METHODS OF CREATING A COMPLEX SEAWAY IN A MODEL BASIN

by

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Fort Collins, Colorado

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ABSTRACT

The wave patterns due to long wave generators which produce displacements uniformly distributed along their length are first investigated. Formulas are developed by which the difraction pattern produced by single sources or long wave generators can be estimated. Approximations have been made when simplified computation procedures could be obtained thereby.

These methods are then used to investigate the types of short crested seas which could be produced in circular or rectangular model basins by a limited number of wave generators. It is found that short crested seas can be produced in this way but that there is difficulty in obtaining uniform conditions over a wide test area. A servo-motor type of wave generator whose performance can be controlled by a cam is described. Such a wave generator has advantages when used for producing irregular types of waves.

The requirements which must be met if a replica of an actual storm sea is to be reproduced over the entire area of a model basin and sustained for a period of time ample for testing purposes are next investigated. This investigation makes use of the mathematical developments which are concerned with the conditions of uniqueness for solutions of the differential equation of Laplace. It is concluded that the replica sea can be realized if the observed undulations are reproduced, to model scale, completely

around the perimeter of the model basin and the driving forces of wind are also supplied.

Since none of the conventional types of wave generators would be satisfactory for this service, because of the waves coming in to their location from other generators, a new type of wave generator is devised. This generator embodies the elements described previously but its action is now controlled by a device which senses the wave height at its location, compares it with a program, and acts to bring the actual wave height to the programmed wave height. The number of wave generators required for production of the replica sea is investigated. With a finite number of wave generators a true replica sea can only be approximated but the approximation grows better as the lengths of the wave generators is shortened and their number correspondingly increased. In any case the length of each single wave generator must be short compared to the length of the model to be tested. The requirements which must be met if wave generators of the new type are to function properly are studied analytically. Because of the stringency of these requirements, it is considered advisable to build and test a pair of them before any large scale installation is attempted.

NOTATION

The following notation is used:

a	a wave amplitude	(feet)
a_3, b_3, c_3	amplitude constants	
a_1	a specified radius	
a_2	an amplitude constant	
A	an area	$(\text{feet})^2$
B_1	an amplitude constant	(feet)
b and b_1	constants	
b_2	width of a basin	(feet)
b_4	a constant	
C	the velocity of wave propagation	(ft/sec)
C_1	a distance	(ft)
C_w	celerity of a wave	(ft/sec)
D_1, D_2	volume displacements	$(\text{ft})^3$
D_m	a maximum volume displacement	$(\text{ft})^3$
$e = 2.71828 +$		
g	the acceleration of gravity	(ft/sec^2)
h	depth of a wave basin	
J_o and Y_o	Bessels functions of order zero as tabulated in reference (6)	
$k = \frac{2\pi}{\lambda}$		
K	a spring constant	(lb/ft)
L	length of a wave generator	

n	a quantity determined from equation 8. This quantity approaches $n = \frac{2\pi}{\lambda}$ at sufficiently great distances to make the product nr large compared to unity.	(lb/ft)
n_1	distance along a normal	
n_2	a ratio	
p	a pressure	(lb/ft^2)
p_o	an initial pressure	(lb/ft^2)
p_1	a pressure change	(lb/ft^2)
P_2	average power requirement	(ft lb/sec)
Q_m	The maximum volume displacement produced by a linear wave generator of length L.	(ft^3)
r	a radius	(ft)
R	The distance from the center of a wave generator to the point at which the wave amplitude is to be computed	(feet)
R_1	a ratio	
S_1	The distance between the centers of two radially symmetrical wave generators	(feet)
S	a surface	
t	time	(seconds)
T	the period of a wave	(seconds)
u	velocity, positive in the direction of r or x	
v	a volume	(ft)^3
v_o	an initial volume	(ft)^3
V	a volume	(ft)^3

w	density of water	(lb/ft^3)
w_1	a compressibility factor	
W	weight	(lb)

x, y, z coordinates. The coordinate x is measured horizontally in the direction of wave propagation, y horizontally at right angles to x and z, z is measured vertically from the undisturbed water surface. The positive direction is up. (feet)

y_1	a departure from an equilibrium position	(feet)
α	an angle between a line passing through two sources and a radial line drawn from one of them	(radians)
α_1	the angle between R and the long dimension of a wave generator	(radians)
η	a distance measured along the length of a wave generator from the center of a linear wave generator.	
λ	the wave length	(feet)
$\sigma = \frac{2\pi}{T}$		(1/sec)
ϕ	a velocity potential	(ft^2/sec)
ζ	the height of a wave above the stable level	(feet)
ϵ	a constant specifying phase	(sec)
$\alpha_3, \beta_3, \gamma_3, \zeta_3$	constants relating to the operation of a wave generator (See Fig. 13)	
α_3	ratio of the spool valve displacement to the port width (dimensionless)	
β_3	ratio of the sleeve displacement to the port width (dimensionless)	

σ_3	displacement of the plunger from a neutral position	(ft)
ζ_3	departure from normal level in the wave basin	(ft)

I. WAVE MOTION PRODUCED BY LINEAR WAVE GENERATORS

Wave Patterns

The developments described herein are directed to the development of ready means for calculating the characteristics of wave trains produced by wave generators which act by producing volume displacements along a linear interval. It has been accepted that the utility of the final results is of paramount importance and for this reason approximations have been used whenever computation procedures could be simplified by their adoption.

A word of explanation may be worthwhile concerning the procedure followed in this development. A first simple fundamental solution of the hydrodynamic equations was sought which could be used as a basis for constructing the more elaborate cases required for the present purposes. This simple solution relates to the train of waves diverging from an isolated point source in water of any depth when the disturbance at the source is of a simple, harmonic type. The case of a wave generator of length L was then obtained by summing the effects produced by simple solutions of uniform strength and distributed uniformly along the length of the wave generator. The amplitude of the wave train produced is related to the displacement produced by the wave generator. The solution thus obtained relates to the behavior of an isolated generator from which waves are free to progress in any direction. A different case is presented, however, by a wave generator operating in an experimental wave basin since the walls of the basin interfere with the progress of the waves in certain directions. It is proposed to account for these boundaries by using the method of images. In this way the effect of operating the wave generator adjacent to a wall, in a corner, within a rectangular strip or within a rectangular basin can be found. In all these cases the same set of simple formulas can be used.

Mathematical Basis

Consider the velocity potential :

$$\phi = \frac{B_1 g}{\sigma}\left[J_0(nr)\sin\sigma t - Y_0(nr)\cos\sigma t\right]\frac{\cosh n(z+h)}{\cosh nh} \quad \cdots (1)$$

This expression satisfies LaPlace's equation (Ref. 7, p. 76)

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad \cdots\cdots (2)$$

and to be acceptable for our purposes it must also satisfy the conditions. (Ref. 7, pp. 73 & 74).

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{when } z = -h.$$

$$\frac{\partial\zeta}{\partial t} = -\frac{\partial\phi}{\partial z} \quad \text{on the free surface} \qquad \cdots (3)$$

$$\zeta = \frac{1}{g}\frac{\partial\phi}{\partial t} \quad \text{on the free surface.}$$

From (1), by differentiation

$$\frac{\partial \phi}{\partial z} = \frac{B_1 g}{\sigma}\left[J_o(nr)\sin\sigma t - Y_o(nr)\cos\sigma t\right]\frac{n\sinh n(z+h)}{\cosh nh} \qquad \text{---- (4)}$$

Since $\frac{\partial \phi}{\partial z}$ is the vertical component of velocity and this expression becomes zero at the bottom, when $z = -h$, the first condition is satisfied.

The second condition yields

$$\zeta = B_1\left[J_o(nr)\cos\sigma t + Y_o(nr)\sin\sigma t\right] \qquad \text{---- (5)}$$

To satisfy the third condition with

$$\frac{\partial \zeta}{\partial t} = B_1\sigma\left[-J_o(nr)\sin\sigma t + Y_o(nr)\cos\sigma t\right] \qquad \text{----- (6)}$$

and

$$-\frac{\partial \phi}{\partial z} = -\frac{B_1 g}{\sigma}\left[J_o(nr)\sin\sigma t - Y_o(nr)\cos\sigma t\right]\frac{n\sinh nh}{\cosh nh} \qquad \text{--- (7)}$$

It is required that

$$B_1\sigma = \frac{B_1 g}{\sigma}\,\frac{n\sinh nh}{\cosh nh}$$

By rearrangement and multiplication of both sides of the expression by the factor h this relation can be put into the form

$$\frac{h\sigma^2}{g} = nh \tanh nh \qquad \text{- - - - -(8)}$$

This is the basic relation connecting the values of σ and n. If the frequency is to be specified by selecting a value for σ the corresponding value of n can be determined immediately through use of Fig. 1.

At a distance from the source sufficient to make the quantity nr large compared to unity, the Bessel functions Jo(nr) and Yo(nr) take the approximate forms: (Ref. 10, pp. 158 and 161)

$$J_o(nr) \cong \sqrt{\frac{2}{\pi nr}} \cos\left(nr - \frac{\pi}{4}\right)$$

$$\text{- - - - -(9)}$$

$$Y_o(nr) \cong \sqrt{\frac{2}{\pi nr}} \sin\left(nr - \frac{\pi}{4}\right)$$

if $nr \gg 1$.

Under these conditions a wave length λ can be specified since

$$n = \frac{2\pi}{\lambda} \qquad \text{- - - - - -(10)}$$

If a wave length is chosen the value of n obtained from this relation can be substituted into Eq. (8) to find the quantity σ which will, in turn, determine the frequency.

The Eq. (9) can be used to establish some additional important relationships. Under the condition stated, Eq. (5) for the wave height takes the form

$$\zeta \cong B_1 \sqrt{\frac{2}{\pi n r}}\left[\cos\left(nr - \frac{\pi}{4}\right)\cos\sigma t + \sin\left(nr - \frac{\pi}{4}\right)\sin\sigma t\right]$$

If $nr \gg 1$.

and then can be written

$$\zeta \cong B_1 \sqrt{\frac{2}{\pi n r}}\left[\cos\left(nr - \sigma t - \frac{\pi}{4}\right)\right] \qquad \text{------} (11)$$

If we fix attention on a certain part of the wave this form of the expression shows that r must increase with time if the same phase position is to be maintained. Suppose we choose to watch the crest. This will require that the cosine term will have a maximum value and this imposes the relation

$$nr - \sigma t - \frac{\pi}{4} = m\,2\pi$$

where m is some whole number. By differentiation of this expression we can obtain the relation

$$\frac{dr}{dt} = \frac{\sigma}{n}$$

which can be interpreted as the rate at which the crest progresses. This is the wave velocity C. This relation can be combined with expressions (8) and (10) to obtain a formula for the wave propagation velocity.

$$c^2 = \frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda}. \qquad \text{------} (12)$$

If $nr \gg 1$.

Although these relations are approximate, a scrutiny of the roots of the J_o and Y_o Bessel functions will indicate that they should hold closely enough for most engineering purposes beyond two wave lengths distance from the source.

In order to produce waves of a specific amplitude it is necessary to know how much displacement is needed at the source to maintain them. The displacement volume is given by an integral of the type :

$$D_1 = 2\pi r_o \int \int_{-h}^{0} u\,dz\,dt .$$

This can be evaluated in the form

$$D_1 = \frac{2\pi r_o B_1 g}{\sigma^2}\left[-J_o'(nr)\cos\sigma t - Y_o'(nr)\sin\sigma t\right]\frac{\sinh nh}{\cosh nh} . \qquad ---(13)$$

This displacement reaches a maximum value of

$$D_m = \frac{2\pi r_o B_1 g}{\sigma^2}\left[(J_o'(nr_o))^2 + (Y_o'(nr_o))^2\right]\frac{\sinh nh}{\cosh nh} . \qquad ---(14)$$

The energy fed into the wave motion can be obtained by integrating the product of the pressure changes

$$p_1 = \frac{w}{g}\frac{\partial\phi}{\partial t}$$

and the velocity u over the surface of a cylinder of radius r and extending from the surface of the water to the bottom of the basin. The development is somewhat lengthy and it seems desirable therefore to give only the final result. By making use of the relation (See reference 10, page 156.)

$$Y_0(nr_0)J_0'(nr_0) - Y_0'(nr_0)J_0(nr_0) = -\frac{2}{\pi n r_0}$$

The average power requirement will be

$$P_2 = \frac{wB^2g}{n\sigma}\left[\frac{\sinh nh \cosh nh + nh}{\cosh^2 nh}\right] \qquad \text{- - - (15)}$$

The velocity head was neglected in the development of this formula. It may be commented that the selection of a motor to drive a wave generator may depend more upon the starting torque requirements than the power needed to maintain the wave train. The device used to make the displacement should not extend to too great a depth. It is suggested to create the displacement within the depth

$$h_e = \frac{1}{n}\tanh nh. \qquad \text{- - - - - - - (16)}$$

These displacement and power requirements are sufficient to maintain a wave train which completely surrounds the source. If the wave train is bounded by a straight vertical wall which passes through the source then only one half of these displacements and power requirements are needed.

In some cases it is desired to propagate a wave train of limited width and this may be accomplished with some effectiveness by making use of the possibilities of interference. Suppose, for example, that two equal, in phase sources are operating at a separation S_1 of one half a wave length. It can be expected that the wave motion in the direction of the line joining the sources would be almost completely annuled because the waves from the two sources would be of almost equal amplitude and would be 180 degrees out of phase. Along a normal to this line, drawn from a point midway between the sources, however, the wave motion would be enhanced because the waves from the two sources would be nearly in phase. The wave crests would be nearly circular in form but their height would vary in each quadrant from a maximum to nearly zero. In this way a very definite concentration of the wave motion into a portion of the surface area can be accomplished. It will be of interest to develop the case of the two sources somewhat more fully.

If α represents an angle between a radius drawn from one of the sources and the line which passes through both of them, a point at radius r from one source will lie at the distance C_1 from the other source where, by the cosine law

$$C_1 = \sqrt{r^2 + S_1^2 - 2rS_1 \cos\alpha}$$

If r is large compared to the separation S_1 the square of S_1 may be discarded so that approximately

$$C_1 = r \sqrt{1 - \frac{2S_1}{r} \cos\alpha}$$

If $r \gg S_1$

Since the quantity S_1/r is, under these conditions, small compared to unity it will be permissible to expand the radical by the binominal theorem and, again as an approximation, discard all of the terms except the first two, then approximately

$$C_1 \cong r - S_1 \cos \alpha \qquad \text{If } r \gg S_1.$$

Since $(r - C_1)$ is the phase separation of the waves from the two sources, expressed as a length

$$(r - C_1) \cong S_1 \cos \alpha \qquad \text{If } r \gg S_1. \qquad \text{------ (17)}$$

The angular phase separation is

$$\frac{2\pi(r - C_1)}{\lambda} \cong \frac{2\pi S_1}{\lambda} \cos\alpha \qquad \text{If } r \gg S_1. \qquad \text{----- (18)}$$

The amplitude of a resultant wave formed by the superposition of two waves of amplitude Ao and separated by the phase angle β is, again by the cosine law

$$R_m = \sqrt{2}\ Ao \sqrt{1 + \cos\beta} \qquad \text{----- (19)}$$

The combination of (18) and (19) yields

$$R_m = \sqrt{2}\ Ao \sqrt{1 + \cos \frac{2\pi S_1 \cos\alpha}{\lambda}} \qquad r \gg S_1 \qquad \text{----- (20)}$$

The following table shows a computation of wave heights following a circular path with its center at a point midway between the two sources. The computation extends through one quadrant. The other three quadrants are similar.

Table 1.

Computation of relative wave amplitudes produced by two, in phase, sources one half wave length apart.

