7	π_7	C_5	π_5	P_2	$h_3 \rightarrow C_6 = 0$
	C_8	π_8	C_6	π_6	P_3	$r_1 \rightarrow C_1 = 0$
	C_9	π_9	S_1	γ_1	P_3	$r_2 \rightarrow \dot{S}_1 = 0$
	C_{10}	π_{10}	S_3	γ_3	P_3	$r_3 \rightarrow \dot{S}_3 = 0$
	S_2	γ_2				

TABLE A3.1 continued

$t_8 > t > t_7$	C_2	π_2	C_1	π_1	P_4	$h_1 \rightarrow C_4 = 0$
	C_3	π_3	C_4	π_4	P_1	$h_2 \rightarrow C_5 = 0$
	C_6	π_6	C_5	π_5	P_2	$r_1 \rightarrow C_1 = 0$
	C_7	π_7	S_1	γ_1	P_6	$r_2 \rightarrow \dot{S}_3 = 0$
	C_8	π_8	S_2	γ_2	P_5	$r_3 \rightarrow \dot{S}_1 = 0$
	C_9	π_9	S_3	γ_3	P_3	$h_3 \rightarrow \dot{S}_3 = 0$
	C_{10}	π_{10}				
$t_7 > t > t_6$	C_2	π_2	C_1	π_1	P_4	$h_1 \rightarrow C_4 = 0$
	C_5	π_5	C_3	π_3	P_6	$r_1 \rightarrow C_1 = 0$
	C_6	π_6	C_4	π_4	P_1	$r_3 \rightarrow C_3 = 0$
	C_7	π_7	S_1	γ_1	P_5	$r_2 \rightarrow \dot{S}_1 = 0$
	C_8	π_8	S_2	γ_2	P_2	$h_2 \rightarrow \dot{S}_2 = 0$
	C_9	π_9	S_3	γ_3	P_3	$h_3 \rightarrow \dot{S}_3 = 0$
	C_{10}	π_{10}				
$t_6 > t > t_5$	C_4	π_4	C_1	π_1	P_4	$r_1 \rightarrow C_1 = 0$
	C_5	π_5	C_2	π_2	P_5	$r_2 \rightarrow C_2 = 0$
	C_6	π_6	C_3	π_3	P_6	$r_3 \rightarrow C_3 = 0$
	C_7	π_7	S_1	γ_1	P_1	$h_1 \rightarrow S_1 = 0$
	C_8	π_8	S_2	γ_2	P_2	$h_2 \rightarrow S_2 = 0$
	C_9	π_9	S_3	γ_3	P_3	$h_3 \rightarrow S_3 = 0$
	C_{10}	π_{10}				

TABLE A3.1 continued

$t_5>t>t_4$	C_4	π_4	C_1	π_1	P_4	$r_1 \to C_1 = 0$
	C_5	π_5	C_2	π_2	P_5	$r_2 \to C_2 = 0$
	C_7	π_7	C_3	π_3	P_6	$r_3 \to C_3 = 0$
	C_8	π_8	C_6	π_6	P_3	$h_3 \to C_6 = 0$
	C_9	π_9	S_1	γ_1	P_1	$h_1 \to \dot{S}_1 = 0$
	C_{10}	π_{10}	S_2	γ_2	P_2	$h_2 \to \dot{S}_2 = 0$
	S_3	γ_3				
$t_4>t>t_3$	C_5	π_5	C_1	π_1	P_4	$r_1 \to C_1 = 0$
	C_7	π_7	C_2	π_2	P_5	$r_2 \to C_2 = 0$
	C_8	π_8	C_3	π_3	P_6	$r_3 \to C_3 = 0$
	C_9	π_9	C_4	π_4	P_1	$h_1 \to C_4 = 0$
	C_{10}	π_{10}	C_6	π_6	P_3	$h_3 \to C_6 = 0$
	S_1	γ_1	S_2	γ_2	P_2	$h_2 \to \dot{S}_2 = 0$
	S_3	γ_3				
$t_3>t>t_2$	C_7	π_7	C_1	π_1	P_4	$r_1 \to C_1 = 0$
	C_8	π_8	C_2	π_2	P_5	$r_2 \to C_2 = 0$
	C_9	π_9	C_3	π_3	P_6	$r_3 \to C_3 = 0$
	C_{10}	π_{10}	C_4	π_4	P_1	$h_1 \to C_4 = 0$
	S_1	γ_1	C_5	π_5	P_2	$h_2 \to C_5 = 0$
	S_2	γ_2	C_6	π_6	P_3	$h_3 \to C_6 = 0$
	S_3	γ_3				

TABLE A3.1 continued

$t_2>t>t_1$	C_7	π_7	C_1	π_1	P_4	$r_1 \rightarrow C_1 = 0$
	C_8	π_8	C_2	π_2	P_5	$r_2 \rightarrow C_2 = 0$
	C_9	π_9	C_3	π_3	P_6	$r_3 \rightarrow C_3 = 0$
	C_{10}	π_{10}	C_4	π_4	P_1	$h_1 \rightarrow C_4 = 0$
	S_1	γ_1	C_5	π_5	P_2	$h_2 \rightarrow C_5 = 0$
	S_2	γ_2	C_6	π_6	P_3	$h_3 \rightarrow C_6 = 0$
	S_3	γ_3				
$t_1>t>t_0$	C_7	π_7	C_1	π_1	P_4	$r_1 \rightarrow C_1 = 0$
	C_8	π_8	C_2	π_2	P_5	$r_2 \rightarrow C_2 = 0$
	C_9	π_9	C_3	π_3	P_6	$r_3 \rightarrow C_3 = 0$
	C_{10}	π_{10}	C_4	π_4	P_1	$h_1 \rightarrow C_4 = 0$
	S_1	γ_1	C_5	π_5	P_2	$h_2 \rightarrow C_5 = 0$
	S_2	γ_2	C_6	π_6	P_3	$h_6 \rightarrow C_6 = 0$
	S_3	γ_3				

Notes:

1. Time intervals are those shown on Figure A3.2.
2. Lagrange multipliers in this column are those associated with non zero constraints on the same line in the previous column.
3. Lagrange multipliers in this column are those associated with the binding constraints on the same lines in the previous column.
4. Control equations listed in this column are satisfied by the Lagrange multipliers on the same line in the previous column.
5. $r_1 \rightarrow C_4 = 0$ should be read as "r_1 is determined from $C_4 = 0$". The controls in this column are listed in the order required for solution.

TABLE A3.2

THE EQUATIONS OF THE λ MULTIPLIERS AND THE CORNER CONDITIONS FOR THE PROBLEM OF THREE RESERVOIRS IN THE "Y" CONFIGURATION

Time	Values of λ_i or $\dot{\lambda}_i$
t_f	$\lambda_i = 0 \qquad i = 1,2,3$
$t_f > t > t_{10}$	$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$
	$\dot{\lambda}_2 = \{Z_5 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}}$
	$\dot{\lambda}_3 = \{Z_6 + \frac{\lambda_3}{A_3(d_3)} - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_3} r_3^2}{2\sqrt{d_3}}$
Corner at t_{10}	All variables are continuous. Therefore the corner conditions yield
	$\lambda_i/_{t_{10-}} = \lambda_i/_{t_{10+}} \qquad i = 1,2,3$
$t_{10} > t > t_9$	$\dot{\lambda}_1 = \{Z_4 + Z_6 + \frac{\lambda_3}{A_3(d_3)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$
	$\dot{\lambda}_2 = \{Z_5 - Z_6 - \frac{\lambda_3}{A_3(d_3)} + \frac{\lambda_2}{A_2(d_2)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}}$
	$\dot{\lambda}_3 = 0 \qquad \rightarrow \lambda_3 = \text{const.}$
Corner at t_9	All variables are continous. Therefore the corner conditions yield
	$\lambda_1/_{t_{9-}} = \lambda_1/_{t_{9+}}$

TABLE A3.2 cont'd

$t_9 > t > t_8$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_i = 0 \quad \rightarrow \quad \lambda_i = \text{const.} \qquad i = 2,3$$

Corner at t_8

All variables are continuous. Therefore the corner conditions yield

$$\lambda_i/_{t_{8-}} = \lambda_i/_{t_{8+}} \qquad i = 1,2,3$$

$t_8 > t > t_7$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_i = 0 \quad \rightarrow \quad \lambda_i = \text{const.} \qquad i = 2,3$$

Corner at t_7

All variables are continuous. Therefore the corner conditions yield

$$\lambda_i/_{t_{7-}} = \lambda_i/_{t_{7+}} \qquad i = 1,2,3$$

$t_7 > t > t_6$

$$\dot{\lambda} = \{Z_4 + \frac{\lambda_1 - \gamma_1(2d_1 - D_1)}{A_1(d_1)}\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_i = 0 \quad \rightarrow \lambda_i = \text{const.} \qquad i = 2,3$$

Corner at t_6

All variables are continuous. Therefore the corner conditions yield

$$\lambda_i/_{t_{6-}} = \lambda_i/_{t_{6+}} \qquad i = 1,2,3$$

$t_6 > t > t_5$

$$\dot{\lambda}_1 = \{Z_4 - Z_1\} \frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_i = 0 \quad \rightarrow \quad \lambda_i = \text{const.} \qquad i = 1,2,3$$

TABLE A3.2 cont'd

Corner at t_5

At t_5, h_3 is discontinuous and therefore $\dot{d}_3$ is discontinuous. Therefore the corner conditions yield

$$\lambda_i/_{t_{5-}} = \lambda_i/_{t_{5+}} \qquad i = 1,2$$

$$\lambda_3/_{t_{5-}} = -Z_3A_3(D_3)$$

$t_5 > t > t_4$

$$\dot{\lambda}_1 = \{Z_4 - Z_1\}\frac{C_{D_1} r_1^{\,2}}{2\sqrt{d_1}}$$

$$\dot{\lambda}_1 = 0 \quad \rightarrow \quad \lambda_i = \text{const.} \qquad i = 2,3$$