α	$\cos\alpha$	$\frac{2\pi S_1}{\lambda}\cos\alpha$	$\cos\left(\frac{2\pi S_1 \cos\alpha}{\lambda}\right)$	$\frac{R_m}{A_p}$
0	1.000	3.142	-1.000	0.000
10^o	.985	3.09	-0.999	0.050
20^o	.940	2.95	-0.982	0.194
30^o	.866	2.72	-0.913	0.295
40^o	.766	2.40	-0.738	0.722
50^o	.643	2.02	-0.436	1.05
60^o	.500	1.57	0.000	1.41
70^o	.342	1.07	+0.480	1.72
80^o	.174	0.54	+0.855	1.93
90^o	.000	0.00	+1.000	2.00

Note: $$\frac{2\pi S_1}{\lambda} = \frac{2\pi\lambda}{2\lambda} = \pi$$

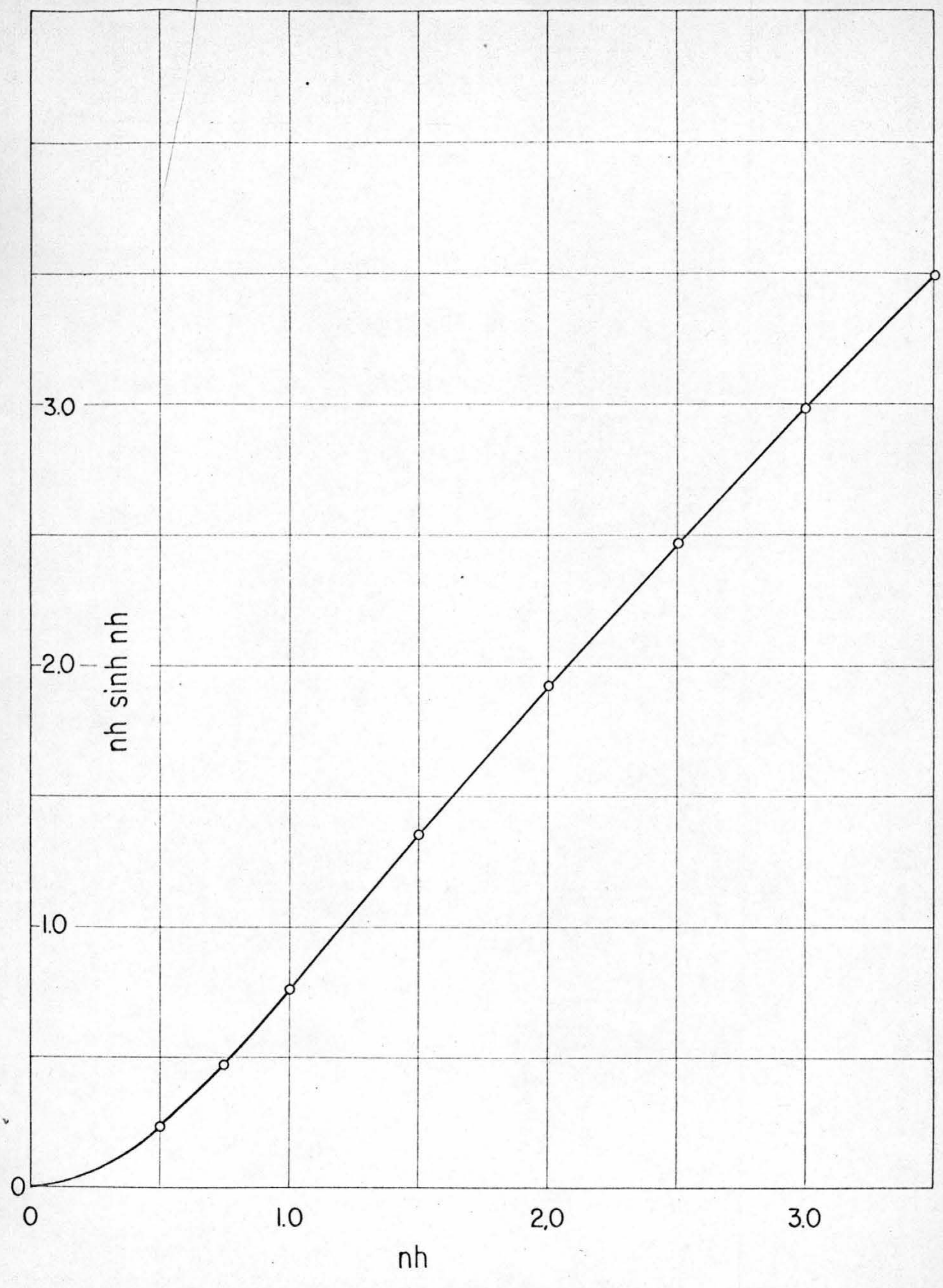


Fig. 1

Wave Motion Propagated from a Long Wave Generator

Let

L	represent the length of the wave generator.
Q_m	the maximum volume displacement produced by the whole generator in generating waves of period T.
R	the distance from the center of the wave generator to a point for which a wave amplitude or phase is to be computed.
T	the period of the wave produced.
x, y	coordinates as shown in the sketch.
C_1	a distance from a point on the generator at a distance η from the center of the generator and a point defined by the coordinates x and y.
α_1	the angle between R and the long direction of the wave generator.
η	a distance along the wave generator.
λ	the wave length.
C	the celerity of the wave.

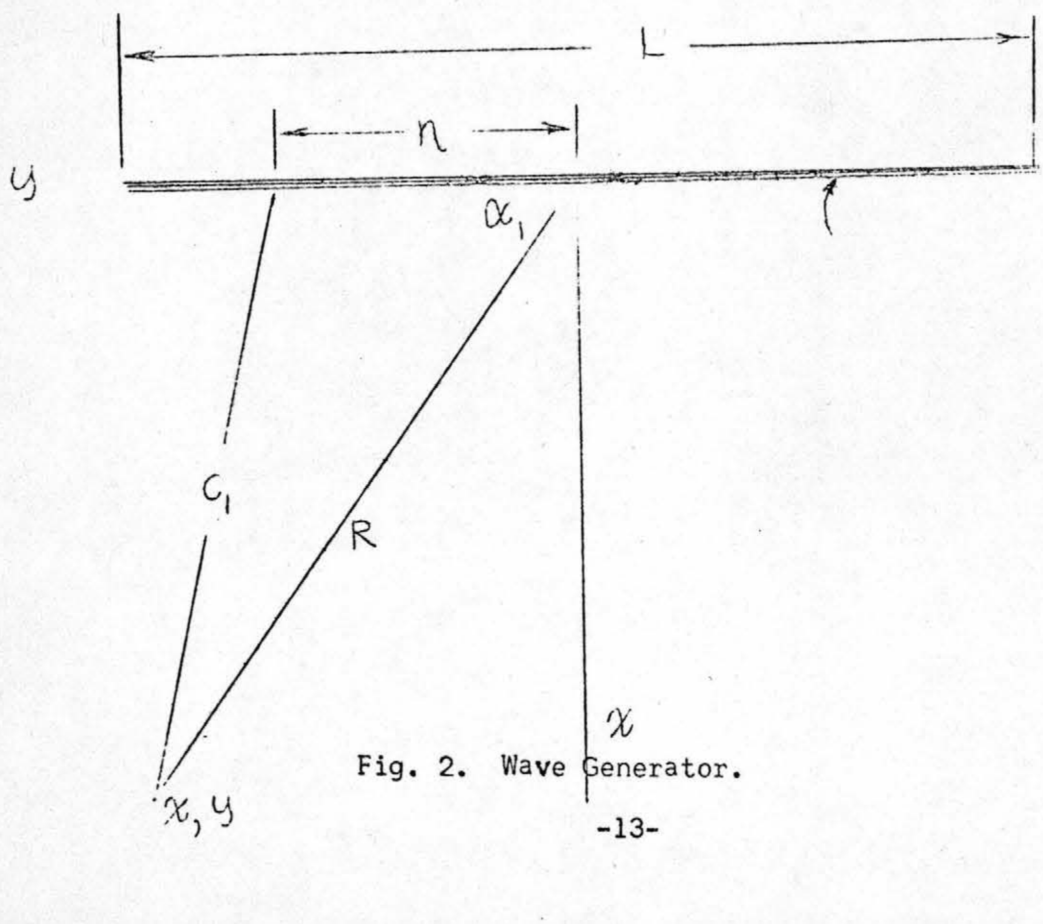


Fig. 2. Wave Generator.

From the paragraph on "Deep Water Waves Propagated from a Center" formula 14.

$$D_m = \frac{2\pi r_0 B_1 g}{\sigma^2}\left[\left(J_0'(nr_0)^2\right) + \left(Y_0'(nr_0)\right)^2\right] \frac{\sinh nh}{\cosh nh} \quad \text{.....(21)}$$

when

$$nr_0 > 1., \quad \left(J_0'(nr_0)\right)^2 + \left(Y_0'(nr_0)\right)^2 \cong \frac{2}{\pi n r_0} \quad \text{....(22)}$$

with

$$\sigma = \frac{2\pi}{T} \qquad T = \frac{\lambda}{c} \qquad \sigma = \frac{2\pi c}{\lambda} \quad \text{...(23)}$$

Then, by substitution

$$D_m \cong \frac{2\pi r_0 B_1 g}{\sigma^2} \frac{2}{\pi n r_0} \frac{\sinh nh}{\cosh nh} = \frac{4 B_1 g}{n\sigma^2} \frac{\sinh nh}{\cosh nh} \quad \text{....(24)}$$

or

$$D_m \cong \frac{B_1 g \lambda^2}{n\pi^2 C^2} \frac{\sinh nh}{\cosh nh} \quad \text{.....(25)}$$

and

$$B_1 \cong \frac{n\pi^2 c^2}{g\lambda^2} \frac{D_m \cosh nh}{\sinh nh} \quad \text{.........(26)}$$

If the wave height ζ due to an amplitude factor B_1 is

$$\zeta = B_1\left[J_0(nr)\cos\sigma t + Y_0(nr)\sin\sigma t\right] \quad \text{.....(27)}$$

Then the increment of wave height ζ_1 at the distance C_1 from an element of displacement $Q_m \frac{d\eta}{L}$ is

$$d\zeta_1 = \frac{n\pi^2 C^2 Q_m}{g\lambda^2 L}\left[J_o(nC_1)\cos\sigma t + Y_o(nC_1)\sin\sigma t\right]\frac{\cosh nh}{\sinh nh}\,d\eta \qquad \text{---}(28)'$$

If R is large compared to $\frac{L}{2}$ then R is large compared to η and, to a first approximation, the cosine law

$$C_1^2 = R^2 + \eta^2 - 2R\eta\cos\alpha \qquad \text{------}(29)'$$

can be expressed as:

$$C_1 \cong \sqrt{R^2 - 2R\eta\cos\alpha} \qquad \text{-----}(30)'$$

by discarding η^2 as small compared to R^2. Then by using the binomial theorem and discarding all but the first two terms

$$C_1 \cong R\left(1 - \frac{\eta}{R}\cos\alpha\right)$$

then

$$(R - C_1) \cong \eta\cos\alpha$$

If we neglect η as being small compared to R and refer phase positions to the phase position at the center of the wave generator, then approximately:

$$d\zeta_1 \cong \frac{n\pi^2 c^2 Q_m}{g\lambda^2 L}\Big[J_0(nR)\cos\left(\sigma t - \frac{2\pi n\cos\alpha}{\lambda}\right) +$$

$$Y_0(nR)\sin\left(\sigma t - \frac{2\pi n\cos\alpha}{\lambda}\right)\Big]\frac{\cosh nh}{\sinh nh}\,d\eta \quad \cdots (33)$$

then

$$\zeta_1 \cong \frac{n\pi^2 c^2 Q_m \cosh nh}{g\lambda^2 L\sinh nh}\int_{-\frac{L}{2}}^{+\frac{L}{2}} J_0(nR)\cos\left(\sigma t - \frac{2\pi\eta\cos\alpha}{\lambda}\right)d\eta$$

$$+\ \frac{n\pi^2 c^2 Q_m \cosh nh}{g\lambda^2 L\sinh nh}\int_{-\frac{L}{2}}^{+\frac{L}{2}} Y_0(nR)\sin\left(\sigma t - \frac{2\pi\eta\cos\alpha}{\lambda}\right)d\eta \quad \cdots (34)$$

or

$$\zeta_1 = \frac{n\pi^2 c^2 Q_m \cosh nh}{g\lambda^2 \sinh nh}\,\frac{\lambda}{2\pi L\cos\alpha}\Big[-J_0(nR)\sin\left(\sigma t - \frac{\pi L}{\lambda}\cos\alpha\right)$$

$$+J_0(nR)\sin\left(\sigma t + \frac{\pi L}{\lambda}\cos\alpha\right)$$

$$+Y_0(nR)\cos\left(\sigma t - \frac{\pi L}{\lambda}\cos\alpha\right)$$

$$-Y_0(nR)\cos\left(\sigma t + \frac{\pi L}{\lambda}\cos\alpha\right) \quad \cdots (35)$$

or

$$\zeta_1 \cong \frac{n\pi^2 c^2 Q_m \cosh nh}{g\lambda^2 \sinh nh}\,\frac{2\lambda}{2\pi L\cos\alpha}\Big[J_0(nR)\cos\sigma t\cdot\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)$$

$$+Y_0(nR)\sin\sigma t\cdot\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right).$$

$$\cdots\cdots (36)$$

The maximum amplitude is

$$\zeta_m \cong \frac{n\pi^2 C^2 Q_m \cosh nh}{g\lambda^2 \sinh nh} \frac{\lambda}{\pi L \cos\alpha} \sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)\sqrt{\left(J_0(nR)\right)^2+\left(Y_0(nR)\right)^2} \quad \text{----} (37)$$

If $R >> \frac{L}{2}$

If (nR) is large compared to unity, then :

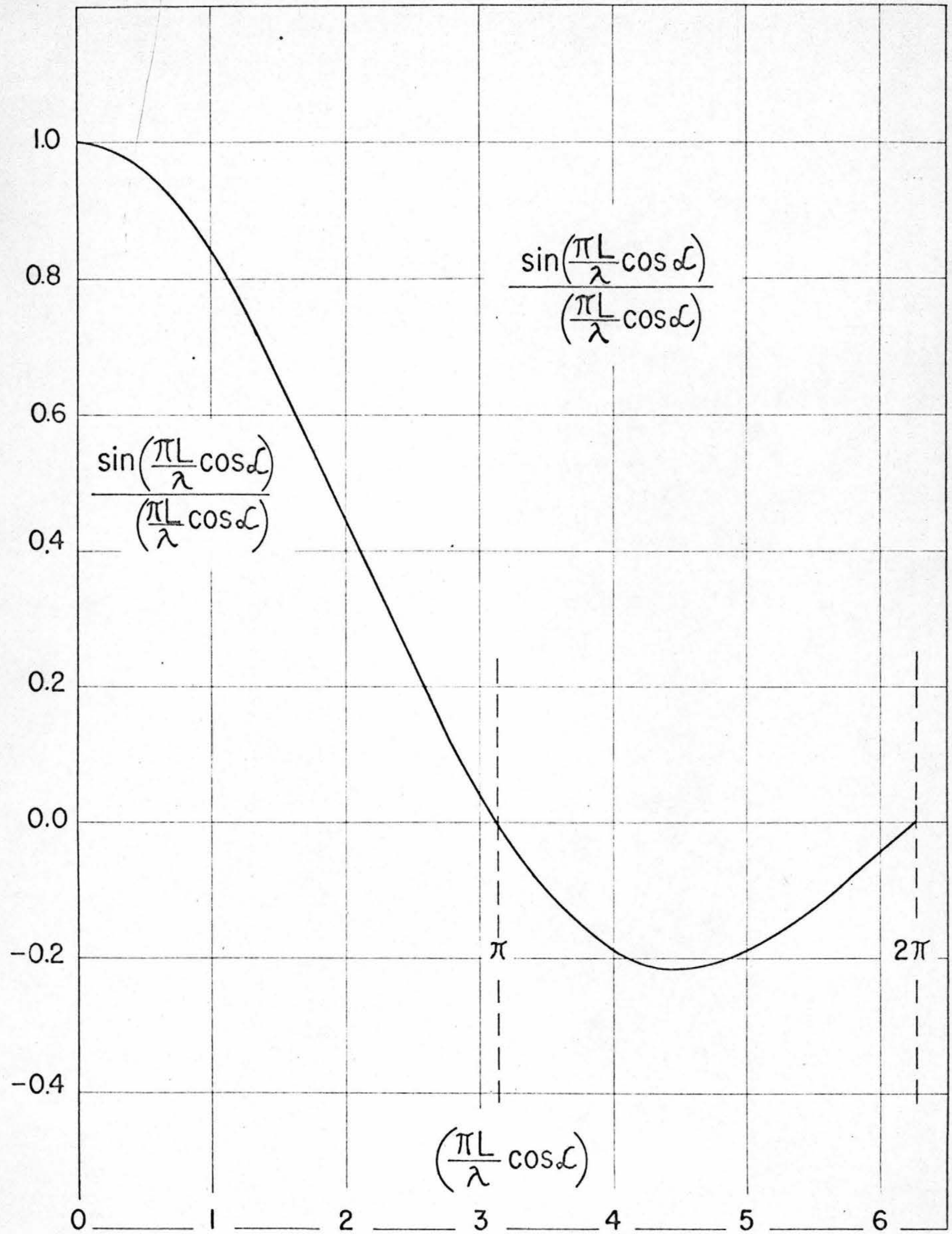
$$\sqrt{\left(J_0(nR)\right)^2+\left(Y_0(nR)\right)^2} \cong \sqrt{\frac{2}{\pi(nR)}} \quad \text{----} (38)$$

And approximately:

$$\zeta_m \cong \frac{n\pi^2 C^2 Q_m \cosh nh}{g\lambda^2 \sinh nh} \frac{\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)}{\left(\frac{\pi L}{\lambda}\cos\alpha\right)} \sqrt{\frac{2}{\pi(nR)}} \quad \text{----} (39)$$

Valid if $R >> \frac{L}{2}$

$(nR) >> 1.$



1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
$\dfrac{\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)}{\left(\frac{\pi L}{\lambda}\cos\alpha\right)}$
$\dfrac{\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)}{\left(\frac{\pi L}{\lambda}\cos\alpha\right)}$
π
2π
$\left(\frac{\pi L}{\lambda}\cos\alpha\right)$
0
1
2
3
4
5
6

Fig. 3

Method of Images

The case of a wave generator operating near a wall as shown in Fig. 4 can be obtained from the solution for the isolated case if a second wave generator, having the characteristics of the first, is located where the image of the first generator would be formed if the wall were a mirror.

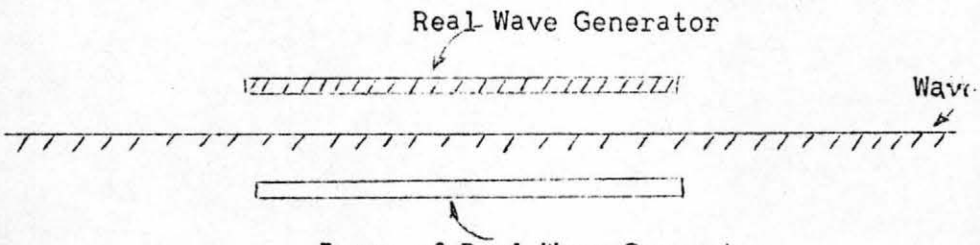


Fig. 4. Wave generator near a wall.

The boundary condition at the wall requires that there be no normal component of velocity at its surface. Two identical wave generators would produce this condition along the line midway between them if they operated in a water surface area of unlimited extent. Two solutions for the isolated case, used in this way, will therefore reproduce the boundary condition imposed by the wall.