Corner at t_4

At t_4, h_1 is discontinuous and therefore $\dot{d}_1$ is discontinuous. Therefore the corner conditions yield

$$\lambda_1/_{t_{4-}} = -Z_1A_1(D_1)$$

$$\lambda_i/_{t_{4-}} = \lambda_i/_{t_{4+}} \qquad i = 2,3$$

$t_4 > t > t_3$

$$\dot{\lambda}_1 = \{Z_4 - \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^{\,2}}{2\sqrt{d_1}}$$

$$\dot{\lambda}_i = 0 \quad \rightarrow \quad \lambda_i = \text{const.} \qquad i = 2,3$$

Corner at t_3

At t_3, h_2 is discontinuous and therefore $\dot{d}_3$ is discontinuous. Therefore the corner doncitions yield

$$\lambda_i/_{t_{3-}} = \lambda_i/_{t_{3+}} \qquad i = 1,3$$

$$\lambda_2/_{t_{3-}} = -Z_2A_2(D_2)$$

TABLE A3.2 cont'd

$t_3 > t > t_2$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_i = 0 \quad \rightarrow \quad \lambda_i = \text{const.} \qquad i = 2,3$$

Corner at t_2

At this corner there is no state variable inequality constraint involved, thus, even though the control r_2 is discontinuous the corner conditions require:

$$\lambda_i/_{t_{2-}} = \lambda_i/_{t_{2+}} \qquad i = 1,2,3$$

$t_2 > t > t_1$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_2 = \{Z_5 - \frac{\lambda_1}{A_1(d_1)} + \frac{\lambda_2}{A_2(d_2)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}}$$

$$\dot{\lambda}_3 = 0 \quad \rightarrow \quad \frac{\lambda_3}{A_3(d_3)} = \text{const.}$$

Corner at t_1

At this corner there is no state variable inequality constraint involved, thus, even though the control r_3 is discontinuous the corner conditions require:

$$\lambda_i/_{t_{1-}} = \lambda_i/_{t_{1+}}$$

$t_1 > t \geq t_0$

$$\dot{\lambda}_1 = \{Z_4 + \frac{\lambda_1}{A_1(d_1)}\}\frac{C_{D_1} r_1^2}{2\sqrt{d_1}}$$

$$\dot{\lambda}_2 = \{Z_5 - \frac{\lambda_1}{A_1(d_1)} + \frac{\lambda_2}{A_2(d_2)}\}\frac{C_{D_2} r_2^2}{2\sqrt{d_2}}$$

$$\dot{\lambda}_3 = \{Z_6 - \frac{\lambda_1}{A_1(d_1)} + \frac{\lambda_2}{A_2(d_2)}\}\frac{C_{D_3} r_3^2}{2\sqrt{d_3}}$$

A.III.7 Possibilities for a Jump in Control

To show that the jumps in controls r_2 and r_3 can occur at times t_2 and t_1 respectively, the corner conditions given by equations II.1-11 and 12 of Chapter II can be applied, or the control equations for r_2 and r_3 (equations A.III-16 and A.III-17) can be examined. Following the latter approach, if a jump in control r_2 is to occur then equation A.III-16 must at some instant in time be zero for any value of r_2. When neither reservoir 1 or reservoir 2 is overflowing this leads to the requirement that

$$Z_5 + \frac{\lambda_2}{A_2(d_2)} - \frac{\lambda_1}{A_1(d_1)} = 0 \qquad \text{A.III-31}$$

Examination of the λ trajectories shown on figure A3.1 shows that this condition can occur at t_2. Similar results can be shown for the jump in control r_3 at t_1. For this case the necessary condition for a jump in r_3 is

$$Z_6 + \frac{\lambda_3}{A_3(d_3)} - \frac{\lambda_1}{A_1(d_1)} = 0 \qquad \text{A.III-32}$$

Other possibilities for jumps in the controls r_2 and r_3 are similar to those discussed in CASES 1, 2 and 3 of Chapter III and the series part of the three reservoir "V" configuration discussed in Appendix II.

APPENDIX IV

EXAMPLES OF OPTIMAL CONTROL

A.IV.1 Introduction

This appendix gives the optimal control results for the ten reservoir control problem for two examples in which the weighting factors are as discussed in Chapter V and shown in Table V.3. The inflow hydrographs for both examples are the same and are shown in figure A4.1. Their peaks are shifted in time to simulate the passage of a storm from top to bottom across the basin shown in figure V.4 of Chapter V.

The values of the overflow and throughput weighting factors are the same for both examples and are listed in Table A4.1.

TABLE A4.1

The Weighting Factors For Examples A4.1 and A4.2

Reservoir Number	Overflow Weighting Factor	Throughput Weighting Factor
1	20	-.060
2	16	-.059
3	8	-.055
4	12	-.057
5	14	-.058
6	10	-.056
7	7	-.054
8	6	-.052
9	6.50	-.053
10	5.50	-.051

The weighting factors were given a large spread, in an attempt to ensure that it would not be advantageous for the overflow from a reservoir to be reduced with a resultant increase in the overflow from a downstream reservoir with a higher overflow weighting factor.

A.IV.2 Example A4.1

In this example only the flow constraint governing the outflows from reservoirs 1 and 5 was binding. The data for the reservoir constraints is shown in Table A4.2 The results presented in figure A4.2 show that in fact there was no advantage to be gained by overflowing from a reservoir with a high overflow weighting factor. The overflows shown for reservoirs 1, 5 and 6 are as small as numerical accuracy would allow. The outflow from reservoir 3 was zero until reservoir 2 stopped overflowing and thus no further reduction could be made in the overflow from reservoir 2. The results for the remaining reservoirs appear to be optimal.

TABLE A4.2

The Reservoir Constraints for Example A4.1

Reservoir Number	C_D	C_W	R_{max}	$D_{Initial}$	D_{max}	A(d)
1	2.50	15.0	3.0	1.0	6.0	50 + 80d
2	2.00	15.0	3.0	1.0	6.0	50 + 80d
3	2.00	15.0	3.0	1.0	6.0	50 + 80d
4	2.00	15.0	3.0	1.0	6.0	50 + 80d
5	2.50	15.0	3.0	1.0	6.0	50 + 80d
6	2.00	15.0	3.0	1.0	6.0	50 + 80d
7	2.00	15.0	3.0	1.0	6.0	50 + 80d
8	2.00	15.0	3.0	1.0	6.0	50 + 80d
9	2.00	15.0	3.0	1.0	6.0	50 + 80d
10	2.00	15.0	3.0	1.0	6.0	50 + 80d

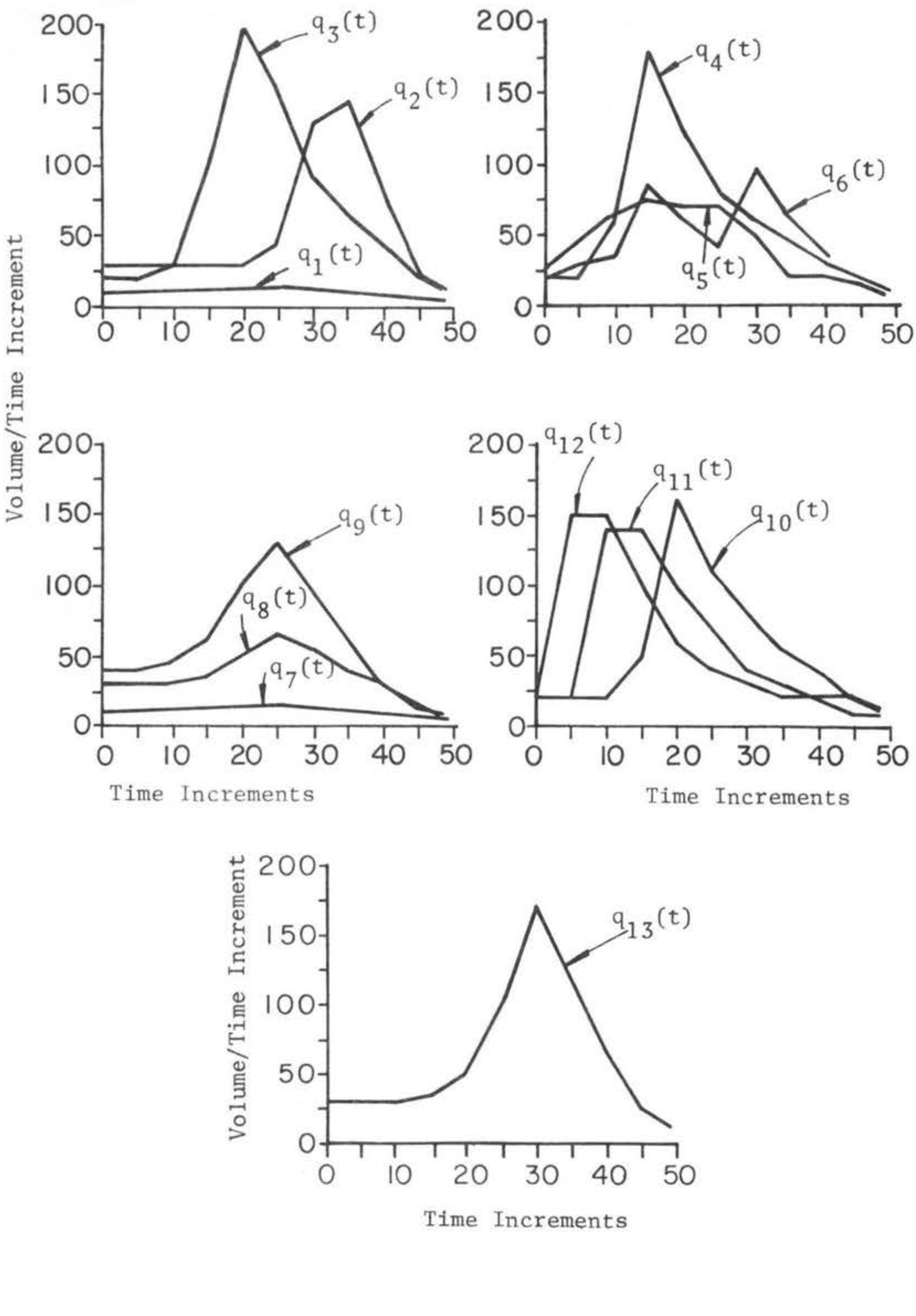

Fig. A4.1 The Inflow Hydrographs for Examples A4.1 and A4.2

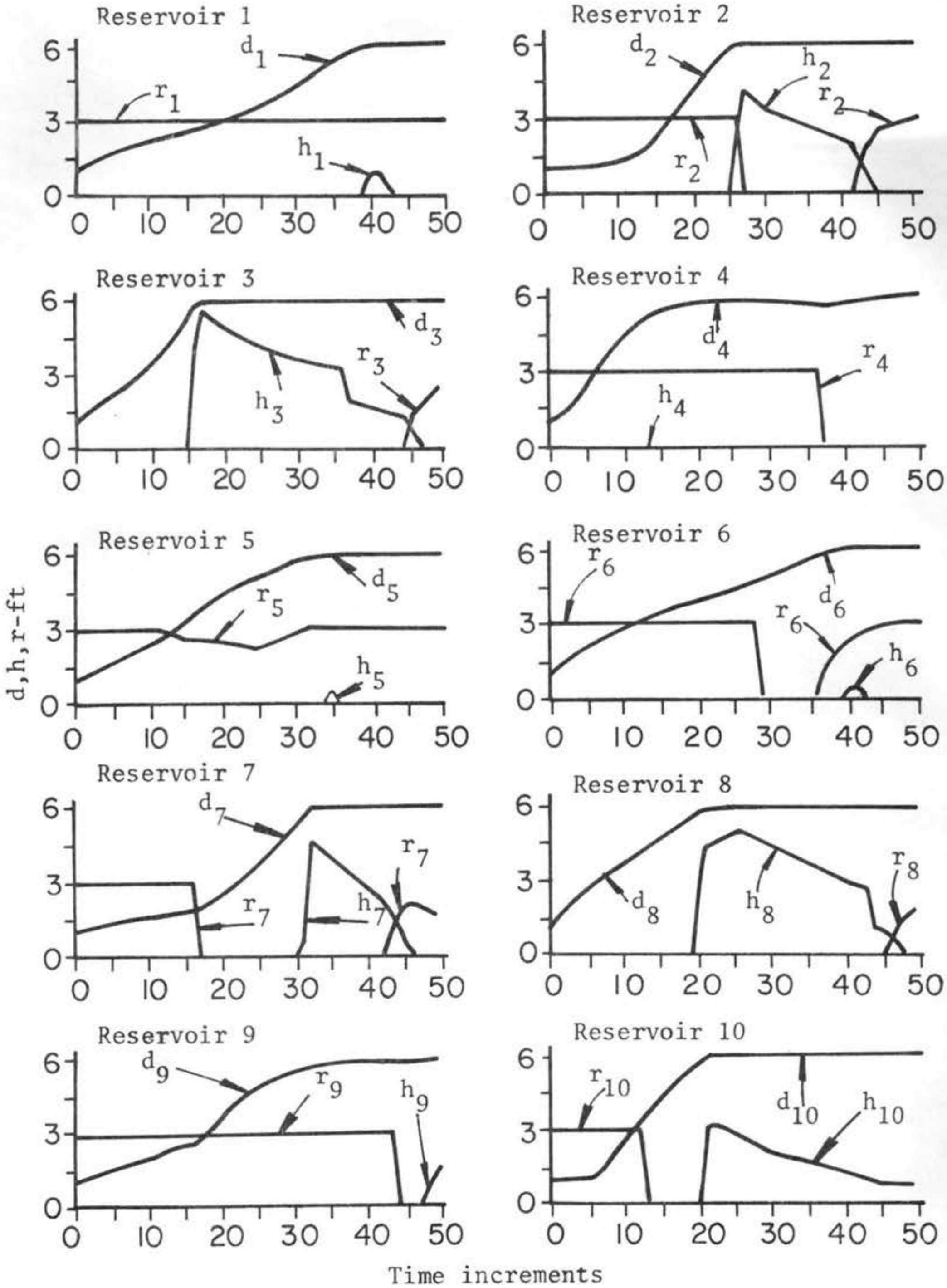

Fig. A4.2 The Computed Optimal State and Control Variable Trajectories for Example A4.1

This run took 86 seconds of IBM 60-67 computer time, including the plotting routines and reduced the objective function from an initial value of 88,712 to a final value of 74,563.

The explanation of this time which was 45% greater than that of any other run, is given in section A.IV.4. Ninety percent of the reduction was obtained within the normal running time of 50 seconds.

A.IV.3 Example A4.2

In this example the area-depth relationships of the reservoirs were changed from the previous example along with the discharge coefficients for orifices 5 and 6. The values of the area depth relationships and the discharge coefficients are listed in Table A4.3

TABLE A4.3

Data for Example A4.2

Reservoir Number	A(d)	C_D
1	30 + 140d	2.5
2	40 + 185d	2.0
3	20 + 175d	2.0
4	30 + 120d	2.0
5	50 + 100d	3.2
6	50 + 180d	3.0
7	50 + 150d	2.0
8	30 + 140d	2.0
9	30 + 185d	2.0
10	50 + 80d	2.0

The intent of this run was to demonstrate what happens when a storm is too small to require the entire capacity of the system. The results of this run, which was stopped before completion, are plotted on figure A4.3. They show that the optimal control tends to make maximum use of the downstream storage capacity. Considering the

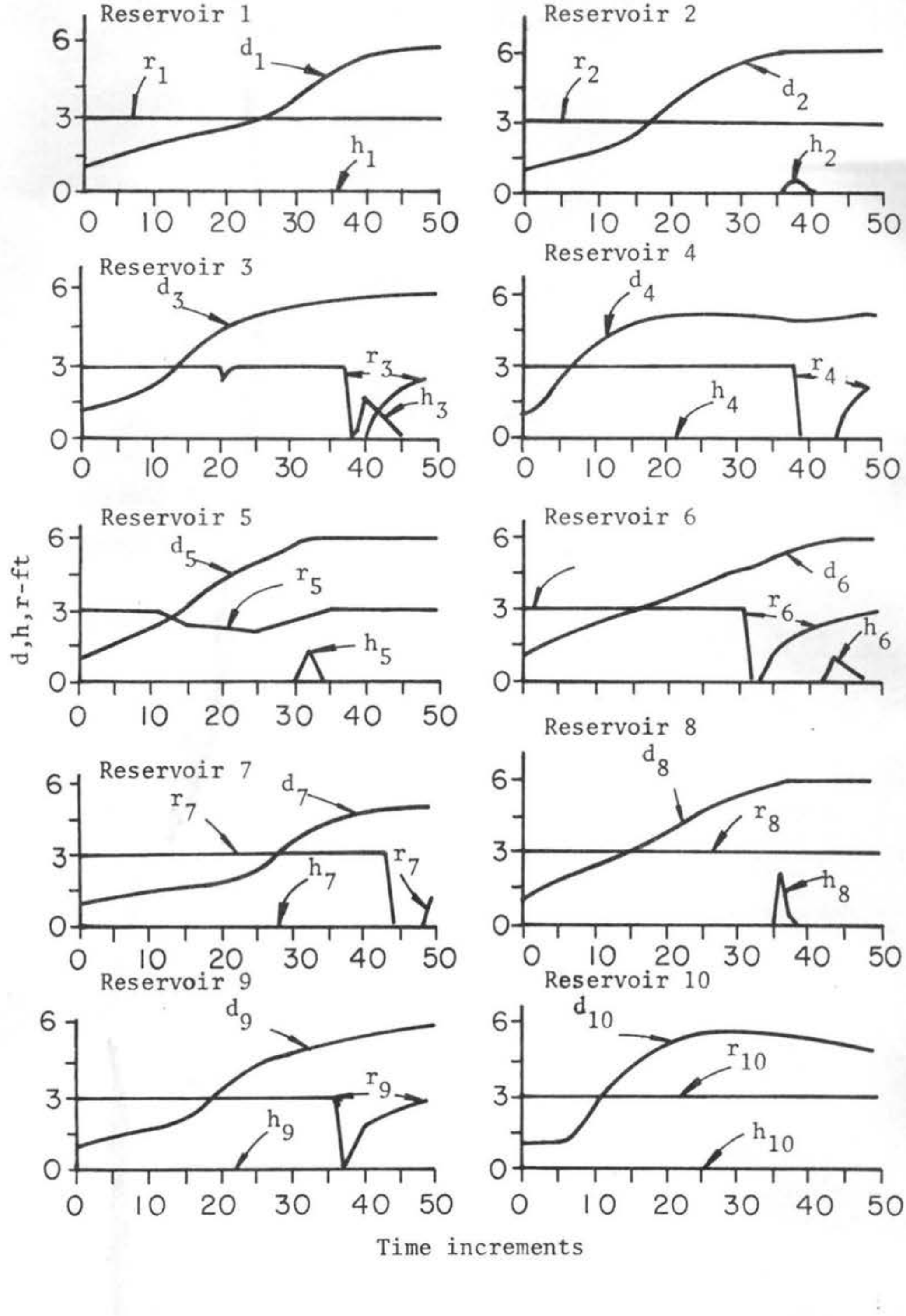

Fig. A4.3 The Computed Optimal State and Control Variable Trajectories for Example A4.2

accuracy of information available for the control determination, this tendency could result in unnecessary overflow from the system when in fact there is adequate storage capacity available. Protection against such an occurrence could be obtained by placing a safety factor on the maximum allowable depth for the most critical downstream reservoirs in the system.