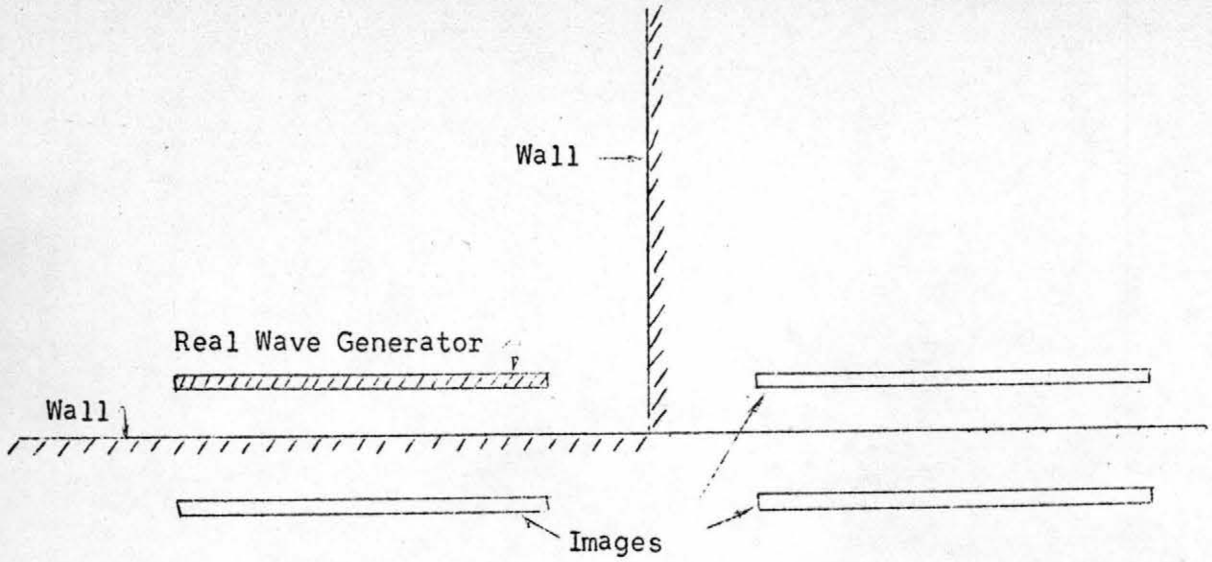


Fig. 5. Wave generator in a corner.

The case of a wave generator in a corner may be reproduced by the arrangement of images shown in Fig. 5. This will insure that there will be no normal component of velocity at either wall.

The case of a wave generator in a rectangular strip is a little more complicated. The condition is shown in Fig. 6. Here the first images in wall (1) will result in the proper boundary conditions being met along wall (1) but will not do so for wall (2). If the real generator, its image in the end, and the two first images in wall (1) are now imaged in wall (2). The required conditions along wall (2) will be met, but at the expense of a slight interference with the boundary condition at wall (1).

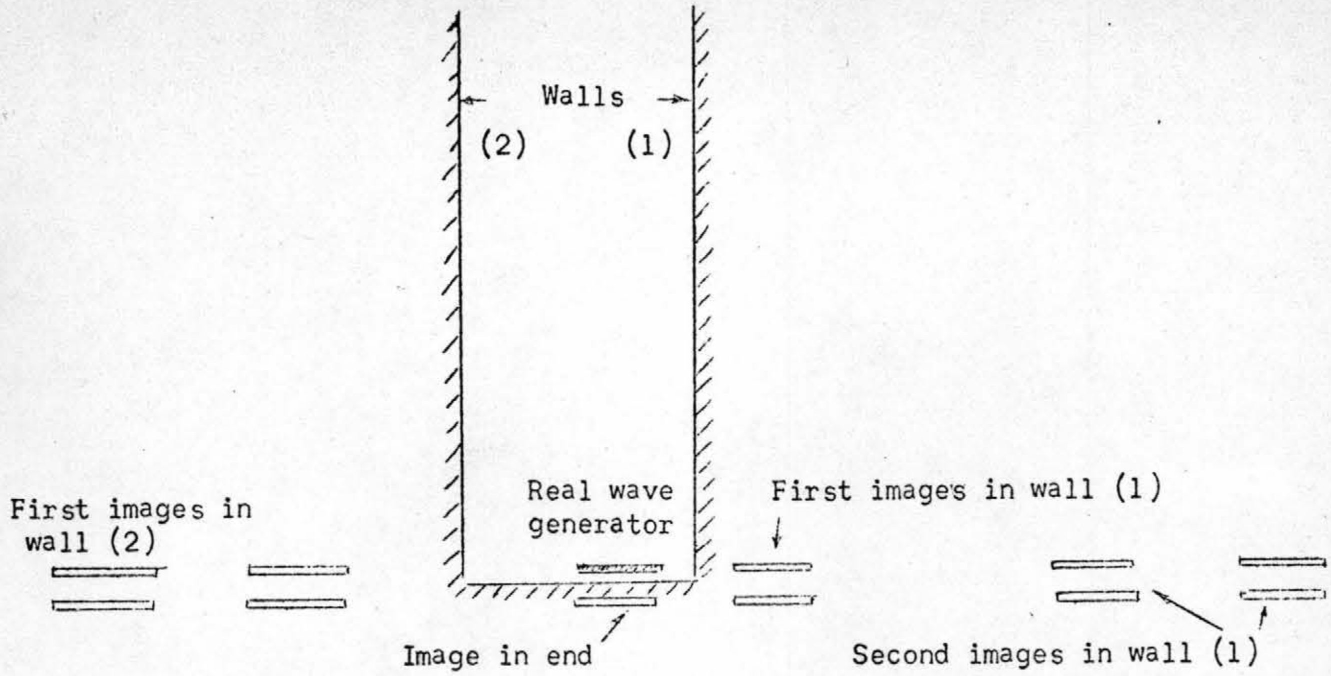


Fig. 6. Wave generator in a rectangular strip.

To remedy this we can introduce the second images in wall (1). These will completely restore the boundary conditions at wall (1) but in turn, upset them slightly at wall (2). Continuation of this process leads to an infinite series of terms. The series is generally, however, rapidly convergent.

If the strip of Fig. 6 had an upper end to convert it into a rectangular tank the real wave generator and its image in the near end could first be imaged in the far end to meet the boundary conditions at the far end. Successive imagings in the side walls (1) and (2) would then permit satisfaction of the boundary conditions along these walls without upsetting the boundary conditions at the ends.

These descriptions have assumed that there is no absorption of energy at the wall. If energy absorbers are arranged along a wall it is believed permissible to consider that wall absent. To the approximation contemplated herein, it is permissible to compute the wave motion by superposing the effects of the real wave generator and its images. For this purpose equation 36 should be used to obtain the sine and cosine amplitudes seperately. When all of these have been obtained the maximum amplitude can be computed by taking the square root of the sum of the sine and cosine amplitude squared. Since the maximum amplitudes coming from the generators and images, as given by equation 37, will be out of phase, a vectorial addition would be required to obtain from them an estimate of the maximum amplitude of the resultant wave motion. The process outlined above avoids this difficulty.

II. FIRST APPROXIMATION TO A CONFUSED SEA IN A CIRCULAR MODEL BASIN

A ship is an expensive piece of equipment and for this reason model studies are of value because they enable the designer to determine many of the performance characteristics of a proposed design before funds for its construction must be committed.

If the evaluation of performance is to be useful, it will not only be necessary to build a scale model of the ship which is true to dimensions, but also to provide a model tank and to produce upon the water surface therein waves which will act upon the model in the same way that waves upon the open sea will act upon its prototype. It seems essential also to try to select from among the myriads of wave configurations to be met in the open ocean those which will test the important strength and stability characteristics of the hull most severely. It is important to identify the characteristics which need to be tested and, if possible, to arrange the tests to isolate them in a manner that will not be obscured by irrelevant factors. A type of sea combined with a manner of navigation which will tax most severely a certain strength or stability factor will hereafter be referred to as a "predicament."

It will be worthwhile to list some well known cases as examples:

Predicament 1:- A sea with parallel crests and a wave length approximately equal to the length of the ship combined with a heading which causes the ship to cross them at right angles. This predicament produces the "sagging" and "hogging" condition which puts a severe strain upon the beam strength of the hull.

<u>Predicament 2</u>:- A sea with parallel crests and a wave length somewhat less than the length of the ship combined with a heading which takes the ship across them at an angle. This predicament tests the torsional strength of the hull. At certain combinations of wave length and ship speed it provides a severe test of the rolling stability of the ship as well as a test of the coupling between roll and pitch.

<u>Predicament 3</u>:- A sea with parallel crests and a wave length which has a period equal to the natural rolling period of the ship combined with a heading transverse to the direction of wave travel. This predicament provides a severe test of the rolling stability of the ship.

The Confused Sea

A type of sea met at times in the open ocean is characterized by a lack of apparent order. It is described as being produced by violent and sustained winds and it is supposed that it is, or can be, produced by a storm of the hurricane type whose winds are violent, sustained and changing in direction. Because of its lack of apparent order it has been described as "confused" although some success has apparently come from efforts to explain its undulations on the basis of a random superposition of waves occupying an entire spectrum of wave length and amplitude as well as a range of orientations. Waves of great height can be encountered in such seas and it is obvious that they can test the structure and stability of a ship severely. It is the belief of this writer, however, that this type of sea can present a predicament which is not met at all or only in milder forms in the regular types of seas. This can be labeled:

<u>Predicament 4</u>:- A confused sea presenting a profusion of surface configurations and a ship heading through them and experiencing violent motions of the type described as pitch, roll, heave, surge, sway, and yaw so that a fortuitous combination of ship motion and sea surface configuration can conspire to produce situations in which the hull of the ship is brought into simultaneous contact over considerable areas with a portion of the sea surface.

When this happens a very great mass of water must be set into motion. The result is a sudden blow of great violence delivered to the ship. Some idea of the violence of these blows can be obtained from the fact that water pressures up to about 147 ft. of water can be developed for each foot per second of relative velocity which must be obliterated at the instant of contact. This is equivalent to about 64 lbs. per sq. in. for each foot per second of relative velocity obliterated. Such pressures enjoy only a very brief existance but this may be long enough to damage the plating and to set the hull into violent vibration.

Estimates based upon the equivalent masses which could reasonably be set into motion in this way indicate that the transient stresses associated with these hull vibrations may approach the magnitude of the bending stress due to the sagging condition previously described.[19] It is possible, of course, that these transient stresses could be added to bending stresses which were already high. An attempt to maintain high ship speeds through such seas could aggravate these difficulties.

Model Tests for the Confused-Sea Condition

It seems obvious that the confused sea condition is an important one and one in which model results would be very valuable if the tests can be properly made. A difficulty arises, however, when an attempt is made to specify and to generate a confused sea in a model tank. Fortunately, it may be unnecessary to try to reproduce the exact sequence of configurations found at some time on the ocean. The requirements for model testing purposes might be met by a wave combination which will produce dynamical situation on a model scale which are comparable to those to be met by the prototype at sea. A wave system that can be clearly and completely specified is also essential for model work in order to permit results to be checked.

A circular wave basin offers advantages for making tests of this kind because of the facility with which wave generators can be arranged to command the test area. A sufficiently confused sea can possibly be obtained by superposition of the waves from three wave generators propagating sinusoidal or irregular patterns across the test area which, in this case, would be a circular area in the middle of a model basin.

Computation of the wave pattern from a single wave generator, Fig. 7, shows the type of coverage attainable.

Fig. 8 shows the approximation to a confused, short-crested sea obtained by superposition of three sinusoidal wave patterns. This is the type of sea surface that could be expected over the center area of a circular basin, assuming the three line source generators occupy about 120° of the periphery of the tank and wave absorbers are located along the walls of the basin. This figure shows only the very simple case of three monochromatic waves, one being generated by each generator. A much more irregular sea surface can be generated by programming a less simple but periodic function through each generator.

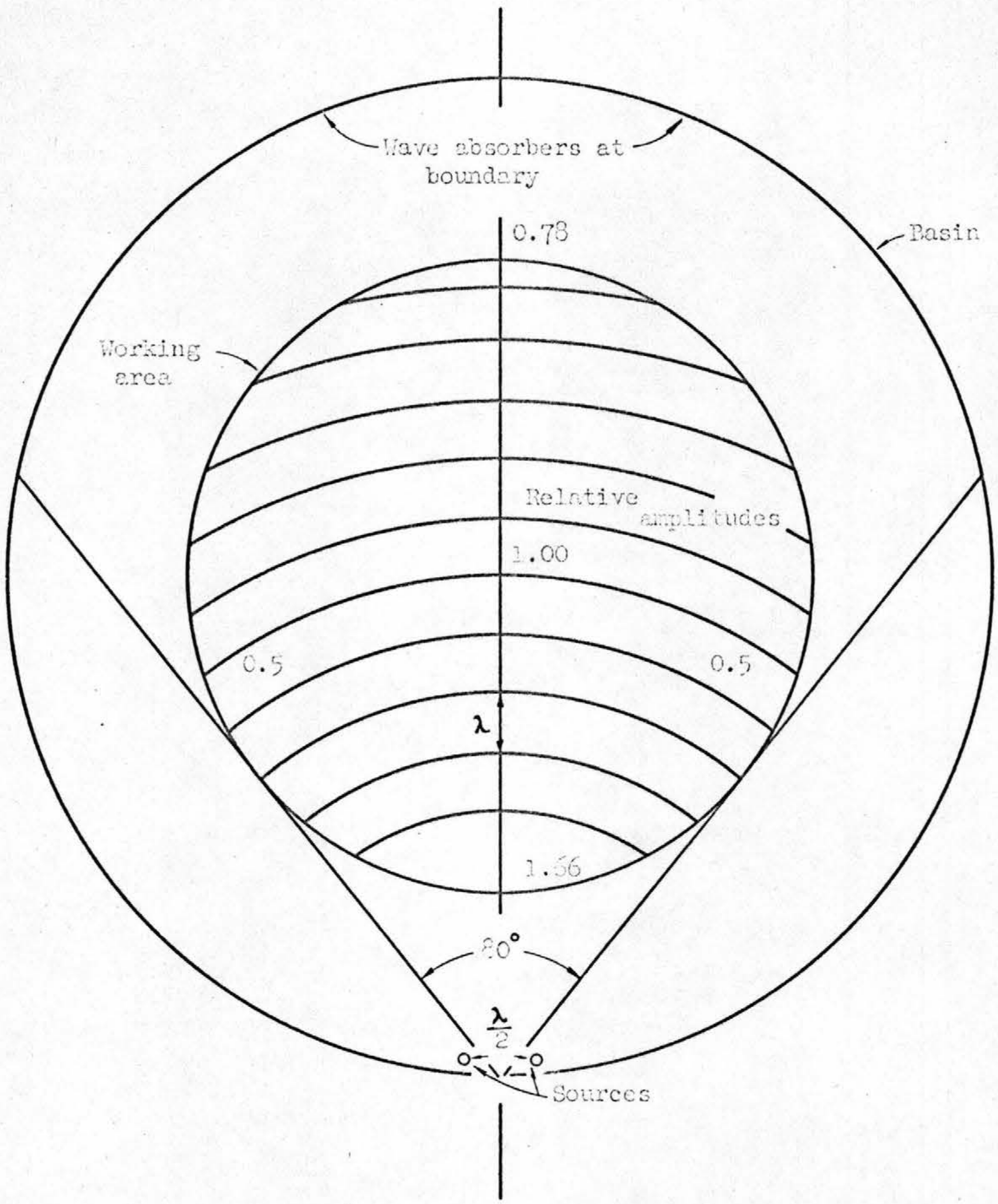


Fig. 7 Wave amplitudes in a circular tank due to a single wave generator of length $\lambda/2$.

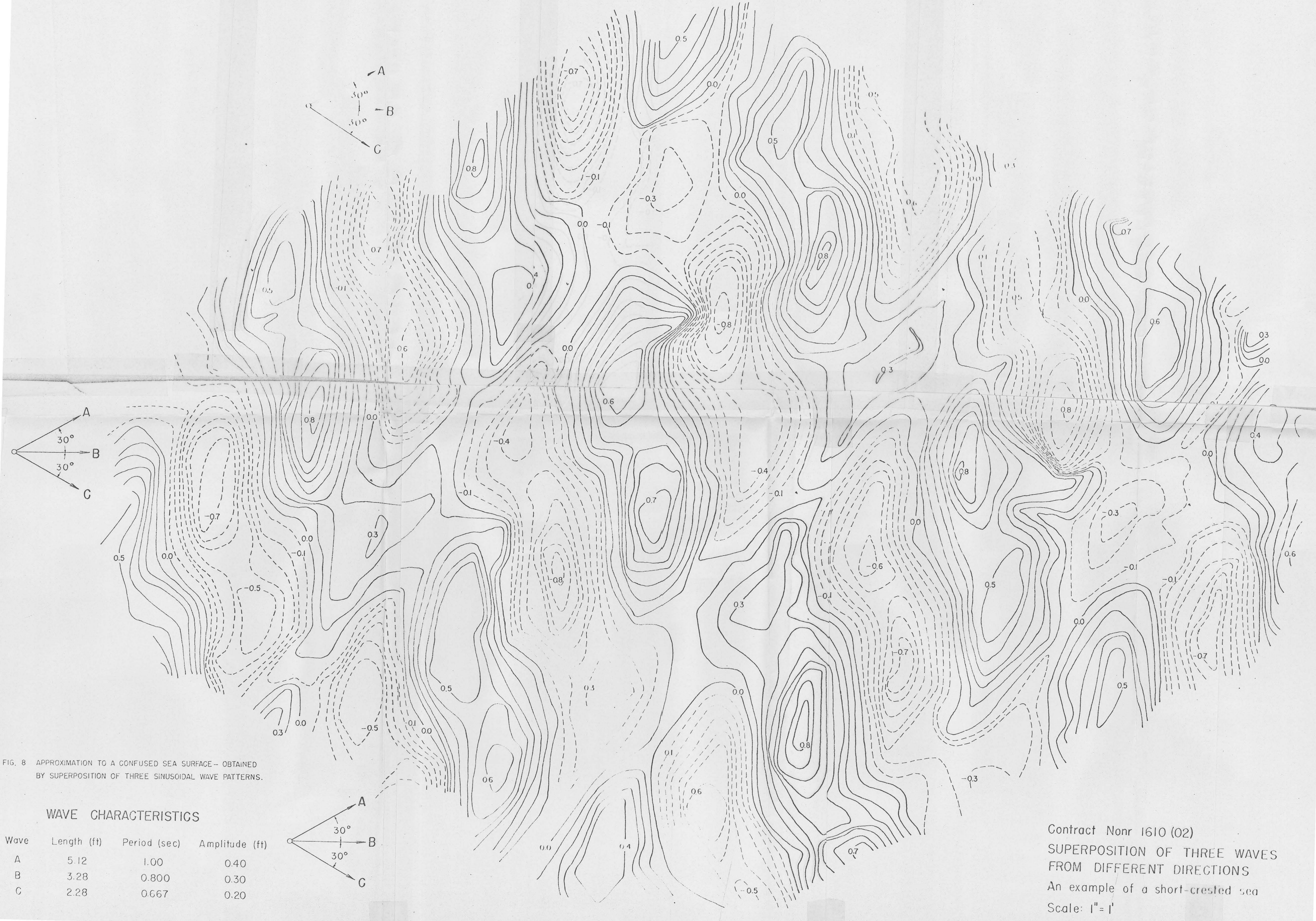


FIG. 8 APPROXIMATION TO A CONFUSED SEA SURFACE – OBTAINED BY SUPERPOSITION OF THREE SINUSOIDAL WAVE PATTERNS.

WAVE CHARACTERISTICS

Wave	Length (ft)	Period (sec)	Amplitude (ft)
A	5.12	1.00	0.40
B	3.28	0.800	0.30
C	2.28	0.667	0.20

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SUPERPOSITION OF THREE WAVES FROM DIFFERENT DIRECTIONS

An example of a short-crested sea

Scale: 1" = 1'

III. A MECHANICAL MEANS OF GENERATING A CONFUSED SEA

Recent studies of wave forms met at sea under storm conditions, and analyses made for purposes of correlation, indicate that such seas are produced by the superposition of elemental waves of varying amplitude and wave length. It may be desirable to reproduce such waves in model tanks in order to permit a more faithful reproduction of the conditions the prototype ship will encounter at sea and these researches provide a clue as to how this may be accomplished.

If the wave generator is of the plunger type, it must be given a motion represented by a sum of the components contributing to the actual wave profile. Since the wave generator will generally be at some distance from the test area, it will be necessary to account for the difference in wave travel time for waves of different wave lengths and the different diffraction characteristics associated with the different wave lengths.

Control of the Motion of the Wave Generator

A mechanism for generating such wave trains would have essentially two parts.

1. A means of generating the sum of the selected wave components.

2. A means of imposing the sum on the plunger of the wave generator.

If the sum of a finite number of components can be used satisfactorily, their sum can be produced by a device of the following sort. If a taut cord is fastened at one end and passes over a series of pulleys of which alternate ones have fixed axles and the remainder are attached to cranks, the free end of the cord will generate the sum of the motions contributed by the cranks. A schematic drawing of this arrangement is shown in Fig. 9. The throw of the cranks can be made adjustable to permit the contribution of each component to be varied. An alternative device is provided by a cam cut to the shape required by the wave profile. This device, (Fig. 10), however, permits no adjustment and can be altered only by the cutting of a new cam.

By making the cams of aluminum sheet and using the outer race of a small ball bearing as a roller, the production of cams can be simplified. The wave profile can be plotted directly on the sheet, corrections made for the roller diameter, and the desired profile roughed out with a scroll saw. Final finishing can be done with rasps and files used for wood working. Since the outer race of the ball bearing is smooth and true and will roll freely without abrasion, such a cam should last a long time. An identification of the wave profile it produces can be permanently attached to it.

The merit of the device of Fig. 9 is the ease with which a new profile can be set up if the Fourier coefficients of the desired wave profile are known. The cams, however, are much more compact, can be laid out directly from the desired wave profile without development into a Fourier series, and should be easily produced.

Having produced the motion to be followed, it remains to impose this motion on the plunger. This will require some form of servo-motor. For the sake of economy of time and funds this device should make use of existing equipment whenever possible. It is proposed to use, for these purposed, a device similar to those interposed between the fly-ball governor head and the relay valve of hydraulic turbine governor mechanisms. This device is shown schematically in Fig. 11. The motion to be followed is communicated to a spool valve with a movable sleeve. The sleeve is connected to the piston rod of the servo-motor cylinder in such a way that the induced movement of the piston rod tends to close the valve. In this way the piston rod follows faithfully the motion of the spool valve.

It is suggested to arrange the wave making plunger and the servo-motor drive as a unit mounted on a frame, to bring the pressure supply to it through a flexible hose and to convey the cam motion to the spool valve by means of a Selsyn motor. In this way the wave generator can be made movable and the cams can be located in any convenient housing. The wave motions produced by these devices will be reproducible.

Wave Computations

If a device such as is shown in Fig. 9 is used, the amplitudes and periods of the wave components will be known but if a cam is to be used to generate an arbitrary wave profile, it will be necessary to develop this sequence as a Fourier series based upon the period of revolution of the cam. The series so obtained will represent the wave motion in the test area. In order to produce these phase and amplitude relationships in the test area, it will be necessary to shape the cam to compensate for differences in the wave velocities of the components and to adjust the plunger amplitude for each component for the diffraction pattern produced by each ratio of plunger length to wave length. In the following paragraphs it will be assumed that the wave generating plunger is narrow in the direction of wave propagation so that it produces displacements essentially along a line.

The length of the wave generator used may depend somewhat upon the shape of the model basin. If the basin is circular a wave generator having a length equal to the wave length of the longest period wave to be propagated may be a good choice. In a rectangular tank the wave generator may extend along the entire length of an end or a side.

The wave amplitude ζ is given approximately by an expression of the form:

$$\zeta \cong \frac{n^2 Q_m}{4} \frac{\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)}{\left(\frac{\pi L}{\lambda}\cos\alpha\right)} \sqrt{\frac{2}{\pi(nR)}} \left[\cos\left(nr - \sigma t - \frac{\pi}{4}\right)\right] \quad \cdots (40)'$$

Valid if $R \gg \frac{L}{2}$, $(nR) \gg 1$ and $h > \frac{\lambda}{2}$.

This expression is derived from formula (36) with the aid of relations (8), (9) and (23), subject to the requirement that tanh nh is close to unity. This requirement is equivalent to specifying that the waves will be deep water waves. They will be waves of this type if the depth of water h in the tank is greater than about one half the wavelength λ.

The maximum amplitude, without regard to phase position is:

$$\zeta_m = \frac{n^2 Q_m}{4} \frac{\sin\left(\frac{\pi L}{\lambda}\cos\alpha\right)}{\left(\frac{\pi L}{\lambda}\cos\alpha\right)} \sqrt{\frac{2}{\pi(nR)}} \quad \cdots\cdots (41)'$$

Valid if $R \gg \frac{L}{2}$, $(nR) \gg 1$ and $h > \frac{\lambda}{2}$.

Example:

Suppose the record has been analyzed and that the longest wave length in the model scale is 10 ft. and that it is desired to provide a wave of amplitude 0.25 ft. and wave length 5.0 ft. as one component. It will be assumed that the wave generator is 10 ft. long and the wave is to be produced at the center of a circular tank of 40 ft. radius and a water depth of 4 ft.

From Eq. 41, with

$\zeta_m = 0.25$ ft.,

$L = 10.0$ ft.,

$\lambda = 5.0$ ft.,

$\alpha = 90^\circ$,

$R = 40$ ft.,

one has $n = \dfrac{2\pi}{\lambda} = \dfrac{6.2832}{5} = 1.257$, $n^2 = 1.576$,

$nR = 50.28$.

From the chart of Fig. 31, with

$$\frac{\pi L}{\lambda} \cos \alpha = 0, \quad \text{one has} \quad \frac{\sin\left(\frac{\pi L}{\lambda} \cos\alpha\right)}{\left(\frac{\pi L}{\lambda} \cos\alpha\right)} = 1.000.$$

Then, in this case,

$$\zeta_m = \frac{n^2 Q_m}{4} \sqrt{\frac{2}{\pi(nR)}} = \frac{1.567}{4} Q_m \sqrt{\frac{2}{158}} = Q_m .0444$$

and

$$Q_m = \frac{0.25}{0.0444} = 5.64 \text{ cubic feet.}$$

With a generator plunger 1 ft. wide and 10 ft. long it would require 0.564 feet of vertical movement to produce this displacement. However, if the wave generator is located near the wall of the tank, wave energy can be propagated only toward the center of the tank and this figure can be cut in half. (In a previous paragraph, it is explained how the effect of the wall can be accounted for by the use of images.) Then the amplitude of the plunger motion needed to maintain this wave component is $\frac{0.564}{2}$ = 0.282 ft. It remains to adjust the phase. This is best done by compensating for the difference in wave travel times. The wave velocity is:

$$C = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}} \qquad \text{- - - - (3)}$$

Valid if $nr \gg 1$.

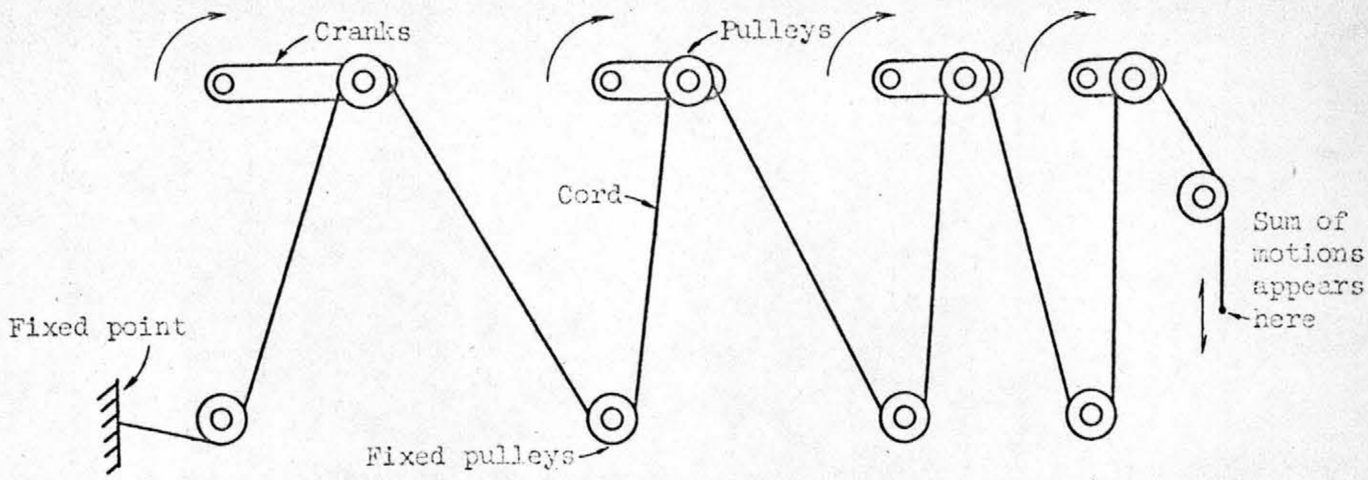
The wave velocities for the two waves described above may be computed as follows:

Wave Length (Feet)	Depth h (feet)	Acceleration of Gravity g (ft/sec^2)	$\frac{g\lambda}{2\pi}$ (ft^2/sec^2)	$\frac{2\pi h}{\lambda}$ ----	$\tanh\frac{2\pi h}{\lambda}$ --------	C (ft/sec)
10	4	32.2	51.3	2.52	0.9871	7.12
5	4	32.2	25.6	5.03	1.0000	5.06

The travel distance is 40 ft. and the travel times and the time shifts to put the shorter wave in its proper position at the center of the wave basin may then be computed as follows:

Wave Length (feet)	Travel Time (Seconds)	Time Shift (Seconds)
10	5.62	0
5	7.91	2.29

Then the 5-ft. wave-length component must be set ahead 2.29 seconds on the cam to give it enough lead on the 10-ft. wave-length component to have it arrive in its proper phase relationship at the center of the tank.



Note: Distance between fixed and movable pulleys to be made large compared to crank throws.

Fig. 9 Device for adding sinusoidal motions.

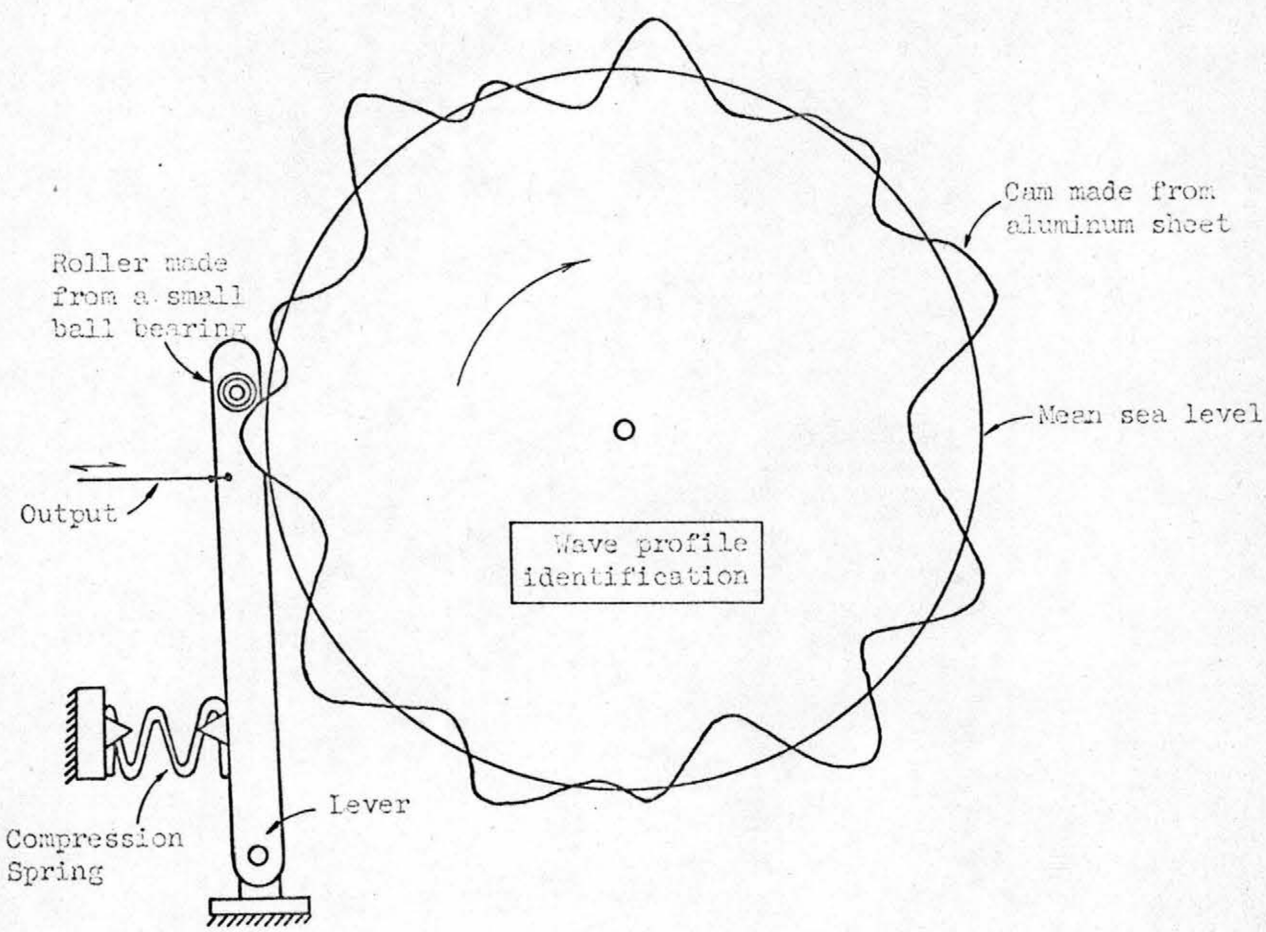


Fig. 10 Cam

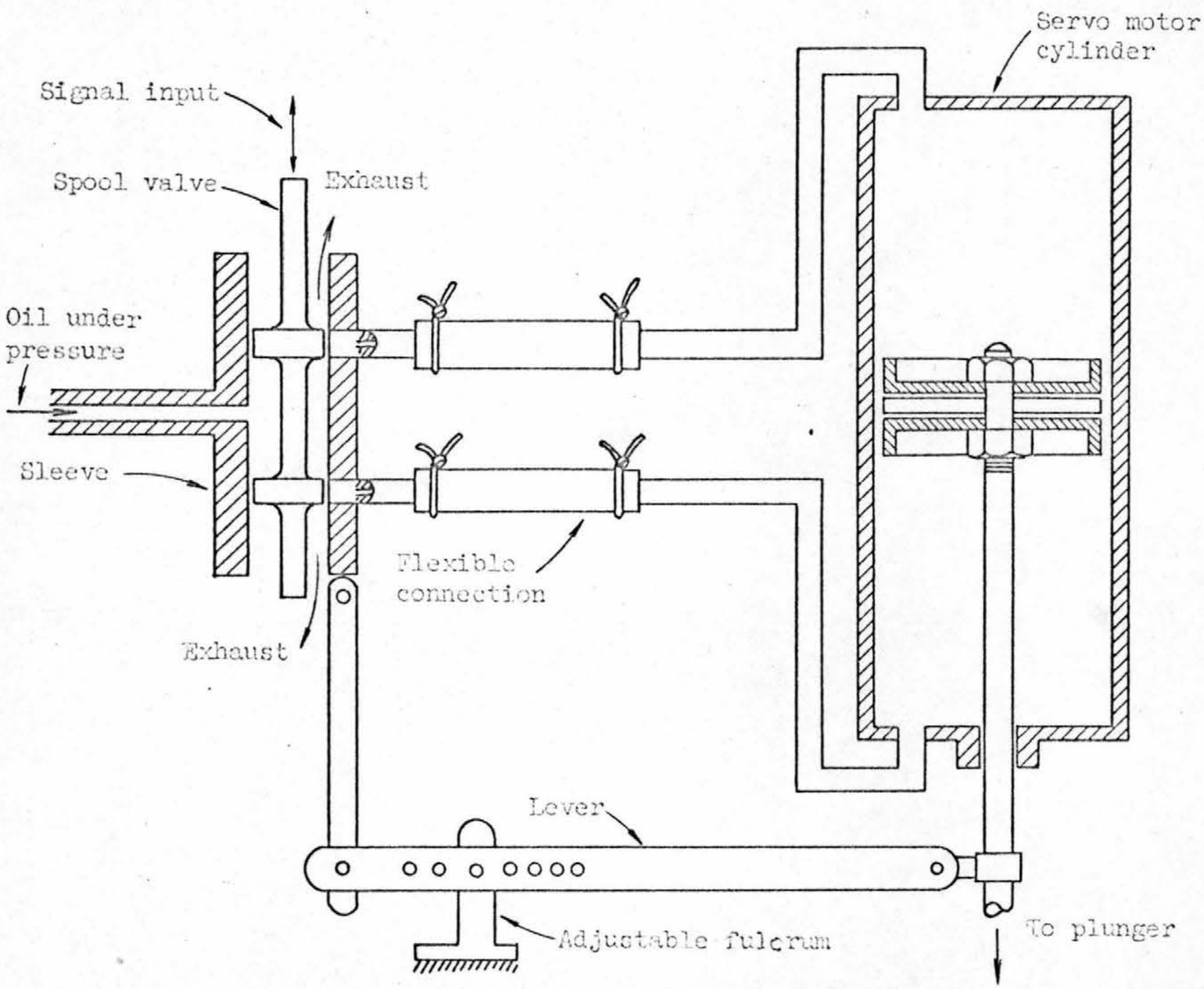


Fig. 11 Wave generator drive.