Ideally, this run should have filled all the downstream reservoirs to their maximum capacity where possible, and yet not have permitted any overflow. Numerical accuracy generally precludes finding exactly zero overflow, and thus in cases where there is more than adequate storage capacity in the system, as there is in this example, the numerical optimum is obtained by filling each reservoir just short of its maximum depth (since the overflow penalties exceed any throughput gains). When this occurs there is no way for the upstream orifices to open once they have shut down. Thus in this example, when run to completion, none of the reservoirs overflowed during the fifty time steps but orifices 3, 4, 7, 9 and 10 were still shut down at $T = 50$. Probably the simplest way to avoid this problem is to allow a small amount of overflow from each reservoir without penalty; the gain in throughput would then not be offset by a penalty against any small overflow and the computed optimal control would then ensure that all downstream reservoirs filled where possible and as a result the orifice controls for each orifice would be opening at t_f.

One final point of interest in this example is the operation of the orifice controls for reservoirs seven and eight. If only one reservoir, say reservoir seven, were upstream of reservoir six, then it would be computationally advantageous during the optimization process to shut down orifice seven as soon as reservoir six overflowed. With two reservoirs in parallel upstream of reservoir six, such a procedure would prevent the gradient search procedure from obtaining the optimal control for reservoirs seven and eight in cases similar to this example, where reservoir six should fill to maximum capacity but not overflow. If the true optimal solution allowed overflow from reservoir six then for the given overflow weighting factors, orifices seven and eight would in fact close before reservoir six overflows; however, there is no way to determine a priori that this will in fact be the case.

A.IV.4 Comments on the Examples

For these two computer runs, the gradient search routine was modified to eliminate the problems created when one reservoir in the system reached its maximum level but did not overflow. Prior to the start of each iteration a check was made to determine if the volume of spill from each reservoir was less than a predetermined limit. If any reservoir filled and had overflow less than the specified limit, the switching time for the upstream reservoir was increased by one time step. In addition, the adjusted switching time for the upstream reservoir was assumed to be an optimum value and was held constant unless in later iterations the overflow from the downstream reservoir again fell below the arbitrary limit in which case the adjustment procedure was again applied to the upstream reservoir. When an upstream reservoir switching time was adjusted an additional check was made to ensure that the adjustment would not cause it to stop overflowing. If this reservoir was itself liable to stop overflowing then an adjustment was applied to the next reservoir upstream. Once all derivatives were found to be zero, then further iterations were made in which no checks or adjustments were made for small spill volumes. (This was similar to the procedure - in fact an additional part of it - discussed in Chapter V to reduce computations by temporarily eliminating any switching times which showed a zero derivative).

The results obtained using the above procedure showed that it worked effectively and did produce some reduction in the objective function when compared to results obtained without the additional routine. The increased reduction in the objective function which amounted to less than 10% of the total reduction, for Example A4.1 was obtained at the expense of a 45% increase in computational time. When it is realized that in this example during the initial iteration, the orifice switching times were adjusted to prevent flow into an overflowing reservoir from an upstream reservoir with a lower overflow weighting factor, (with the exception of reservoir six) and, as a result, the full capacity of nine of the ten storage reservoirs was utilized and all but 200 volume units of the remaining reservoir utilized, then it does not appear that the benefits gained by the additional computations are worth the increase in computational time.

APPENDIX V

EXAMPLES OF OPTIMAL CONTROL - GENERAL PROGRAM

A.V.1 Introduction

This appendix gives the optimal control results for the ten reservoir control problem for two examples in which the relative values of the overflow weighting factors are different from those discussed in Chapter V and shown in Table V.3.

The inflow hydrographs for both examples are the same and are shown in figure A.5.1. Their peaks are shifted in time to simulate the passage of a storm from left to right across the basin shown in figure V.4 of Chapter V.

The data for the reservoir constants was also kept constant for the two examples and is shown in Table A5.1

TABLE A5.1

The Reservoir Constants for the Control Examples

Reservoir Number	C_D	C_W	R_{max}	$D_{initial}$	D_{max}	A(d)
1	2.50	15.0	3.0	1.0	6.0	50 + 80d
2	2.00	15.0	3.0	1.0	6.0	50 + 80d
3	2.00	15.0	3.0	1.0	6.0	50 + 80d
4	2.00	15.0	3.0	1.0	6.0	50 + 80d
5	2.50	15.0	3.0	1.0	6.0	50 + 80d
6	2.00	15.0	3.0	1.0	6.0	50 + 80d
7	2.00	15.0	3.0	1.0	6.0	50 + 80d
8	2.00	15.0	3.0	1.0	6.0	50 + 80d
9	2.00	15.0	3.0	1.0	6.0	50 + 80d
10	2.00	15.0	3.0	1.0	6.0	50 + 80d

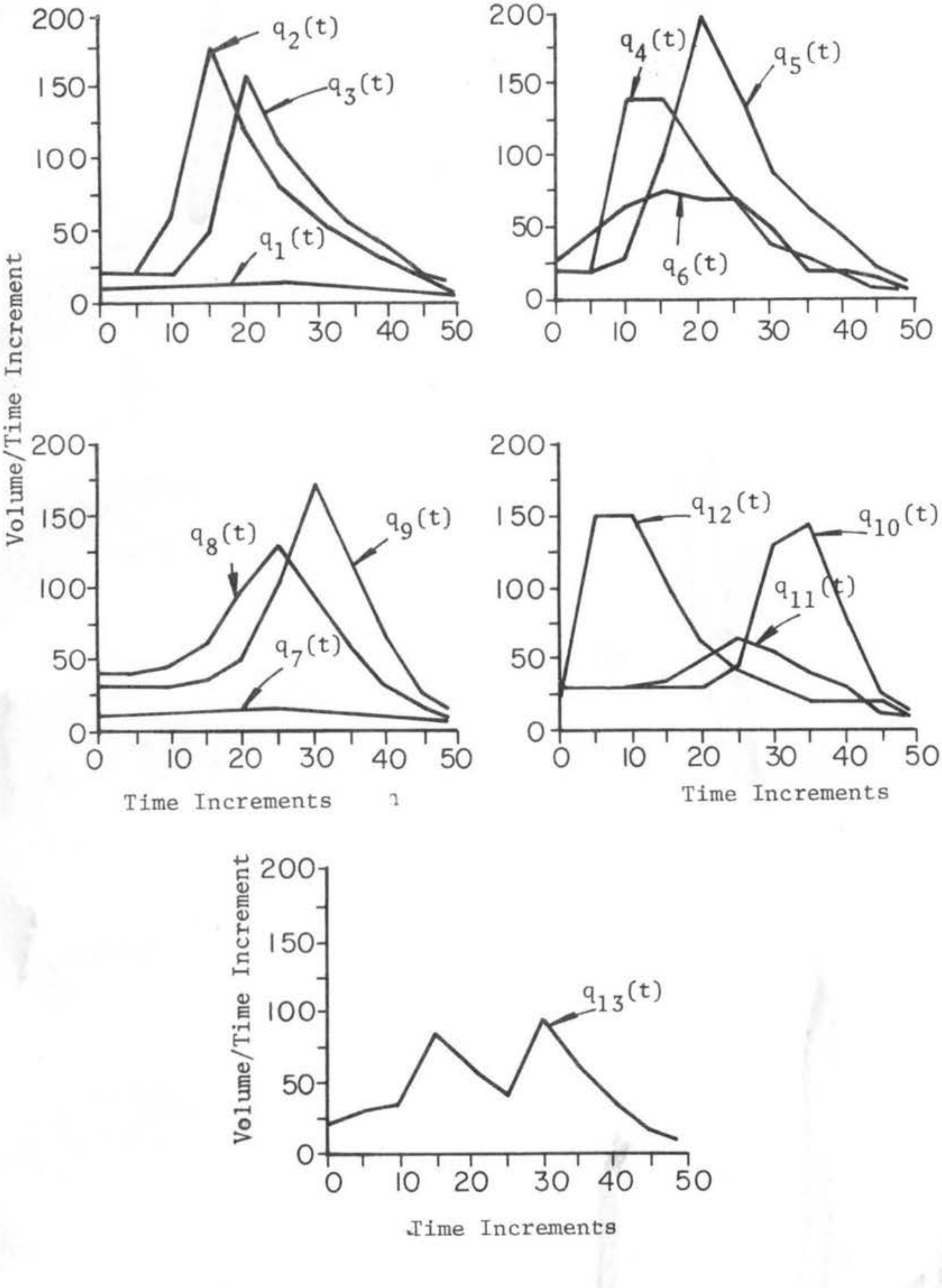

Fig. A5.1 The Inflow Hydrographs for Examples A5.1 and A5.2

A.V.2 Example A5.1

In this example all the flow constraints were non binding and thus the system behaved as if it were two separate series systems. The relative values of the weighting factors are listed in Table A5.2