IV. ARRANGEMENT OF WAVE GENERATORS FOR PRODUCTION OF A CONFUSED SEA IN A RECTANGULAR MODEL BASIN

An arrangement of three cam controlled wave generators is described for producing a sea of the confused type in a model basin in which random wave amplitudes, wave lengths and orientation of propagation directions are possible.

A sufficiently close approximation of a confused sea might be obtained if a wave profile representing a configuration observed under storm conditions at sea were cut on a cam and used to control one or more wave generators. This profile should contain enough crests and troughs to be representative of storm conditions before repetition occurs when the cam makes a complete revolution. The model sea would then be specified in terms of the shape of the cam, its period of revolution, its starting time, and the amplitude of the waves produced. If three wave generators controlled by such cams were oriented to command a test area the factor of orientation could be included.

It would be convenient to make the centerline of the model basin the principal direction of propagation since the central wave generator could then be installed against the end of the tank. In order to command the test area at the proper orientation, the two lateral wave generators could then be mounted on booms hinged near the ends of the central generator and provided with a reflecting panel immediately behind their wave generators. Wave absorbers should be installed around the far end of the basin to prevent reflected waves from running through the test area. Some attenuation of the waves would occur as the waves traversed the test area. This factor could probably be allowed for when the data are analyzed. Due to the greater speed of travel of the long wave-length waves the wave profile would continually change during its passage through the test area. This difficulty seems to be inherent in any wave propagation involving confused sea characteristics. For propagation of waves with parallel crests the two lateral generators could be swung back against the end wall. The suggested arrangement is shown in Fig. 12.

One type of confused sea which could be produced by the wave generator arrangement described above is shown on Fig. 8. In this case also variations of amplitude similar to those illustrated on figure 7 would be present in the test area.

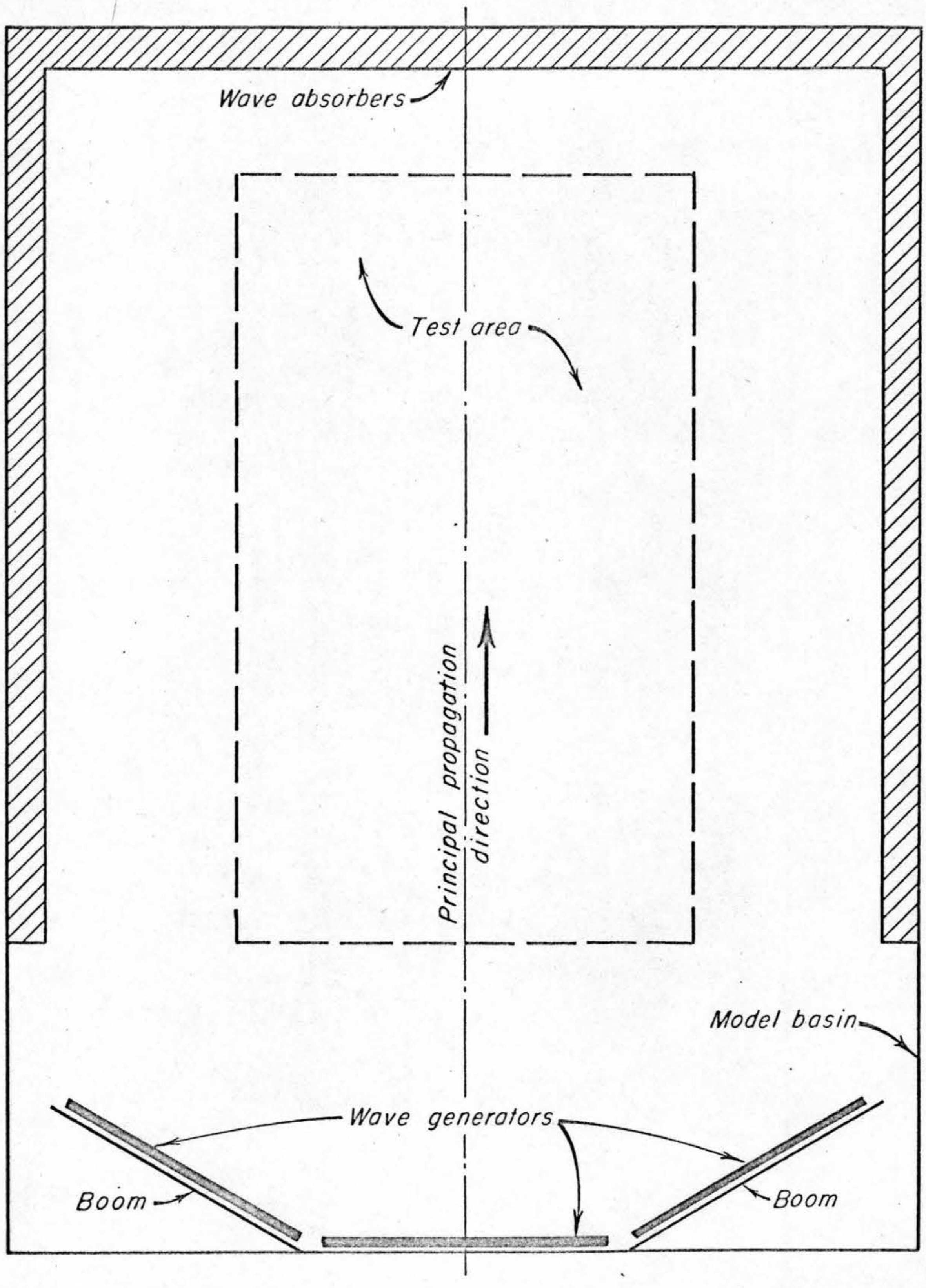


Fig. 12 Confused sea arrangement for a rectangular tank.

V. REQUIREMENTS FOR PRODUCTION OF A REPLICA SEA IN A MODEL BASIN

The following note concerns the possibility of generating, in a model basin, a sea which reproduces, to model scale, a storm sea observed on the open ocean. It is intended that the reproduction should be essentially true in time and space and should persist in the model tank for a sufficient time to permit significant model tests to be made. Such a model sea will be referred to hereafter as a "replica sea." It is possible to produce short crested seas in a model tank but if it is known that these are not identifiable with any actual sea then complications arise relative to the correlation of model test results and prototype behavior. A replica sea would not present these difficulties and if, in addition, it can be made to fill substantially the entire tank area it would provide a design tool of unusual value.

Prototype Data

It would be essential to have prototype data available if a replica sea is to be produced. These data should delineate the surface of a storm sea over a considerable area and for a sufficient period of time. The data should be taken in such a way that a closed boundary could be superimposed on the area mapped and referenced to points fixed with respect to the quiet water below the depths where wave disturbances are present.

Such data might be obtained from a timed sequence of simultaneous stereo-photos taken from two airplanes flying parallel courses upwind over a storm sea and provided with equipment for synchronizing camera shutters and for determining their distances apart as well as their distance above the sea. Reference markers should show on these photographs to indicate position with reference to the quiet water below the waves. The results of such observations could be shown on a series of maps of the area which would describe the sea surface at known instants by means of contours. The wind direction should also be shown on these maps.

Possible Method for Creating the Replica Sea

Suppose we have a model basin, which will be assumed to be rectangular in plan, in which the long dimension represents the downwind direction for the storm sea to be represented and that a model scale ratio has been chosen. It would then be possible to locate within the observed area a boundary representing, to prototype scale, the boundary of the model basin and to plot from the series of contour maps the variation of wave height with time for selected points on the boundary. Suppose now that the model tank is equipped with cam controlled wave generators around all four sides and that the above chosen points represent the centers of the wave generators. It will be assumed that there will be enough of these generators so that the length of each one will be short compared to the distances between crests of the sea to be reproduced. Each of the generators would be controlled by a mechanism to cause it to maintain, at its location, the wave level corresponding to that of the prototype for the corresponding point. The wave height-time sequence to be maintained would be cut on the cam controlling each generator. There would be no two of these alike.

Type of Sea Generated

If the wave generators are put into operation as described, a disturbance will appear in the model basin. The question which it is important to answer is whether, after a sufficient time has elapsed for a wave to traverse the long dimension of the model tank, a replica sea will appear on the surface of the water in the tank.

Proposed Method of Investigation

In order to investigate this question it will be assumed that the disturbance which appears on the surface of the tank represents a solution of the appropriate hydrodynamical differential equation subject to the conditions imposed at the boundaries. If the wave generators impose, to scale, the conditions of the prototype over these boundaries and these are sufficient to render the solution unique then it may be expected that the replica sea will be realized in the model basin. There is an extensive literature on the mathematical aspects of this problem.

This subject is known as Cauchy's problem, after one of the first mathematicians to investigate the combination of conditions required to obtain a unique solution of a differential equation.

In our case, if the water is considered to be both incompressible and the flow irrotational, the applicable differential equation is the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0. \qquad (42)$$

Here x, y and z represent space coordinates and ϕ represents the velocity potential. In a physical sense this equation expresses the "condition of continuity." This is to say that for every element of volume dxdydz the amount of fluid which flows into it in any interval of time is just equal to the amount which flows out. This relationship must be satisfied whether the flow is steady or is changing with time. It is therefore always appropriate.

Boundary Conditions

The boundaries present in this case are represented by the bottom of the wave basin, the free surface at the air-water interface, and the walls of the tank. If the depth of water in the model basin is at least one half the distance between crests of the sea to be generated then the wave motion produced will be a true representation of deep water waves (3). In this case the distance between crests is to be measured in the downwind direction.

With this depth of water the appropriate boundary condition at the bottom of the tank will be met. At the surface of the water the requirement to be met is that the pressure should be atmospheric. It is obvious that we cannot impose any further conditions on this boundary, since any attempt to alter it will impose pressures which will violate the condition which we know must prevail there. This leaves the walls of the tank as the only boundary where control may be exercised. It is proposed to operate the wave generators at the walls to produce undulations which reproduce to model scale those found at the corresponding point in the storm sea. The question which now must be answered is; does this lead to a unique solution?

The Cauchy Problem

In the development of mathematics the question arose as to what requirements are necessary to render the solution of a differential equation unique. That is, what is necessary to be specified so that we get one, and only one, solution. The differential equation itself generally has many solutions. The additional conditions must therefore come from the initial and boundary conditions. If there are too few of these more than one solution can be obtained. If there are too many the problem is over determined and there are no solutions. Only with the proper number is a unique solution obtained.

Early investigators of these questions were Cauchy, Sophie Kowalewsky and Darboux (12). The time was about 1840. Their work involved the restriction that the solution was to be analytic in form. They concluded that both a value of the dependent variable and a gradient had to be imposed at the boundary. In our case this would imply that we must specify both a wave height and a velocity normal to the wall of the tank, along the lines of wave generators, or some equivalent.

These questions have continued to interest mathematicians and an account of some of the developments since the time of these early investigators is contained in Hadamard's Lectures (12). Concerning the early results he has this to say:

> "The reasonings of Cauchy, Kowalewsky, and Darboux, the equivalent of which has been given above, are perfectly rigorous; nevertheless, their conclusion must not be considered as an entirely general one. The reason for this lies in the hypothesis, made above, that Cauchy's data, as well as the coefficients of the equations, are expressed by <u>analytic</u> functions; and the theorem is very often likely to be false when this hypothesis is not satisfied."

In a subsequent paragraph he continues:

> "If, in the first place, we take such a Cauchy Problem as was spoken of in paragraph 4 (Cauchy's problem with respect to $t = o$, for equations (e_1), (e_2) and (e_3)[1]), our above conclusions are valid, as we shall see as these lectures proceed, without the need of the hypothesis of analyticity.
>
> But the conclusions will be altogether different if, for instance, we deal with Laplace's

---

[1] Differential equations for 1, 2 and 3 dimensional wave propagation in a <u>compressible</u> fluid medium.

classic equation of potentials

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

This will be immediately realized by comparison with another classic boundary problem; I mean Dirichlet's problem. This consists as we know, in determining a solution of Laplace's equation within a given volume V, the value of u being given at every point of the boundary surface S of that volume. It is a known fact that this problem is correctly set: i.e. it has one (and only one) solution.

This fact immediately appears as contradictory to Cauchy-Kowalewsky's theorem; for, if the numerical values of u at the points S (together with the partial differential equation) is by itself sufficient to determine the unknown function within V, we evidently have no right to impose upon u any additional condition and we cannot therefore, besides values of u, choose arbitarily those of $\frac{du}{dn}$.[2]

Indeed there is, between those two sets of values, an infinity of relations which must be satisfied in order that a corresponding harmonic function should exist."

Since Laplace's equation applies to our case, it is inferred that when we have imposed the appropriate value of ϕ only around the walls of the tank, we have met the essential requirements. This consideration applies also to the boundary marked out on the prototype as well as at the wave basin walls.

---

2 The derivative with respect to distance along the normal to the boundary.

Because of the boundary condition at the free surface, the wave heights and the potential ϕ are related. In each elementary component which contributes to the storm sea undulations, there is a relationship of the form. (Ref. 7, page 75)

$$\zeta = a \sin m(x-ct) \qquad \text{--------- (43)'}$$

$$\phi = \frac{ga}{mc} \frac{\cosh m(z+h)}{\cosh mh} \cos m(x-ct) \qquad \text{---- (44)'}$$

Where: ζ represents the wave height and ϕ velocity potential

a represents a constant

c the wave propagation velocity defined by

$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{(2\pi h)}{\lambda}$

g the acceleration of gravity

h the depth of the water

$m = \frac{2\pi}{\lambda}$ $\pi = 3.14159+$

λ the wave length

t time, and x, y and z are space coordinates.

The x-y plane is the plane of the undisturbed water surface. The coordinate z is measured positive upward.

To establish ζ for this component it is necessary to fix a. When a is fixed so is ϕ. In view of the investigations described above, this should render the problem unique.

To propagate this wave, alone, across the tank the wave generators would have to be set to maintain the appropriate values of ζ as a function of time along the walls. For a sum of such elements the wave generators would, in the same manner, be arranged to maintain the combined wave height at the walls. A replica sea would represent such a sum.

Since the differential equations designated by Hadamard as (e_1), (e_2) and (e_3),[1] differ from Laplace's equation by the inclusion of a term accounting for a compressibility it may be further inferred that the difference in the boundary requirements for a unique solution reflects the difference in the physical nature of the medium and that incompressibility restricts the freedoms which are present in a compressible medium.

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1 His equation (e_3) is of the form:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\omega^2}\frac{\partial^2 u}{\partial t^2} = 0.$$

Pertinent Abstracts from J. Hadamard's "Lecons par la Propagation Des Ondes" (15).

The following abstracts are of interest for our purposes.

From his paragraph 38*

"We have, up to now, occupied ourselves with the problem in which the normal derivative was given on a limited surface. It should not be necessary to believe that this problem and that of Dirichlet, in which the values of the potential function itself are given on the entire surface, are the only ones which we will be able to solve. In the theory of heat one encounters an analogous problem in which enter the values of $\frac{dv}{dn} + hv$ (h being a negative number). But, even in hydrodynamics, one is in general led, not to the problem of Dirichlet or that which we have just treated, but a mixed problem in which the values of the harmonic function for which we are looking are given on a part of the surface and that of its normal derivative on the remaining part of the surface. As in the preceeding case, this problem, if there is a solution, has only one."

From his paragraph 137*

"Let us suppose now that the liquid has a free surface. In this case one no longer knows dp/dn.** But let us suppose that one is given at each point the value of p. We are thus led, this

* Translation from the original French.

** The symbol p represents the pressure; the symbol n, distance along a normal. This is Hadamard's notation.

time, to the mixed problem presented in paragraphs 38 to 41. That is, p is given on the free surface, dp/dn on the walls. It is then certain that the solution of the problem is unique. . ."

These comments have two features of interest for our case because they show, first, that the case of the free surface has been given due consideration and, secondly, that as in the other cases, only one numerical datum is to be imposed on the boundaries. It is clear also that the conditions at a free surface are adequately accounted for if a pressure alone is prescribed there.

An Alternate Approach

What we need to prove is that if wave heights are specified, as a function of time, around the perimeter of a wave basin, the wave motion which appears in the basin will be the same as that within a similar perimeter marked out on an indefinitely extended water surface if the undulations which are imposed at the wave basin perimeter are the same as those which appear at cooresponding points of the extended surface. For the sake of brevity we will refer to the extended surface as the prototype.

Suppose the wave basin perimeter is provided with a large number of wave generators, arranged with cam control, which will cause each of them to maintain a wave surface elevation at its location according to a prearranged program. It is further assumed that these prearranged levels will be maintained even though waves may arrive at its location from other points.

Consider a selected one of these generators and the corresponding point of the extended surface of the prototype. At the corresponding point we impose a disturbance in the form of a local displacement varying in a prescribed manner with time.