TABLE A5.2

The Weighting Factors for Example A5.1

Reservoir Number	Overflow Weighting Factor	Throughput Weighting Factor
1	10.0	-.060
2	9.30	-.059
3	5.80	-.054
4	6.50	-.055
5	8.60	-.058
6	7.90	-.057
7	7.20	-.056
8	4.40	-.052
9	5.10	-.053
10	3.70	-.051

The results presented in figure A5.2 show that for reservoirs 1 and 2 there was a slight advantage to be gained by reducing overflow from reservoir 2 at the expense of reservoir 1. The same advantage could also be gained by reducing overflow from reservoir 6 at the expense of reservoir 5. In the latter case the late peak of the inflow hydrograph to reservoir 5 resulted in a much larger spill from that reservoir than from reservoir 1. There was also a longer time span between the time orifice 6 closed down and reservoir 5 overflowed than there was from reservoirs 1 and 2, thus increasing the advantage to be gained by decreasing spill from reservoir 6 at the expense of reservoir 5. There

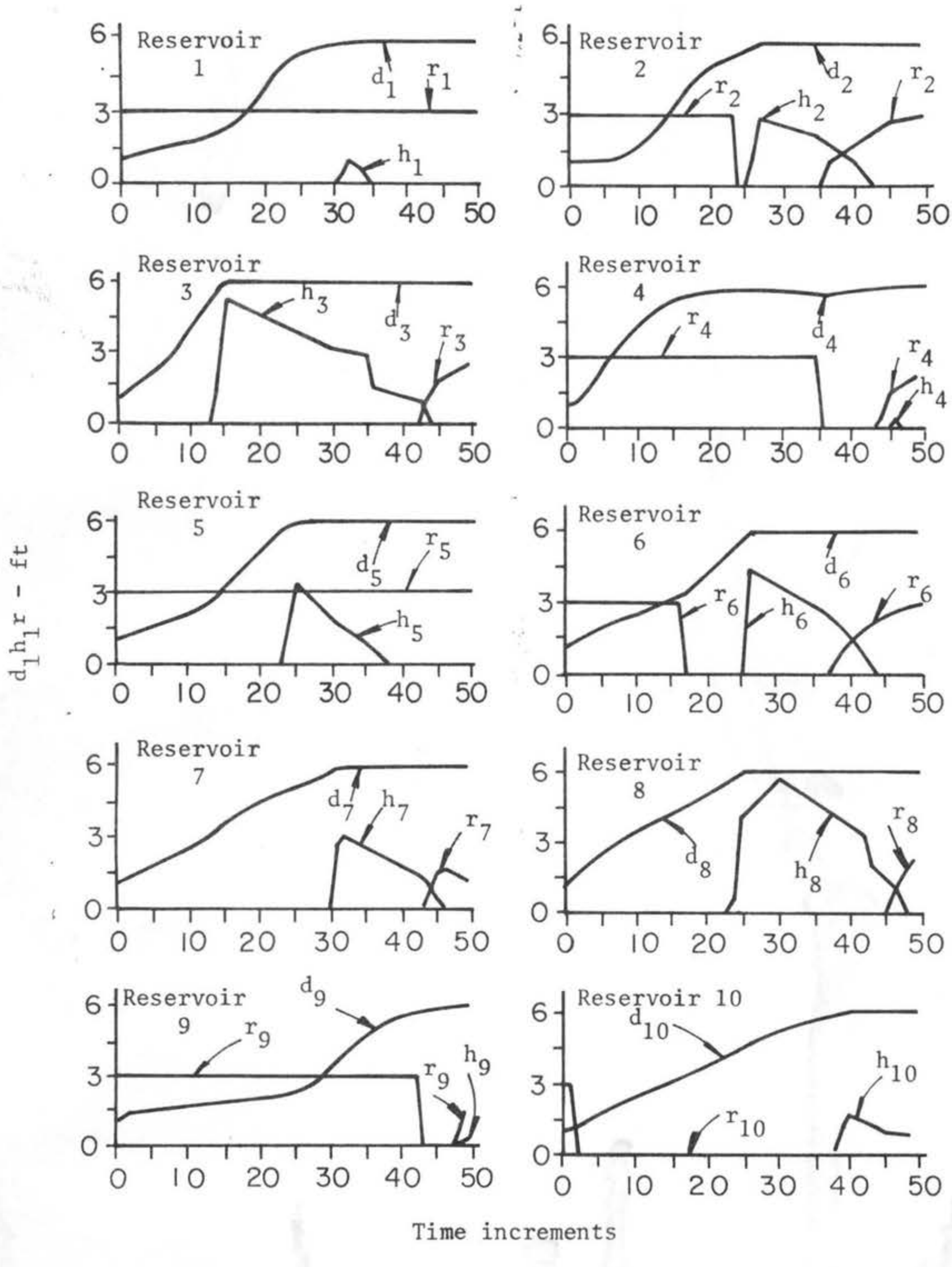

Fig. A5.2 The Computed Optimal State and Control VAriable Trajectories for Example A5.1

does not appear to be any changes that could be made to the controls that would reduce the objective function.

This run took 51 seconds of IBM 60-67 computer time (including the plotting routines) and reduced the objective function from an initial value of 63,634 to a final value of 54,480.

A.V.3 Example A5.2

In this example the flow constraint Q_{max}, was set at 165 vol/time increment, thus ensuring that the ten reservoirs behaved as one system. The relative values of the weighting factors are listed in Table A5.3.

TABLE A5.3

The Weighting Factors for Example A5.2

Reservoir Number	Overflow Weighting Factor	Throughput Weighting Factor
1	5.0	-.054
2	4.0	-.053
3	3.0	-.052
4	2.0	-.051
5	11.0	-.060
6	10.0	-.059
7	9.0	-.058
8	8.0	-.057
9	7.0	-.056
10	6.0	-.055

In this example the overflow weighting factors decreased upstream for each of the two main legs of the system. The results are presented in figure A5.3 and show that because of the flow constraint there was no advantage to be gained by reducing the overflow from

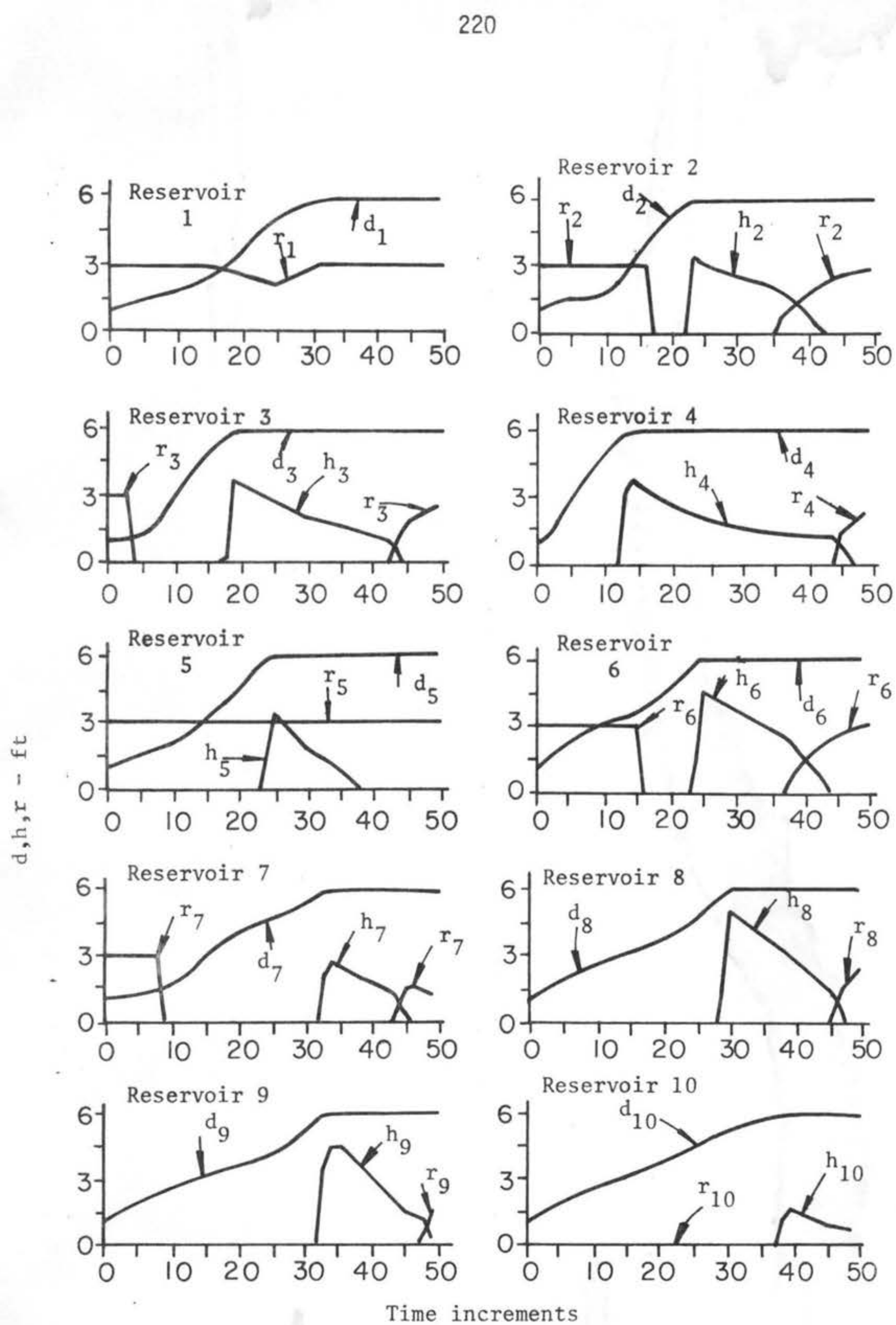

Fig. A5.3 The Computed Optimal State and Control Variable Trajectories for Example A5.2

reservoir 2 at the expense of reservoir 1 and thus there was no overflow from reservoir 1. However as a result of the zero overflow from 1, some initial outflow was required from reservoir 3 to ensure that reservoir 1 filled. Shutting down orifice 2 or orifice 3 one time step earlier would have reduced the inflow to reservoir 1 sufficiently that it would not have filled and by the operating rules would have been unable to open orifice 2 (and hence 3). It is possible that if orifice 2 shut down one time step later and orifice 3 shut down at time zero a slight reduction in the objective function might have been realized (depending upon the effects of numerical accuracy). This same effect is evident in the operation of orifice 7 which possibly should have been closed earlier. In this case a reduction in throughput from reservoirs 6 or 7 would have caused a slight reduction in the throughput from reservoir 5 which would allow a slight increase in throughput from reservoir 1 again preventing its filling. For reservoirs 5, 6 and 7 because there was no constraint on the outflow from reservoir 5 it would, however, have been advantageous to reduce the overflow from an upstream reservoir at the expense of a downstream reservoir but not to the extent shown in this example. (Compare the weighting factors and results of example A5.1 with those for this example). No improvement can be made on the results shown for reservoirs 8-10.