The wave motion produced by such a disturbance is represented by the expressions

$$\phi = \zeta_0 \int_0^\infty \frac{g \sin\sqrt{\frac{gt^2u}{a_1}}}{\sqrt{\frac{gu}{a_1}}} e^{\frac{uz}{a_1}} J_1(u) J_0\left(\frac{r}{a_1}u\right) du \qquad \text{-----(46)}$$

$$\zeta = \zeta_0 \int_0^\infty J_1(u) J_0\left(\frac{r}{a_1}u\right) \cos\sqrt{\frac{gt^2u}{a_1}}\, du \qquad \text{------(47)}$$

Where:

a_1 represents a specified radius

g the acceleration of gravity

J_n a Bessel function of order n

r a radius

t time

u a variable of integration

z distance measured positive upward from the undisturbed surface

ζ a wave height

ζ_0 a specified initial elevation

ϕ the velocity potential

$e = 2.71828+$ The base of the natural system of logarithms.

The solution(46) represents the effect of releasing a right cylinder of water of radius a_1 and height ζ_0 at the origin at time zero. The wave motion which ensues is given by (42) in terms of the wave height at the radius r at the time t.

This solution meets the boundary conditions

$$\zeta = \frac{1}{g}\frac{\partial \phi}{\partial t} \quad \text{at} \quad t = 0.$$

$$\frac{\partial \zeta}{\partial t} = -\frac{\partial \phi}{\partial z} \quad \text{at} \quad z = 0. \qquad \text{----}(48)$$

$$\zeta = \zeta_0 \quad \text{for } 0 < r < a_1 \text{ when } t = 0.$$

$$\zeta = 0 \quad \text{for } r > a_1 \quad \text{when } t = 0.$$

The potential ϕ satisfies Laplace's equation:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

The volume of water $\pi a_1^2 \zeta_0$ can be considered as an elementary displacement produced by a wave generator. We may use these formulas to compute the wave motion at any point of the prototype water surface and, in particular, we may compute the wave motion at each of the points corresponding to the location of a wave generator on the perimeter of the wave basin. Suppose we do this and program each of the generators to maintain the computed levels at its location and at the same time we impose the arbitrary prescribed disturbance on the selected wave generator previously described. Under these conditions we may expect that the wave motion which appears in the wave basin will be exactly the same as that which appeared within the corresponding area of the prototype because the wave generators of the wave basin maintain the conditions which prevail at the corresponding points of the prototype and there is therefore nothing at the perimeter of the wave basin which departs from the conditions of the prototype.

If we select two generators and impose prescribed motions at the corresponding points of the prototype surface, we may compute the wave heights produced by each separately and superimpose them to find the disturbance produced when they act simultaneously. In this manner we may compute the wave heights at each of the generator locations as before and if we program each of them to maintain the computed wave heights at its location, we will again observe that the wave motion in the wave basin duplicates the prototype wave motion in the corresponding area. In this case, however, the motion imposed at the selected wave generators must be the sum of the prescribed wave motion and the undulations arriving at its location from the other selected generator. Then the wave heights at the boundary of the wave-basin again duplicate those at the corresponding prototype points and there is nowhere any departure from prototype conditions.

In this same manner we may impose on the prototype an arbitrary disturbance at each of the points corresponding to the position of a wave generator and compute the wave heights at any one generator as a sum of the wave heights produced by each separately. The computed wave heights will in each case include those produced by the arbitrary displacements imposed at its location. This is to say that the computed wave height at any location includes the effects of all of the generators. If we now compare the prototype and wave-basin wave motions, we find them again identical because the prototype conditions are maintained around the perimeter of the wave basin and there is nowhere any departure from prototype conditions.

In this case we have seen how a solution over an area may be constructed by imposing wave heights over the boundary of a closed perimeter and we note that the wave heights alone are sufficient for this purpose. Any attempt to add a flow across the boundary would have altered the boundary wave heights and, through them, the wave motion on the enclosed area. If the wave generators are set to reproduce the wave height at the boundaries of the wave basin which would be produced by a simple or complex wave train traveling over the prototype surface, we should expect to find a wave motion in the wave basin which would duplicate that of the original wave train of the prototype. This would be the case because the boundary wave heights alone are sufficient to determine the wave motion over the enclosed area.

For the sake of simplicity, these arguments have been presented as though the wave basin and the prototype had the same scale. If the appropriate similarity conditions are met, these arguments should apply where the model basin and prototype have different scales.

To complete this approach it may be worthwhile to consider whether the solution constructed in the manner described is unique. We may do this by supposing that there might be another solution. If this is possible it could be obtained by adding a solution of Laplace's equation to the one we now possess. The solution added would represent the difference between the solution obtained and the supposed alternate solution. The solution added must, of course, conform to the pressure conditions at the free surface. Because the proper boundary conditions were imposed in obtaining the first solution, it will not be permissible to add anything to them. This means that a condition of zero wave height would have to be imposed on the second solution around the walls of the model basin. This last restriction insures that the second solution must be zero everywhere because the boundary condition at the free surface and at the bottom of the tanks are inherently appropriate. If no wave motion can be generated around the walls of the tank, then no wave motion can be created anywhere. The second solution must therefore be nil. The solution originally obtained is unique because no alternate solution can be obtained which differs from it. It goes without saying, of course, that no pre-existing wave motion in the basin would be permissible. Any attempt to produce a specific type of wave motion in the presence of such interference would be futile.

Model Prototype Relationships

If the storm sea can be considered to be composed of elementary waves which are distributed in a random manner with respect to amplitude, length, phase and orientation, then each elementary deep water wave will be of the type: (7)

$$\zeta_p = a_p \sin \frac{2\pi}{\lambda_p}\left(x_p - c_p t_p\right) \qquad \text{------}(49)$$

$$\phi_p = \frac{a_p g \lambda_p}{2\pi c_p} e^{\frac{2\pi}{\lambda_p} z_p} \cos \frac{2\pi}{\lambda_p}\left(x_p - c_p t_p\right) \qquad \text{------}(50)$$

where the subscript p represents a prototype quantity. For model representation a scale ratio n_2, less than unity, is chosen so that the model wave amplitudes and wave lengths are n_2 times those of the prototype. Then if the subscript m indicates a model quantity substitution of the relations:

$$n_2 = \frac{\lambda_m}{\lambda_p} \qquad a_m = \frac{\lambda_m}{\lambda_p} a_p \qquad C_m = \sqrt{\frac{g\lambda_m}{2\pi}} \qquad C_p = \sqrt{\frac{g\lambda_p}{2\pi}}$$

$$t_m = \sqrt{\frac{\lambda_m}{\lambda_p}}\, t_p \qquad \zeta_m = \frac{\lambda_m}{\lambda_p} \zeta_p \qquad x_m = \frac{\lambda_m}{\lambda_p} x_p \qquad z_m = \frac{\lambda_m}{\lambda_p} z_p$$

$$\phi_m = \left(\frac{\lambda_m}{\lambda_p}\right)^{\frac{3}{2}} \phi_p \qquad \frac{C_m t_m}{\lambda_m} = \frac{C_p t_p}{\lambda_p} \qquad \text{------}(51)$$

into the above equations will yield an identical set of equations with m subscripts. Note that these relationships apply to all of the components regardless of amplitude or wave length.

In both cases the Laplace differential equation applies. Then if we impose in the wave basin boundary conditions which correspond with those of the prototype and these boundary conditions render the solution unique, the wave motion in the wave basin must be a replica of the prototype sea.

Example

It is of interest to see how this would work out in a simple case. Suppose we wish to propagate a wave of the type of equation (43) down the length of the wave basin. This we know we can do because it has been done many times.

The horizontal velocity is:

$$u = -\frac{\partial\phi}{\partial x} = \frac{g a_2}{c}\,\frac{\cosh m(z+h)}{\cosh mh}\sin m(x-ct) \qquad \text{---}(52)$$

If b_2 represents the width of the wave basin the volume displacement D_2 at any cross section is obtained by integration of the velocities in the manner:

$$D_2 = b_2\int_{-h}^{0} u\,dz \qquad \text{----}(53)$$

If this integration is performed we obtain,

$$D_2 = \frac{g a_2 B_2}{mc}\,\frac{\sinh mh}{\cosh mh}\sin(x-ct) \qquad \text{---}(54)$$

In this case we can generate the wave by generating the velocities u of Eq. 52 or by creating the displacement D_2 of Eq. 54. These are equivalent because of the incompressibility factor. These are both equivalent to a generation of the wave by imposing a gradient $-\frac{\partial\phi}{\partial x}$.

But this gradient, the potential ϕ and the wave height are connected through the factor a_2. They are not independent. We may not, therefore, impose both ϕ and $-\frac{\partial\phi}{\partial x}$ at the boundary separately. Note that in this case a unique result is obtained by imposing a wave height alone at the boundary. This is in accord with the mathematical work which indicates that one boundary condition only can be imposed when Laplace's differential equation applies.

It is also of interest to consider the behavior of the wave generators in this case. The generators along the walls would not move because the water surface elevations called for by their cams would be satisfied. The generators at the downstream end of the tank would absorb the wave motion. This is because while the wave motion would correspond to the cam requirements at the time of arrival, the reflection of the wave at the end of the tank would tend to increase the water level above that called for. The wave generator would then act to prevent this.

Conclusions

The evidence available leads to the conclusion that the fixing of wave heights completely around the boundary of an area will uniquely determine the wave motion over the enclosed area. This statement applies to model and prototype alike. If the wave motion around the walls of a model basin is made to be a replica of that observed at corresponding points of a storm sea the wave motion in the area within the wave basin boundaries also will reproduce the storm sea after a sufficient time has elapsed, after starting the wave generators, for a wave to travel the long dimension of the model basin. The replica sea so produced will represent the storm sea in both space and time if the time scale in the model is made appropriate for the scale factor chosen. The uniqueness considerations apply with equal force to the model and prototype conditions. If the proper similitude relationships are maintained then the mathematical studies would indicate that the replica sea must appear on the wave basin.

Comments

A person who saw the original storm sea and the replica sea produced in the way described might find the model sea somewhat unconvincing. This is because the driving effect of the wind would be absent. This factor produces many of the most impressive phenomena in an actual storm. The lack of the wind driving force would, in fact, produce an error in the replica sea of some magnitude. It can only be assumed that the storm sea observed had developed over a reach which was long compared to the prototype area equivalent to the model basin dimensions and that therefore the absence of a wind drive will not distort the waves seriously during the travel across the model basin. The wind drive could, of course, be arranged.

The accuracy of the replica sea could be checked by taking stereophotos at specific times and comparing them with the original stereos from which the storm sea data were derived.

VI. A NEW TYPE OF WAVE GENERATOR

Wave Generator Requirements

Reproduction of an observed storm sea, to model scale, in a wave basin would provide a design tool of unusual value because it would permit model tests to be made under conditions truly representative of the seas to be encountered on the open ocean. Previous studies have indicated that such a reproduction can be made if the storm sea undulations are imposed, to model scale, around the entire perimeter of the wave basin. The older types of wave generators would not be adaptable for these uses because of the programing difficulties presented by reflected waves. The new type of wave generator described here will automatically compensate for the effects of reflected waves while maintaining the programmed surface levels at its location. The studies described in the following paragraphs relate to stability and to the closeness of attainment of the programmed levels.

Wavemaker Arrangement

The general arrangement of the wavemaker is shown schematically in Fig. 13. Alternative spool valve arrangements are shown in Fig. 14. A plunger type of wavemaker is illustrated in Fig. 15 and the spool valve suitable for this case would be as shown in Fig. 14a. A pneumatic type of wavemaker would need a spool valve similar to the one shown in Fig. 14b. The program input is represented by a cam but an electrical signal would probably be used if a true storm sea representation was to be made.

Preliminary Considerations

Even though an analysis may indicate certain capabilities, actual realization may still be an impossiblility because of the limitations of the types of mechanism which may have to be used. It is considered to be justified, therefore, to investigate an actual design to see whether such impossibilities may be encountered. Such computations would also serve as a guide for future use in preparing designs for this type of wavemaker. Since the expense of equipping a wave basin with wave generators for production of a replica sea would be large, the design construction and testing of an experimental wavemaker would be desirable. This design will relate to an experimental plunger type of wavemaker 2 feet in length, driven by an oil operated servo-cylinder to produce waves of wave length λ = 5.0 feet and amplitude a = 0.25 feet. The design will incorporate commercially available parts when they are obtainable.

For effective operation it will be necessary to have the natural period of the plunger short compared to the period of any wave to be produced. A ratio of 1 to 4 would be desirable. The period of a deep water wave of 5 foot wave length is given by a relation of the type.

$$T = \sqrt{\frac{2\pi\lambda}{g}} \qquad (55)$$

In this case the period comes out to be 0.988 seconds. For our present purposes this may be rounded out to 1 second. If 5 feet is the shortest wave length to be produced then a natural period of 1/4 second would be satisfactory. It is unlikely that a plunger type wave generator with a hydraulic drive would have too long a period but a pneumatic type might give trouble. For purposes of illustration we may digress long enough to estimate the period of such a wave generator having a capacity suitable for the conditions of our example. We must first select some dimensions. The volume displacement of the water particles from an initial position of rest, due to the passage of a wave is (Lamb, p. 368)

$$x = ae^{kz} \cos(kx - \sigma t) \qquad \text{---(56)}$$

where

$$\sigma^2 = gk \qquad k = \frac{2\pi}{\lambda}$$

and λ represents the wave length. The horizontal and vertical coordinates are x and z. The positive direction for z is up. The coordinate x is measured in the direction of wave travel. Each particle describes a circle with a constant angular velocity σ. The symbol t represents time. The volume displacement which the wave generator must produce to create the wave amplitude a and length λ is

$$d = \pm a \int_{-\infty}^{0} e^{kz} dz = \pm \frac{a\lambda}{2\pi} \qquad \text{----(57)}$$

This volume d is the displacement required per unit length of wave generator. Both d and a are reckoned as departures from a mean value. In the present case

$$d = \pm \frac{a\lambda}{2\pi} = \pm \frac{(0.25)(5)}{2\pi} = \pm 0.2 \text{ cubic feet per foot(nearly)}$$

This could be amply provided for by a wave generator 0.5 feet wide, in the direction of wave travel, and immersed to a depth of 0.75 foot as shown in figure 15. It will be assumed that the top of the wave generator is 1.5 feet above the equilibrium level. For each unit of length of the generator the spring constant, for small oscillations, is provided by an air cushion 1.5 feet deep and the mass is represented by the 0.75 depth of water in the wave generator plus an addition due to the hydrodynamic effect of the water which must be set into motion outside the wave generator. The equivalent mass of the water outside the wave generator may be visualized as a half cylinder of water which moves with the water in the wave generator. The arrangement is illustrated in Fig. 15.

If a weight W departs from a position of rest by the amount y_1 and this departure is resisted by some sort of a restoring force proportional to y_1 the differential equation of motion is

$$\frac{W}{g}\frac{d^2y_1}{dt^2} + Ky_1 = 0 \qquad \text{--- (59)}$$

where g represents the acceleration of gravity and K is the spring constant. The product Ky_1 is the force tending to restore the weight W to the position of rest.

For our case the spring constant K for small oscillations, can be obtained from the isothermal pressure law

$$pv = p_o v_o \qquad \text{--- - - (60)}$$

where p and v are the absolute pressure and volume respectively and the subscripts o indicate the conditions at the position of rest. Then

$$p = \frac{p_o v_o}{v} \qquad \text{- - - - (61)}$$

and

$$\frac{dp}{dv} = - \frac{p_o v_o}{v^2} \qquad \text{- - - -(62)}$$

If the area is A the restoring force is Adp and the increment of volume is $dv = -Ady_1$, we may, for small displacements y_1 take $v = v_o$ so that

$$Adp = \frac{p_o A^2 \, dy_1}{v_o} \qquad \text{- - - - -(63)}$$

The spring constant is then

$$K = \frac{Adp}{dy} = \frac{p_o A^2}{v_o} \qquad \text{- - - - -(64)}$$

At an altitude where the atmospheric pressure is about 12.2 lb/in^2 or 1750 lb/ft^2 we would have, for a unit length of wave generator

$$K = \frac{p_o A^2}{v_o} = \frac{(1750)(0.5)^2(1)^2}{(0.5)(1)(1.5)} = 585 \text{ lb/ft} \qquad ---(65)$$

The weight, based upon 62.4 lb/ft^3 as the density of water is

$$W = (62.4)(0.5)(1)\left[\ 0.75 + \frac{\pi}{8}\ 0.5\ \right]$$
$$= 29.5 \text{ lbs.}$$

A solution of the differential equation is

$$y_1 = A \sin\sqrt{\frac{Kg}{W}}\,t + B \cos\sqrt{\frac{Kg}{W}}\,t \qquad ----(66)$$

A complete cycle will occur in the period T when

$$\sqrt{\frac{Kg}{W}}\,T = 2\pi \qquad ----(67)$$

the period is then

$$T = 2\pi\sqrt{\frac{W}{Kg}} \qquad -----(68)$$

In our case the period will be

$$T = 2\pi\sqrt{\frac{29.5}{(585)(32.2)}}$$
$$= \frac{\pi\ 2}{25.25} = 0.249 \text{ seconds}$$

This would do for wave lengths longer than 5 feet but would be too slow for wave lengths shorter than this.

An alternative form of wave generator is shown in Fig. 13. This is in the form of a wedge which is moved vertically by means of a hydraulic cylinder. If the wedge is 2 feet high, 1 foot wide at the top and the plunger is 2 feet long, the cross-sectional area at mid height will be 1 sq. foot. The previously estimated displacement required to produce the wave of amplitude 0.25 feet and wave length 5 feet is 0.20 cu. foot per foot of length of wave generator. The total displacement required for the 2 foot length of plunger is then $(2)(0.2) = 0.4$ cu. feet. This can be obtained with a vertical movement of approximately 0.4 feet in either direction. The natural period of a plunger of this type would be determined by its own mass, plus the equivalent mass of water, and the spring constant provided by the rigidity of the hydraulic drive. These rigidities would provide a very large value for the restoring force K and it seems quite certain that its period would be short compared to that of any wave it would be desired to produce.