This run took 51 seconds of IBM 60-67 computer time (including plotting routines) and reduced the objective function from an initial value of 67,470 to a final value of 57,566.

A.V.4 Comments on the Examples

In the two examples presented in this appendix the initial guesses of the switching times were made quite large, thus ensuring that nearly all the reservoirs filled to their maximum values initially. At the start of each iteration these switching times are automatically adjusted to ensure that if a reservoir is spilling there is no inflow from an upstream reservoir with a lower overflow weighting factor. Thus the initial guess is immediately adjusted to a reasonably good guess. As a result, a large portion of the reduction of the objective function can be attributed to the redistribution of the total spill amongst the ten reservoirs of the system. On this basis it would appear, therefore, that any further improvement in the results in Example A5.2 would have been small.

APPENDIX VI

THE CONTROL PROGRAM

Figure A6.1 is a listing of the subroutine used to determine the value of the objective function for a given set of values of the switching times X_i. The listing is for the most general relationships between the overflow weighting factors as discussed in Appendix V.

The neumonics of the program and the required data inputs are noted in the comment statements. The logic follows the flow chart shown in figure V.5.

Fig. A6.1 Listing of the Control Subroutine for the Ten Reservoir Example

```
      SUBROUTINE PHI(JPRINT,X,PHIZ,IOVER,QTOT)
C THIS PROGRAM COMPUTES THE MINIMUM OVERFLOW FOR A TEN RESERVOIR PROBLEM
C GIVEN THE SWITCHING POINTS X(I). LEFT ARM HAS RESERVOIRS 1,2,3 AND 4
C RIGHT ARM HAS RESERVOIRS 5,6,8,9 AND 10, RESERVOIR 7 IS PARALLEL WITH
C RESERVOIR 8 AND FEEDS RESERVOIR 6, RESERVOIRS 1 AND 5 ARE DOWNSTREAM
C THE OVERFLOW WEIGHTING FACTORS ARE ORDERED  Z1 .GT. Z2  .GT. Z5 .GT.
C Z4 .GT. Z6 .GT. Z3 .GT. Z7 .GT. Z9 .GT. Z8 .GT. Z10 , THE THROUGHPUT
C WEIGHTING FACTORS ARE PROPORTIONAL TO (BUT VERY MUCH SMALLER IN
C MAGNITUDE THAN THE OVERFLOW FACTORS. Z21 AND Z22 ARE DUMMY
C
C  * *   *  *  *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * **
C  THE DATA REQUIREMENTS ARE:-                                              *
C  CO(I) - ORIFICE COEFFICIENTS          CW(I) - WEIR COEFFICIENTS           *
C  RM(I) - MAX ORIFICE RADIUS            DM(I) - MAX DEPTH BEHIND WEIR       *
C  DI(I) - INITIAL RES. DEPTHS           SA(I) AND SB(I) RESERVOIR STORAGE  *
C  QMAX(I)- MAX ALLOWABLE FLOW                   PARAMETERS                  *
C      DOWNSTREAM OF THE RESERVOIR      DT  - INTEGRATION TIME INTERVAL     *
C  QI(J) - POINTS ON THE INFLOW         TMAX - STOPPING TIME (=XMAX IN      *
C     HYDROGRAPH  DTQ  MIN APART                GRADIENT)                    *
C      *  **  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * * * * *
C
      DIMENSION  CO(10),COM(10),RM(10),CW(10),DM(10),QMAX(10),SA(10),
     1SB(10),DI(10),Z(22),QI(11),H(10),R(10),D(10),DN(10),X(10),
     2QII(13,101),ICLOSE(10),QTHRU(13),QOVER(10),INEXT(10),IDUM(10)
     3,IOVER(10),KFILL(10)
      DIMENSION ALPH(20) ,DPD(55,10),DPR(55,10),DPH(55,10),DPY(55),
     1DPT(55)
      DIMENSION QTOT(10)
    1 FORMAT(1X,41HTHE INPUT DATA TO THE CONTROL PROGRAM IS- )
    2 FORMAT(10F8.2)
    3 FORMAT(1X,5HDEPTH,2X,10(2X,F10.2))
    4 FORMAT(1X,5HQOVER,2X,10(2X,F10.2))
    5 FORMAT(1X,5HQTHRU,2X,10(2X,F10.2))
    6 FORMAT(1X,7HORIFICE,10(2X,F10.2))
    7 FORMAT(1X,4HWEIR,3X,10(2X,F10.2))
    8 FORMAT(1X,6HINFLOW,1X,13(1X,F8.2))
    9 FORMAT(5X,6HTIME =,F8.2)
   10 FORMAT(1X,23HTHE FINAL RESULTS ARE -,F10.2,6H = PHI)
   11 FORMAT(10I5)
      GO TO (770,880,880,880),JPRINT
C SECTION A * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *
C READ DATA, COMPUTE HYDROGRAPH POINTS FOR EACH DT INTERVAL ,COMPUTE *
C MISCELLANEOUS PARAMETERS, AND WRITE OUT DATA     * *  * *  * *  * * *
  770 WRITE(6,1)
      DO 71 I=1,10
      READ(5,2) CO(I),CW(I),DI(I),DM(I),RM(I),SA(I),SB(I)
      WRITE(6,2) CO(I),CW(I),DI(I),DM(I),RM(I),SA(I),SB(I)
   71 CONTINUE
      READ(5,2) (QMAX(I),I=1,10)
      READ(5,2) (Z(I),I=1,20)
      READ(5,2) DT,TMAX
      READ(5,11) JGRAPH
      WRITE(6,2) (QMAX(I),I=1,10)
      WRITE(6,2) (Z(I),I=1,20)
      WRITE(6,2) DT,TMAX
      WRITE(6,11) JGRAPH
C COMPUTE MISCELLANEOUS PARAMETERS  COMMON TO ALL ITERATIONS
```

```
      Z(21) = 0.00
      Z(22) = 100.00
      DO 72 I=1,10
   72 COM(I) =CO(I)*(RM(I)**2)
      NT = TMAX/DT + 1.01
      NTM = NT - 1
      ND = NTM/10
      DTQ = ND
      DO 73 I=1,13
      READ(5,2)(QI(L),L=1,11)
      DO 74 J=1,10
      K = J + 1
      DIFF = (QI(K) - QI(J))/DTQ
      M = (J-1)*ND + 1
      N = J*ND
      DO 74 L=M,N
      P = L-M
   74 QII(I,L) = QI(J) + P*DIFF
      QII(I,NT) = QI(11)
      WRITE(6,2) (QI(L),L=1,11)
   73 CONTINUE
C * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * *
C START SECTION B - EVALUATION OF THE OBJECTIVE FUNCTION
C  COMPUTE THROUGHPUTS AND OVERFLOWS FOR THE GIVEN SWITCHING POINTS
C
C  COMPUTE INITIAL CONDITIONS FOR T=0.0
880   PHIZ = 0.0
      T = 0.0
      LONE = X(1) + DT
      XONE = LONE
      DO 5001 KJ=1,10
5001  QTOT(KJ) = 0.0
      DO 51 I=1,10
      ICLOSE(I)=1
      IF(X(I).LE.0.0) ICLOSE(I)=2
      IOVER(I) = 1
      INEXT(I)=I
      IF(I.NE.5) GO TO 51
      ICLOSE(5)=5
      IF(ICLOSE(1).NE.2) GO TO 51
      ICLOSE(1)=5
      INEXT(1) =5
      ICLOSE(5)=1
      INEXT(5)=1
   51 D(I) = DI(I)
      DO 52 I=11,13
   52 QTHRU(I) = 0.0
C BEGIN OUTER LOOP ON TIME STEPS  * *  * *  * *  * *  * *  * *  * *  * *
      DO 500 J=1,NTM
      LOOP = 0
      JZLOOP = 0
C COMPUTE ORIFICE THROUGHPUTS   * *  * *  * *  * *  * *  * *  * *  * *
  202 DO 118 KJ=1,10
  118 QTHRU(KJ) = 0.0
      IF(XONE.GT.X(5)) GO TO 2021
      IF(X(5).LT.T-DT.OR.X(5).GT.T) GO TO 2021
      ICLOSE(1) = 1
      ICLOSE(5) = 5
      DO 2022 KI=1,10
      IF(INEXT(KI).EQ.1) GO TO 2023
```

```
            IF(INEXT(KI).NE.5) GO TO 2022
            INEXT(KI) = 1
            GO TO 2022
     2023   INEXT(KI) = 5
     2022   CONTINUE
     2021   CONTINUE
            DO 100 KI = 1,10
            K =INEXT(KI)
     C  CLOSE ORIFICES IF T.GT.X(K) OR IF DOWNSTREAM RESERVOIR OVERFLOWS
     C  PROVIDED ZU.LT.ZD
            IF(ICLOSE(K).GT.1.OR.K.EQ.5) GO TO 111
            IF(K.EQ.1) GO TO 115
            IF(T.LE.X(K)) GO TO 111
            ICLOSE(K) = 2
        111 IF(JPRINT.EQ.2.OR.JPRINT.EQ.3) GO TO 112
            IF(D(K).GE.DM(K).AND.IOVER(K).LT.3) GO TO 114
            GO TO 112
     114    IF(K.EQ.4.OR.K.EQ.6.OR.K.EQ.7.OR.K.EQ.10) GO TO 112
            LK = K+1
            IF(ICLOSE(LK).GE.2) GO TO 112
            IF(Z(LK).GT.Z(K)) GO TO 112
            ICLOSE(LK) = 2
            IF(T.GT.X(LK)) GO TO 112
            X(LK) = T - DT/2.0
            IF(ICLOSE(K).EQ.5) X(LK) = T + DT/2.0
            GO TO 112
        115 IF(T-DT.LE.X(1).AND.T.GT.X(1))GO TO 116
            GO TO 111
        116 ICLOSE(1)=5
            ICLOSE(5)=1
            INEXT(KI)=5
            K = 5
            M = KI+1
            DO 117 LK=M,10
            IF(INEXT(LK).EQ.5) INEXT(LK) = 1
        117 CONTINUE
            GO TO 111
        112 M = ICLOSE(K)
            GO TO (60,70,80,90,60),M
     C COMPUTE THROUGHPUT ON THE BASIS ORIFICE AT MAX ALLOWABLE
         60 QTHRU(K) = COM(K)*SQRT(D(K))
            IF(K.EQ.1.OR.K.EQ.5) GO TO 61
            IF(QTHRU(K) + QII(K,J).GT.QMAX(K)) QTHRU(K) = QMAX(K) - QII(K,J)
            IF(K.EQ.7.OR.K.EQ.8) GO TO 63
            GO TO 100
         61 L = 5
            N = 1
            IF(K.EQ.5) L=1
         62 IF(QTHRU(K)+QII(K,J)+QTHRU(L)+QII(L,J).GT.QMAX(N)) QTHRU(K) =
           1QMAX(N)- QTHRU(L)-QII(K,J) -QII(L,J)
            GO TO 100
         63 N=5
            L=7
            IF(K.EQ.7) L=8
            GO TO 62
     C COMPUTE THROUGHPUT ON THE BASIS ORFICE IS FULLY CLOSED
         70 QTHRU(K) =0.0
            GO TO 100
     C COMPUTE THROUGHPUT ON THE BASIS DDOT(DOWNSTREAM) = 0
         80 JL = K - 1
```

```
      IF(K.EQ.8) JL = 6
      IF(JL.EQ.6) GO TO 85
      QTHRU(K) = QTHRU(JL) - QII(K,J)
      IF(QTHRU(K).LT.0.0) GO TO 70
   81 IF(QTHRU(K)+QII(K,J).GT.QMAX(K)) QTHRU(K) =QMAX(K) - QII(K,J)
      QDUM = COM(K)*SQRT(D(K))
      IF(QTHRU(K).GT.QDUM) QTHRU(K) = QDUM
      GO TO 100
   85 L = 7
      IF(K.EQ.7) L=8
      QSUM = QII(L,J) + QII(K,J) + QTHRU(L)
      QTHRU(K) = QTHRU(JL) - QSUM
      IF(QTHRU(K).LE.0.0) GO TO 70
      IF(QTHRU(K) + QSUM.GT.QMAX(5)) QTHRU(K) = QMAX(5)-QSUM
      GO TO 81
C COMPUTE THROUGHPUT ON THE BASIS DDOT(K) = 0
   90 IF(K.EQ.6) GO TO 95
      L = K + 1
      IF(K.EQ.4) L = 12
      IF(K.EQ.7) L = 13
      QTHRU(K) = QII(L,J) + QTHRU(L)
      IF(K.EQ.5.OR.K.EQ.1) GO TO 91
      GO TO 81
   91 QDUM = COM(K)*SQRT(D(K))
      IF(QTHRU(K).GT.QDUM) QTHRU(K) = QDUM
      GO TO 61
   95 QTHRU(K) = QII(7,J) + QII(8,J) + QTHRU(7) +QTHRU(8)
      GO TO 81
  100 CONTINUE
C  CHECK TO SEE THAT THE FULL CAPACITY OF THE LINE FROM RESERVOIRS
C  1 AND 5 IS BEING USED TO CAPACITY.  NOTE THAT IF  THE HYDROGRAPHS
C  INCREASED SHARPLY, SOME RESERVOIRS MIGHT OVERFLOW THAT HAD PREVIOUSLY
C  CEASED TO OVERFLOW
      JZCNT = 0
      IF(ICLOSE(1).NE.4.AND.ICLOSE(5).NE.4) GO TO 101
      IF(QTHRU(1) + QTHRU(5) + QII(1,J) + QII(5,J) + .001.GE.QMAX(1))
     1 GO TO 101
      IF(ICLOSE(5).EQ.4) GO TO 1031
      ICLOSE(1) = 5
      ICLOSE(5) = 1
      KL = 2
      GO TO 1032
 1031 ICLOSE(1) = 1
      ICLOSE(5) = 5
      KL = 6
 1032 DO 102 KJ = KL,10
      IF(ICLOSE(KJ).EQ.4) GO TO 103
      IF(ICLOSE(KJ).EQ.1) GO TO 104
      ICLOSE(KJ) = 3
      GO TO 104
  103 IF(KJ.EQ.7) GO TO 102
      ICLOSE(KJ) = 3
      IF(KJ.EQ.4) GO TO 104
  102 CONTINUE
  104 JZCNT = 10
  101 CONTINUE
C COMPUTE RESERVOIR DEPTHS AND WEIR OVERFLOWS - MAKE CHECKS FOR CHANGES
C IN COMPUTATION ORDER  *  * *  * *  * *  * *  * *  * *  * *  * *  * *  * * *
      DO 200 K=1,10
      IF(JZCNT.EQ.10) GO TO 190
```

```
          L=K+1
          IF(K.EQ.4) L=12
          IF(K.EQ.7) L=13
          QIN = QII(L,J) + QTHRU(L)
          IF(K.EQ.6) QIN = QIN+ QII(8,J) + QTHRU(8)
          AREA = SA(K) + SB(K)*D(K)
          DDOT = (QIN - QTHRU(K))/AREA
          DN(K) = D(K) +DDOT*DT
          IF(DN(K)+.002  .GE.DM(K))GO TO 110
          IF(D(K)+.002   .GE. DM(K) .AND.DN(K)+.002   .LT.DM(K)) GO TO 120
          QOVER(K) = 0.00
          GO TO 200
      110 QOVER(K) = QIN - QTHRU(K) - (DM(K) - D(K))*AREA/DT
          IF(DN(K).GT.DM(K)) DN(K) = DM(K)
          IF(QOVER(K).LT.0.0) QOVER(K) = 0.0
          IF(IOVER(K).NE.3.AND.QOVER(K).GT..002) IOVER(K) = 2
          GO TO 200
C COMPUTE CHANGES IN THE ORDER OF CALCULATION - FIND RESERVOIR WITH
C NEXT LARGEST Z THAT IS OVERFLOWING OR MAY OVERFLOW
      120 IMAX = 0
          QOVER(K) = 0.00
          IOVER(K) = 3
C  DEFINE THE LIMITS OF THE SEARCH
C  FIND LIMITS UPSTREAM
          NBL = K + 1
          NBU = 4
          IF(K.GE.5) NBU = 10
          KU = K
          IF(NBL.EQ.5.OR.NBL.EQ.11) GO TO 1502
          DO 1501 KJ=NBL,NBU
          IF(KJ.EQ.7) GO TO 1504
          IF(ICLOSE(KJ).EQ.1) GO TO 1502
          KM = KJ - 1
          IF(ICLOSE(KJ).EQ.3.AND.DN(KM)+.002.LT.DM(KM)) GO TO 1502
          GO TO 1501
 1504     IF(ICLOSE(7).EQ.1.AND.ICLOSE(8).EQ.1) GO TO 1502
          IF(ICLOSE(7).EQ.3.AND.ICLOSE(8).EQ.3.AND.DN(6)+.002.LT.DM(6)) GO
        1 TO 1502
 1501     KU = KJ
C SEARCH FOR LIMITS DOWNSTREAM
 1502     NBU = K
          NBL = 1
          IF(K.GE.5) NBL = 5
          KD = K
          KR = NBU + 1
          DO 1503 KJ=NBL,NBU
          KR = KR - 1
          IF(KR.EQ.5) GO TO 1503
          IF(KR.EQ.7.AND.(T+DT.LT.X(6).OR.ICLOSE(6).EQ.2)) GO TO 1505
          IF((T+DT.LT.X(KR).OR.ICLOSE(KR).EQ.2).AND.KR.NE.7) GO TO 1505
 1503     KD = KR
 1505     IF(KD.EQ.1.OR.KD.EQ.5) GO TO 1550
          GO TO 1520
C  SEARCH FOR PARALLEL RESERVOIRS THAT MAY BE AIDED