A similar problem is present with the float system because good performance of the wave generator could not be obtained if the float responded sluggishly to level changes. Suppose we try a float made of a foam plastic block 18 in. long, 4 in. wide and 2 in. high, to float with the 2 in. dimension [illegible] vertical. If the density of the plastic is 12 lbs. per cu. foot the weight of this float will be 1 lb. The equivalent mass of water will be nearly that contained in one half of a cylinder of water 4 in. in diameter and 18 in. long. The volume of the float is 1/12 cu. ft. and the volume of the equivalent mass is

$$\frac{1}{12} \; \frac{\pi}{4} = 0.0655 \text{ ft}^3$$

The density of water is 62.4 lb/ft^3 and the weight of the equivalent mass is therefore

$$(62.4)(.0655) = 4.08 \text{ lbs}$$

If the attachments weigh an additional pound the total weight would be

$$1 + 4.08 + 1 = 6.08 \text{ lbs.}$$

The restoring force would be produced by floatation in this case. The horizontal cross-section of the float is 0.5 sq. ft. Then the K value is

$$K = (62.4)(0.5) = 31.2 \text{ lb/ft}$$

The natural period is then, by use of equation 68

$$T = 2\pi \sqrt{\frac{W}{Kg}}$$

$$= 6.2832 \sqrt{\frac{6.08}{(31.2)(32.2)}} = 0.489 \text{ seconds.}$$

This is too slow to follow satisfactorily a wave of even 1 sec. period since about a 4 to 1 ratio is needed for close following. A cylinder with its axis vertical, as shown in fig. 16, does somewhat better because the equivalent mass of water (Lamb, p. 124) is 1/2 of that contained in a hemisphere having the same diameter as the float but the improvement is not sufficient to make it a satisfactory device. We can overcome these difficulties by increasing the restoring force. A cantilever spring mounting as shown in Fig. 16 could be made to accomplish this.

To complete the design we chose to operate the plunger with a 1-1/2 in. diameter double acting hydraulic cylinder with 6 in. stroke, connected to the float by a lever having a 5 to 1 ratio. At 500 lbs. per sq. in. oil pressure this will exert 176 lbs. of thrust at the plunger. This is ample to completely submerge the plunger.

For a sinusoidal displacement of the plunger of 0.4 ft. and a frequency of 1.0 cycle per second required to produce the wave of wave length 5 ft. and amplitude 0.25 ft. the displacement would be of the form:

$$\gamma_3 = 0.40 \sin \frac{2\pi}{T}(t + \epsilon) \qquad \text{----(69)}$$

Where ϵ is a constant specifying a phase position. The corresponding velocity would be

$$\frac{d\gamma_3}{dt} = 0.40 \frac{(2\pi)}{T} \cos \frac{2\pi}{T} (t + \epsilon) \quad ---(70)$$

The maximum rate will occur when $\cos \frac{2\pi}{T} (t + \epsilon) = 1$. Then with $T = 1$ second

$$\left(\frac{d\gamma 3}{dt}\right)_{max} = 0.40 \frac{(2\pi)}{T} = \frac{(0.40)(6.2832)}{1} = 2.51 \frac{ft}{sec} \quad --(71)$$

After taking the lever ratio into account the cylinder rate would be $\frac{30.12}{5} = 6.02$ in/sec. With a 1-1/2 in. diameter drive cylinder of area 1.77 sq. in., the maximum oil supply rate will be

$$(6.02)(1.77) = 10.63 \text{ cu. in/sec}$$

This is equivalent to

$$\frac{(10.63)(60)}{231} = 2.76 \text{ gallons per minute}$$

Solenoid operated hydraulic servo-valves are commercially available. One such valve* which would be suitable for our purposes has the following characteristics:

---

* Pegasus Model 120-F Electro Hydraulic Servo Valve. Pegasus Laboratories Inc., 3690 Eleven Mile Road, Berkeley, Michigan. The price quotation for one of these valves is $462.00. It will be assumed for computation purposes later that the capacity at 500 lb/in^2 is 3.50 g.p.m.

Capacity (1000 lb/in^2) 5.0 gpm

Operating pressure 200 to 1000 lb/in^2

Differential current 40 milliamperes

Coil resistance 1200 ohms

Coil inductance 2 Henries

This valve will follow 20 cycles per second with less than 10 degrees phase lag.

Two pair of resistance type strain gage elements are shown mounted on the float restoring spring of Fig. 16. These can be arranged in a temperature compensated bridge which will produce an electrical signal indicative of the float position. A similar signal would be obtained from an electrically recorded program. These two signals would be compared by a vacuum tube circuit comprising a power supply and an amplifier. The amplifier would be connected to the solenoid of the electro-hydraulic servo-valve and would cause it to act if the wave height, as indicated by the float, disagreed with the program wave height. If the maximum flow rate for the valve is 3.5 gallons per minute at 500 pounds per square inch pressure it would be within the capacity of the wave generator and its controls to make the servo-piston travel at its maximum rate if the actual wave height departed from the program wave height by 0.1 ft. An analyses of the performance of such a system will be carried through as an example. A reference to figures 13 and 14 will show that if a positive motion α_3, as in Fig. 13, is imposed on the spool valve of Fig. 14a the upper port will be opened to the pressure supply and the lower port will be opened to the exhaust. Then the piston of the servo-cylinder will move down and the plunger will also move down. This will produce a local rise of the water surface ζ_3 in the wave basin (Fig. 13). The float will respond to this and raise the sleeve (figures 13 and 14). When the level specified by the motion α_3 (Fig. 13) is reached the ports will be closed (Fig. 14) and motion of the plunger will cease.

Let

α_3	represent the ratio of the spool valve displacement to the port width-positive if it produces a positive motion of the plunger. (See figures 13 and 14a).
β_3	The ratio of the sleeve valve displacement to the port width-positive if it produces a negative motion of the plunger. (See figures 13 and 14a).
γ_3	The displacement of the plunger from the neutral position-positive down. (See Fig. 13)
ζ_3	Departure from normal level in the wave basin-positive up. (See Fig. 13).
m_1, m_2, m_3, etc.	Positive constants.
t	time

The equations of motion are

$$
\begin{aligned}
\frac{d\gamma_3}{dt} &= m_1(\alpha_3-\beta_3) - m_2\gamma_3 \\
\zeta_3 &= m_3\frac{d\gamma_3}{dt} \qquad \text{- - - - (72)} \\
\beta_3 &= m_4\zeta_3
\end{aligned}
$$

In these expressions the constant m_1 relates to the relation between the rate of plunger movement and the departure from the closed position of the valve. In our case with $(\alpha_3-\beta_3) = 1$, which would indicate a wide open valve, there would be a flow of oil to the servo-cylinder at the rate of 3.5 gallons per minute. This will give a float travel of 3.18 ft/sec which, it will be noted, is not greatly in excess of the 2.51 ft/sec previously computed as being required for the production of a wave of amplitude 0.25 ft. and wave length 5 ft. Then with $(\alpha_3-\beta_3) = 1$, for a wide open valve and $d\gamma_3/dt = 3.18m_1 = 3.18$ ft/sec. The constant m_2 relates to a leakage which is provided so that the plunger may be restored to its neutral position. This leak permits oil to pass from one end of the servo-cylinder to the other. If this were not provided a slight maladjustment would cause the machine to run until the plunger were either submerged or out of the water, thereby rendering the wave generator inoperative. We will return to a consideration of the value of this constant later. The value of m_3 can be obtained from previous computations which indicate that a wave amplitude 0.25 ft. can be generated by a plunger motion of 2.51 feet per second. Then

$$m_3 = \frac{0.25}{2.51} = 0.10 \text{ seconds.}$$

The constant m_4 can be evaluated from a choice such that a rise of 0.1 ft. in the water surface should, in the absence of other changes, open the valve ports wide then

$$1 = m_4 \, 0.1$$

and

$$m_4 = 10 \text{ 1/ft.}$$

Then we have, as a set of trial values

$$m_1 = 3.18 \text{ ft/sec}$$

$$m_2 = \text{(to be determined later) 1/sec}$$

$$m_3 = 0.10 \text{ seconds}$$

$$m_4 = 10.0 \text{ 1/ft.}$$

Elimination of the variable γ_3 from the equations 72 yields the ordinary differential equation

$$\frac{d\zeta_3}{dt} + a_3\zeta_3 = b_3\frac{d\alpha_3}{dt} \qquad ---(73)$$

where

$$a_3 = \left(\frac{m_2}{1 + m_1 m_3 m_4}\right) \qquad b_3 = \left(\frac{m_1 m_3}{1 + m_1 m_3 m_4}\right) \qquad \text{---} (74)$$

If the program demands, for example, that

$$\alpha_3 = \alpha_o \sin \frac{2\pi}{T} t \qquad \text{----}(75)$$

then

$$\frac{d\alpha_3}{dt} = \left(\frac{\alpha_o 2\pi}{T}\right) \cos \frac{2\pi}{T} t \qquad \text{----}(76)$$

and a solution of equation 73 is

$$\theta_3 = \frac{b_3 \alpha_o 2\pi}{T} \frac{\left(a_3 \cos \frac{2\pi}{T} t + \frac{2\pi}{T} \sin \frac{2\pi}{T} t\right)}{\left(a_3^2 + \frac{4\pi^2}{T^2}\right)} + C_3 e^{-a_3 t} \qquad \text{---}(77)$$

We may now return to a consideration of the constant m_2. We note that it occurs in the numerator of the quantity a_3 and that this is related to the rate at which the plunger will return to its neutral position. Suppose, for example, we adjust the leak so that the plunger will return through 0.1 of an initial displacement in one period. Then we could write

$$0.9 = e^{-a_3 T}$$

In our case, with $T = 1.0$ and, from tables of the exponential function, $a_3 T = 0.1$ approximately, we can conclude that $a_3 = 0.1/1.0 = 0.1$. Then

$$a_3 = \left(\frac{m_2}{1 + m_1 m_3 m_4}\right) = 0.1$$

$$m_2 = 0.1\left[1 + (3.18)(0.10)(10)\right] = 0.418$$

We may now evaluate the performance of the wave generator.

$$a_3^{\,2} = 0.01$$

$$\frac{4\pi^2}{T^2} = \frac{(4)(9.8696)}{1.00} = 39.5$$

$$\left(a_3^{\,2} + \frac{4\pi^2}{T^2}\right) = 39.51$$

$$b_3 = \left(\frac{m_1 m_2}{1 + m_1 m_3 m_4}\right) = \frac{(-3.18)(0.10)}{4.18} = 0.0762$$

$$\frac{a_3 \frac{2\pi}{T}}{\left(a_3^2 + \frac{4\pi^2}{T^2}\right)} = \frac{(0.1)(6.2832)}{39.51} = 0.0159$$

$$\frac{\left(\frac{2\pi}{T}\right)^2}{\left(a_3^2 + \frac{4\pi^2}{T^2}\right)} = \frac{39.5}{39.51} = 1.00$$

We notice several things. The program demands an amplitude

$$\alpha_3 = \alpha_0 \sin \frac{2\pi}{T} t \qquad ---(78)$$

After the transient, as represented by the exponential term has died away, the machine yields a wave having both a sine and a cosine term. We may consider the cosine term as a distortion since we seek a sine wave. If we measure the distortion by the ratio of the amplitudes of the cosine and sine terms we obtain a distortion of approximately

$$\frac{a_3 T}{2\pi} \qquad \text{if} \quad a_3 \ll \frac{2\pi}{T} \qquad ---(79)$$

In the present case the distortion is only about 0.0159/1.00. This would be an acceptable ratio. The distortion is due to the leak since the expression for a_3 has the quantity m_2 in the numerator. The response we get is

$$\zeta_3 = b_3 \alpha_0 (0.0159 \cos \frac{2\pi}{T} t + 1.00 \sin \frac{2\pi}{T} t \qquad ----(80)$$

If b_3 were nearly 1.00 instead of 0.242, the performance would be satisfactory. It would not be satisfactory with the constants chosen. We must change the adjustments to improve the operation of the wavemaker. We need, approximately,

$$b_3 = \left(\frac{m_1 m_3}{1 + m_1 m_3 m_4}\right) = 1.0$$

We cannot change m_3 but we can change m_1 and m_4. We solve this equation for the product $m_1 m_3$ to obtain

$$m_1 m_3 = 1 + m_1 m_3 m_4$$

$$m_1 m_3 \ (1 - m_4) = 1$$

$$m_1 m_3 = \left(\frac{1}{1 - m_4}\right)$$

It is apparent that m_4 must be less than 1. If we make $m_4 = 0.5$, then $m_1 = 20$. Then $m_1 m_3 = (20)(0.1) = 2$ $m_1 m_3 m_4 = 1$.

$$a_3 = 0.1 \qquad\qquad b_3 = \frac{2}{1 + 1} = 1.00$$

This would be a satisfactory performance. The high amplifier gain implied by $m_1 = 20$ would not cause trouble unless the capacity of the servo- valve was exceeded. This will not occur in the present case. The distortion is less than 2 percent. The leak would need to be adjusted to restore the value of a_3 to 0.1. This is accomplished if $m_2 = 0.2$.

It remains to investigate the performance of the machine as a wave absorber. Reference to figures 13 and 14a will show that the incoming wave crest would raise the float which, in turn, would raise the sleeve. With α_3 adjusted to zero, so that the machine is set to maintain the normal water level, the lower port will be opened to pressure and the upper one to exhaust. The servo-cylinder piston will move up and the plunger will also move up. This will create a local water surface depression which the float will follow. When normal level is restored the machine ceases to move. The capacities required for a wave absorber should be comparible to those for a wave maker if the wave amplitudes and wave lengths are comparible. Suppose an incoming wave makes

$$\beta_3 = m_4 \left(\zeta_3 + 2 \zeta_0 \sin \frac{2\pi}{T} t \right). \quad \text{while } \alpha_3 = 0 \qquad ---(81)$$

The factor 2 takes care of the doubling of the amplitude by reflection.* The differential equation derived from equation 72 now takes the form

$$\frac{d\zeta_3}{dt} + a_3 \zeta_3 = - b_4 2 \zeta_0 \frac{2\pi}{T} \cos \frac{2\pi}{T} t \qquad ----(82)$$

where

$$b_4 = - \frac{(m_1 m_3 m_4)}{1 + m_1 m_3 m_4}$$

A solution is

$$\zeta_3 = b_4 2 \zeta_0 \frac{2\pi}{T} \frac{\left(a_3 \cos \frac{2\pi}{T} t + \frac{2\pi}{T} \sin \frac{2\pi}{T} t \right)}{\left(a_3^2 + \frac{4\pi^2}{T^2} \right)} + C_4 e^{-a_3 t} \qquad ---(83)$$

with the new constants

$m_1 = 20$

$m_2 = 0.20$

$m_3 = 0.10$

$m_4 = 0.50$

---

* If the wave is reflected from the surface of the plunger then the distance between float and plunger must be short compared to a wavelength if this relation is to hold. A short distance is needed for effective operation as a wave absorber.

$m_1m_3 = 2.0 \quad \cdot \quad m_1m_3m_4 = 1.0 \quad b_4 = -0.5$ then

$2b_4 = -1.0$

We then have an acceptable wave absorption, but the absorption is not complete. The amplitude of the outgoing wave is about $aT/2\pi$ times that of the incoming wave. In this case the ratio of amplitudes of the outgoing and incoming waves is $aT/2\pi = 0.1/6.2832 = 0.0159$.

Arrangement of Wave Generators

The arrangement of wave generators around a wave basin for producing a replica of an observed storm sea must be such that the program wave height at each generator can be continually compared with the wave height existing at its location. If there is a difference the wave generator must move in a direction and at a speed which will bring the wave height to that called for by the program. The total number of wave generators must be enough to reproduce the storm sea in sufficient detail for testing purposes.

As an example of the considerations which would be involved in the planning of a wave generator installation of this kind the requirements for an installation on the Colorado State University wave basin will be outlined. This basin is circular and is 80 ft. in diameter. The factors involved when a wave moves at an angle with the wall may be presented more clearly with a basin of this shape than with a rectangular basin. To fix ideas we may suppose that the direction north to south represents the downwind direction with reference to the storm sea to be reproduced.

There is evidence that a storm sea is composed of elementary wave profiles which are, to a degree, random in amplitude, phase, wave length and direction of propagation. We may consider one of these wave elements which runs toward the south. The generators on the north and south portions of the rim will have only the task of generating and absorbing the wave. Those on the east and west sides will have nothing to do because the wave runs transverse to them but those on the northwest and northeast portions of the rim must contribute to the generation of the wave. Those on the southwest and southeast portions must similarly contribute to its absorption. For these generators the wave profile will appear along the wall and there must be enough generators in the wave length to define the wave. We may assume that models of approximately 5 ft. length are to be used in this basin and that it will be important to generate waves of a wave length equal to the model length and that some shorter components should be included. Along the northwest portion of the rim a 5 ft. wave length, of a wave being propagated toward the south, would cover about 7 ft. A one foot length for an individual wave generator would place 7 generators in this length. This should prove adequate since the wave profile should be well reproduced by a control at 7 evenly spaced points in the profile. All resolution would be lost, however, for waves less than 2 ft. length along the wall or 1.4 ft. wave length as measured in the direction of propagation toward the south. We may conclude that this resolution will be satisfactory but we will realize at the same time that some of the sharp-crestedness present in the actual storm sea will be absent from the model reproduction because of the limitations imposed by the individual wave generator lengths. We could only improve the resolution by using shorter wave generators and more of them. We may decide to use 60 wave generators in each quadrant or 240 for the entire perimeter.

An effective arrangement could be built around an electrically recorded program. The programner, which reads this record, could be placed near the rim of the basin and the program wave height could be transmitted to each individual wave generator by a pair of wires. At each generator there would be a power supply and a vacuum tube amplifier. Each generator would have a float just on the wave basin side of it to sense the wave height and convert it to an electrical signal, and a solenoid operated hydraulic servo-valve. The program and wave height signals could be compared by a Wheatstone bridge arrangement whose output would be fed to the grids of the amplifier. The amplifier output would be connected to the solenoid of the hydraulic servo-valve which would control the oil supply to the hydraulic servo-cylinder attached to the wave generator plunger. A small leak in a bypass connecting the two ends of the hydraulic drive cylinder would permit the wave generator plunger to creep back toward its neutral position to prevent the wave generator from exhausting itself as a result of small errors in adjustment to the wave basin level. A cable tray would carry a power line, to supply the vacuum-tube power supplies, and the pairs running from the programmer to the individual wave generators. This power line should have a capacity of about 10 kilowatts. An oil pressure line and an oil return line would encircle the wave basin to supply oil to the drive cylinders under the control of the hydraulic servo-valves and to convey the used oil back to the pump. This oil pressure line would need to be fitted with air tanks at intervals to prevent sluggishness of oil flow due to the inertia of long supply lines. We will assume one of these to be located in the middle of each quadrant. Oil under pressure could be supplied by a pump adjacent to the programmer. This pump would probably consume about 10 kilowatts also.