 1550     CONTINUE
          IF(K.GE.5) GO TO 1506
          NBL = 5
          NBU = 10
          GO TO 1510
 1506     NBL = 1
```

```
      NBU = 4
 1510 KP = 0
      DO 1507 KJ=NBL,NBU
      IF(KJ.EQ.7) GO TO 1509
      IF(ICLOSE(KJ).EQ.1) GO TO 1515
      IF(KJ.EQ.1.OR.KJ.EQ.5) GO TO 1508
      KM = KJ - 1
      IF(ICLOSE(KJ).EQ.3.AND.DN(KM)+.002.LT.DM(KM)) GO TO 1515
      GO TO 1507
 1508 IF(ICLOSE(KJ).EQ.5.AND.QTHRU(KJ).GE.COM(KJ)*SQRT(D(KJ)))GO TO 1515
      GO TO 1507
 1509 IF(ICLOSE(7).EQ.1.AND.ICLOSE(8).EQ.1) GO TO 1515
      IF(ICLOSE(7).EQ.3.AND.ICLOSE(8).EQ.3.AND.DN(6)+.002.LT.DM(6))
     1 GO TO 1515
 1507  KP = KJ
C  LIMITS OF SEARCH ARE NOW DEFINED
 1515 IF(KP.EQ.0) GO TO 1520
      NBL = KP + 1
      NBU = 4
      IF(K.LT.5) NBU = 10
      GO TO 1530
 1520 IF(K.GT.4) GO TO 1521
      NBL = 5
      NBU = 10
      GO TO 1530
 1521 NBL = 1
      NBU = 4
C  SEARCH FOR RESERVOIR THAT IS OR MAY BE OVERFLOWING
 1530 ZMAX = 0.0
      IMAX = 0
      DO 1531  KJ = 1,10
      IF(IOVER(KJ).EQ.3) GO TO 1531
      IF(KJ.LT.KD.OR.KJ.GT.KU) GO TO 1531
      IF(KJ.GE.NBL.AND.KJ.LE.NBU) GO TO 1531
      IF(Z(KJ).LT.ZMAX) GO TO 1531
      ZMAX = Z(KJ)
      IMAX = KJ
 1531 CONTINUE
      IF(ZMAX.GT.0.0) GO TO 1540
      IF(K.EQ.4.OR.K.EQ.7.OR.K.EQ.10) GO TO 200
      IR = K + 1
      IF(IR.EQ.7) GO TO 1532
      IF(ICLOSE(IR).EQ.3) GO TO 200
 1533 ICLOSE(IR) = 3
      GO TO 190
 1532 IF(ICLOSE(7).NE.3) GO TO 1533
      IR = 8
      IF(ICLOSE(IR).NE.3) GO TO 1533
      GO TO 200
 1540 CONTINUE
C  RESET RESERVOIR CONTROLS
      IF((K.GE.5.AND.IMAX.GE.5).OR.(K.LT.5.AND.IMAX.LT.5)) GO TO 122
      IF(K.GE.5.AND.IMAX.LT.5) GO TO 124
C  CASE K.LT.5 AND IMAX.GE.5
      MK = 1
      NK = 6
      LK = 5
  142 DO 140 KJ=MK,K
  140 ICLOSE(KJ) = 4
      ICLOSE(LK) = 5
```

```
 143  DO 141 KJ=NK,IMAX
      ICLOSE(KJ) = 3
      IF(KJ.EQ.7.AND.IMAX.GT.7) ICLOSE(KJ) = 4
 141  CONTINUE
      GO TO 190
C  CASE KGT.5 AND IMAX.LT.5
 124  MK = 5
      NK = 2
      LK = 1
      GO TO 142
C  CASE K AND IMAX BOTH GT. OR LT. 5
 122  IF(K.GT.IMAX) GO TO 145
      NK = K + 1
      IF(K.EQ.7) NK = K
      GO TO 143
 145  ICLOSE(K) = 4
      GO TO 190
  190 LOOP = LOOP + 1
      JZCNT = 0
      IF(LOOP.GT.20) GO TO 666
      LK=0
C  COMPUTE THOSE RESERVOIRS WITH ICLOSE = 1
      KM = 1
 1192 DO 191 KJ=1,10
      IF(ICLOSE(KJ).NE.KM) GO TO 191
 1193 LK = LK + 1
      INEXT(LK) = KJ
  191 CONTINUE
      IF(KM.EQ.5) GO TO 197
      IF(KM.EQ.2) GO TO 192
C  COMPUTE THOSE RESERVOIRS WITH ICLOSE = 2
      KM = 2
      GO TO 1192
C  COMPUTE THOSE RESERVOIRS WITH ICLOSE = 4.  DO IN ORDER OF DECREASING
C  RESERVOIR NUMBER
  192 KM = 4
  193 NK = 0
      DO 194 KJ=1,10
      IF(ICLOSE(KJ).NE.KM) GO TO 194
      NK = NK + 1
      IDUM(NK) = KJ
  194 CONTINUE
      IF(NK.EQ.0) GO TO 197
      DO 196 KJ=1,NK
      LK = LK + 1
      JMAX = 0
      IF(KM.EQ.3) JMAX = 100
      DO 195 KL=1,NK
      MN = IDUM(KL)
      IF((KM.EQ.3.AND.MN.GT.JMAX).OR.MN.EQ.21) GO TO 195
      IF((KM.EQ.4.AND.MN.LT.JMAX).OR.MN.EQ.22) GO TO 195
      JMAX = MN
      MP = KL
  195 CONTINUE
      INEXT(LK) = JMAX
      IDUM(MP) = 18 + KM
  196 CONTINUE
  197 IF(KM.EQ.3) GO TO 202
      IF(KM.EQ.4) GO TO 1197
C  COMPUTE THOSE RESERVOIRS WITH ICLOSE = 3. DO IN ORDER OF INCREASING
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      C  RESERVOIR NUMBER
            KM = 3
            GO TO 193
      C  COMPUTE THOSE RESERVOIRS WITH ICLOSE = 5
       1197 KM = 5
            GO TO 1192
        666 WRITE(6,11) (INEXT(I),I=1,10      )
            WRITE(6,11) (ICLOSE(I),I=1,10)
            WRITE(6,11) K
            WRITE(6,2) (X(I),I=1,10)
            GO TO 661
        200 CONTINUE
            IF(JPRINT.NE.3) GO TO 250
        661 CONTINUE
            WRITE(6,9) T
            WRITE(6,3) (D(KM),KM=1,10)
            WRITE(6,4) (QOVER(KM),KM=1,10)
            WRITE(6,5) (QTHRU(KM),KM=1,10)
            WRITE(6,8) (QII(KM,J),KM=1,13)
            DPT(J) = T
            JKM = J
      2402  DO 240 KM=1,10
            DPD(JKM,KM) = D(KM)
            R(KM) = SQRT(QTHRU(KM)/(CO(KM)*SQRT(D(KM))))
            DPR(JKM,KM) = R(KM)
            H(KM) = (QOVER(KM)/CW(KM))**.6667
            DPH(JKM,KM) = H(KM)
      240   CONTINUE
            WRITE(6,6) (R(KM),KM=1,10)
            WRITE(6,7) (H(KM),KM=1,10)
            IF(LOOP.GT.20) STOP 2
        250 CONTINUE
            DO 5002 KJ=1,10
      5002  QTOT(KJ) = QTOT(KJ) + QOVER(KJ)
            DO 260 KM=1,10
            LK =KM + 10
            PHIZ = PHIZ +QTHRU(KM)*Z(LK) + QOVER(KM)*Z(KM)
        260 D(KM) = DN(KM)
        500 T = T + DT
            PHIZ = PHIZ*DT
            IF(JPRINT.EQ.3) WRITE(6,10) PHIZ
      C  PLOT RESULTS
            IF(JPRINT.NE.3) GO TO 3301
            IF(JGRAPH.NE.0) GO TO 3301
            READ(5,11) ND,KA,KB,KC,KD
            READ(5,2) HA,HB,HC,VA,VB,VC
            IU = 6
            DO 3303 JKM=1,10
            READ(5,3310) (ALPH(I),I=1,20)
      3310  FORMAT(20A4)
            DO 3303 NF=1,3
            DO 3304 NPD=1,51
            GO TO (3305,3306,3307),NF
      3305  DPY(NPD) = DPD(NPD,JKM)
            GO TO 3304
      3306  DPY(NPD) = DPR(NPD,JKM)
            GO TO 3304
      3307  DPY(NPD) = DPH(NPD,JKM)
      3304  CONTINUE
            CALL CGPL(DPT,DPY,DPY,ND,NF,KA,KB,KC,KD,HA,HB,HC,VA,VB,VC,ALPH,IU)
```

107

12

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3303 CONTINUE
C  PLOT HYDROGRAPHS
     READ(5,11) ND,KA,KB,KC,KD
     READ(5,2) HA,HB,HC,VA,VB,VC
     NI = -2
     NII = 0
     DO 4301 JKM=1,5
     NI = NI + 3
     NII = NII + 3
     IF(NII.GT.13) NII = 13
     NF = 0
     READ(5,3310) (ALPH(I),I=1,20)
     DO 4302 JK=NI,NII
     NF = NF + 1
     DO 4303 JL=1,51
4303 DPY(JL) = QII(JK,JL)
     CALL CGPL(DPT,DPY,DPY,ND,NF,KA,KB,KC,KD,HA,HB,HC,VA,VB,VC,ALPH,IU)
4302 CONTINUE
4301 CONTINUE
     NF = 0
     CALL CGPL(DPT,DPY,DPY,ND,NF,KA,KB,KC,KD,HA,HB,HC,VA,VB,VC,ALPH,IU)
3301 CONTINUE
     RETURN
     END
```