Some inquiries were sent out to firms who manufacture computing equipment to learn whether some of their devices would be adaptable for use as a programmer. The desired characteristics were outlined as follows:

1. To have up to 240 channels
2. To run for 2 minutes
3. To repeat at the end of the program, if possible
4. To permit an easy substitution of one program for another

5. To be readily reset to the starting position
6. To permit the program to be made up from data plotted in the form of curves
7. To have the program records in a form which is easily stored and not easily damaged.
8. To have the records remain essentially unchanged over a period of several years

The replies received indicate that there is no device presently available which is exactly suited to these needs. This is not surprising since they were made for other purposes. One gets the impression that they either run too fast or do not have enough channels or both. The capacity of these is expressed in "bits" of information. To obtain a rough idea of the capacity which would be required for our purposes we may suppose that the program will run 2 minutes and that the 5 ft. wave length, mentioned previously, would be adequately defined by 10 bits of information per second. The period of this wave is about one second. This decision is somewhat arbitrary but we can get some idea of the limitations of such a choice by noting that the limit of resolution is reached when a wave is defined by two "bits." The period of such a wave would be 0,2 second. The wave length would be given by

$$\lambda = \frac{T^2 g}{2\pi} = \frac{(04)(32.2)}{6.2832} = 0.20 \text{ feet}$$

Although the resolution for a wave of 0.2 second period would be poor, we can accept this quality of resolution as adequate for the wave lengths of primary interest. The total of these "bits" for 240 generators operating for 120 seconds (2 minutes) at 10 bits per second is

$$(240)(120)(10) = 288{,}000 \text{ bits}$$

At least one of the devices described in manufacturers literature had a capacity of 20,000,000 bits. It would appear therefore that presently available devices are potentially capable of storing the amount of data which we would need for our program.

While no device was found which was immediately applicable for our purposes it would be surprising if presently available parts could not be assembled into a suitable programming device. Consider for example, the possibilities of a magnetic tape looped over a series of 20 pulleys equipped with reading heads. Each loop would be long enough to run the full 2 minutes. A magnetic tape 1 inch wide will accommodate up to 14 channels and can be run at as low speeds as 1-7/8 inches per second. A 2 minute program would require

$$(2)(60)(1.875) = 225 \text{ inches or } 18.75 \text{ feet of tape}$$

We could use 20 reading heads with 12 channels each. If we allow 20 ft. of tape between reading heads to provide for some adjustment we would need

$$(20)(20) = 400 \text{ feet of tape to contain the entire program}$$

At this tape speed a response from 0 to 312 cycles per second can be obtained, so it is obvious that there is ample resolution for our purposes. A standard 10-1/2 inch reel will hold 2400 ft. of this tape and since we would need only about 400 feet for our program, we could store 6 programs on a single reel of this size. It therefore seems certain that available devices and parts can be assembled into a programming device suitable for our purposes. A 14 track recorder-reproducer consumes about 1600 to 2100 watts. An elevation of the power requirements for a programmer with 240 reading heads would probably require the making of a detailed design. The required information could also be stored on cams controlling the individual wave generators.

Summary

The results of these studies can be summarized in the following way.

1. It is possible to reproduce an observed storm sea to model scale in a model basin.

2. The replica sea will appear in the model basin if the observed wave heights are reproduced, to model scale, around the entire perimeter of the model basin.

3. Because of the level changes produced by incoming waves a type of wave generator is needed which will sense the wave height, compare it with a programmed height, and act to bring the wave height at its location, to the programmed height.

4. The length of individual wave generators will set a lower limit to the wave length which may be reproduced. In this sense all resolution is lost when the wave length is less than two wave generator lengths.

5. It has been shown how a suitable wave generator may be built and controlled and an application to an 80 ft. diameter wave basin has been outlined.

6. Devices now available could be adapted to the construction of a programmer suitable for the production of a "replica sea."

7. It is probable that pneumatic type wave generators will have too low natural frequencies for wave generators of this type.

8. It may be necessary to provide spring restoring forces for the floats to give them a sufficiently high natural frequency for these purposes.

9. The large amount of information necessary for the production of a replica sea could be stored in an electrically recorded program or on cams controlling the individual wave generators.

10. Suitable solenoid operated hydraulic servo-valves are available commercially.

11. Where an electrically recorded program is used the electric circuit may need close adjustment for satisfactory operation.

12. The mechanism described could be used to produce waves of many patterns and these would be reproducible.

13. The analysis indicates that effective wave absorbing characteristics are possible.

14. Wind forces provided on the model basin would contribute to the faithful representation of storm sea conditions in a model basin.

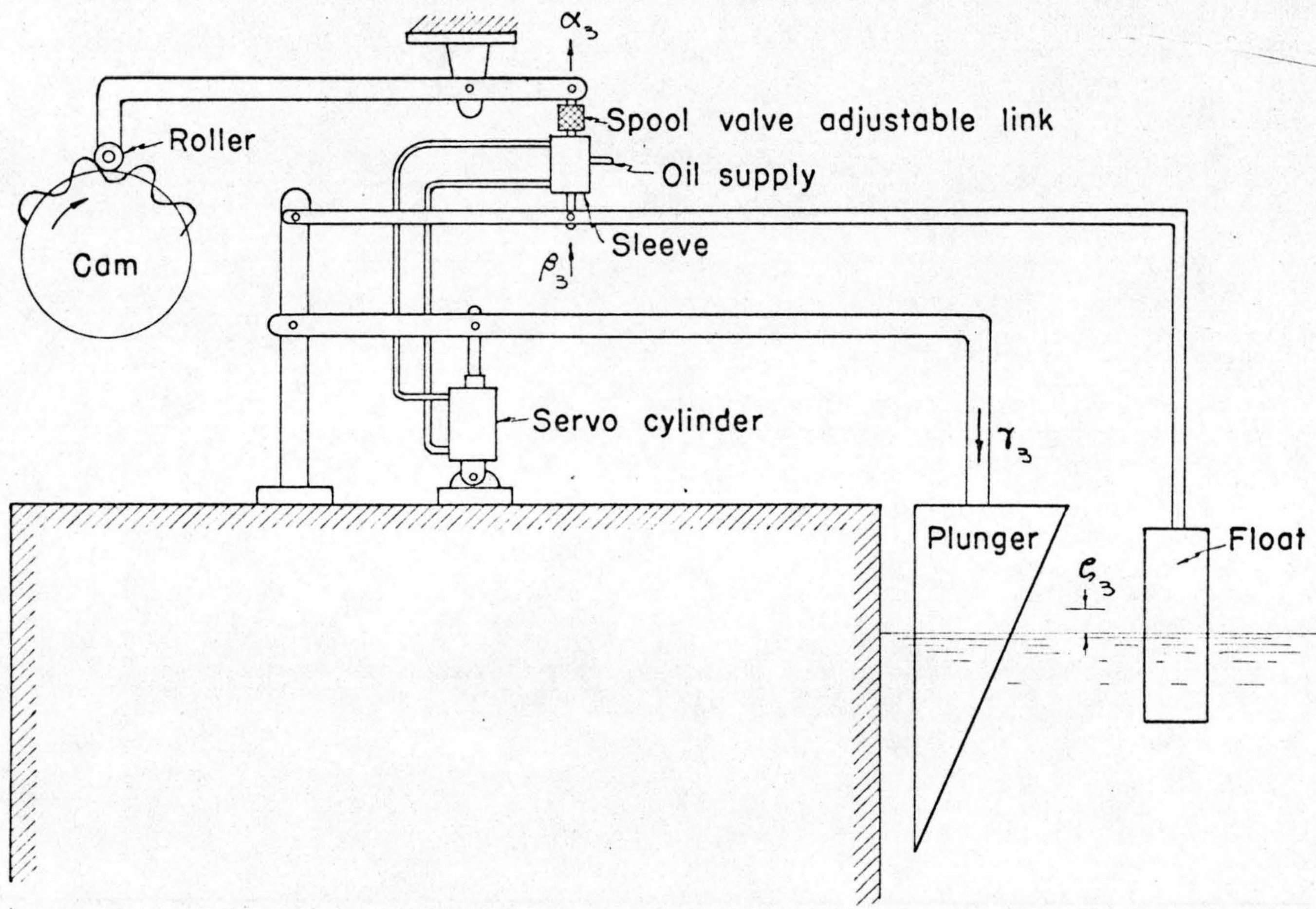
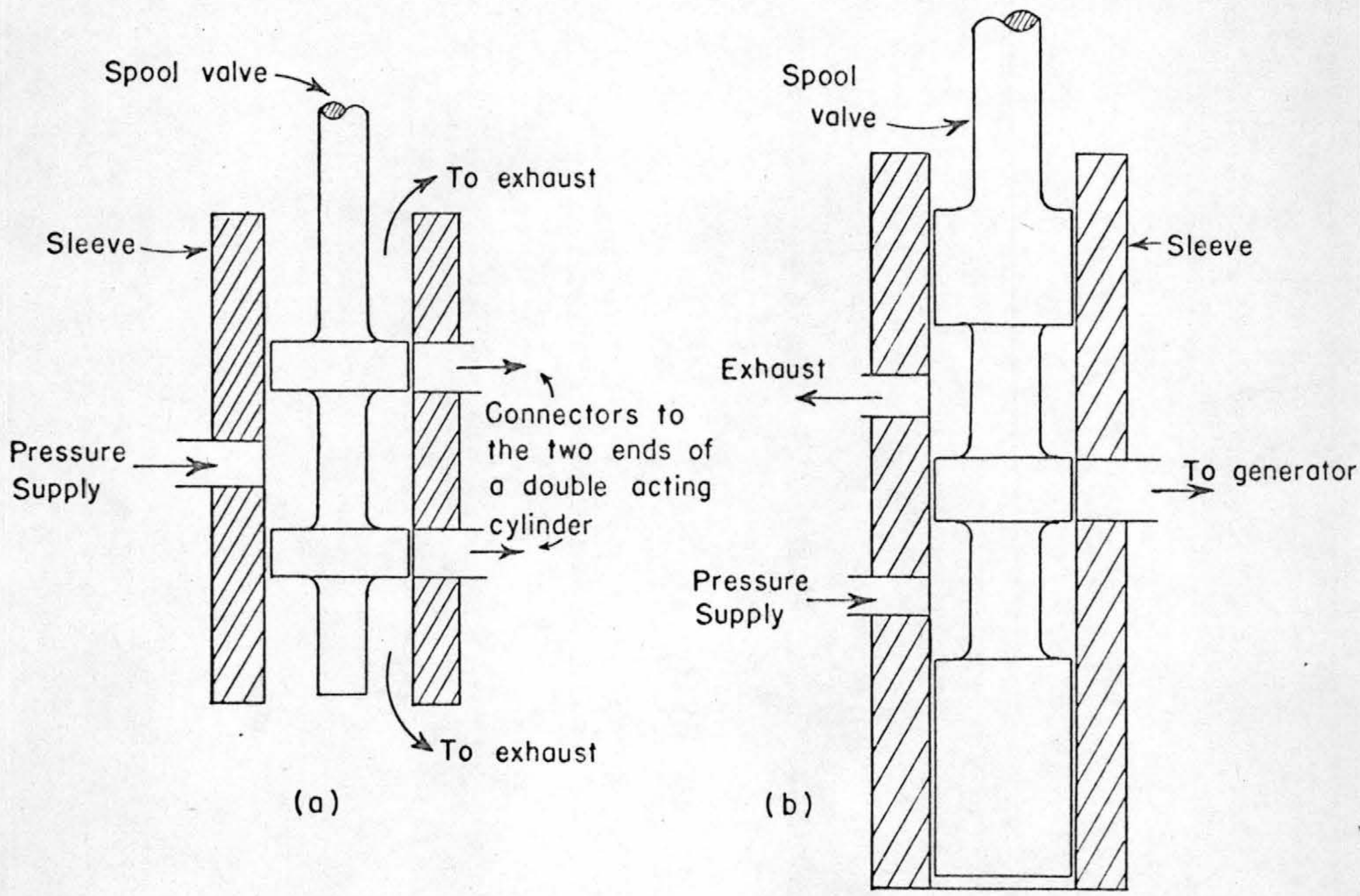


Fig. 13 Wave generator arrangement



(a) For a double acting oil cylinder

(b) For pneumatic wave generator

Fig. 14 Spool valve arrangement

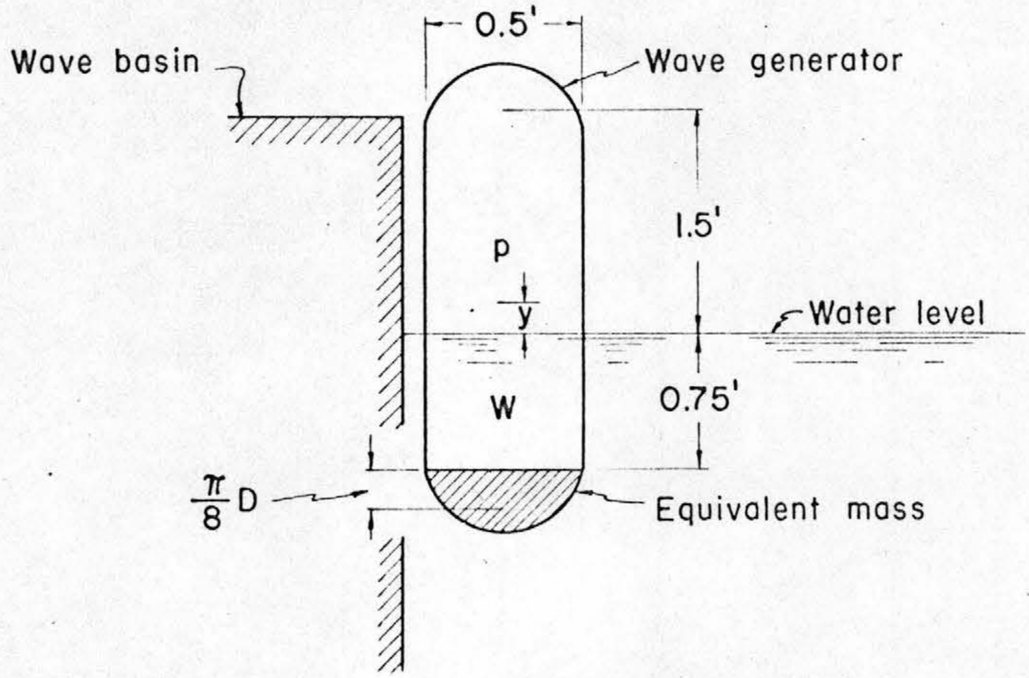


Fig. 15 Pneumatic wave generator

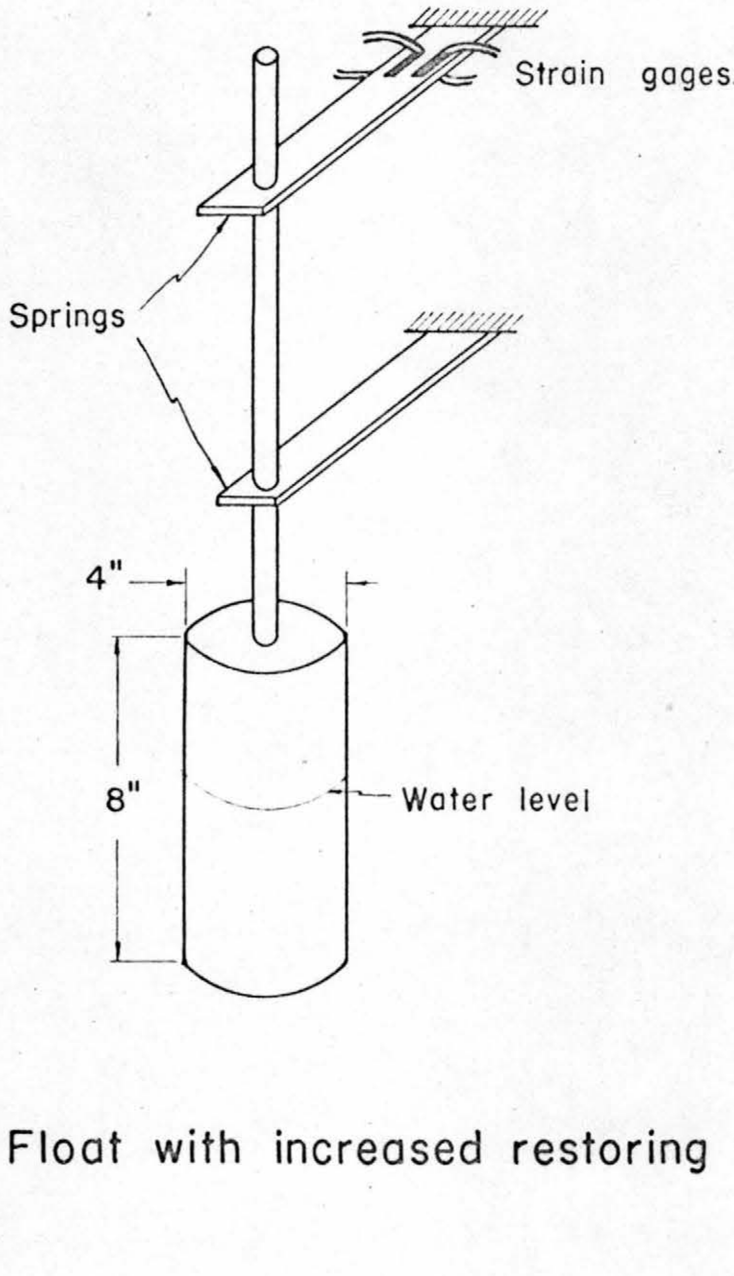


Fig. 16 Float with increased restoring force

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